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A REMARKABLE INTEGER SEQUENCE RELATED TO $\pi$ AND $\sqrt{2}$

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Abstract
We prove that five ways to define entry \textbf{A086377} in the OEIS do lead to the same integer sequence.

– Dedicated to Jeff Shallit on the occasion of his 60th birthday

1. Introduction

In September of 2003 Benoît Cloitre contributed a sequence to the On-Line Encyclopedia of Integer Sequences \cite{OEIS}, defined by him as $a_1 = 1$, and for $n \geq 2$ by

$$a_n = \begin{cases} 
  a_{n-1} + 2 & \text{if } n \text{ is in the sequence,} \\
  a_{n-1} + 2 & \text{if } n \text{ and } n-1 \text{ are not in the sequence,} \\
  a_{n-1} + 3 & \text{if } n \text{ is not in the sequence, but } n-1 \text{ is in the sequence.}
\end{cases} \quad (1)$$

The first 25 values of this sequence are

$$1, 4, 6, 8, 11, 13, 16, 18, 21, 23, 25, 28, 30, 33, 35, 37, 40, 42, 45, 47, 49, 52, 54, 57, 59.$$
2. The Theorem

**Theorem 1.** The following five definitions produce the same integer sequence:

1. \((a_n)\) defined by \(a_1 = 1\) and for \(n \geq 2:\)

   \[
   a_n = \begin{cases} 
   a_{n-1} + 2 & \text{if } n \text{ is in the sequence,} \\
   a_{n-1} + 2 & \text{if } n \text{ and } n-1 \text{ are not in the sequence,} \\
   a_{n-1} + 3 & \text{if } n \text{ is not in the sequence, but } n-1 \text{ is in the sequence;}
   \end{cases}
   \]

2. \((b_n)\) defined by \(b_1 = 1\) and for \(n \geq 2:\)

   \[
   b_n = \begin{cases} 
   b_{n-1} + 2 & \text{if } n-1 \text{ is not in the sequence,} \\
   b_{n-1} + 3 & \text{if } n-1 \text{ is in the sequence;}
   \end{cases}
   \]

3. \((c_n)\) for \(n \geq 1\) defined as the position of the \(n\)-th zero in the fixed point of the morphism 
   \[
   \phi : \begin{cases} 
   0 \mapsto 011 \\
   1 \mapsto 01
   \end{cases}
   \]

4. \((d_n)\) defined by \(d_n = \left\lfloor (1 + \sqrt{2}) \cdot n - \frac{1}{2}\sqrt{2} \right\rfloor\) for \(n \geq 1;\)

5. \((e_n)\) defined by \(e_n = \lfloor r_n \rfloor = \lfloor r_n + \frac{1}{2} \rfloor,\) with \(r_1 = \frac{4}{\pi}\) and \(r_{n+1} = \frac{n^2}{r_n - (2n-1)},\) for \(n \geq 1.\)

At first we found it hard to believe the equivalence of these definitions, but a verification of the first 130000 terms \((a_{130000} = 313847)\) convinced us to look for proofs.

3. Simplification and a Morphic Sequence

To show that \((b_n)\) defines the same sequence as \((a_n),\) simply note that \(a_n - a_{n-1} \geq 2\) for all \(n:\) hence if \(n\) is in the sequence then \(n - 1\) is not, and we can combine the first two cases in Equation (1).

In a comment to sequence **A086377**, Clark Kimberling asked if the integers in this sequence coincide with the positions of the zeroes in sequence **A189687**, which is the fixed point of the substitution

\[
\phi : \begin{cases} 
0 \mapsto 011 \\
1 \mapsto 01
\end{cases}
\]
defining the sequence \((c_n)\) in the Theorem. It is not hard to see that this indeed produces the same as sequence \((b_n)\); repeatedly applying the morphism \(\phi\) to 0 produces after a few steps the initial segment

\[
01101011011011011010111011011011011010110110110110110111\ldots .
\]

The position \(c_n\) of the \(n\)-th zero is 2 ahead of \(c_{n-1}\) precisely when the latter is followed by a single 1, that is, when there is a 1 at position \(n - 1\), and it is 3 ahead of \(c_{n-1}\) if that zero is followed by 11, which means that there was a 0 at position \(n - 1\). Thus the rule is exactly that defining \((b_n)\).

