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SUBSTITUTION INVARIANT STURMIAN WORDS AND BINARY TREES

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Abstract

We take a global view at substitution invariant Sturmian sequences. We show that homogeneous substitution invariant Sturmian sequences $s_{\alpha,\alpha}$ can be indexed by two binary trees, associated directly with Johannes Kepler's tree of harmonic fractions from 1619. We obtain similar results for the inhomogeneous sequences $s_{\alpha,1-\alpha}$ and $s_{\alpha,0}$.

1. Introduction

A Sturmian word w is an infinite word $w = w_0w_1w_2\ldots$, in which occur only $n+1$ subwords of length n for $n = 1, 2\ldots$. It is well known (see, e.g., [17]) that the Sturmian words w can be directly derived from rotations on the circle as

$$w_n = s_{\alpha,\rho}(n) = [(n+1)\alpha+\rho] - [n\alpha+\rho], \quad n = 0, 1, 2, \ldots. \tag{1}$$

or as

$$w_n = s'_{\alpha,\rho}(n) = \lceil(n+1)\alpha+\rho\rceil - \lceil n\alpha+\rho\rceil, \quad n = 0, 1, 2, \ldots. \tag{2}$$

Here $0 < \alpha < 1$ and ρ are real numbers, $[\cdot]$ is the floor function, and $\lceil\cdot\rceil$ is the ceiling function.

Sturmian words have been named after Jacques Charles François Sturm, who never studied them. A whole chapter is dedicated to them in Lothaire's book 'Algebraic combinatorics on words' ([17]). There is a huge literature, in particular on the *homogeneous* Sturmian words

$$c_\alpha := s_{\alpha,\alpha},$$

which have been studied since Johann III Bernoulli. The homogeneous Sturmian words are also known as *characteristic words*, see Chapter 9 in [2].

Interestingly, for certain α and ρ the Sturmian word w is a fixed point $\sigma(w) = w$ of a morphism[1] σ of the monoid of words over the alphabet $\{0, 1\}$. For example, for $\alpha = \rho = (3 - \sqrt{5})/2$ one obtains the Fibonacci word $c_\alpha = 0100101\ldots$, fixed point of the Fibonacci morphism φ given by $\varphi(0) = 01$, $\varphi(1) = 0$. Another example is the Pell word $c_\alpha = 0010010001\ldots$ obtained for $\alpha = \rho = (2 - \sqrt{2})/2$, with morphism given by $0 \to 001,\ 1 \to 0$.

It is well known for which α one obtains a substitution invariant word c_α. This was first obtained in [9], and an extensive treatment can be found in [17, Section 2.3.6]. See also [22]. The result is that $\alpha \in (0, \frac{1}{2})$ gives a fixed point if and only if there exists a natural number k such that α has continued fraction expansion

$$\alpha = [0; 1 + a_0, \overline{a_1 \ldots a_k}], \qquad a_k \geq a_0 \geq 1, \tag{3}$$

and $\alpha \in (\frac{1}{2}, 1)$ gives a fixed point if and only if there exists a natural number k such that α has continued fraction expansion

$$\alpha = [0; 1, a_0, \overline{a_1 \ldots a_k}], \qquad a_k \geq a_0. \tag{4}$$

The Fibonacci word is obtained for $k = 1, a_0 = a_1 = 1$, and the Pell word for $k = 1, a_0 = 2, a_1 = 3$.

Any α that gives a substitution invariant c_α is called a *Sturm number*. In terms of their continued fraction expansions these are characterized in equations (3) and (4). There is however a simple algebraic way to describe them, given in [1]:

an irrational number $\alpha \in (0, 1)$ is a Sturm number if and only if it is a quadratic irrational number whose algebraic conjugate $\overline{\alpha}$, defined by the equation $(x - \alpha)(x - \overline{\alpha}) = 0$, satisfies

$$\overline{\alpha} \notin [0, 1].$$

Let E be the 'exchange' morphism

$$E: \begin{cases} 0 \to 1 \\ 1 \to 0 \end{cases}.$$

Then $Es_{\alpha,\rho} = s'_{1-\alpha,1-\rho}$, as shown in [17, Lemma 2.2.17]. Note that this implies that if the word $s_{\alpha,\rho}$ is a fixed point of the morphism σ, then the word $s'_{1-\alpha,1-\rho}$ is fixed point of the morphism $E\sigma E$. Because of this duality we will confine ourselves often to α with $0 < \alpha < \frac{1}{2}$ in the sequel.

The first question we will consider is: what are the morphisms that leave a homogeneous Sturmian word c_α invariant? The answer in [9] is: they are compositions of the infinitely many morphisms $G_k :\ 0 \to 1^k 0,\ 1 \to 1$ and $H_k = G_k E$. The answer in [2] is: they are compositions of the infinitely many morphisms $h_k :\ 0 \to 0^k 1,\ 1 \to 0 \to 0^k 10$ (actually only for α's with a purely periodic continued fraction expansion). See [16] for yet another infinite family of morphisms.

[1] We interchangeably use the terms morphisms and substitutions.

In the paper [11] the authors call the inhomogeneous word $s_{\alpha,0}$ a characteristic sequence, and do actually derive a result close to our Theorem 3, using completely different techniques with continued fractions and extensive matrix multiplications.

