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# System safety assessment under epistemic uncertainty: Using imprecise probabilities in Bayesian network

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ABSTRACT

System safety and reliability assessment relies on historical data and experts opinion for estimating the required failure probabilities. When data comes from different sources, be it different databases or subject domain experts, the estimation of accurate probabilities would be very challenging, if not impossible, and subject to high epistemic uncertainty. In such cases, the use of imprecise probabilities to reflect the incomplete knowledge of analysts and their epistemic uncertainty is inevitable.

Evidence theory is an effective tool for manipulating imprecise probabilities. However, challenges in the assignment of prior belief masses and the lack of effective inference algorithms for combining and updating the belief masses have impeded the widespread application of evidence theory.

To address the foregoing issues, in the present study, (i) an innovative heuristic approach is developed to determine the prior belief masses based on the prior imprecise probabilities, and (ii) it is demonstrated how Bayesian network can be used for both propagating and updating the belief masses. In a nutshell, the developed methodology converts the prior imprecise probabilities into prior belief masses, propagates and updates the belief masses using Bayesian network, and back-transforms the predicted/updated belief masses to posterior imprecise probabilities.

## 1. Introduction

Uncertainty is an integral part of system safety and reliability assessment. It can be in the form of structural uncertainty, reflecting the indeterminacy in the selection of a model to represent an engineering system, or in the form of parameter uncertainty, reflecting the uncertainty in the data used as model input. Having a sufficiently large and reliable dataset, both types of the foregoing uncertainty can reasonably be accounted for in the modeling and assessment of engineering systems. For instance, in the case of structural uncertainty, a modeler may use a number of metrics such as Bayesian Information Criterion (BIC) (Schwarz, 1978) to select among a finite set of models the model which is simpler (having less parameters) yet results in the maximum likelihood of given data (Neapolitan, 2003). In the case of parametric uncertainty, which is the scope of the present study, probability distributions are the most common way to characterize the randomness of events.

However, in absence of sufficient data or data of sufficient accuracy, for example due to the rarity of an event, identification of point probabilities or probability distributions to characterize parameter uncertainty would be subject to degrees of imprecision, if not practically impossible. In such cases, for instance, the analyst may be able to express his prior assessment of an event in the form of interval or imprecise probabilities (Walley, 1991). Imprecise probabilities characterize the uncertainty of an event A through a lower probability $\underline{P}(A)$ and an upper probability $\bar{P}(A)$, resulting in less specific yet more credible outcomes (Kozine and Filimonov, 2000). Imprecise probabilities have effectively been used in system safety and reliability assessment to tackle epistemic uncertainty arising from data scarcity, data incoherency, data incompleteness, and prior ignorance (Coolen and Newby, 1994; Penmetsa and Grandhi, 2002; Utkin and Kozine, 2010).

Dempster-Shafer Theory (DST) – also known as evidence theory (Dempster, 1967; Shafer, 1976) – has been employed as a promising technique for manipulating imprecise probabilities (Guth, 1991; Xu and Smets, 1996; Denoeux, 1999; Kozine and Filimonov, 2000; Smets, 2002; Rakowsky, 2007; Simon et al., 2008; Zhang et al., 2017). In DST, the propagation of uncertainty is based on belief masses rather than probability masses. The application of DST to reasoning with imprecise probabilities consists of three steps: (i) obtaining the belief masses (degrees of belief) from imprecise probabilities, (ii) combining the belief masses using Dempster's rule of combination and propagating the

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beliefs, and (iii) converting the resultant belief masses back into imprecise probabilities.

Nevertheless, compared to probability theory, the application of evidence theory to the field of system safety and reliability has not been so widespread mainly due to the drawbacks of Dempster's rule of combination and also the lack of effective inference (propagation) algorithms:

- Dempster's rule can generate counterintuitive results when used to combine belief masses derived from inconsistent (or even consistent) probability intervals (Zadeh, 1986; Kozine and Filimonov, 2000). Besides its inefficiency in producing consistent results when combining the belief masses (forward reasoning), Dempster's rule cannot be joined with Bayes' rule for belief updating (backward reasoning).
- The inference algorithms developed based on evidence theory for combining joint/disjoint belief masses are not so effective as those based on probability theory (Simon et al., 2008). This in turn has hindered the application of evidence theory to complicated and interdependent processes and systems. However, the development of evidential networks (ENs), which are directed acyclic graphs for reasoning based on belief masses (Xu and Smets, 1996), has largely facilitated the application of evidence theory to complex systems (Zhang et al., 2017).

To address the foregoing issues and enhance the efficiency of DST in handing imprecise probabilities, Simon et al. (2008, 2009) developed an EN based on Bayesian network (BN), hereafter BN-based EN. This way, they managed to use the junction tree algorithm – an algorithm used in BN for belief propagation (Jensen, 1996) – to propagate belief masses, ridding the need of Dempster's rule of combination.

In the present study, we modify the BN-based EN to make it more intuitive and readily applicable to complex and interdependent systems. We also demonstrate that the BN-based EN can be used to update belief masses. More importantly, we introduce an innovative technique for identifying belief masses from imprecise probabilities, which is the first and the most challenging step in the application of DST.

Section 2 recapitulates the basics of DST; it is also shown how the application of Dempster's rule of combination may produce inconsistent results. In Section 3, an innovative heuristic technique is introduced to obtain belief masses from imprecise point probabilities. In Section 4, BN and EN are briefly explained. In Section 5, the BN-based EN originally introduced in Simon et al. (2008) is modified and shown to be applicable to both belief propagating and updating. In Section 6, an application of the methodology to safety assessment under uncertainty is demonstrated. The main outcomes of the study are summarized in Section 6.

