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A REMARKABLE INTEGER SEQUENCE RELATED TO $\pi$ AND $\sqrt{2}$

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Abstract

We prove that five ways to define entry **A086377** in the OEIS do lead to the same integer sequence.

-Dedicated to Jeff Shallit on the occasion of his 60th birthday-

1. Introduction

In September of 2003 Benoit Cloitre contributed a sequence to the On-Line Encyclopedia of Integer Sequences [4], defined by him as $a_1 = 1$, and for $n \geq 2$ by

$$a_n = \begin{cases} 
  a_{n-1} + 2 & \text{if } n \text{ is in the sequence}, \\
  a_{n-1} + 2 & \text{if } n \text{ and } n - 1 \text{ are not in the sequence}, \\
  a_{n-1} + 3 & \text{if } n \text{ is not in the sequence, but } n - 1 \text{ is in the sequence}.
\end{cases}$$

(1)

The first 25 values of this sequence are

$$1, 4, 6, 8, 11, 13, 16, 18, 21, 23, 25, 28, 30, 33, 35, 37, 40, 42, 45, 47, 49, 52, 54, 57, 59.$$ 

The purpose of this paper is to prove the equivalence of five ways to define this integer sequence, most of them already conjecturally stated in the OEIS article on **A086377**. Besides a simplified recursion, the alternatives involve statements in terms of a morphic sequence, of a Beatty sequence, and of approximation properties linking a classical continued fraction of $\frac{4}{\pi}$ to that of $\sqrt{2}$.

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2. The Theorem

**Theorem 1.** The following five definitions produce the same integer sequence:

(a\_n) defined by \( a_1 = 1 \) and for \( n \geq 2 \):

\[
a_n = \begin{cases} 
  a_{n-1} + 2 & \text{if } n \text{ is in the sequence}, \\
  a_{n-1} + 2 & \text{if } n \text{ and } n-1 \text{ are not in the sequence}, \\
  a_{n-1} + 3 & \text{if } n \text{ is not in the sequence, but } n-1 \text{ is in the sequence}.
\end{cases}
\]

(b\_n) defined by \( b_1 = 1 \) and for \( n \geq 2 \):

\[
b_n = \begin{cases} 
  b_{n-1} + 2 & \text{if } n-1 \text{ is not in the sequence}, \\
  b_{n-1} + 3 & \text{if } n-1 \text{ is in the sequence}.
\end{cases}
\]

(c\_n) for \( n \geq 1 \) defined as the position of the \( n \)-th zero in the fixed point of the morphism 

\[
\phi : \begin{cases} 
  0 \mapsto 011 \\
  1 \mapsto 01
\end{cases}
\]

(d\_n) defined by \( d_n = \lfloor (1 + \sqrt{2}) \cdot n - \frac{1}{2} \sqrt{2} \rfloor \) for \( n \geq 1 \);

(e\_n) defined by \( e_n = \lfloor r_n \rfloor = \lfloor r_n + \frac{1}{2} \rfloor \), with \( r_1 = \frac{4}{\pi} \) and \( r_{n+1} = \frac{n^2}{r_n - (2n - 1)} \) for \( n \geq 1 \).

At first we found it hard to believe the equivalence of these definitions, but a verification of the first 130000 terms (\( a_{130000} = 313847 \)) convinced us to look for proofs.

3. Simplification and a Morphic Sequence

To show that \((b\_n)\) defines the same sequence as \((a\_n)\), simply note that \( a_n - a_{n-1} \geq 2 \) for all \( n \): hence if \( n \) is in the sequence then \( n-1 \) is not, and we can combine the first two cases in Equation (1).

