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Online Identification of Continuous Bimodal and Trimodal Piecewise Affine Systems

Le Quang Thuan[1], Ton van den Boom[2] and Simone Baldi[3]

Abstract— **This paper investigates the identification of continuous piecewise affine systems in state space form with jointly unknown partition and subsystem matrices. The partition of the system is generated by the so-called centers. By representing continuous piecewise affine systems in the max-form and using a recursive Gauss-Newton algorithm for a suitable cost function, we derive adaptive laws to online estimate parameters including both subsystem matrices and centers. The effectiveness of the proposed approach is demonstrated with a numerical example.**

I. Introduction

A piecewise affine (PWA) system is a special kind of finite-dimensional, nonlinear input-state-output systems, with the distinguishing feature that the functions representing the systems differential equations and output equations are piecewise affine functions of state and input [20]. Any piecewise affine system can be considered as a collection of finite-dimensional affine input-state-output systems, together with a partition of the product of the state space and input space into polyhedral regions. Each of these regions is associated with one particular affine system from the collection. Depending on the region in which the state and input vector are contained at a certain time, the dynamics is governed by the affine system associated with that region. Thus, the dynamics switches if the state-input vector changes from one polyhedral region to another. In recent decades, the analysis and control of PWA systems have been extensively investigated due to their modeling capabilities. PWA systems form a subclass of hybrid systems and they can be used to approximate nonlinear systems [11], [16]. They are widely used in engineering and applied science to model complex systems. Many physical systems appearing in theoretical engineering can be modeled by means of piecewise affine systems such as relay systems, hysteresis systems, and system with saturation phenomena, etc. In addition, PWA systems have become more popular thanks to their equivalence with many classes of hybrid systems such as mixed logical dynamical systems [6], linear complementary systems and other model classes [9]. Thus, PWA systems provide the powerful means for analysis and design of hybrid systems. This paper focuses on identifying continuous piecewise affine systems.

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The system identification has been a long-standing problem in control theory and received much attentions. For PWA systems, identification is composed of two ingredients: estimation of the subsystem parameters and the hyperplanes defining the partition. In the case that one of the ingredients is assumed to be known, various contributions have been presented in the literature. Identifying PWA systems with known partitions can be carried out by standard linear identification techniques in a local manner. When both subsystems and the partition are unknown, to identify PWA systems, the partition must be estimated together with the subsystems. This has been known a very challenging problem and the main difficulty lies in the fact that the identification problem includes a classification problem to determine in which region each data point must be associated. Despite of the difficulty, there are recently proposed techniques dealing with the issue: Bayesian procedure [12], the bounded-error procedure [5], the clustering-based procedure [8] and the Mixed-Integer Programming procedure [18]. Further results on identification of subclasses of PWA systems can be found in [1], [22]. Most of the work in the area of identifying PWA systems focuses on the development of identification algorithms for discrete-time piecewise affine functions in regression form and the algorithms are offline.

A different study on system identification has been performed with continuous-time PWA system in state space form and known partition [7], [13]–[15]. In [13], [15], the authors proposed a method to identify the sub-models of PWA system online and under persistence of excitation condition ensures the asymptotic convergence of parameters to true parameters. These results are generalized in [14] with the use of concurrent learning. The paper shown that the concurrent use of recorded and instantaneous data leads to exponential convergence of all subsystem parameters under verifiable conditions on the recorded data. Ealier work on this direction dates back to the paper [7] dealing with continuous bimodal piecewise affine systems. Summarizing, to the best of our knowledge, there is no online identification method developed for continuous-time PWA systems in state space form with joint subsystem and partition estimation.

In this paper, we develop a method to identify online continuous-time PWA systems in state space form where both the partition and the subsystems are unknown. The system partition is assumed to be generated by the so-called centers [1]. The advantage of such partitions is that one can represent continuous-time PWA systems in more compact form. Then, a cost function depending on the estimation error can be defined, and the derivative of the cost function with respect to all parameters can be taken. This can be used to develop a recursive Gauss-Newton algorithm, thus obtaining a set of adaptive laws for the estimated parameters, including both subsystem matrices and centers.

This paper is organized as follows. Section II introduces the considered PWA systems and its identification problem. The main results of this paper are presented in Section III and Section IV with online identification of bimodal and trimodal piecewise affine systems, respectively. A numerical example will follow in Section V. Finally, Section VI is the conclusions and future work.

