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Competition between Cooperative Projects

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Abstract. A paper needs to be good enough to be published; a grant proposal needs to be sufficiently convincing compared to the other proposals, in order to get funded. Papers and proposals are examples of cooperative projects that compete with each other and require effort from the involved agents, while often these agents need to divide their efforts across several such projects. We aim to provide advice how an agent can act optimally and how the designer of such a competition (e.g., the program chairs) can create the conditions under which a socially optimal outcome can be obtained. We therefore extend a model for dividing effort across projects with two types of competition: *a quota* or *a success threshold*. In the quota competition type, only a given number of the best projects survive, while in the second competition type, only the projects that are better than a predefined success threshold survive. For these two types of games we prove conditions for equilibrium existence and efficiency. Additionally we find that competitions using a success threshold can more often have an efficient equilibrium than those using a quota. We also show that often a socially optimal Nash equilibrium exists, but there exist inefficient equilibria as well, requiring regulation.

1 Introduction

Cooperative projects often compete with each other. For example, a paper needs to have a certain quality, or to be among a certain number of the best papers to be published, and a grant needs to be one of the best to be awarded. Either the projects that achieve a certain *minimum level*, or those that are among a certain *quota* of the best projects attain their value. Agents endowed with a resource budget (such as time) need to divide this resource across several such projects. We consider so-called public projects where agents contribute resources to create something together. If such a project survives the competition, its rewards are typically divided among the contributors based on their individual investments.

Agents often divide effort across competing projects. In addition to co-authoring articles or books [6, 7, 10] and research proposals, examples include participating in crowdsensing projects [8] and online communities [9]. Examples of quotas for successful projects include investing effort in manufacturing several products, where the market becomes saturated with a certain number of products. Examples of success thresholds are investing in start-ups, where a

** Most of this work was done at Delft University of Technology.

minimum investment is needed to survive, or funding agencies contributing to social projects, where a minimum contribution is required to make the project succeed. Another example is students investing effort in study projects.

The ubiquity and the complexity of such competing projects calls for a decision-support system, helping agents to divide their efforts wisely. Assuming rationality of all the others, an agent needs to know how to behave given the behavior of the others, and the designer of the competition would like to know which rules lead to better results. In the terms of non-cooperative game theory, the objective of this work is to find the equilibria and their efficiency.

Analyzing the NE and their efficiency helps characterizing the influence of a quota or a success threshold on how efficient the stable strategies are for the society and thus increase the efficiency of investing time in the mentioned enterprises. For example, Batchelor [4] suggests increasing the publication standards. However, in addition to maximizing the total value of the published papers, he considers goals such as reducing the noise (number of low quality publications).

To make things clear, we employ this running example:

Example 1. Consider scientists investing time from their time budget in writing papers. A paper attains its value (representing the acknowledgment and all the related rewards) if it stands up to the competition with other papers. The competition can mean either being one of the q best papers, or achieving at least the minimum level of δ, depending on the circumstances. A scientist is rewarded by a paper by becoming its co-author if she has contributed enough to that paper.

Here, the submitters need to know how to split their efforts between the papers, and the conference chairs need to properly organize the selection process, e.g. by defining the quota or threshold on the papers to get accepted.

There were several studies of contributing to projects but the projects did not compete. For example, in the all-pay auction model, only one contributor benefits from the project, but everyone contributes. Its equilibria are analyzed in [5], etc. A famous example is the colonel Blotto game with two players [14], where these players spread their forces among the battlefields, winning a battle if allocating it more forces than the opponent does. The relative number of won battles determines the player's utility. Anshelevich and Hoefer [2] model two-player games by an undirected graph where nodes contribute to the edges. A project, being an edge, obtains contributions from two players. They study minimum-effort projects, proving the existence of an NE and showing that the price of anarchy (PoA)[3] is at most 2.

The effort-dividing model [13] used the model of a shared effort game [3], where each player has a budget to divide among a given set of projects. The game possesses a contribution threshold θ, and the project's value is equally shared among the players who invest above this threshold. They analyzed Nash

[3] The social welfare is the sum of the utilities of all the players. The price of anarchy [11, 12] is the ratio of the minimum social welfare in an NE to the maximum possible social welfare. The price of stability [15, 1] is the ratio of the maximum social welfare in an NE to the maximum possible social welfare.

equilibria (NE) and their price of anarchy (PoA) and stability (PoS) for such games. However, they ignored that projects may compete for survival. We fill this gap, extending their model by allowing the projects only to obtain their modeled value if they stand up to a competition. To conclude, we study the yet unanswered question of strategic behavior with multiple competing projects.

Compared to the contribution in [10], we model contributing to multiple projects by an agent, and concentrate on the competition, rather than on sharing a project's utility. Unlike devising division rules to make people contribute properly, studied in cooperative game theory (see *Shapley value* [16] for a prominent example), we model given division rules and analyze the obtained game, using non-cooperative game theory.