In a comment to sequence [A086377](https://oeis.org/A086377), Clark Kimberling asked if the integers in this sequence coincide with the positions of the zeroes in sequence [A189687](https://oeis.org/A189687), which is the fixed point of the substitution

\[
\phi : \begin{cases} 
  0 \mapsto 011 \\
  1 \mapsto 01
\end{cases}
\]
defining the sequence \((c_n)\) in the Theorem. It is not hard to see that this indeed produces the same as sequence \((b_n)\); repeatedly applying the morphism \(\phi\) to 0 produces after a few steps the initial segment

\[01101010110110110101101101101011011010110110101011010101101011010101101010101101010101101010101010101010101011\ldots .\]

The position \(c_n\) of the \(n\)-th zero is 2 ahead of \(c_{n-1}\) precisely when the latter is followed by a single 1, that is, when there is a 1 at position \(n - 1\), and it is 3 ahead of \(c_{n-1}\) if that zero is followed by 11, which means that there was a 0 at position \(n - 1\). Thus the rule is exactly that defining \((b_n)\).

4. Beatty Sequence

Every pair of real numbers \(\alpha\) and \(\beta\) determines a Beatty sequence by

\[B(\alpha, \beta)_n := \lfloor n\alpha + \beta \rfloor, \quad n = 1, 2, \ldots .\]

The numbers \(\alpha\) and \(\beta\) also determine sequences by

\[\text{St}(\alpha, \beta)_n := \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor, \quad n = 1, 2, \ldots ,\]

which is a Sturmian sequence (of slope \(\alpha\)), over the alphabet \(\{0, 1\}\), provided that \(0 \leq \alpha < 1\).

Thus Sturmian sequences are first differences of Beatty sequences (when \(0 \leq \alpha < 1\), but Beatty sequences and Sturmian sequences are also linked in another way.

**Lemma 1.** Let \(\alpha > 1\) be irrational, and let \((s_n)_{n \geq 1}\) be given by \(s_n = \text{St}(\frac{1}{\alpha}, -\frac{\beta}{\alpha})_n\), for some real number \(\beta\) with \(\alpha + \beta > 1\) and such that \(k\alpha + \beta \notin \mathbb{Z}\) for all positive integers \(k\). Then \(B(\alpha, \beta)\) is the sequence of positions of 1 in \((s_n)\).

**Proof.** This is a generalization of Lemma 9.1.3 in [1], from homogeneous to inhomogeneous Sturmian sequences. The proof also generalizes:

- there exists \(k \geq 1\) : \(n = \lfloor k\alpha + \beta \rfloor\) if and only if
- there exists \(k \geq 1\) : \(n \leq k\alpha + \beta < n + 1\) if and only if
- there exists \(k \geq 1\) : \(\frac{n - \beta}{\alpha} \leq k < \frac{n + 1 - \beta}{\alpha}\) if and only if
- there exists \(k \geq 1\) : \(\left\lfloor \frac{n - \beta}{\alpha} \right\rfloor = k - 1\) and \(\left\lfloor \frac{n - \beta + 1}{\alpha} \right\rfloor = k\) if and only if
- \(\left\lfloor \frac{n + 1}{\alpha} - \frac{\beta}{\alpha} \right\rfloor - \left\lfloor \frac{n}{\alpha} - \frac{\beta}{\alpha} \right\rfloor = 1\) if and only if
- \(\text{St}(\frac{1}{\alpha}, -\frac{\beta}{\alpha})_n = 1\). \(\square\)
Our goal in this section is to prove that \((c_n) = (d_n)\). Let \(\psi\) be the morphism \(\psi : \{0 \rightarrow 10, 1 \rightarrow 100\}\), and let \(w\) be the fixed point. Then
\[
w = 1001010100101001010010010100101001010010100100\cdots,
\]
which is obtained by exchanging 0s and 1s in the fixed point of \(\phi\), i.e., \(\psi = E\phi E\), with \(E\) the exchange morphism given by \(E(0) = 1, E(1) = 0\). So the positions of 0 in the fixed point of \(\phi\) correspond to the positions of 1 in the fixed point \(w\) of \(\psi\).

Let \(\alpha_d = 1 + \sqrt{2}\) and \(\beta_d = -\frac{1}{2}\sqrt{2}\); then \(d_n = B(\alpha_d, \beta_d)n\), for \(n \geq 1\).

