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Shortest Paths in Networks with Correlated Link Weights

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***Abstract*—Solving the shortest path problem is important in achieving high performance or to efficiently utilize resources in various kinds of networks, e.g., data communication networks and transportation networks. Fortunately, under independent additive link weights, this problem is solvable in polynomial time. However, in many real-life networks, the link weights (e.g., delay, bandwidth, failure probability) are often correlated due to spatial or temporal dependencies. These correlated link weights together might behave in a different manner and are not always additive.**

In this paper, we first propose two correlated link-weight models, namely (i) the deterministic correlated model and (ii) the (log-concave) stochastic correlated model. Subsequently, we study the shortest path problem under these two correlated models. We prove that the shortest path problem is NP-hard under the deterministic correlated model, and even cannot be approximated to arbitrary degree in polynomial time. On the other hand, we show that the shortest path problem is polynomial-time solvable under a nodal deterministic correlated model. Finally, we show that the shortest path problem under the (log-concave) stochastic correlated model can be solved by convex optimization.

I. Introduction

The shortest path problem is undoubtedly one of the most fundamental and important problems in the network research community, since it plays a crucial role in achieving high performance or efficient resource/cost utilization in networks. For instance, a shortest path algorithm can return the minimum delay route in vehicle networks, the most energy-efficient path in energy networks, etc. In networks where the link weights are independent and additive, the shortest path problem is solvable in polynomial time, e.g., see [1]. However, often correlations or (inter-)dependencies exist among link weights. For example, in overlay [2] or multilayer networks [3], the abstract links in the logical layer are mapped to different physical links in the physical layer. In this context, two or more abstract links that contain the same underlying physical links may have correlated latencies, bandwidth usage or geographical failures. Another example is interdependent networks [4], where for instance the electricity network and Internet network are coupled and inter-connected, and a node or link failure in one network may cause failures of nodes or links in the other network. A similar case is reflected by optical Shared-Risk Link Group (SRLG) networks [5], where fiber links in the same duct will fail simultaneously in case their duct fails. The dependencies in interdependent and SRLG networks can also be seen as correlations, so we use the term correlation throughout this paper. Our key contributions are as follows:

- In Section II, we propose two correlated link weight models, namely a *deterministic correlated model* and a *stochastic correlated model*.
- In Section III, we study the shortest path problem under the deterministic correlated model and we prove that it cannot be approximated in polynomial time. We further show that this problem is polynomial-time solvable under a nodal deterministic correlated model.
- In Section IV, we propose a convex optimization formulation to solve the shortest path problem under the stochastic correlated model.

Related work is presented in Section V and we conclude in Section VI. In [6] we present additional results.

II. Correlated Link Weight Models

A network having node and link weights can be transformed to a directed network with only link weights, as done in [7]. Therefore, we assume nodes are unweighted and only consider correlated link weights. Throughout this paper, we use the term "correlated model" to represent "correlated link weight model".

A. Deterministic Correlated Model

Without loss of generality, we use $w(l)$ to represent the weight of link l. For simplicity, in this paper we call $w(l)$ the cost of l, although it could also reflect other metrics such as delay, energy, etc. In the deterministic correlated model, for any two links l_i and l_j, their joint total cost is represented by $w(l_i) \oplus w(l_j)$, where the operator $\oplus$ indicates the joint total cost of the links, which may be unequal to the $+$ operator when they are *correlated*. In this model, the use of one link may influence the cost of another. For example, in Fig. 1 where the cost is shown above each link, it is assumed that only link (s, a) and (b, t) are correlated with joint cost of 11 for simplicity, and all the other links have uncorrelated costs. We can see that in path s-b-t, the cost of link (b, t) is 10 since another link (s, b) in this path is not correlated with it. Therefore this path's cost is equal to 18. However, in path s-a-b-t, the cost of link (b, t) should be calculated together with another link (s, a) with a joint cost of 11, since they are correlated and both appear in this path. Therefore, this path's total cost is equal to $11 + 4 = 15$, which is not equal to the sum of the individual link costs ($6 + 4 + 10 = 20$).