We formally define the following models:

1. Given a *quota* q, only q projects receive their value. This models the limit on the number of papers to be accepted to a conference, the number of politicians in a city council, the lobbyists being the agents and the politicians being the projects, or the number of projects an organization can fund.
2. There exists a *success threshold* δ, such that only the projects that have a value of at least δ actually receive their value. This models a paper or proposal acceptance process that is purely based on quality.

Our contributions are as follows: We analyze existence and efficiency of NE in these games. In particular, we demonstrate that introducing a quota or a success threshold can sometimes kill existing equilibria, but sometimes allow for new ones. We study how adjusting a quota or a success threshold influences the contribution efficiency, and thereby the social welfare of the participants. We derive that competitions using a success threshold have efficient equilibria more often than those with a quota. We also prove that characterizing the existence of an NE would require more parameters than just the quota or the threshold and the number of the agents and the projects.

We formalize our models in Section 2, analyze the Nash equilibria of the first model and their efficiency in Section 3, and analyze the second model in Section 4. Theorems 2, 3, 5 and 6 are inspired by the existence and efficiency results for the model without competition. Having analyzed both models of competition between projects, Section 5 compares their characteristics, the possibility to influence the authors' behavior through tuning the acceptance criteria, and draws further conclusions. Some proofs are deferred to the appendix (Section A).

2 Model

We build our model on that from [13], since that is a model of investment in common projects with a general threshold. We first present their model for *shared effort games*, which also appears in [3]. From Definition 1 on, we introduce competition among the projects.

There are n players $N = \{1, \ldots, n\}$ and a set Ω of m projects. Each player $i \in N$ can contribute to any of the projects in Ω_i, where $\emptyset \subsetneq \Omega_i \subseteq \Omega$;

the contribution of player i to project $\omega \in \Omega_i$ is denoted by $x^i_\omega \in \mathbb{R}_+$. Each player i has a budget $B_i > 0$, so that the strategy space of player i (i.e., the set of her possible actions) is defined as $\left\{x^i = (x^i_\omega)_{\omega\in\Omega_i} \in \mathbb{R}_+^{|\Omega_i|} | \sum_{\omega\in\Omega_i} x^i_\omega \leq B_i\right\}$. Denote the strategies of all the players except i by x^{-i}.

The next step to define a game is defining the utilities. Let us associate each project $\omega \in \Omega$ with its *project function*, which determines its *value*, based on the total contribution $x_\omega = (x^i_\omega)_{i\in N}$ that it receives; formally, $P_\omega(x_\omega) \colon \mathbb{R}^n_+ \to \mathbb{R}_+$. The assumption is that every P_ω is increasing in every parameter. The increasing part stems from the idea that receiving more effort does not make a project worse off. When we write a project function as a function of a single parameter, like $P_\omega(x) = \alpha x$, we assume that project functions P_ω depend only on the $\sum_{i\in N}(x^i_\omega)$, which is denoted by x_ω as well, when it is clear from the context. The project's value is distributed among the players in $N_\omega \triangleq \{i \in N | \omega \in \Omega_i\}$ according to the following rule. From each project $\omega \in \Omega_i$, each player i gets a share $\phi^i_\omega(x_\omega) \colon \mathbb{R}^n_+ \to \mathbb{R}_+$ with free disposal:

$$\forall \omega \in \Omega : \sum_{i\in N_\omega} \phi^i_\omega(x_\omega) \leq P_\omega(x_\omega). \tag{1}$$

We assume the sharing functions are non-decreasing. The non-decreasing assumption fits the intuition that contributing more does not get the players less.

Denote the vector of all the contributions by $x = (x^i_\omega)^{i\in N}_{\omega\in\Omega}$. The utility of a player $i \in N$ is defined to be

$$u^i(x) \triangleq \sum_{\omega\in\Omega_i} \phi^i_\omega(x_\omega).$$

Consider the numerous applications where a minimum contribution is required to share the revenue, such as paper co-authorship and homework. To analyze these applications, define a specific variant of a shared effort game, called a *θ-sharing mechanism*. This variant is relevant to many applications, including co-authoring papers and participating in crowdsensing projects. For any $\theta \in [0, 1]$, the players who get a share are defined to be $N^\theta_\omega \triangleq \left\{i \in N_\omega | x^i_\omega \geq \theta \cdot \max_{j\in N_\omega} x^j_\omega\right\}$, which are those who bid at least θ fraction of the maximum bid size to ω. Define the θ-equal sharing mechanism as equally dividing the project's value between all the users who contribute to the project at least θ of the maximum bid to the project.

The *θ-equal sharing mechanism*, denoted by M^θ_{eq}, is

$$\phi^i_\omega(x_\omega) \triangleq \begin{cases} \frac{P_\omega(x_\omega)}{|N^\theta_\omega|} & \text{if } i \in N^\theta_\omega, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider θ-equal sharing, where all the project functions are linear, i.e. $P_\omega(x_\omega) = \alpha_\omega(\sum_{i\in N} x^i_\omega)$. W.l.o.g., $\alpha_m \geq \alpha_{m-1} \geq \ldots \geq \alpha_1$. We denote the number of projects with the largest coefficient project functions by $k \in \mathbb{N}$,

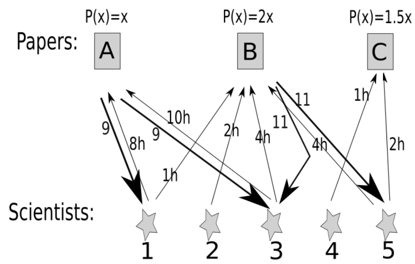

Fig. 1. Scientists contribute time to papers (arrows up), and share the value of the accepted ones (arrows down).

i.e. $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$. We call those projects *steep*. Assume w.l.o.g. that $B_n \geq \ldots \geq B_2 \geq B_1$.

