A new approach to determine the concentration distribution in slurry transport

Miedema, Sape

Publication date
2017

Document Version
Final published version

Published in
Proceedings Dredging Summit and Expo 2017

Citation (APA)

Important note
To cite this publication, please use the final published version (if applicable).
Please check the document version above.
A NEW APPROACH TO DETERMINE THE CONCENTRATION DISTRIBUTION IN SLURRY TRANSPORT

Sape A. Miedema

ABSTRACT

Most concentration distribution models are based on the advection diffusion equation for two-dimensional (2D) channel flow. These models may or may not incorporate hindered settling based on the cross-sectional averaged spatial volumetric concentration. These models also use a diffusivity based on the turbulent momentum diffusivity, sometimes corrected for the particle diameter (Kaushal & Tomita (2000)). The result is a 2D concentration profile. To make this applicable for pipe flow, the integrated concentration profile must match the cross-sectional averaged spatial volumetric concentration, which requires a correction. Still the result is based on a constant hindered settling velocity, which is not correct considering the concentration over the vertical coordinate is not a constant. An iterative method has been developed to correct the concentration profile for the local hindered settling velocity. About 4 iterations are necessary to find the resulting concentration profile.

The paper explains the new method and gives a comparison with experimental data.

Keywords: Dredging, Newtonian settling slurry transport, concentration profile.

INTRODUCTION

The Advection Diffusion Equation.

In literature, the concentration distribution is often determined with the solution of the advection diffusion differential equation for 2D channel flow. The advection diffusion equation is derived and solved for low concentrations and for high concentrations including the effect of upwards liquid flow. Wasp et al. (1977) and Doron et al. (1987) use the solution for low concentrations, while Karabelas (1977) and Kaushal & Tomita (2002B) use the Hunt (1954) approach with upwards liquid flow. Hindered settling is not yet included in the basic solutions, but added by replacing the terminal settling velocity with the hindered terminal settling velocity. For the diffusivity and the relation between the sediment diffusivity and the turbulent eddy momentum diffusivity different approaches are possible.

If we assume the diffusivity is a constant, the advection diffusion equation can be solved. Giving the differential equation in the equilibrium situation with hindered settling, but without the upwards liquid velocity, since this is assumed to be part of the hindered settling (with \( r \) the vertical distance from the bottom of the pipe):

\[
C_{vs}(r) \cdot v_{th} + \varepsilon_s \cdot \frac{dC_{vs}(r)}{dr} = C_{vs}(r) \cdot v_{th} + \beta_{sm} \cdot \varepsilon_m \cdot \frac{dC_{vs}(r)}{dr} = 0
\]

The coordinate \( r \) ranges from 0 to \( D_p \), the pipe diameter. Now the variables must be separated according to:

\[
\frac{dC_{vs}(r)}{C_{vs}(r)} = -\frac{v_{th}}{\beta_{sm} \cdot \varepsilon_m} \cdot dr \Rightarrow \ln(C_{vs}(r)) = -\frac{v_{th}}{\beta_{sm} \cdot \varepsilon_m} \cdot r + C
\]

With \( C_{vs}(0) = C_{vB} \), the bottom concentration, the integration constant can be determined, giving:

\[
C_{vs}(r) = C_{vB} \cdot e^{-\frac{v_{th}}{\beta_{sm} \cdot \varepsilon_m} \cdot r}
\]

1 Associate Professor, Delft University of Technology, Mekelweg 2, 2628 CD, Delft, The Netherlands, T: ++31-15-2788359, Email: s.a.miedema@tudelft.nl.
This basic solution is still equal to the solution for open channel flow. Although this is just an indicative equation for open channel flow, Doron et al. (1987) and Doron & Barnea (1993) used it in their 2 and 3 layer models. The difference between pipe flow and open channel flow is in the determination of the diffusivity. Assuming the Law of the Wall, one can also determine the average diffusivity by integration (Lane & Kalinske (1941)):

$$\varepsilon_s = \beta_{sm} \cdot \kappa \cdot u_s \cdot \left( \frac{R-r}{R} \right) = \beta_{sm} \cdot \kappa \cdot u_s \cdot \left( 1 - \frac{r}{R} \right)$$  \hspace{1cm} (4)

Integration over the cross-section of the pipe gives:

$$\varepsilon_s = \frac{1}{\pi \cdot R^2} \int_0^{2\pi} \int_0^R \varepsilon_s \cdot r \cdot d\phi = \beta_{sm} \cdot \kappa \cdot u_s \cdot \frac{R^3}{\pi \cdot R^2} \int_0^{2\pi} \int_0^1 (1 - \bar{r}) \cdot d\bar{r} \cdot d\phi$$

$$\varepsilon_s = \beta_{sm} \cdot \kappa \cdot u_s \cdot \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (1 - \bar{r}) \cdot d\bar{r} \cdot d\phi = \beta_{sm} \cdot \kappa \cdot u_s \cdot \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (1 - \bar{r}) \cdot d\bar{r} \cdot d\phi$$