4. Beatty Sequence

Every pair of real numbers \(\alpha\) and \(\beta\) determines a Beatty sequence by

\[
B(\alpha, \beta)_n := \lfloor n\alpha + \beta \rfloor, \quad n = 1, 2, \ldots .
\]

The numbers \(\alpha\) and \(\beta\) also determine sequences by

\[
St(\alpha, \beta)_n := \lfloor (n + 1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor, \quad n = 1, 2, \ldots ,
\]

which is a Sturmian sequence (of slope \(\alpha\)), over the alphabet \(\{0, 1\}\), provided that \(0 \leq \alpha < 1\).

Thus Sturmian sequences are first differences of Beatty sequences (when \(0 \leq \alpha < 1\)), but Beatty sequences and Sturmian sequences are also linked in another way.

**Lemma 1.** Let \(\alpha > 1\) be irrational, and let \((s_n)_{n \geq 1}\) be given by \(s_n = St(\frac{1}{\alpha}, -\frac{\beta}{\alpha})_n\), for some real number \(\beta\) with \(\alpha + \beta > 1\) and such that \(k\alpha + \beta \not\in \mathbb{Z}\) for all positive integers \(k\). Then \(B(\alpha, \beta)\) is the sequence of positions of 1 in \((s_n)\).

**Proof.** This is a generalization of Lemma 9.1.3 in [1], from homogeneous to inhomogeneous Sturmian sequences. The proof also generalizes:

- there exists \(k \geq 1\) : \(n = \lfloor k\alpha + \beta \rfloor\) if and only if
- there exists \(k \geq 1\) : \(n \leq k\alpha + \beta < n + 1\) if and only if
- there exists \(k \geq 1\) : \(\frac{n - \beta}{\alpha} \leq k < \frac{n + 1 - \beta}{\alpha}\) if and only if
- there exists \(k \geq 1\) : \(\left\lfloor \frac{n - \beta}{\alpha} \right\rfloor = k - 1\) and \(\left\lfloor \frac{n - \beta}{\alpha} + \frac{1}{\alpha} \right\rfloor = k\) if and only if
- \(\left\lfloor \frac{n + 1}{\alpha} - \frac{\beta}{\alpha} \right\rfloor - \left\lfloor \frac{n}{\alpha} - \frac{\beta}{\alpha} \right\rfloor = 1\) if and only if
- \(St(\frac{1}{\alpha}, -\frac{\beta}{\alpha})_n = 1\). \(\square\)
Our goal in this section is to prove that \((c_n) = (d_n)\). Let \(\psi\) be the morphism \(\psi : \begin{cases} 0 \rightarrow 10 \\ 1 \rightarrow 100 \end{cases}\), and let \(w\) be the fixed point. Then
\[ w = 100101010010100101001010010100101001010010010100101001001010010010100\cdots, \]
which is obtained by exchanging 0s and 1s in the fixed point of \(\phi\), i.e., \(\psi = E\phi E\), with \(E\) the exchange morphism given by \(E(0) = 1, E(1) = 0\). So the positions of 0 in the fixed point of \(\phi\) correspond to the positions of 1 in the fixed point \(w\) of \(\psi\).

Let \(\alpha_d = 1 + \sqrt{2}\) and \(\beta_d = -\frac{1}{2}\sqrt{2}\); then \(d_n = B(\alpha_d, \beta_d)n\), for \(n \geq 1\).

Applying Lemma 1, we deduce that \(d_n\) also equals the position of the \(n\)-th 1 in the Sturmian sequence \(St(\alpha, \beta)\), generated by
\[ \alpha = \frac{1}{\alpha_d} = \sqrt{2} - 1, \quad \beta = \frac{-\beta_d}{\alpha_d} = 1 - \frac{1}{2}\sqrt{2}. \]

**Lemma 2.** \(St(\sqrt{2} - 1, 1 - \frac{1}{2}\sqrt{2}) = w\).

**Proof.** This was already proved by Nico de Bruijn in 1981 ([2]), where it is the main example. See also Lemma 2 in [6]. Note, however, that our Sturmian sequences start at \(n = 1\).

For a ‘modern’ proof as suggested by [3, Section 4], let \(\psi_1\) and \(\psi_2\) be the elementary morphisms given by \(\psi_1(0) = 01, \psi_1(1) = 0\), and \(\psi_2(0) = 10, \psi_2(1) = 0\). Then \(\psi = \psi_2\psi_1 E\). This implies that the fixed point \(w\) of \(\psi\) is a Sturmian word (see [5, Corollary 2.2.19]). To find its parameters \((\alpha, \beta)\), use the 2D fractional linear maps that describe how the parameters of a Sturmian word change when one applies an elementary morphism. For Sturmian words starting at \(n = 0\), the maps for \(E, \psi_1\) and \(\psi_2\) are\(^2\) respectively (see [5, Lemma 2.2.17, Lemma 2.2.18, Exercise 2.2.6])
\[ T_0(x, y) = (1-x, 1-y), \quad T_1(x, y) = \left(\frac{1-x}{2-x}, \frac{1-y}{2-x}\right), \quad T_2(x, y) = \left(\frac{1-x}{2-x}, \frac{2-x-y}{2-x}\right). \]