II. PWA SYSTEMS AND IDENTIFICATION PROBLEM

Consider continuous piecewise affine dynamical systems of the form

$$\dot{x} = \begin{cases} A_1 x + B_1 u + e_1 & \text{if } (x,u) \in \mathcal{X}_1 \\ \quad\vdots & \qquad\vdots \\ A_N x + B_N u + e_N & \text{if } (x,u) \in \mathcal{X}_N \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}, e_i \in \mathbb{R}^n$ are given matrices, and $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\}$ is a polyhedral partition of $\mathbb{R}^{n+m}$. In the piecewise systems literature, several kinds of polyhedral partitions of $\mathbb{R}^{n \times m}$ have been considered (see e.g. [1], [17], [19], [20]). Since this paper aims at the identification of piecewise affine dynamical systems, we will work with the partitions generated by the so-called centers due to its minimality of parameters. As defined in [1], given N vectors $c_1, c_2, \ldots, c_N$ of $\mathbb{R}^{n+m}$ called centers, one can generate polyhedral regions

$$\begin{aligned} \mathcal{X}_j &= \left\{ z \in \mathbb{R}^{n+m} \mid ||z - c_j||_2 \leqslant ||z - c_k||_2, \forall k \neq j \right\} \\ &= \left\{ z \in \mathbb{R}^{n+m} \mid \mathcal{A}_j z \leqslant q_j \right\} \end{aligned} \tag{2}$$

where

$$\mathcal{A}_j = 2 \begin{bmatrix} c_1 - c_j & \ldots & c_{j-1} - c_j & c_{j+1} - c_j & \ldots & c_N - c_j \end{bmatrix}^T$$
$$q_j = \begin{bmatrix} \beta_{1,j} & \ldots & \beta_{j-1,j} & \beta_{j+1,j} & \ldots & \beta_{N,j} \end{bmatrix}^T$$

with $\beta_{k,j} = c_k^T c_k - c_j^T c_j$ for $j = 1, 2, \ldots, N$.

Identification problem. Consider the system (1) where $\mathcal{X}_j$ is defined by (2). Suppose that the subsystem matrices A_j, B_j, e_j and the centers c_j are unknown. Based on the measured state $x(t)$ and input $u(t)$, we want to find the update laws for the estimated parameters $\hat{A}_j(t), \hat{B}_j(t), \hat{e}_j(t)$ and $\hat{c}_j(t)$ such that $\hat{A}_j(t) \to A_j,\ \hat{B}_j(t) \to B_j,\ \hat{e}_j(t) \to e_j,\ \hat{c}_j(t) \to c_j$ as $t \to \infty$.

III. ONLINE IDENTIFICATION OF BIMODAL PWA SYSTEMS

In this section, we first consider the identification of a particular class of the system (1) where $N = 2$, called bimodal piecewise affine systems. As $N = 2$, the system (1) reads as

$$\dot{x} = \begin{cases} A_1 x + B_1 u + e_1 & \text{if } (x,u) \in \mathcal{X}_1 \\ A_2 x + B_2 u + e_2 & \text{if } (x,u) \in \mathcal{X}_2 \end{cases} \tag{3}$$

where

$$\mathcal{X}_1 = \{(x,u) \mid 2(c_2 - c_1)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_2^T c_2 - c_1^T c_1) \leqslant 0\},$$
$$\mathcal{X}_2 = \{(x,u) \mid 2(c_2 - c_1)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_2^T c_2 - c_1^T c_1) \geqslant 0\}.$$

A. Max-form presentation of bimodal PWA systems

The system (3) is nonlinear in the parameters. However, by invoking the well-known properties of continuous bimodal piecewise affine functions, one can split the right-hand side into two parts: one part is linear and the other is nonlinear in the parameters. In fact, the continuity of the system (3) is equivalent to the unique existence of the $h \in \mathbb{R}^n$ such that (see, [21])

$$\begin{bmatrix} A_1 & B_1 \end{bmatrix} - \begin{bmatrix} A_2 & B_2 \end{bmatrix} = 2h(c_2 - c_1)^T, \tag{4a}$$
$$e_1 - e_2 = -h(c_2^T c_2 - c_1^T c_1). \tag{4b}$$

In view of (4), one can rewrite the bimodal system (3) as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} A_2 & B_2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + e_2 \\ &\quad - h \max\left\{ 2(c_1 - c_2)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_1^T c_1 - c_2^T c_2), 0 \right\}. \end{aligned} \tag{5}$$

This system is called the max-form representation of the bimodal piecewise affine system (3).