This gives for the concentration as a function of the vertical distance from the pipe bottom:

$$C_{vs}(r) = C_{vB} \cdot e^{-12 \cdot \frac{v_{th}}{\beta_{sm} \cdot \kappa \cdot u_s} \cdot \frac{r}{D_p}}$$  \hspace{1cm} (6)

Based on integration, assuming open channel flow and settling in the carrier liquid, the bottom concentration $C_{vB}$ can be found, at the LDV $C_{vB} = C_{vb}$:

$$C_{vB} = C_{vs} \cdot \left( \frac{12 \cdot v_{tv}}{\beta_{sm} \cdot \kappa \cdot u_s} \right) \cdot \frac{1 - e^{-12 \cdot \frac{v_{th}}{\beta_{sm} \cdot \kappa \cdot u_s}}}{1 - e^{-12 \cdot \frac{v_{tv}}{\beta_{sm} \cdot \kappa \cdot u_s}}}$$  \hspace{1cm} (7)

**THE DIFFUSIVITY BASED ON THE LIMIT DEPOSIT VELOCITY (LDV).**

To have a consistent model, the bottom concentration should be equal to the bed concentration at the LDV, since this is the definition of the LDV. Now most concentration profile equations are not related to the LDV, but make use of a modified diffusivity to match experiments. Here an attempt is made to make the concentration profile match the LDV. At the Limit Deposit Velocity, the bottom concentration $C_{vB}$ equals the bed concentration $C_{vb}$ giving:

$$C_{vb} = C_{vs} \cdot \left( \frac{12 \cdot v_{tv,ldv}}{\beta_{sm,ldv} \cdot \kappa \cdot u_{s,ldv}} \right) \cdot \frac{1 - e^{-12 \cdot \frac{v_{tv,ldv}}{\beta_{sm,ldv} \cdot \kappa \cdot u_{s,ldv}}}}{1 - e^{-12 \cdot \frac{v_{tv,ldv}}{\beta_{sm,ldv} \cdot \kappa \cdot u_{s,ldv}}}}$$  \hspace{1cm} (8)

Neglecting the denominator at low concentrations, since it’s close to unity (say the denominator equals a factor $a_{sm}$), the diffusivity can be derived.
\[
\beta_{\text{sm,ldv}} = 12 \frac{C_{\text{vs}}}{C_{\text{vb}}} \frac{v_{r,\text{ldv}}}{\alpha_{\text{sm}} \cdot \kappa \cdot u_{s,\text{ldv}}} = 12 \cdot C_{\text{vr}} \frac{v_{r,\text{ldv}}}{\alpha_{\text{sm}} \cdot \kappa \cdot u_{s,\text{ldv}}}
\]

This gives for the concentration distribution in the pipe:

\[
C_{\text{vs}}(r) = C_{\text{vB}} \cdot e^{-\frac{12}{12} \left( \frac{v_{r}}{v_{r,\text{ldv}}} \right) \frac{r}{D_{p}} - \frac{\alpha_{\text{sm}} u_{s,\text{ldv}}}{\alpha_{\text{sm}} \cdot \kappa \cdot u_{s,\text{ldv}}} \frac{v_{r}}{v_{r,\text{ldv}}} \frac{r}{D_{p}}}
\]

(9)

\[
C_{\text{vB}} = C_{\text{vs}} \cdot \frac{\left( \frac{\alpha_{\text{sm}}}{C_{\text{vs}}} \cdot \frac{u_{s,\text{ldv}}}{u_{s}} \cdot \frac{v_{r}}{v_{r,\text{ldv}}} \right) \left( 1 - e^{-\frac{\alpha_{\text{sm}}}{C_{\text{vs}}} \cdot \frac{u_{s,\text{ldv}}}{u_{s}} \cdot \frac{v_{r}}{v_{r,\text{ldv}}} \frac{r}{D_{p}}} \right)}{\left( \frac{\alpha_{\text{sm}}}{C_{\text{vs}}} \cdot \frac{u_{s,\text{ldv}}}{u_{s}} \cdot \frac{v_{r}}{v_{r,\text{ldv}}} \right) \left( 1 - e^{-\frac{\alpha_{\text{sm}}}{C_{\text{vs}}} \cdot \frac{u_{s,\text{ldv}}}{u_{s}} \cdot \frac{v_{r}}{v_{r,\text{ldv}}} \frac{r}{D_{p}}} \right)}
\]

(10)

The bottom concentration is now, using the fact that the denominator equals the factor \(\alpha_{\text{sm}}\):

\[
\alpha_{\text{sm}} = \left( 1 - e^{-\frac{\alpha_{\text{sm}}}{C_{\text{vs}}} \cdot \frac{u_{s,\text{ldv}}}{u_{s}} \cdot \frac{v_{r}}{v_{r,\text{ldv}}} \frac{r}{D_{p}}} \right) = 0.9847 + 0.304 \cdot C_{\text{vr}} - 1.196 \cdot C_{\text{vr}}^2 - 0.5564 \cdot C_{\text{vr}}^3 + 0.47 \cdot C_{\text{vr}}^4
\]