## 2. Imprecise probabilities

### 2.1. Evidence theory

There are many techniques to manipulate and propagate imprecise probabilities, including fuzzy sets, interval analysis, second-order probabilities, and DST (Eldred et al., 2011). DST, which is also known as evidence theory, was originally introduced by Dempster (1967) and further developed by Shafer (1976) as a means to express lower and upper bounds probabilities. DST has since been used as an effective tool for handling imprecise probabilities and reasoning under uncertainty (Holmberg et al., 1989; Guth, 1991; Xu and Smets, 1996; Denoeux, 1999; Smets, 2002; Yager, 2004; Bae et al., 2004; Rakowsky, 2007; Simon et al., 2008; Riley, 2015; Misuri et al., 2018).

According to DST, all the possible states (mutually exclusive and collectively exhaustive) of an event are presented as singletons in a set known as the frame of discernment Ω. Based on available information (objective data or experts' opinion), to each subset of Ω such as $A_i$ an evidential weight $0 \le m(A_i) \le 1$ can be assigned to indicate the degree of evidence in favor of the claim that a specific state in Ω belongs to $A_i$ (Rakowsky, 2007). $m(A_i)$ is also known as the belief mass function (or belief mass, in short):

$$\Omega = \{H_1, H_2, \cdots, H_n\} \tag{1}$$

$$A_i \in 2^{\Omega} = \{\varnothing, \{H_1\}, \{H_2\}, \cdots, \{H_n\}, \{H_1, H_2\}, \{H_1, H_3\}, \cdots, \{H_1, H_2, \cdots, H_n\}\} \tag{2}$$

Each $A_i$ which satisfies $m(A_i) > 0$ is called a focal set. If all the states of an event are known, then $m(\varnothing) = 0$, and the null hypothesis (state) can be eliminated from the set of focal sets for simplicity. Otherwise, the null can be maintained as a focal set with a positive belief mass to show the uncertainty about the possible states of the event of interest (Simon et al., 2008). Further, it must always hold that:

$$\sum_{2^{\Omega}} m(A_i) = 1 \tag{3}$$

Having all $m(A_i)$ determined, the amounts of belief (Bel) and plausibility (Pls) of each focal set $A_i$, which are equivalent to the lower and upper probabilities of $A_i$, respectively, can be determined as (Shafer, 1976):

$$Bel(A_i) = \sum_{B|B \subseteq A_i} m(B) \tag{4}$$

$$Pls(A_i) = \sum_{B|B \cap A_i \neq \varnothing} m(B) \tag{5}$$

$$Bel(A_i) \le P(A_i) \le Pls(A_i) \tag{6}$$

According to Eqs. (4) and (5), $Bel(\Omega) = Pls(\Omega) = 1.0$.

Further, as can be noted from Eq. (4), $Bel(A_i)$ can be interpreted as the degree of evidence that the state of the event belongs to $A_i$ or to any of its subsets, i.e., $B \subseteq A_i$. Similarly, according to Eq. (5), $Pls(A_i)$ can be interpreted as the degree of evidence that the state of the event belongs to $A_i$ or any set such as B whose intersection with $A_i$ is not null, i.e., $B \cap A_i \neq \varnothing$. The difference between $Pls(A_i)$ and $Bel(A_i)$ represents the epistemic uncertainty about $A_i$ as shown in Fig. 1 (Rakowsky, 2007).

Since $Bel(A_i)$ and $Pls(A_i)$ are non-additive functions, that is, $Bel(A_i) + Pls(A_i) \neq 1$ (Simon et al., 2008):

$$Bel(A_i^c) = 1 - Pls(A_i) \tag{7}$$

$$Pls(A_i^c) = 1 - Bel(A_i) \tag{8}$$

where $A_i^c$ is the complement of $A_i$, i.e., $A_i^c = \Omega - A_i$. Having the Bel and Pls functions, the belief mass of a focal set can be determined using the möbius transformation as (Smets, 2002):

$$m(A_i) = \sum_{B|B \subseteq A_i} (-1)^{|A_i| - |B|} Bel(B) \tag{9}$$

where $|A_i| - |B|$ refers to the difference between the number of elements in $A_i$ and B. As can be noted from Eq. (9), for singleton focal sets, which are the focal sets with only one element, $m(A_i) = Bel(A_i)$. Further, the amount of uncertainty in a focal set can be expressed as:

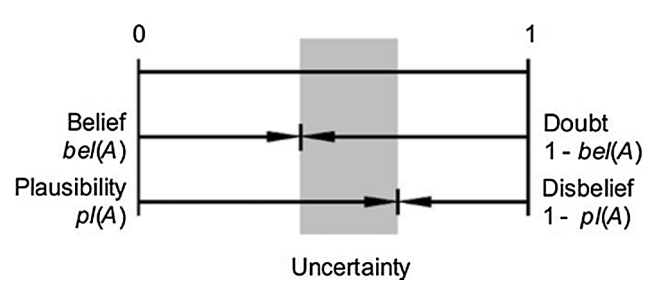


**Fig. 1.** Presentation of epistemic uncertainty using belief and plausibility functions (Rakowsky, 2007).

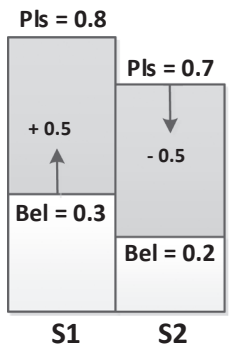


**Fig. 2.** Relationship between amounts of uncertainty in a binary event. The amount of belief mass shared between the states is equal to the contribution of one state's uncertainty to the other's.

$$Unc(A_i) = Pls(A_i) - Bel(A_i) \tag{10}$$

Considering an event with two states S1 and S2; the frame of discernment and the focal sets of the event can be identified as $\Omega = \{S1, S2\}$ and $A_i = 2^{\Omega} = \{\emptyset, \{S1\}, \{S2\}, \{S1, S2\}\}$, respectively. Since the states of the event are known with certainty, $m(\emptyset) = 0$, and thus $\emptyset$ is no longer considered as a focal set: $A_i = \{\{S1\}, \{S2\}, \{S1, S2\}\}$. Among the remaining focal sets, {S1} and {S2} are singletons, referring to the exact states of the system whereas {S1, S2} refers to the uncertainty of the analyst about the state of the event.

