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On Strong Markov Property of Solutions to Stochastic Differential Equations on Hybrid State Spaces

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Abstract

In this paper we study a class of strong Markov solutions to stochastic differential equations on a hybrid state spaces. We construct stochastic hybrid processes as solutions to Ito-Skorohod type stochastic differential equations. Then we present strong existence and uniqueness results and show that under weak conditions these solutions are strong Markov processes.

Keywords: Stochastic hybrid systems; Strong Markov property; Hybrid jumps; Strong solutions; Existence and Uniqueness.

Mathematics Subject Classification: Primary 60G20; Secondary 60Hxx.

1 Introduction

Currently, the theory of stochastic differential equations (SDEs) has become a powerful tool for constructive description of various classes of stochastic processes including the processes which are Markov. Continuous time Markov processes such as diffusions, point processes and diffusions with jumps have been successfully used for years in stochastic modelling of various continuous time real-world dynamical systems with the Euclidean phase space. However, for realistic modelling of complex dynamical systems as, for example, in Air Traffic Management problems [1, 2, 3], a more general continuous time stochastic processes with hybrid state spaces containing both Euclidean and discrete valued components are required. Euclidean and discrete valued components may interact, i.e. Euclidean valued components may influence the dynamics of
discrete valued component and vice versa. This makes the modelling and the analysis of stochastic hybrid processes quite involved and challenging. Several classes of stochastic hybrid processes have been studied in the literature, e.g. counting processes with diffusion intensity [4, 5], diffusion processes with Markovian switching parameters [6, 7], switching diffusions [8, 9], piecewise deterministic Markov processes [10, 11, 12], Markov decision drift processes [13], stochastic hybrid systems [14, 15] and more recent SDE models on hybrid state spaces [16, 17, 18, 19, 20]. All these stochastic hybrid processes arise in various kind of applications, have different degree of modelling power and have different properties inherent to the problems that they have been developed for. In this paper we restrict our attention to modelling by Markov processes which are defined as solutions to SDE [19, 20]. Our main goal of this paper is to prove that under weak conditions the solutions to SDEs on hybrid state spaces are strong Markov processes. This result is of great importance in many applications. One of the applications, which actually motivated this research, is development of efficient rare event estimation algorithms based on Interacting Particle Systems (IPS) simulation approach [21, 22, 23]. The key requirement in order to apply an IPS based simulation approach is that stochastic process under consideration is a strong Markov process.

This paper is organized as follows. The construction of an SDE on a hybrid state space and prove of existence and uniqueness of solutions are presented in Section 2. The strong Markov property of solutions is proven in Section 3.

2 Stochastic Hybrid Processes as Solutions of SDE

In this section we construct a switching jump diffusion \( \{X_t, \theta_t\} \) taking values in \( \mathbb{R}^n \times M \), where \( M = \{e_1, e_2, \ldots, e_N\} \) is a finite set. We assume that for each \( i = 1, \ldots, N \), \( e_i \) is the \( i \)-th unit vector, \( e_i \in \mathbb{R}^N \). Let \( \{X_t, \theta_t\} \) be an \( \mathbb{R}^n \times M \)-valued process given by the following stochastic differential equation of Ito-Skorohod type (see Appendix A).

\[
\begin{align*}
    dX_t &= a(X_t, \theta_t)dt + b(X_t, \theta_t)dW_t + \int_{\mathbb{R}} g_1(X_{t-}, \theta_{t-}, u)g_1(dt, du) + \int_{\mathbb{R}} g_2(X_{t-}, \theta_{t-}, u)p_2(dt, du), \\
    d\theta_t &= \int_{\mathbb{R}} c(X_{t-}, \theta_{t-}, u)p_2(dt, du).
\end{align*}
\]

(2.1) (2.2)

Here:

(i) for \( t = 0 \), \( X_0 \) is a prescribed \( \mathbb{R}^n \)-valued random variable.

(ii) for \( t = 0 \), \( \theta_0 \) is a prescribed \( M \)-valued random variable.

(iii) \( W \) is an \( m \)-dimensional standard Wiener process.
(iv) \( q_1(dt, du) \) is a martingale random measure associated to a Poisson random measure \( p_1 \) with intensity
\( dt \times m_1(du) \), where \( m_1 \) is the Lebesgue measure on \( \mathbb{R} \).

(v) \( p_2(dt, du) \) is a Poisson random measure with intensity \( dt \times m_2(du) \), where \( dt \times m_2(du) \) is the Lebesgue measure on \( \mathbb{R} \).

(vi) Wiener process \( W \) and Poisson random measures \( p_1 \) and \( p_2 \) are mutually independent.

The coefficients are assumed to be measurable.

\[
a : \mathbb{R}^n \times M \to \mathbb{R}^n
\]
\[
b : \mathbb{R}^n \times M \to \mathbb{R}^{n \times m}
\]
\[
g_1 : \mathbb{R}^n \times M \times \mathbb{R}^d \to \mathbb{R}^n
\]
\[
g_2 : \mathbb{R}^n \times M \times \mathbb{R} \to \mathbb{R}^n
\]
\[
c : \mathbb{R}^n \times M \times \mathbb{R} \to \mathbb{R}^N.
\]

