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Hoekstra, Jaap

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A Solution of the Two-Capacitor Problem Through Its Similarity to Single-Electron Electronics

JAAP HOEKSTRA (Member, IEEE)
Department of EEMCS, Delft University of Technology, 2628 CD Delft, The Netherlands
CORRESPONDING AUTHOR: J. HOEKSTRA (e-mail: j.hoekstra.nl@ieee.org)

ABSTRACT The purpose of this paper is to investigate the two-capacitor paradox using circuit models developed in the analysis of circuits that include nanoelectronic single-electron tunneling devices. The two-capacitor paradox, in which it seems that energy is not conserved in a simple circuit consisting of two capacitors in parallel separated by an ideal switch, is resolved by applying linear circuit theory utilizing a current—described by a (Dirac) delta function—and stepping voltages across all *three* elements. Based on a similar description, successfully used for tunneling of electrons through metal junctions in nanoelectronics, the switch is modeled as a device across which—upon closing—the voltage steps down while the current through it is an impulse. The model distinguishes three intervals in describing the ideal switch: $t < 0, t = 0$, and $t > 0$. As a consequence, the ideal switch dissipates energy *during* the switching action at $t = 0$ in zero time. Although the solution of the two-capacitor problem looks like a theoretical curiosity, the application of nanoelectronic concepts allow a physical explanation based on electron tunneling; it shows that the ideal switch is best described by the tunneling of many electrons. In such a context, some of those electrons loose energy and the v-i characteristic shows Ohm's law.

INDEX TERMS Two-capacitor problem, two-capacitor paradox, energy paradox, energy dissipation, circuit theory, nanoelectronics, single-electron tunneling, (Dirac) delta function, ideal switch, unbounded current.

I. INTRODUCTION

In CIRCUIT theory, according to Tellegen's theorem, energy (or power) must be conserved in electrical circuits that obey the Kirchhoff current law (KCL) and Kirchhoff voltage law (KVL). However, both in linear circuit theory and in nanoelectronics, basic circuits appear in which energy seems not to be conserved.

Well known is the two-capacitor problem or two-capacitor paradox. Two capacitors are separated by a make contact in a configuration shown in Fig. 1. Keeping the switch open, one of them is charged with a charge Q while the other capacitor remains uncharged. Then, the switch is closed at the instant $t = 0$. Charge will be immediately redistributed among the two capacitors. As an example, we assume both capacitors having the same value ($C_1 = C_2 = C$). After redistribution the charge on both capacitors is equal: $q_1(t > 0) = q_2(t > 0) = Q/2$. Before closing the switch, the total stored energy in the circuit $w_{se}(t)$ is

$$w_{se}(t < 0) = \frac{q_1(t < 0)^2}{2C} = \frac{Q^2}{2C} \tag{1}$$

while after closing the total stored energy is

$$\begin{aligned} w_{se}(t > 0) &= \frac{q_1(t > 0)^2}{2C} + \frac{q_2(t > 0)^2}{2C} \\ &= \frac{(Q/2)^2}{2C} + \frac{(Q/2)^2}{2C} = \frac{Q^2}{4C} \end{aligned} \tag{2}$$

The stored energy after closing the switch is only half of that before closing; it seems that half of the energy is missing.

The usual solution [1], presented in most textbooks on linear circuits and basic physics, adds a resistor to obtain energy conservation in the steady state, as is shown later. In other solutions the circuit oscillates and does not reach a final steady state [2]. In [3] the oscillation is described as a RCL-circuit considered in the limit ($R \to 0, L \to 0$)

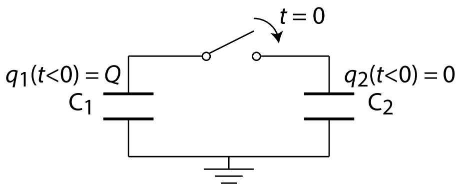

FIGURE 1. Two-capacitor problem; is energy conservation violated?

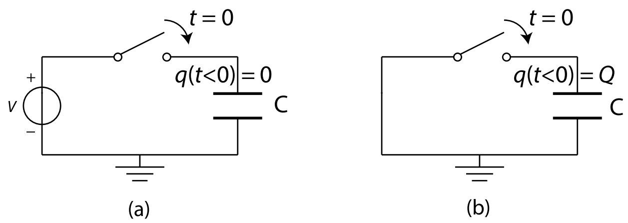

FIGURE 2. (a) The single capacitor problem. (b) The capacitor shorting problem.