Equivalently, we could formulate $w(l_i) \oplus w(l_j) = \rho_{i,j} \cdot (w(l_i)+w(l_j))$, where $\rho_{i,j}$ stands for the correlation coefficient

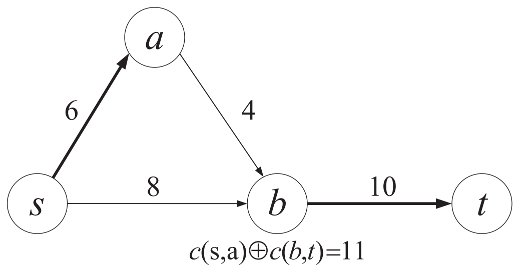

Fig. 1: An example of the deterministic correlated model.

between links l_i and l_j, and its value varies in the range $(0, \infty)$, since we do not consider negative costs. When $\rho_{i,j}$ is equal to 1, it indicates that l_i and l_j are uncorrelated, when $\rho_{i,j}$ is greater than 1, it indicates that l_i and l_j have an increasing correlation, otherwise we say l_i and l_j have a decreasing correlation.

Analogously, for given $m > 1$ links $l_1, l_2, ..., l_m$ in the deterministic correlated model, their joint total cost can be expressed as follows:

$$w(l_1) \oplus w(l_2) \cdots \oplus w(l_m) = \rho_{1,2,...,m} \cdot (w(l_1) + w(l_2) + \cdots + w(l_m)) \tag{2.1}$$

Similarly, if the link l's weight is multiplicative (e.g., failure probability), then by using $-log(w(l))$ to represent its weight value, Eq. (2.1) also applies. The decreasing correlation case can also reflect SRLG networks. For instance, in SRLG networks, each link is associated with several SRLG events with their respective failure occurring probabilities. Hence, the total failure occurring probabilities (represented by P_{SRLG}) of two correlated links that have at least one SRLG in common will be equal to the product of the failure occurring probabilities of all the distinct SRLG events that belong to these two links. Let us denote $P_{l_1} = P_{s_1} \cdot P_s$ and $P_{l_2} = P_{s_2} \cdot P_s$ as the failure probability of these two links, respectively, where P_s denotes the overlapping SRLGs' failure occurring probability between l_1 and l_2, and P_{s_1} (P_{s_2}) is the non-overlapping SRLGs' failure probability of l_1 (l_2). Then we can have $P_{l_1} \cdot P_{l_2} < P_{SRLG} = P_{s_1} \cdot P_{s_2} \cdot P_s < min(P_{l_1}, P_{l_2})$. By taking the $-log$ on this inequality, we have:

$$\max(-log(P_1), -log(P_2)) < -log(P_{SRLG}) < (-log(P_{l_1})) + (-log(P_{l_2}))$$

Or equivalently,

$$-log(P_{SRLG}) = \rho \cdot (-log(P_{l_1})) + (-log(P_{l_2}))$$

where $\rho < 1$ denotes their correlation coefficient.

In probability theory, given two random variables X and Y with respective expected values μ_X and μ_Y, and respective standard deviations σ_X and σ_Y, their *linear* correlation coefficient $\rho(X, Y)$ is defined as:

$$\rho(X, Y) = \frac{Cov[X, Y]}{\sigma_X \sigma_Y} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y} \tag{2.2}$$

where $Cov[X, Y]$ represents the covariance of X and Y.

However, the *linear* correlation coefficient in probability theory is different from and cannot be transformed to the one defined in the deterministic correlated model because: the variances of X and Y in Eq. (2.2) must be nonzero and finite. However, in the deterministic correlated model, for any link l, when none of its correlated links simultaneously belong to a path, the cost of l is fixed/deterministic with variance of 0.