Applying Lemma 1, we deduce that \(d_n\) also equals the position of the \(n\)-th 1 in the Sturmian sequence \(St(\alpha, \beta)\), generated by
\[
\alpha = \frac{1}{\alpha_d} = \sqrt{2} - 1, \quad \beta = \frac{-\beta_d}{\alpha_d} = 1 - \frac{1}{2}\sqrt{2}.
\]

**Lemma 2.** \(St(\sqrt{2} - 1, 1 - \frac{1}{2}\sqrt{2}) = w\).

**Proof.** This was already proved by Nico de Bruijn in 1981 ([2]), where it is the main example. See also Lemma 2 in [6]. Note, however, that our Sturmian sequences start at \(n = 1\).

For a ‘modern’ proof as suggested by [3, Section 4], let \(\psi_1\) and \(\psi_2\) be the elementary morphisms given by \(\psi_1(0) = 01, \psi_1(1) = 0\), and \(\psi_2(0) = 10, \psi_2(1) = 0\). Then \(\psi = \psi_2\psi_1E\). This implies that the fixed point \(w\) of \(\psi\) is a Sturmian word (see [5, Corollary 2.2.19]). To find its parameters \((\alpha, \beta)\), use the 2D fractional linear maps that describe how the parameters of a Sturmian word change when one applies an elementary morphism. For Sturmian words starting at \(n = 0\), the maps for \(E, \psi_1\), and \(\psi_2\) are\(^2\) respectively (see [5, Lemma 2.2.17, Lemma 2.2.18, Exercise 2.2.6])
\[
T_0(x, y) = (1-x, 1-y), \quad T_1(x, y) = \left(\frac{1-x}{2-x}, \frac{1-y}{2-x}\right), \quad T_2(x, y) = \left(\frac{1-x}{2-x}, \frac{2-x-y}{2-x}\right).
\]
The change of parameters by applying \(\psi\) is therefore the composition
\[
T_{210}(x, y) := T_2T_1T_0(x, y) = \left(\frac{1}{2+x}, \frac{2+x-y}{2+x}\right).
\]

But the parameters \(\alpha\) and \(\beta\) of \(w\) do not change when one applies \(\psi\). This means that \((\alpha, \beta)\) is a fixed point of \(T_{210}\), and one easily computes \(\alpha = \sqrt{2} - 1\), and then \(\beta = \frac{1}{2}\sqrt{2}\). Since our Sturmian words start at \(n = 1\), we have to subtract \(\alpha\) from \(\beta\) and obtain that \(w = St(\sqrt{2} - 1, 1 - \frac{1}{2}\sqrt{2})\). \(\square\)

\(^2\)Actually there is a subtlety here involving the ceiling representation of a Sturmian sequence, but that does not apply in our case since \(\beta \notin \mathbb{Z}\alpha + \mathbb{Z}\).
5. Converging Recurrence

In a comment to entry A086377, Joseph Biberstine conjectured a beautiful connection with the infinite continued fraction expansion

\[
\frac{4}{\pi} = 1 + \frac{1^2}{3 + \frac{2^2}{5 + \frac{3^2}{7 + \frac{4^2}{9 + \frac{5^2}{11 + \ddots}}}}},
\]

derived from the arctangent function expansion. If we define \( R_n \) for \( n \geq 1 \) by

\[
R_n = 2n - 1 + \frac{n^2}{2n + 1 + \frac{(n+1)^2}{2n + 3 + \frac{(n+2)^2}{2n + 5 + \frac{(n+3)^2}{\ddots}}}},
\]

then \( R_1 = 4/\pi \) and \( R_n = 2n - 1 + \frac{n^2}{R_{n+1}} \). We see that

\[
\frac{R_n}{n} \frac{R_{n+1}}{n+1} - \frac{2n-1}{n} \frac{R_{n+1}}{n+1} - \frac{n^2}{n(n+1)} = 0.
\]

This implies that if \( R_n/n \) converges, for \( n \to \infty \), then it does so to a (positive) zero of \( x^2 - 2x - 1 \), that is, to \( 1 + \sqrt{2} \); cf. Lemma 3 below.