More satisfactory is the answer in the book [17] or the paper [4], where only two generating morphisms are used, namely the exchange morphism E and the morphism G given by $G(0) = 0$, $G(1) = 01$. What we propose are also only two generators, which we denote by φ_0 and φ_1:

$$\varphi_0 : \begin{cases} 0 \to 0 \\ 1 \to 01 \end{cases} \qquad \varphi_1 : \begin{cases} 0 \to 01 \\ 1 \to 0 \end{cases}.$$

Note that $\varphi_0 = G$, and that $\varphi_1 = GE$, the Fibonacci morphism. Obviously, this proposal is very close to the one in [17], but what we gain is a natural way to index all the morphisms that leave homogeneous Sturmian words invariant by a binary tree—actually two binary trees, one for $\alpha \in (0, \frac{1}{2})$, and a dual version for $\alpha \in (\frac{1}{2}, 1)$. The dual tree is labelled by the compositions of the two morphisms

$$E\varphi_0 E : \begin{cases} 0 \to 10 \\ 1 \to 1 \end{cases} \qquad E\varphi_1 E : \begin{cases} 0 \to 1 \\ 1 \to 10 \end{cases}.$$

In Section 2.1 we treat some preliminaries to give in Section 2.2 our main result.

We remark that a similar tree associated with the rational numbers appears in the work of de Luca [18, 19]. The labeling there is not with morphisms, but with words.

The second question we will consider is: what are the substitution invariant Sturmian words that can only be obtained via the ceiling function, i.e., the Sturmian words that can only be obtained as in equation (2)? In this respect the homogeneous Sturmian words are regular, in that for all α

$$c_\alpha = s_{\alpha,\alpha} = s'_{\alpha,\alpha}.$$

So these 'strictly ceiling' Sturmian words have to be sought among the inhomogeneous Sturmian words, what we do in Section 3.

The short Section 4 is more or less independent of the remainder of the paper, but its contents have been very useful in our research.

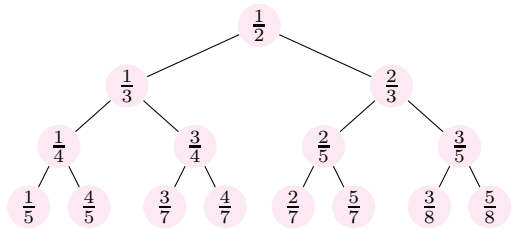
2. Homogeneous Sturmian Words

2.1. The Binary Tree of Harmonic Fractions

The binary tree is a graph with 2^n nodes $i_1 \dots i_n$ at level n for $n = 1, 2, \dots$, where the i_k are 0 or 1. At level 0 there is the root node Λ.

As early as 1619 Johannes Kepler defined in [12] a binary tree with fractions $\frac{p}{q}$ at the nodes. In the root there is $\frac{1}{2}$, and if $\frac{p}{q}$ is at a node, then the two children nodes receive the fractions

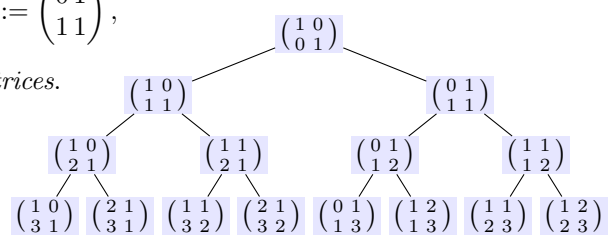
$$\frac{p}{p+q}, \quad \frac{q}{p+q}.$$



Rather surprisingly, every rational number p/q with $(p,q) = 1$ in the interval (0,1) occurs exactly once in the tree. This is not hard to prove, see, e.g., the paper [23]. We remark that the paper [14] consider this problem for larger classes of trees, but regretfully the rules for what the authors call the Kepler tree are different from Kepler's, but rather like those for the Calkin-Wilf tree (see [7]).
We introduce the two 2×2 matrices

$$K_0 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad K_1 := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

which we call the *Kepler matrices*.



It is clear that the fraction at the node $\underline{i} = i_1 \dots i_n$ in Kepler's tree of fractions is equal to p/q, where

$$\begin{pmatrix} p \\ q \end{pmatrix} = K_{i_n} \cdots K_{i_1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We claim that all the matrices $K_{i_n} \cdots K_{i_1}$ are different when n ranges over the natural numbers, and $i_1 \dots i_n$ is a string of 0's and 1's. Formulated slightly differently we have the following.

Lemma 1. *The monoid of matrices generated by K_0 and K_1 is free.*

Proof. If $K_{\underline{i}} = K_{\underline{j}}$, then $K_{\underline{i}} \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) = K_{\underline{j}} \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$, contradicting uniqueness on the Kepler tree. □

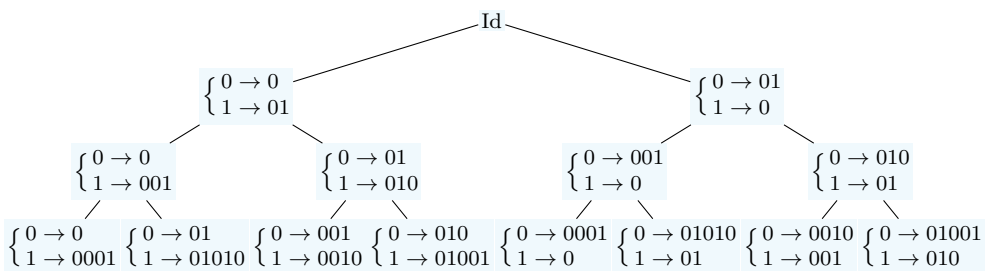
We remark that in general it is hard to determine freeness of matrix monoids. It is for instance an undecidable problem for 3×3 nonnegative integer matrices ([15], see also [8]). We mention also that K_0 and K_1 are unimodular matrices, but that they do not satisfy the criteria in [20].

2.2. A Tree of Morphisms

Let φ_0 and φ_1 be the two morphisms given by

$$\varphi_0 : \begin{cases} 0 \to 0 \\ 1 \to 01 \end{cases} \qquad \varphi_1 : \begin{cases} 0 \to 01 \\ 1 \to 0 \end{cases}.$$

We form a tree of morphisms $\mathcal{T}_\varphi$ by putting $\mathrm{Id}\colon 0 \to 0,\ 1 \to 1$ at Λ, and $\varphi_{i_n} \cdots \varphi_{i_1}$ at node $\underline{i} = i_1 \dots i_n$ for all n and all $i_k \in \{0,1\}$, $k = 1, \dots, n$.



