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EFFICIENT SIZING OF STRUCTURES UNDER STRESS CONSTRAINTS

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Abstract. *Optimisation algorithms used to automatically size structural members commonly involve stress constraints to avoid material failure. Therefore the cost of optimisation grows rapidly as the number of structural members is increased due to the corresponding increase in the number of constraints. In this work, an efficient method for large scale stress constrained structural sizing optimisation problems is proposed. A convex, separable, and scalable approximation for stress constraints which splits the approximation into a local fully stressed term and a global load distribution term is introduced. Predictor-corrector interior point method, an excellent option for large scale optimization problem, is chosen to solve the approximate subproblems. The core idea in this work is to achieve computational efficiency in the optimization procedure by avoiding the construction and the solution of the Schur complement system generated by the interior point method. Avoiding the Schur complement, and explicit sensitivity analysis, eliminates the high cost of solving stress constrained problems within the interior point optimisation. This is achieved using the preconditioned conjugate gradient method, and a new preconditioner is proposed specifically for stress constrained problems. The proposed method is applied to a number of beam sizing problems. Numerical results show that optimal complexity is achieved by the algorithm, the computational cost being linearly proportional to the number of sizing variables.*

1 INTRODUCTION

Optimisation methods have been extensively used to size structural components [1][2][3] [4][5][6] — mostly in a finite element environment. The use of optimisation algorithms allows for automated sizing and for a more thorough search of the design space leading more efficient structures. Strength constraints, which define the failure limits for the structure, have to be satisfied in every part of the structure during the sizing optimization. This is enforced numerically by applying strength constraints per finite element. For a large and complex structure with hundreds of thousands of degree of freedom, the computational cost of strength constraints becomes prohibitive for large scale optimization problems.

One simple optimization algorithm to handle stress constraints is the fully stressed method [7]. However, it turns out that the fully stressed design is not the minimum-weight design for statically indeterminate structures [8] and works well only with discrete structures. From then on, the structure optimization based on optimality condition starts to be investigated. In 1974, by introducing stress approximations, L.A. Schmit and B. Farshi [9] first achieve a significant improvement in computational efficiency of sizing optimization. Different approximation methods have been investigated thereafter [10][11][12][13][14][15]. Higher order approximations, even though they demonstrate high accuracy, are complicated when calculating the coefficients for higher order terms. After numerous attempts, first order and diagonal second order approximation are regarded as good options when both accuracy and computational complexity have been taken into account [10][14][16]. Therefore, they are commonly applied in structural optimization.

Interior point method stands out among gradient-based algorithms because of its robustness and efficiency. The convergence rate of interior point method is not sensitive to the size of optimization problem. Originally proposed for linear programming, it has been extended to different types of problems. Boggs et al.[17] use interior point method to solve sequential quadratic programming (SQP) for large scale nonlinear programming problems. Robert and David[18] extend the method to solve nonconvex nonlinear programming, and show the robustness and efficiency of the method. In [19], Mehrotra has shown that by the predictor-corrector pattern, the number of iterations decreases further to solve linear programming problem. By employing a Taylor polynomial of second order to approximate a primal-dual trajectory, Mehrotra[20] successfully improve the efficiency of the algorithm. Andersen, Roos and Terlaky[21], utilize Mehrotra's predictor-corrector interior point method to solve large-scale sparse conic quadratic optimization problems robustly and efficiently. Zillober[22] demonstrates the advantages of predictor-corrector interior point method over the dual approach in solving large scale nonlinear programming with applications to structural optimisation problems.

The Preconditioned Conjugate Gradient (PCG) method is an efficient iterative technique for the solution of large sets linear equations. This iterative method converges rapidly if the preconditioner manages to reduce the condition number of coefficient matrix or when a good initial guess is available. Schmit [23] employes the PCG to solve finite element method in structural optimization and discuss the potential parallel computation of the method. Another advantage of PCG is that the coefficient matrix does not need to be stored explicitly. This property will help greatly to save computational effort in the proposed algorithm.

The objective of this paper is to demonstrate an efficient optimization method to optimize statically indeterminate beam structures. The method is designed to be fast for typical sizing scenarios under stress constraints. The computational complexity of of the algorithm is reduced from approximately cubic dependence on the number of elements to approximately linear dependence.

The structure in this paper is outlined as follows: In Section 2 the framework of the conventional optimization approach for stress constrained problems is described. A brief description of the Finite Element model, successive convex approximations for stress constraints, and sensitivity analysis is given. The predictor-corrector interior point method used in this paper is also described and the computational complexity of the conventional approach estimated. Section 3 introduces an efficient and cheap preconditioner to solve the dense linear system generated by the interior point method. The matrix-vector multiplications needed by the PCG method are reduced to an implicit sensitivity analysis leading to a reduction in the computational cost of sensitivity analysis . Two numerical examples will be used to demonstrate the efficiency of the optimization method in Section 4. Finally, Section 5 provides the conclusions.

2 Formulation

Before describing the proposed method, an overview of the optimisation process, stress approximation, sensitivity analysis and the iterative optimization is given in this section. The purpose is to familiarise the reader with the relevant concepts and formulations first, and to identify the computational bottlenecks in the procedure.

The, more or less, standard approach in structural optimisation is to use successive convex and separable approximations. The optimisation starts at a given design point, convex separable approximations of all involved response are constructed based on sensitivity information, then the approximate subproblem based on the approximations is solved to find the next design iterate. The procedure is repeated until a suitable convergence tolerance is achieved. The solution of the convex subproblem itself is obtained using an iterative convex optimisation technique. In this work, an predictor-corrector interior point method is used. The various elements of the procedure are explained in more details in the following sections.

2.1 Successive Convex Approximation Method

In this section, a cheap and efficient convex approximation for stress constraints is described. Vanderplaats [12, 13] has shown that, approximating the internal forces first as a *linear* function of the design variables, is more accurate than approximating the stress constraints directly. In this approach, the linear approximations are used to calculate the internal forces from which the stresses and hence the failure constraint(s) (e.g., von Mises stress) may be calculated. However, these so-called force approximations are not necessarily separable or convex. The motivation for the present research is to design a high quality approximation method which overcomes this disadvantage.

The basic idea is to use a zeroth order force approximation rather than a first order approximation. This zeroth order approximation is then corrected to attain first order accuracy by the addition of linear terms.

The model problem is an Euler-Bernoulli beam designed for minimum volume under material failure constraints. For simplicity a rectangular cross section, width w and height h, is considered. Furthermore, either only the width or only the height would be used as a design variable which is denoted by x. The stress constraint takes the form,

$$\frac{|\sigma|}{\sigma_{all}} - 1 \leq 0 \tag{1}$$

where σ_{all} is the allowable normal stress, and the normal stress σ is related to the intermediate internal force M, representing the bending, moment by,

$$\sigma = \frac{|M|h}{2I}, \tag{2}$$

where I is the moment of inertia of the cross section given by

$$I = \frac{wh^3}{12}. \tag{3}$$

The continuous problem would be solved using methods of the calculus of variations. The numerical treatment starts by discretising the beam using n finite elements. Each element is assigned a design variable. The stress constraint for the i^{th} element may be written as a function of M_i and the design variables,

$$r_i(M_i, x_i) = \pm\frac{6M_i}{\sigma_{all} w_i h_i^2} \leq 1., x_i = w_i \text{ or } h_i, \quad i = 1, 2, \ldots n \tag{4}$$

Note that the bending moment in any given element is generally a function of all the design variables $M_i = M_i(\mathbf{x})$. The volume of the beam is the integral over the length of the cross sectional area wh.

Let the current design iterate be denoted by $\mathbf{x}_k$. The proposed approximation then takes the form,

$$r_i \approx \frac{b_i^{(k)}}{x_i^q} + \sum_{j=1}^{n} A_{ji}^{(k)}(x_j - x_j^{(k)}). \tag{5}$$

For width optimisation $q = 1$ and $b_i^{(k)} = 6|M_i^{(k)}|/\sigma_{all}h^2$, while for height optimisation $q = 2$ and $b_i^{(k)} = 6|M_i^{(k)}|/\sigma_{all}w$. The first term represents the zeroth order force approximation for the stress constraint. This first term would be exact only for statically determinate structures where the bending moment is independent of the design variables. The second term provides a linear correction and represents the effect of *load redistribution* present in statically indeterminate structures. The coefficients $A_{ji}^{(k)}$ are obtained by matching the derivatives of the approximation Eq.(5) with respect to design variables $\mathsf{G}(\mathbf{x}_k)$ to the derivative values obtained from sensitivity analysis G_e at the approximation point $\mathbf{x}_k$.

$$\mathsf{G}(\mathbf{x}_k) = \mathsf{G}_s(\mathbf{x}_k) + A^{(k)} \tag{6}$$

$$\mathsf{G}_e(\mathbf{x}_k) = \mathsf{G}(\mathbf{x}_k) \tag{7}$$

Where,

$$\mathsf{G}_e(\mathbf{x}_k) = \left.\begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_2}{\partial x_1} & \cdots & \frac{\partial r_n}{\partial x_1} \\ \frac{\partial r_1}{\partial x_2} & \frac{\partial r_2}{\partial x_2} & \cdots & \frac{\partial r_n}{\partial x_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial r_1}{\partial x_n} & \frac{\partial r_2}{\partial x_n} & \cdots & \frac{\partial r_n}{\partial x_n} \end{bmatrix}\right|_{\mathbf{x}_k} \tag{8}$$

$$\mathsf{G}_s(\mathbf{x}_k) = \begin{bmatrix} -\frac{qb_1^{(k)}}{x_1^{q+1}} & 0 & \cdots & 0 \\ 0 & -\frac{qb_2^{(k)}}{x_2^{q+1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{qb_n^{(k)}}{x_n^{q+1}} \end{bmatrix} \tag{9}$$

Therefore the coefficient $A^{(k)}$ is,

$$A^{(k)} = \mathsf{G}_e(\mathbf{x}_k) - \mathsf{G}_s(\mathbf{x}_k) \tag{10}$$

The proposed stress approximation here is thus convex and separable.

