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Sequential Convex Relaxation for Robust Static Output Feedback Structured Control ⋆

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Abstract: We analyse the very general class of uncertain systems that have Linear Fractional Representations (LFRs), and uncertainty blocks in a convex set with a finite number of vertices. For these systems we design static output feedback controllers. In the general case, computing a robust static output feedback controller with optimal performance gives rise to a bilinear matrix inequality (BMI). In this article we show how this BMI problem can be efficiently rewritten to fit in the framework of sequential convex relaxation, a method that searches simultaneously for a feasible controller and one with good performance. As such, our approach does not rely on being supplied with a feasible initial solution to the BMI. This sets it apart from methods that depend on a good initial, feasible starting point to progress from there using an alternating optimization scheme. In addition to using the proposed method, the controller matrices can be of a predetermined fixed structure. Alternatively, an ℓ_1 constraint can be easily added to the optimization problem as a convex variant of a cardinality constraint, in order to induce sparsity on the controller matrices.

Keywords: Output feedback control, Robust control, Robust controller synthesis, Relaxations

1. INTRODUCTION

There are several advantages of feedback control with a static gain between measurements and inputs. Examples are the simplicity of the implementation or perhaps economic advantages that come along with the absence of controller dynamics that have to be computed. Especially if the feedback matrix is sparse, the implementation features only connections between certain inputs and outputs of the system. Hence, there is active research interest in structured (output) feedback control. For a recent comprehensive review, see (Sadabadi and Peaucelle, 2016).

For example, in (Lin et al., 2013; Lin, 2012) optimal sparse *state* feedback controllers are computed, see (Jovanović and Dhingra, 2016) for a recent overview. However, if the system states are unavailable, these methods are not applicable.

The problem of sparse static *output* feedback is analysed in (Arastoo et al., 2014). Using a reformulation of the problem into a rank constraint problem, they find an optimal controller using the Alternating Direction Method of Multipliers. Their method is flexible enough to handle constraints on the input and output signal norms. (Arastoo et al., 2015) uses a similar rank-constrained reformulation as (Arastoo et al., 2014), but assumes that there is no measurement noise and a stabilizing (non-sparse) controller has already been computed.

The previously mentioned methods do not consider the case were there is uncertainty present in the system. In (Dong and Yang, 2013) robust static output feedback was considered for systems whose system matrices lie in a convex polytope, an important class of uncertain systems. However, measurement noise was not included in the analysis. In another recent paper, (Chang et al., 2015), this measurement noise was included in an analysis for the same type of uncertain systems. Their resulting algorithm was able to compute output feedback controllers, but not able to impose a structure on the controller matrix.

We will analyse systems with a Linear Fractional Representation (LFR), and an uncertainty block in some convex polytope. This is a very general class of systems.

Analysis of the stability of this system and the $\mathcal{H}_\infty$ norm of the transfer function reveals that the optimization problem is a Linear Matrix Inequality (LMI) when the controller is known, but a BMI if a controller has to be found as well. Computing a solution to such a BMI is NP-hard in general (Toker and Özbay, 1995). In special cases computing a solution to the output feedback control problem becomes a combination of LMI's and grid search over a few parameters, (Dong and Yang, 2013; Chang et al., 2015), or an LMI when some parameters are assumed to be known (Xu and Chen, 2004). If a feasible solution to the BMI is known, one could do alternating optimization to find a controller with improved performance, (Iwasaki, 1999),

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but the resulting solution may not be globally optimal. If no feasible controller is known that robustly stabilizes the system, one could start by searching for a controller that stabilizes the nominal system, and during alternating optimization slightly increase the uncertainty set until a feasible solution is found (Massioni, 2014).

The main contribution of this paper is a convex relaxation of the robust static output feedback structured control problem. A proposed sequential algorithm uses this convex relaxation to optimize closed-loop performance in an iterative manner. Our approach does not assume a feasible solution is known in advance, nor do we fix variables during the optimizations in our proposed solution. This allows us to constrain the controller matrix to be in an arbitrary convex set. A fixed structure of the controller matrix is one example of such a set, inducing sparsity with ℓ_1 norm regularization is another. Our different approach to solving the BMI problem might not only return a feasible solution to the BMI, but also a better solution than could be found using alternating minimization from a known solution as described above. This is an advantage compared to standard BMI solvers for structured control, where the choice of initial guess is crucial (Sadabadi and Peaucelle, 2016).