We consider now, conversely and slightly more generally, for any real \( h \geq 1 \), a sequence of positive numbers \( r_n \) satisfying

\[
r_n = hn - 1 + \frac{n^2}{r_{n+1}} \tag{2}
\]

for \( n \geq 1 \). We first show that this sequence is unique, i.e., there is a unique \( r_1 > 0 \) such that \( r_n > 0 \) for all \( n \geq 1 \), and give estimates for its terms.

**Lemma 3.** For each \( h \geq 1 \), there is a unique sequence of positive real numbers \( (r_n)_{n \geq 1} \) satisfying the recurrence (2). Moreover, we have for this sequence, for all \( n \geq 1 \),

\[
0 < r_n - \alpha n + c < \frac{(\alpha - c)(c - 1)}{\alpha n} \tag{3}
\]

with \( \alpha = \frac{h + \sqrt{h^2 + 4}}{2} \) and \( c = \frac{1 + \alpha}{2\alpha - h} = \frac{1}{2} + \frac{h + 2}{2\sqrt{h^2 + 4}} \).
Proof. Let \( f_n(x) = hn - 1 + n^2/x \). Suppose that a sequence of positive numbers \( r_n \) satisfies (2), i.e., that \( f_n(r_{n+1}) = r_n \) for all \( n \geq 1 \). Then we have \( r_n > hn - 1 \) and thus \( r_n < (h + 1/h)n \) for all \( n \geq 1 \). We deduce that there exists some \( \delta > 0 \) and \( N \geq 1 \) such that \( r_n > (h + \delta)n \) for all \( n \geq N \). Suppose that there is another sequence of positive numbers \( \tilde{r}_n \) satisfying (2). Since \( |f'_n(x)| = |n/x|^2 < 1/(h + \delta) \) for all \( x > (h + \delta)n \), we have

\[
|r_N - \tilde{r}_N| = |f_N f_{N+1} \cdots f_{n-1}(r_n) - f_N f_{N+1} \cdots f_{n-1}(\tilde{r}_n)|
< \frac{|r_n - \tilde{r}_n|}{(h + \delta)^n - N} < \frac{n/h}{(h + \delta)^n - N}
\]

for all \( n \geq N \), hence \( r_N = \tilde{r}_N \), which implies that \( r_n = \tilde{r}_n \) for all \( n \geq 1 \).

Next we show that

\[
f_n \left( \alpha(n+1) - c \right) < \alpha n - c + \frac{(\alpha - c)(c - 1)}{\alpha n}
\]

and

\[
f_n \left( \alpha(n+1) - c + \frac{(\alpha - c)(c - 1)}{(n+1)\alpha} \right) > \alpha n - c.
\]

Indeed, using that \( \alpha^2 = h\alpha + 1 \) and \( 2\alpha c - hc = 1 + \alpha \), we have

\[
(\alpha n + \alpha - c) f_n (\alpha(n+1) - c) = (hn - 1)(\alpha n + \alpha - c) + n^2
= (h\alpha + 1)n^2 + (h\alpha - hc - \alpha)n - (\alpha - c)
< \alpha^2 n^2 + (\alpha^2 - 2\alpha c)n - (\alpha - c) + \frac{(\alpha - c)^2(c - 1)}{\alpha n}
= (\alpha n + \alpha - c) \left( \alpha n - c + \frac{(\alpha - c)(c - 1)}{\alpha n} \right),
\]