B. Online identification of bimodal PWA systems

Since every bimodal piecewise affine system (3) can be equivalently represented in the form (5), its identification can be performed by identifying the system (5). Moreover, for the hyperplane $\mathcal{X}_1 \cap \mathcal{X}_2$, there are infinitely many pairs of centers (c_1, c_2) which generates the hyperplane. However, when we fix one center by an arbitrary vector, the other one is uniquely determined. Thus, without loss of generality, we suppose that the center c_2 is known, $c_2 = \tilde{c}$. For convenience, we use the notations c, A, B, e instead of c_1, A_2, B_2, e_2, respectively. Then, the system (5) becomes

$$\begin{aligned} \dot{x} &= \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + e \\ &\quad - h \max\left\{ 2(c - \tilde{c})^T \begin{bmatrix} x \\ u \end{bmatrix} - (c^T c - \tilde{c}^T \tilde{c}), 0 \right\}. \end{aligned} \tag{6}$$

We now solve the identification issue of the system (6) with unknown parameter

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \\ c \end{bmatrix}, \text{ where } \theta_i = \begin{bmatrix} \mathrm{r}_i(A)^T \\ \mathrm{r}_i(B)^T \\ e_i \\ h_i \end{bmatrix} \text{ for } i = 1, 2, \ldots, n,$$

and $\mathrm{r}_i(M)$ denotes the i^{th} row of the matrix M. Let us now suppose that the state $x(t)$ and output $u(t)$ are available from measurements. The identification of the system (6) with these measurements is carried out by minimizing the following integral cost function

$$J(t,\hat{\theta}) = \frac{1}{2}\int_0^t e^{-\lambda(t-s)}||x(s)-\hat{x}(s,\hat{\theta})||^2 ds \tag{7}$$
$$= \frac{1}{2}\int_0^t e^{-\lambda(t-s)}\sum_{j=1}^{n}\big(\hat{x}_j(s,\hat{\theta})-x_j(s)\big)^2 ds.$$

Here, $\lambda > 0$ is a forgetting factor decided by the designer, $\hat{\theta}$ is the estimated values of θ, and $\hat{x}(s,\hat{\theta})$ denotes the estimated state of the system (6) with the observer

$$\dot{\hat{x}}(s,\hat{\theta}) = D\hat{x}(s,\hat{\theta}) + \begin{bmatrix}\hat{A}-D & \hat{B}\end{bmatrix}\begin{bmatrix}x(s)\\u(s)\end{bmatrix} + \hat{e} - \hat{h}\max\Big\{\Lambda(\hat{c},x(s),u(s)),0\Big\} \tag{8}$$

where $\Lambda(\hat{c},s) = 2(\hat{c}-\tilde{c})^T\begin{bmatrix}x(s)\\u(s)\end{bmatrix} - (\hat{c}^T\hat{c}-\tilde{c}^T\tilde{c})$ and D is a stable matrix of the form $D = \mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_n)$ with $\mu_j < 0$ for all $j = 1,2,\ldots,n$. Note that the solution of the system (8) for the initial state $x(0)$ can be component-wisely written as

$$\hat{x}_i(s,\hat{\theta}) = e^{\mu_i s}x_i(0) + \int_0^s e^{\mu_i(s-\tau)}\Big\{\mathrm{r}_i(\hat{A}-D)x(\tau) + \mathrm{r}_i(\hat{B})u(\tau) + \hat{e}_i - \hat{h}_i\max\big\{\Lambda(\hat{c},s),0\big\}\Big\}d\tau$$

for $i = 1,2,\ldots,n$.