A project that receives no contribution in a given profile is called a *vacant* project. A player is *dominated* at a project ω, if it belongs to the set $D_\omega \triangleq N_\omega \setminus N_\omega^\theta$. A player is *suppressed* at a project ω, if it belongs to the set $S_\omega \triangleq \left\{i \in N_\omega : x_\omega^i > 0\right\} \setminus N_\omega^\theta$. That is, a player who is contributing to a project but is dominated there.

We now depart from [13] and model competition in two different ways.

Definition 1 *In the* quota model*, given a natural number $q > 0$, only the q highest valued projects actually obtain a value to be divided between their contributors. The rest obtain zero. In the case of ties, all the projects that would have belonged to the highest q under some tie breaking rule receive their value; therefore, more than q projects can receive their value in this case. Formally, project ω is* in the quota *if $|\{\omega' \in \Omega | P_{\omega'}(x_{\omega'}) > P_\omega(x_\omega)\}| < q$, and ω is* out of the quota *otherwise, and, effectively, $P_\omega(x_\omega) = 0$.*

The second model is called the success threshold model.

Definition 2 *In the* success threshold model*, given a threshold δ, only the projects with value at least δ, meaning that $P_\omega(x_\omega) \geq \delta$, obtain a value, while if $P_\omega(x_\omega) < \delta$, then, effectively, $P_\omega(x_\omega) = 0$.*

Example 1 (Continued). Figure 1 depicts a success threshold model, where paper C does not make it to the success threshold, and is, therefore, unpublished. The other two papers are above the success threshold, and get published; such a paper's recognition is equally divided between the contributors who contribute at least θ of the maximum contribution to the paper, and become co-authors.

3 The Quota Model

In this section, we study the equilibria of shared effort games with a quota and their efficiency. We first give an example of an NE, and generalize it to a sufficiency theorem. Then, we provide equilibrium existence and efficiency theorems for the quota model. Finally, we show that no simple setting of parameters guarantees the existence of an equilibrium or the lack thereof.

Intuitively, introducing a quota can make previously unstable profiles become NE, by making deviations non-profitable. This would increase the price of stability but decrease the price of anarchy. On the other hand, a profile that is an NE without a quota can cease being so in our model, since some projects may obtain no value because of the quota.

Having a quota can lead to counter-intuitive results. In the following example, there can be an NE where no steep project obtains a contribution. The idea is that any deviation from the project where everyone contributes is non-profitable, because it would still leave the other projects out of quota.

Example 2. Given projects 1 and 2, such that $\alpha_2 > \alpha_1$, assume that all the players contribute all their budgets to project 1. If $\alpha_2 B_n < \alpha_1 \sum_{i=1}^{n-1} B_i$ and $q = 1$, then no player can deviate to project 2, as this would still leave that project out of the quota, and therefore, this profile is an NE.

In this NE, the social welfare is equal to $\alpha_1 \sum_{i \in N} B_i$. The optimal social welfare, achieved if and only if all the players contribute all their budgets to project 2, is equal to $\alpha_2 \sum_{i \in N} B_i$. The ratio between the social welfare in this NE and the optimal one is $\frac{\alpha_1}{\alpha_2}$. That ratio is an upper bound on the price of anarchy of this game. In addition, since the optimal profile is also an NE, the price of stability is 1.

The price of anarchy is smaller than $\frac{\alpha_1}{\alpha_2}$ if and only if some agents do not contribute all their budgets. This can only happen in an NE if θ is positive, and if this is the case, then we can have arbitrarily low price of anarchy, down to the case when only agent n contributes, if $\theta B_n > B_{n-1}$, and then, $\mathrm{PoA} = \frac{\alpha_1 B_n}{\alpha_2 \sum_{i \in N} B_i}$.

We now generalize these ideas to the following theorem about possible NE.

Theorem 1. *Consider a θ-equal sharing game with $n \geq 2$ players with budgets $B_n \geq \ldots \geq B_2 \geq B_1$ (the order is w.l.o.g.), $0 < \theta < 1$, linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g.), and quota q.*

This game has a pure strategy NE, if $q = 1$ and $B_n < \sum_{i=1}^{n-1} B_i$. Additionally, $\mathrm{PoA} = \frac{\alpha_l \sum_{i\colon B_i \geq \theta B_n} B_i}{\alpha_m \sum_{i \in N} B_i} \leq \frac{\alpha_l}{\alpha_m}$*, for* $l \triangleq \min\left\{ j \in \Omega : \alpha_m B_n < \alpha_j \sum_{i=1}^{n-1} B_i \right\}$, $\mathrm{PoS} = 1$.