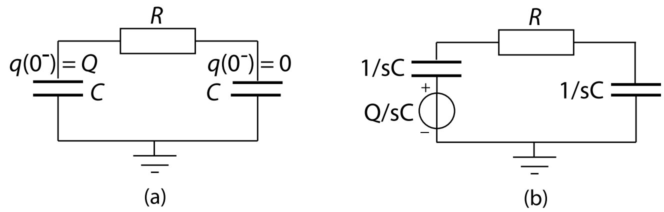

FIGURE 3. "Solving" the two-capacitor problem by adding a resistor. The switch is modeled by a short circuit in series with a resistor for $t \geq 0$; energy is conserved. (a) Circuit for $t = 0$. (b) Equivalent circuit in Laplace domain.

and the energy loss should be interpreted only in the limit of small resistance. More research papers, including [4]–[7], consider damped oscillations and associated energy loss by radiation. And even in some cases, energy loss is proposed without considering any dissipating element [8], which might be occasionally fine in a pure physical description but does not work in circuit theory. In circuit theory, being only a subset of physics (assuming KCL and KVL), one needs a circuit element to model dissipation; in physics it is allowed to consider the system striving to minimal energy.

The author is not aware of papers distinguishing three time intervals in describing the constitutive relation of the ideal switch. In [9] an energy-based jump rule is introduced, the voltages across the capacitors jump instantaneously after closing the switch, describing the constitutive relation with two time intervals. However, the stepping voltage is based on energy minimization, so energy is not conserved (though charge is conserved). Reference [10] is the first paper addressing the probability of absorbing energy by the switch itself. Although the paper investigates stepping voltages together with a delta current pulse, it describes the switch as $i = 0$ for $t < 0$, $v = 0$ for $t > 0$, and adds a constant, with unit Ohm, in the switch's constitutive equation. By doing this, it assumes a nonzero switching time, deviating from the definition of the ideal switch.

There are many variants of the two-capacitor problem, such as the single capacitor problem and the capacitor shorting problem (consider the switch as a one-port), shown in Fig. 2. In (a) the voltage source provides energy $w_{\text{source}} = VQ = CV^2$ to charge the capacitor with charge $Q = CV$, while the stored energy on the capacitor is only $w_{\text{C}} = (1/2)CV^2$. Again half of the energy is missing. In (b) the capacitor is shortened and the energy $w_{\text{C}} = (1/2)Q^2/C$ seems to entirely disappear from the system.

Recently, the two capacitor paradox gained renewed interest due to research in applications of switched capacitor circuits, see, i.e., [11], and in nanoelectronics [12]. Results in this paper are of importance for understanding switched-capacitor circuits, Coulomb blockade, and tunneling in nanoelectronics. Some nanoelectronic topics will be discussed later in this paper. Modern switched capacitor circuits have many applications today. However, the simplest model with only capacitors and ideal switches is unable to account for the loss of energy during the switching.

The paper is organized in the following way. First in Section II the "standard" solution of the paradox is presented by adding a resistor in the circuit. As is shown in this paper, adding a resistor is not necessary to solve the paradox. In Section III Tellegen's theorem and Henderson's theorem are briefly introduced to motivate a solution without a resistor. In Section IV the two-capacitor paradox is solved without introducing any other elements. Energy efficiency of the various circuits is presented in Section V. It is useful to derive a general expression for the dissipated energy in case of arbitrary capacitor values and initial charges on both capacitors, this is done in Section VI. The solution of the paradox is based on similarities between the description of currents and voltages in single-electron electronics and in the two-capacitor problem; this is discussed in Section VII. By exploring the analogy between nanoelectronics and the dissipating switch a possible physical interpretation can be investigated. To do this first a brief introduction to single-electron electronics is given in Section VIII. The interpretation is presented in Section IX. Section X presents the conclusions. An Appendix is included to define some of the relations and concepts used in this paper.

II. STANDARD SOLUTION

In general, the two-capacitor paradox is "solved" by introducing a resistor *after* the switch is closed (it takes time to dissipate energy through a resistor). The switch is modeled as a short circuit in series with a resistor. The new circuit is solved using the initial voltage model of the charged capacitor, see the Appendix, in the s-domain.

The two-capacitor circuit including an additional resistor is shown in Fig. 3; (a) in the time domain, and (b) in the Laplace domain. In the Laplace domain we find easily by inspection

$$I_{\text{R}}(s) = I(s) = \frac{Q/sC}{1/sC + 1/sC + R} = \frac{Q/RC}{s + 2/RC} \tag{3}$$

Taking the inverse transform yields

$$i(t) = \frac{Q}{RC}e^{-2t/RC}, t > 0 \tag{4}$$

The energy dissipation by the resistor w_R is calculated using

$$w_{\mathrm{R}}(t) = R\int_0^t i^2(\tau)d\tau \tag{5}$$

$$= R\int_0^t \frac{Q^2}{R^2C^2}e^{-4\tau/RC}d\tau$$

$$= -\frac{Q^2}{4C}\left(e^{-4t/RC}-1\right) \tag{6}$$

In the limit of $t \to \infty$, the steady state, the amount of energy dissipated by the resistor is exactly the missing amount $Q^2/4C$. A similar analysis can also be done in the time domain, based on Heaviside operators, see for example [13].