For the case under consideration, the volume of the beam is an exact linear function of the design variables. Thus, the objective function may be written as,

$$V = \sum_{i=1}^{n} c_i x_i. \tag{11}$$

2.2 Sensitivity Analysis

Sensitivity analysis is required first to provide the gradient of the objective and constraints in gradient-based optimisation. Then the gradient is employed to guide the optimisation method in the optimisation procedure, which pushes the design variables towards the optimal solution in the feasible domain. The well-established and efficient way to calculate sensitivity is through the Finite Element Method (FEM) which estimates the mechanical response of structures.

Consequently, FEM is employed to analyse structures and provide information required by the proposed optimization method. Due to the ease-to-implement and powerful capability of FEM to analyse complex structures, it ensures the propose to solve real engineering problems in the very beginning. According to minimum total potential energy principal, we can come to the equation for static analysis:

$$\mathsf{K}\mathbf{u} = \mathbf{f} \tag{12}$$

where K is the global stiffness matrix attained by assembling stiffness matrix of each element according to their degree of freedom. $\mathbf{u}$ is the global displacement of each node. $\mathbf{f}$ is the external force loaded on the structure.

Before we get the internal force for i^{th} element in finite element method, the displacement corresponding to the DOF of i^{th} element in local coordinate system should be obtained first by,

$$\tilde{\mathbf{u}}_i = \mathsf{T}_i \mathbf{u} \tag{13}$$

Here, T_i is the transformation matrix which transfer the displacement of the i^{th} element from global coordinate to local element coordinate system.

The equation for the internal force, i.e., the bending moment in this case, is,

$$M_i = \mathsf{D}(x_i)\mathsf{B}_i\tilde{\mathbf{u}}_i. \tag{14}$$

Here, $\mathsf{D}(x_i)$ is the material stiffness, B_i is the strain-displacement matrix, and $\mathbf{u}_i$ contains the nodal degrees of freedom connected to the i^{th} element. In the case of beams under pure bending D_i for the i^{th} element is scalar given by the bending stiffness $EI(x_i)$, and the strain-displacement matrix B_i reduced a row vector relating the average curvature to the nodal degrees of freedom.

Substituting Eq. (13) into Eq. (14), the sensitivity of the beding moment M_i with respect to the design variable x_j is given by,

$$\frac{\partial M_i}{\partial x_j} = \frac{\partial D_i}{\partial x_i}\delta_{ij}\mathsf{B}_i\mathsf{T}_i\mathbf{u}_i + \mathbf{z}_i^T\frac{\partial \mathbf{u}_i}{\partial x_j} \tag{15}$$

where δ_{ij} is Kronecker's delta, and $\mathbf{z}_i = \mathsf{T}_i^T\mathsf{B}_i^T\mathsf{D}_i^T$.

There are two common methods to proceed with the sensitivity analysis: the direct method and the adjoint method [24].

The direct method starts from the assembled finite element equations,

$$\mathsf{K}(\mathbf{x})\,\mathbf{u} = \mathbf{f}, \tag{16}$$

where K is the assembled stiffness matrix, $\mathbf{u}$ the vector of degrees of freedom and $\mathbf{f}$ is the vector of external forces.

Differentiating Eq.(16),

$$\mathsf{K}\frac{\partial \mathbf{u}}{\partial x_j} = \frac{\partial \mathbf{f}}{\partial x_j} - \frac{\partial \mathsf{K}}{\partial x_j}\mathbf{u}, \tag{17}$$

The solution to Eq.(17) allows the computation of the derivative of the displacements with respect to every design variable.

The adjoint method attempts to avoid the explicit calculation of the displacement derivatives. To this end, the vector $\mathbf{z}_i$ is pseudo-load, and the following adjoint system is introduced,

$$\mathsf{K}\,\mathbf{t}_i = \mathbf{z}_i. \tag{18}$$

Then the sensitivity of the bending moments is given by,

$$\frac{\partial M_i}{\partial x_j} = \frac{\partial D_i}{\partial x_i}\delta_{ij}\mathsf{B}_i\mathsf{T}_i\mathbf{u}_i - \mathbf{t}_i^T\frac{\partial \mathsf{K}}{\partial x_j}\mathbf{u} \tag{19}$$

The derivatives of the bending moment in all elements then follows from Eq.(15). The derivatives of the stress constraint itself are then easily computed as follows,

$$\frac{\partial r_i}{\partial x_j} = \left.\frac{\partial r_i}{\partial x_i}\right|_{M_i}\delta_{ij} + \left.\frac{\partial r_i}{\partial M_i}\right|_{\mathbf{x}_i}\frac{\partial M_i}{\partial x_j} \tag{20}$$

Using the direct method, Eq.(17) has to be solved once for each design variable. By contrast, Eq. (18) for the adjoint method needs to be solved only as many times as there are stress constraints.

2.3 Predictor-Corrector Interior Point Method

The predictor-corrector interior point method is a good option for large scale optimization problems because of its great efficiency [25]. In this paper, the interior point method is therefore employed in structural optimisation. In interior point methods a logarithmic penalty is used to enforce the constraints. The penalty parameter is then driven towards zero to obtain the optimal solution. At each step of optimisation Newton's method is used to update primal and dual (Lagrange multiplier) variables. The predictor corrector method of Mehrotra[20] is used to accelerate convergence.

When minimizing the volume of a beam with stress and design variables constrained, the problem can be formulated as follows.

objective:

$$\text{Min:} f_0(x) = \sum_{i=1}^{n} x_i h_i l_i \tag{21}$$

subject to:

$$f_i(x) \leq \sigma_{all} \quad i = 1 \ldots \text{m, m is the number of stress constraints}$$
$$x_j^l \leq x_j \leq x_j^u \quad j = 1 \ldots \text{n, n is the number of design variables}$$

In this paper, $m = n$ for the beam structure. x_i and l_i are the thickness and the length of the i^{th} beam, respectively. x_j^l and x_j^u are the lower bound and upper bounds of the design variable. Since we confine the optimization problem to be sizing optimization, $x_j^l > 0$.

The Lagrangian function of the problem is

$$\begin{aligned} L(\mathbf{x}, \mathbf{s}, \mathbf{y}^s, \mathbf{s}^u, \mathbf{y}^u, \mathbf{s}^l, \mathbf{y}^l) =& f_0 + (\mathbf{y}^s)^T(\mathbf{f}(x) + \mathbf{s}) + (\mathbf{y}^u)^T(\mathbf{x} - \overline{\mathbf{x}} + \mathbf{s}^u) + (\mathbf{y}^l)^T(\underline{\mathbf{x}} - \mathbf{x} + \mathbf{s}^l) \\ & - \mu \sum_{i=1}^{n} \ln s_i - \mu \sum_{i=1}^{n} \ln s_i^u - \mu \sum_{i=1}^{n} \ln s_i^l \end{aligned} \tag{22}$$

f_0: normalized objective function; $\mathbf{f}(\mathbf{x})$: normalized stress constraints (stress: $f_i(\mathbf{x}) = sgn(\sigma_i)\sigma_i/\sigma_{allow} - 1, i = 1 \ldots n$); s: the slack of the stress constraints which transform inequality constraints stress constraints into equality constraints; $\mathbf{s}^u$,$\mathbf{s}^l$: the slack of the design variables with respect to the upper bound and lower bound which transform inequality constraints of design variables into equality constraints; $\mathbf{y}^s$: Lagrange multipliers of the stress constraints; $\mathbf{y}^u$,$\mathbf{y}^l$: the Lagrange multipliers of the design variable with respect to the upper bound and the lower bound respectively; $\overline{\mathbf{x}}$: upper bound of the design variables ($\overline{x}_i = x_i^u$); $\underline{\mathbf{x}}$: lower bound of the design variables ($\underline{x}_i = x_i^l$); μ: penalty parameter.

According to Karush-Kuhn-Tucker conditions, a set of equations will be:

$$\frac{\partial L}{\partial \mathbf{x}} = \nabla \mathbf{f}(\mathbf{x}) + \mathsf{G}(\mathbf{x}) \cdot \mathbf{y}^s + \mathbf{y}^u - \mathbf{y}^l = \mathbf{0} \tag{23}$$
$$\frac{\partial L}{\partial \mathbf{y}^l} = \underline{\mathbf{x}} - \mathbf{x} + \mathbf{s}^l = \mathbf{0} \tag{24}$$
$$\frac{\partial L}{\partial \mathbf{y}^u} = \mathbf{x} - \overline{\mathbf{x}} + \mathbf{s}^u = \mathbf{0} \tag{25}$$
$$\frac{\partial L}{\partial \mathbf{y}^s} = \mathbf{f}(\mathbf{x}) + \mathbf{s} = \mathbf{0} \tag{26}$$
$$\frac{\partial L}{\partial \mathbf{s}} = \mathsf{Y}^s \cdot \mathbf{s} - \mu \cdot \overline{\mathbf{e}} = \mathbf{0} \tag{27}$$
$$\frac{\partial L}{\partial \mathbf{s}^u} = \mathsf{Y}^u \cdot \mathbf{s}^u - \mu \cdot \overline{\mathbf{e}} = \mathbf{0} \tag{28}$$
$$\frac{\partial L}{\partial \mathbf{s}^l} = \mathsf{Y}^l \cdot \mathbf{s}^l - \mu \cdot \overline{\mathbf{e}} = \mathbf{0} \tag{29}$$

Here $\mathsf{G}(\mathbf{x})$ is the gradient of the stress approximation (Eq.(5)); $\overline{\mathbf{e}}^T = \overbrace{[1, 1, \ldots, 1]}^{n}$; $\mathsf{Y}^s = diag(y_1^s, \cdots, y_n^s)$; Y^u and Y^l are similar.