The organization of this article is as follows. In Section 2 we discuss the system types and controller types of interest. Section 3 explains how a bilinear equality constraint can be relaxed in a sequential manner. We then show in Section 4 how the robust static output feedback controller problem can be written as an optimization problem subject to such a constraint. In Section 5 an example from (Chang et al., 2015) is analysed and we show that for this example our approach outperforms the one in (Chang et al., 2015) *and* allows for structured control analysis.

The specific notation is as follows. We use subscripts to indicate dimensions of certain matrices, $0_{m\times n}$ and I_n are respectively a zero matrix of size $m \times n$ and an identity matrix of size $n \times n$. Subscripts with parentheses denote matrix elements: $X_{(1,2)}$ is the element on the first row, second column of the matrix X.

2. ROBUST STATIC OUTPUT FEEDBACK CONTROL

We are interested in static output feedback for the continuous time system Σ depicted in Figure 1, with a system description as follows

$$\Sigma : \begin{pmatrix} \dot{x}(t) \\ q(t) \\ z(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B_p & B_w & B_u \\ C_q & D_{qp} & D_{qw} & D_{qu} \\ C_z & D_{zp} & D_{zw} & D_{zu} \\ C_y & D_{yp} & D_{yw} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \\ w(t) \\ u(t) \end{pmatrix}, \qquad (1)$$
$$p(t) = \Delta q(t),$$

where the uncertainty Δ is an element in the convex hull of the vertices in the set $\{\Delta_1, \ldots, \Delta_n\}$, $\Delta_1 = 0$. $x(t) \in \mathbb{R}^{n_s}$ is the system state, $w(t) \in \mathbb{R}^{m_w}$ is a disturbance, $u(t) \in \mathbb{R}^{m_u}$ is the input, $z(t) \in \mathbb{R}^{r_z}$ is the output, $y(t) \in \mathbb{R}^{r_y}$ is the available measurement, and $p(t) \in \mathbb{R}^{m_p}$, $q(t) \in \mathbb{R}^{r_q}$ are the signals used to describe how the uncertainty Δ influences the system dynamics. This is a general Linear Fractional Representation (LFR) of a dynamical system.

This system is stable, and has an $\mathcal{H}_\infty$ norm of the transfer function of performance channel $w \to z$ lower than γ, if the

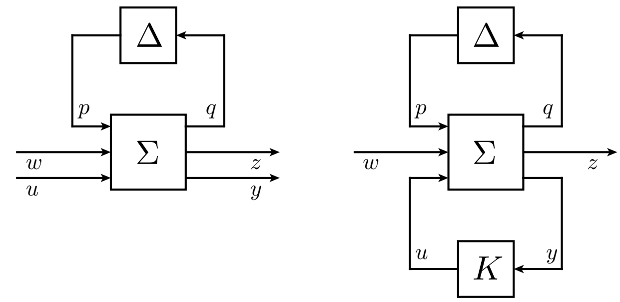

Fig. 1. A schematic depiction of the system under consideration with indicated uncertainty block Δ (left) and including controller K (right).

LMI's in the following lemma are feasible. The methods we propose can be generalized to other quadratic performance criteria. This lemma uses a full-block multiplier P,

$$P = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix},$$

to guarantee robust stability and performance, see for example (Scherer and Weiland, 2000).

Lemma 1. The Full-block S-procedure (Scherer, 2001). The system Σ in (1) is robustly stable and the transfer function $T_{w\to z}(j\omega)$ has an $\mathcal{H}_\infty$ norm less than γ if there exist a $Q = Q^T \in \mathbb{R}^{m_p \times m_p}$, $S \in \mathbb{R}^{m_p \times r_q}$, $R = R^T \in \mathbb{R}^{r_q \times r_q}$, $Y = Y^T \in \mathbb{R}^{n_s \times n_s}$ and $\gamma^2 \in \mathbb{R}$ such that the following LMI's are feasible:

$$Q \prec 0, \ Y \succ 0, \ R \succ 0,$$
$$\begin{pmatrix} I \\ -\Delta_i^T \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I \\ -\Delta_i^T \end{pmatrix} \prec 0, \quad i = 1, \ldots, n,$$
$$\begin{pmatrix} G \\ I \end{pmatrix}^T \begin{pmatrix} L(Q) & W(Y,S) \\ W(Y,S)^T & N(R,\gamma^2) \end{pmatrix} \begin{pmatrix} G \\ I \end{pmatrix} \succ 0,$$

where the following definitions and abbreviations are used:

$$G^T := -\begin{pmatrix} A & B_p & B_w \\ C_q & D_{qp} & D_{qw} \\ C_z & D_{zp} & D_{zw} \end{pmatrix},$$
$$L(Q) := \begin{pmatrix} 0_{n_s \times n_s} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & -I_{m_w} \end{pmatrix},$$
$$W(Y,S) := \begin{pmatrix} Y & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0_{m_w \times r_z} \end{pmatrix},$$
$$N(R,\gamma^2) := \begin{pmatrix} 0_{n_s \times n_s} & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & \gamma^2 I_{r_z} \end{pmatrix}.$$

For structured static output feedback we are interested in feedback of the form

$$u(t) = Ky(t), \qquad (2)$$

see also Figure 1. For any algorithm for structured output feedback control, it is desirable to allow for influence on the structure of the feedback matrix K. We denote the convex set of feedback gains that we are interested in as $\mathcal{K}$. The structure of K could be a prescribed structure or one that is partly determined by optimization, see (Lin, 2012; Jovanović and Dhingra, 2016). For example, if K is sparse, this would have advantages in terms of implementation. Only selected outputs need to be connected to selected inputs, which could lead to economic advantages. Or if we have a set of index pairs $\bar{I} = \{(i_1, j_1), \ldots, (i_n, j_n)\}$ of controller elements that should be equal to zero (a fixed controller structure), our set $\mathcal{K}$ would be

$$\mathcal{K}_{\text{fixed str.}} = \left\{K \ : \ K_{(i,j)} = 0, \ (i,j) \in \bar{I}\right\}.$$

If $\bar{I} = \emptyset$, then $\mathcal{K}_{\text{fixed str.}} = \mathcal{K}_{\text{no str.}} = \mathbb{R}^{m_u \times r_y}$. Another example is a controller with induced sparsity using an ℓ_1 norm, like in (Tibshirani, 1996):

$$\mathcal{K}_{\text{sparse}} = \left\{K \ : \ \sum_{i,j} |K_{(i,j)}| \leq \tau \right\}$$

for some $\tau > 0$.

The final interesting structure we mention would be an empty row or empty column of the matrix K, indicating that either an actuator or sensor respectively is not a design necessity for robust stabilization and performance, see for example (Dhingra et al., 2014).

Applying control law (2) to system (1) gives the closed-loop system description

$$\begin{aligned}
\begin{pmatrix} \dot{x}(t) \\ q(t) \\ z(t) \end{pmatrix} &= \begin{pmatrix} \mathcal{A} & \mathcal{B}_p & \mathcal{B}_w \\ \mathcal{C}_q & \mathcal{D}_{qp} & \mathcal{D}_{qw} \\ \mathcal{C}_z & \mathcal{D}_{zp} & \mathcal{D}_{zw} \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \\ w(t) \end{pmatrix}, \\
\begin{pmatrix} \mathcal{A} \\ \mathcal{C}_q \\ \mathcal{C}_z \end{pmatrix} &= \begin{pmatrix} A + B_u K C_y \\ C_q + D_{qu} K C_y \\ C_z + D_{zu} K C_y \end{pmatrix}, \\
\begin{pmatrix} \mathcal{B}_p \\ \mathcal{D}_{qp} \\ \mathcal{D}_{zp} \end{pmatrix} &= \begin{pmatrix} B_p + B_u K D_{yp} \\ D_{qp} + D_{qu} K D_{yp} \\ D_{zp} + D_{zu} K D_{yp} \end{pmatrix}, \\
\begin{pmatrix} \mathcal{B}_w \\ \mathcal{D}_{qw} \\ \mathcal{D}_{zw} \end{pmatrix} &= \begin{pmatrix} B_w + B_u K D_{yw} \\ D_{qw} + D_{qu} K D_{yw} \\ D_{zw} + D_{zu} K D_{yw} \end{pmatrix}, \\
p(t) &= \Delta q(t),
\end{aligned} \tag{3}$$

where Δ is a convex combination of the vertices in the set $\{\Delta_1, \ldots, \Delta_n\}$.