(12)

At low relative concentrations, \(C_{\text{vr}} < 0.3\), this factor is about 1. Based on the diffusivity derived, the portion of the solids in the carrier liquid according to the Wasp criterion can be determined by:

\[
\frac{C_{\text{vs,v}}}{C_{\text{vs}}} = e^{-0.92 - 0.5 \left( \frac{\alpha_{\text{sm}}}{C_{\text{vs}}} \cdot \frac{u_{s,\text{ldv}}}{u_{s}} \cdot \frac{v_{r}}{v_{r,\text{ldv}}} \frac{r}{D_{p}} \right)}
\]

(13)

**SIMPLIFICATION OF THE EQUATIONS.**

Now in reality, the concentration distribution depends on more than just the advection diffusion processes. At line speeds below the LDV there is a bed with possibly sheet flow and also collisions play a dominant role in the heterogeneous regime. Above the LDV the Darcy Weisbach friction factor is not changing much at higher line speeds. The power of the line speed is about -0.15 for larger pipe diameters. The power of the pipe diameter is about -0.175 (see Miedema (June 2016)).

\[
\lambda_1 = 0.01216 \cdot \left( v_{ls} \right)^{-0.15} \left( D_{p} \right)^{-0.175}
\]

(14)

Substituting this for the friction velocities gives:

\[
C_{\text{vs}}(r) = C_{\text{vB}} \cdot e^{-\frac{\alpha_{\text{sm}} \sqrt{\frac{v_{r,\text{ldv}}}{8} \cdot \frac{v_{s,\text{ldv}}}{v_{r,\text{ldv}}} \cdot \frac{r}{D_{p}}}}{\left( \frac{\alpha_{\text{sm}}}{C_{\text{vs}}} \cdot \frac{v_{r,\text{ldv}}}{v_{r,\text{ldv}}} \right) \left( 1 - e^{-\frac{\alpha_{\text{sm}}}{C_{\text{vs}}} \cdot \frac{u_{s,\text{ldv}}}{u_{s}} \cdot \frac{v_{r}}{v_{r,\text{ldv}}} \frac{r}{D_{p}}} \right)}}
\]

(15)

The bottom concentration is now for line speeds at the LDV:
\[ C_{v_{B}} = C_{v_{B}} \cdot \left( \frac{v_{ls,ldv}}{v_{ls}} \right)^{0.925} \cdot \frac{v_{tv}}{v_{tv,ldv}} \quad \text{or} \quad C_{v_{B}} = C_{v_{B}} \cdot \left( \frac{v_{ls,ldv}}{v_{ls}} \right)^{0.925} \cdot \frac{v_{thv}}{v_{thv,ldv}} \] (16)

For line speeds below the LDV there is a bed, so the concentration profile has to be adjusted for the presence of the bed. The portion of the solids in the carrier liquid according to the Wasp criterion can be determined by:

\[ \frac{C_{v_{s,v}}}{C_{v_{s}}} = e^{-\left(0.92-0.5\right)\frac{\alpha_{sm}}{C_{vr}} \left( \frac{v_{ls,ldv}}{v_{ls}} \right)^{0.925} \cdot \frac{v_{tv}}{v_{tv,ldv}}} \quad \text{or} \quad \frac{C_{v_{s,v}}}{C_{v_{s}}} = e^{-\left(0.92-0.5\right)\frac{\alpha_{sm}}{C_{vr}} \left( \frac{v_{ls,ldv}}{v_{ls}} \right)^{0.925} \cdot \frac{v_{thv}}{v_{thv,ldv}}} \] (17)

The above equations give the solution for both hindered settling and non-hindered settling. If the hindered settling is based on the total volumetric concentration, it has no influence, since it gives a constant reduction factor. However, if the hindered settling is based on the local volumetric concentration at a certain location in the pipe, it will influence the outcome.

**NUMERICAL IMPLEMENTATION.**

The concentration in the pipe can be described according to:

\[ C_{v_{s}}(r) = C_{v_{B}} \cdot e^{-\left(0.92-0.5\right)\frac{\alpha_{sm}}{C_{vr}} \left( \frac{u_{s,ldv}}{u_{s}} \right)^{0.925} \cdot \frac{v_{tv}}{v_{tv,ldv}}} \quad \text{or} \quad C_{v_{s}}(r) = C_{v_{B}} \cdot e^{-\left(0.92-0.5\right)\frac{\alpha_{sm}}{C_{vr}} \left( \frac{v_{ls,ldv}}{v_{ls}} \right)^{0.925} \cdot \frac{v_{thv}}{v_{thv,ldv}}} \] (18)

The bottom concentration is now for line speeds above the LDV:

\[ C_{v_{B}} = C_{v_{B}} \cdot \left( \frac{v_{ls,ldv}}{v_{ls}} \right)^{0.925} \cdot \frac{v_{tv}}{v_{tv,ldv}} \quad \text{or} \quad C_{v_{B}} = C_{v_{B}} \cdot \left( \frac{v_{ls,ldv}}{v_{ls}} \right)^{0.925} \cdot \frac{v_{thv}}{v_{thv,ldv}} \] (19)