In order to solve the above equations, Newton's method is employed. Then a linear system of the subproblem is obtained:

$$\mathsf{G}(\mathbf{x}) \cdot \Delta \mathbf{y}^s + \mathsf{H}(\mathbf{x}) \cdot \mathbf{y}^s \cdot \Delta \mathbf{x} + \Delta \mathbf{y}^u - \Delta \mathbf{y}^l = -\nabla \mathbf{f} - \mathsf{G}(\mathsf{x}) \cdot \mathbf{y}^s - \mathbf{y}^u + \mathbf{y}^l \tag{30}$$
$$-\Delta \mathbf{x} + \Delta \mathbf{s}^l = \mathbf{x} - \underline{\mathbf{x}} - \mathbf{s}^l \tag{31}$$
$$\Delta \mathbf{x} + \Delta \mathbf{s}^u = \overline{\mathbf{x}} - \mathbf{x} - \mathbf{s}^u \tag{32}$$
$$\mathsf{G}^T(\mathbf{x}) \Delta \mathbf{x} + \Delta \mathbf{s} = -\mathbf{f}(\mathbf{x}) - \mathbf{s} \tag{33}$$
$$\mathsf{S} \cdot \Delta \mathbf{y}^s + \mathsf{Y}^s \cdot \Delta \mathbf{s} = \mu \cdot \overline{\mathbf{e}} - \mathsf{Y}^s \cdot \mathbf{s} \tag{34}$$
$$\mathsf{S}^u \cdot \Delta \mathbf{y}^u + \mathsf{Y}^u \cdot \Delta \mathbf{s}^u = \mu \cdot \overline{\mathbf{e}} - \mathsf{Y}^u \cdot \mathbf{s}^u \tag{35}$$
$$\mathsf{S}^l \cdot \Delta \mathbf{y}^l + \mathsf{Y}^l \cdot \Delta \mathbf{s}^l = \mu \cdot \overline{\mathbf{e}} - \mathsf{Y}^l \cdot \mathbf{s}^l \tag{36}$$

Here $\mathsf{H}(\mathbf{x})$ are the Hessian matrixs of the stress approximation with respect to the design variables. Since the approximation is separable, $\mathsf{H}(\mathbf{x}) \cdot \mathbf{y}_s$ is diagonal matrix which is very cheap to obtain as shown in [22]. To signify the right hand side of the equations above, we have:

$$\begin{bmatrix} f_x \\ f_{y^l} \\ f_{y^u} \\ f_{y^s} \\ f_s \\ f_{s^u} \\ f_{s^l} \end{bmatrix} = \begin{bmatrix} -\nabla \mathbf{f} - \mathsf{G}(\mathbf{x}) \cdot \mathbf{y}^s - \mathbf{y}^u + \mathbf{y}^l \\ \mathbf{x} - \underline{\mathbf{x}} - \mathbf{s}^l \\ \overline{\mathbf{x}} - \mathbf{x} - \mathbf{s}^u \\ -\mathbf{f}(\mathbf{x}) - \mathbf{s} \\ \mu \cdot \overline{\mathbf{e}} - \mathsf{Y}^s \cdot s \\ \mu \cdot \overline{\mathbf{e}} - \mathsf{Y}^u \cdot s^u \\ \mu \cdot \overline{\mathbf{e}} - \mathsf{Y}^l \cdot s^l \end{bmatrix} \tag{37}$$

From Eq.(32) and Eq.(35), we can get:

$$\Delta \mathbf{s}^u = f_{y^s} - \mathsf{G}^T(x) \Delta \mathbf{x} \tag{38}$$
$$\Delta \mathbf{y}^u = (\mathsf{S}^u)^{-1} \cdot (\mathbf{f}_{s^u} - \mathsf{Y}^u \cdot \Delta \mathbf{s}^u) \tag{39}$$

From Eq.(31) and Eq.(36), we can get:

$$\Delta \mathbf{s}^l = f_{y^l} + \Delta \mathbf{x} \tag{40}$$
$$\Delta y^l = (\mathsf{S}^l)^{-1} \cdot (\mathbf{f}_{s^l} - \mathsf{Y}^l \cdot (\mathbf{f}_{y^l} + \Delta \mathbf{x})) \tag{41}$$

From Eq.(33), we can get:

$$\Delta \mathbf{s} = f_{y^s} - \mathsf{G}^T(x) \cdot \Delta \mathbf{x} \tag{42}$$

Substitute $\Delta \mathbf{s}^u$ into $\Delta \mathbf{y}^u$ and $\Delta \mathbf{s}^l$ into $\Delta \mathbf{y}^l$, then substitute $\Delta \mathbf{y}^u$ and $\Delta \mathbf{y}^l$ into Eq.(30) yields:

$$\mathsf{G}(x) \cdot \Delta \mathbf{y}^s + \mathsf{H}(x) \cdot \mathbf{y}^s \cdot \Delta \mathbf{x} + \mathsf{Y}^u \cdot (\mathsf{S}^u)^{-1} \cdot \Delta \mathbf{x} + \mathsf{Y}^l \cdot (\mathsf{S}^l)^{-1} \cdot \Delta \mathbf{x} = \mathbf{f}_{sys_x} \tag{43}$$
$$\mathbf{f}_{sys_x} = \mathbf{f}_x - (\mathsf{S}^u)^{-1} \cdot \mathbf{f}_{s^u} + Y^u \cdot (\mathsf{S}^u)^{-1} \cdot \mathbf{f}_{y^u} + (\mathsf{S}^l)^{-1} \cdot \mathbf{f}_{s^l} - Y^l \cdot (\mathsf{S}^l)^{-1} \cdot \mathbf{f}_{y^l}$$

with Eq.(34) substituting into Eq.(33) it comes to:

$$\mathsf{G}^T(x) \cdot \Delta \mathbf{x} + (\mathsf{Y}^s)^{-1} \cdot (\mathbf{f}_s - \mathsf{S} \cdot \Delta \mathbf{y}_s) = \mathbf{f}_{y^s} \tag{44}$$

Combining Eq.(43) and Eq.(44) we can get the final equation of the subproblem with only $\Delta \mathbf{y}^s$:

$$[(\mathsf{Y}^s)^{-1} \cdot \mathsf{S} + \mathsf{G}^T(x) \cdot \mathsf{C}^{-1} \cdot \mathsf{G}(x)] \Delta \mathbf{y}^s = \mathbf{f} \tag{45}$$

Here, $\mathsf{C} = \mathsf{H}(x) \cdot \mathbf{y}^s + Y^u \cdot (\mathsf{S}^u)^{-1} + \mathsf{Y}^l \cdot (\mathsf{S}^l)^{-1}$

Since all these three parts are diagonal matrix, C is a diagonal matrix.

$$\mathbf{f} = -\mathbf{f}_{y^s} + (\mathsf{Y}^s)^{-1} \cdot \mathbf{f}_s + \mathsf{G}^T(x) \cdot \mathsf{C}^{-1} \cdot \mathbf{f}_{sys_x} \tag{46}$$

From Eq.(45) we can get $\Delta \mathbf{y}^s$. Back substitute it into Eq.(30 – 36) we can get: $\Delta \mathbf{x}$, $\Delta \mathbf{s^u}$, $\Delta \mathrm{s}$, $\Delta \mathrm{s}^l$, $\Delta \mathbf{y}^u$, and $\Delta \mathbf{y}^l$.

Algorithm 1 Algorithm of Predictor-Corrector Interior Point Method

1: Set $k = 1$, initialize the parameters $\mu, \mathbf{x}, \mathbf{s}^u, \mathbf{s}^l, \mathbf{s}, \mathbf{y}^s, y^u, y^l$.
2: **while** $d_{gap} \geq tol$ **do** ▷ stop if duality gap $< tol$
3: Update Eq.(45) by setting $\mu = 0$
4: $\mu_{(k+1)} \leftarrow$ solve Eq.(45) in predictor step
5: Update Eq.(45) by setting $\mu = \mu_{(k+1)}$
6: $\mathbf{x}, \mathbf{s}, \mathbf{y}^s, \mathbf{s}^u, \mathbf{y}^u, \mathbf{s}^l, \mathbf{y}^l \leftarrow$ solve Eq.(45) in corrector step
7: Update $d_{(k+1)}$
8: **end while**

- Initialize the variables (Line 1) in the predictor-corrector interior point method:

  Set the design variables $\mathbf{x}$ a set of feasible design. Make $\mathbf{s}^u = \overline{\mathbf{x}}$, $\mathbf{s}^l = \underline{\mathbf{x}}$, $\mathbf{s} = \mathbf{1}$, $\mu = 1/(dim(\mathbf{y}^s) + 2dim(\mathbf{x}))$, $\mathbf{y}^s = \mu/\mathbf{s}$, $\mathbf{y}^u = \mu/\mathbf{s}^u$, $\mathbf{y}^l = \mu/\mathbf{s}^l$.