Proof. If all the players contribute to any single project $j \geq l$, then since $B_n < \sum_{i=1}^{n-1} B_i$, no player can deviate to any project, because this would still leave this project out of the quota. Therefore, this profile is an NE.

In particular, when all the players invest all their budgets in project m, it is an NE, and thus, $\mathrm{PoS} = 1$.

To find the price of anarchy, notice that the worst equilibrium for the social welfare is when everyone contributes to the least profitable possible project, i.e. l, and only those who have a reason to do so contribute. Having an incentive means being not below the threshold amount, θB_n. This equilibrium yields $\alpha_l \sum_{i\colon B_i \geq \theta B_n} B_i$. ∎

This theorem, in accord with the intuition above, shows that reducing the quota can either facilitate an optimal NE, or a very inferior NE. Actually, every efficiency of the form $\frac{\alpha_i}{\alpha_m}$ is possible at equilibrium, which brings us to the question of equilibria appearing and disappearing, which we treat next.

Example 3. A game with NE can cease having equilibria after introducing a quota. For example, consider $\theta \in (0, 1)$, 2 players with budgets $B_1 = \theta B_2$ and 2 projects with the coefficients $\alpha, (1-\epsilon)\alpha$. This game has an NE if no quota exists, by Theorem 3 from [13]. However, introducing the quota of $q = 1$ implies there is no NE. Indeed, the only candidate profile for an equilibrium is both agents contributing everything to the same project or when both projects obtain the same value. In the former case, agent 2 would like to deviate, to avoid sharing, since the projects are close for small enough an ϵ. In the latter case, agent 2 contributes to both projects, since for small enough an ϵ, agent 1 alone would make the project be out of quota. Since there exists at least one project where agent 1 contributes less than θB_2, say project i, agent 2 would benefit from contributing to that project all its budget. This is because she would gain at least $(1-\epsilon)\alpha x_\omega^2$ while losing at most $\alpha(B_1 + x_\omega^2)/2$, which is smaller, for small enough ϵ and θ.

A game can also start having equilibria after introducing a quota. For instance, consider a game with two players, $B_2 = B_1$, $\alpha_m = 1.9\alpha_{m-1}$. Then Theorem 3 from [13] implies that no NE exists, but if we introduce the quota of 1, then both agents contributing to project m is an NE, since a deviator would be out of quota.

We now present an existence theorem. The theorem presents possible equilibria, providing advice on possible stable states. Afterwards, we study efficiency.

Theorem 2. *Consider an equal θ-sharing game with $n \geq 2$ players with budgets $B_n \geq \ldots \geq B_2 \geq B_1$ (the order is w.l.o.g.), $0 < \theta < 1$, linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g.), and quota q.*

This game has a pure NE if one of the following holds.[4]

1. *$B_1 \geq k\theta B_n$, $k \leq q$ and $\frac{1}{n}\alpha_{m-k+1} \geq \alpha_{m-k}$,*
2. *$B_1 \geq q\theta B_n$, $k \geq q$ and $B_n < \sum_{j=1}^{n-1} B_i/q$;*
3. *$B_{n-1} < \frac{\theta}{|\Omega|} B_n$ and all the project functions are equal, i.e. $\alpha_m = \alpha_1$.*

The proof provides a profile and shows that no deviation is profitable.

Proof. To prove part 1, distinguish between the case where $k \leq q$ and $k > q$. If $k \leq q$, then the profile where all the players allocate $1/k$th of their respective budgets to each of the steep projects is an NE for the same reasons that were given for the original model, since here, the quota's existence can only reduce the motivation to deviate.

[4] If α_{m-k} does not exist, consider the containing condition to be vacuously true.

As for the part 2, consider the profile where all the players allocate $1/q$th of their respective budgets to each of the q steep projects $m, m-1, \ldots, m-q+1$. This is an NE, since the only deviation that is possibly profitable, besides reallocating between the non vacant projects, is a player moving all of her contributions from some projects to one or more of the vacant projects. This cannot bring profit, because these previously vacant projects will be outside of the quota, since $B_n < \sum_{j=1}^{n-1} B_i/q$. As for reallocating between the non-vacant projects, this is not profitable, since $B_1 \geq q\theta B_n$ means that suppressing is impossible. Therefore, this is an NE.

We now prove part 3. Let every player divide her budget equally among all the projects. No player wants to deviate, for the following reasons. All the projects obtain equal value, and therefore are in the quota. Player n suppresses all the rest and obtains her maximum possible profit, $\alpha_m(\sum_{i \in N} B_i)$. The rest obtain no profit, since they are suppressed whatever they do. ■

We now prove an efficiency result, based on Theorem 2.