Suppose that based on the available evidence, the lower and upper probabilities of the event being in state {S1} has been determined as $0.3 \le P(S1) \le 0.8$. According to Eq. (6), the lower and upper probabilities can be taken as Bel(S1) = 0.3 and Pls(S1) = 0.8. Since {S1} is a singleton, based on Eq. (9): m(S1) = Bel(S1) = 0.3.

Since the focal sets {S1} and {S2} are complements (see the system's Ω), using Eqs. (7) and (9): m(S2) = Bel(S2) = 1 − Pls(S1) = 0.2. As the belief masses of all the focal sets have to add up to unity (Eq. (3)), m(S1, S2) = 1 − m(S1) − m(S2) = 0.5. Obviously, Bel(S1, S2) = Pls(S1, S2) = 1.0, showing that at a time the state of the event will certainly be one of {S1} or {S2}. Having m(S2) = 0.2 and m(S1, S2) = 0.5, using Eqs. (4) and (5), the belief and plausibility measures of {S2} can be calculated as Bel(S2) = m(S2) = 0.2 and Pls(S2) = m(S2) + m(S1, S2) = 0.7; thus: $0.2 \le P(S2) \le 0.7$.

The amounts of Bel and Pls of {S1} and {S2} have been depicted in Fig. 2, where the amount of uncertainty for each state has been shown as the difference between the respective Pls and Bel amounts (the numbers inside the gray areas). For the sake of clarity, the amount of uncertainty for each state has been denoted by positive and negative signs to show that the increase in one's probability is compensated for by the decrease in the other's. As can be noted from Fig. 2, m(S1, S2) is equal to the contribution of the uncertainty of one of the states to the uncertainty of the other state, which in this case is m(S1, S2) = Unc (S1) = Unc(S2) = 0.5.

Having the lower and upper probabilities of the states[1], the belief masses of all the focal sets can be determined for an event with two states (e.g., the foregoing example) or three states (will be shown in the next section). To calculate the belief masses of events with more than three states, the abovementioned equations will result in a system of equations with more unknowns than the equations (an undetermined system of equations) and thus an infinite number of solutions. In Section 3 we will introduce a heuristic approach to assign belief masses to events with more than three states.

[1] Each state is equivalent to a singleton focal set.

### 2.2. Dempster's rule of combination

Dempster's rule of combination is a technique to aggregate the belief masses assigned to focal sets by multiple independent sources of information (e.g., different databases, or experts) (Shafer, 1976). This rule takes into account common shared believes among the sources while discarding the conflicting believes through a normalization factor. Having $m_1(A)$ and $m_2(A)$ as belief masses estimated by two sources of information for an identical frame of discernment, the joint belief mass $m_{1,2}(A)$ can be calculated as:

$$m_{1,2}(\emptyset) = 0 \tag{11}$$

$$m_{1,2}(A) = (m_1 \oplus m_2)(A) = \frac{1}{1-K} \sum_{B \cap C = A \neq \emptyset} m_1(B) m_2(C) \tag{12}$$

$$K = \sum_{B \cap C = \emptyset} m_1(B) m_2(C) \tag{13}$$

where K is a measure of conflict between the believes of source 1 and source 2. $m_1(B)$ and $m_2(C)$ are the masses of the subsets of the same frame of discernment according to the two different sources of information, e.g., two experts.

Dempster's rule of combination, however, has been criticized for generating inconsistent and counterintuitive results (Zadeh, 1986; Voorbraak, 1991; Kozine and Filimonov, 2000). For instance, consider a system with two states $\Omega_{System}$ = {up, down}; asking the opinion of two experts about the probability of the system being in the down state, the first expert expresses his opinion as $0.1 < P_1(down) < 0.3$ whereas the second expert as $0.4 < P_2(down) < 0.7$.

According to the 1st expert: $m_1(down) = Bel_1(down) = 0.1$, $m_1(up) = 1 - Pls_1(down) = 1 - 0.3 = 0.7$, and $m_1(up, down) = 1 - m_1(down) - m_1(up) = 0.2$. According to the 2nd expert: $m_2(down) = Bel_2(down) = 0.4$, $m_2(up) = 1 - Pls_2(down) = 1 - 0.7 = 0.3$, and $m_2(up, down) = 1 - m_2(down) - m_2(up) = 0.3$.

To find the joint belief mass $m_{1,2}(down)$, the measure of conflict K should first be calculated. Since {up} ∩ {down} = ∅, $K = m_1(down).m_2(up) + m_1(up).m_2(down) = (0.1 \times 0.3) + (0.7 \times 0.4) = 0.31$. Thus:

$$m_{1,2}(down) = \frac{1}{1-K}\{m_1(down).m_2(down) + m_1(down).m_2(up, down) + m_1(up, down).m_2(down)\} = 0.22;$$

$$m_{1,2}(up) = \frac{1}{1-K}\{m_1(up).\ m_2(up) + m_1(up).\ m_2(up, down) + m_1(up, down).\ m_2(up)\} = 0.70, \quad \text{and}$$

$$m_{1,2}(up, down) = 1 - m_{1,2}(down) - m_{1,2}(up) = 0.08.$$

As a result, $Bel_{1,2}(down) = m_{1,2}(down) = 0.22$ and $Pls_{1,2}(down) = m_{1,2}(down) + m_{1,2}(up, down) = 0.30$, resulting in $0.22 \le P_{12}(down) \le 0.30$. As can be seen from this example, the aggregation of belief masses via Dempster's rule of combination has resulted in a joint probability interval for the down state of the system which is a subset of the probability interval estimated by the first expert despite the fact that the opinions of the both experts were equally taken into account. An application of Dempster's rule to a ternary event can be found in Rakowsky (2007).