If K is a decision variable, then substituting the matrices in (3) into the LMI's of Lemma 1 results in a BMI. In the next section we discuss how such a BMI problem can be transformed and relaxed to a convex problem.

3. SEQUENTIAL CONVEX RELAXATION OF BILINEAR EQUALITY CONSTRAINTS

In (Doelman and Verhaegen, 2016) the following optimization problem was analyzed:

$$\begin{aligned}
\min_{x, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}} \quad & f(x, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \\
s.t. \quad & g(x, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \succeq 0, \\
& \boldsymbol{APB} = \boldsymbol{C},
\end{aligned} \tag{4}$$

where x is a decision variable appearing affinely in the problem, $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are decision variables in matrix form that appear affinely in f and g, and the bilinearity is contained in the equality constraint $\boldsymbol{APB} = \boldsymbol{C}$. The matrix $\boldsymbol{P}$ is not a decision variable, but can be any (non-zero) matrix of appropriate dimensions. In general, such a bilinearity causes the problem to be NP-hard (Toker and Özbay, 1995).

There are two problems with the last constraint. First, there are the bilinearly appearing decision variables $\boldsymbol{A}$ and $\boldsymbol{B}$. The second problem is the equality constraint, which cannot just be relaxed: otherwise the solution to a relaxed problem is not a solution to the original problem. The constraint can be transformed in such a way that it turns the bilinear constraint into an equivalent rank constraint.

Require: A randomly chosen $X_{1,0}$ and $X_{2,0}$, regularization parameter λ, iterator $i = 0$.
while not converged **do**
Minimize (6) using the matrix

$$M(\boldsymbol{A}_i, \boldsymbol{P}, \boldsymbol{B}_i, \boldsymbol{C}_i, X_{1,i}, X_{2,i})$$

Using the optimal values for $\boldsymbol{A}_i^*$ and $\boldsymbol{B}_i^*$ in the previous step, set

$$X_{1,i+1} \leftarrow -\boldsymbol{A}_i^*, \quad X_{2,i+1} \leftarrow -\boldsymbol{B}_i^*,$$

$i \leftarrow i + 1$.
end while

Algorithm 1. The sequential convex relaxation algorithm.

Lemma 2. Rank equivalence (Doelman and Verhaegen, 2016) The constraint $\boldsymbol{APB} = \boldsymbol{C}$ is equivalent to the rank constraint

$$\operatorname{rank}\left(M(\boldsymbol{A}, \boldsymbol{P}, \boldsymbol{B}, \boldsymbol{C}, X_1, X_2)\right) = \operatorname{rank}\left(\boldsymbol{P}\right), \tag{5}$$

where $M(\cdot)$ is defined as

$$M(\boldsymbol{A}, \boldsymbol{P}, \boldsymbol{B}, \boldsymbol{C}, X_1, X_2) := \begin{pmatrix} \boldsymbol{C} + X_1 \boldsymbol{P} X_2 + \boldsymbol{AP} X_2 + X_1 \boldsymbol{PB} & (\boldsymbol{A} + X_1)\boldsymbol{P} \\ \boldsymbol{P}(\boldsymbol{B} + X_2) & \boldsymbol{P} \end{pmatrix}$$

for *any* matrices $X_1, \ X_2$ of appropriate size.

The variables $\boldsymbol{A}$ and $\boldsymbol{B}$ no longer appear bilinearly in the matrix M. Instead of the equality constraint, we now have a rank constraint.

