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Deterministic Creation and Braiding of Chiral Edge Vortices

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Majorana zero modes in a superconductor are midgap states localized in the core of a vortex or bound to the end of a nanowire. They are anyons with non-Abelian braiding statistics, but when they are immobile one cannot demonstrate this by exchanging them in real space and indirect methods are needed. As a real-space alternative, we propose to use the chiral motion along the boundary of the superconductor to braid a mobile vortex in the edge channel with an immobile vortex in the bulk. The measurement scheme is fully electrical and deterministic: edge vortices (π-phase domain walls) are created on demand by a voltage pulse at a Josephson junction and the braiding with a Majorana zero mode in the bulk is detected by the charge produced upon their fusion at a second Josephson junction.

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Introduction.—Non-Abelian anyons have the property that a pairwise exchange operation may produce a different state, not simply related to the initial state by a phase factor [1]. Because such “braiding” operations are protected from local sources of decoherence they are in demand for the purpose of quantum computations [2]. Charge $e/4$ quasi-particles in the $\nu = 5/2$ quantum Hall effect were the first candidates for non-Abelian statistics [3], followed by vortices in topological superconductors [4,5].

Since experimental evidence for non-Abelian anyons in the quantum Hall effect [6,7] has remained inconclusive, the experimental effort now focuses on the superconducting realizations [8]. While the mathematical description of the braiding operation (the Clifford algebra) is the same in both realizations, the way in which braiding is implemented is altogether different: In the quantum Hall effect one uses the chiral motion along the edge to exchange pairs of non-Abelian anyons and demonstrate non-Abelian statistics [9–11]. In contrast, in a superconductor the non-Abelian anyons are midgap states (“zero modes”) bound to a defect (a vortex [12,13] or the end-point of a nanowire [14–16]). Because they are immobile, existing proposals to demonstrate non-Abelian statistics do not actually exchange the zero modes in real space [17–21].

Topological superconductors do have chiral edge modes [4], and recent experimental progress [22] has motivated the search for ways to use the chiral motion for a braiding operation [23]. The obstruction one needs to overcome is that the Majorana fermions which propagate along the edge of a superconductor have conventional fermionic exchange statistics. In the quantum Hall effect each charge $e/4$ quasiparticle contains a zero mode and the exchange of two quasiparticles is a non-Abelian operation on a topological qubit encoded in the zero modes. However, Majorana fermions contain no zero mode which might encode a topological qubit, one needs vortices for that.

In this Letter we show how one can exploit the chiral motion along the edge of a topological superconductor to exchange zero modes in real space. The key innovative element of our design, which distinguishes it from Ref. [23], is the use of a biased Josephson junction to on demand inject a pair of isolated vortices into chiral edge channels. Previous studies of such “edge vortices” relied on quantum fluctuations of the phase to create a vortex pair in the superconducting condensate [24–27], but here the injection is entirely deterministic. When the two mobile edge vortices encircle a localized bulk vortex their fermion parity switches from even to odd, as a demonstration of non-Abelian braiding statistics. The entire operation, injection braiding detection, can be carried out fully electrically, without requiring time-dependent control over Coulomb interactions or tunnel probabilities.

Edge vortex injection.—Figure 1 shows different ways in which the edge vortex can be injected: driven by a flux bias or by a voltage bias over a Josephson junction. We show two possible physical systems that support chiral edge channels moving in the same direction on opposite boundaries of the superconductor. Both are hybrid systems, where a topologically trivial superconductor (spin-singlet s-wave pair potential $\Delta_0$) is combined with a topologically nontrivial material: a 2D Chern insulator (quantum anomalous Hall insulator) [22,28] [panel (a)] or a 3D topological insulator gapped on the surface by ferromagnets with opposite magnetization $M_{1,\perp}$ [24,29] [panel (b)].

The superconducting phase difference $\phi(t)$ across the Josephson junction is incremented with $2\pi$ by application of a voltage pulse $V(t)$ (with $\int V(t)dt = h/2e$), or by an $h/2e$ increase of the flux $\Phi(t)$ through an external
superconducting loop. If the width $W$ of the superconductor is large compared to the coherence length $\xi_0 = \hbar v/\Delta_0$, the edge channels at $x = \pm W/2$ are not coupled by the Josephson junction—except when $\phi$ is near $\pi$, as follows from the junction Hamiltonian [13,29]

$$H_J = v p_x \sigma_z + \Delta_0 \sigma_y \cos(\phi/2).$$

The Pauli matrices act on excitations moving in the $\pm x$ direction with velocity $v$, in a single mode for $\xi_0$ large compared to the thickness of the junction in the $y$ direction.

At $\phi = \pi$ a Josephson vortex passes through the superconductor [30,31]. A Josephson vortex is a $2\pi$ phase winding for the pair potential, so a $\pi$ phase shift for an unpaired fermion. As explained in Ref. [32], the passage of the Josephson vortex leaves behind a pair of edge vortices: a phase boundary $\sigma(y)$ on each edge, at which the phase of the Majorana fermion wave function $\psi(y)$ jumps by $\pi$. Because of the reality constraint on $\psi$, a $\pi$ phase jump (a minus sign) is stable: it can only be removed by merging with another $\pi$ phase jump. And because the phase boundary is tied to the fermion wave function, it shares the same chiral motion, $\sigma(y, t) = \sigma(y - vt)$.