Theorem 3. *Consider an equal θ-sharing game with $n \geq 2$ players with budgets $B_n \geq \ldots \geq B_2 \geq B_1$, $0 < \theta < 1$ (the order is w.l.o.g.), linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g).[5], and quota q.*

1. *If at least one of the following holds.*
   (a) $B_1 \geq k\theta B_n$, $k \leq q$ and $\frac{1}{n}\alpha_{m-k+1} \geq \alpha_{m-k}$;
   (b) $B_1 \geq q\theta B_n$, $k \geq q$ and $B_n < \sum_{j=1}^{n-1} B_i/q$;
   Then, there exists a pure strategy NE and there holds: PoS = 1.
2. *Assume $B_{n-1} < \frac{\theta}{|\Omega|} B_n$ and all the project functions are equal, i.e. $\alpha_m = \alpha_1$. Then, there exists a pure strategy NE and the following holds:* $\mathrm{PoS} = 1, \mathrm{PoA} = \frac{B_n}{\sum_{i \in \{1,2,\ldots,n\}} B_i}$.

Proof. We first prove part 1a and 1b. According to the proof of parts 1 and 2 of Theorem 2, equally dividing all the budgets among $\min\{k, q\}$ steep projects is an NE. Therefore, PoS = 1.

For part 2, recall that in the proof of part 3 of Theorem 2, we show that everyone equally dividing the budgets between all the projects is an NE. This is optimal for the social welfare, and so PoS = 1. We turn to find the price of anarchy now. If player n acts as just mentioned, while the other players do not contribute anything, then this is an NE, since all the projects are equal and therefore, in the quota, and players $1, \ldots, n-1$ will be suppressed at any contribution. An NE cannot have a lower social welfare, since n gets at least $\alpha_m B_n$ in any NE, since this is obtainable alone. Therefore, the fraction between the two social welfare values, namely $\frac{\alpha_m B_n}{\alpha_m \sum_{i \in \{1,2,\ldots,n\}} B_i}$, is the PoA. ■

The condition "$k \geq q$ and $B_n < \sum_{j=1}^{n-1} B_i/q$" in Theorem 3 does not hold if the largest budget can be much larger than the rest, implying that we shall

[5] If α_{m-k} does not exist, consider the containing condition to be vacuously true.

ask whether our optimum NE is guaranteed by part 1a, which requires that the quota has to be at least k. When there are many equally glorious projects to contribute to, meaning that k is large, this constraint becomes non-trivial to implement. The condition "$k \leq q$ and $\frac{1}{n}\alpha_{m-k+1} \geq \alpha_{m-k}$" in Theorem 3 does not hold if the difference between the two largest projects is not big enough, and then the quota has to be at most k if one wants our optimum NE to follow from part 1b. This is non-trivial when we have few most glorious (steep) projects.

We do not know a full characterization of the existence of equilibria; we do know that it would require many parameters. We prove now that the quota with the number of agents and projects do not determine existence.

Proposition 1. *For any quota $q \geq 1$, any number of agents $n \geq 2$ and projects $m \geq 2$, there exists a game which possesses an NE, and a game which does not.*

The proof engineers games with the given parameters with and without NE.
Proof. A game that satisfies the conditions of Theorem 2 provides evidence for the existence.

To find a game without an NE, we first treat the case of $q = 1$. Let all the project coefficient be equal to one another and let

$$B_n > \sum_{i=1}^{n-1} B_i, \tag{2}$$

$$\text{and } B_n > \frac{\sum_{i=1}^{n} B_i}{|\{i \in N : B_i \geq \theta B_n\}|}. \tag{3}$$

Because of the equality of all the project coefficients and of (2), in an equilibrium, all the agents with budgets at least θB_n will be together with n. Then, (3) implies agent n will deviate, contradictory to having an equilibrium.

For quota $q \geq 2$, let $B_{n-1} < \frac{\theta}{m} B_n$. In any NE, agent n dominates all the rest in the sense that it invests (strictly) more than $B_{n-1}\theta$ in any project that is in the quota, because otherwise, the other agents could get a share at some projects, and assuming $\alpha_l(x + \frac{x}{\theta}) > \alpha_m(\frac{x}{\theta})$ for every project l, agent n would prefer to suppress that. However, if $\alpha_m > \alpha_{m-1}$, n would always prefer to move a bit more contribution to project m, contradictory to the assumption of an NE. ∎

4 The Success Threshold Model

In this section, we consider the NE of shared effort games with a success threshold. We allow success thresholds δ be at most the sum of all the budgets times α_m, to let at least one project to obtain its value, in at least one strategy profile. We begin with an example, which inspires a theorem, and then we study existence and efficiency with a given success threshold.

In a profile, we call a project that has a value of at least the threshold an *accepted* project, and we call it *unaccepted* otherwise. In Example 1, the accepted papers are A and B.

Success threshold can cause counter-intuitive results, as follows.

Example 4. Given the projects 1 and 2, such that $\alpha_2 > \alpha_1$, assume that all the players contribute all their budgets to project 1. If $\delta > \alpha_2 B_n$, then no player can deviate to project 2, as this would leave that project unaccepted, and therefore, this profile is an NE.

The conclusions about the prices of anarchy and stability are the same as in Example 2, besides that the price of anarchy can be even zero if $\alpha_1 \sum_{i=1}^{n} B_i < \delta$.

The exemplified ideas yield the following theorem.