The relaxation of (4) uses the nuclear norm to induce solutions with a low rank matrix $M(\cdot)$:

$$\begin{aligned}
\min_{x, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}} \quad & f(x, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) + \lambda \left\| M(\boldsymbol{A}, \boldsymbol{P}, \boldsymbol{B}, \boldsymbol{C}, X_1, X_2) \right\|_* \\
s.t. \quad & g(x, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \succeq 0,
\end{aligned} \tag{6}$$

where λ is a regularization parameter. Let a superscript * denote the optimal value of a parameter for the convex problem. If after solving the convex problem (6), we check the rank of $M(\boldsymbol{A}^*, \boldsymbol{P}, \boldsymbol{B}^*, \boldsymbol{C}^*, X_1, X_2)$ and have $\operatorname{rank}\left(M(\cdot)\right) = \operatorname{rank}\left(\boldsymbol{P}\right)$, then this solution is a feasible solution for the two constraints in (4). We expect the matrix $M(\cdot)$ to not be of full rank, due to the rank minimizing property of the nuclear norm.

As mentioned before, the matrices X_1 and X_2 can be any matrices of appropriate size. The recommended way to choose their values is in an iterative manner, using the steps outlined in Algorithm 1, see also (Doelman and Verhaegen, 2016). For a version of Algorithm 1 that incorporates additional constraints on the decision variables, it can be shown that the value of

$$f(x_i, \boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{C}_i) + \lambda \left\| \boldsymbol{C}_i - \boldsymbol{A}_i \boldsymbol{P} \boldsymbol{B}_i \right\|_* \tag{7}$$

converges. However, this modified version is often outperformed by Algorithm 1. Convergence of (7) does not guarantee a convergence of $\left\| \boldsymbol{C}_i - \boldsymbol{A}_i \boldsymbol{P} \boldsymbol{B}_i \right\|_*$ to 0, and that (4) is solved. A feasible solution for (4) is found if and only if for any iteration of Algorithm 1 (5) holds for the optimal $\boldsymbol{A}_i^*, \boldsymbol{B}_i^*$ and $\boldsymbol{C}_i^*$.

4. ROBUST STATIC OUTPUT FEEDBACK AS A BILINEARLY CONSTRAINED PROBLEM

If we straightforwardly substitute the closed loop system matrices of (3) into the LMI's in Lemma 1, then for a known feedback matrix K this results again in LMI's, but for a decision variable K there will be three terms where decision variables appear bilinearly. Notice that using a Schur complement argument, the following two inequalities are equivalent.

$$\begin{pmatrix}\mathcal{G}(K)\\ I\end{pmatrix}^T\begin{pmatrix}L(Q) & W(Y,S)\\ W^T(Y,S) & N(R,\gamma^2)\end{pmatrix}\begin{pmatrix}\mathcal{G}(K)\\ I\end{pmatrix}\succ 0$$
$$\Leftrightarrow$$
$$\begin{pmatrix}-\bar{L} & \bar{L}\bar{\mathcal{G}}\\ \bar{\mathcal{G}}^T\bar{L} & N+W^T\mathcal{G}+\mathcal{G}^TW\end{pmatrix}\succ 0.$$

where $\mathcal{G}^T, \bar{L}$ and $\bar{\mathcal{G}}^T$ are defined as follows

$$\mathcal{G}^T(K):=-\begin{pmatrix}\mathcal{A} & \mathcal{B}_p & \mathcal{B}_w\\ \mathcal{C}_q & \mathcal{D}_{qp} & \mathcal{D}_{qw}\\ \mathcal{C}_z & \mathcal{D}_{zp} & \mathcal{D}_{zw}\end{pmatrix},$$
$$\bar{L}(Q):=\begin{pmatrix}Q & 0\\ 0 & -I_{m_w}\end{pmatrix},$$
$$\bar{\mathcal{G}}^T(K):=-\begin{pmatrix}\mathcal{B}_p & \mathcal{B}_w\\ \mathcal{D}_{qp} & \mathcal{D}_{qw}\\ \mathcal{D}_{zp} & \mathcal{D}_{zw}\end{pmatrix}.$$