The cost function $J(t,\hat{\theta})$ has a global minimum at the real system parameters $\hat{\theta} = \theta$. In order to try to find θ, the recursive Gauss-Newton algorithm would be employed in this paper. By this algorithm, we first choose an initial value $\hat{\theta}_0$ which is assumed to be in a small neighborhood of θ. Then, the sequence $\hat{\theta}(t)$ is updated online via the following adaptive law

$$\dot{\hat{\theta}}(t) = -\Gamma(U(t))^{-1}\Phi(t)\begin{bmatrix}\dfrac{\partial J(t,\hat{\theta})}{\partial\hat{x}_1}\\ \vdots \\ \dfrac{\partial J(t,\hat{\theta})}{\partial\hat{x}_n}\end{bmatrix}\Bigg|_{\hat{\theta}=\hat{\theta}(t)}, \tag{9a}$$

$$\hat{\theta}(0) = \hat{\theta}_0 \tag{9b}$$

where $\Gamma > 0$ is decided by designer, and

$$\dot{U}(t) = -\Gamma U(t) + \Phi(t)\Phi(t)^T, U(0) = 0 \tag{10}$$

with

$$\Phi(t) = \begin{bmatrix}\dfrac{\partial\hat{x}_1(t,\hat{\theta})}{\partial\hat{\theta}_1} & 0 & \cdots & 0\\ 0 & \dfrac{\partial\hat{x}_2(t,\hat{\theta})}{\partial\hat{\theta}_2} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \dfrac{\partial\hat{x}_n(t,\hat{\theta})}{\partial\hat{\theta}_n}\\ \dfrac{\partial\hat{x}_1(t,\hat{\theta})}{\partial\hat{c}} & \dfrac{\partial\hat{x}_2(t,\hat{\theta})}{\partial\hat{c}} & \cdots & \dfrac{\partial\hat{x}_n(t,\hat{\theta})}{\partial\hat{c}}\end{bmatrix}$$

It can be verified that

$$\frac{d}{dt}\Big(\frac{\partial J(t,\hat{\theta})}{\partial\hat{x}_i}\Big) = -\lambda\frac{\partial J(t,\hat{\theta})}{\partial\hat{x}_i} + \hat{x}_i(t,\hat{\theta}) - x_i(t), \tag{11a}$$
$$\frac{\partial J(t,\hat{\theta})}{\partial\hat{x}_i}(0) = 0 \tag{11b}$$

for $i = 1,2,\ldots,n$. Furthermore, since one can write $\hat{x}_i(t,\hat{\theta})$ in the form

$$\hat{x}_i(t,\hat{\theta}) = g_{0,i}(t) + \begin{bmatrix}g_{1,i}^T(t) & g_{2,i}^T(t) & g_{3,i}(t) & -g_{4,i}(t)\end{bmatrix}\hat{\theta}_i$$

with

$$g_{0,i}(t) = e^{\mu_i t}x_i(0) - \mu_i\int_0^t e^{\mu_i(t-\tau)}x_i(\tau)d\tau,$$
$$g_{1,i}(t) = \int_0^t e^{\mu_i(t-\tau)}x(\tau)d\tau, g_{2,i}(t) = \int_0^t e^{\mu_i(t-\tau)}u(\tau)d\tau,$$
$$g_{3,i}(t) = \int_0^t e^{\mu_i(t-\tau)}d\tau,$$
$$g_{4,i}(t) = \int_0^t e^{\mu_i(t-\tau)}\max\Big\{\Lambda(\hat{c},\tau),0\Big\}d\tau,$$

it is easy to verify that

$$\frac{\partial\hat{x}_i(t,\hat{\theta})}{\partial\hat{\theta}_i} = \begin{bmatrix}g_{1,i}(t)\\g_{2,i}(t)\\g_{3,i}(t)\\-g_{4,i}(t)\end{bmatrix} =: \tilde{g}_i(t)$$

$$\hat{x}(t,\hat{\theta}) - x(t) = \begin{bmatrix}g_{0,1}(t)-x_1(t)\\ \vdots\\ g_{0,n}(t)-x_n(t)\end{bmatrix} + \begin{bmatrix}\tilde{g}_1(t)^T\hat{\theta}_1\\ \vdots\\ \tilde{g}_n(t)^T\hat{\theta}_n\end{bmatrix}$$

and

$$\frac{\partial\hat{x}_i(t,\hat{\theta})}{\partial\hat{c}} = -\hat{h}_i\int_0^t e^{\mu_i(t-\tau)}\begin{bmatrix}w_1(\tau)\\ \vdots\\ w_{n+m}(\tau)\end{bmatrix}d\tau$$

where

$$w_j(\tau) = \begin{cases}2\mathbf{x}_j(\tau) - 2\hat{c}_j, & \Lambda(\hat{c},\tau) = \max\Big\{\Lambda(\hat{c},\tau),0\Big\}\\ 0, & \text{otherwise}\end{cases} \tag{12}$$

for $j = 1,2,\ldots,n$ where

$$\mathbf{x}_j(\tau) = \begin{cases}x_j(\tau), & j = 1,2,\ldots,n\\ u_{j-n}(\tau), & j = n+1,\ldots,n+m.\end{cases}$$