Now assuming that the terminal (hindered) settling velocity in the suspension hardly depends on the line speed, these equations can be written as:

The concentration in the pipe can be described according to:

\[ C_{v_{s}}(r) = C_{v_{B}} \cdot e^{-\left(0.92-0.5\right)\frac{\alpha_{sm}}{C_{vr}} \left( \frac{u_{s,ldv}}{u_{s}} \right)^{0.925} \cdot \frac{v_{tv}}{v_{tv,ldv}}} \] (20)

The bottom concentration is now for line speeds above the LDV:

\[ C_{v_{B}} = C_{v_{B}} \cdot \left( \begin{array}{c} \alpha_{sm} \frac{u_{s,ldv}}{u_{s}} \frac{v_{tv}}{v_{tv,ldv}} \\ 1 - e \frac{\alpha_{sm}}{C_{vr}} \frac{u_{s,ldv}}{u_{s}} \frac{v_{tv}}{v_{tv,ldv}} \end{array} \right) \] (21)

These equations describe the concentration distribution well for 2D channel flow above the LDV. However, for a circular pipe and below the LDV some adjustments have to be made. When the concentration found is integrated over the circular cross section of the pipe, the cross-sectional averaged concentration has to be equal to the average concentration that is input to the calculations, which might be the case for a symmetrical concentration distribution, but certainly not for an asymmetrical concentration distribution.
The concentration in the pipe can be described according to:

\[ C_{ys}(r) = C_{vB} \cdot e^{-\frac{\alpha_{sm}}{C_{vr}} \left( \frac{v_{h,ldv}}{v_{ls}} \right)^{0.925} \frac{r}{D_p} \cdot \alpha_{c1}} \]  

(22)

Now in the case of a circular pipe the vertical coordinate \( r/D_p \) should be replaced by the fraction of the cross-section \( f \). This gives a much better match with the cross-sectional averaged concentration in case there is no bed, so above the LDV. This fraction can be determined by the angle \( \beta \) matching a certain vertical coordinate, like the angle \( \beta \) for the stationary and sliding bed.

\[ \beta = \arccos \left( \frac{0.5 - \frac{r}{D_p}}{0.5} \right) \]  

(23)

The fraction \( f \) is now:

\[ f = \frac{\beta - \sin(\beta) \cdot \cos(\beta)}{\pi} \]  

(24)

The concentration at \( r/D_p \) is now:

\[ C_{ys}(r) = C_{vB} \cdot e^{-\frac{\alpha_{sm}}{C_{vr}} \left( \frac{v_{h,ldv}}{v_{ls}} \right)^{0.925} \cdot f \cdot \alpha_{c1}} \]  

(25)

Introduction of a correction factor. This correction factor is chosen in such a way that at the LDV line speed the bottom concentration is exactly the bed concentration and the cross-sectional averaged concentration equals the input concentration. For line speeds, higher than the LDV the correction factors of the LDV are applied. For line speeds below the LDV, first a correction on the bottom concentration must be applied, since the bottom concentration can never be higher than the bed concentration. In other words, the relative concentration can never be higher than 1. So, the concentration is limited to the bed concentration and thus the relative concentration is limited to 1. At the LDV the correction factor almost equals unity because of the choice of using the fraction \( f \). If the concentration distribution would be symmetrical in the vertical direction the factor would be 1.

**HINDERED SETTLING NUMERICAL IMPLEMENTATION.**

The concentration profiles found match well except for the influence of local hindered settling, which will decrease the concentration at the top of the pipe and increase the concentration at the bottom of the pipe.

The concentration in the pipe without the local hindered settling effect can be described according to:

\[ C_{ys,0}(r) = C_{vB} \cdot e^{-\frac{\alpha_{sm}}{C_{vr}} \left( \frac{v_{h,ldv}}{v_{ls}} \right)^{0.925} \cdot f \cdot \alpha_{c1}} \]  

(26)

Hindered settling is not explicitly present in this equation. However, the LDV depends strongly on the concentration and on hindered settling, so implicitly hindered settling is present. The LDV increases with the concentration up to a concentration of 17.5%. For higher concentrations, the LDV decreases with increasing concentration. This of course influences the concentration distribution.
In the derivation of the concentration distribution it is assumed that the terminal settling velocity or the hindered terminal settling velocity is a constant, determined by the cross-sectional average concentration. In reality the concentration is not a constant, but decreases from top to bottom. So, the hindered terminal settling velocity at the top of the pipe will be higher than at the bottom, since at the top the concentration is lower. This could be implemented in the basic advection diffusion equation and solved numerically by iteration. However, one can also add a correction to adjust for this hindered settling effect. Such a correction should take the cross-sectional averaged concentration into account, since the effect of hindered settling is larger at higher concentrations. After trial and error the following method is found to take the effect of hindered settling into account:

\[ C_{\text{vs}}(r) \cdot \nu_{\text{th}}(r) + \beta_{\text{sm}} \cdot \epsilon_m \cdot \frac{dC_{\text{vs}}(r)}{dr} = 0 \quad \Rightarrow \quad \frac{dC_{\text{vs}}(r)}{dr} = - \frac{C_{\text{vs}}(r) \cdot \nu_{\text{th}}(r)}{\beta_{\text{sm}} \cdot \epsilon_m} = - \frac{C_{\text{vs}}(r) \cdot \nu_1 \cdot (1 - C_{\text{vs}}(r))^\beta}{\beta_{\text{sm}} \cdot \epsilon_m} \]  