  $dim(\mathbf{x})$ is the length of vector $\mathbf{x}$; $dim(\mathbf{y}^s)$ is the length of vector $\mathbf{y}^s$

- The way to calculate penalty parameter μ (Line 4) in predictor step is as follows:

  First we solve Eq.(45) with $\mu = 0$, where we get slacks ($\Delta \mathrm{s}$, $\Delta \mathrm{s}^u$, $\Delta \mathrm{s}^l$) and Lagrange multipliers ($\Delta \mathbf{y}^s$, $\Delta \mathbf{y}^u$, $\Delta \mathbf{y}^l$). Then we update the duality gap by,

  $d\_gap_{(k+1)} = (\mathbf{s} + \delta_{primal} \cdot \Delta \mathbf{s})^T \cdot (\mathbf{y}^s + \delta_{dual} \cdot \Delta \mathbf{y}^s) + (\mathbf{s}^u + \delta_{primal} \cdot \Delta \mathbf{s}^u)^T \cdot (\mathbf{y}^u + \delta_{dual} \cdot \Delta \mathbf{y}^u) + (\mathbf{s}^l + \delta_{primal} \cdot \Delta \mathbf{s}^l)^T \cdot (\mathbf{y}^l + \delta_{dual} \cdot \Delta \mathbf{y}^l)$

  Meanwhile we calculate the current duality gap by,

  $d\_gap_k = \mathbf{s}^T \cdot \mathbf{y}^s + (\mathbf{s}^u)^T \cdot \mathbf{y}^u + (\mathbf{s}^l)^T \cdot \mathbf{y}^l$

  The penalty parameter is obtained by,

  $\mu_{(k+1)} = min\{max((d\_gap_{(k+1)}/d\_gap_k)^2, 0.1), 1\} \cdot \mu_k$

- Update the design variables, Lagrange multipliers and slacks in Line 6:

  $\mathbf{x} = \mathbf{x} + \delta_{primal} \cdot \Delta \mathbf{x}$; $\mathbf{s} = \mathbf{s} + \delta_{primal} \cdot \Delta \mathbf{s}$;

  $\mathbf{s}^u = \mathbf{s}^u + \delta_{primal} \cdot \Delta \mathbf{s}^u$; $\mathbf{s}^l = \mathbf{s}^l + \delta_{primal} \cdot \Delta \mathbf{s}^l$;

  $\mathbf{y}^s = \mathbf{y}^s + \delta_{dual} \cdot \Delta \mathbf{y}^s$; $\mathbf{y}^u = \mathbf{y}^u + \delta_{dual} \cdot \Delta \mathbf{y}^u$;

  $\mathbf{y}^l = \mathbf{y}^l + \delta_{dual} \cdot \Delta \mathbf{y}^l$;

The way to obtain δ_{primal}, δ_{dual} in the previous steps are all based on the following rule:
$\delta_{primal} = 0.95 \cdot min\{max\{-s_i^l/\delta s_i^l\}, max\{-s_i/\delta s_i\}, max\{-s_i^u/\delta s_i^u\}, 0.95\}$
$\delta_{dual} = 0.95 \cdot min\{max\{-y_i^l/\delta y_i^l\}, max\{-y_i^u/\delta y_i^u\}, max\{-y_i^s/\delta y_i^s\}, 0.95\}$

2.4 Computational Complexity

The computational complexity of a numerical method is directly relevant to the efficiency of the method. In sizing optimization, the computationally expensive operations are FEM analysis, stress sensitivity analysis and iterative optimization with the interior point method. For a numerical structure, the number of elements and/or nodes is denoted by n. The order of magnitude of each of these operations will be estimated as a function of n.

Since the stiffness matrix for FEM analysis is a symmetric and positive definite equation, the Cholesky decomposition is employed to save computational effort. The computational work for Cholesky decomposition of the finite element equations is:

$$O(nm^2)$$

where m is the bandwidth of the equations. For the beam problem under consideration the number of neighbours for each node is fixed and the bandwidth is constant, thus the computational cost for FEM is,

$$C_1 = O(n),$$

the computational cost for FEM analysis is linear with respect to the number of elements in the numerical model.

For stress sensitivity analysis, the most expensive part is to solve the Eq.(18) for the adjoint displacements. This stiffness matrix is the same as that for equilibrium equation in FEM analysis. The order of computational effort for each adjoint calculation is therefore $O(n)$. Nevertheless, in a beam structure with n elements, the number of stress constraints is n. Therefore Eq.(18) is solved n times to get G_e. The computational work for stress sensitivity analysis is:

$$C_2 = O(n^2)$$

The number of iteration inside the interior point method does not vary much when the size of optimization problem increases. The largest cost in each interior point iteration is the construction and solution of the system of equations Eq.(45) repeated here,

$$[(\mathsf{Y}^s)^{-1}\mathsf{S} + \mathsf{G}^T(x)\mathsf{C}^{-1}\mathsf{G}(x)]\Delta\mathbf{y}^s = \mathbf{f}$$

Since Y_s and S are diagonal, the cost to obtain $\mathsf{Y}_s^{-1}\mathsf{S}$ is $O(n)$. The cost to obtain $\mathsf{G}^T(\mathbf{x})\mathsf{C}^{-1}\mathsf{G}(\mathbf{x})$ is $O(n^3)$ because of the cost of full matrix multiplication. The solution of Eq.(45) can be obtained using Cholesky decomposition. The matrix is full, thus the computational cost is $O(n^3)$. As a result, the computational complexity of the interior point method is,

$$C_3 = O(n^3)$$

Since the number of iterations for the method of successive approximations (successive optimization) is almost independent of the problem size. The computational complexity for the entire optimisation is dominated by the interior point method and would be,

$$C = O(n^3)$$

Consequently reducing the computational complexity of the optimisation should start by addressing the cost of Eq.(45). If this is reduced, the second target would be reducing the cost of the adjoint sensitivity analysis. The main contribution of the paper is in reducing both of them to $O(n)$ operations.

3 Numerical Improvements

Since the computational work for the above optimisation method increases sharply while the size of optimisation increases. Methods to reduce the computational complexity of the optimisation procedure will be introduced in this section.

The first priority is to accelerating the iterative optimisation which dominates the overall computational cost. In the proposed method, to alleviate the main cost for the iterative optimisation, the preconditioned conjugate gradient method (PCG) is used to solve system of equations Eq.(45) for the dual variables. The PCG is an efficient solver for large scale linear equations. The storage of the full coefficient matrix is not require as well. Therefore the cost for the matrix matrix multiplication in building the coefficient matrix of Eq.(45) is avoided. Additionally, since the efficiency of the method is crucially dependent on the quality of the preconditioner, a diagonal preconditioner based on the concept of fully stressed design is proposed for the optimisation problem in Section 3.1. This is an optimal preconditioner from the point of view of the computational cost of using the preconditioner for one step ($O(n)$). Numerical results (Section 4) will confirm that it is also optimal from the point of view of the number of the PCG iterations.

The PCG method requires the ability to form the matrix-vector product of the coefficient matrix with an arbitrary vector. To avoid having to explicitly construct the gradient matrix G, a method is proposed to merge the stress sensitivity analysis into the optimisation procedure, which leads to an implicit sensitivity analysis as shown in Section 3.2. Thus, the computational cost of sensitivity analysis is significantly reduced by creating an direct inline link between the FEM and the optimiser. With the implicit sensitivity analysis, the most expensive cost, the production of the gradient matrix and an arbitrary vector, in the PCG can be accordingly reduced to $O(n)$. Combined with PCG and the interior point method, the entire optimisation cost are finally reduced to $O(n)$ from $O(n^3)$. The proposed method therefore achieves powerful computational savings for large scale optimisation problems.

3.1 Preconditioner for Conjugate Gradient Method

The efficiency of the PCG depends greatly on the effectiveness of the preconditioner. A preconditioner M is a matrix that approximates the coefficient matrix of the equations to be solved. It is important to make sure that the preconditioner is cheap to invert and can effectively reduce the condition number of the preconditioned coefficient matrix. In this section, an effective preconditioner for the subproblem Eq.(45) satisfying the two requirements is developed.

Let us recap the system of equations in Eq.(45),

$$\left[(\mathsf{Y}^s)^{-1}\mathsf{S} + \mathsf{G}^T\mathsf{C}^{-1}\mathsf{G}\right]\Delta\mathbf{y} = \mathbf{f}. \tag{47}$$

The gradient matrix G is, in general, a fully populated dense matrix. Thus the coefficient matrix is also dense. From Eq.(5), the gradient matrix G takes the form,

$$\mathsf{G} = \mathsf{G}_s(\mathbf{x}) + A^{(k)}, \tag{48}$$

Where G_s is a diagonal matrix containing the gradient of the statically determinate part of the stress approximation. $A^{(k)}$ is the gradient matrix of the linear part representing load redistribution. As we have derived in Section 2.1, $A^{(k)}$ is a full matrix for statically indeterminate structures. It is expensive to obtain because of the calculation procedure to obtain $\mathsf{G}_e(\mathbf{x}_k)$.

However to create a cheap preconditioner the following approximation is invoked,

$$\mathsf{G} \approx \mathsf{G}_s. \tag{49}$$

Physically this is to assume that the effect of load redistribution is negligible. In other words, that the internal loads in the structure are *frozen* at the value at the latest design point. With these assumption the preconditioner takes the form,

$$\mathsf{M} = (\mathsf{Y}^s)^{-1}\mathsf{S} + \mathsf{G}_s^T \mathsf{C}^{-1}\mathsf{G}_s, \tag{50}$$

where M is a diagonal matrix which is easy to invert. The effectiveness of the preconditioner in terms of the condition number cannot be demonstrated theoretically but will be assessed in the numerical experiments in Section 4.

3.2 Implicit Sensitivity Analysis

While solving the subproblem (Eq.(45)) with the PCG iteratively in the interior point method, it requires the forming of products of the coefficient matrix of Eq.(45) by an arbitrary vector $\mathbf{v}$,

$$\left[(\mathsf{Y}^s)^{-1}\mathsf{S} + \mathsf{G}^T(\mathbf{x})\mathsf{C}^{-1}\mathsf{G}(\mathbf{x})\right]\mathbf{v}. \tag{51}$$

The first term of the coefficient matrix, $(\mathsf{Y}^s)^{-1}\mathsf{S}$ is diagonal and forming products of the form $\mathbf{w}_f = (\mathsf{Y}^s)^{-1}\mathsf{S} \cdot \mathbf{v}$ is cheap ($O(n)$).