## 3. Identifying belief masses from imprecise probabilities: A heuristic technique

In this section, we introduce a heuristic technique for obtaining the joint belief masses from imprecise point probabilities with no need for applying Dempster's rule of combination. In Section 5, we will demonstrate how the identified belief masses can be used in a BN formalism for belief propagation.

### 3.1. Ternary event

Suppose that we seek the opinion of the experts about the

**Table 1**
Experts opinion about a ternary event in the form of point probabilities.

| State | Expert 1 | Expert 2 |
|---|---|---|
| S1 | $P_1 = 0.2$ | $P_2 = 0.5$ |
| S2 | $P_1 = 0.5$ | $P_2 = 0.3$ |
| S3 | $P_1 = 0.3$ | $P_2 = 0.2$ |

probabilities of the states of a ternary event. This time the experts express their believes in the form of point probabilities as reported in Table 1.

Hence, the experts' uncertainty about the states of the system can be expressed using probability bounds (Simon et al., 2008) as:

$$\begin{cases} P_1 = 0.2 \le \mathrm{P(S1)} \le P_2 = 0.5 \\ P_2 = 0.3 \le \mathrm{P(S2)} \le P_1 = 0.5 \\ P_2 = 0.2 \le \mathrm{P(S3)} \le P_1 = 0.3 \end{cases}$$

Note that these imprecise probabilities can also be interpreted as the imprecise probabilities of the states estimated by one expert (instead of two) based on available evidence. Since {S1}, {S2}, and {S3} are singleton focal sets: m(S1) = Bel(S1) = 0.2, m(S2) = Bel(S2) = 0.3, and m(S3) = Bel(S3) = 0.2.

Now consider the focal set {S1, S2}. According to Eq. (7): Bel(S1, S2) = 1 − Pls(S3) = 0.7. Furthermore, since {S1}, {S2}, and {S1, S2} are the subsets of {S1, S2}, using Eq. (9): m(S1, S2) = Bel(S1, S2) − Bel(S1) − Bel(S2) = 0.2. Following the same procedure, m(S1, S3) = 0.1, m(S2, S3) = 0, and m(S1, S2, S3) = 0. As such, {S1, S3} and {S1, S2, S3} would no longer be considered as focal sets. This approach has been adopted from Simon and Weber (2009) where the lower and upper bound probabilities (or failure rates) have been derived from different reliability databases instead of subject domain experts.

Consider the previous ternary event with the same point probabilities in Table 1. Having the lower and upper probability bounds of the states, the amounts of shared belief masses can be identified using the similar schematic approach as in Fig. 2.

Fig. 3 depicts the amounts of uncertainty (numbers inside gray areas) about the probabilities of S1, S2, and S3. Again, the uncertainties have been denoted by positive and negative signs merely to indicate the direction of changes (increase or decrease) in the probabilities when moving from the first expert (denoted as $P_1$) to the second expert (denoted as $P_2$).

As can be seen from Fig. 3, the uncertainty in S1 is equal to an increase from $P_1(S1) = 0.2$ to $P_2(S1) = 0.5$, i.e., Unc(S1) = $P_2(S1) - P_1(S1) = +0.3$. This increase in the probability of S1 has been compensated for by both the decrease in the probability of S2 as Unc(S2) = $P_1(S2) - P_2(S2) = -0.2$ and the decrease in the probability of S3 as Unc(S3) = $P_1(S3) - P_2(S3) = -0.1$.

Since the amount of belief masses shared between the two states is equal to the contribution of one state's uncertainty to the other state's uncertainty and vice versa, m(S1, S2) will be equal to the increase in P(S1) due to the decrease in P(S2), or simply the contribution of Unc(S2) to Unc(S1). Considering the absolute values of uncertainties (disregarding their positive or negative sign), such contribution can be quantified as:

$$\mathrm{m(S1, S2)} = \mathrm{Unc(S1)}\frac{\mathrm{Unc(S2)}}{\mathrm{Unc(S2)} + \mathrm{Unc(S3)}} = 0.3 \times \frac{0.2}{0.2 + 0.1} = 0.2$$

Likewise, having Unc(S1) = +0.3 and Unc(S3) = −0.1:

$$\mathrm{m(S1, S3)} = \mathrm{Unc(S1)}\frac{\mathrm{Unc(S3)}}{\mathrm{Unc(S2)} + \mathrm{Unc(S3)}} = 0.3 \times \frac{0.1}{0.2 + 0.1} = 0.1$$

As can be seen from Fig. 3, both P(S2) and P(S3) decrease due to an increase in P(S1), making S2 and S3 co-directional focal sets, that is, they both are associated with negative (direction-wise) uncertainties. Since S2 and S3 both experience negative uncertainties, they do not seem to contribute to each other's uncertainty (a negative uncertainty cannot be compensated for by another negative uncertainty), and thus m(S2, S3) = 0. Having the belief masses of single and binary focal sets, the belief mass of Ω can thus be calculated using Eq. (3) as m(S1, S2, S3) = 0.

As can be seen, the belief masses calculated using the heuristic approach in Fig. 3 are the same as the ones calculated using the approach adopted from Simon and Weber (2009). Thus, in the next example, we only demonstrate the application of the heuristic technique.

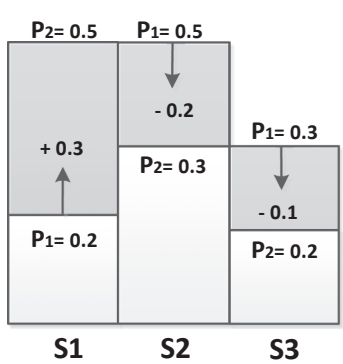


**Fig. 3.** Presentation of uncertainty in a three-state event. The probabilities are listed in Table 1.

### 3.2. Quaternary event

Now suppose an event with four states with a frame of discernment as Ω = {S1, S2, S3, S4} and the power set $A_i$ = {(S1}, …, {S4}, {S1, S2}, …, {S3, S4}, {S1, S2, S3}, …, (S2, S3, S4}, {S1, S2, S3, S4}}. The two experts' opinions about the states probabilities are presented in Table 2.