(27)

This is the advection diffusion equation with location dependent hindered settling. To determine the location dependent hindered settling, the local concentration has to be known. In the zero step (index 0) the concentration profile is determined without local hindered settling based on equation (26). The location dependent concentration is already part of the solution and should not be taken into account in the first iteration (correction) step. Now at each level in the pipe the corrected concentration gradient can be determined according to:

\[
\left( \frac{dC_{\text{vs},1}(r)}{dr} \right) = \left( \frac{dC_{\text{vs},0}(r)}{dr} \right) \cdot \left( \frac{1 - C_{\text{vs},0}(r)}{1 - \alpha_{2,1} \cdot C_{\text{vs}}} \right)^{(4C_{\text{vs}} \beta)}
\]

(28)

The left-hand side is the corrected concentration gradient, the right term on the right-hand side gives the correction factor according to the Richardson & Zaki (1954) equation, but with a relative concentration dependent power. The power \( \beta \) is the Richardson & Zaki (1954) power with a value of 4.7 for very small particles and 2.4 for large particles. After determining the corrected concentration gradient at each level in the pipe, the new concentration profile is found by integrating the concentration gradient from bottom to top. It is assumed that the bottom concentration is unchanged. The factor \( \alpha_{2,1} \) is required to ensure that the cross-sectional averaged concentration is unchanged. This process can be repeated several times until there is no significant change in the concentration profile. Here 4 iteration steps are used. For the next iteration steps (starting at index 2) also the location dependent concentration ratio is also added, because it influences the concentration gradient, giving:

\[
\left( \frac{dC_{\text{vs},i}(r)}{dr} \right) = \left( \frac{dC_{\text{vs},i-1}(r)}{dr} \right) \cdot \left( \frac{C_{\text{vs},i-1}(r)}{C_{\text{vs},i-2}(r)} \right) \cdot \left( \frac{1 - C_{\text{vs},i-1}(r)}{1 - \alpha_{2,1} \cdot C_{\text{vs},i-2}(r)} \right)^{(4C_{\text{vs}} \beta)}
\]

(29)

For the next iteration steps (2, 3 and 4) the factor \( \alpha_{2,1} \) is very close to unity.

Although this method gives good results, there are still issues. One of the main issues is that the Richardson & Zaki (1954) hindered settling equation is based on the spatial volumetric concentration \( C_{\text{vs}} \) and not on the relative spatial volumetric concentration \( C_{\text{vs}} = C_{\text{vs}}/C_{\text{vb}} \).

\[
\frac{C_{\text{vs}}}{C_{\text{vb}}} = (1 - C_{\text{vs}})^\beta
\]

(30)

So even when the spatial volumetric concentration reaches a concentration where a bed with maximum porosity occurs, for sand at about \( C_{\text{vs}} = 50\% \), still a hindered settling velocity is determined, while in reality this hindered settling velocity will be zero. Normal sands will have a porosity of about 40%, so \( C_{\text{vs}} = 60\% \). A fixed bed may have a porosity of 40%, but a sliding bed will have a higher porosity in between 40% and 50%. The porosities mentioned here depend on the type of sand, but are mentioned to give a feeling of the order of magnitude. An equation that may work better is:
\[
\frac{v_{th}}{v_t} = e^{-\beta c_{vr}^{1.25}} \cdot \left(1 - c_{vr}^3\right)
\]  

(31)

For small concentrations, this equation gives the same result as the original equation, but for concentrations approaching the bed concentration, this equation approaches a zero-settling velocity. This would describe the bed behavior much better. So, for small concentrations this equation describes hindered settling, while for large relative concentrations approaching 1, the behavior is closer to consolidation behavior. The power \( \beta \) in this equation is equal to the original power \( \beta \).