Therefore, one can write (11a) as

$$\frac{d}{dt}\Big(\frac{\partial J(t,\hat{\theta})}{\partial\hat{x}_i}\Big) = -\lambda\frac{\partial J(t,\hat{\theta})}{\partial\hat{x}_i} + g_{0,i}(t) - x_i(t) + \tilde{g}_i(t)^T\hat{\theta}_i \tag{13}$$

for $i = 1,2,\ldots,n$ and write $\Phi(t)$ in the form

$$\Phi(t) = \begin{bmatrix}\tilde{g}_1(t) & 0 & \cdots & 0\\ 0 & \tilde{g}_2(t) & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \tilde{g}_n(t)\\ \dfrac{\partial\hat{x}_1}{\partial\hat{c}} & \dfrac{\partial\hat{x}_2}{\partial\hat{c}} & \cdots & \dfrac{\partial\hat{x}_n}{\partial\hat{c}}\end{bmatrix}.$$

To update $g_{0,i}, \tilde{g}_i$ and Φ, we use the fact that

$$\dot{\tilde{g}}_i(t) = \mu_i \tilde{g}_i(t) + \begin{bmatrix} x(t) \\ u(t) \\ 1 \\ -\max\{\Lambda(\hat{c}(t), t), 0\} \end{bmatrix}, \tilde{g}_i(0) = 0 \tag{14a}$$

$$\dot{g}_{0,i}(t) = \mu_i g_{0,i}(t) - \mu_i x_i(t),\ g_{0,i}(t) = x_i(0) \tag{14b}$$

$$\frac{d}{dt}(\frac{\partial \hat{x}_i}{\partial \hat{c}}) = \mu_i \frac{\partial \hat{x}_i}{\partial \hat{c}} - \hat{h}_i \begin{bmatrix} w_1(t) \\ \vdots \\ w_{n+m}(t) \end{bmatrix}, \frac{\partial \hat{x}_i}{\partial \hat{c}}(0) = 0 \tag{14c}$$

for $i = 1, 2, \dots, n$ and $w_1, w_2, \dots, w_{n+m}$ defined in (12). In summary, one can update the parameters $\hat{\theta}(t)$ as follows:

Theorem 1: The parameters $\hat{\theta}(t)$ can be updated by the differential equations (9), (10), (13) and (14).

C. Special cases: Affine systems

In the continuous bimodal piecewise affine system (3), by taking $A_1 = A_2 = A$, $B_1 = B_2 = B$ and $e_1 = e_2 = e$, the system boils down to the affine system $\dot{x} = Ax + Bu + e$ for any given centers c_1 and c_2. The online identification of the system with $e = 0$ has been well-studied by Lyapunov function approach; see e.g. [10]. In this section, by reducing our results to affine systems, we obtain an online update law to identify affine systems. It seems that these formulation is unavailable anywhere else.

To derive the formulation, note that one can take any centers c_1, c_2 and $h = 0$. Thus, we only need to identify the parameters

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}, \text{ where } \theta_i = \begin{bmatrix} \mathrm{r}_i(A)^T \\ \mathrm{r}_i(B)^T \\ e_i \end{bmatrix} \text{ for } i = 1, 2, \dots, n,$$

and the observer (8) just simply is

$$\dot{\hat{x}}(s, \hat{\theta}) = D\hat{x}(s, \hat{\theta}) + (\hat{A} - D)x(s) + \hat{B}u(s) + \hat{e}. \tag{15}$$