B. Stochastic Correlated Model

In many real-life networks, the link weights are uncertain because of inaccurate Network State Information (NSI) [8], [9]. For instance, Papagiannaki *et al.* [10] showed that the queuing delay distribution can be approximated by a Weibull distribution. Since the Cumulative Density Function (CDF) of a Weibull distribution is log-concave and the CDFs of many common distributions (e.g., Exponential distribution, Uniform distribution, etc.) are log-concave [11], [12], we make, as in [8], [13], a mild (general) assumption that the link l's weight follows a log-concave distribution.

We first define the *Correlated Group* (CG):

Definition 1: Given is a network $G(\mathcal{N}, \mathcal{L})$ where $\mathcal{N}$ represents a set of N links and $\mathcal{L}$ denotes a set of L links. A Correlated Group (CG) is a subset of links $L_{CG} \subseteq \mathcal{L}$, and $\forall l \in L_{CG}$, $\exists l' \in L_{CG} \backslash \{l\}$, such that l and l' are correlated ($\rho_{l,l'} \neq 1$).

Accordingly, the *Maximum Correlated Group* (MCG) is defined as a CG with the maximum number of correlated links. If a link l is uncorrelated/independent with all the other links, then we say $\{l\}$ is a single element MCG. Suppose there are Ω Maximum Correlated Groups (MCGs), and there are $m_i > 0$ links (denoted as l_1^i, l_2^i,...,$l_{m_i}^i$) in the i-th MCG, where $1 \leq i \leq \Omega$. Each link l has an upper bound of allocating cost $w_l^{\max}$. In the i-th MCG, a multivariate m_i dimensional log-concave Cumulative Density Function $CDF_i(x_1, x_2, ..., x_{m_i})$ is given to allocate cost x_1, x_2,..., x_{m_i} for links l_1^i, l_2^i,..., $l_{m_i}^i$, respectively.

Therefore, if the possible cost of link l ranges from 0 to $w_l^{\max}$ ($0 < w_l^{\max}$), then the probability of allocating a cost value out of this range is 0. Hence, we have $CDF_i(w_l^{\max}) = 1$ for a single element MCG i, and $CDF_j(w_{l_1^j}^{\max}, w_{l_2^j}^{\max}, ..., w_{l_{m_j}^j}^{\max}) = 1$ for a multi-element MCG j.

For example, Fig. 2 shows a 2-dimensional multivariate Normal distribution, where both variables are in the range $[0, 4]$ with mean 2, and the corresponding covariance matrix is $\begin{bmatrix} 0.9 & 0.4 \\ 0.4 & 0.3 \end{bmatrix}$. Similarly to Eq. (2.2), the correlation matrix (composed of linear correlation coefficients) can be derived from the covariance matrix and the variables' standard variances in the multivariate Normal distribution. However, we do not explicitly use the *linear* correlation coefficient in the stochastic correlated model, since we will later prove that via the log-concave property of this model, the shortest path problem under this model can be solved by convex optimization.

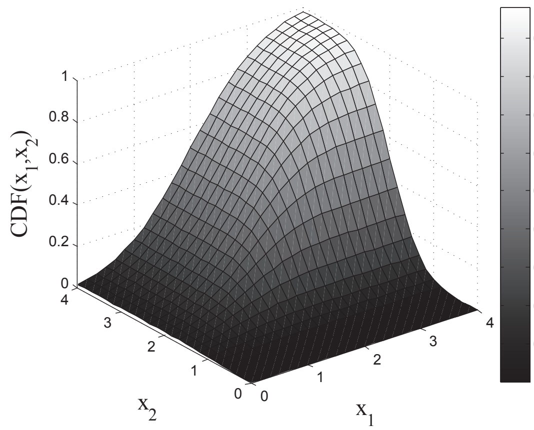

Fig. 2: A 2-dimensional multivariate Normal distribution.

III. Shortest Path under the Deterministic Correlated Model

A. Problem Definition and Complexity Analysis

Definition 2: Given is a directed network $G(\mathcal{N}, \mathcal{L})$, and each link $l \in \mathcal{L}$ has a cost $w(l)$ following the deterministic correlated model. The Shortest Path under the Deterministic Correlated Model (SPDCM) problem is to find a path from the source s to the destination t with minimum cost.