It is interesting to investigate the current in the limit $R \to 0$. Using the delta sequence $\delta_n(t)$

$$\delta_n(t) = \begin{cases} 0 & t < 0 \\ ne^{-nt} & t > 0 \end{cases}$$

$n = 1, 2, 3, \ldots$

An expression for the current can be found for $R = 0$ by defining $\delta(t)$ as

$$\delta(t) = \lim_{a\to 0}\frac{1}{a}e^{-t/a} \tag{7}$$

using (4) and defining $a = RC/2$:

$$i(t)|_{R=0} = \lim_{R\to 0}\frac{Q}{RC}e^{-2t/RC} = \lim_{a\to 0}\frac{Q}{2}\frac{1}{a}e^{-t/a}$$

$$= \frac{Q}{2}\delta(t) \tag{8}$$

The delta-pulse current is also called an unbounded current [14]. This result, $i(t) = (Q/2)\delta(t)$, is also appearing in Section IV, where the circuit without any resistor is considered.

III. THEOREMS OF TELLEGEN AND HENDERSON

Tellegen's theorem is important in circuit theory. It only depends on the topology of the directed circuit graph, which shows the interconnection properties of the circuit. It holds for lumped circuits build up with both linear and nonlinear elements, and follows from the KCL and KVL equations; see for example [15] and refs. therein. Tellegen's theorem is briefly introduced.

Consider a circuit graph consisting of n nodes and b branches. For a given circuit graph we can write all linearly independent KCL equations as a matrix equation:

$$\mathbf{Ai} = \mathbf{0} \tag{9}$$

where $\mathbf{i} = [\mathbf{i_1 i_2} \cdots \mathbf{i_b}]^{\mathbf{T}}$ is the branch current vector, and $\mathbf{A}$ is the incidence matrix of all independent KCL equations. Using the $\mathbf{A}$ matrix the linearly independent KVL equations can also be written in a systematic way:

$$\mathbf{v} = \mathbf{A^T e} \tag{10}$$

where $\mathbf{v} = [\mathbf{v_1 v_2} \cdots \mathbf{v_b}]^{\mathbf{T}}$ is the branch voltage vector, and $\mathbf{e} = [\mathbf{e_1 e_2} \cdots \mathbf{e_{n-1}}]^{\mathbf{T}}$ is the node-to-datum voltage vector, also called node potential vector.

Tellegen's theorem can now be stated:

$$\sum_{k=1}^{b} v_k i_k = 0 \text{ or equivalently } \mathbf{v^T i} = \mathbf{0} \tag{11}$$

This can easily be proved: using (10) we write

$$\mathbf{v}^T = \left(\mathbf{A}^T\mathbf{e}\right)^T = \mathbf{e}^T\left(\mathbf{A}^T\right)^T = \mathbf{e}^T\mathbf{A}$$

and, using (9) we find:

$$\mathbf{v}^T\mathbf{i} = \mathbf{e}^T(\mathbf{Ai}) = \mathbf{0}$$

If we now consider a circuit at some time t Tellegen's theorem becomes

$$\mathbf{v}(t)^T\mathbf{i}(t) = \sum_{k=1}^{b} v_k(t)i_k(t) = 0 \tag{12}$$

When $v_k(t)i_k(t)$ is the power delivered, at time t, to branch k by the remainder of the circuit; and equivalently, $v_k(t)i_k(t)$ is the *rate* at which energy is delivered, at time t, to branch k, then (12) says that energy is conserved [14].

Henderson's theorem [16] predicts when a voltage across a capacitor in a circuit containing switches is continuous, that is, the voltage across a capacitor is the same immediately before and after switching, when one or more switches have changed their state. It is only in circuits in which capacitor voltages change discontinuously where the energy conservation paradox appears. As a consequence, Henderson's theorem limits the number of circuits in which the paradox may appear.

More precisely, Henderson's theorem can be stated as follows: in a network consisting of resistors, inductors, capacitors, switches operating simultaneously, d.c. voltage and current sources, one is sure that capacitor voltages are continuous except in those capacitors which, together with at least one make contact, form a loop in the circuit derived from the original by reducing all source intensities to zero.

Only an outline of the proof is given here. All switches operate at time $t = 0$. We call the instant immediately before operating $t = 0^-$ and the moment immediately after operating $t = 0^+$. Consider the make contact in Fig. 1. At the instant $t = 0^-$ the voltage across the make contact is $v(0^-)$. The current is zero. All currents and voltages in the circuit remain the same at $t = 0^-$ if this contact is replaced by a voltage source with the same time dependency as the voltage across the contact, and in particular with intensity $v(0^-)$ at $t = 0^-$. At $t = 0$ these source intensities are reduced to zero. That is, the make contact can be modeled by a stepping voltage source. Henderson calls these sources "switch voltage sources". If a circuit also contains other sources as energy sources, voltage sources can be replaced by short circuits and current sources can be replaced by open branches because, in the absence of switches, they do not cause any discontinuities in the network. Now, the proof shows that all capacitor voltages are continuous for non-zero but otherwise arbitrary chosen intensities of the switch sources, solving the circuit in the Laplace domain.