The change of parameters by applying \(\psi\) is therefore the composition
\[ T_{210}(x, y) := T_2T_1T_0(x, y) = \left(\frac{1}{2+x}, \frac{2+x-y}{2+x}\right). \]

But the parameters \(\alpha\) and \(\beta\) of \(w\) do not change when one applies \(\psi\). This means that \((\alpha, \beta)\) is a fixed point of \(T_{210}\), and one easily computes \(\alpha = \sqrt{2} - 1\), and then \(\beta = \frac{1}{2}\sqrt{2}\). Since our Sturmian words start at \(n = 1\), we have to subtract \(\alpha\) from \(\beta\) and obtain that \(w = St(\sqrt{2} - 1, 1 - \frac{1}{2}\sqrt{2})\). \(\square\)

\(^2\)Actually there is a subtlety here involving the ceiling representation of a Sturmian sequence, but that does not apply in our case since \(\beta \not\in \mathbb{Z} + \mathbb{Z}\).
5. Converging Recurrence

In a comment to entry A086377, Joseph Biberstine conjectured a beautiful connection with the infinite continued fraction expansion

$$\frac{4}{\pi} = 1 + \frac{1^2}{3 + \frac{2^2}{5 + \frac{3^2}{7 + \frac{4^2}{9 + \frac{5^2}{11 + \ddots}}}}}$$

derived from the arctangent function expansion. If we define $R_n$ for $n \geq 1$ by

$$R_n = 2n - 1 + \frac{n^2}{2n + 1 + \frac{(n+1)^2}{2n+3 + \frac{(n+2)^2}{2n+5 + \frac{(n+3)^2}{\ddots}}}}$$

then $R_1 = 4/\pi$ and $R_n = 2n - 1 + \frac{n^2}{R_{n+1}}$. We see that

$$\frac{R_n}{n} \frac{R_{n+1}}{n+1} - \frac{2n - 1}{n} \frac{R_{n+1}}{n+1} - \frac{n^2}{n(n+1)} = 0.$$

This implies that if $R_n/n$ converges, for $n \to \infty$, then it does so to a (positive) zero of $x^2 - 2x - 1$, that is, to $1 + \sqrt{2}$; cf. Lemma 3 below.

We consider now, conversely and slightly more generally, for any real $h \geq 1$, a sequence of positive numbers $r_n$ satisfying

$$r_n = hn - 1 + \frac{n^2}{r_{n+1}}$$  \hspace{1cm} (2)

for $n \geq 1$. We first show that this sequence is unique, i.e., there is a unique $r_1 > 0$ such that $r_n > 0$ for all $n \geq 1$, and give estimates for its terms.

**Lemma 3.** For each $h \geq 1$, there is a unique sequence of positive real numbers $(r_n)_{n \geq 1}$ satisfying the recurrence (2). Moreover, we have for this sequence, for all $n \geq 1$,

$$0 < r_n - \alpha n + c < \frac{(\alpha - c)(c - 1)}{\alpha n}$$  \hspace{1cm} (3)

with $\alpha = \frac{h + \sqrt{h^2 + 4}}{2}$ and $c = \frac{1 + \alpha}{2\alpha - h} = \frac{1}{2} + \frac{h + 2}{2\sqrt{h^2 + 4}}$. 
Proof. Let $f_n(x) = hn - 1 + n^2/x$. Suppose that a sequence of positive numbers $r_n$ satisfies (2), i.e., that $f_n(r_{n+1}) = r_n$ for all $n \geq 1$. Then we have $r_n > hn - 1$ and thus $r_n < (h + 1/h)n$ for all $n \geq 1$. We deduce that there exists some $\delta > 0$ and $N \geq 1$ such that $r_n > (h + \delta)n$ for all $n \geq N$. Suppose that there is another sequence of positive numbers $\tilde{r}_n$ satisfying (2). Since $|f_n'(x)| = |n/x|^2 < 1/(h + \delta)$ for all $x > (h + \delta)n$, we have

$$|r_N - \tilde{r}_N| = |f_N f_{N+1} \cdots f_{n-1}(r_n) - f_N f_{N+1} \cdots f_{n-1}(\tilde{r}_n)|$$

$$< \frac{|r_n - \tilde{r}_n|}{(h + \delta)^{n-N}} < \frac{n/h}{(h + \delta)^{n-N}}$$

for all $n \geq N$, hence $r_N = \tilde{r}_N$, which implies that $r_n = \tilde{r}_n$ for all $n \geq 1$.