In conventional deterministic networks, a subpath of a shortest path is also the shortest. We refer to this property of the shortest path as the dominance of the subpath. However, this is not the case in networks with deterministic correlated link weights, which means a dominated path may also lead to an optimal solution. For instance, in the example of Fig. 1, we can see that although subpath s-b has a smaller cost than subpath s-a-b, path s-a-b-t (instead of path s-b-t) has the minimum cost. In the following, we will study the complexity of the SPDCM problem.

Theorem 1: The SPDCM problem is NP-hard.

Proof: When the correlation coefficient is equal to 1, the SPDCM problem can be solved by a conventional shortest path algorithm in polynomial time. We therefore prove in the following that the SPDCM problem is NP-hard for "increasing correlation"> 1 as well as "decreasing correlation"< 1.

Increasing Correlation:

When the correlation coefficient is greater than 1, we make a reduction to the forbidden pairs shortest path problem, which is known to be NP-hard [14]. In a given network and for given set of node pairs ζ, the forbidden pairs shortest path problem looks for the shortest path between s and t such that at most one node from each pair in the set ζ lies on this path. Let us consider a network with deterministic correlated link weights, where two nodes i and j form a forbidden pair, their costs are correlated such that $w(i,.) \oplus w(j,.) = +\infty$, where $(i,.)$ and $(j,.)$ represent any link that contains an end node of i and j, respectively. In all the other cases, the link costs are uncorrelated and finite. Since $w(i,.) \oplus w(j,.) = +\infty$, if the two forbidden nodes appear in the same path then the cost of this path will be $+\infty$, so it will never be the minimum cost path. Now, the SPDCM problem is equivalent to the forbidden pairs shortest path problem.

Decreasing Correlation:

When the correlation coefficient is less than 1, we make a reduction to the Minimum Color Single-Path (MCSiP) problem, which is NP-hard [15]. Given a network $G(\mathcal{N}, \mathcal{L})$, and given the set of colors $C = \{c_1, c_2, ..., c_g\}$ where g is the total number of colors in G, and given the color set $\{c_l\}$ associated to each link $l \in \mathcal{L}$, the Minimum Color Single-Path (MCSiP) problem is to find one path from source node s to destination node t such that it uses the least amount of colors.

Assume each color c_i is associated with cost 1, where $1 \leq i \leq z$. We further assume that $w(l_1) \oplus w(l_2) \oplus \cdots \oplus w(l_m) = x$, where x is the total number of distinct colors belonging to these m links. Hence, the SPDCM problem is equivalent to the MCSiP problem. ∎

Theorem 2: The SPDCM problem cannot be approximated to arbitrary degree in polynomial time, unless P=NP.

Proof: We prove it by contradiction.

Increasing Correlation:

Assume a polynomial-time approximation algorithm exists that can find a path with a cost at most $\alpha \cdot opt$, where $\alpha > 1$ is an approximation ratio. For a pair of forbidden nodes i and j, we further assume $w(i,.) \oplus w(j,.) > \alpha * opt$. Therefore, if an approximation algorithm can find a path ψ with cost at most $\alpha * opt$ from s to t, then i and j cannot be simultaneously traversed by this path ψ, which means that the forbidden pairs shortest path problem can be solved in polynomial time, which results in a contradiction.

Decreasing Correlation:

We first introduce the Disjoint Connecting Path problem [16]. Given a directed network $G(\mathcal{N}, \mathcal{L})$, a collection of disjoint node pairs $(s_1,\ t_1)$, $(s_2,\ t_2)$, ..., $(s_z,\ t_z)$, does G contain z mutually link-disjoint paths, one connecting s_i and t_i for each i, $1 \leq i \leq z$. This problem is NP-hard when $z \geq 2$. Assume a polynomial-time approximation algorithm exists that can find a path with a cost at most $\alpha \cdot opt$, where $\alpha > 1$ is an approximation ratio. Assuming all the links in the network have weight 1, and link (u, v) and any $m > 0$ links in $\mathcal{L}\backslash\{(u, v)\}$ are correlated, with a total cost of $\frac{1}{\beta} \cdot m$. Moreover, any two or more links in $\mathcal{L}\backslash\{(u, v)\}$ are assumed to be uncorrelated/independent.