TABLE 1. **Current and voltages in the two-capacitor circuit $C_1 = C_2 = C$.**

t	$i(t)$	$v_{sw}(t)$	$v_{C_1}(t)$	$v_{C_2}(t)$
< 0	0	Q/C	Q/C	0
0	$(Q/2)\delta(t)$	undefined	undefined	undefined
> 0	0	0	Q/2C	Q/2C

The theorem predicts that the voltages across the capacitors in the two-capacitor problem will be continuous as soon as any resistor is added and, in absence of a resistor, will be discontinuous as soon as a loop exist with the switch and the other capacitor, provided there is a difference in initial voltage on the capacitors.

IV. THE TWO-CAPACITOR PROBLEM RESOLVED

There are three equivalent ways to resolve the two-capacitor problem. The first one involves basically only Tellegen's theorem. We consider the two-capacitor circuit consisting of three elements (three one-ports): two capacitors and the switch. Because Tellegen's theorem holds for linear elements, the capacitors, as well as nonlinear elements, the switch, energy in the circuit must be conserved. So, if the total energy of the two capacitors after switching is less than before, then the switch, considered as a one-port, must absorb this energy during the switching action; without modeling an additional resistor.

Secondly, to investigate whether this is indeed possible we do an explicit calculation of the current through and the voltage across the switch. To find the current, consider the circuit of Fig. 3 (b) with the resistor replaced by a short circuit representing the circuit in the closing action of the switch. By inspection

$$I(s) = \frac{Q/sC}{1/sC + 1/sC} = \frac{Q}{2} \tag{13}$$

Taking the inverse transform yields

$$i(t) = \frac{Q}{2}\delta(t) \tag{14}$$

half of the charge is transported to the uncharged capacitor between $t = 0^-$ and $t = 0^+$. The analysis can be done in the time domain too, see [17].

The voltage across the switch, v_{sw}, during the closing action steps from Q/C to zero, see also Table 1, and can be modeled as stepping voltage source (switch voltage source)

$$v_{sw}(t) = \frac{Q}{C}(1 - u(t)) \tag{15}$$

$u(t)$ being the unit step function. (For its definition see the Appendix.)

Now, consider the switch as a device (a one-port) then the energy dissipation, after the switch is closed, can be calculated; note that the direction of the current is into the switch, when described as a one-port, and the voltage across is positive or zero:

$$w_{sw}(t) = \int_{-\infty}^{t} v_{sw}(\tau)i(\tau)d\tau \tag{16}$$

$$= \frac{Q^2}{2C}\int_{-\infty}^{t}(1 - u(\tau))\delta(\tau)d\tau \tag{17}$$

The integral can be evaluated by noting that the delta function is nonzero only at $t = 0$.

$$w_{sw} = \frac{Q^2}{2C}\left(1 - \int_{0^-}^{0^+} u(t)\delta(t)dt\right) \tag{18}$$

The integral in (18) can be evaluated using the definition of the Dirac delta pulse being the derivative of the unit step function:

$$\int_{0^-}^{0^+} u(t)\delta(t)dt = \int_{0^-}^{0^+} u(t)\frac{du(t)}{dt}dt \tag{19}$$

$$= \left.\frac{u(t)^2}{2}\right|_{0^-}^{0^+} = \frac{1}{2} \tag{20}$$

And we obtain for the dissipation of the (ideal) switch *during* the switching action:

$$w_{sw} = \frac{Q^2}{2C}\left(1 - \frac{1}{2}\right) = \frac{Q^2}{4C} \tag{21}$$

exactly the missing energy.

We can use the initial condition models of both capacitors in the time domain, see the Appendix, to do energy calculations and combine the result with the dissipation of the switch from (21). Looking at Table 1, v_{C_1} is stepping from Q/C at $t = 0^-$ to $Q/2C$ at $t = 0^+$, while the current through the circuit is $i(t) = \frac{Q}{2}\delta(t)$, see (14). Following the passive sign convention we obtain:

$$\Delta w_{C_1} = \int_{0_-}^{0^+}\left(\frac{Q}{C} - \frac{Q}{2C}u(t)\right)\left(-\frac{Q}{2}\right)\delta(t)dt = -\frac{3}{8}\frac{Q^2}{C} \tag{22}$$

and v_{C_2} is stepping from 0 at $t = 0^-$ to $Q/2C$ at $t = 0^+$, while the current through the circuit is $i(t) = \frac{Q}{2}\delta(t)$

$$\Delta w_{C_2} = \int_{0_-}^{0^+}\left(\frac{Q}{2C}u(t)\right)\frac{Q}{2}\delta(t)dt = \frac{1}{8}\frac{Q^2}{C} \tag{23}$$

We see that energy is released at capacitor C_1, and absorbed at capacitor C_2 and the switch:

$$\Delta w_{C_1} + \Delta w_{C_2} + w_{sw} = \frac{Q^2}{C}\left(\frac{1}{8} - \frac{3}{8} + \frac{2}{8}\right) = 0 \tag{24}$$

In accordance with Tellegen's theorem, energy is always conserved in the two-capacitor circuit.