**Braiding of an edge vortex with a bulk vortex.**—Two vortices may be in a state of odd or even fermion parity, meaning that when they fuse they may or may not leave behind an unpaired electron. The fermion parity of vortices $\sigma_1$ and $\sigma_2$ is encoded in the $\pm 1$ eigenvalue of the parity operator $P_{12} = \gamma_2 \gamma_1$, where $\gamma_n$ is the Majorana operator associated with the zero mode in vortex $n$ [33]. The two edge vortices are created at the Josephson junction in a state of even fermion parity, $P_{12} = +1$, but as illustrated in Fig. 1(a) that may change as they move away from the junction. If one of the edge vortices, say $\sigma_1$, crosses the branch cut of the phase winding around a bulk vortex, $\gamma_1$ picks up a minus sign and the fermion parity $P_{12} \rightarrow -1$ switches from even to odd [5]. This is the essence of the non-Abelian braiding statistics of vortices. Overall fermion parity is conserved, because a second branch cut crossing [see Fig. 1(c)] also switches the fermion parity of the bulk vortices.

**Detection of the fermion-parity switch.**—Figure 2 shows the voltage-biased layout for a fully electrical measurement. The fermion parity of the edge vortices cannot be detected if they remain separated on opposite edges, so we first fuse them at a second Josephson junction. The characteristic time scale of the injection process [29] is the time $t_{\text{inj}} = (\xi_0/W)(d\phi/dt)^{-1}$ when $\phi(t)$ is within $\xi_0/W$ from $\pi$, and if the distance $L$ between the two Josephson junctions is less than $v t_{\text{inj}}$ we can neglect the time delay between the injection at the first junction $J_1$ and the fusion at the second junction $J_2$. This is convenient, because then the whole process can be driven by a single voltage pulse $V(t)$ applied to the region $|y| < L/2$ between the two junctions, relative to the grounded regions $y < -L/2$ and $y > L/2$ outside.

Both these grounded regions are connected to normal metal electrodes $N_1$ and $N_2$ and the electrical current $I(t)$ between them is measured. As we will now show, the transferred charge $Q = \int I(t) dt$ is quantized at unit electron charge if the region between the Josephson junctions contains a bulk vortex, while $Q = 0$ if it does not.

**Mapping onto a scattering problem.**—Tunneling of edge vortices driven by quantum fluctuations of the phase is a many-body problem of some complexity [32]. We avoid this because we rely on an external bias to inject the edge vortices; hence the phase $\phi(t)$ can be treated as a classical variable with a given time dependence.

The dynamics of the Majorana fermions remains fully quantum mechanical, governed by the Hamiltonian

$$H = i \begin{pmatrix} -\nu \partial / \partial y & -\mu[y, \phi(t)] \\ \mu[y, \phi(t)] & -\nu \partial / \partial y \end{pmatrix} \equiv v p_x \sigma_0 + \mu \sigma_y.$$
two Josephson junctions

(We set $\hbar = 1$.) The $2 \times 2$ Hermitian matrix $H$ acts on the Majorana fermion wave functions $\Psi = (\psi_1, \psi_2)$ at opposite edges of the superconductor, both propagating in the $+y$ direction (hence the unit matrix $\sigma_0$). The interedge coupling $\mu$ multiplies the $\sigma_y$ Pauli matrix to ensure that $H$ is purely imaginary and the wave equation $\partial \Psi / \partial t = -i H \Psi$ is purely real (as it should be for a Majorana fermion).

For low-energy, long-wavelength wave packets the $y$ dependence of the interedge coupling may be replaced by a delta function, $\mu[y, \phi(t)] = v \delta(y) \eta(t)$. This “instantaneous scattering approximation” [34] is valid if the transit time $t_{\text{trans}} \approx L / v$ of the wave packet through the system is short compared to the characteristic time scale $t_{\text{inj}}$ of the vortex injection, hence if $d\phi / dt \ll v \xi / A_{\text{junction}}$, where $A_{\text{junction}} = WL$ is the area of the region between $J_1$ and $J_2$. In this regime there is no need to explicitly consider the vortex dynamics in between the Josephson junctions, instead we can treat this as a scattering problem “from the outside.”

Incoming and outgoing states are related by

$$\Psi_{\text{out}}(E) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(\omega) \Psi_{\text{in}}(E - \omega),$$  

where $S(\omega)$ is the Fourier transform of the adiabatic (or “frozen”) scattering matrix $S(t)$,

$$S(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} S(t), \quad S(t) = \exp(-i\eta(t) \sigma_y),$$  

describing the scattering at $E = 0$ for a fixed $\phi(t)$. Note that $S(t)$ is unitary but $S(\omega)$ is not.