Function \( c(\cdot, \cdot, \cdot) \) is defined by
\[
c(x, e_i, u) = \begin{cases} 
e_j - e_i & \text{if } u \in \Delta_{ij}(x), \\ 0 & \text{otherwise,} \end{cases}
\] (2.3)

where for \( i, j \in \{1, \ldots, N\}, i \neq j, x \in \mathbb{R}^n \), \( \Delta_{ij}(x) \) are the intervals of the real line defined as:

\[
\Delta_{12}(x) = [0, \lambda_{12}(x))
\]
\[
\Delta_{13}(x) = [\lambda_{12}(x), \lambda_{12}(x) + \lambda_{13}(x))
\]
\[\vdots\]
\[
\Delta_{1N}(x) = \left[ \sum_{j=2}^{N-1} \lambda_{1j}(x), \sum_{j=2}^{N} \lambda_{1j}(x) \right)
\]
\[
\Delta_{21}(x) = \left[ \sum_{j=2}^{N} \lambda_{1j}(x), \sum_{j=2}^{N} \lambda_{1j}(x) + \lambda_{21}(x) \right)
\]

and so on. In general,

\[
\Delta_{ij}(x) = \left[ \sum_{j'=1}^{i-1} \sum_{j''=1}^{N} \lambda_{ij'}(x), \sum_{j'=1}^{i-1} \sum_{j''=1}^{N} \lambda_{ij'}(x) + \sum_{j'=1}^{i} \sum_{j''=1}^{N} \lambda_{ij'}(x) + \sum_{j'=1}^{j} \lambda_{ij'}(x) \right).
\]

For fixed \( x \) these are disjoint intervals, and the length of \( \Delta_{ij}(x) \) is \( \lambda_{ij}(x) \).

\( \lambda_{ij} : \mathbb{R}^n \to \mathbb{R}, i, j = 1, \ldots, N, i \neq j. \)
Let $K_1$ be the support of $g_2(\cdot, \cdot, \cdot)$ and let $U_1$ be the projection of $K_1$ on $\mathbb{R}$. It is assumed that $U_1$ is bounded. Let $K_2$ denote the support of $c(\cdot, \cdot, \cdot)$ and $U_2$ the projection of $K_2$ on $\mathbb{R}$. By definition of $c$, $U_2$ is a bounded set. One can define function $g_2(\cdot, \cdot, \cdot)$ so that the sets $U_1$ and $U_2$ form three nonempty sets: $U_1 \setminus U_2$, $U_1 \cap U_2$ and $U_2 \setminus U_1$ (see Figure 1). Then, we have the following:

(i) For $u \in U_1 \cap U_2$

\[
\begin{align*}
g_2(\cdot, \cdot, u) &\neq 0 \\
c(\cdot, \cdot, u) &\neq 0
\end{align*}
\]

i.e., simultaneous jumps of $X_t$ and switches of $\theta_t$ are possible.

(ii) For $u \in U_2 \setminus U_1$

\[
\begin{align*}
g_2(\cdot, \cdot, u) &= 0 \\
c(\cdot, \cdot, u) &\neq 0
\end{align*}
\]

i.e., only random switches of $\theta_t$ are possible.

(iii) For $u \in U_1 \setminus U_2$

\[
\begin{align*}
g_2(\cdot, \cdot, u) &\neq 0 \\
c(\cdot, \cdot, u) &= 0
\end{align*}
\]

i.e., only random jumps of $X_t$ are possible.

We make the following assumptions on the coefficients of SDE (2.1)-(2.2).

(A1) There exists a constant $l$ such that for all $i = 1, 2, \ldots, N$

\[
|a(x, e_i)|^2 + |b(x, e_i)|^2 + \int_{\mathbb{R}} |g_1(x, e_i, u)|^2 m_1(du) \leq l(1 + |x|^2).
\]
(A2) For any \( r > 0 \) one can specify constant \( l_r \) such that for all \( i = 1, 2, \ldots, N \)

\[
|a(x, e_i) - a(y, e_i)|^2 + |b(x, e_i) - b(y, e_i)|^2 + \int_{\mathbb{R}} |g_1(x, e_i, u) - g_1(y, e_i, u)|^2 m_1(du) \leq l_r|x - y|^2
\]

for \( |x| \leq r, |y| \leq r. \)

(A3) Function \( c \) is defined by (2.3) where for all \( i, j = 1, 2, \ldots, N \), functions \( \lambda_{ij}(\cdot) \) are bounded and Lipschitz, \( \lambda_{ij}(\cdot) \geq 0 \) for \( i \neq j \) and \( \sum_{j=1}^{N} \lambda_{ij}(\cdot) = 0 \) for any \( i \in \{1, \ldots, N\} \).

(A4) \( U_1 \), the projection of support of \( g_2(\cdot, \cdot, \cdot) \) on \( \mathbb{R} \), is bounded and

\[
\int_0^t \int_{\mathbb{R}} |g_2(X_{s-}, \theta_{s-}, u)|p_2(ds, du) < \infty, \text{ P-a.s.}
\]

Theorem 2.1. Assume (A1)-(A4). Let \( p_1, p_2, W, X_0 \) and \( \theta_0 \) be independent. Then SDE (2.1)-(2.2) has a unique strong solution.

Proof. See Appendix B.

Our next goal is to prove Markov and Strong Markov properties, but before that we will need some additional results which will be useful in the subsequent section.

Lemma 2.2. Suppose functions \( \lambda_{ij}(\cdot) \) \( i, j = 1, \ldots, N \) satisfy (A3). Then there exist a constant \( C_c \) such that

\[
\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz \leq C_c(|x - y| + 1) \text{ for } i \neq k
\]

and

\[
\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz \leq C_c(|x - y|) \text{ for } i = k
\]

for all \( x, y \in \mathbb{R}^n \) and \( e_i, e_k \in \mathbb{M} \). In other words,

\[
\int_{\mathbb{R}} |c(\xi, z) - c(\zeta, z)|^2 dz \leq C_c(|\xi - \zeta|),
\]

where \( \xi = (x, \theta) \in \mathbb{R} \times \mathbb{M} \) and \( \zeta = (y, \eta) \in \mathbb{R} \times \mathbb{M} \).