V. ENERGY EFFICIENCY

Energy efficiency of switched capacitors circuits is frequently discussed in papers. Review of literature shows that there are contradictory viewpoints on this issue, see [11] and refs. therein. Energy efficiency, η, determines the fraction of incoming energy that is passed to the output. In case of the two-capacitor circuit:

$$\eta_{C_2} = \frac{|\Delta w_{C_2}|}{|\Delta w_{C_1}|} = \frac{1}{3} \tag{25}$$

This result shows that $\frac{2}{3}|\Delta w_{C_1}|$ is lost (in the switch).

In a similar way the efficiencies of the single capacitor circuits, of Fig. 2, can be calculated. For the single switched capacitor circuit excited by a d.c. voltage source of Fig. 2(a) the current through the circuit, upon closing the switch, is $i(t) = CV\delta(t)$. The voltage across the switch is described by $v_{\mathrm{sw}}(t) = V(1-u(t))$. After the switching action the following equations are obtained:

$$\Delta w_C = \frac{CV^2}{2} \tag{26}$$

$$w_{\mathrm{source}} = -V\int_{0^-}^{0^+} i(\tau)d\tau = -CV^2 \tag{27}$$

$$w_{\mathrm{sw}} = CV^2\int_{0^-}^{0^+}(1-u(t))\delta(t)dt = \frac{CV^2}{2} \tag{28}$$

The energy efficiency can be calculated:

$$\eta_C = \frac{|\Delta w_C|}{|w_{\mathrm{source}}|} = \frac{1}{2} \tag{29}$$

and shows that exactly half of the energy is dissipated by the switch.

VI. GENERALIZATION TO ARBITRARY VALUES OF C_1, C_2, $Q_1(0^-)$, AND $Q_2(0^-)$

An expression for dissipation of the switch in case of arbitrary capacitor and initial charge values can be derived. Therefore (14) and (15) are generalized to:

$$i(t) = q_{\mathrm{tr}}\delta(t) \tag{30}$$

where q_{tr} is the transported charge through the switch at $t = 0$, and

$$v_{\mathrm{sw}}(t) = v_{\mathrm{sw}}\left(0^-\right)(1-u(t)) \tag{31}$$

and (21) for the energy dissipated by the switch becomes

$$w_{\mathrm{sw}} = \frac{1}{2}q_{\mathrm{tr}}v_{\mathrm{sw}}\left(0^-\right) \tag{32}$$

First, assuming $v_1(0^-) \geq v_2(0^-)$, the voltage across the switch is

$$v_{\mathrm{sw}}\left(0^-\right) = \frac{q_1\left(0^-\right)}{C_1} - \frac{q_2\left(0^-\right)}{C_2} \tag{33}$$

The transported charge is found by charge conservation and KVL:

$$q_1\left(0^+\right) + q_2\left(0^+\right) = q_1\left(0^-\right) + q_2\left(0^-\right) \tag{34}$$

$$\frac{q_1\left(0^+\right))}{C_1} = \frac{q_2\left(0^+\right))}{C_2} \tag{35}$$

Solving these equations gives:

$$q_{\mathrm{tr}} = \frac{C_2}{C_1+C_2}q_1\left(0^-\right) - \frac{C_1}{C_1+C_2}q_2\left(0^-\right) \tag{36}$$

We obtain for the dissipation:

$$w_{\mathrm{sw}} = \frac{\frac{q_1\left(0^-\right)^2 C_2}{2C_1} - q_1\left(0^-\right)q_2\left(0^-\right) + \frac{q_2\left(0^-\right)^2 C_1}{2C_2}}{C_1+C_2} \tag{37}$$

Assuming $v_1(0^-) < v_2(0^-)$, the voltage across the switch is negative but the current, flowing into the opposite direction, is negative too; the dissipation is described with the same formula (37). The circuit is symmetrical in the position of the capacitors.

VII. SIMILARITIES BETWEEN SINGLE-ELECTRON ELECTRONICS AND THE TWO-CAPACITOR PROBLEM

Equation (31) describing the voltage across the switch in the two-capacitor problem is, in fact, a special case of a more general expression successfully used in the description of the voltage across a tunneling junction in single-electron electronics. In single-electron electronics, a junction between two metal leads results in a nanoelectronic tunneling capacitor. If the voltage just before tunneling is $v(0^-)$ and the voltage just after tunneling is $v(0^+)$, then the more general equation that describes the voltage, within $t = [0^- \cdots 0^+]$, is

$$v(t) = v\left(0^-\right) + \left(v\left(0^+\right) - v\left(0^-\right)\right)u(t) \tag{38}$$

During tunneling through the tunnel barrier, that is at $t = 0$, the voltage is not defined; the probability to find the electron is just shifting from one side of the tunnel junction to the other. Filling in the values of the switch in the two-capacitor problem: $v(0^-) = Q/C$, $v(0^+) = 0$ immediately gives (15). Because the tunneling current of a single electron through a metallic tunnel junction is described by $i = q\delta(t)$ [18]—with q the elementary electron charge, dissipation of the tunneling junction is determined by

$$w_{\mathrm{TJ}} = q\int_{0^-}^{0^+}\left[v\left(0^-\right) + \left(v\left(0^+\right) - v\left(0^-\right)\right)u(t)\right]\delta(t)dt \tag{39}$$

performing the integration [19],

$$w_{\mathrm{TJ}} = \frac{q}{2}\left(v\left(0^+\right) + v\left(0^-\right)\right) \tag{40}$$

Equation (40) is generally accepted in theories on single-electron tunneling [19]–[23].