Theorem 4. *Consider an equal θ-sharing game with $n \geq 2$ players with budgets $B_n \geq \ldots \geq B_2 \geq B_1$ (the order is w.l.o.g.), $0 < \theta < 1$, linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g.), and success threshold δ.*

This game has a pure NE, if $\alpha_m B_n < \delta$. In addition, $\mathrm{PoA} \leq \frac{\alpha_1}{\alpha_m}$ and $\mathrm{PoS} = 1$. If $\alpha_1 \sum_{i=1}^{n} B_i < \delta$, then $\mathrm{PoA} = 0$.

Proof. If all the players contribute to any single project, then since $\alpha_m B_n < \delta$, no player can deviate to any project, because this would still leave that project unaccepted. Therefore, this profile is an NE.

In particular, when all the players invest all their budgets in project m, it is an NE, and thus, $\mathrm{PoS} = 1$. When all the players invest in 1, it also is an NE, showing that $\mathrm{PoA} \leq \frac{\alpha_1}{\alpha_m}$, and if $\alpha_1 \sum_{i=1}^{n} B_i < \delta$, then $\mathrm{PoA} = 0$. ∎

This theorem, in accord with the intuition above, shows that increasing the success threshold can either facilitate an optimal NE, or an inferior NE. Actually, every efficiency of the form $\frac{\alpha_j}{\alpha_m}$, for $j \geq \min \{i : \alpha_i \sum_{l=1}^{n} B_l \geq \delta\}$, is possible at an equilibrium.

Example 5 (Introducing a success threshold can kill or create new NE). The game with $\theta \in (0, 0.5)$, 2 players with budgets $B_1 = 2\theta B_2$ and 2 projects with the coefficients α, α has an NE if no success threshold exists, by Theorem 3 from [13]. If we introduce the success threshold of αB_2, then in any NE both agents have to contribute to the same project. Then, agent 2 will deviate. For an emerging NE, consider a game with two players, $B_2 = B_1$, $\alpha_m = 1.9\alpha_{m-1}$. Then Theorem 3 from [13] implies that no NE exists, but if we introduce the success threshold of $2B_1\alpha_m$, then both agents contributing to project m constitute an NE, since a deviator would be at a project below the success threshold.

Next, we provide sufficient conditions for the existence of an NE.

Theorem 5. *Consider an equal θ-sharing game with $n \geq 2$ players with budgets $B_n \geq \ldots \geq B_2 \geq B_1$ (the order is w.l.o.g.), $0 < \theta < 1$, linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g.), and success threshold δ.*

This game has a pure NE, if one of the following holds.[6] *Define* $p \triangleq \left\lfloor \frac{\alpha_m \sum_{i \in N} B_i}{\delta} \right\rfloor$*; intuitively, it is the number of the projects that can be accepted.*

1. $B_1 \geq k\theta B_n$, $k \leq p$ *and* $\frac{1}{n}\alpha_{m-k+1} \geq \alpha_{m-k}$,
2. $B_1 \geq p\theta B_n$, $k \geq p \geq 1$ *and* $\alpha_m B_n < \delta$*;*
3. $B_{n-1} < \frac{\theta}{|\Omega|} B_n$, *all the project functions are equal, i.e.* $\alpha_m = \alpha_1$.

Proof. We first prove part 1. The profile where all the players allocate $1/k$th of their respective budgets to each of the steep projects is an NE for the same reasons that were given for the original model, since here, the requirement to be not less than the success threshold can only reduce the motivation to deviate.

In part 2, consider the profile where all the players allocate $1/p$th of their respective budgets to each of the p steep projects $m, m-1, \ldots, m-p+1$. This is an NE, since the only deviation that is possibly profitable, besides moving budgets between the non vacant projects, is a player moving all of her contributions from some projects to one or more of the vacant projects. This cannot bring profit, because these previously vacant projects will be unaccepted, since $\alpha_m B_n < \delta$. Additionally, any reallocating between the non-vacant projects is not profitable, since $B_1 \geq p\theta B_2$ means that suppressing is impossible. Therefore, the current profile is an NE.

We now prove part 3. We distinguish between the case where the condition $p \geq |\Omega|$ holds or not. If $p \geq |\Omega|$, then the proof continues as in the case of part 3 of Theorem 2, where every player divides her budget equally among all the projects. All the projects are accepted, so no new deviations become profitable.

In the case that $p < |\Omega|$, consider the profile where all the players allocate $1/p$th of their respective budgets to each of the p projects $m, m-1, \ldots, m-p+1$. This is an NE, since the only deviation that is possibly profitable is some player $j < n$ moving all her budget to a vacant project. However, this is not profitable, since the project would be unaccepted, because $B_j \leq B_{n-1} < \frac{\theta}{|\Omega|} B_n < \theta\delta/\alpha_m \leq \delta/\alpha_m$. The penultimate inequality stems from $p < |\Omega| \iff \frac{\alpha_m \sum_{i \in N} B_i}{|\Omega|} < \delta$. Therefore, this is an NE. ∎

We now provide an efficiency result, proven in the appendix.