To determine the location dependent hindered settling, the local concentration must be known. In the zero step (index 0) the concentration profile is determined without local hindered settling based on equation (31). The location dependent concentration is already part of the solution and should not be considered in the first iteration (correction) step. Now at each level in the pipe the corrected concentration gradient can be determined according to:

\[
\frac{dC_{vs,i}(r)}{dr} = \frac{dC_{vs,0}(r)}{dr} \cdot \frac{e^{-\beta c_{vr,0}^{1.25}} \cdot \left(1 - c_{vr,0}^3\right)}{e^{-\alpha_{i,0} \beta c_{vr}^{1.25}} \cdot \left(1 - c_{vr}^3\right)}
\]

(32)

The left-hand side is the corrected concentration gradient, the first term on the right-hand side the concentration gradient determined with equation (31). The second term on the right-hand side gives the correction factor according to the Richardson & Zaki (1954) equation, but with a relative concentration dependent power. The power \( \beta \) is the Richardson & Zaki (1954) power with a value of 4.7 for very small particles and 2.4 for large particles. After determining the corrected concentration gradient at each level in the pipe, the new concentration profile is found by integrating the concentration gradient from bottom to top. It is assumed that the bottom concentration is unchanged. The factor \( \alpha_{i,0} \) is required to ensure that the cross-sectional averaged concentration is unchanged. This process can be repeated a number of times until there is no significant change in the concentration profile. Here 4 iteration steps are used. For the next iteration steps (starting at index 2) also the location dependent relative concentration ratio is added, because it influences the concentration gradient, giving:

\[
\frac{dC_{vs,i}(r)}{dr} = \frac{dC_{vs,i-1}(r)}{dr} \cdot \frac{C_{vr,i-1}(r)}{C_{vr,i-2}(r)} \cdot \frac{e^{-\beta c_{vr,i-1}^{1.25}} \cdot \left(1 - c_{vr,i-1}^3\right)}{e^{-\beta c_{vr,i-2}^{1.25}} \cdot \left(1 - c_{vr,i-2}^3\right)}
\]

(33)

For the next iteration steps (2, 3 and 4) the factor \( \alpha_{i,2} \) is very close to unity. Using this new equation gives significant different concentration profiles. The concentration profile of a sliding bed with sheet flow is simulated well with this equation. The powers used in this equation (1.25 and 3) are a first attempt and may be changed slightly in the future.

Figure 1 shows a concentration profile for a line speed of 1 times the LDV and a relative concentration of 50% of the bed concentration. The red line shows the concentration profile as determined with equation (31). This results in the 4 iteration steps, where above the bottom the concentration is increased, while at the top the concentration has decreased. It is clear from this figure that 4 iteration steps give enough convergence.

At high concentrations, the concentration profile at the top of the pipe may not be accurate, especially at small line speeds. This is caused by the high concentrations at the bottom of the pipe. The standard graphs are determined with an average of \( \beta = 3 \) for medium sized particles. To use these graphs, one should first determine the LDV at different concentrations for a specific particle diameter. Based on the LDV, the line speed to LDV ratio can be determined. Once this ratio is known, the graph closest to the ratio should be chosen.

The graphs are constructed for spatial volumetric concentration. The factor 3 in the hindered settling power is a default value. Based on experiments this factor is between 2 and 4. The value of this factor is still subject of research.
EXAMPLES HINDERED SETTLING NUMERICAL IMPLEMENTATION.

Figure 2, Figure 3 and Figure 4 show the concentration distributions for different concentrations and relative velocities. The graphs are determined with a maximum bed concentration of 60%, which is reasonable for dense sand. For line speeds below the LDV sheet flow is observed by many researchers. Sheet flow is a layer of fast moving particles on top of a stationary or sliding bed. The particles closest to the bed move with the speed of the bed, while the particles at the top of the sheet flow layer move with a speed related to the speed of the liquid above the bed. The concentration is assumed to decrease almost linear, starting at the bed with the bed concentration, to almost zero at the top of the sheet flow layer. The start of the sheet flow layer is not at the maximum bed concentration, but at a concentration where the particles still form a solid bed, which would be at about 50% bed concentration, a relative concentration of about 0.83 or 83%. The graphs show an almost linear concentration profile from a relative concentration of 83% to a relative concentration of 10%, except for the highest overall relative concentrations of 90% and 95%. The steepness of the almost linear part of the concentration profiles increases with increasing overall (cross-sectional averaged) relative concentration and decreases with decreasing line speed.

The sheet flow behavior is simulated well with the new hindered settling equation and the method used. Above the LDV the bottom concentration is reduced depending on the line speed to LDV ratio. The concentration profiles become steeper with increasing line speed. In the range of the line speed to LDV ratio of 1-1.5 the concentration profiles change significantly. The sheet flow behavior completely disappears. Also, the stationary or sliding bed completely disappears. There is not much difference between the use of the original Richardson & Zaki (1954) equation and equation (31), since they give the same hindered settling velocity at lower concentrations.

Since the graphs are constructed for a hindered settling power of 3, one should only apply the graphs for medium sands. Fine sands will have a higher power, gravel a lower power. Richardson & Zaki (1954) derived their equation for spheres, so non-spherical particles will have a different power. Usually non-spherical particles have a higher power than spherical particles. So, although the graphs are constructed for medium sands, they do give a good impression of the development of the concentration profiles as a function of the line speed to LDV ratio.
Figure 2. The concentration profiles at $v_{ls} = 0.50 \cdot v_{ls,ldv}$ with hindered settling.

Figure 3. The concentration profiles at $v_{ls} = 1.00 \cdot v_{ls,ldv}$ with hindered settling.
Figure 4. The concentration profiles at $v_{ls} = 2.00 \cdot v_{k,ldv}$ with hindered settling.