Proof. See Appendix C

Lemma 2.3. Function \( c(\xi, u), (\xi = (x, \theta) \in \mathbb{R}^n \times \mathbb{M} \subset \mathbb{R}^{n+N}) \), is continuous in measure \( m_2 \) w.r.t. \( \xi \) for almost all \( u \).
Proof. Let \( \xi_n \) be a sequence which converges to \( \xi \), i.e. \( \lim_{n \to \infty} \xi_n = \xi \). Then from Lemma 2.2 follows that
\[
||c(\xi_n, \cdot) - c(\xi, \cdot)||_{L^2(m_2)} \xrightarrow{n \to \infty} 0,
\]
i.e. function \( c(\xi, u) \) is continuous in \( L^2(m_2) \) w.r.t. \( \xi \). This implies continuity in measure \( m_2 \) w.r.t. \( \xi \) for almost all \( u \). \( \square \)

Remark 2.4. Suppose function \( g_2(\xi, u) \) is continuous in measure \( m_2 \) w.r.t. \( \xi \) for almost all \( u \). Then a vector function
\[
\tilde{f}_2(\xi, u) = \begin{bmatrix} g_2(\xi, u) \\ c(\xi, u) \end{bmatrix}
\]
is continuous in measure \( m_2 \) w.r.t. \( \xi \) for almost all \( u \).

Theorem 2.5. Assume (A1)-(A4) and that \( g_2(\xi, u) \) is continuous in measure \( m_2(du) \) w.r.t. \( \xi \) for almost all \( u \). Then for all \( t > 0 \) and \( \varepsilon > 0 \)
\[
\lim_{n \to \infty} P(\sup_{s \leq t} |\xi_s^{0,\eta_n} - \xi_s^{0,\eta}| > \varepsilon) = 0,
\]
where \( \eta_n \) denotes a converging sequence with limit \( \eta \) in \( \mathbb{R}^{n+N} \).

Proof. See [24]. \( \square \)

3 Markov and Strong Markov Properties

Now we are ready to prove Markov and Strong Markov properties.

Assume we are given the following objects:

- a measurable space \( (S, \mathcal{S}) \);
- a measurable space \( (\Omega, \mathcal{G}) \) and a family of \( \sigma \)-algebras \( \{\mathcal{G}_t^s, 0 \leq s \leq t \leq \infty\} \), such that \( \mathcal{G}_t^s \subset \mathcal{G}_u^s \subset \mathcal{G} \) provided \( 0 \leq u \leq s \leq t \leq v \); \( \mathcal{G}_t^s \) denotes a \( \sigma \)-algebra of events on time interval \([s, t]\); we write \( \mathcal{G}_t \) in place of \( \mathcal{G}_t^0 \) and \( \mathcal{G}^s \) in place of \( \mathcal{G}_t^\infty \);
- a probability measure \( P_{s,x} \) for each pair \( (s, x) \in [0, \infty) \times S \) on \( \mathcal{G}^s \);
- a function (stochastic process) \( \xi_t(\omega) = \xi(t, \omega) \) defined on \([0, \infty) \times \Omega\) with values in \( S \).

The system consisting of these four objects will be denoted by \( \{\xi_t, \mathcal{G}_t^s, P_{s,x}\} \) [25].
Definition 3.1. A system of objects \( \{ \xi_t, \mathcal{G}_t^s, P_{s,x} \} \) is called a Markov process provided:

(i) for each \( t \in [0, \infty) \) \( \xi_t(\omega) \) is measurable mapping of \( (\Omega, \mathcal{G}) \) into \( (S, \mathcal{I}) \);

(ii) for arbitrary fixed \( s, t \) and \( B (0 \leq s \leq t, B \in \mathcal{I}) \) the function \( Q(s, x, t, B) = P_{s,x}(\xi_t \in B) \) is \( \mathcal{I} \)-measurable with respect to \( x \);

(iii) \( P_{s,x}(\xi_s = x) = 1 \) for all \( s \geq 0 \) and \( x \in S \);

(iv) \( P_{s,x}(\xi_u \in B \mid \mathcal{G}_s^u) = P_{t,s}(\xi_u \in B) \) for all \( s, t, u, 0 \leq s \leq t \leq u < \infty, x \in S \) and \( B \in \mathcal{I} \).

The measure \( P_{s,x} \) should be considered as a probability law which determines the probabilistic properties of the process \( \xi(\omega) \) given that it starts at point \( x \) at the time \( s \). Condition (iv) in Definition 3.1 expresses the Markov property of the processes. Let \( E_{s,x} \) denote the expectation with respect to measure \( P_{s,x} \). For \( \mathcal{G}^s \)-measurable random variable \( \xi(\omega) \)

\[
E_{s,x}[\xi(\omega)] = \int \xi(\omega)P_{s,x}(d\omega).
\]

It is not difficult to show that the Markov property (iv) in Definition 3.1 can be rewritten in terms of expectations as follows:

\[
E_{s,x}[f(\xi_u) \mid \mathcal{G}_s^u] = E_{t,s}[f(\xi_u)], \quad 0 \leq s \leq t \leq u < \infty,
\]

where \( f \) is an arbitrary \( \mathcal{I} \)-measurable bounded function.