The figure shows the first 3 levels of this tree, labeled with the morphisms. Note that the left edge of $\mathcal{T}_\varphi$ with nodes $\underline{i} = 0^n$ contains the morphisms φ_0^n, which do not generate infinite words.

Theorem 1. *The tree $\mathcal{T}_\varphi$ contains all morphisms that have homogeneous Sturmian words c_α as fixed point, for any α with $0 < \alpha < \frac{1}{2}$. Each such morphism occurs exactly once.*

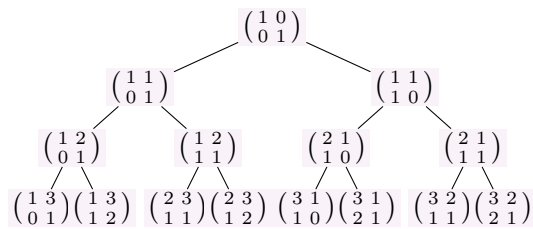
Proof: In [17] it is proved that for $\alpha \in (0,1)$, any morphism f fixing a homogeneous Sturmian word is a composition of the two morphisms E and G, excluding $f = E^n$, $f = G^n$ and $f = EG^nE$ for $n \geq 1$. Moreover, if $0 < \alpha < \frac{1}{2}$, then the first element in the composition of f is G. But since $E^2 = \mathrm{Id}$, f can then be written as a composition of $G = \varphi_0$ and $GE = \varphi_1$. This finishes the existence part of the proof.

For the uniqueness part, we remark first that it is shown in [17, Corollary 2.3.15], that in the monoid generated by the two morphisms E and GE the only relation is $E^2 = \mathrm{Id}$. This implies, of course, that the monoid generated by $G = \varphi_0$ and $GE = \varphi_1$, is free, but here we prefer to give a short self-contained proof, proving something stronger, which yields the emergence of the binary tree.

Consider the incidence matrices of the morphisms φ_0 and φ_1 :

$$M_0 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_1 := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Obviously morphisms with different incidence matrices are different. Let $\mathcal{T}_M$ be the tree with the matrix product $M_{i_n} \cdots M_{i_1}$ at node $i_1 \dots i_n$.



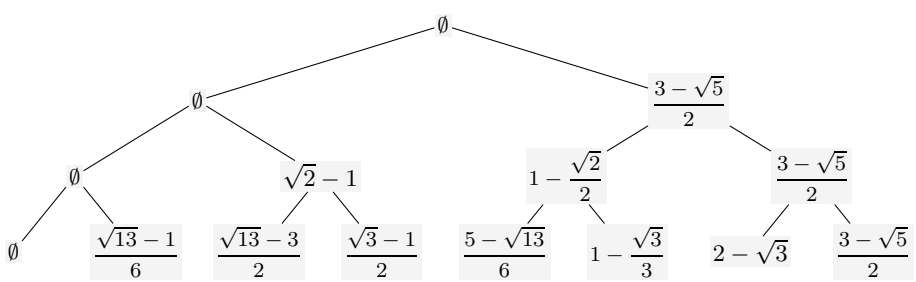
The matrices M_0 and M_1 are conjugate to the matrices K_0 and K_1 by the same conjugation matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. It follows that for *any* node $\underline{i}$ one has the equation

$$M_{\underline{i}} = \begin{pmatrix}0\,1\\1\,0\end{pmatrix} K_{\underline{i}} \begin{pmatrix}0\,1\\1\,0\end{pmatrix}.$$

But then Lemma 1 implies that all the $M_{\underline{i}}$ on $\mathcal{T}_M$ are different, and so each morphism occurs exactly once on $\mathcal{T}_\varphi$. $\square$

2.3. A Tree of Sturm Numbers

The tree of morphisms $\mathcal{T}_\varphi$ has a left edge with morphisms that do not generate infinite words. Below we display the first three levels of the tree of Sturm numbers α with $\alpha < \frac{1}{2}$ associated with the morphisms of $\mathcal{T}_\varphi$. Each such α will occur infinitely many times, since the powers of a morphism generate the same Sturmian word. In particular we will see on the right edge the number $(3-\sqrt{5})/2$ associated with the powers of the Fibonacci morphism φ_1^n.



3. Inhomogeneous Sturmian Words

In this section we consider *all* substitution invariant Sturmian words. There is again a simple algebraic characterization given by Yasutomi in ([24]):

Let $0 < \alpha < 1$ and $0 \le \rho \le 1$. Then $s_{\alpha,\rho}$ is substitution invariant if and only if the following two conditions are satisfied:

$$(i) \quad \alpha \text{ is an irrational quadratic number and } \rho \in \mathbb{Q}(\alpha);$$

$$(ii) \quad \overline{\alpha} > 1,\ 1-\overline{\alpha} \le \overline{\rho} \le \overline{\alpha} \quad \text{or} \quad \overline{\alpha} < 0,\ \overline{\alpha} \le \overline{\rho} \le 1-\overline{\alpha}.$$

3.1. The Eight Elementary Morphisms

As in [21] we define the eight morphisms ψ_i by

$$\psi_1 : 0 \to 01,\ 1 \to 0,\ \psi_2 : 0 \to 10,\ 1 \to 0,\ \psi_3 : 0 \to 0,\ 1 \to 01,\ \psi_4 : 0 \to 0,\ 1 \to 10.$$
$$\psi_5 : 0 \to 1,\ 1 \to 10,\ \psi_6 : 0 \to 1,\ 1 \to 01,\ \psi_7 : 0 \to 10,\ 1 \to 1,\ \psi_8 : 0 \to 01,\ 1 \to 1.$$