As we shall see in a moment, the transferred charge is independent of how $\eta(t) = \eta[\phi(t)]$ is varied as a function of time, only the net increment $\delta\eta = \eta(t \to \infty) - \eta(t \to -\infty)$ matters. When there is no vortex in the region between the two Josephson junctions $J_1$ and $J_2$ there is no difference between $\phi = 0$ and $\phi = 2\pi$, hence $\delta\eta = 0$. On the contrary, when there is a bulk vortex in this region we find [35]

$$\eta = 2 \arccos \left( \frac{\cos(\phi/2) + \tanh \beta}{1 + \cos(\phi/2) \tanh \beta} \right), \quad \beta = \frac{W}{\xi_0} \cos \frac{\phi}{2},$$  

hence $\delta\eta = 2\pi$. More generally, when there are $N_{\text{vortex}}$ vortices between $J_1$ and $J_2$ the phase increment is

$$\delta\eta = \pi [1 - (-1)^{N_{\text{vortex}}}].$$  

In Fig. 3 we show that the analytical result Eq. (5) agrees well with a computer simulation (using KWANT [35,36]) of a lattice model of a quantum anomalous Hall insulator with induced $s$-wave superconductivity [28].

**Transferred charge.**—The expectation value of the transferred charge [38],

$$Q = e \int_0^\infty \frac{dE}{2\pi} \langle \Psi_{\text{out}}(E) \sigma_y \Psi_{\text{in}}(E) \rangle,$$  

given at zero temperature, when

$$\langle \Psi_{\text{in},n}(E) \Psi_{\text{in},m}(E') \rangle = \delta_{nm} \delta(E - E') \theta(-E).$$  

FIG. 2. Starting from the layout of Fig. 1(a), we have inserted a second Josephson junction ($J_2$) and we have added normal metal contacts ($N_1, N_2$) to measure the current $I(t)$ carried by the edge modes in response to the voltage $V(t)$ applied to the superconductor. A unit charge per $2\pi$ increment of $\phi$ is transferred from the superconductor into the normal metal contact. The counterpropagating Dirac edge mode along the upper edge of the Chern insulator isdecoupled from the superconductor and plays no role in the analysis.

FIG. 3. Bottom panel: Scattering phase $\eta(\phi) - \eta(0)$ according to Eq. (5) (solid curve) and as obtained numerically (blue data points) from a lattice model [28] of the system shown in Fig. 2. There are no fit parameters in the comparison, the ratio $W/\xi_0 = 4.04$ was obtained directly from the simulation [35,37]. The grey data points show the result without vortices, when there is no net increment as $\phi$ advances from 0 to $2\pi$. Top panel: Current density in the lattice model. The two vortices are faintly visible.
by an integral over positive excitation energies,

$$Q = \frac{e}{4\pi^2} \int_0^\infty d\omega \omega \text{Tr} S^\dagger(\omega) \sigma_y S(\omega).$$

(The factor $\omega = \int_0^\infty dE \delta(\omega - E)$ appears from the integration over the step function.) Because $S(-\omega) = S^\dagger(\omega)$ the integrand in Eq. (9) is an even function of $\omega$ and the integral can be extended to negative $\omega$,

$$Q = \frac{ie}{4\pi^2} \int_{-\infty}^\infty d\omega \omega \text{Tr} S^\dagger(\omega) \sigma_y S(\omega)
= i\frac{e}{4\pi^2} \int_{-\infty}^\infty dt \text{Tr} S^\dagger(t) \sigma_y \frac{\partial}{\partial t} S(t).$$

This is the superconducting analogue of Brouwer’s charge-pumping formula [39] (see Ref. [40] for an alternative derivation).

Substitution of $S(t) = \exp(-i\eta(t)\sigma_y)$ results in

$$Q = (e/2\pi)\delta\eta = e$$

if $N_{\text{vortex}}$ is odd, while $Q = 0$ if $N_{\text{vortex}}$ is even.

Transferred particle number.—This quantized transfer of one electron charge may be accompanied by the non-quantized transfer of neutral electron-hole pairs. To assess this we calculate the expectation value of the transferred particle number, given by Eq. (9) upon substitution of the fermion parity. The deterministic voltage-induced injection of edge vortices that we have proposed here could become a key building block for such applications.

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[29] L. Fu and C. L. Kane, Probing Neutral Majorana Fermion Edge Modes with Charge Transport, Phys. Rev. Lett. 102, 216403 (2009).


[35] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.146803 for more details. The calculation of the scattering phase shift $\eta(\phi)$ is given in Appendix A. Equation (5) for $0 \leq \phi \leq 2\pi$ repeats periodically modulo $2\pi$. Details of the numerical simulation are given in Appendix B.


[37] The code that can be used to reproduce the numerical results is available at https://doi.org/10.5281/zenodo.2556947.

[38] The charge operator $\hat{Q} = e\sigma_z$ in the electron-hole basis transforms into $\hat{Q} = e\sigma_z$ in the basis of Majorana fermions.


[41] The special time dependence $\eta(t) = \pi + 2\arctan(t/2f_{\nu\nu})$ produces precisely one particle with charge $e$.