Next, following the approach used in [24] we can prove Markov property of solutions of SDEs on a hybrid state space.

Theorem 3.2. Assume that conditions of Theorem 2.5 are satisfied. Then solutions of SDE (2.1-2.2) are Markov.

Proof. Let \( \xi^{s,\eta}_t = (X_t^{s,x}, \theta_t^{s,\eta}) \) denote the solution of SDE (2.1-2.2) on \( [s, \infty) \) satisfying initial condition \( \xi^{s,\eta}_s = \eta = (X_s^{s,x}, \theta_s^{s,\eta}) \). Note that now \( S = \mathbb{R}^n \times M \) and \( \mathcal{I} = \mathcal{B}_{\mathbb{R}^n \times M} \) is the \( \sigma \)-algebra of Borel sets on \( \mathbb{R}^n \times M \). Let \( \mathcal{F}_t^{s} \), \( s < t \) be the \( \sigma \)-algebras generated by \( \{ W_u - W_s, p_1([s, u], dz), p_2([s, u], dz), u \in [s, t] \} \), \( \mathcal{F}_0^{s} = \mathcal{F}_s, \mathcal{F}_\infty^{s} = \mathcal{F}^s \). For \( s \leq t \) the \( \sigma \)-algebras \( \mathcal{F}_s \) and \( \mathcal{F}^s \) are independent. Process \( \xi^{s,\eta}_t \) is \( \mathcal{F}^s \)-measurable, hence, it is independent of \( \sigma \)-algebra \( \mathcal{F}_s \). From Theorem 2.5 follows that \( \xi^{s,\eta}_t \) is stochastically continuous w.r.t. initial value \( \eta \) at time \( s \). Thus, \( \xi^{s,\eta}_t \) depends on \( \eta \) in measurable way, i.e. for fixed \( s \) it is measurable w.r.t. \( \mathcal{B}_{\mathbb{R}^n \times M} \times \mathcal{F}^s \). Let \( \eta_s \) be an arbitrary \( \mathbb{R}^n \times M \)-valued \( \mathcal{F}_s \) measurable random variable. Then \( \xi^{s,\eta}_t, t \geq s \), is unique \( \mathcal{F}_t \)-measurable solution of SDE (2.1-2.2) on \( [s, \infty) \) satisfying the initial condition \( \xi^{s,\eta}_s = \eta_s \).
Since for \( u < s \) process \( \xi_{t}^{u,y} \) is \( \mathcal{F}_{t} \)-measurable on \([s, \infty)\) with initial condition \( \xi_{s}^{u,y} \) then the following equality holds
\[
\xi_{t}^{u,y} = \xi_{t}^{s,\xi_{s}^{u,y}}, \ u < s < t. \tag{3.1}
\]

Let \( \varphi \) be a bounded measurable function on \( \mathbb{R}^{n} \times \mathbb{M} \), let \( \zeta_{s} \) be an arbitrary bounded \( \mathcal{F}_{s} \)-measurable quantity.
The independence of \( \mathcal{F}_{s} \) and \( \mathcal{F}_{s} \) and the Fubini theorem imply that measure \( P \) on \( \mathcal{F}_{\infty} \) is a product of measures \( P_{s} \) and \( P^{s} \), where \( P_{s} \) is a restriction of \( P \) on \( \mathcal{F}_{s} \), where \( P^{s} \) is a restriction of \( P \) on \( \mathcal{F}^{s} \), and
\[
\mathbb{E}[\varphi(\xi_{t}^{u,y})\zeta_{s}] = \mathbb{E}[\varphi(\xi_{s}^{x,y})\zeta_{s}] = \mathbb{E}[(\mathbb{E}[\varphi(\xi_{t}^{s,x})])_{x=\xi_{s}^{u,y}}].
\]

Since \( \xi_{s}^{u,y} \) is \( \mathcal{F}_{s} \)-measurable then
\[
\mathbb{E}[\varphi(\xi_{t}^{u,y}) | \mathcal{F}_{s}] = \mathbb{E}[\varphi(\xi_{s}^{x,y})]_{x=\xi_{s}^{u,y}}. \tag{3.2}
\]

If \( \xi_{t} \) is an arbitrary process defined by (2.1-2.2), by the same reasoning with help of which equalities (3.1) and (3.3) have been obtained, one can show that \( \xi_{t} = \xi_{t}^{s,\xi_{s}} \) for \( s < t \) and that
\[
P(\xi_{t} \in B | \mathcal{F}_{s}) = Q(s, \xi_{s}^{u,y}, t, B). \tag{3.3}
\]

Hence, the process defined by (2.1-2.2) is a Markov process with transition probability \( Q(s, x, t, B) \) defined by (3.3). To be precise, we have shown that the system of objects \( \{(X_{t}, \theta_{t}), \mathcal{F}_{t}, P_{s(x, \theta)}\} \) , where \( P_{s(x, \theta)}(\{(X_{t}, \theta_{t}) \in B\}) = Q(s, (x, \theta), t, B) = P(\{(X_{t}^{s,x}, \theta_{t}^{s,x}) \in B\}, B \in \mathcal{B}_{\mathbb{R}^{n} \times \mathbb{M}}, \) is a Markov process. \( \square \)

In what follows we prove the Markov property
\[
P_{s,x}(\xi_{u} \in B | \mathcal{F}_{t}^{u}) = P_{t,\xi_{t}}(\xi_{u} \in B), \ s \leq t \leq u
\]
remains valid also when a fixed time moment \( t \) is replaced by a stopping time.