According to this assumption, the minimum value of a shortest path is 1 if link (u, v) is not traversed, i.e., it traverses only one link from s to t. However, the optimal solution which traverses link (u, v) has a total cost of $opt = \frac{1}{\beta} \cdot c$, where c is the sum of minimum hops from s to u and from v to t. For any given α, let $\frac{\beta}{c} > \alpha$, then $1 > \alpha \cdot \frac{1}{\beta} \cdot c$, which means $1 > \alpha \cdot opt$. To find a path with cost at most $\alpha \cdot opt$, the polynomial-time algorithm must find a path which traverses link (u, v). In that case the algorithm can, in polynomial time, find two link-disjoint paths from s to u, and from v to t, which results in a contradiction. ∎

To solve the SPDCM problem exactly, we can apply a modified Dijkstra's algorithm by letting each node store as many subpaths as possible (at the cost of exponential running time), which is similar to the exact algorithm for multi-constrained routing [17]. We start with some notations used in the algorithm:

$sus[u][h]$: the parent node of node u for its stored h-th subpath from s to it.

$dist[u][h]$: the cost value stored at node u for its stored h-th subpath from s to it.

$counter[u]$: the number of stored subpaths of node u.

$u[m]$: node u's stored m-th subpath from s to it.

$adj(u)$: the set of adjacent nodes of node u.

The pseudo-code of the exact algorithm is given in Algorithm 1.

Algorithm 1 $SPDCM(G, s, t)$

$Q \leftarrow s$, $P \leftarrow \emptyset$, $dist[s][1] \leftarrow 0$, $dist[i][h] \leftarrow +\infty$, $sus[i][h] \leftarrow i$, $counter[s] \leftarrow 1$, $counter[i] \leftarrow 0$, $\forall i \in \mathcal{N} \backslash \{s\}$.
While $Q \neq \emptyset$
  $u[m] \leftarrow$ Extract-min(Q)
  If ($u == t$) **do**
    Insert (P, $u[m]$)
  Else
    Foreach $v \in adj(u)$ **do**
      $counter(v) = counter(v) + 1$
      Calculate the total cost of subpath $u[m] \rightarrow v$ and assign it to $dist[v][counter(v)]$
      $sus[v][counter(v)] \leftarrow u$
      Insert $(Q, v, counter(v))$
return min(P).

Next, we study the performance of a conventional shortest path algorithm running on a graph where each link has an "uncorrelated" weight value.

Lemma 1: When all the correlation coefficients are greater than 1, a conventional shortest path ψ has a total cost at most $\frac{\rho_{\max}}{\rho_{opt}} \cdot opt$, where $\rho_{\max}$ and ρ_{opt} are the largest correlation coefficient and optimal solution's correlation coefficient, respectively, and *opt* is the cost of the optimal solution.

Proof: Let $U(\psi) = \sum_{l \in \psi} w(l)$ and let $C(\psi) = \rho_u \cdot U(\psi) = \rho_u \cdot \sum_{l \in \psi} w(l)$ reflect the total joint cost of path ψ considering their correlation, where ρ_u indicates the correlation coefficient of path ψ. On one hand, a conventional shortest path ψ should satisfy $U(\psi) \leq \frac{opt}{\rho_{opt}}$. On the other hand, $C(\psi) \leq \rho_{\max} \cdot U(\psi)$ considering ρ_{max} is the largest correlation coefficient. Hence, $C(\psi) \leq \rho_{\max} \cdot U(\psi) \leq \frac{\rho_{\max}}{\rho_{opt}} \cdot opt$. ■

Lemma 2: When all the correlation coefficients are less than 1, a conventional shortest path ψ has a cost at most $\frac{1}{\rho_{\min}} \cdot opt$, where $\rho_{\min}$ is the smallest correlation coefficient among all the correlation coefficients.