By making an analogy between nanoelectronics and the dissipating ideal switch, a possible physical interpretation can be investigated.

VIII. BRIEF INTRODUCTION TO SINGLE-ELECTRON ELECTRONICS

The observed experimental behavior of tunneling of single electrons through metal-insulator-metal junctions, two metals separated by a barrier, is basically threefold: electrons tunnel through a thin barrier, Coulomb blockade—electrons do not tunnel through the junctions—although the barrier is thin, and Coulomb oscillations. In this last case Coulomb blockade exists until a critical voltage is reached, then an electron tunnels and by doing so changing the polarity of the voltage across the metal-insulator-metal junction; a new blockade will be following. Key understanding in all three cases is the static or dynamic behavior of the Fermi-levels at both metal sides.

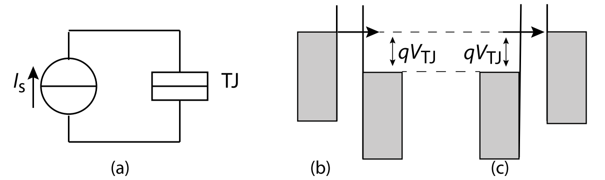

FIGURE 4. **(a) Tunnel junction excited by an ideal current source. (b) Energy-band diagram at $t = 0^-$. (c) Energy-band diagram at $t = 0^+$; $q = e/2$.**

The difference of Fermi-levels across a junction is direct proportional to the voltage across it, the constant being the electron charge. During closing the switch, the difference of Fermi-levels changes stepwise. As the Fermi-level across a junction is determined by the voltage across it and the tunnel current is described by a delta pulse, the description of stepping Fermi-levels during tunneling is directly related to the description of stepping voltages and delta currents as described in the first sections of this paper. An interpretation of dissipation in an ideal switch because of tunneling is worth being investigated.

Circuit theories predicting tunneling, Coulomb blockade, and Coulomb oscillation make the following assumptions. 1) electron energy quantization is ignored inside the conductors; 2) charge quantization is only considered near tunneling junctions (that is: single electrons are only visible during tunneling); 3) charge is considered to be continuous throughout the circuit with the only exceptions being the tunneling junctions; 4) in metals conduction electrons move on the Fermi-level ($T = 0$ K); 5) tunneling is only possible when electrons can tunnel to empty energy levels (just as is the case in a tunnel diode); 6) when many electrons will be tunneling at the same time it is assumed that the probability of tunneling during a given interval is statistically independent of the number of electrons that tunneled previously (that is, a Poisson distribution is assumed)—this assumption leads to a linear resistance in describing tunneling of many electrons; and 7) the time τ being the time for tunneling through a barrier and moving to the Fermi-level in the metal (often just called tunneling time) is assumed to be negligible small in comparison with other time scales, that is, we assume $\tau = \Delta t = 0$. (For tunnel junction of practical interest $\tau \approx 10^{-15}$s.)

Before coming to a physical interpretation of the energy loss in the switch during switching the following two topics are reviewed: tunneling under dynamic changing of the Fermi-levels in a circuit described with a bounded current, and tunneling when the Fermi-levels do not change in a circuit described with an unbounded (delta pulse) current. In fact, the interpretation of energy loss in the switch is based on dynamic changing of Fermi-levels in combination with unbounded currents and is akin to the interpretation of energy dissipation in single-electron transistors (SETs) excited by an ideal voltage source. However, a full explanation of SET circuits is beyond the scope of this paper and unnecessary for understanding dissipation of the switch.

The prototype circuit describing dynamic changing of Fermi-levels together with the existence of a bounded current is a circuit in which a tunnel junction is excited by an ideal current source. Figure 4 shows both the circuit and energy-band diagrams just before and just after tunneling of an electron. For an electron to tunnel an empty energy level must be available *after* tunneling. This is a well-known phenomenon in electronics because the operation of the tunnel diode is explained in this way. If this empty energy level is not present then the junction is in Coulomb blockade.

Because charge cannot be transported through the current source, in zero time, there is no current through the circuit and charge cannot be transferred outside the junction. The tunneling of an electron causes the Fermi-levels to change locally at the tunnel junction during tunneling. The difference in Fermi-levels just before and just after tunneling will decrease or even become negative (the negative electron tunnels towards the side that was positive before tunneling).