However, the second term, for $\mathsf{G} \cdot \mathbf{v}$ and $\mathsf{G}^T \cdot \mathbf{v}$, is extremely expensive for the large scale optimisation problems because of the computational work to obtain the gradient matrix G. As we have shown the cost to form stress sensitivity G_e, which forms part of G, increases quadratically when n increases in Section 2.4. Therefore we attempts to avoid the cost of calculating the gradient matrix G explicitly before the optimisation starts. Instead we merge the gradient matrix inside the interior point method by substituting the derivative of stress approximation Eq.(5) into $\mathsf{G} \cdot \mathbf{v}$, $\mathsf{G}^T \cdot \mathbf{v}$ directly. This forms an implicit sensitivity analysis which will be described below.

The product $\mathbf{w}_s = \mathsf{G}^T\mathsf{C}^{-1}\mathsf{G} \cdot \mathbf{v}$ can be computed in three steps.

$$\begin{aligned} \mathbf{w}_1 &= \mathsf{G}(\mathbf{x})\mathbf{v}, \\ \mathbf{w}_2 &= \mathsf{C}^{-1}\mathbf{w}_1, \\ \mathbf{w}_s &= \mathsf{G}^T(\mathbf{x})\mathbf{w}_2. \end{aligned} \tag{52}$$

The matrix C is diagonal, hence the matrix vector product involving it is $O(n)$. Thus the cost is dominated by products of the form $\mathsf{G}\mathbf{v}$ and $\mathsf{G}^T\mathbf{v}$. Note that the matrix $\mathsf{G}(\mathbf{x})$ contains the gradients of the stress approximations at the current iterate of the interior point method which is different from the gradient matrix evaluated at the approximation point $\mathsf{G}(\mathbf{x}_k)$ ($\mathsf{G}_e(\mathbf{x}_k)$). Recalling the decomposition of $\mathsf{G}(x)$, Eq.(6) and Eq.(10), G is expressed as,

$$\mathsf{G}(\mathbf{x}) = \mathsf{G}_s(\mathbf{x}) - \mathsf{G}_s(\mathbf{x}_k) + \mathsf{G}(\mathbf{x}_k). \tag{53}$$

Noting that $\mathsf{G}_s(\mathbf{x})$ and $\mathsf{G}_s(\mathbf{x}_k)$ are diagonal and cheap to explicitly construct and store, then the cost is dominated by products of the form $\mathsf{G}(\mathbf{x}_k) \cdot \mathbf{v}$ and $\mathsf{G}^T(\mathbf{x}_k) \cdot \mathbf{v}$. The purpose of this section is to show that these products may be formed without the explicit calculation and storage of the matrix $\mathsf{G}(\mathbf{x}_k)$ and in $O(n)$ operations.

To compute $\mathsf{G}(\mathbf{x}_k) \cdot \mathbf{v}$ start from Eq.(20) for the stress sensitivity to obtain,

$$\mathsf{G}(\mathbf{x}_k) \cdot \mathbf{v} = \left.\begin{bmatrix} \sum_i \left(\mathsf{D}_i\mathsf{B}_i\mathsf{T}_i\right) v_i(-\mathsf{K}^{-1})\frac{\partial \mathsf{K}}{\partial x_1}\mathbf{u} \\ \sum_i \left(\mathsf{D}_i\mathsf{B}_i\mathsf{T}_i\right) v_i(-\mathsf{K}^{-1})\frac{\partial \mathsf{K}}{\partial x_2}\mathbf{u} \\ \vdots \\ \sum_i \left(\mathsf{D}_i\mathsf{B}_i\mathsf{T}_i\right) v_i(-\mathsf{K}^{-1})\frac{\partial \mathsf{K}}{\partial x_n}\mathbf{u} \end{bmatrix}\right|_{\mathbf{x}_k} \tag{54}$$

The first part $\sum_i (\mathsf{D}_i\mathsf{B}_i\mathsf{T}_i)\, v_i(-\mathsf{K}^{-1})$ is the same in each term of the above equation. Since the stiffness matrix K is symmetric, this term can be obtained by the adjoint method.

$$\mathsf{K}\mathbf{t} = \mathbf{z}, \tag{55}$$

where, $\mathbf{z} = \sum_i (\mathsf{D}_i\mathsf{B}_i\mathsf{T}_i)^T v_i$ is the dummy load.

The final equation becomes,

$$G(\mathbf{x}_k)\cdot\mathbf{v} = \left.\begin{bmatrix} \mathbf{t}^T\frac{\partial\mathsf{K}}{\partial x_1}\mathbf{u} \\ \mathbf{t}^T\frac{\partial\mathsf{K}}{\partial x_2}\mathbf{u} \\ \vdots \\ \mathbf{t}^T\frac{\partial\mathsf{K}}{\partial x_n}\mathbf{u} \end{bmatrix}\right|_{\mathbf{x}_k} \tag{56}$$

The number nonzero elements in $\frac{\partial\mathsf{K}}{\partial x_i}$ depends on the number of degrees of freedom of the element which is a small *fixed* number. Hence, $\frac{\partial\mathsf{K}}{\partial x_i}\mathbf{u}$ has a fixed cost to compute. Therefore the most expensive cost for the above equation is the solution of the adjoint system. The computational cost correspondingly is $O(n)$ to evaluate $\mathsf{G}(\mathbf{x}_k)\cdot\mathbf{v}$ once.

Next calculating $\mathsf{G}^T(\mathbf{x}_k)\cdot\mathbf{v}$ is considered.

$$\mathsf{G}^T(\mathbf{x}_k)\cdot\mathbf{v} = \left.\begin{bmatrix} (\mathsf{D}_1\mathsf{B}_1\mathsf{T}_1)\,(-\mathsf{K}^{-1})\sum_i v_i\frac{\partial\mathsf{K}}{\partial x_i}\mathbf{u} \\ (\mathsf{D}_2\mathsf{B}_2\mathsf{T}_2)\,(-\mathsf{K}^{-1})\sum_i v_i\frac{\partial\mathsf{K}}{\partial x_i}\mathbf{u} \\ \vdots \\ (\mathsf{D}_n\mathsf{B}_n\mathsf{T}_n)\,(-\mathsf{K}^{-1})\sum_i v_i\frac{\partial\mathsf{K}}{\partial x_i}\mathbf{u} \end{bmatrix}\right|_{\mathbf{x}_k} \tag{57}$$

It is important to notice $(-\mathsf{K}^{-1})\sum_i v_i\frac{\partial\mathsf{K}}{\partial x_i}\mathbf{u}$ is the same in each component of the vector. Actually, we may consider the scalar v_i as the perturbation of design variable x_i. Consequently we can assemble the $\sum_i v_i\frac{\partial\mathsf{K}}{\partial x_i}$ into a single matrix $\Delta\mathsf{K}$. $(-\mathsf{K}^{-1})\sum_i v_i\frac{\partial\mathsf{K}}{\partial x_i}\mathbf{u}$ is actually $(-\mathsf{K}^{-1})\Delta\mathsf{K}\mathbf{u}$. Then by reanalysis method [24],

$$\Delta\mathsf{K}\mathbf{u} + \mathsf{K}\Delta\mathbf{u} = \mathbf{0}, \tag{58}$$

from which the displacement increment $\Delta\mathbf{u}$ may be calculated.

Consequently, Eq.(57) is estimating $\Delta r_i = \mathsf{D}_i\mathsf{B}_i\mathsf{T}_i\Delta\mathbf{u}$:

$$\mathsf{G}^T(\mathbf{x}_k)\cdot\mathbf{v} = \left.\begin{bmatrix} \Delta r_1 \\ \Delta r_2 \\ \vdots \\ \Delta r_n \end{bmatrix}\right|_{\mathbf{x}_k} \tag{59}$$

The most expensive part in this computation is to inverse the stiffness matrix K which the computational cost is $O(n)$. In this method, we need to inverse K only *once* in the reanalysis method Eq.(58) to obtain Eq.(59). Therefore the computational work is $O(n)$ to calculate $\mathsf{G}^T\cdot\mathbf{v}$ once.

Both the cost for $\mathsf{G}\cdot\mathbf{v}$ and $\mathsf{G}^T\cdot\mathbf{v}$ is $O(n)$ now. By contrast, previously we have to repeat n times to inverse K in adjoint method to get stress sensitivity first and then to complete gradient matrix $\mathsf{G}(\mathbf{x})$. The computational work is $O((n)^2)$. Therefore the cost to get $\mathsf{G}\cdot\mathbf{v}$ and $\mathsf{G}^T\cdot\mathbf{v}$ has been decreased effectively.

In the procedure to set up the subproblem and stress approximate, the implicit sensitivity analysis procedure mentioned above can also be applied when $\mathsf{G}(\mathbf{x})\cdot\mathbf{v}$ or $\mathsf{G}(\mathbf{x})^T\cdot\mathbf{v}$ are required.

The explicit stress sensitivity analysis which is a very expensive part is successfully removed inside the optimisation framework. The structure of the proposed method are simplified to only FEM and iterative optimisation with the implicit sensitivity analysis. Rather than completely separated, the FEM and the iterative optimisation are linked by the reanalysis and adjoint method in the implicit sensitivity analysis. Inside the optimisation, adjoint method and the reanalysis employ the stiffness matrix of FEM to update the Lagrange multipliers and design variables, the stress approximation updates the mechanical responses $(\mathbf{u}, \sigma)$ by reanalysis method with the updated design.

3.3 Summary of the Optimization Algorithm

The computational cost of the iterative optimisation has been efficiently reduced by the PCG with the proposed preconditioner. Meanwhile, the computational efficiency for the sensitivity analysis has also been improved by the implicit sensitivity analysis. Therefore the entire optimisation procedure can be accelerated. In this subsection, the overall optimisation algorithm is summarised.

The outermost loop has the design point changed according to the method of successive approximations.