and

\[
\left( \alpha n + \alpha - c + \frac{(\alpha - c)(c - 1)}{\alpha(n+1)} \right) (\alpha n - c)
< \alpha^2 n^2 + (\alpha^2 - 2\alpha c)n - (\alpha - c) - \frac{c(\alpha - c)(c - 1)}{\alpha(n+1)}
< (h\alpha + 1)n^2 + (h\alpha - hc - \alpha)n - (\alpha - c) - \frac{(\alpha - c)(c - 1)}{\alpha(n+1)}
< (hn - 1) \left( \alpha n + \alpha - c + \frac{(\alpha - c)(c - 1)}{\alpha(n+1)} \right) + n^2
= \left( \alpha n + \alpha - c + \frac{(\alpha - c)(c - 1)}{\alpha(n+1)} \right) f_n \left( \alpha(n+1) - c + \frac{(\alpha - c)(c - 1)}{\alpha(n+1)} \right),
\]

As \( f_n \) is monotonically decreasing for \( x > 0 \), we deduce that

\[
0 < f_n(x) - \alpha n + c < \frac{(\alpha - c)(c - 1)}{\alpha n}
\]
for all $x$ with $0 \leq x - \alpha(n + 1) + c \leq \frac{(\alpha-c)(c-1)}{\alpha(n+1)}$. Then we also have
\[
0 < f_n f_{n+1} \cdots f_{n+k-1} (\alpha(n+k) - c + x) - \alpha n + c < \frac{(\alpha-c)(c-1)}{\alpha n}
\]
for all $k, n \geq 1$, $0 \leq x - \alpha(n+k) + c \leq \frac{(\alpha-c)(c-1)}{\alpha(n+k)}$. As $f_n$ is contracting for $x \geq \alpha(n+1) - c$, the intervals $[f_1 f_2 \cdots f_n(\alpha(n+1) - c), f_1 f_2 \cdots f_n(\alpha(n+1) - c + (\alpha-c)(c-1)/\alpha(n+1))]$ converge to a point $r_1$. Then the numbers $r_n$ given by (2) satisfy (3) for all $n \geq 1$. By the first paragraph of the proof, this is the unique sequence of positive numbers satisfying (2).

Now consider when $\alpha n - c + \frac{1}{2}$ is close to $[\alpha n - c + \frac{1}{2}]$. Let $p_k/q_k$ be the convergents of the regular continued fraction $\alpha = [h; h, \ldots]$, i.e., $q_- = 0$, $q_0 = 1$, $q_{k+1} = h q_k + q_{k-1}$ for $k \geq 1$, $p_k = q_{k+1}$. Then we have
\[
q_k = \frac{\alpha^{k+1} + (-1)^k}{\alpha + 1/\alpha}
\]
and thus
\[
q_k \alpha - p_k = \frac{(-1)^k}{\alpha^{k+1}}.
\]

**Lemma 4.** Let $h$ be a positive integer and $\alpha = \frac{h + \sqrt{h^2 + 4}}{2}$. Then we have
\[
[\alpha n] - \alpha n = \begin{cases} 
\frac{j}{\alpha^{2k}} & \text{if } n = j q_{2k-1}, k \geq 1, 1 \leq j < \alpha^{2k}, \\
\frac{(\alpha-1)/\alpha^{2k+1}}{\alpha^{2k+1}} & \text{if } n = q_{2k-1} + q_{2k}, k \geq 0, \\
\frac{(\alpha+1)/\alpha^{2k+2}}{\alpha^{2k+2}} & \text{if } n = q_{2k+1} - q_{2k}, k \geq 0,
\end{cases}
\]
and $n([\alpha n] - \alpha n) \geq 1$ for all other $n \geq 1$.