Theorem 6. *Consider an equal* θ*-sharing game with* $n \geq 2$ *players with budgets* $B_n \geq \ldots \geq B_2 \geq B_1$, $0 < \theta < 1$ *(the order is w.l.o.g.), linear project functions with coefficients* $\alpha_m = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ *(the order is w.l.o.g).*[7]*, and success threshold* δ*. Define* $p \triangleq \left\lfloor \frac{\alpha_m \sum_{i \in N} B_i}{\delta} \right\rfloor$*, as in Theorem 5.*

1. *If at least one of the following holds.*
   (a) $B_1 \geq k\theta B_n$, $k \leq p$ *and* $\frac{1}{n}\alpha_{m-k+1} \geq \alpha_{m-k}$,
   (b) $B_1 \geq p\theta B_n$, $k \geq p \geq 1$ *and* $\alpha_m B_n < \delta$*;*
   Then, there exists a pure NE and there holds: $\mathrm{PoS} = 1$.

[6] If α_{m-k} does not exist, consider the containing condition to be vacuously true.
[7] If α_{m-k} does not exist, consider the containing condition to be vacuously true.

2. *Assume* $B_{n-1} < \frac{\theta}{|\Omega|} B_n$*, all the project functions are equal, i.e.* $\alpha_m = \alpha_1$*. Then, there exists a pure NE and* PoS $= 1$*. If, an addition,* $\alpha_m B_n \geq \delta$*, then* PoA $= \frac{B_n}{\sum_{i \in \{1,2,\ldots,n\}} B_i}$.

Condition 1a of Theorem 6 implies that if the second best project is close to a best one, then the threshold should be big enough, for condition 1b to guarantee our optimum NE. The contrapositive of the condition 1b implies that if the biggest player is able to make a project succeed on her own, then the threshold should be small enough so that p is at least the number of the most profitable projects, for our optimum NE to be guaranteed by condition 1a.

There exists no simple characterization for the NE existence when $\delta \leq \alpha_m B_n$.

Proposition 2. *For any success threshold* $\delta \in [0, \alpha_m B_n]$ *and any number of agents* $n \geq 2$ *and projects* $m \geq 2$*, there exists a game which possesses an NE, and a game which does not.*

The proof appears in Section A.

5 Conclusions and Further Research

We analyze the stable investments in projects, where a project has to comply to certain requirements to obtain its value. This models paper co-authorship, investment in firms, etc. The goal is to advise which investments are individually and socially preferable. Each agent freely divides her budget of time or effort between the projects. A project that succeeds in the competition obtains a value, which is divided between the contributors who have contributed at least a given fraction of the maximum contribution to the project. We model succeeding in a competition either by a quota of projects that actually obtain their value, or by a success threshold on the value of projects that do.

For purposes like organizing a conference, we ask which quota or success threshold would make the behavior of the players better for the social welfare. Theorem 1 implies that if no player has a budget as large as the total budget of all the other players times the ratio between the least and the most efficient project coefficient, then the quota of 1 makes many equilibria, including an optimal one, possible. Theorem 4 promises the same by choosing a success threshold that disables any player to make a project successful on her own. The first problem of this approach is that it also allows very inefficient profiles constitute equilibria, asking for some coordination. The second problem is that the discussed equilibria have all the players investing in the same project, which is understandable because of the linear project functions but practically unreasonable in conferences, though possible in other applications, such as sponsorship of large projects like Uber, Lyft, Facebook and VKontakte.

Comparing these models, we see from Theorems 1 and 4 that the success threshold allows ensuring that there exists a socially optimal equilibrium while the quota requires also assuming that the largest effort budget is less than the sum of the other ones times the ratio of the least to the most profitable project coefficients. In addition, comparing Theorems 3 and 6 shows that provided the smallest budget is at least a certain fraction of the largest one, choosing large enough a threshold or small enough a quota guarantees that an optimal profile will be an equilibrium. Unlike in the described cases, where success threshold seems stronger than quota, we notice that the second part of Theorem 6 actually contains an additional condition, relatively to the second part of Theorem 3, but since the second parts of these theorems refer to the case of a single agent being able to dominate everyone everywhere and all the projects being equally rewarding, this is less practical. To conclude the comparison, sometimes, choosing success threshold has more power, since choosing quota needs to assume an additional relation between the budgets, in order to guarantee that socially optimal equilibria exist. Intuitively, this stems from a quota needing an assumption on what the players are able to do to increase their utility, given the quota, while providing a success threshold can be done already with the budgets in mind.

Both a quota and a success threshold have a concentrating effect: equilibria where the agents contribute to less projects than without any of these conditions.

Many directions to expand the research exist. First, some common projects like papers and books have an upper bound on the maximal number of participants. Also a person has an upper bound on the maximal number of projects she can contribute to. The model should account for these bounds. Second, competition can be of many sorts. For instance, a project may need to have a winning coalition of contributors, in the sense of cooperative games. The fate of the projects that fail the competition can also vary; for example, their value can be distributed between the winning projects. We have extended the sufficiency results for existence from [13], and proven the necessity to be harder for analytical analysis. Simulations or other analytical approaches may be tried to delineate the set of Nash equilibria more clearly. Naturally, project functions do not have to be linear, so there is a clear need to model various non-linear functions. Such a more general model will make the conclusions on scientific investments, paper co-authorship, and the many other application domains more precise, and enable us to further improve the advice to participants as well as organizers. We can look at submitting a paper to a highly-ranked conference and reducing the conference level till the paper gets accepted as on a series of shared effort games with various quotas, success thresholds and participants. If we model the cost of each submission, then the question is to which conference to submit first.