Let \( \{\xi_{t}(\omega), \mathcal{F}_{t}^{s}, P_{s,x}\} \) be a Markov process in the space \((S, \mathcal{F})\). Let \( \mathcal{T} \) denote the \( \sigma \)-algebra of Borel sets on \([0, \infty)\).

**Definition 3.3.** A Markov process is called strong Markov if:
(i) the transition probability $P(s, x, t, B)$ for a fixed $B$ is a $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$-measurable function of $(s, x, t)$ on the set $0 \leq s \leq t < \infty, x \in S$;

(ii) it is progressively measurable;

(iii) for any $s \geq 0, t \geq 0, \mathcal{S}$-measurable function $f(x)$ and arbitrary stopping time $\tau$,

\[ E_{s,x}[f(\xi_{t+\tau}) \mid \mathcal{G}_\tau^s] = E_{\tau,\xi_\tau}[f(\xi_{t+\tau})]. \]  

(3.4)

Remark 3.4. For Equation (3.4) to be satisfied, it is necessary that the random variable $g(\xi_\tau, \tau, t + \tau) = E_{\tau,\xi_\tau}[f(\xi_{t+\tau})]$ be $\mathcal{G}_\tau^s$-measurable. For this reason assumptions (i) and (ii) make part of the definition of the strong Markov property [25].

Now we return to the process $\xi_t = (X_t, \theta_t)$ defined by (2.1-2.2). We have shown that it is a Markov process. The following theorem proves that it is also a Strong Markov process.

**Theorem 3.5.** Assume that conditions of Theorem 2.5 are satisfied. Then for any bounded Borel function $f : \mathbb{R}^n \times M \to \mathbb{R}$ and any $\mathcal{F}_t^s$-stopping time $\tau$

\[ E_{s,x}[f(\xi_{t+\tau}) \mid \mathcal{F}_\tau^s] = E_{\tau,\xi_\tau}[f(\xi_{t+\tau})], \]

i.e. strong Markov property holds.

**Proof.** Let us introduce on function space $C_{\mathbb{R}^n \times M}$ the following operators:

\[ Q_t^s \varphi(\eta) = \int Q(s, \eta, t, d\xi)\varphi(\xi). \]  

(3.5)

We will show that $Q_t^s$ maps $C_{\mathbb{R}^n \times M}$ to $C_{\mathbb{R}^n \times M}$ (this means that Markov process, defined by solution of SDE (2.1-2.2), is Feller process). Theorem 2.5 implies that, for fixed $s$ and $t$, $\xi_t^{\eta,n}$ is stochastically continuous w.r.t. $\eta$. Hence $E[\varphi(\xi_t^{\eta,n})]$ is also continuous w.r.t. $\eta$ for all $\varphi \in C_{\mathbb{R}^n \times M}$, and by (3.2)

\[ E[\varphi(\xi_t^{\eta,n})] = \int Q(s, \eta, t, d\xi)\varphi(\xi) = Q_t^s \varphi(\eta). \]

We finish the proof of the theorem by using the property that the natural filtration of a càdlàg Feller process is right continuous when completed [26, Thm. 47, pp. 36], i.e. $\mathcal{F}_t^s = \mathcal{F}_t^{s^+}$, and the fact that a càdlàg Markov process satisfying the Feller property, given $\mathcal{F}_t^{s^+} = \mathcal{F}_t^s$, also satisfies the strong Markov property [27, Thm. 2.4, pp. 26], i.e.

\[ E_{s,x}[f(\xi_{t+\tau}) \mid \mathcal{F}_\tau^s] = E_{\tau,\xi_\tau}[f(\xi_{t+\tau})], \]
for any bounded Borel function $f : \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}$ and any $\mathcal{F}_t^\tau$-stopping time $\tau$.

\section{Ito-Skorohod type SDE}

Throughout this section we assume that a probability space $(\Omega, \mathcal{F}, P)$ is equipped with a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a “completed” stochastic basis endowed with:

(i) $W = (W^i)_{i \leq m}$, an $m$-dimensional standard Wiener process (i.e., each $W^i$ is a standard Wiener process, and the $W^i$’s are independent);

(ii) $p_i$ are Poisson random measures on $\mathbb{R}_+ \times U$ with intensity measure $dt \cdot m_i(du)$, $i = 1, 2$; here, $(U, \mathcal{U})$ is an arbitrary Blackwell space (one may take $U = \mathbb{R}^d$ for practical applications), and $m_i$, $i = 1, 2$, is a positive $\sigma$-finite measure on $U, \mathcal{U}$; We denote the compensated Poisson random measure by $q_i(dt, du) = p_i(dt, du) − dt \cdot m_i(du)$, $i = 1, 2$;

(iii) We assume that Wiener process $W$ and Poisson random measures $p_1$ and $p_2$ are mutually independent.