Proof: Let $V(\psi) = \sum_{l \in \psi} w(l)$ and let $C(\psi) = \rho_u \cdot \sum_{l \in \psi} w(l)$ reflect the total joint cost of path ψ considering their correlation. Since all the correlations are decreasing ($\rho < 1$), we have $C(\psi) \leq V(\psi)$. On the other hand, $\rho_{\min} \cdot V(\psi) \leq opt$ considering that $\rho_{\min}$ is the smallest correlation coefficient. Hence, $C(\psi) \leq V(\psi) \leq \frac{1}{\rho_{\min}} \cdot opt$. ■

Via Lemmas 1 and 2, we obtain Theorem 3.

Theorem 3: In a network with links following the deterministic correlated model, a conventional shortest path can have cost at most $max(\frac{\rho_{\max}}{\rho_{opt}}, \frac{1}{\rho_{\min}}) \cdot opt$.

Theorem 3 reveals that a conventional shortest path may have arbitrary bad performance, since either $\frac{\rho_{\max}}{\rho_{opt}}$ can be infinitely large or ρ_{min} can be infinitely small.

B. Shortest Path under the Nodal Deterministic Correlated Model

In some real-world networks (e.g., SRLG networks), the links that are spatially (geographically) close to each other are usually correlated whereas the links that are located spatially far from each other are usually uncorrelated. We make an additional assumption, which is that only the links sharing the same node can be correlated. We name this kind of correlation as *nodal* correlation.

Although the SPDCM problem is NP-hard, we will show that, by transforming the original graph to an auxiliary graph, the Shortest Path under the Nodal Deterministic Correlated Model (SPNDCM) problem is polynomial-time solvable. For any node a, there are generally two cases of nodal correlation, namely (1) links in the form of (a, b) and (a, c), and (2) links in the form of (a, b) and (b, c) are correlated. When (a, b) and (a, c) are correlated, a simple path cannot traverse both of them since looping is not allowed. In this sense, any simple path only traverses at most one of them, which means that the links' correlation will not affect the cost calculation of any simple path. Therefore, we only need to consider the case when (a, b) and (b, c) are correlated. We first define that if (a, b) and (b, c) are correlated, then a and b are called *correlated nodes*, which is represented by C_n, else they are *uncorrelated nodes*, which is denoted by U_n. Subsequently, based on the original graph $G(\mathcal{N}, \mathcal{L})$, the auxiliary graph $G^A(\mathcal{N}^A, \mathcal{L}^A)$ can be constructed as follows:

1) For any two links $(u, v) \in \mathcal{L}$ and $(v, w) \in \mathcal{L}$ that are correlated in G, create new nodes u_v, v^{uw} and v_w in G^A if they do not exist.
2) For any node $a \in \mathcal{N}$ and if it is an *uncorrelated node* (in U_n), create node a_a in G^A.
3) For any two correlated links (u, v) and (v, w) in G, create links (u_v, v^{uw}), (v^{uw}, v_w) and (v_w, w_w) in $\mathcal{G}^A$. Assign the links (u_v, v^{uw}) and (v_w, w_r) with the weights of $w(u, v)$ and $w(v, w)$, respectively, and the link (v^{uw}, v_w) with the weight of $(\rho_{(u,v)(v,w)} - 1) \cdot (w(u, v) + w(v, w))$, where $\rho_{(u,v)(v,w)}$ is the correlation coefficient of links (u, v) and (v, w).
4) For each link $(a, b) \in \mathcal{L}$ such that both node a and node b are not *correlated nodes*, create the link (a_a, b_b) also in G^A with the link weight of $w(a, b)$.