Similarly to the previous section, the experts' uncertainty about the states of the event can be expressed using probability intervals as:

$$\begin{cases} P_1 = 0.1 \le \mathrm{P(S1)} \le P_2 = 0.3 \\ P_1 = 0.4 \le \mathrm{P(S2)} \le P_2 = 0.45 \\ P_2 = 0.15 \le \mathrm{P(S3)} \le P_1 = 0.3 \\ P_2 = 0.1 \le \mathrm{P(S4)} \le P_1 = 0.2 \end{cases}$$

The lower bound probabilities can be taken as the belief functions whereas the upper bound probabilities as the plausibility functions. Owing to the fact that the belief function of a singleton focal set is equal to its belief mass: m(S1) = 0.1, m(S2) = 0.4, m(S3) = 0.15, and m(S4) = 0.1. Fig. 4 depicts the lower and upper probability bounds of the states as well as the amounts of uncertainty, where $P_1$ and $P_2$ refer, respectively, to the estimates made by the first and the second experts.

The belief masses of the binary focal sets can be calculated based on the contribution of positive and negative (direction-wise) uncertainties as:

$$\begin{aligned} m(S1, S3) &= Unc(S1)\frac{Unc(S3)}{Unc(S3) + Unc(S4)} \\ &= Unc(S3)\frac{Unc(S1)}{Unc(S1) + Unc(S2)} = 0.2 \times \frac{0.15}{0.15 + 0.1} = 0.12. \end{aligned}$$

**Table 2**
Experts opinion about the states of a quaternary event.

| State | Expert 1 | Expert 2 |
|---|---|---|
| S1 | $P_1 = 0.1$ | $P_2 = 0.3$ |
| S2 | $P_1 = 0.4$ | $P_2 = 0.45$ |
| S3 | $P_1 = 0.3$ | $P_2 = 0.15$ |
| S4 | $P_1 = 0.2$ | $P_2 = 0.1$ |

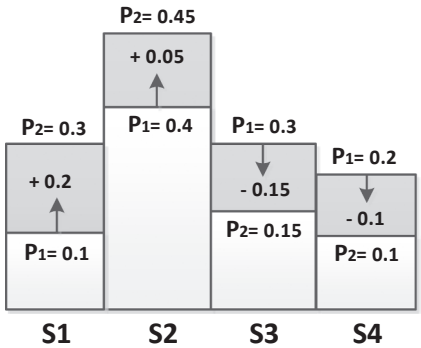


**Fig. 4.** Presentation of uncertainty in a four-state event. Probabilities are listed in Table 2.

$$m(S1,\ S4) = Unc(S1)\frac{Unc(S4)}{Unc(S3) + Unc(S4)} = Unc(S4)\frac{Unc(S1)}{Unc(S1) + Unc(S2)} = 0.2 \times \frac{0.1}{0.15 + 0.1} = 0.08.$$

$$m(S2,\ S3) = Unc(S2)\frac{Unc(S3)}{Unc(S3) + Unc(S4)} = Unc(S3)\frac{Unc(S2)}{Unc(S1) + Unc(S2)} = 0.05 \times \frac{0.15}{0.15 + 0.1} = 0.03.$$

$$m(S2,\ S4) = Unc(S2)\frac{Unc(S4)}{Unc(S3) + Unc(S4)} = Unc(S4)\frac{Unc(S2)}{Unc(S1) + Unc(S2)} = 0.05 \times \frac{0.1}{0.15 + 0.1} = 0.02.$$

Likewise, for the co-directional singleton focal sets, we will have m (S1, S2) = m(S3, S4) = 0. The summation of the belief masses of the single and binary focal sets determined this way should add up to unity, making the belief masses of the other multi-state focal sets (e.g., focal sets with three states) amount to zero, according to Eq. (3).

In order to examine the accuracy of the foregoing approach in identifying the belief masses, consider the belief function of the focal set {S2, S3, S4} which can be calculated using Eqs. (4) or (7). Considering Eq. (4), for instance, Bel(S2, S3, S4) = 1 − Pls(S1) = 1 − 0.3 = 0.7. Similarly, using Eq. (7), Bel(S2, S3, S4) = m(S2) + m(S3) + m(S4) + m (S2, S3) + m(S2, S4) + m(S3, S4) + m(S2, S3, S4) = 0.7, showing that the belief masses identified using the heuristic approach are correct.

The results obtained from the graphical presentation of the uncertainties and belief masses can be summarized as:

Suppose that based on the lower and upper bounds probabilities, the singleton focal sets (focal sets with one state, or simply all the states of the system) can be split into two subsets $S^+$ and $S^-$ to indicate the sets of states associated with positive and negative uncertainties, respectively.

- If $S_i$ is a state of the system with positive uncertainty, i.e., $S_i \in S^+$, and $S_j$ is a state of the system with negative uncertainty, i.e., $S_j \in S^-$:

$$m(S_i,\ S_j) = Unc(S_i)\frac{Unc(S_j)}{\sum_{S_k \in S^-} Unc(S_k)} = Unc(S_j)\frac{Unc(S_i)}{\sum_{S_q \in S^+} Unc(S_q)} \tag{14}$$

- If $S_i$ and $S_j$ are co-directional, i.e., both belong to $S^+$ or both belong to $S^-$:

$$m(S_i,\ S_j) = 0. \tag{15}$$

- Having the belief masses of the binary focal sets determined using Eqs. (14) and (15), the belief masses of the other multi-state focal sets such as $m(S_i,\ S_j,\ S_k)$ will be zero.