Let us also be given the coefficients:

\begin{equation}
\begin{aligned}
a = (a^i)_{i \leq n}, & \quad \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \\
b = (b^i)_{i \leq n, j \leq m}, & \quad \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m \\
f_1 = (f_1^i)_{i \leq n}, & \quad \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \times U \to \mathbb{R}^n, \\
f_2 = (f_2^i)_{i \leq n}, & \quad \text{a Borel function: } \mathbb{R}_+ \times \mathbb{R}^n \times U \to \mathbb{R}^n.
\end{aligned}
\end{equation}

Let the initial variable be an $\mathcal{F}_0$-measurable $\mathbb{R}^n$-valued random variable $X_0$. The stochastic differential equation is as follows:

\begin{equation}
dX_t = a(t, X_t)dt + b(t, X_t)dW_t + \int_U f_1(t, X_{t-}, u)q_1(dt, du) + \int_U f_2(t, X_{t-}, u)p_2(dt, du). \tag{A.2}
\end{equation}

By a solution of the SDE (A.2) we mean a càdlàg $\mathcal{F}_t$-adapted process $\{X_t\}$ such that the following equation is satisfied with probability one for every $t \in \mathbb{R}_+$

\begin{equation}
X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t \int_U f_1(s, X_{s-}, u)q_1(ds, du) \\
+ \int_0^t \int_U f_2(s, X_{s-}, u)p_2(ds, du). \tag{A.3}
\end{equation}
In what follows we state strong existence and strong uniqueness theorem for SDE (A.2). The proof is similar to that of [24, pp.223-245] and can be also found in [28, 20].

**Theorem A.1.** Assume that the following conditions hold:

(i) There exists a constant \( l \) such that

\[
|a(t, x)|^2 + |b(t, x)|^2 + \int_U |f_1(t, x, u)|^2 m_1(du) \leq l(1 + |x|^2),
\]

(ii) for each \( r > 0 \) there exist a constant \( l_r \), for which

\[
|a(s, x) - a(s, y)|^2 + |b(s, x) - b(s, y)|^2 + \int_U |f_1(s, x, u) - f_1(s, y, u)|^2 m_1(du) \leq l_r|x - y|^2,
\]

for all \( |x| \leq r, |y| \leq r, s \leq r \).

(iii) Function \( f_2(s, x, u) \) is bounded.

(iv) \( m_2(S_u) < \infty \), where \( S_u \) is the projection on space \( U \) of the support of function \( f_2(s, x, u) \).

(v) \( X_0 \) is independent of \( \{W_s, q_1(ds, du), p_2(ds, du)\} \).

(vi) Let \( \mathcal{F}_t \) be the \( \sigma \)-algebra generated by \( X_0 \) and \( \{W_s, q_1([0, s], du), p_2([0, s], du), s \leq t\} \).

Then there exists a unique \( \mathcal{F}_t \)-measurable solution of SDE (A.2).

**B Existence and uniqueness of Itô-Skorohod SDE on hybrid state space**

**Theorem B.1.** Assume (A1)-(A4). Let \( p_1, p_2, W, X_0 \) and \( \theta_0 \) be independent. Then SDE (2.1)-(2.2) has a unique strong solution.

Proof. The switching jump diffusion \( \{X_t, \theta_t\} \) governed by Equations (2.1)-(2.2) can be seen as the \( \mathbb{R}^{n+N} \)-valued jump diffusion \( \{\xi_t \triangleq (X_t, \theta_t)^T\} \) governed by the stochastic differential equation

\[
d\xi_t = \tilde{a}(\xi_t)dt + \tilde{b}(\xi_t)dW_t + \int_{\mathbb{R}} \tilde{f}_1(\xi_t, u)q_1(dt, du) + \int_{\mathbb{R}} \tilde{f}_2(\xi_t, u)p_2(dt, du) \quad (B.1)
\]
with the following coefficients:

\[ \tilde{a} : \mathbb{R}^{n+N} \to \mathbb{R}^{n+N} \quad \tilde{a}(\cdot) \triangleq [a(\cdot), O^N]^T \]

\[ \tilde{b} : \mathbb{R}^{n+N} \to \mathbb{R}^{(n+N) \times m} \quad \tilde{b}(\cdot) \triangleq [b(\cdot), O^{N \times m}]^T \]

\[ \tilde{f}_1 : \mathbb{R}^{n+N} \times \mathbb{R} \to \mathbb{R}^{n+N} \quad \tilde{f}_1(\cdot, \cdot) \triangleq [g_1(\cdot, \cdot), O^N]^T \]

\[ \tilde{f}_2 : \mathbb{R}^{n+N} \times \mathbb{R} \to \mathbb{R}^{n+N} \quad \tilde{f}_2(\cdot, \cdot) \triangleq [g_2(\cdot, \cdot), c(\cdot, \cdot)]^T \]

where by \( O^k \) and \( O^{k \times s} \) we denote the \( k \)-dimensional zero vector and \( k \times s \)-dimensional zero matrix correspondingly.

Next we show that conditions (A1)-(A4) imply the conditions of Theorem A.1 thus the Equation (B.1) has an a.s. unique strong solution which implies that SDE (2.1)-(2.2) has an a.s. unique strong solution.

Let us verify all conditions.

**Growth condition:** by (A1) for every \( \xi = (x, e_i)^T \in \mathbb{R}^{n+N} \quad i = 1, \ldots, N \) we have

\[
|\tilde{a}(\xi)|^2 + |\tilde{b}(\xi)|^2 + \int_{\mathbb{R}} |\tilde{f}_1(\xi, u)|^2 m_1(du) \\
= |\tilde{a}(x, e_i)|^2 + |\tilde{b}(x, e_i)|^2 + \int_{\mathbb{R}} |\tilde{f}_1(x, e_i, u)|^2 m_1(du) \\
= |a(x, e_i)|^2 + |b(x, e_i)|^2 + \int_{\mathbb{R}} |g_1(x, e_i, u)|^2 m_1(du) \\
\leq l(1 + |x|^2) \leq l(1 + |x|^2 + |e_i|^2) = l(1 + |\xi|^2). 
\]