5) For each link $(a,b) \in \mathcal{L}$ such that $a \in U_n$ and $b \in C_n$, draw links (a_a, b_r) in G^A, where $r \in \mathcal{N}$ and $b_r \in G^A$.
6) For each link $(a,b) \in \mathcal{L}$ such that $a \in C_n$ and $b \in U_n$, draw links (a^{rp}, b_b) in G^A, where $r, p \in \mathcal{N}$ and $a^{rp} \in \mathcal{N}^A$.

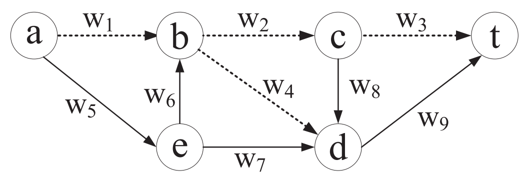

Fig. 3: An example network with dotted links following the nodal deterministic correlated model.

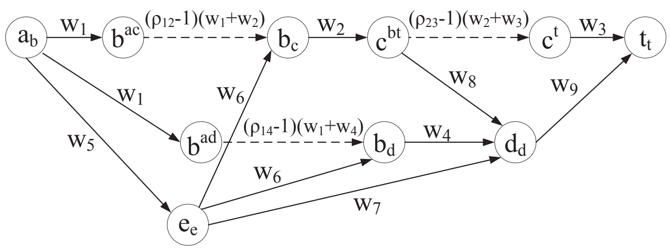

Fig. 4: Auxiliary graph of Fig. 3 for the SPDCM problem under the nodal deterministic correlated model.

The idea of the auxiliary graph is that if two links (u,v) and (v,w) are correlated, we create four corresponding nodes u_v, v^{uw}, v_w, w_w and then draw three links (u_v, v^{uw}),(v^{uw}, v_w) and (v_w, w_w). We use (u_v, v^{uw}) and (v_w, w_w) to indicate their uncorrelated value, respectively, if only one of these two links is traversed, and (v^{uw}, v_w) to represent the correlated loss (decreasing correlation)/gain (increasing correlation), respectively, if they are traversed simultaneously. For instance, in Fig. 3 where the link weight is labeled above each link, assuming links (a,b), (b,c), (c,t) and (b,d) are nodal correlated, then Fig. 4 is its corresponding auxiliary graph with the assigned weight shown on each link. In particular, since (a,b), (b,c) and (a,b), (b,d) may have different correlation coefficients, in Fig. 4 we use (b^{ac}, b_c) and (b^{ad}, b_d) to represent their correlation value, respectively. Meanwhile, when there is a link from an *uncorrelated node* to a *correlated node* in the original graph, e.g., (e,b) in Fig. 3, we draw links (e_e, b_c) and (e_e, b_d) to reflect it (Step 5); When there is a link from a *correlated node* to an *uncorrelated node* in the original graph, e.g., (c,d) in Fig. 3, we draw link (c^{bt}, d_d) to represent it (Step 6). Considering that there are at most $N(N-1)$ nodal links in a graph, the original graph can be transferred to the auxiliary graph in polynomial time.

Consequently, running a shortest path algorithm on the auxiliary graph can return a minimum cost path under the nodal deterministic correlated model. Our auxiliary graph can deal with both decreasing and increasing correlation cases. Considering that $(\rho_{(u,v)(v,w)} - 1) \cdot (w(u,v) + w(v,w)) < 0$ in the auxiliary graph under the decreasing correlation case, and Dijkstra's algorithm cannot handle negative link weights, we could for instance run Bellman-Ford's algorithm on the auxiliary graph.

IV. Shortest Path under the Stochastic Correlated Model

The Shortest Path under the Stochastic Correlated Model (SPSCM) problem is defined as follows:

Definition 3: The Shortest Path under the Stochastic Correlated Model (SPSCM) problem: In a given directed graph $\mathcal{G}(\mathcal{N}, \mathcal{L})$ where the link costs follow the stochastic correlated model, it is assumed that there are in total Ω Maximum Correlated Groups (MCGs). The SPSCM problem is to find a path from the source s to the destination t such that its total cost is minimized and the probability to realize this value is no less than P_s.