Reducing from the bimodal case, we come up with an update law to updating $\hat{A}$, $\hat{B}$ and $\hat{e}$ as follows:

$$\dot{\hat{\theta}}(t) = -\Gamma(U(t))^{-1}\Phi(t)\frac{\partial J(t, \hat{\theta})}{\partial \hat{x}}\Big|_{\hat{\theta} = \hat{\theta}(t)} \tag{16a}$$

$$\hat{\theta}(0) = \hat{\theta}_0 \tag{16b}$$

where $\Gamma > 0$ is chosen by designer, and

$$\dot{U}(t) = -\Gamma U(t) + \Phi(t)\Phi(t)^T,$$

$$\Phi(t) = \begin{bmatrix} \tilde{g}_1(t) & 0 & \cdots & 0 \\ 0 & \tilde{g}_2(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{g}_n(t) \end{bmatrix},$$

$$\frac{d}{dt}\Big(\frac{\partial J(t, \hat{\theta})}{\partial \hat{x}_i}\Big) = -\lambda \frac{\partial J(t, \hat{\theta})}{\partial \hat{x}_i} + g_{0,i}(t) - x_i(t) + \tilde{g}_i(t)^T \hat{\theta}_i,$$

$$\dot{\tilde{g}}_i(t) = \mu_i \tilde{g}_i(t) + \begin{bmatrix} x(t) \\ u(t) \\ 1 \end{bmatrix}, \tilde{g}_i(0) = 0,$$

$$\dot{g}_{0,i}(t) = \mu_i g_{0,i}(t) - \mu_i x_i(t), g_{0,i}(0) = x_i(0),$$

for $i = 1, 2, \dots, n$.

IV. Online identification of trimodal PWA systems

This section aims at solving the identification of the system (1) for $N = 3$, trimodal piecewise affine systems. In this case, the system (1) reads as

$$\dot{x} = \begin{cases} A_1 x + Bu + e_1 & \text{if } (x, u) \in \mathcal{X}_1 \\ A_2 x + Bu + e_2 & \text{if } (x, u) \in \mathcal{X}_2 \\ A_3 x + Bu + e_3 & \text{if } (x, u) \in \mathcal{X}_3 \end{cases} \tag{17}$$

where

$$\begin{aligned} \mathcal{X}_1 = \{(x, u) \mid\ & 2(c_2 - c_1)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_2^T c_2 - c_1^T c_1) \leqslant 0, \\ & 2(c_3 - c_1)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_3^T c_3 - c_1^T c_1) \leqslant 0\}, \\ \mathcal{X}_2 = \{(x, u) \mid\ & 2(c_2 - c_1)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_2^T c_2 - c_1^T c_1) \geqslant 0, \\ & 2(c_3 - c_2)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_3^T c_3 - c_2^T c_2) \leqslant 0\}, \\ \mathcal{X}_3 = \{(x, u) \mid\ & 2(c_3 - c_2)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_3^T c_3 - c_2^T c_2) \geqslant 0, \\ & 2(c_3 - c_1)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_3^T c_3 - c_1^T c_1) \geqslant 0\}. \end{aligned}$$

A. Max-form presentation of trimodal PWA systems

Once the system (17) is continuous, one can write it in the max-form. Indeed, for any c_1, c_2 and c_3, two following possible cases may occur:

a) Case 1: c_1, c_2 and c_3 lie on a line. Without loss of generality, we assume that c_2 is in the segment $[c_1, c_3]$. In this case the continuity of system (17) is equivalent to the existence of the vectors h and k in $\mathbb{R}^n$ such that one can write the system (17) as

$$\begin{aligned} \dot{x} =& A_1 x + B_1 u + e_1 \\ & - h \max\{2(c_2 - c_1)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_2^T c_2 - c_1^T c_1), 0\} \\ & - k \max\{2(c_3 - c_2)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_3^T c_3 - c_2^T c_2), 0\}. \end{aligned} \tag{18}$$

b) Case 2: c_1, c_2 and c_3 do not lie on a line. The continuity of system (17) is equivalent to the existence of the vectors $h_1, h_2, h_3 \in \mathbb{R}^n$ such that

$$\begin{bmatrix} A_1 & B_1 \end{bmatrix} - \begin{bmatrix} A_2 & B_2 \end{bmatrix} = 2h_1(c_2 - c_1)^T, \quad e_1 - e_2 = -h_1(c_2^T c_2 - c_1^T c_1), \tag{19a}$$

$$\begin{bmatrix} A_2 & B_2 \end{bmatrix} - \begin{bmatrix} A_3 & B_3 \end{bmatrix} = 2h_2(c_3 - c_2)^T, \quad e_2 - e_3 = -h_2(c_3^T c_3 - c_2^T c_2), \tag{19b}$$

$$\begin{bmatrix} A_3 & B_3 \end{bmatrix} - \begin{bmatrix} A_1 & B_1 \end{bmatrix} = 2h_3(c_3 - c_1)^T, \quad e_3 - e_1 = -h_3(c_3^T c_3 - c_1^T c_1). \tag{19c}$$