Working out the product $\bar{\mathcal{G}}^T(K)\bar{L}(Q)$ gives us

$$\begin{aligned}&-\begin{pmatrix}\mathcal{B}_p & \mathcal{B}_w\\ \mathcal{D}_{qp} & \mathcal{D}_{qw}\\ \mathcal{D}_{zp} & \mathcal{D}_{zw}\end{pmatrix}\begin{pmatrix}Q & 0\\ 0 & -I_{m_w}\end{pmatrix}\\ &=-\begin{pmatrix}B_pQ+B_uKD_{yp}Q & -B_w-B_uKD_{yw}\\ D_{qp}Q+D_{qu}KD_{yp}Q & -D_{qw}-D_{qu}KD_{yw}\\ D_{zp}Q+D_{zu}KD_{yp}Q & -D_{zw}-D_{zu}KD_{yw}\end{pmatrix},\end{aligned}\tag{8}$$

with the bilinear term $E_2 := KD_{yp}Q$. For the term $\mathcal{G}(K)^TW(Y,S)$ we obtain

$$\begin{aligned}&-\begin{pmatrix}\mathcal{A} & \mathcal{B}_p & \mathcal{B}_w\\ \mathcal{C}_q & \mathcal{D}_{qp} & \mathcal{D}_{qw}\\ \mathcal{C}_z & \mathcal{D}_{zp} & \mathcal{D}_{zw}\end{pmatrix}\begin{pmatrix}Y & 0 & 0\\ 0 & S & 0\\ 0 & 0 & 0_{m_w\times r_z}\end{pmatrix}\\ &=-\begin{pmatrix}AY+B_uKC_yY & B_pS+B_uKD_{yp}S & 0\\ C_qY+D_{qu}KC_yY & D_{qp}S+D_{qu}KD_{yp}S & 0\\ C_zY+D_{zu}KC_yY & D_{zp}S+D_{zu}KD_{yp}S & 0\end{pmatrix}\end{aligned}\tag{9}$$

with the bilinear terms $E_1 := KC_yY$ and $E_3 := KD_{yp}S$.

We have three bilinear terms that can be collected into the single bilinear constraint

$$\underbrace{\begin{pmatrix}K & K\end{pmatrix}}_{\boldsymbol{A}}\underbrace{\begin{pmatrix}C_y & 0\\ 0 & D_{yp}\end{pmatrix}}_{\boldsymbol{P}}\underbrace{\begin{pmatrix}Y & 0 & 0\\ 0 & Q & S\end{pmatrix}}_{\boldsymbol{B}}=\underbrace{\begin{pmatrix}E_1 & E_2 & E_3\end{pmatrix}}_{\boldsymbol{C}},\tag{10}$$

with the understanding that E_1, E_2 and E_3 are substituted for the corresponding bilinear terms in equations (8) and (9).

The inequalities of Lemma 1 with the closed loop system matrices of (3) can thus be written in the form of (4) through the use of additional variables E_1, E_2 and E_3 and by using the bilinear equality constraint in (10), i.e. the expressions

$$\begin{cases}\bar{\mathcal{G}}^T(K)\bar{L}(Q),\\ \mathcal{G}(K)^TW(Y,S)\end{cases}$$

and

$$\begin{cases}\bar{\mathcal{G}}^T\bar{L}(E_2,Q),\\ \mathcal{G}^TW(Y,E_1,S,E_3),\\ \boldsymbol{APB}=\boldsymbol{C}\end{cases}$$

are equivalent.

The full BMI problem for robust static output feedback structured control is now

$$\begin{aligned}\min_{Q,S,R,Y,K,E_{1,2,3}}\quad & \gamma^2\\ s.t.\quad & Q\prec 0,\ Y\succ 0,\ R\succ 0,\\ & \begin{pmatrix}I\\ -\Delta_i^T\end{pmatrix}^T\begin{pmatrix}Q & S\\ S^T & R\end{pmatrix}\begin{pmatrix}I\\ -\Delta_i^T\end{pmatrix}\prec 0,\\ & \quad i=1,\ldots,n,\\ & \begin{pmatrix}-\bar{L} & \bar{L}\bar{\mathcal{G}}\\ \bar{\mathcal{G}}^T\bar{L} & N+W^T\mathcal{G}+\mathcal{G}^TW\end{pmatrix}\succ 0,\\ & K\in\mathcal{K},\\ & \boldsymbol{APB}=\boldsymbol{C}.\end{aligned}\tag{11}$$