*Proof.* The formulas for $n = j q_{2k-1}$, $n = q_{2k-1} + q_{2k}$ and $n = q_{2k+1} - q_{2k}$ are immediate from (4). By [7, Ch. 2, §5, Theorem 2], we have $n([\alpha n] - \alpha n) \geq 1$ for all $n \geq 1$ that are not of the form $j q_k$, $1 \leq j < \alpha/\sqrt{h}$, $q_k + q_{k-1}$ or $q_k - q_{k-1}$. Since $\alpha q_{2k} = [\alpha q_{2k}] = 1/\alpha^{2k+1}, \alpha(q_{2k} + q_{2k+1}) - [\alpha(q_{2k} + q_{2k+1})] = (\alpha-1)/\alpha^{2k+2}$ and $\alpha(q_{2k} - q_{2k-1}) - [\alpha(q_{2k} - q_{2k-1})] = (\alpha+1)/\alpha^{2k+1}$, we have $[\alpha n] - \alpha n > 1/2$ for $n = j q_{2k}, n = q_{2k} + q_{2k+1}$ and $n = q_{2k} - q_{2k-1}$. If moreover $n \geq 2$, then we have thus $n([\alpha n] - \alpha n) \geq 1$ for these $n$ as well. Since $q_0 + q_{-1} = 1$, the case $n = 1$ has already been treated.

We obtain that
\[
n([\alpha n] - \alpha n) = \begin{cases} 
\frac{j^2(1 - 1/\alpha^{2k})}{\sqrt{h^2 + 4}} & \text{if } n = j q_{2k-1}, k \geq 1, 1 \leq j < \alpha^{2k}, \\
\frac{h - (\alpha-1)^2/\alpha^{4k+2}}{\sqrt{h^2 + 4}} & \text{if } n = q_{2k-1} + q_{2k}, k \geq 0, \\
\frac{h - (\alpha+1)^2/\alpha^{4k+4}}{\sqrt{h^2 + 4}} & \text{if } n = q_{2k+1} - q_{2k}, k \geq 0.
\end{cases}
\]
The worst case for \( n = q_{2k-1} + q_{2k} \) or \( n = q_{2k+1} - q_{2k} \) is given by \( n = q_{-1} + q_0 = 1 \), hence
\[
n([an] - an) > h + 1 - \alpha = 1 - \frac{1}{\alpha}
\]
for all \( n \geq 1 \) such that \( n \neq q_{2k-1} \) for all \( k \geq 1 \).

Now we come back to the case \( h = 2 \) and consider the distance of \( an - c + \frac{1}{2} \) to the nearest integer above \( an - c + \frac{1}{2} \). Note that \( c - \frac{1}{2} = \frac{1}{\sqrt{2}} \). We have
\[
2\left(\left\lfloor an - \frac{1}{\sqrt{2}} \right\rfloor - an + \frac{1}{\sqrt{2}}\right) = 2\left\lfloor an - \frac{1}{\sqrt{2}} \right\rfloor - 1 - \alpha(2n - 1) \\
\geq \left\lfloor \alpha(2n - 1) \right\rfloor - \alpha(2n - 1) > \frac{\alpha - 1}{2\alpha n},
\]
where we have used that \( q_{2k-1} \) is even for all \( k \geq 1 \); thus
\[
\left\lfloor an - \frac{1}{\sqrt{2}} \right\rfloor - an + \frac{1}{\sqrt{2}} > \frac{\alpha - 1}{4\alpha n}.
\]
Since \( (\alpha - c)(c - 1) = \frac{1}{4\alpha} \), we have
\[
an - \frac{1}{\sqrt{2}} < r_n + \frac{1}{2} < an - \frac{1}{\sqrt{2}} + \frac{1}{4\alpha n} < an - \frac{1}{\sqrt{2}} + \frac{\alpha - 1}{4\alpha n} < \left\lfloor an - \frac{1}{\sqrt{2}} \right\rfloor
\]
for all \( n \geq 1 \), thus \( d_n = e_n \). This completes the proof of Theorem 1.

We remark that \( h = 2 \) cannot be replaced by an arbitrary positive integer in the previous paragraph. For example, for \( h = 1 \), we have \( \alpha = \frac{1 + \sqrt{5}}{2}, \ c = \frac{\alpha^2}{\sqrt{5}}, \ [137\alpha - c + \frac{1}{2}] = 220 \) and \( [r_{137} + \frac{1}{2}] = 221 \). However, computer simulations suggest that (for any \( h \)) we always have \( [an - c] = [r_n] \).

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