In the previous section the two morphisms $\psi_1 = \varphi_1$, and $\psi_3 = \varphi_0$ were used. The first four morphisms are linked to Sturmian words with slope $\alpha < 1/2$, the last four to Sturmian words with slope $\alpha > 1/2$. In the columns there is duality: $\psi_{i+4} = E\psi_i E$ for $i = 1, 2, 3, 4$. We also have $\psi_{2i} = \widetilde{\psi}_{2i-1}$ for $i = 1, 2, 3, 4$, where $\widetilde{\sigma}$ is the time reversal of a morphism σ.
The notation used in [17] is:

$\psi_1 = \varphi, \psi_2 = \widetilde{\varphi}, \psi_3 = \varphi E, \psi_4 = \widetilde{\varphi} E, \psi_5 = E\varphi E, \psi_6 = E\widetilde{\varphi} E, \psi_7 = E\varphi, \psi_8 = E\widetilde{\varphi}$.
Let $\mathcal{M}_{i,j} = \langle \psi_i, \psi_j \rangle$ denote the monoid generated by the morphisms ψ_i and ψ_j. We will also need $\mathcal{M}_i = \langle \psi_i \rangle$, the set of powers of ψ_i.

3.2. The Floor-Ceiling Structure of Sturmian Words

For most α's and ρ's the floor and the ceiling representation of a Sturmian word in equations (1) and (2) are equal. Rather surprisingly, *if* they are not equal, then they only differ in at most two consecutive indices ([17]). If there exists a natural number $m_\diamond$ such that

$$s_{\alpha,\rho}(m_\diamond - 1) \neq s'_{\alpha,\rho}(m_\diamond - 1) \text{ and } s_{\alpha,\rho}(m_\diamond) \neq s'_{\alpha,\rho}(m_\diamond),$$

then we call $(s_{\alpha,\rho}, s'_{\alpha,\rho})$ a *lozenge pair* with index $m_\diamond$. In case $m_\diamond = 0$, there is actually only the index 0 where they differ. As indicated in [17] right after Figure 2.3, $(s_{\alpha,\rho}, s'_{\alpha,\rho})$ is a lozenge pair with index $m_\diamond$ if and only if

$$\alpha m_\diamond + \rho \in \mathbb{N}. \tag{5}$$

Example. Let $\alpha = (3 - \sqrt{5})/2$ and $\rho = (\sqrt{5} - 1)/2$. Then $\alpha + \rho = 1$, so $(s_{\alpha,\rho}, s'_{\alpha,\rho})$ is a lozenge pair with index 1. Here $s_{\alpha,\rho} = 1001001010010010\ldots$ and $s'_{\alpha,\rho} = 0101001010010010, \ldots$. Both words are substitution invariant for the substitution $0 \to 010, 1 \to 10$, as evidenced in [5].

The following result is related to Corollary 1.4. in [6].

Proposition 1. *For substitution invariant lozenge pairs $m_\diamond = 0$ or $m_\diamond = 1$.*

Proof. This follows directly from equation (5) and Yasutomi's characterization. Suppose $m_\diamond \geq 2$ and $\alpha m_\diamond + \rho = k$ for an integer k. Since $0 < \alpha m_\diamond < m_\diamond$ and $0 \leq \rho \leq 1$ we must have

$$\alpha m_\diamond + \rho = k, \text{where } k \in \{1, 2, \ldots, m_\diamond\}.$$

One easily checks that $\overline{\rho} = k - \overline{\alpha} m_\diamond$. Now if $\overline{\alpha} > 1$, then it should hold that

$$1 - \overline{\alpha} \leq \overline{\rho} = k - \overline{\alpha} m_\diamond \;\Rightarrow\; (m_\diamond - 1)\overline{\alpha} \leq k - 1 \;\Rightarrow\; \overline{\alpha} \leq 1,$$

yielding a contradiction. Similarly the case $\overline{\alpha} < 0$ yields a contradiction. □

We undertake the task of determining all substitution invariant Sturmian words that are lozenge pairs. We start with the case $m_\diamond = 1$, which is simpler than $m_\diamond = 0$.

3.3. Substitution Invariant Sturmian Words With $\mathbf{m_\diamond = 1}$

Note that $m_\diamond = 1$ implies

$$\alpha + \rho = 1.$$

We first give a very simple way to obtain the lozenge pair.

Proposition 2. *For $\alpha \in (0,1)$ let $(s_{\alpha,1-\alpha}, s'_{\alpha,1-\alpha})$ be the lozenge pair with $m_\diamond = 1$. Then*

$$s_{\alpha,1-\alpha} = 10\, c_\alpha \quad \text{and} \quad s'_{\alpha,1-\alpha} = 01\, c_\alpha.$$

Proof. Since $\alpha < 1$, we have $s_{\alpha,1-\alpha}(0) = [\alpha+\rho] - [\rho] = 1$ and $s_{\alpha,1-\alpha}(1) = [2\alpha+\rho] - [\alpha+\rho] = 0$. Also, $s'_{\alpha,1-\alpha}(0) = \lceil\alpha+\rho\rceil - \lceil\rho\rceil = 0$ and $s'_{\alpha,1-\alpha}(1) = \lceil 2\alpha+\rho\rceil - \lceil\alpha+\rho\rceil = 1$. Let S be the shift: $S(w_0 w_1 w_2 \dots) = w_1 w_2 \dots$. Adding α to ρ shifts a Sturmian word by one: $s_{\alpha,\rho+\alpha} = S(s_{\alpha,\rho})$. So $S^2(s_{\alpha,1-\alpha}) = s_{\alpha,1+\alpha} = s_{\alpha,\alpha} = c_\alpha$. □

We still have to investigate whether $s_{\alpha,1-\alpha}$ and $s'_{\alpha,1-\alpha}$ are substitution invariant. This can be directly derived from the results in [5], but we give here a short and more global proof. We start with a combinatorial lemma.