Next we show that

$$f_n(\alpha(n+1) - c) < an - c + \frac{(\alpha - c)(c - 1)}{\alpha n}$$

and

$$f_n(\alpha(n + 1) - c + \frac{(\alpha - c)(c - 1)}{(n+1)\alpha}) > an - c.$$ 

Indeed, using that $\alpha^2 = h\alpha + 1$ and $2ac - hc = 1 + \alpha$, we have

$$(\alpha n + \alpha - c) f_n(\alpha(n+1) - c) = (hn - 1)(\alpha n + \alpha - c) + n^2$$

$$= (h\alpha + 1)n^2 + (h\alpha - hc - \alpha)n - (\alpha - c)$$

$$< \alpha^2 n^2 + (\alpha^2 - 2ac)n - (\alpha - c) + \frac{(\alpha - c)^2(c - 1)}{\alpha n}$$

$$= (an + \alpha - c)(an - c + \frac{(\alpha - c)(c - 1)}{\alpha n}),$$

and

$$\left(\alpha n + \alpha - c + \frac{(\alpha - c)(c - 1)}{\alpha(n+1)}\right)(an - c)$$

$$< \alpha^2 n^2 + (\alpha^2 - 2ac)n - (\alpha - c) - \frac{c(\alpha - c)(c - 1)}{(\alpha n + 1)}$$

$$< (h\alpha + 1)n^2 + (h\alpha - hc - \alpha)n - (\alpha - c) - \frac{(\alpha - c)(c - 1)}{\alpha(n + 1)}$$

$$< (hn - 1)\left(\alpha n + \alpha - c + \frac{(\alpha - c)(c - 1)}{\alpha(n+1)}\right) + n^2$$

$$= \left(\alpha n + \alpha - c + \frac{(\alpha - c)(c - 1)}{\alpha(n+1)}\right) f_n(\alpha(n+1) - c + \frac{(\alpha - c)(c - 1)}{\alpha(n+1)}).$$

As $f_n$ is monotonically decreasing for $x > 0$, we deduce that

$$0 < f_n(x) - an + c < \frac{(\alpha - c)(c - 1)}{\alpha n}.$$
for all $x$ with $0 \leq x - \alpha(n + 1) + c \leq \frac{(\alpha-c)(c-1)}{\alpha(n+1)}$. Then we also have

$$0 < f_n f_{n+1} \cdots f_{n+k-1} (\alpha(n+k) - c + x) - \alpha n + c < \frac{(\alpha-c)(c-1)}{\alpha n}$$

for all $k, n \geq 1$, $0 \leq x - \alpha(n+k) + c \leq \frac{(\alpha-c)(c-1)}{\alpha(n+k)}$. As $f_n$ is contracting for $x \geq \alpha(n+1) - c$, the intervals $[f_1 f_2 \cdots f_n (\alpha(n+1) - c), f_1 f_2 \cdots f_n (\alpha(n+1) - c + \frac{(\alpha-c)(c-1)}{\alpha(n+1)})]$ converge to a point $r_1$. Then the numbers $r_n$ given by (2) satisfy (3) for all $n \geq 1$. By the first paragraph of the proof, this is the unique sequence of positive numbers satisfying (2).

Now consider when $\alpha n - c + \frac{1}{2}$ is close to $[\alpha n - c + \frac{1}{2}]$. Let $p_k/q_k$ be the convergents of the regular continued fraction $\alpha = [h; h, \ldots]$, i.e., $q_{-1} = 0$, $q_0 = 1$, $q_{k+1} = h q_k + q_{k-1}$ for $k \geq 1$, $p_k = q_k$. Then we have

$$q_k = \frac{\alpha^{k+1} + (-1)^k}{\alpha + 1/\alpha},$$

and thus

$$q_k \alpha - p_k = \frac{(-1)^k}{\alpha^{k+1}}. \tag{4}$$

**Lemma 4.** Let $h$ be a positive integer and $\alpha = \frac{h + \sqrt{h^2 + 4}}{2}$. Then we have

$$[\alpha n] - \alpha n = \begin{cases} j/\alpha^{2k} & \text{if } n = jq_{2k-1}, k \geq 1, 1 \leq j < \alpha^{2k}, \\ (\alpha - 1)/\alpha^{2k+1} & \text{if } n = q_{2k-1} + q_{2k}, k \geq 0, \\ (\alpha + 1)/\alpha^{2k+2} & \text{if } n = q_{2k+1} - q_{2k}, k \geq 0, \end{cases}$$

and $n([\alpha n] - \alpha n) \geq 1$ for all other $n \geq 1$.