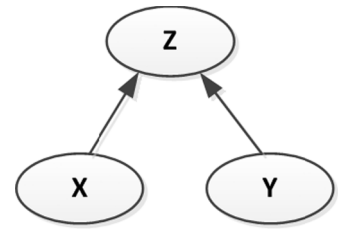


**Fig. 5.** A typical Bayesian network.

## 4. Reasoning under uncertainty

### 4.1. Bayesian network

Bayesian network BN = (G, θ) (Pearl, 1988) is a directed acyclic graph for knowledge representation and probabilistic inference. G is the structure of the graph in which the random variables are presented as nodes and dependencies among the random variables are denoted as directed arcs connecting the nodes (Fig. 5). The graph G satisfies Markovian condition in that each variable in G is independent of its nondescendents given its immediate parents. As a result, the associated joint probability distribution of the random variables can be factorized as the multiplication of conditional probabilities of the nodes (variables) given their parents as:

$$P(X_1, X_2, \cdots, X_n) = \prod_{i=1}^{n} P(X_i | pa(X_i)) \tag{16}$$

The conditional probabilities $\theta_i = P(X_i|pa(X_i))$ are known as the network parameters which can either be elicited from subject matter experts or be learned from data. Considering the BN in Fig. 5, the joint probability of the variables can be presented as: P(X, Y, Z) = P(X) P(Y) P(Z|X, Y). BN can be used for predictive analysis (prediction of the symptom based on the cause), e.g., $P(Z = z_2|X = x_1)$, and for diagnostic analysis (diagnosis of the cause given the symptom), e.g., $P(X = x_1|Z = z_2)$.

Having the marginal probabilities of the root nodes (nodes with no parents, e.g., X and Y in Fig. 5) and the conditional probabilities of child nodes (e.g., Z in Fig. 5), a number of exact inference algorithms such as bucket elimination (or variable elimination) (Dechter, 1996) and junction tree (Lauritzen and Spiegelhalter, 1988; Jensen, 1996) as well as approximate inference techniques such as belief propagation (also known as sum-product message passing) (Pearl, 1982), Monte Carlo Markov Chain (MCMC) (Cheng and Druzdzel, 1999), dynamic discretization and MCMC[2] (Fenton and Neil, 2019) and rank correlation[3] (Hanea and Kurowicka, 2008) can be used to calculate the marginal probabilities of the child nodes.

### 4.2. Evidential network

The main assumption in BN formalism is that all data can be represented by probability functions. That is, in BN the uncertain knowledge is to be modeled by probability theory. There have been attempts to develop network-based systems for modeling uncertain knowledge using other theories such as possibility theory (Zadeh, 1978) and evidence theory (Shafer, 1976). Shenoy (1989, 1992) proposed a framework, so-called valuation-based system (VBS), for modeling uncertainty in expert systems, applicable to probability theory, possibility theory, and evidence theory. In other words, BN can be deemed as a

[2] The algorithm has been used in AGENARISK software: https://www.agenarisk.com/

[3] The algorithm has been used in UNINET software: https://www.tudelft.nl/ewi/over-de-faculteit/afdelingen/applied-mathematics/applied-probability/risk/software/uninet/

VBS based on probability theory (Simon et al., 2008; Simon and Weber, 2009; Benavoli et al., 2009).

Likewise, an evidential network (EN) can be seen as a specification of VBS based on evidence theory (Xu and Smets, 1996). Similarly to BN, EN is a directed acyclic graph composed of nodes and arcs to propagate uncertainty based on belief masses and conditional belief tables rather than probability masses and conditional probability tables. Lauritzen and Jensen (1997) generalized the algorithm originally developed by Jensen (1996) for BN so that it could be applied for inference in other domains such as evidence theory.

## 5. Propagation of belief masses using Bayesian network

The belief masses allocated to the focal sets of an event must add up to unity; thus, if the event is presented as a root node in BN, the belief masses can be taken as marginal probabilities of the node's states. As a result, inference algorithms developed for BN can be used to propagate uncertainty based on evidence theory (i.e., using belief masses) rather than probability theory (i.e., using probability masses).

In this regard, Simon et al. (2008) developed an innovative EN by combining DST with BN to take advantage of the junction tree algorithm (Jensen, 1996) in propagating and computing the belief masses of child nodes based on the belief masses of their parent nodes. This way, the developed EN (herein, the BN-based EN), can simply be modeled and analyzed using BN modeling software. In Section 5.1, the BN-based EN developed by Simon et al. (2008) is first described using a simple system made of binary components/events. In Section 5.2, we modify the BN-based EN to become more intuitive and less complex. In Section 5.3, we demonstrate that the BN-based EN can be applied to both combining and updating the belief masses.

### 5.1. BN-based EN (Simon et al., 2008)

For the sake of exemplification, consider a system Z comprising two binary components X and Y as shown in Fig. 5. Since the components and the system are binary, they can each be in one of S1 = {up} or S2 = {down} states (see Fig. 2). The frame of discernment of X (or Y or Z) and its focal sets can be presented as $\Omega_X$ = {up, down} and $A_X$ = {{up}, {down}, {up, down}}, respectively, where {up, down} = {up} ⊕ {down}. Among the focal sets of X, {up, down} models the epistemic uncertainty about the state of X (it does not mean that X can be both in {up} and {down} states).

The combination of the belief masses of the components (nodes) can be done by means of Boolean algebra (Simon et al., 2008). For instance, consider a case where X = {up} and Y = {up, down} are connected to Z by an AND[4] gate; using Boolean algebra, the state of Z can be identified as {up} ∩ {up, down} = {up} ∩ ({up}⊕{down}) = ({up} ∩ {up}) ⊕ ({up} ∩ {down}) = {up} ⊕ {up} = {up}. Likewise, in the case of an OR[5] gate, the state of Z can be identified as {up} ∪ {up, down} = ({up}∪{up}) ⊕ ({up}∪{down}) = {up} ⊕ {down} = {up, down}. The results of AND and OR gates in the form of a truth table have been presented in Table 3.

For the system shown in Fig. 5, assume that the analyst, based on his degree of belief, has assigned the marginal belief mass distributions to the focal sets of X and Y as $m(A_X)$ = {0.5, 0.4, 0.1} and $m(A_Y)$ = {0.4, 0.4, 0.2}. Fig. 6 displays the resulting EN in which X and Y are connected to Z via OR gate.