Figure 5. The concentration profiles as a function of the line speed for $C_{vr} = 0.40$. 
EXPERIMENTS.

Figure 6, Figure 7, Figure 8 and Figure 9 are based on experiments carried out by Gillies (1993). The experiments were carried out at two line speeds for each particle diameter. The bed concentrations are estimated at 0.55-0.60. In general, the concentration profiles are similar for the different particle diameters, however the \(d=0.55\) mm particle shows a very low concentration at the top of the pipe, which cannot be predicted.

It is remarkable that the LDV approach, relating the concentration profile to the LDV instead of some diffusivity, gives such good results for the different particle diameters. The only difference between the particle diameters is the Richardson & Zaki power \(\beta\). The downside of this approach is, that each concentration has a different LDV, so a different line speed to LDV ratio has to be used for each concentration. Here constant ratios are used for convenience.

CONCLUSIONS & DISCUSSION.

The concentration profiles are based on an exponential power, containing the hindered terminal settling velocity. This hindered settling velocity is implemented as a constant over the vertical coordinate. Since in reality and also following from the above graphs, the concentration decreases with an increasing vertical coordinate, one may expect that the hindered settling velocity increases with the vertical coordinate. So, in the top part, the hindered settling velocity is higher than in the bottom part. This will result in slightly lower concentrations at the top and higher concentrations at the bottom. The graphs with local hindered settling velocity have a correction for this effect. Also, the repulsive effects near the pipe wall will influence the concentration profiles, which is not taken into account here.

It is also assumed that the concentration profile is 1 dimensional with constant concentration in the transverse direction of the pipe. This is also questionable, since the repulsive effect near the pipe wall is effective at the whole pipe circumference and the local velocities are also influenced by the pipe wall. Recent Computational Fluid Dynamics (CFD) simulations have shown both effects (Ofei & Ismail (August 2016)).
Figure 7. Gillies (1993) data, $d=0.38$ mm, $D_p=0.263$ m, $\nu_{LS}=4.1-4.8$ m/s, $\beta=3.21$, LDV=3.55-4.22 m/s.

Figure 8. Gillies (1993) data, $d=0.55$ mm, $D_p=0.263$ m, $\nu_{LS}=3.9-4.4$ m/s, $\beta=2.95$, LDV=3.89-4.27 m/s.
Figure 9. Gillies (1993) data, \(d=2.40 \text{ mm}, D_p=0.263 \text{ m}, v_{ls}=3.3-4.0 \text{ m/s}, \beta=2.47, \text{LDV}=3.59-3.61 \text{ m/s}.

Since the model assumes a constant concentration in the transverse direction and is not corrected for the repulsive effect at the pipe wall, experimental data will never match exactly with the model. Usually the experimental data only show the measured concentrations on the vertical axis.

Figure 6, Figure 7, Figure 8 and Figure 9 show the calculated curves, including the local hindered settling effect with data of Gillies (1993) in a \(D_p=0.263 \text{ m} \) pipe. The experiments were carried out with particles of \(d=0.29 \text{ mm}, d=0.38 \text{ mm}, d=0.55 \text{ mm} \) and \(d=2.4 \text{ mm} \). The line speeds varied from line speeds just below the LDV to line speeds just above the LDV. Since the LDV differs based on the particle diameter and the volumetric concentrations, these are given in the caption of each figure. In general, the correlation is good, especially in the center of the pipe. At the top and bottom of the pipe the model may deviate from the experimental data.

The current model gives a good estimate of the concentration distribution and is corrected for the cross-sectional averaged concentration and the local hindered settling velocity, which is not the case for most models from literature.

The choice of using the fraction of the cross-sectional area of the pipe instead of the vertical coordinate seems to be a good choice. Even with the constant hindered settling velocity the concentration profiles match the experimental data much better. Using the adjustment of local hindered settling gives an even better correlation with the experimental data. The choice of an LDV based diffusivity gives good results. Of course, this depends on the correctness of the determination of the LDV, but based on the comparison with experimental data, the model is close.

It should be mentioned that the concentration profiles are determined with 100 steps, so a step size of \(0.01 \cdot r/D_p\). It should also be mentioned that the LDV values used are an average based on the Delft Head Loss & Limit Deposit Velocity (DHLLDV) Framework, while the LDV is concentration dependent. The DHLLDV Framework is a bit conservative giving high values for the LDV.

To validate the model, two issues have to be addressed. The first issue is the value of the LDV. If the value of the LDV is not correct, the wrong line speed to LDV ratio will be used. The second issue is the shape of the concentration profile. Using the correct LDV, the shape of the concentration profile should match the experimental data. In general,
the shapes of the concentration profiles match well given the LDV values used. The LDV values used are a bit conservative, giving slightly to high values in order to have safe operations. Especially for the smallest particles of \( d=0.18 \text{ mm} \) in a \( D_p=0.0532 \text{ m} \) pipe, and LDV of 1.2 m/s is applied, while the theoretical LDV varies from 1.2 m/s to 1.8 m/s.