We present a convex optimization formulation to solve the SPSCM problem. Convex optimization problems can usually be solved quickly and accurately with convex optimization solvers [18]. Let us first introduce how to develop a Linear Programming (LP) formulation to solve the shortest path problem in deterministic networks:

Objective:

$$\max \ d_t \tag{4.1}$$

Constraints:

$$d_s = 0 \tag{4.2}$$

$$d_v - d_u \leq w(u,v) \quad \forall (u,v) \in \mathcal{L} \tag{4.3}$$

where d_u is a value between 0 and 1. Similarly, the SPSCM problem can be solved by the following convex formulation:

Objective:

$$\max \ d_t \tag{4.4}$$

Constraints:

$$d_s = 0 \tag{4.5}$$

$$d_v - d_u \leq f(u,v) \quad \forall (u,v) \in \mathcal{L} \tag{4.6}$$

$$\sum_{i \in \Omega} -\log\left(CDF_i(f(l_1^i), f(l_2^i), ..., f(l_{m_i}^i))\right) \leq -\log(P_s) \tag{4.7}$$

where the variables $f(u,v), f(l_1^i), f(l_2^i), ..., f(l_{m_i}^i)$ indicate the cost of links (u,v), l_1^i, l_2^i,..., $l_{m_i}^i$, respectively. Constraint (4.7) ensures that the total probability of realizing the total cost is no more than P_s. In Eq. (4.7), for each MCG we apply the multi-dimensional CDF functions to calculate the probability of realizing a cost. Since the multi-dimensional CDF function is log-concave,

$-\log\left(CDF_i(f(l_1^i), f(l_2^i), ..., f(l_{m_i}^i))\right)$ is convex, and by summing all the MCGs' CDFs together, it remains convex, which indicates that Eq. (4.7) is convex. The other constraints are also convex, which proves that the above formulation is a convex optimization model.

V. Related Work

In a network with each link having multiple additive link weight metrics (e.g., delay, cost, jitter, etc.), the Quality of Service (QoS) routing problem is to find a path that satisfies a given metric vector constraint. Kuipers and Van Mieghem [19] study the QoS routing problem under uncorrelated link weights. Another common source of correlation is Shared-Risk Link Groups (SRLGs). Sometimes one SRLG can also be represented by one color, but they share the same meaning in terms of reliability. In this context, Yuan *et al.* [15] prove that the Minimum Color Single-Path (MCSiP) problem is NP-hard. Yuan *et al.* also prove that finding two link-disjoint paths with total minimum distinct amount of colors or least amount of coupled/overlapped colors is NP-hard. Lee *et al.* [20] propose a probabilistic SRLG framework to model correlated link failures and develop an Integer Nonlinear Programming Programming (INLP) to find one unprotected path or two link-disjoint paths with the lowest failure probability.

There is also some literature dealing with correlated routing problems in stochastic networks [21]. For example, in [22], only two possible states are assumed, which are congested and uncongested, and each state corresponds to a cost value. A probability matrix $P_{a,b}^{u,v,w}$, which represents the probability that if (u, v) is in state a then (v, w) is in state b, is given. Two similar link weight models called link-based congestion model and node-based congestion model are proposed in [23]. Based on these models, [22], [23] define and solve the least expected routing problem, which is to find a path from the source to the destination with the minimum expected costs. However, in [22], [23] there are only two possible states for each link and only the correlation of the adjacent links is known/assumed. We assume a more general (and different) stochastic correlated model, where as long as the links (not necessarily adjacent) are correlated, their joint CDF for allocating costs is known.

VI. Conclusion

In this paper, we have studied the shortest path problem under two correlated link weight models, namely (i) the deterministic correlated model and (ii) the (log-concave) stochastic correlated model. We have proved that the shortest path problem is NP-hard under the deterministic correlated model, and cannot be approximated to arbitrary degree, unless P=NP. In particular, we have shown that this problem is polynomial-time solvable under the nodal deterministic correlated model. For the stochastic correlated model, we have proposed a convex optimization formulation to find the shortest path.

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