Algorithm 2 **Algorithm for the proposed optimisation method**

```
1: Initialize the design variable x0
2: while vol ≥ tol do                                   ▷ stop if the volume variation tol
3:     Finite element analysis Eq. (12)
4:     Update x ← the solution of predictor-corrector interior point method ▷ see Section 2.3
5: end while
```

In the proposed optimisation method, there are only two main operations, the FEM and the iterative optimisation. The procedure iterates until the relative change of the volume of the structure is smaller than the predefined tolerance. The explicit sensitivity analysis module is removed from the optimisation procedure. The solution of the approximate subproblem remains the same as in Section 2.3. The only difference is that the PCG method is employed to solve the system of equations in Eq. (45) iteratively in the predictor and corrector steps of the algorithm. Since no explicit gradient matrix is available in the proposed method, the implicit sensitivity analysis is deployed every iteration step in the interior point method (Eqs. (56) and (59)).

The number of iteration steps for both the interior point method and PCG is small and not influenced by the size of optimization problems [20][25]. The computational cost for the optimisation will be reduced significantly.

In the next section, the computational complexity of the proposed method will be demonstrated.

3.4 Computational Complexity for the Efficient Optimization Method

The computational complexity of the proposed efficient optimization method will be analysed in this section to estimate how much improvement can be obtained. The most expensive parts in the optimisation module are the implicit sensitivity analysis and the PCG iterations.

For the implicit sensitivity analysis, the most expensive cost is for the reanalysis method and the adjoint method (Eq. (56,59)). Thus the cost of implicit sensitivity analyses is,

$$C_2' = O(n),$$

per iteration.

In the PCG, since the full coefficient matrix is not required, the computational work $(O(n^3))$ for the matrix matrix product is saved. The most costly part is the use of implicit sensitivity analysis, already discussed, and the inversion of the preconditioner. Since the preconditioner is a diagonal matrix then the cost of one PCG iteration is,

$$C_3' = O(n).$$

The number of iteration steps for both the interior point method and the PCG is small and is *assumed* not to depend on the size of optimization problem. Therefore, the total number of iterations of the PCG method $n_{iter} \ll n$ when n is large enough. The computational cost for the complete optimisation is therefore $O(n)$. The proposed method, successfully reduces the computational complexity for stress constrained optimization problems from $O(n^3)$ to $O(n)$.

4 Computational Result

In this section, two numerical cases will be utilized to show how much do we achieve with the proposed method. The target structures are statically indeterminate beams. In both cases, we repeat the optimization procedure with increasing number of elements in FEM model. All tests are implemented on an DEll pc with 8.0 GB installed memory, Intel Core i5-4670 CPU @ 3.40 GHz 3.40 GHz. The CPU time for the second test optimization procedure is recorded and compared between the original method and the proposed method in this paper. The numerical cases show that the proposed method not only produce feasible optimal solution but also successfully decreases the CPU time of optimization procedure by one order.

4.1 Two-span Continuous Beam

We employ the proposed method to complete two numerical cases. The first numerical case is the two-span continuous beam in Figure 1. The width w in the rectangular cross-section A-A is the design variable while the height h of the cross-section is fixed in the optimization problem. In this numerical case, apart from the stress constraints, the displacement in y direction at $7370mm$ is also constrained apart from the stress constraints and the lower and upper bound of the design variables.

The optimization problem is defined as:

Objective:

$$\text{Min:} f_0(\mathbf{w}) = h \sum_{i=1}^{n} l_i w_i \tag{60}$$

subject to:

$f_i(\mathbf{w}) \le \sigma_{all}$ $\quad$ i = 1 ... n, n is the number of stress constraints

$w_l \le w_i \le w_u$ $\quad$ i = 1 ... n, n is the number of design variables

$u_{con}(\mathbf{w}) \le u_a$ $\quad$ $u_{con}(\mathbf{w})$ is the displacement at 7370mm from the left

The parameters of the optimization problem is listed in Table 1.

Table 1: Parameters of Two-span continuous beam

Young's Modulus	$E = 20Gpa$
Allowable stress	$\sigma_{all} = 5Mpa$
Allowable displacement/span ratio	$r = 1/3500$
Initial design width	$t_i = 1000mm$
Minimum width	$t_l = 100mm$
Maximum width	$t_u = 1000mm$
Distributed load	$p1 = 35KN/m, p2 = 25KN/m$

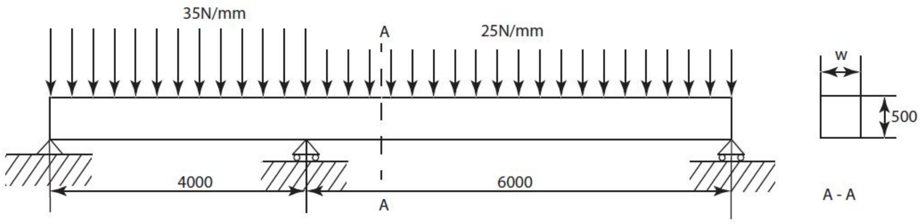

Figure 1: Two-span Continuous Beam

In this optimisation, the relative error of PCG is $err_{PCG} = 1e^{-4}$, the tolerance of duality gap is $tol_d = 1e^{-5}$, the tolerance of the volume variation is $tol_v = 1e^{-3}$. The optimal solution of the width is shown in Figure 2. It keeps at the minimum where both stress and the displacements constraints are not active. Then compared with Figure 34, the width of the beam at $1272mm$ and $4000mm$ increases because of the active of stress constraints. Then the width of the beam increases again at $7334mm$. However the increment of width in this part is due to the active of displacement constraints as demonstrated in Figure 5.