**Lipschitz condition:** From (A1) and (A2) it follows that for any \( r > 0 \) one can specify a constant \( L_r \) such that for all \( \xi = (x, e_i)^T \in \mathbb{R}^{n+N}, \xi = (y, e_j)^T \in \mathbb{R}^{n+N} \quad i, j = 1, \ldots, N \), and for \( |x| < r, |y| < r \), i.e. \( |\xi| \leq \sqrt{r^2 + 1}, |\zeta| \leq \sqrt{r^2 + 1} \), we have
\[|\tilde{a}(\xi) - \bar{a}(\zeta)|^2 + |\tilde{b}(\xi) - \bar{b}(\zeta)|^2 + \int_{\mathbb{R}} |\tilde{f}_1(\xi, u) - \bar{f}_1(\zeta, u)|^2 m_1(du)\]

\[= |a(x, e_i) - a(y, e_j)|^2 + |b(x, e_i) - b(y, e_j)|^2 + \int_{\mathbb{R}} |g_1(x, e_i, u) - g_1(y, e_j, u)|^2 m_1(du)\]

\[\leq 2(|a(x, e_i) - a(y, e_j)|^2 + |b(x, e_i) - b(y, e_j)|^2 + \int_{\mathbb{R}} |g_1(x, e_i, u) - g_1(y, e_j, u)|^2 m_1(du))\]

\[\leq 2(l_r |x - y|^2 + 4(|a(y, e_i)|^2 + |b(y, e_i)|^2 + \int_{\mathbb{R}} |g_1(y, e_i, u)|^2 m_1(du)))\]

\[\leq 2(l_r |x - y|^2 + 4l(1 + |y|^2)) \leq 2(l_r |x - y|^2 + 4l(1 + r^2))\]

\[= 2(l_r |x - y|^2 + 2l(1 + r^2)|e_i - e_j|^2) \leq L_r (|x - y|^2 + |e_i - e_j|^2) = L_r |\xi - \zeta|^2,\]

where \(L_r = \max(2l_r, 4l(1 + r^2)).\)

Let \(S\) be the support of \(\tilde{f}_2\) and \(S_u\) be the projection of \(S\) on \(U = \mathbb{R}\). By (A3) and (A4) we have that \(m_2(S_u) < \infty\), where \(m_2\) is the Lebesgue measure.

By (A4) and definition of function \(c\) we have that for all \(t > 0, i = 1, \ldots N\)

\[\int_0^t \int_{\mathbb{R}} |\tilde{f}_2(x, e_i, u)| p_2(du, ds) < \infty, \ P\text{-a.s.}\]

We have shown that coefficients of Equation (B.1) satisfy the conditions of Theorem A.1, thus Equation (B.1) (correspondingly (2.1)-(2.2)) has an a.s. unique strong solution. \(\square\)

C  Proof of Lemma 2.2

Lemma C.1. Suppose functions \(\lambda_{ij}(\cdot), i, j = 1, \ldots, N\) satisfy (A3). Then there exist a constant \(C_c\) such that

\[\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz \leq C_c(|x - y| + 1) \text{ for } i \neq k\]  \hfill (C.1)

and

\[\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz \leq C_c(|x - y|) \text{ for } i = k\] \hfill (C.2)

for all \(x, y \in \mathbb{R}^n\) and \(e_i, e_k \in \mathbb{M}\).
Proof. Define the following interval:

\[ U(x) \triangleq \bigcup_{i=1}^{N} \left( \bigcup_{j=1 \atop j \neq i}^{N} \Delta_{ij}(x) \right), \]

it includes all intervals \( \Delta_{ij}(x), i, j = 1, \ldots, N, i \neq j. \) Since the length of each interval \( \Delta_{ij}(x) \) is \( \lambda_{ij}(x) \), and this is continuous and bounded function for \( i, j = 1, \ldots, N, i \neq j, \) it follows that the length of interval \( U(x) \) (we denote it by \( l(U(x)) \)) is bounded and is a continuous function of \( x \). Therefore, it has a maximum at some point \( x^* \):

\[ l(U(y)) \leq l(U(x^*)) \text{ for all } y \in \mathbb{R}^n. \]

Let \( U_{\max} \triangleq U(x^*) \) denote the interval of maximum length and let \( \lambda_{\max} \triangleq l(U_{\max}) \) denote its length.

Then,

\[
\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz = \int_{\mathbb{R}} \left| \sum_{j=1 \atop j \neq i}^{N} 1_{\Delta_{ij}(x)}(z) \cdot e_j - \sum_{j=1 \atop j \neq k}^{N} 1_{\Delta_{kj}(y)}(z) \cdot e_k \right|^2 dz
\]

\[
= \int_{U_{\max}} \left| (e_k - e_i) + \left( \sum_{j=1 \atop j \neq i}^{N} 1_{\Delta_{ij}(x)}(z) \cdot e_j - \sum_{j=1 \atop j \neq k}^{N} 1_{\Delta_{kj}(y)}(z) \cdot e_j \right) \right|^2 dz
\]

\[
\leq 2\lambda_{\max}|e_k - e_i|^2 + 2\int_{U_{\max}} \left| \sum_{j=1 \atop j \neq i}^{N} 1_{\Delta_{ij}(x)}(z) \cdot e_j - \sum_{j=1 \atop j \neq k}^{N} 1_{\Delta_{kj}(y)}(z) \cdot e_j \right|^2 dz.
\]

Let us consider two cases:

1) suppose \( i = k, \) then

\[
\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz \leq 2\int_{U_{\max}} \left| \sum_{j=1 \atop j \neq i}^{N} 1_{\Delta_{ij}(x)}(z) \cdot e_j - \sum_{j=1 \atop j \neq k}^{N} 1_{\Delta_{kj}(y)}(z) \cdot e_j \right|^2 dz
\]

\[
= 2\int_{U_{\max}} \left| \sum_{j=1 \atop j \neq i}^{N} (1_{\Delta_{ij}(x)}(z) - 1_{\Delta_{ij}(y)}(z)) \cdot e_j \right|^2 dz
\]

\[
\leq 2N\int_{U_{\max}} \sum_{j=1 \atop j \neq i}^{N} |1_{\Delta_{ij}(x)}(z) - 1_{\Delta_{ij}(y)}(z)|^2 dz
\]

\[
= 2N\sum_{j=1 \atop j \neq i}^{N} \left( \int_{1_{\Delta_{ij}(x)} \setminus 1_{\Delta_{ij}(y)}} 1 dz + \int_{1_{\Delta_{ij}(y)} \setminus 1_{\Delta_{ij}(x)}} 1 dz \right). \quad \text{(C.3)}
\]
(1a) suppose $\Delta_{ij}(x) \cap \Delta_{ij}(y) \neq \emptyset$. Then

$$
\int_{\Delta_{ij}(x) \setminus \Delta_{ij}(y)} 1dz + \int_{\Delta_{ij}(y) \setminus \Delta_{ij}(x)} 1dz
$$

$$
= \left| \sum_{i' = 1}^{i-1} \sum_{j' \neq i}^{N} \lambda_{i'j'}(x) + \sum_{j' = 1}^{j-1} \lambda_{ij'}(x) - \sum_{i' = 1}^{i-1} \sum_{j' \neq i}^{N} \lambda_{i'j'}(y) - \sum_{j' = 1}^{j-1} \lambda_{ij'}(y) \right|
$$

$$
= \left| \sum_{i' = 1}^{i-1} \sum_{j' \neq i}^{N} \lambda_{i'j'}(x) + \sum_{j' = 1}^{j-1} \lambda_{ij'}(x) - \sum_{i' = 1}^{i-1} \sum_{j' \neq i}^{N} \lambda_{i'j'}(y) - \sum_{j' = 1}^{j-1} \lambda_{ij'}(y) \right|
$$

$$
\leq 2N^2C_\lambda |x - y|.
$$

(1b) now suppose $\Delta_{ij}(x) \cap \Delta_{ij}(y) = \emptyset$. We denote by $\Delta_{ij}^{x,y}$ the interval that is contiguous to intervals $\Delta_{ij}(x)$ and $\Delta_{ij}(y)$. Then

$$
\int_{\Delta_{ij}(x)} 1dz + \int_{\Delta_{ij}(y)} 1dz \leq \int_{\Delta_{ij}(x) \cup \Delta_{ij}^{x,y}} 1dz + \int_{\Delta_{ij}(y) \cup \Delta_{ij}^{x,y}} 1dz
$$

$$
= \left| \sum_{i' = 1}^{i-1} \sum_{j' \neq i}^{N} \lambda_{i'j'}(x) + \sum_{j' = 1}^{j-1} \lambda_{ij'}(x) - \sum_{i' = 1}^{i-1} \sum_{j' \neq i}^{N} \lambda_{i'j'}(y) - \sum_{j' = 1}^{j-1} \lambda_{ij'}(y) \right|
$$

$$
= \left| \sum_{i' = 1}^{i-1} \sum_{j' \neq i}^{N} \lambda_{i'j'}(x) + \sum_{j' = 1}^{j-1} \lambda_{ij'}(x) - \sum_{i' = 1}^{i-1} \sum_{j' \neq i}^{N} \lambda_{i'j'}(y) - \sum_{j' = 1}^{j-1} \lambda_{ij'}(y) \right|
$$

$$
\leq 2N^2C_\lambda |x - y|.
$$

Now we can proceed with expression (C.3):

$$
2N \sum_{j = 1}^{N} \left( \int_{\Delta_{ij}(x) \setminus \Delta_{ij}(y)} 1dz + \int_{\Delta_{ij}(y) \setminus \Delta_{ij}(x)} 1dz \right) \leq 2N \sum_{j = 1}^{N} (2N^2C_\lambda |x - y|) \leq 4N^4C_\lambda |x - y|.
$$

2) suppose $i \neq k$, then

$$
\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz \leq 4\lambda_{\max} + 2 \int_{U_{\max}} \left| \sum_{j = 1}^{N} 1_{\Delta_{ij}(x)}(z) \cdot e_j - \sum_{j = 1}^{N} 1_{\Delta_{kj}(y)}(z) \cdot e_j \right|^2 dz
$$

$$
\leq 4\lambda_{\max} + 2 \cdot 4N^2\lambda_{\max}
$$

$$
\leq \lambda_{\max}(4 + 8N^2) + |x - y|.
$$
From the above estimations follows that there exists a constant $C_c$ such that

$$
\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz \leq C_c(|x - y| + 1) \text{ for } i \neq k,
$$

and

$$
\int_{\mathbb{R}} |c(x, e_i, z) - c(y, e_k, z)|^2 dz \leq C_c(|x - y|) \text{ for } i = k.
$$

We can summarize it in one expression

$$
\int_{\mathbb{R}} |c(\xi, z) - c(\zeta, z)|^2 dz \leq C_c(|\xi - \zeta|), \quad (C.4)
$$

where $\xi = (x, \theta) \in \mathbb{R} \times \mathbb{M}$ and $\zeta = (y, \eta) \in \mathbb{R} \times \mathbb{M}$.

References