**Proof.** The formulas for $n = jq_{2k-1}$, $n = q_{2k-1} + q_{2k}$ and $n = q_{2k+1} - q_{2k}$ are immediate from (4). By [7, Ch. 2, §5, Theorem 2], we have $n([\alpha n] - \alpha n) \geq 1$ for all $n \geq 1$ that are not of the form $jq_k$, $1 \leq j < \alpha/\sqrt{h}$, $q_k + q_{k-1}$ or $q_k - q_{k-1}$. Since $\alpha q_{2k} - [\alpha q_{2k}] = 1/\alpha^{2k+1}$, $\alpha(q_{2k} + q_{2k+1}) - [\alpha(q_{2k} + q_{2k+1})] = (\alpha - 1)/\alpha^{2k+2}$ and $\alpha(q_{2k} - q_{2k-1}) - [\alpha(q_{2k} - q_{2k-1})] = (\alpha + 1)/\alpha^{2k+1}$, we have $[\alpha n] - \alpha n > 1/2$ for $n = jq_{2k}$, $n = q_{2k} + q_{2k+1}$ and $n = q_{2k} - q_{2k-1}$. If moreover $n \geq 2$, then we have thus $n([\alpha n] - \alpha n) \geq 1$ for these $n$ as well. Since $q_0 = q_{-1} = 1$, the case $n = 1$ has already been treated.

We obtain that

$$n([\alpha n] - \alpha n) = \begin{cases} j^2(1 - 1/\alpha^{4k})/\sqrt{h^2 + 4} & \text{if } n = jq_{2k-1}, k \geq 1, 1 \leq j < \alpha^{2k}, \\ h - (\alpha - 1)^2/\alpha^{4k+2} & \text{if } n = q_{2k-1} + q_{2k}, k \geq 0, \\ h - (\alpha + 1)^2/\alpha^{4k+4} & \text{if } n = q_{2k+1} - q_{2k}, k \geq 0. \end{cases}$$
The worst case for \( n = q_{2k-1} + q_{2k} \) or \( n = q_{2k+1} - q_{2k} \) is given by \( n = q_{-1} + q_0 = 1 \), hence
\[
n([an] - an) \geq h + 1 - \alpha = 1 - \frac{1}{\alpha}
\]
for all \( n \geq 1 \) such that \( n \neq q_{2k-1} \) for all \( k \geq 1 \).

Now we come back to the case \( h = 2 \) and consider the distance of \( an - c + \frac{1}{2} \) to the nearest integer above \( an - c + \frac{1}{2} \). Note that \( c - \frac{1}{2} = \frac{1}{\sqrt{2}} \). We have
\[
2\left([an - \frac{1}{\sqrt{2}}] - an + \frac{1}{\sqrt{2}}\right) = 2\left([an - \frac{1}{\sqrt{2}}]\right) - 1 - \alpha(2n - 1)
\]
\[
\geq [\alpha(2n - 1)] - \alpha(2n - 1) > \frac{\alpha - 1}{2an},
\]
where we have used that \( q_{2k-1} \) is even for all \( k \geq 1 \); thus
\[
\left([an - \frac{1}{\sqrt{2}}]\right) - an + \frac{1}{\sqrt{2}} > \frac{\alpha - 1}{4an}.
\]
Since \( (\alpha - c)(c - 1) = \frac{1}{4\alpha} \), we have
\[
an - \frac{1}{\sqrt{2}} < r_n + \frac{1}{2} < an - \frac{1}{\sqrt{2}} + \frac{1}{4an} < an - \frac{1}{\sqrt{2}} + \frac{\alpha - 1}{4an} < \left[an - \frac{1}{\sqrt{2}}\right]
\]
for all \( n \geq 1 \), thus \( d_n = e_n \). This completes the proof of Theorem 1.

We remark that \( h = 2 \) cannot be replaced by an arbitrary positive integer in the previous paragraph. For example, for \( h = 1 \), we have \( \alpha = \frac{1 + \sqrt{5}}{2}, \ c = \frac{\alpha^2}{\sqrt{5}} \), \( [137\alpha - c + \frac{1}{2}] = 220 \) and \( [r_{137} + \frac{1}{2}] = 221 \). However, computer simulations suggest that (for any \( h \)) we always have \( [an - c] = [r_n] \).

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**References**