Lemma 2. *For any $\gamma \in \mathcal{M}_{1,3}$ the words $01\gamma^2(0)$ and $10\gamma^2(1)$ are palindromes.*

Proof. This relies on the notions and results of [17, Section 2.2.1]. The standard morphisms are the elements of $\langle \varphi, E\rangle = \langle \psi_1, E\rangle$. Since $\psi_3 = \varphi E$, all γ from $\mathcal{M}_{1,3}$ are standard. By Proposition 2.2.2 and Proposition 2.3.11 of [17], the words $\gamma(0)$ and $\gamma(1)$ are two standard words, which differ in their last two letters. But then $\gamma^2(0)$ and $\gamma^2(1)$ are standard words so that $\gamma^2(0)$ ends in 0, and $\gamma^2(1)$ ends in 1. Moreover, according to [17, Theorem 2.2.4], a word w is standard if and only if it has length 1 or there exists a palindrome word p such that $w = p\,01$ or $w = p\,10$. So the words

$$01\gamma^2(0) = 01p\,10 \quad \text{and } 10\gamma^2(1) = 10p'01$$

are palindromes. The length 1 case may occur, but then 010 is a palindrome. □

Theorem 2. *Let $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ be substitution invariant Sturmian words with $m_\diamond = 1$ and $\alpha < 1/2$. Then these two words are fixed points of ψ^2, where $\psi \in \mathcal{M}_{2,4} \setminus \mathcal{M}_4$. Here $\psi = \widetilde{\gamma}$, where $\gamma(c_\alpha) = c_\alpha$.*

Proof. The condition $m_\diamond = 1$ means that $\rho = 1 - \alpha$. What we will show is that $s_{\alpha,1-\alpha}$ and $s'_{\alpha,1-\alpha}$ are fixed points of $\widetilde{\gamma}^2$, where γ fixes c_α. Recall that in general $\widetilde{\psi}$ is the time reversal of a morphism ψ, and that $\widetilde{\psi}_1 = \psi_2$, and $\widetilde{\psi}_3 = \psi_4$. In general one has $\widetilde{\sigma\tau} = \widetilde{\sigma}\widetilde{\tau}$ for two morphisms σ and τ. It thus follows from Lemma 2 that for $\gamma \in \mathcal{M}_{1,3}$ and all $n \geq 1$

$$01\,\gamma^{2n}(0) = \widetilde{\gamma}^{2n}(0)\,10 \quad \text{and} \quad 10\,\gamma^{2n}(1) = \widetilde{\gamma}^{2n}(1)\,01.$$

For such γ, not from $\mathcal{M}_3$, when $n \to \infty$, the left sides converge to $01c_\alpha$, respectively $10c_\alpha$, and thus by Proposition 2, the right sides converge to $s_{\alpha,1-\alpha}$ respectively $s'_{\alpha,1-\alpha}$. □

It is easily seen via duality that Theorem 2 also applies in case $\alpha > 1/2$, where $\mathcal{M}_{2,4} \setminus \mathcal{M}_4$ has to be replaced by $\mathcal{M}_{6,8} \setminus \mathcal{M}_8$.

3.4. Substitution Invariant Sturmian Words With $\mathbf{m_\diamond = 0}$

Note that $m_\diamond = 0$ implies $\rho = 0$. As in the case of $m_\diamond = 1$ there is a simple way to obtain the lozenge pair:

$$s_{\alpha,0} = 0\,c_\alpha \quad \text{and} \quad s'_{\alpha,0} = 1\,c_\alpha.$$

It is well known that $0\,c_\alpha$ is substitution invariant, see, e.g., Corollary 1.4. in [6]. However, it is now less simple to determine the substitution fixing $0\,c_\alpha$ for a given α.

Example. Let $\alpha = (\sqrt{13}-1)/6$. Then $\gamma(c_\alpha) = c_\alpha$ for γ given by $\gamma(0) = 01,\ \gamma(1) = 01010$.
(The same morphism is considered in [5] on page 262.)
Here $\psi(s_{\alpha,0}) = s_{\alpha,0}$ for ψ given by

$$\psi(0) = 0010101, \quad \psi(1) = 0010101001010101.$$

A recipe is given in [5]. The recipe depends strongly on the last letter of $\gamma(0)$, so it is useful to characterize the morphisms γ with $\gamma(0)$ ending in 0.

Lemma 3. *Let $\gamma = \psi_{i_1} \dots \psi_{i_m}$ be a morphism from $\mathcal{M}_{1,3}$. Then $\gamma(0)$ ends in 0 if and only if the number of 1 in $i_1 \dots i_m$ is even.*

Proof. For any morphism $\sigma : \{0,1\}^* \to \{0,1\}^*$, let $\lambda[\sigma] : \{0,1\} \to \{0,1\}$ be the map that maps j to the last letter of $\sigma(j)$, $j = 0, 1$. Then $\lambda[\sigma\tau] = \lambda[\sigma]\lambda[\tau]$ for two morphisms σ and τ. Since $\lambda[\psi_1] = E$, and $\lambda[\psi_3] = \mathrm{Id}$, this implies the lemma. □

We denote the set of γ from $\mathcal{M}_{1,3}$ such that $\gamma(0)$ ends in 0 by $\mathcal{M}^0_{1,3}$.