As can be seen in Fig. 6, the inference algorithm of BN can be used to calculate the marginal belief mass distribution of Z as $m(A_Z)$ = {0.2, 0.64, 0.16} based on the marginal mass distributions of X and Y. Having the belief mass distribution of Z, the belief and plausibility of Z = {up} can be calculated using Eqs. (4) and (5) as Bel({up}) = m({up}) = 0.2 and Pls({up}) = m({up}) + m({up, down}) = 0.36. Thus, according to Eq. (6): 0.2 ≤ P(Z = up) ≤ 0.36. Likewise, the belief and plausibility of Z = {down} can be calculated as: 0.64 ≤ P(Z = down) ≤ 0.80.

The procedure of calculating belief and plausibility can be carried out directly using the developed BN (which in fact is an EN) by adding the Bel and Pls nodes for each state to the network (Fig. 7). For the sake of clarity, the conditional belief tables used to connect nodes Bel (Z = down) and Pls(Z = down) to Z are presented in Tables 4 and 5, respectively. To avoid an unnecessarily large EN, the analyst may decide to eliminate nodes Bel(Z = up) and Pls(Z = up) from the EN and only keep Bel(Z = down) and Pls(Z = down) if the failure of the system is of interest (similar to fault tree analysis) or the opposite if the reliability of the system is of interest (similar to reliability block diagram analysis). Nevertheless, having the belief and plausibility of the down state, for instance, those of the up state can readily be calculated using Eqs. (7) and (8).

**Table 3**
Truth table to combine the focal sets of components X and Y via AND and OR gates (Simon et al., 2008).

| | | Z | |
|---|---|---|---|
| X | Y | OR | AND |
| {up} | {up} | {up} | {up} |
| {up} | {down} | {down} | {up} |
| {up} | {up, down} | {up, down} | {up} |
| {down} | {up} | {down} | {up} |
| {down} | {down} | {down} | {down} |
| {down} | {up, down} | {down} | {up, down} |
| {up, down} | {up} | {up, down} | {up} |
| {up, down} | {down} | {down} | {up, down} |
| {up, down} | {up, down} | {up, down} | {up, down} |

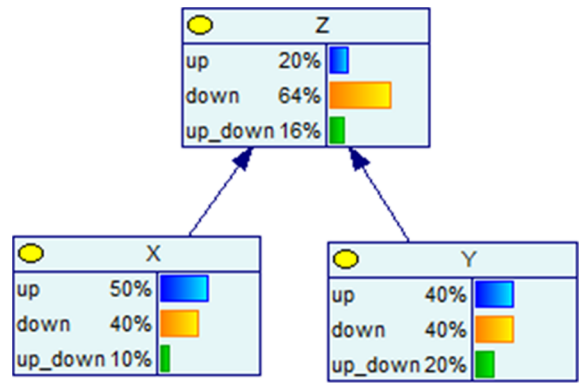


**Fig. 6.** EN for uncertainty assessment using BN inference algorithm.

### 5.2. Modified BN-based EN

As can be seen in Fig. 7, since Bel and Pls functions are non-additive, e.g., Bel(up) + Pls(up) ≠ 1.0, they have been presented as two separate nodes in the EN (Simon et al., 2008). The EN in Fig. 7 can, however, be modified as Fig. 8 so that the uncertainty in Z = {up} can be expressed within Z without resorting to additional nodes. The EN in Fig. 8 has been developed taking into account the relationship among the belief, plausibility, and disbelief functions as shown in Fig. 1:

$$Bel(A_i) + Unc(A_i) + Dis(A_i) = 1.0 \tag{17}$$

$$Dis(A_i) = 1 - Pls(A_i) \tag{18}$$

where $Unc(A_i)$ and $Dis(A_i)$, respectively, refer to the uncertainty and disbelief about the focal set $A_i$ (see Fig. 1).

[4] AND gate implies that both X and Y must be in the down state for Z to be in the down state.

[5] OR gate implies that either X or Y must be in the down state for Z to be in the down state.

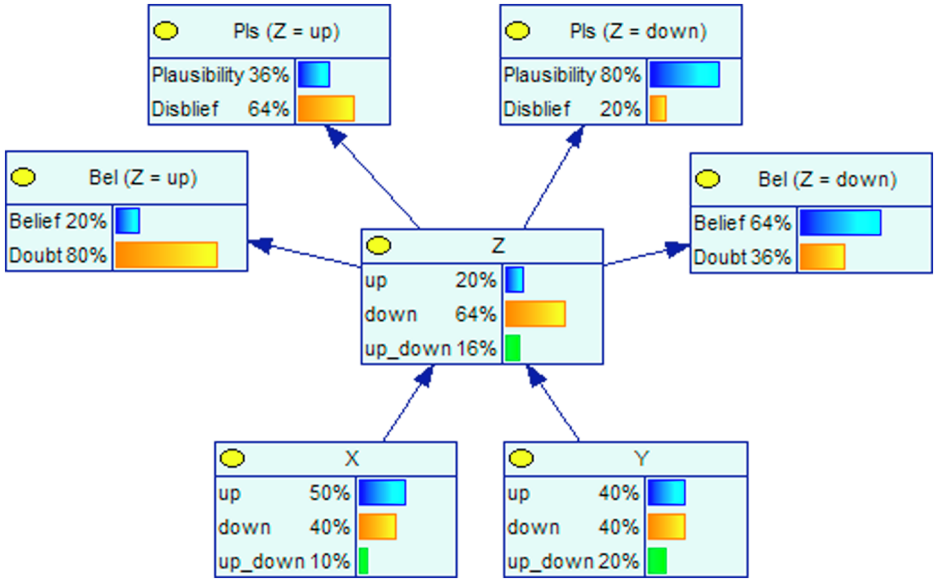


**Fig. 7.** Calculating the interval probabilities of Z = {up} and Z = {down} using Bel and Pls nodes.

**Table 4**
Conditional belief table to calculate the states of Bel(Z = down) based on the states of Z in Fig. 7.

| | Bel (Z = down) | |
|---|---|---|
| Z | Belief | Doubt |
| {up} | 0 | 1 |
| {down} | 1 | 0 |
| {up,down} | 0 | 1 |