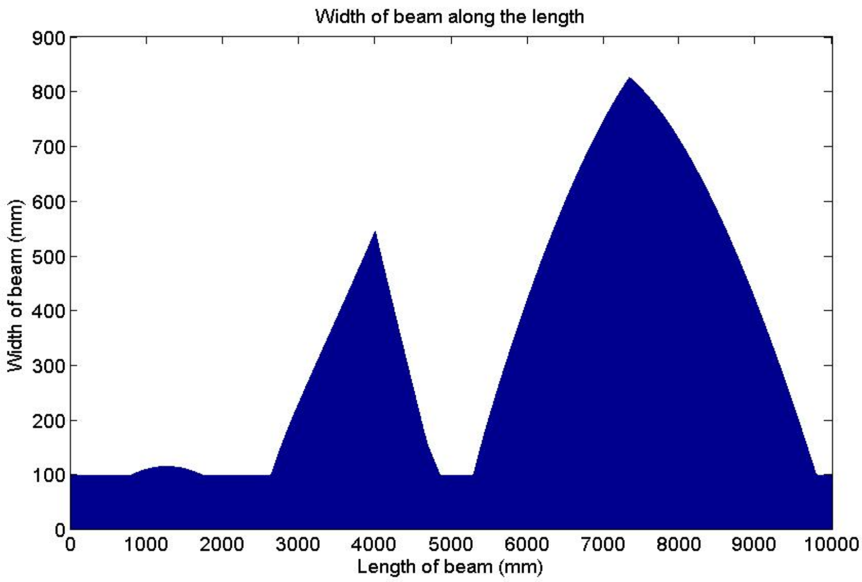

Figure 2: The width along the length of the optimal beam

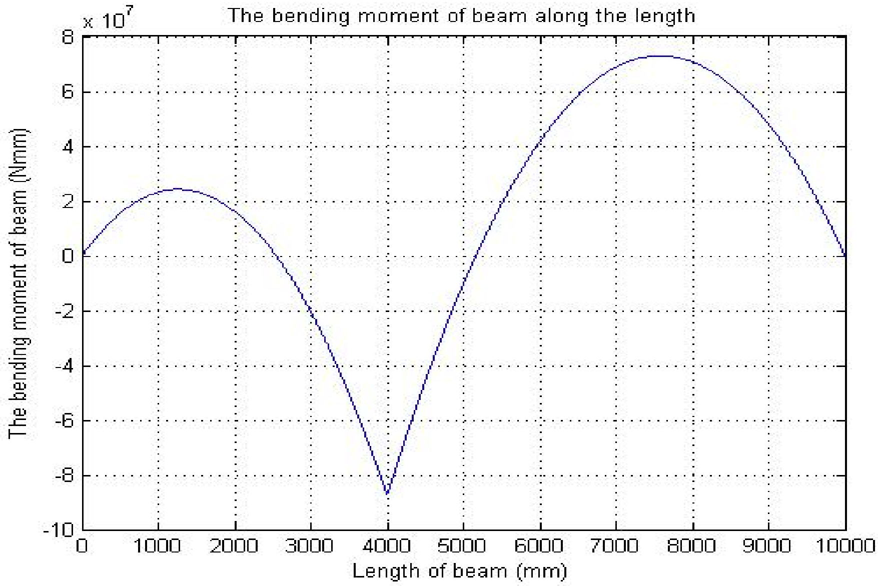

Figure 3: the bending moment in the optimal beam

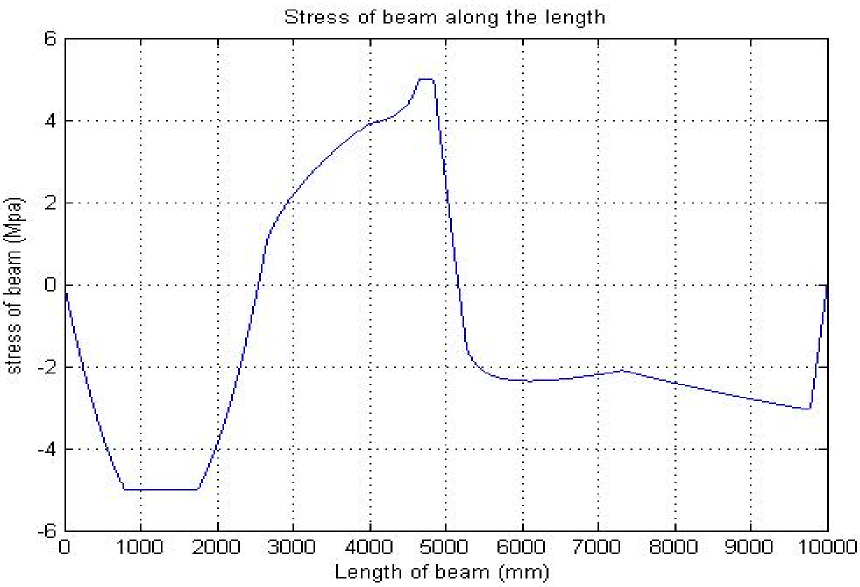

Figure 4: the normal stress on the top of the optimal beam

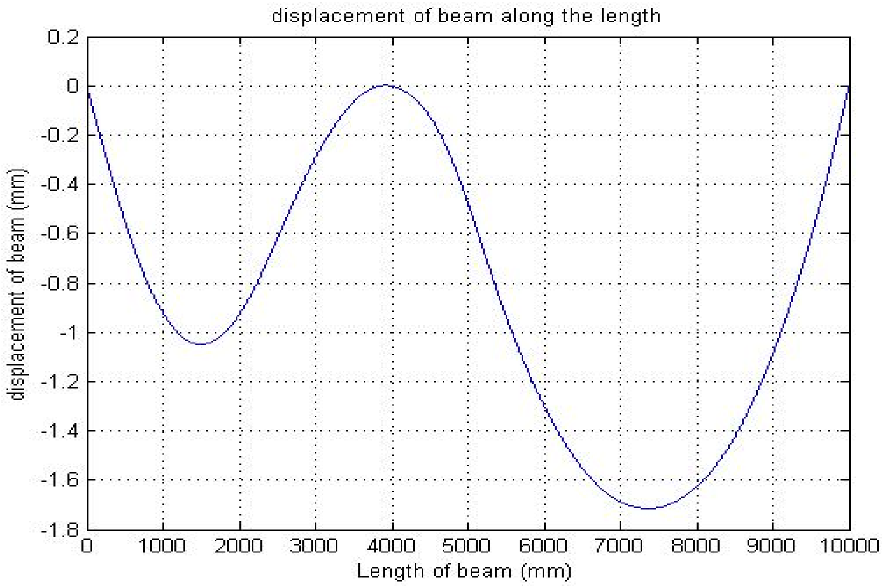

Figure 5: the displacement of the optimal beam

The volume of the beam decreases during the optimization as shown in Figure 6. It drops rapidly in the first three steps before it converges at eighth steps.

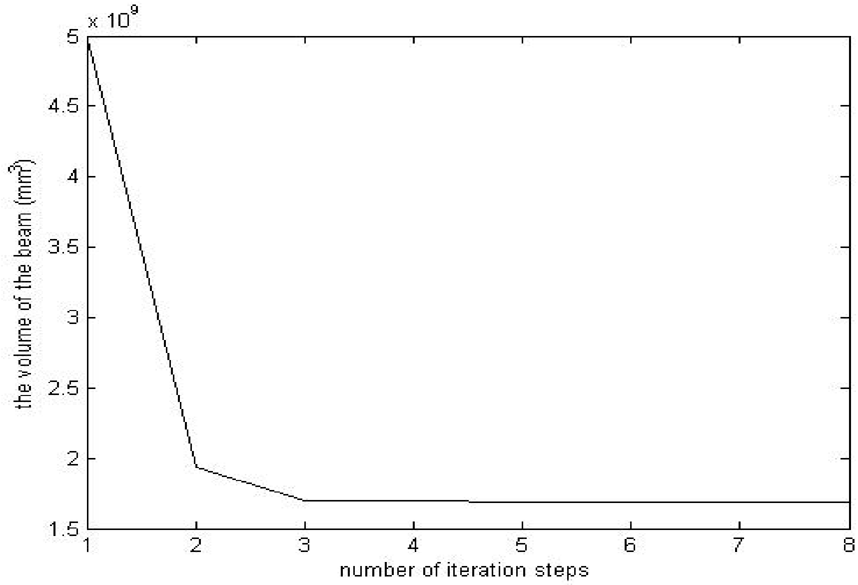

Figure 6: The volume of the beam in each iteration

4.2 Hollow Beam Structure

In this case, the design variables are the thickness t of the hollow beam shown in Figure 7. The left edge of the beam is fixed and the right edge is simply supported. Uniform distributed load is applied on the top. While design the beam with minimum volume of material, the stress within each element is constrained and the design variables are confined within the predefined range. The optimization model is defined:

Objective:

$$\textit{Min:} f_0(t) = S \sum_{i=1}^{n} l_i t_i \quad S \text{ is the perimeter of the cross-section area} \tag{61}$$

subject to:

$$f_i(t) \leq \sigma_{all} \quad i = 1 \ldots \text{n, n is the number of stress constraints}$$
$$t_l \leq t_i \leq t_u \quad i = 1 \ldots \text{n, n is the number of design variables}$$

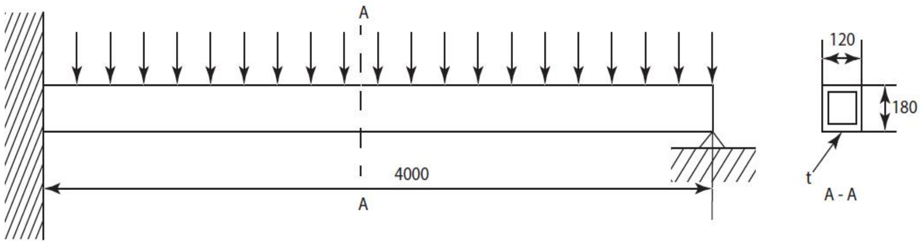

Figure 7: Hollow Beam Structure

The optimization starts with uniform thickness distribution. The parameters of the optimization problem is listed in Table 2

Table 2: Parameters of hollow beam structure

Young's Modulus	$E = 200Gpa$
Allowable stress	$\sigma_{all} = 172.36Mpa$
Initial design thickness	$t_i = 20mm$
Minimum thickness	$t_l = 3mm$
Maximum thickness	$t_u = 20mm$
Distributed load	$p = 40N/mm$

In this case, the relative error of PCG is $err_{PCG} = 1e^{-4}$, the tolerance of duality gap is $tol_d = 1e^{-5}$, the tolerance of the volume variation is $tol_v = 1e^{-3}$. The optimal thickness along the length of the beam is depicted in Figure 8. It first decreases from $16.7mm$ from the left edge to $3mm$ at around $1000mm$. Then it increases to $7.2mm$ at around $2600mm$ before it reduces to $3mm$ again.

The internal force of the beam, i.e., the bending moment here, is illustrated in Figure 9. On the left edge, it reaches $-9.31 \times 10^7 Nmm$, then it increases to $0Nmm$ and then reaches the $4 \times 10^7 Nmm$ at about $2500mm$. It goes back to $0Nmm$ again on the right edge.

Therefore the distribution of the thickness matches the trend of bending moment to confine the stress. From Figure 10, which shows the normal stress on the top edge of the beam along the length, the normal stress reaches the allowable value where the stress constraints are active.