Proposition 3. *Let $\gamma \in \mathcal{M}^0_{1,3}$ such that $\gamma(c_\alpha) = c_\alpha$. Let Ψ_γ be conjugate to γ, with conjugating word equal to $u = \gamma(0)0^{-1}$. Then*

$$\Psi_\gamma(0c_\alpha) = 0c_\alpha.$$

For a proof of this proposition, see [5, Theorem 3.1].

Proposition 4. *Let $\gamma, \delta \in \mathcal{M}^0_{1,3}$. Then $\Psi_{\gamma\delta} = \Psi_\gamma \Psi_\delta$.*

Proof. We assume that $|\gamma(0)| < |\gamma(1)|$; the proof of the other case is quite similar. It is well known that there exist words u, v, x, and y such that

$$\gamma : \begin{cases} 0 \to u\,0 \\ 1 \to u\,0\,v \end{cases} \quad \Psi_\gamma : \begin{cases} 0 \to 0\,u \\ 1 \to 0\,v\,u \end{cases} \qquad \delta : \begin{cases} 0 \to x\,0, \\ 1 \to x\,0\,y \end{cases} \quad \Psi_\delta : \begin{cases} 0 \to 0\,x \\ 1 \to 0\,y\,x. \end{cases}$$

The product $\gamma\delta$ also generates a characteristic word, and we have

$$\gamma\delta : \begin{cases} 0 \to \gamma(x)\,\gamma(0) = \gamma(x)\,u\,0 \\ 1 \to \gamma(x)\,\gamma(0)\,\gamma(y) = \gamma(x)\,u\,0\,\gamma(y) \end{cases} \qquad \Psi_{\gamma\delta} : \begin{cases} 0 \to 0\,\gamma(x)\,u \\ 1 \to 0\,\gamma(y\,)\gamma(x)\,u. \end{cases}$$

This is a slightly extended version of [17, Lemma 2.3.17 (iii)], which leads to $\Psi_{\gamma\delta} = \Psi_\gamma \Psi_\delta$, since

$$\Psi_\gamma \Psi_\delta : \begin{cases} 0 \to \Psi_\gamma(0)\,\Psi_\gamma(x) = 0\,u\,\Psi_\gamma(x) = 0\,\gamma(x)\,u \\ 1 \to \Psi_\gamma(0)\,\Psi_\gamma(y)\,\Psi_\gamma(x) = 0\,u\,\Psi_\gamma(y)\,\Psi_\gamma(x) = 0\,\gamma(y\,)\gamma(x)\,u, \end{cases}$$

where we use the conjugation relation $u\,\Psi_\gamma(w) = \gamma(w)\,u$ for $w = x, yx$. □

For a further description, we need next to ψ_3 yet another elementary morphism $\psi_8 = E\,\widetilde{\varphi}$:

$$\psi_3 :\ 0 \to 0,\ 1 \to 01, \qquad \psi_8 :\ 0 \to 01,\ 1 \to 1.$$

Lemma 4. *Let $\gamma = \psi_1 \psi_3^n \psi_1$ for some $n \geq 0$. Then $\Psi_\gamma = \psi_3 \psi_8^{n+1}$.*

Proof. One easily finds that for all $n \geq 0$

$$\psi_1\psi_3^n\psi_1(0) = (01)^{n+1}0,\ \psi_1\psi_3^n\psi_1(1) = 01; \quad \psi_3\psi_8^{n+1}(0) = 0(01)^{n+1},\ \psi_3\psi_8^{n+1}(1) = 01.$$

This implies the statement of the lemma. □

We need more details on the structure of the map $\gamma \mapsto \Psi_\gamma$.

Proposition 5. *Let γ be a morphism from $\mathcal{M}^0_{1,3}$. Then*

$$\Psi_{E\gamma E} = E\,\widetilde{\Psi}_\gamma\,E.$$

Moreover, $\Psi_{E\gamma E} = \Psi^*_\gamma$, where $\cdot^*$ is the homomorphism defined by $\psi_3^* = \psi_8$, $\psi_8^* = \psi_3$.

Proof. We assume that $|\gamma(0)| < |\gamma(1)|$, the proof of the other case is quite similar. We use the same facts on γ from [17] as in the proof of Lemma 2. Since $(\gamma(0), \gamma(1))$ is a standard pair, and all $\gamma(0)$ from $\mathcal{M}_{1,3}$ start with 0, there exist[2] palindromes p and q such that

$$\gamma : \begin{cases} 0 \to 0\,p\,0\,1\,0 \\ 1 \to 0\,p\,0\,1\,0\,q\,0\,1, \end{cases} \qquad \Psi_\gamma : \begin{cases} 0 \to 0\,0\,p\,0\,1 \\ 1 \to 0\,q\,0\,1\,0\,p\,0\,1. \end{cases}$$

So we have

$$\widetilde{\Psi}_\gamma : \begin{cases} 0 \to 1\,0\,p\,0\,0 \\ 1 \to 1\,0\,p\,0\,1\,0\,q\,0, \end{cases} \qquad E\widetilde{\Psi}_\gamma E : \begin{cases} 0 \to 0\,1\,\overline{p}\,1\,0\,1\overline{q}\,1 \\ 1 \to 0\,1\,\overline{p}\,1\,1. \end{cases}$$

On the other hand,

$$E\gamma E : \begin{cases} 0 \to 1\,\overline{p}\,1\,0\,1\overline{q}\,1\,0 \\ 1 \to 1\,\overline{p}\,1\,0\,1, \end{cases} \qquad \Psi_{E\gamma E} : \begin{cases} 0 \to 0\,1\,\overline{p}\,1\,0\,1\overline{q}\,1 \\ 1 \to \dots. \end{cases}$$