Figure 5 shows the concentration profiles as a function of the line speed. A new term is introduced, the Maximum Limit Deposit Velocity. The reason for this is, that the observed LDV is based on the line speed where the bed completely disappeared. This will occur for a bed concentration close to 50%. In reality however, bed concentrations up to 60% are possible at lower line speeds. The line speed where this maximum bed concentration (60%) occurs is named the MLDV and the line speed where the 50% bed concentration occurs is named the LDV (the observed LDV). The LDV always occurs at a line speed higher than the MLDV. For a relative cross-sectional averaged concentration of 0.4 (Figure 5) the LDV will occur at about 1.4 times the MLDV. The higher the relative cross-sectional averaged concentration, the larger this factor. The minimum value of this factor is about 1.25 for very low concentrations.

Relating the concentration profile to the LDV is possible, but should be corrected for the fact that there is not just one bed concentration. A bed will start with the lowest possible bed concentration of about 50% or even a few percent lower and increases with decreasing line speed up to about 60% or even a few percent higher, depending on the type of sand, the PSD and the angularity.

### NOMENCLATURE

<table>
<thead>
<tr>
<th><strong>vehicle</strong></th>
<th>Pseudo liquid formed by carrier liquid and suspended particles</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>Integration constant</td>
</tr>
<tr>
<td>( C_{vs}(r) )</td>
<td>Volumetric concentration as a function of vertical coordinate in pipe</td>
</tr>
<tr>
<td>( C_{vs,v} )</td>
<td>Volumetric concentration in vehicle</td>
</tr>
<tr>
<td>( C_{vb} )</td>
<td>Volumetric concentration bed</td>
</tr>
<tr>
<td>( C_{vB} )</td>
<td>Volumetric concentration bottom of channel or pipe</td>
</tr>
<tr>
<td>( C_{vR} )</td>
<td>Relative volumetric concentration</td>
</tr>
<tr>
<td>( d )</td>
<td>Particle diameter</td>
</tr>
<tr>
<td>( D_p )</td>
<td>Pipe diameter</td>
</tr>
<tr>
<td>( g )</td>
<td>Gravitational constant (9.81 m/s²)</td>
</tr>
<tr>
<td>( r )</td>
<td>Vertical coordinate in pipe</td>
</tr>
<tr>
<td>( \tilde{r} )</td>
<td>Dimensionless vertical coordinate in pipe</td>
</tr>
<tr>
<td>( R )</td>
<td>Radius of pipe</td>
</tr>
<tr>
<td>( R_{sd} )</td>
<td>Relative submerged density (about 1.65 for sand)</td>
</tr>
<tr>
<td>( u^* )</td>
<td>Friction velocity</td>
</tr>
<tr>
<td>( u^*,ldv )</td>
<td>Friction velocity at LDV</td>
</tr>
<tr>
<td>( v_{ls} )</td>
<td>Line speed</td>
</tr>
<tr>
<td>( v_{ls,ldv} )</td>
<td>Limit Deposit Velocity</td>
</tr>
<tr>
<td>( v_t )</td>
<td>Terminal settling velocity</td>
</tr>
<tr>
<td>( v_{tv} )</td>
<td>Terminal settling velocity in vehicle</td>
</tr>
<tr>
<td>( v_{tv,ldv} )</td>
<td>Terminal settling velocity in vehicle at LDV</td>
</tr>
<tr>
<td>( v_{th} )</td>
<td>Terminal settling velocity hindered</td>
</tr>
<tr>
<td>( v_{th,ldv} )</td>
<td>Terminal settling velocity hindered at LDV</td>
</tr>
<tr>
<td>( v_{thv} )</td>
<td>Terminal settling velocity hindered in vehicle</td>
</tr>
<tr>
<td>( v_{thv,ldv} )</td>
<td>Terminal settling velocity hindered in vehicle at LDV</td>
</tr>
<tr>
<td>( \alpha_{sm} )</td>
<td>Correction factor</td>
</tr>
<tr>
<td>( \alpha_{c1} )</td>
<td>Correction factor</td>
</tr>
<tr>
<td>( \alpha_{c2} )</td>
<td>Correction factor</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Richardson &amp; Zaki power hindered settling</td>
</tr>
<tr>
<td>( \beta_{sm} )</td>
<td>Relation sediment mass diffusivity to eddy momentum diffusivity</td>
</tr>
<tr>
<td>( \beta_{sm,ldv} )</td>
<td>Relation sediment mass diffusivity to eddy momentum diffusivity at LDV</td>
</tr>
<tr>
<td>( \varepsilon_m )</td>
<td>Eddy momentum diffusivity</td>
</tr>
</tbody>
</table>
ε_s  Sediment diffusivity  m/s

\bar{ε_s}  Average sediment diffusivity  m/s

λ_d  Darcy Weisbach friction factor  -

λ_{d,ldv}  Darcy Weisbach friction factor at LDV  -

κ  von Karman constant  -

φ  Pipe angle  °

REFERENCES


CITATION