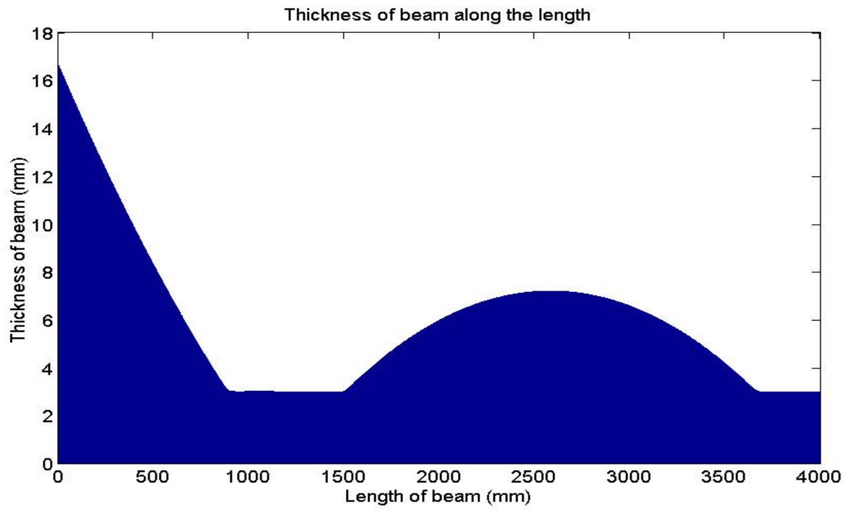

Figure 8: thickness along the length of the optimal beam

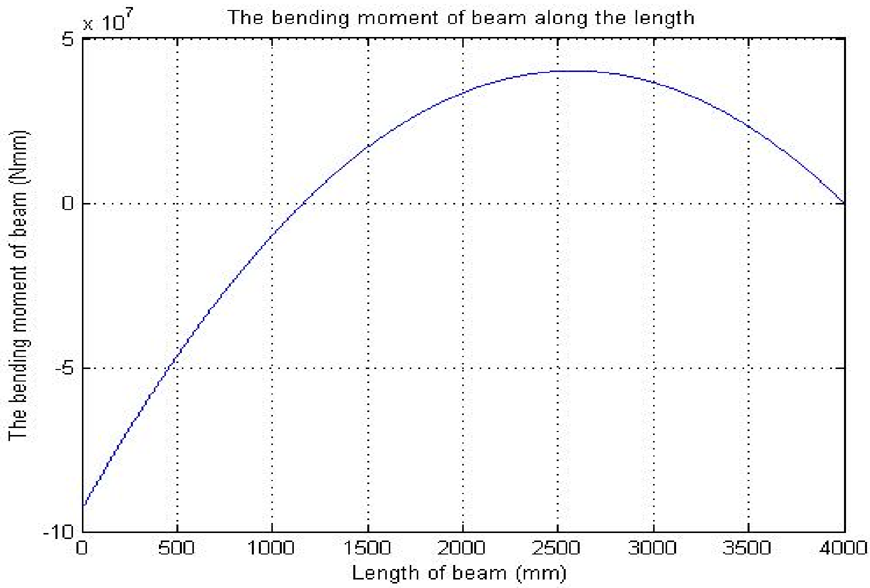

Figure 9: the bending moment in the optimal beam

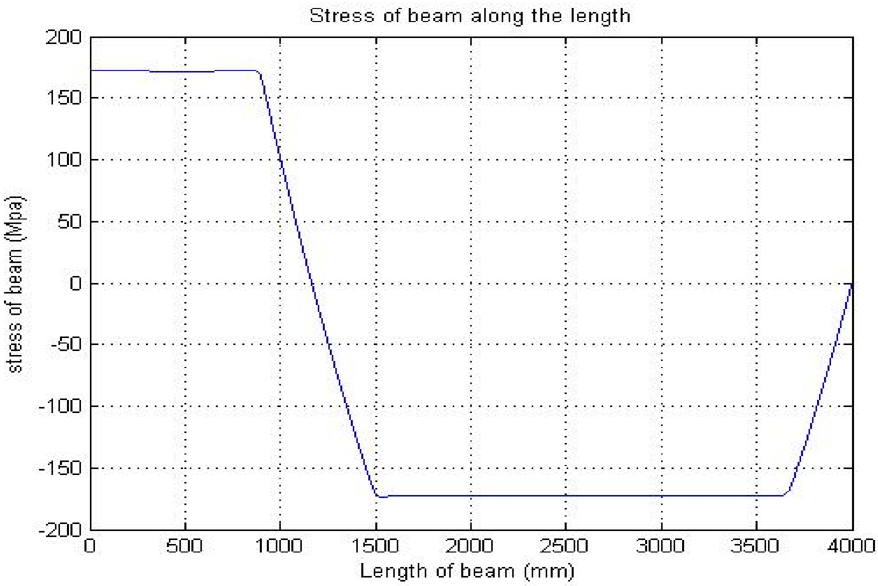

Figure 10: the normal stress on the top of the optimal beam

The variation of the volume during optimization is shown in Figure 11. The volume drops rapidly in the first two loops. It takes merely 6 steps in total to converge to the optimal solution.

In this case, we record the number of iteration steps in the PCG, the interior point method and the entire optimisation when the structure is discretised by different number of elements. We start the procedure from 20 elements and double the number of elements each time. From Figure 12, there is no obvious change for the number of steps in the entire optimization iteration, the interior point and the PCG when the number of elements increases from 20 to 10240. The number of steps for the PCG is the averaged total number of the PCG iterations per interior point method step. The CPU time for the optimization procedure with the original method and the proposed method is portrayed in Figure 13. The dashed line in the figure demonstrates the linear trend between the CPU time and the number of elements. As we can see from the figure, the trend of the solid line is almost parallel to the dashed line. It proves that computational complexity for the whole procedure has been decreased from more than $O(n^2)$ to $O(n)$. Therefore great efficiency is achieved by the proposed method is for large scale stress constrained optimization.

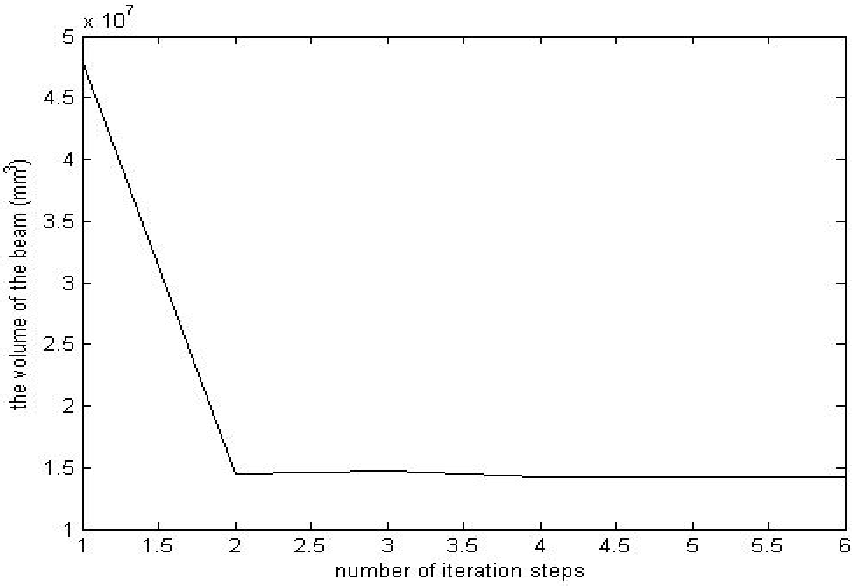

Figure 11: The volume of the beam in each iteration

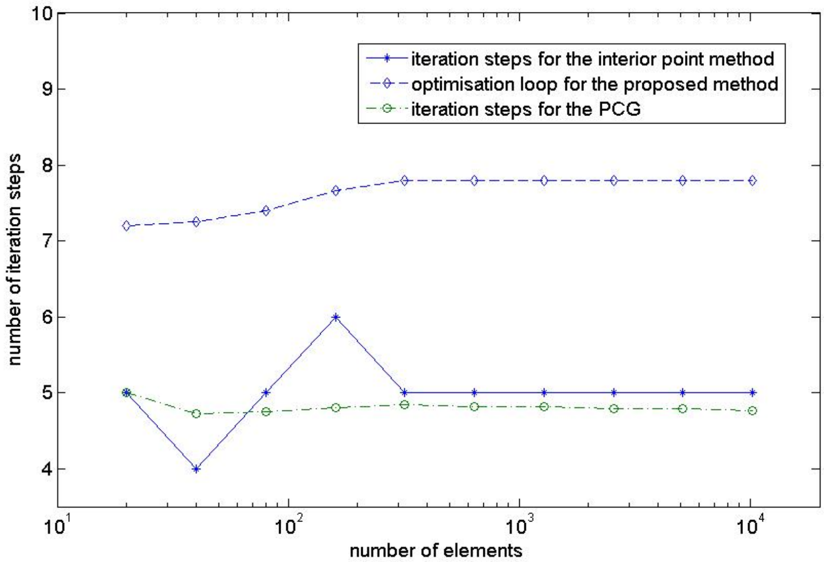

Figure 12: the Number of steps for entire optimization,interior point method and PCG

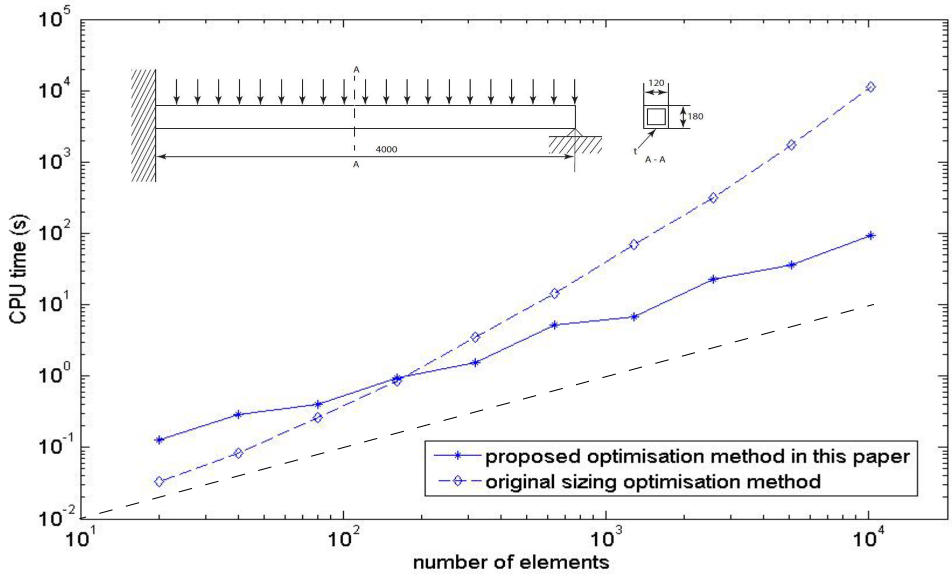

Figure 13: the log-log form of the CPU time vs number of elements for the optimization with original method and the proposed method

5 Conclusion

An efficient method for large-scale beam sizing under stress constraints is introduced in this paper. A novel successive convex stress approximation which is homogenous and separable is described first. This proposed approximation not only provides sufficient accuracy but also makes the subproblem cheap to build in the optimisation module.

The PCG is employed to enhance the efficiency of solving the subproblem in the optimisation module for the large scale optimisation problems. An effective preconditioner based on the fully stressed concept is introduced. Numerical test in Section 4.2 proves that the PCG converges only a few steps to solve the linear equation with the proposed preconditioner.

An efficient implicit sensitivity analysis method is proposed in this method to replace the expensive sensitivity analysis by the adjoint method. The structure of the optimisation framework therefore is simplified to only FEM analysis and iterative optimisation with the interior point method. The computational effort for sensitivity related calculation has been decreased from $O(n^2)$ to $O(n)$.

The implicit sensitivity analysis is deployed each time the gradient matrix G is required in the iterative optimisation procedure. Since the total number of iterations of the PCG method in the interior point method is $n_{iter} \ll n$ when n is large enough. The computational cost for the entire optimisation drops from $O(n^3)$ to $O(n)$.

Great efficiency for stress constrained large scale optimisation is achieved with the proposed method. Numerical tests have validated the conclusion from the theoretical analysis of computational complexity. For the future research, the method is supposed to be extended to plate elements to strength the problem solving capacity of the proposed method.

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