In this case, we claim that

$$h_1 = h_2 = -h_3 =: h. \tag{20}$$

In fact, it follows from (19) that

$$(h_3 + h_2)c_3^T + (h_1 - h_2)c_2^T - (h_1 + h_3)c_1^T = 0. \tag{21}$$

Once c_1, c_2 and c_3 are linearly independent, the equality (21) immediately yields $h_1 = h_2 = -h_3$. If c_1, c_2 and c_3 are linearly dependent, one can express one of them as a linear combination of the others, says for instance $c_3 = \alpha c_1 + \beta c_2$. Then, since c_1, c_2 and c_3 do not belong to a line, c_1 and c_2 must be linearly independent and $\alpha + \beta \neq 1$. Moreover, one has

$$(h_3 + h_2)c_3^T = \alpha(h_3 + h_2)c_1^T + \beta(h_3 + h_2)c_2^T,$$
$$(h_3 + h_2)c_3^T = (h_1 + h_3)c_1^T + (h_2 - h_1)c_2^T.$$

This implies that $h_2 + h_3 = (\alpha + \beta)(h_2 + h_3)$, and hence $h_2 + h_3 = 0$, i.e. $h_2 = -h_3$. Substituting this in (21) and due to the linear independence of c_1 and c_2, one gets $h_1 = h_2$. Therefore, the claim (20) has been proved.

In view of (20) and (19), one now can rewrite the system (17) in the max-form as

$$\dot{x} = A_3 x + B_3 u + e_3 - h \max \Big\{ 2(c_2 - c_3)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_2^T c_2 - c_3^T c_3), 2(c_1 - c_3)^T \begin{bmatrix} x \\ u \end{bmatrix} - (c_1^T c_1 - c_3^T c_3), 0 \Big\}$$

for some $h \in \mathbb{R}^n$.

B. Adaptive update laws

As it has been shown, the max-form presentations of tri-modal PWA systems have two different forms depending on whether or not the centers are in a line. Once the form is determined, we can develop the corresponding adaptive update laws in a similar fashion as bimodal case. However, due to the page limitation, it will be not shown in this paper.

Remark 1: Due to the fact that the system (3) is nonlinear in the parameters, the cost function (13) is nonconvex with respect to $\hat{\theta}$ and can eventually present some local minima. Thus, the convergence of the estimated parameters to their true values cannot be guaranteed for every initial condition.

Remark 2: In case of linear systems, persistence of excitation is an important factor when dealing with parameter identification. This condition guarantees the convergence of the estimated parameters to the true parameters. Persistence of excitation can be achieved by choosing sufficiently rich input signals. However, in our case, the system is nonlinear in the parameters. To the best of our knowledge, no convergence result is available, a part from a class of bilinear parametrizations [10] and PWA systems with known partition [13]–[15]). Convergence can only be demonstrated via simulations.

V. Numerical examples

Consider the continuous bimodal piecewise linear system

$$\dot{x} = \begin{cases} -2x + u \,, x \leqslant 0 \\ \;\;3x + u \,, x \geqslant 0. \end{cases} \tag{22}$$

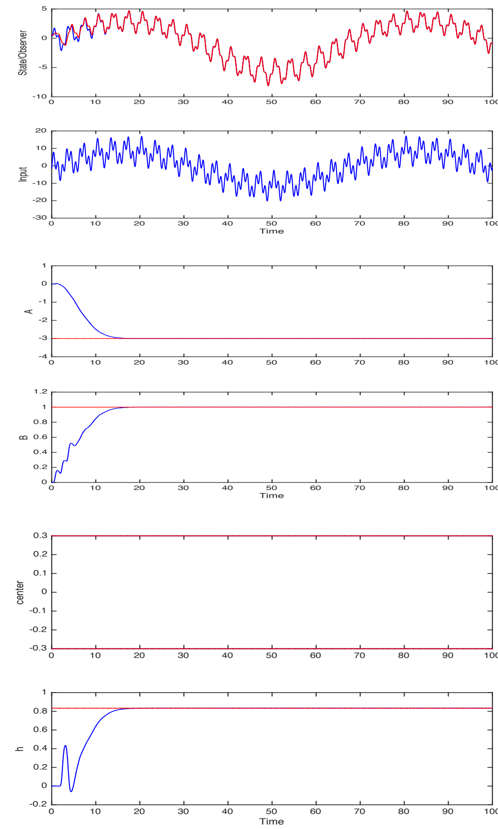

Fig. 1: Online identification of A, B and h when c_1 and c_2 are known (The state of true system and the true parameters are shown in red)