If Algorithm 1 is applied to find a controller K that robustly stabilizes the system and performs (locally) optimal, the extension of the unstructured static output feedback control problem, where $\mathcal{K}=\mathcal{K}_{\text{no str.}}$, to structured control is trivial. One simply changes the set of allowed controllers $\mathcal{K}$ to the set of interest. The ease with which different variants of the structured control problem can be analysed, indicates that our approach of the problem, using Algorithm 1, is very generic. For the sake of brevity we will analyse in the next section two cases: we will compute a $K\in\mathcal{K}_{\text{no str.}}$ and a $K\in\mathcal{K}_{\text{fixed str.}}$ that both stabilize the same uncertain system.

We would like to note that for discrete-time systems the derivation of the bilinear equality constraint problem is mostly similar, but due to space constraints we will not include this analysis here.

5. NUMERICAL EXAMPLE

To demonstrate the capability of Algorithm 1 to find a structured controller with robust performance, we analyse the example problem in (Chang et al., 2015) (Example 1, Section 4), where the same problem was used as in (Benton and Smith, 1999). In the referred article systems are considered with polytopic uncertainties. The system matrices are convex combinations of matrices, determined by the same parameter. The numerical example in (Chang et al., 2015) gives system matrices in a convex set with 2 vertices, an example that can be shown to fit into the system description of (1). To be concrete, the system matrices of the example problem (subscript ep) are of the following form:

$$\begin{aligned}A_{ep}&=(1-\alpha)A_1+\alpha A_2,\\ B_{w,ep}&=(1-\alpha)B_{w,1}+\alpha B_{w,2},\\ B_{u,ep}&=\ldots\end{aligned}$$

for an unknown $\alpha\in[0,1]$. For numerical values of $A_1, A_2, \ldots$, please see (Chang et al., 2015).

This is equivalent to a system of the form (1) by taking

$$q(t)=\begin{pmatrix}x(t)\\ w(t)\\ u(t)\end{pmatrix},$$

$\Delta = \alpha I_{m_p}$ and thus $\Delta \in \operatorname{conv}\left(\{0, I_{m_p}\}\right)$. The system matrices in (1) are

$$\begin{aligned} A &= A_1, \\ B_w &= B_{w,1}, \\ B_u &= B_{u,1}, \\ B_p &= \left(A_2 - A_1 \;\; B_{w,2} - B_{w,1} \;\; B_{u,2} - B_{u,1}\right), \end{aligned}$$

and the rest of the system matrices are defined similarly.

For this example we have $n_s = 4, m_p = 6, m_w = 1, m_u = 1, r_q = 6, r_z = 1, r_y = 2$. The feedback matrix K is therefore of dimension $m_u \times r_y = 1 \times 2$.

The controller in (Chang et al., 2015),

$$K_1 = (-2.7375 \;\; -0.8618)$$

gives according to the article a worst-case $\mathcal{H}_\infty$ norm of $\gamma_1 = 0.6581$.

Applying Lemma 1 to the reformulated system description in this section, and using controller K_1, we obtain a lower bound on the worst-case $\mathcal{H}_\infty$ norm of $\hat{\gamma}_1 = 0.65308$. This difference could be due to using a less conservative LMI formulation to guarantee robust performance.

Applying Algorithm 1 to solve the problem with the bilinear equality constraint (11) that we derived in the previous section, we obtain the following.

The unstructured static output feedback controller

$$K_2 = (-2.1609 \;\; -0.9597)$$

has an upper bound on the $\mathcal{H}_\infty$ norm of $\gamma_2 = 0.64189$. This is a better guaranteed performance than γ_1.

Remarkably, the second measurement channel ($r_y = 2$ in this example) is hardly necessary for robust performance. The controller

$$K_3 = (-0.9505 \;\; 0)\,,$$

designed by adding the constraint $K_{(1,2)} = 0$ to the optimization problem, has a guaranteed performance of $\gamma_3 = 0.71123$. On the other hand, we were not able to find a controller in the form $K = \left(0 \;\; K_{(1,2)}\right)$ with comparable performance using Algorithm 1.