Note that we showed $\Psi_{E\gamma E}(0) = E\widetilde{\Psi}_\gamma E(0)$, but for 1 this is trickier. A way to look at conjugation is by simultaneous repeated rotation. Here by *rotation* we mean the map ρ on words defined by

$$\rho(w_1 w_2 \dots w_m) = w_2 \dots w_m w_1.$$

Moreover, words may only be rotated simultaneously if their first letters are equal. From this viewpoint,

$$\Psi_{E\gamma E}(0) = \rho^{|E\gamma E(0)|-1} E\gamma E(0) = \rho^{|p|+|q|+5} E\gamma E(0).$$

Since the length of $E\gamma E(1)$ equals $|p| + 4$, and $E\gamma E(1)$ is a prefix of $E\gamma E(0)$, we may first rotate $|p| + 4$ times, obtaining

$$\rho^{|p|+4}\Psi_{E\gamma E}(0) = \overline{q}\,1\,0\,1\,\overline{p}\,1\,0\,1, \quad \rho^{|p|+4}\Psi_{E\gamma E}(1) = 1\,\overline{p}\,1\,0\,1.$$

To continue rotating $|q| + 1$ times, , we must see that $\overline{q}\,1$ is a prefix of $1\,\overline{p}\,1\,0\,1$, *and* we want to see that the outcome is

$$\rho^{|q|+1}(1\,\overline{p}\,1\,0\,1) = 0\,1\,\overline{p}\,1\,1, \quad \text{or equivalently,} \quad \rho^{|q|+1}(0\,p\,0\,1\,0) = 1\,0\,p\,0\,0.$$

This requires for the time being that $|q| \le |p| + 4$.

According to [17, Theorem 2.2.4] the word $0\,p\,0\,1\,0\,q$ is a palindrome, and so

$$0\,p\,0\,1\,0\,q = q\,0\,1\,0\,p\,0,$$

hence $\rho^{|q|+1}(0\,p\,0\,1\,0)$ has prefix $1\,0\,p\,0$. We also see from this equation that the first letter of q is a 0, so the letter following $1\,0\,p\,0$ is a 0, as claimed. Note that this palindrome equation also implies that $\overline{q}\,1$ is a prefix of $1\,\overline{p}\,1\,0\,1$.

[2]Here we admit $p = 0^{-1}$, with length $|p| = -1$.

In the case that $|q| > |p| + 4$, suppose that $|q| = a(|p| + 4) + b$, where $b < |p| + 4$. Then one rotates with $\rho^{|p|+4}$ $a + 1$ times, followed by a rotation ρ^{b+1}. That this is possible, one can deduce from [17, Proposition 2.2.2], which states that only the last two letters of the words $\gamma(0)\gamma(1)$ and $\gamma(1)\gamma(0)$ are different. A zig-zag argument then gives that $\gamma(1) = \gamma(0)^{a+1}01$.

For the second part of the proposition, note that $\psi_3^* = E\widetilde{\psi}_3 E = \psi_8$, and $\psi_8^* = E\widetilde{\psi}_8 E = \psi_3$. Let $\Psi_\gamma = \psi_{i_1} \dots \psi_{i_m}$. Then we have, using the first part of the proposition,

$$\Psi_{E\gamma E} = E\,\widetilde{\Psi}_\gamma\,E = E\widetilde{\psi}_{i_1} \dots \widetilde{\psi}_{i_m} E = E\widetilde{\psi}_{i_1} E E \widetilde{\psi}_{i_2} E \dots E\widetilde{\psi}_{i_m} E$$
$$= \psi_{i_1}^* \psi_{i_2}^* \dots \psi_{i_m}^* = \Psi_\gamma^*.$$

□

Theorem 3. *Let α be a Sturmian number, with $0 < \alpha < 1$. Then $s_{\alpha,0}$ is a fixed point of some $\psi \in \mathcal{M}_{3,8}$. Conversely, any $\psi \in \mathcal{M}_{3,8} \smallsetminus \{\mathcal{M}_3 \cup \mathcal{M}_8\}$ fixes an $s_{\alpha,0}$. The same statements hold for $s'_{\alpha,0}$, but then with $\mathcal{M}_{3,8}$ replaced by $\mathcal{M}_{4,7}$.*

Proof. We have $s_{\alpha,0} = 0\,c_\alpha$. Suppose $\gamma \in \mathcal{M}_{1,3}$ satisfies $\gamma(c_\alpha) = c_\alpha$. Then $\gamma^2(c_\alpha) = c_\alpha$ and $\gamma^2 \in \mathcal{M}_{1,3}^0$, so by Proposition 3, $s_{\alpha,0}$ is fixed point of Ψ_{γ^2}.

We claim that any Ψ_γ, where γ is from $\mathcal{M}_{1,3}^0$, is an element of $\mathcal{M}_{3,8}$. We prove this claim by induction on m where $\gamma = \psi_{i_1} \dots \psi_{i_m}$. For $m = 2$, $\gamma = \psi_1^2$, and the claim is true by Lemma 4. Suppose the claim is true for all γ from $\mathcal{M}_{1,3}^0$ with length m or less. An arbitrary $\gamma = \psi_{i_1} \dots \psi_{i_{m+1}}$ from $\mathcal{M}_{1,3}^0$, can be written as $\gamma = \gamma'\gamma''$, where γ' and γ'' are non-trivial elements of $\mathcal{M}_{1,3}^0$, *unless* γ has the form

$$\gamma = \psi_1 \psi_3^{m-1} \psi_1,$$

but then $\Psi_\gamma \in \mathcal{M}_{3,8}$ according to Lemma 4. The first part of the theorem is proved.