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A Omitted Proofs

We now prove Theorem 6.

Theorem 6. *Consider an equal θ-sharing game with $n \geq 2$ players with budgets $B_n \geq \ldots \geq B_2 \geq B_1$, $0 < \theta < 1$ (the order is w.l.o.g.), linear project functions with coefficients $\alpha_m = \alpha_{m-1} = \ldots = \alpha_{m-k+1} > \alpha_{m-k} \geq \alpha_{m-k-1} \geq \ldots \geq \alpha_1$ (the order is w.l.o.g).[8], and success threshold δ.*

Define $p \stackrel{\Delta}{=} \left\lfloor \frac{\alpha_m \sum_{i \in N} B_i}{\delta} \right\rfloor$, as in Theorem 5.

1. *If at least one of the following holds.*

[8] If α_{m-k} does not exist, consider the containing condition to be vacuously true.

(a) $B_1 \geq k\theta B_n$, $k \leq p$ *and* $\frac{1}{n}\alpha_{m-k+1} \geq \alpha_{m-k}$*;*
(b) $B_1 \geq p\theta B_n$, $k \geq p \geq 1$ *and* $\alpha_m B_n < \delta$*;*
Then, there exists a pure NE and there holds: $\mathrm{PoS} = 1$.

2. *Assume* $B_{n-1} < \frac{\theta}{|\Omega|} B_n$*, all the project functions are equal, i.e.* $\alpha_m = \alpha_1$*. Then, there exists a pure NE and* $\mathrm{PoS} = 1$*. If, an addition,* $\alpha_m B_n \geq \delta$*, then* $\mathrm{PoA} = \frac{B_n}{\sum_{i \in \{1,2,\ldots,n\}} B_i}$.

Proof. We first prove parts 1a and 1b. According to proof of parts 1 and 2 in Theorem 5, equally dividing all the budgets among $\min\{k, p\}$ steep projects is an NE. Therefore, $\mathrm{PoS} = 1$.

Part 2 is proven as follows. Since all the players dividing their budgets equally between any $\min\{p, m\}$ projects constitutes an NE, we have $\mathrm{PoS} = 1$.

To treat the PoA, we define the number of projects player n can make accepted on her own, $r \stackrel{\Delta}{=} \lfloor \alpha_m \frac{B_n}{\delta} \rfloor$, and distinguish between the case where $m \leq r$ and $m > r$. If $m \leq r$, consider the profile where player n divides her budget equally between all the projects, while the other players contribute nothing at all. This is an NE, because all the projects are accepted, player n cannot increase her profit and any other player will be suppressed, if she contributes anything anywhere. On the other hand, if $m > r$, consider the profile where player n divides her budget equally between $m, m-1, \ldots, m-r+1$, while the other players contribute nothing at all. The only possible deviation is player $j < n$ contributing to a vacant project. However, we have $B_j \leq B_{n-1} < \frac{\theta}{|\Omega|} B_n < \theta\delta/\alpha_m \leq \delta/\alpha_m$. This means that the project would be unaccepted. Therefore, this is an NE.

Therefore, $\mathrm{PoA} \leq \frac{\alpha_m B_n}{\alpha_m(\sum_{i \in N} B_i)}$. Since $\alpha_m B_n \geq \delta$, in any NE, player n receives at least $\alpha_m B_n$, and therefore, $\mathrm{PoA} = \frac{B_n}{\sum_{i \in \{1,2,\ldots,n\}} B_i}$. ∎

We finally prove Proposition 2.

Proposition 2. *For any success threshold* $\delta \in [0, \alpha_m B_n]$ *and any number of agents* $n \geq 2$ *and projects* $m \geq 2$*, there exists a game which possesses an NE, and a game which does not.*

Proof. For $\delta = 0$, which means for no threshold, the theorem follows from Theorem 3 from [13]. Therefore, we assume henceforth a positive success threshold.

A game that satisfies the conditions of Theorem 5 provides an example of the existence. Notice that the p they define is positive, since $\delta \leq \alpha_m B_N$.

To find a game that does not possess an equilibrium, let all the project coefficient be equal to one another and let

$$B_n > \sum_{i=1}^{n-1} B_i, \tag{4}$$

$$B_1 = \ldots = B_{n-1} = \theta B_n \text{ and } \delta = \alpha B_n. \tag{5}$$

Because of the equality of all the project coefficients, of (4) and of the choice of the success threshold, in an equilibrium, all the agents with budgets at least θB_n (which are $1, \ldots, B_{n-1}$ here) will be together with n on the same single project. Then, agent n will deviate, contradictory to being in an equilibrium. ∎