The optimal γ and constraint violation $\|\boldsymbol{APB} - \boldsymbol{C}\|_F$ during the iterations of Algorithm 1 can be found in Figure 2. As can be seen in this figure, the algorithm finds a feasible solution to both problems in very few iterations.

To verify the resulting bounds on the $\mathcal{H}_\infty$ norm for the computed controllers K_2 and K_3, we computed the resulting closed loop systems for 5000 different values of α, and verified the $\mathcal{H}_\infty$ norm of the system using MATLAB's build-in functions. The highest $\mathcal{H}_\infty$ norms found in this way were 0.6347 for the unstructured controller K_2 and 0.6785 for the structured controller K_3. The resulting system norms are displayed in Figure 3.

The experiments were run on a standard desktop PC using MATLAB, Yalmip (Lofberg, 2004), and the MOSEK (MOSEK ApS, 2016) SDP solver.

6. CONCLUSION

We have shown how a very generic uncertain system, described with its Linear Fractional Representation in (1), gives rise to a BMI when we try to compute an optimal static output feedback controller. The bilinear terms are substituted using additional variables, resulting in an optimization problem where the bilinear terms are contained in a bilinear equality constraint. This equality constraint is subsequently transformed into a rank constraint. In this rank constraint the decision variables no longer appear in a bilinear way. The rank constraint is relaxed using the nuclear norm. Algorithm 1 shows how in a sequential manner we try to find a feasible controller with good performance. Since in the relaxed problem the controller gain matrix can be manipulated, unlike in many other approaches, we can constrain this matrix to a certain convex set. This results in structured static output feedback matrices, that robustly stabilize a system and have a (locally) optimal

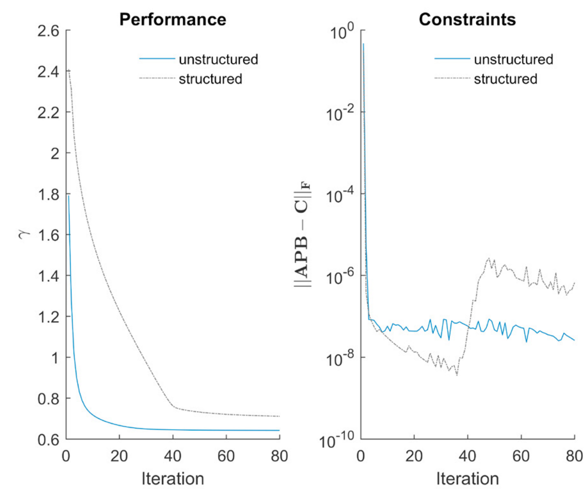

Fig. 2. On the left: the $\mathcal{H}_\infty$ norm γ for the iterations of Algorithm 1. The unstructured controller gives slightly better performance than the structured controller. On the right, the bilinear equality constraint violation. Due to numerical precision of the solver this violation is small, but not equal to zero.

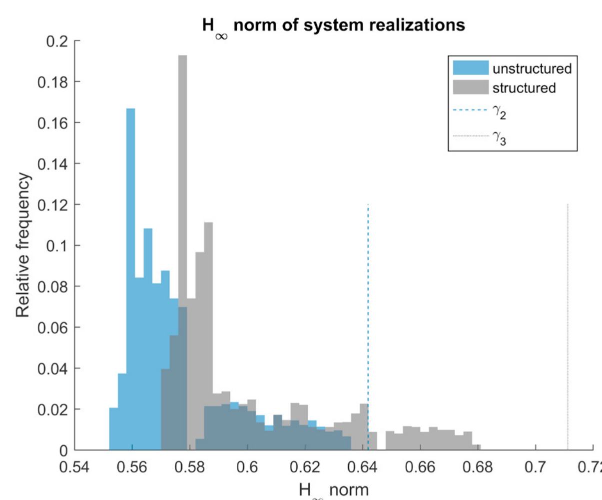

Fig. 3. A histogram of closed loop system $\mathcal{H}_\infty$ norms for 5000 different realizations of α. For the unstructured and structured controllers, all systems have $\mathcal{H}_\infty$ norms below the computed upper bounds γ_2 and γ_3 respectively.

performance guarantee. The numerical experiments verify that our approach outperforms current state-of-the-art.

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