For the second part, we divide the morphisms in $\mathcal{M}_{3,8}$ into two types: the ones starting with ψ_3 and the ones starting with ψ_8. A density argument shows that first type corresponds to ψ with $\alpha < 1/2$, and the second type to ψ with $\alpha > 1/2$. Moreover, by Proposition 5 these are in 1-to-1 correspondence with each other by replacing all ψ_3 by ψ_8 and conversely. It suffices therefore, to show that any ψ from $\mathcal{M}_{3,8} \smallsetminus \mathcal{M}_3$ starting with ψ_3 fixes an $s_{\alpha,0} = 0c_\alpha$. This can be done with an argument similar to the one above. Let $\psi = \psi_3 \psi_{i_2} \dots \psi_{i_m}$. When $m = 2$, $\psi = \psi_3\psi_8$, and we know that $\psi(0c_\alpha) = c_\alpha$, where c_α is the fixed point of ψ_1^2. Proceed by induction, using Proposition 4. Now ψ can be written as $\psi'\psi''$ with $\psi' = \psi_3 \dots$ and $\psi'' = \psi_3 \dots$ *unless* ψ has the form $\psi_3\psi_8^m$, but then we can use Lemma 4.

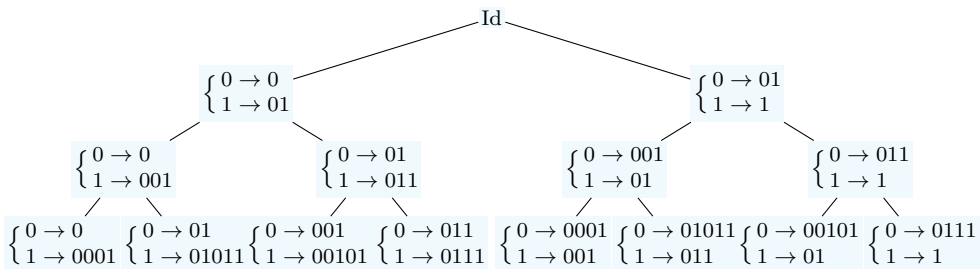
To handle $s'_{\alpha,0} = 1c_\alpha$, we use the property that in general $s'_{\alpha,\rho} = E\,s_{1-\alpha,1-\rho}$ (see [17, Lemma 2.2.17]). This yields

$$s'_{\alpha,0} = E\,s_{1-\alpha,1} = E\,s_{1-\alpha,0}.$$

Since in general $E(w)$ is a fixed point of $E\sigma E$ when w is a fixed point of σ, we obtain that the $s'_{\alpha,0}$ are generated by the morphisms from the monoid $\mathcal{M}_{4,7}$, since $E\psi_3 E = \psi_7$ and $E\psi_8 E = \psi_4$. □

Remark 1. There is an interesting coding from the morphisms starting with ψ_3 in $\mathcal{M}_{3,8}$ to $\mathcal{M}^0_{1,3}$. Let + be binary addition with 3 and 8: 3+3=3, 8+8=3, 3+8=8, 8+3=8. Add $i_2 i_3 \dots i_m 3$ to $3\, i_2 \dots i_m$, and replace 8 by 1. For example: $38...88 + 88...83 = 83...38 \mapsto 13...31$.

We display the first three levels of the binary tree $\mathcal{T}_{3,8}$, where the nodes are labeled with the morphisms given by Theorem 3.



Remark 2. Just as in Theorem 1, each morphism generating an $s_{\alpha,0}$ occurs exactly once on the tree $\mathcal{T}_{3,8}$. This can be deduced from the fact that we have a coding between $\mathcal{M}_{3,8}$ and $\mathcal{M}^0_{1,3}$, but also because the monoid generated by the incidence matrices of ψ_3 and ψ_8 is free. Arnoux remarks that this can be derived in an elementary way ([3, Lemma 6.5.14]).

4. Generating Substitution Invariant Sturmian Words

There is a direct, more analytic way to find substitution invariant Sturmian words. We use an idea already considered by self-similarity expert Douglas Hofstadter in 1963 ([10]). To solve the fixed point equation $\psi(s_{\alpha,\rho}) = s_{\alpha,\rho}$ for ψ, we can equivalently solve the fixed point equation

$$T_\psi(x, y) = (x, y) \quad \text{for } 0 < x, y < 1,$$

where $T_\psi = T_{i_1} \dots T_{i_n}$ if $\psi = \psi_{i_1} \dots \psi_{i_n}$ with the i_k from some subset of $\{1, \dots, 8\}$. Here the T_i are two-dimensional fractional linear functions, such that

$$\psi_i(s_{\alpha,\rho}) = s_{T_i(\alpha,\rho)}.$$

Some T_i are given by [17, Lemma 2.2.18], and the others can be computed in a similar way. We have, for example,

$$T_1(x, y) = \left(\frac{1-x}{2-x}, \frac{1-y}{2-x}\right), \quad T_3(x, y) = \left(\frac{x}{1+x}, \frac{y}{1+x}\right), \text{ and}$$

$$T_8(x, y) = \left(\frac{1}{2-x}, \frac{y}{2-x}\right).$$

Note that both T_3 and T_8 leave the line $y = 0$ invariant; this suggests the use of products of ψ_3 and ψ_8 to solve the equation $\psi(s_{\alpha,0}) = s_{\alpha,0}$, as we did in Theorem 3. We mention that the triple T_1, T_3, T_8 occurs in [11], where they are used to connect two-dimensional continued fraction expansions to substitution invariant Sturmian words.

Solving the equation $T_\psi(x, y) = (x, y)$ is straightforward: there is a one-dimensional fractional linear function fixed point equation for x, which is quadratic, and then there is a linear equation for y, since one can show by induction on the number of ψ_i in ψ that only $\pm y$ will occur in the second component of $T_\psi(x, y)$.

We mention that in some cases the equation is actually $\psi(s_{\alpha,\rho}) = s'_{T_\psi(\alpha,\rho)}$, but this can be dealt with by passing to the square of ψ, or by using Proposition 1.

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