Let c_1, c_2 be a pair of centers such that $||c_1|| = ||c_2||$ and that generates the hyperplane $x + 0 \cdot u = 0$, i.e.

$$\{x \mid x \leqslant 0\} = \{x \mid 2(c_2 - c_1) \begin{bmatrix} x \\ u \end{bmatrix} \leqslant 0\},$$

$$\{x \mid x \geqslant 0\} = \{x \mid 2(c_2 - c_1) \begin{bmatrix} x \\ u \end{bmatrix} \geqslant 0\}.$$

For such c_1, c_2, one can write the system (22) in the form

$$\dot{x} = Ax + Bu - h \max\{2(c_1 - c_2)x, 0\} \tag{23}$$

where $A = -3$, $B = 1$ and h is uniquely determined from $\begin{bmatrix} -3 & 1 \end{bmatrix} = 2h(c_2 - c_1)^T$.

a) First, we suppose that the centers are known with $c_1 = \begin{bmatrix} -0.3 & 0.3 \end{bmatrix}^T$ and $c_2 = \begin{bmatrix} 0.3 & 0.3 \end{bmatrix}^T$. We need to identify A, B and h. Note that the real values of A, B and h are

$$A = -3, B = 1, h = 1/(1.2) = 0.8333.$$

Using the developed algorithm, we get the simulation result which is shown in Fig. 1. The parameter h is estimated correctly while the system is in region 2 and $\max\{\Lambda, 0\} = 0$. This happens because of the retrospective cost (13) (that exploits the past estimation errors) and the fact that the system was in region 1 approximately on the time interval $[2, 3]$.

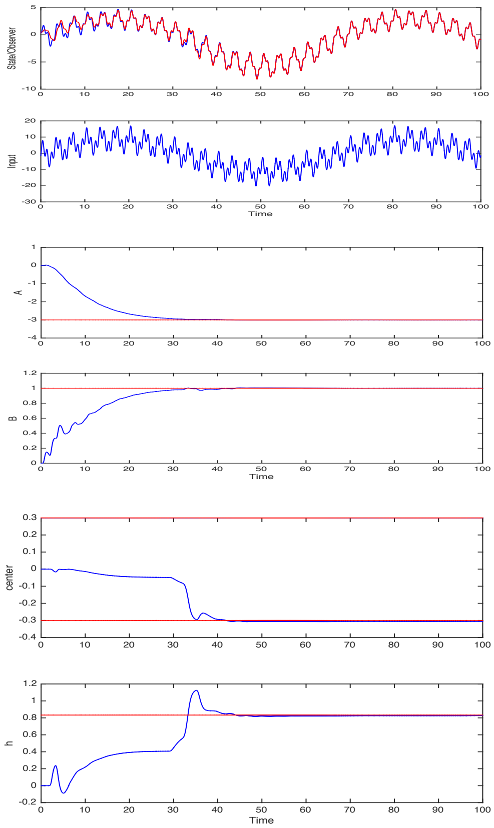

Fig. 2: Online identification of A, B, h and c_1 when only c_2 is known (The state of true system and the true parameters are shown in red)

b) Now, let us assume that only $c_2 = \begin{bmatrix} 0.3 & 0.3 \end{bmatrix}^T$ is known. Initializing in a small neighborhood of real values A, B, h and c_1 with $\hat{A}(0) = 0, \hat{B}(0) = 0, \hat{h}(0) = 0, \hat{c}_1(0) = \begin{bmatrix} 0.0 & 0.3 \end{bmatrix}$ and again using our algorithm developed for bimodal systems, we get the simulation result shown in Fig. 2. Due to the increase number of parameters and nonlinearities, we observe that the convergence to the true values is slower.

VI. Conclusions and Future Works

This paper provided a method for online identifying continuous-time PWA systems in state space form with both the partition and subsystems are unknown. The system partition is assumed to be generated by the centers. In this work, we restrict our consideration to the case where the number of centers is at most three. In the future, we will generalize the results to general piecewise affine systems and study its applications to adaptive optimal control problems of large-scale systems studied in [2]–[4].

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