The requirement that there must be an empty energy level available after tunneling causes an electron to tunnel as soon as the charge q on the tunnel junction exceeds $q = e/2$, causing a single electron tunnel event. Continuously charging of the tunnel junction by the current source gives rise to Coulomb oscillations. In case of Coulomb oscillations, during tunneling, energy in the circuit is always conserved: the current source cannot deliver any energy in zero time; the tunnel junction, according to (40), doesn't dissipate energy because the voltage after tunneling is equal but opposite to the voltage before tunneling.

The prototype circuit describing fixed Fermi-levels together with the existence of a unbounded current is a circuit in which a tunnel junction is excited by an ideal voltage source (a delta-pulse current during tunneling through the voltage source in zero time is possible). Figure 5 shows the circuit and energy-band diagrams. Because the ideal voltage source fixes the Fermi-level, many electrons can and will tunnel for any non-zero voltage source. Energy conservation in this circuit is understood by modeling a *single* tunneling electron as a delta current through the circuit $i = e\delta(t)$. During tunneling a constant voltage source V_s provides energy w_s

$$w_s = \int_{0^-}^{0^+} V_s e\delta(t)dt = eV_s \tag{41}$$

while the junction dissipates according to (40) also eV_s, the voltage across the junction being the same before and after tunneling of an electron. A physical picture based on the band-energy diagram is possible. Suppose an electron is tunneling from the Fermi-level at the negative side of the tunnel junction towards the Fermi-level at the positive side, drawn as the dotted line in Fig. 5(b) to (c). Being a hot electron after tunneling it loses exactly eV_s energy, by collisions. Electrons below the Fermi-level can tunnel too, if they are facing empty energy levels. At the receiving side they will lose less energy. However, at the side they leave, the empty position will be filled causing energy loss (dissipation) at this side. The overall picture being that every

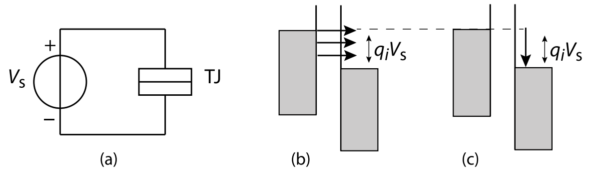

FIGURE 5. (a) Tunnel junction excited by an ideal voltage source. (b) Energy-band diagram at $t = 0^-$. (c) Energy-band diagram at $t = 0^+$; $q_i = e$, total charge $Q = \sum q_i$, and energy loss in the tunnel junction is Q^2/C.

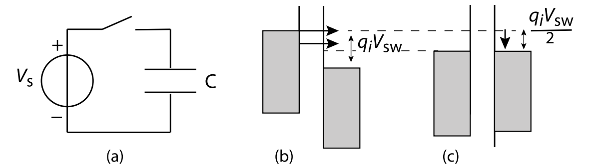

FIGURE 6. (a) Single capacitor circuit. (b) Energy-band diagram of the switch at $t = 0^-$; $q_i = e$, total charge $Q = \sum q_i$. (c) Energy-band diagram of the switch at $t = 0^+$; energy loss in the ideal switch is $Q^2/2C$.

electron is losing an amount of energy of eV_s. If all electrons that tunnel make up the total charge Q then the junction dissipates $QV_s = Q^2/C$. Due to the Poisson distribution describing the tunneling of *many* electrons the measurement of the (tunnel) current as a function of the applied voltage shows a linear (ohmic) behavior.

IX. PHYSICAL INTERPRETATION OF ENERGY LOSS OF A SWITCH DURING SWITCHING, $\Delta T = 0$

The ideal switch can be understood as dissipation by tunneling of electrons. Therefor we first consider the single capacitor circuit, see Fig. 6, with an initially uncharged capacitor.

Upon closing the switch, the current i through the circuit is $i = CV_s\delta(t) = Q\delta(t)$, Q being the amount of charge that is transported through the circuit, while the voltage across the switch is stepping from V_s to 0. The energy-band diagrams of the switch show that, due to the stepping voltage, the Fermi-levels change stepwise. The consequence of this is that only half of the electrons before closing the switch face empty energy levels *after* closing the switch, see Fig. 6 (b) and (c). It is only half of the charge that contributes to the dissipation, $w_{sw} = Q^2/2C$; a result in agreement with (40):

$$w_{sw} = \frac{Q}{2}(V_s + 0) = \frac{Q^2}{2C} \tag{42}$$

The other half of the charge tunneled under the condition that the Fermi-levels align, and do not contribute to dissipation.

Now, consider the two capacitor paradox. In this case the voltage across the switch steps from Q/C to $Q/2C$, see Table 1. Again only half of the electrons contribute to the dissipation; however the total charge that is transported from the initially charged capacitor to the initially uncharged capacitor is only $Q/2$. The (delta) current being only half of the value of the previously discussed single capacitor circuit; and conclude that the total dissipation is only $Q^2/4C$; in agreement with (21).

It should be noted that in both cases the tunneling of many electrons is considered. As a consequence of this, the switch shows ohmic behavior after all.

X. CONCLUSION

The well-known two-capacitor problem, in which energy conservation seems to be violated, can be solved by describing the current through the ideal switch a delta pulse and the voltage across it as a stepping voltage. As a consequence the ideal switch dissipates energy. This is in agreement with Tellegen's theorem if the ideal switch is considered to be a third element, one that dissipates energy. The dissipation can be described with a formalism that is used in the description of nanoelectronic single-electron tunneling devices. Although the solution of the paradox looks like a theoretical curiosity, a physical interpretation based on the tunneling of electrons across the closing switch is possible.

APPENDIX

In describing the switching action at $t = 0$ in the switched capacitor circuits, we distinguish $t = 0^-$ as the instant just before switching and $t = 0^+$ as the instant just after switching. This is in agreement with the description of the single-electron tunneling event, in which the voltage across the tunneling junction is well define just before tunneling and immediately after tunneling; during the tunneling action the voltage is not defined at all. The following definitions and relations are used in this paper.

The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t.

$$u(t) = \begin{cases} 0, & t \le 0^- \\ [0 \cdots 1] & t = 0 \\ 1, & t \ge 0^+ \end{cases} \tag{43}$$

At $t = 0$ the unit step function is not well defined, but bounded between 0 and 1.

The unit impulse function—or delta function—$\delta(t)$ is zero everywhere except at $t = 0$, where it is unlimited but has a finite nonzero area.

$$\delta(t) = \begin{cases} 0, & t \le 0^- \\ \text{unbounded}, & t = 0 \text{ with } \int_{0^-}^{0^+} \delta(t)dt = 1 \\ 0, & t \ge 0^+ \end{cases} \tag{44}$$

Due to the fact that the values $\delta(0)$ and $u(0)$ are only specified vaguely, the unit step and unit impulse functions are generalized functions. They define each other, by differentiation as

$$\delta(t) = \frac{du(t)}{dt} \tag{45}$$

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{46}$$

The one-sided Laplace transform of the unit impulse function is

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = e^{-0} = 1 \tag{47}$$

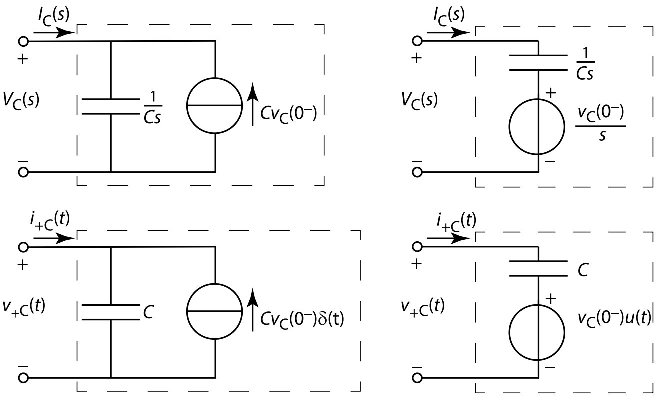

FIGURE 7. Parallel and serial initial condition models for the capacitor in Laplace and time domain.

that for the unit step function can be found using integration by parts

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} u(t)e^{-st}dt$$
$$= -\frac{1}{s}e^{-st}u(t)\Big|_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty}\delta(t)e^{-st}dt = \frac{1}{s} \quad (48)$$

Charged capacitors and tunnel junctions can be described by initial condition models in the Laplace or time domain. The initial condition model for the capacitor is obtained from the constitutional relation

$$i(t) = C\frac{dv(t)}{dt} \quad (49)$$

which transforms into the Laplace domain as

$$I(s) = sCV(s) - Cv\left(0^-\right) \quad (50)$$

the so-called parallel initial condition model, because two current terms are combined: a current term for an uncharged capacitor and a current source accounting for the initial charge. Or in voltage terms

$$V(s) = \frac{1}{sC}I(s) + \frac{v\left(0^-\right)}{s} \quad (51)$$

the so-called serial initial condition model, because two voltage terms are added. We find the initial condition models in the time domain by taking the inverse transforms [17]:

$$i_+(t) = i(t)u(t) = C\frac{dv(t)}{dt}u(t) - Cv\left(0^-\right)\delta(t) \quad (52)$$

and

$$v_+(t) = v(t)u(t) = \frac{1}{C}\int_{0^-}^{t} i(\tau)u(\tau)d\tau + v\left(0^-\right)u(t) \quad (53)$$

If only bounded currents are assumed, the continuity property of capacitor voltages hold and this last equation can be written as

$$v_+(t) = v(t)u(t) = \frac{1}{C}\int_{0^+}^{t} i(\tau)u(\tau)d\tau + v\left(0^-\right)u(t) \quad (54)$$

The models are shown in Fig. 7. If an unbounded current exist in the circuit at $t = 0$—as is the case in describing circuits only consisting of capacitors, switches, and voltage sources or tunneling of a single electron in circuits only consisting of tunnel junctions, capacitors, and ideal voltage sources (sometimes called low-ohmic environment)—then an additional term modeling the change in voltage across the capacitor or tunnel junction at $t = 0$ has to be added.

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