**Table 5**
Conditional belief table to calculate the states of Pls(Z = down) based on the states of Z in Fig. 7.

| | Pls (Z = down) | |
|---|---|---|
| Z | Plausibility | Disbelief |
| {up} | 0 | 1 |
| {down} | 1 | 0 |
| {up,down} | 1 | 0 |

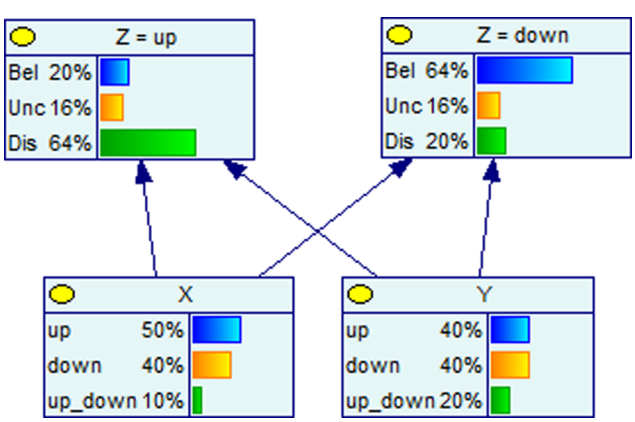


**Fig. 8.** Calculating the epistemic uncertainty of Z = {up} and Z = {down} without additional nodes of Bel and Pls functions.

**Table 6**
Conditional belief table of node "Z = down" in the case of OR gate (Fig. 8).

| | | Z = down (OR gate) | | |
|---|---|---|---|---|
| X | Y | Bel | Unc | Dis |
| up | up | 0 | 0 | 1 |
| up | down | 1 | 0 | 0 |
| up | up, down | 0 | 1 | 0 |
| down | up | 1 | 0 | 0 |
| down | down | 1 | 0 | 0 |
| down | up, down | 1 | 0 | 0 |
| up, down | up | 0 | 1 | 0 |
| up, down | down | 1 | 0 | 0 |
| up, down | up, down | 0 | 1 | 0 |

**Table 7**
Conditional belief table of node "Z = down" in the case of AND gate.

| | | Z = down (AND gate) | | |
|---|---|---|---|---|
| X | Y | Bel | Unc | Dis |
| up | up | 0 | 0 | 1 |
| up | down | 0 | 0 | 1 |
| up | up, down | 0 | 0 | 1 |
| down | up | 0 | 0 | 1 |
| down | down | 1 | 0 | 0 |
| down | up, down | 0 | 1 | 0 |
| up, down | up | 0 | 0 | 1 |
| up, down | down | 0 | 1 | 0 |
| up, down | up, down | 0 | 1 | 0 |

As can be seen from Fig. 8, Bel(Z = {down}) = 0.64; having the amounts of uncertainty as Unc(Z = {down}) = 0.16 and disbelief as Dis (Z = {down}) = 0.20, the amount of plausibility can be calculated either using Eq. (10) as Pls(Z = {down}) = Bel(Z = {down}) + Unc (Z = {down}) = 0.64 + 0.16 = 0.80 or using Eq. (18) as Pls (Z = {down}) = 1 − Dis(Z = {down}) = 1 – 0.20 = 0.80. The results are the same as those calculated via separate nodes of Bel(Z = down) and Pls(Z = down) in Fig. 7. As previously mentioned, if the failure of the system (Z) is of interest, the modeler may decide to keep the EN simple by merely focusing on node "Z = down" in Fig. 8. The conditional belief tables to calculate the states of node "Z = down" in case of OR gate (Fig. 8) or AND gate have been reported in Tables 6 and 7,

respectively.

### 5.3. Belief updating in BN-based EN

In Section 2.2, it was illustrated via an example how Dempster's rule of combination may produce inconsistent results when joining the belief masses. Likewise, conventional ENs which rely on Dempster's rule of combination for propagating belief masses would inevitably inherit the same drawback in combining or updating the belief masses, casting doubt on the credibility of predicted (in forward analysis) and/or updated (in backward analysis) belief masses and corresponding imprecise probabilities.

Mapping EN to credal network (CN) – as an extension to BN in which probability functions are replaced with credal sets (Cozman, 2000) – has been proposed in some studies as a potentially better solution especially when it comes to belief updating. Application of CN to probability updating, however, can produce excessively wide posterior probability intervals which are not always so informative (Seidenfeld and Wasserman, 1993). In addition, as shown in Misuri et al. (2018), the inference algorithms developed for CN can generate different and inconsistent results[6].

In the present study, we will demonstrate that the BN-based EN can reliably be used for belief updating, resulting in consistent updated belief masses from which the posterior probability intervals can be obtained. The reason is that the BN-based EN, as opposed to conventional EN, takes advantage of BN inference algorithms for manipulating the belief masses and propagating the uncertainty rather than relying on Dempster's rule of combination.

#### 5.3.1. Results of evidential network

To make the discussion more concrete, the EN in Fig. 9 presents the same EN in Fig. 6 in which the belief masses of X and Y have been updated given $m_Z(\text{down}) = 1.0$, which simply implies P (Z = down) = 1.0, as evidence.

In order to examine the accuracy of the updated belief masses, a comparison between the results of EN and a Monte Carlo simulation is performed for both the forward (Fig. 6) and backward (Fig. 9) analyses. In Fig. 6, given the belief masses of nodes X, Y, and Z, the corresponding prior probability intervals of down states can be calculated as: $0.4 \le P(X = \text{down}) \le 0.5$, $0.4 \le P(Y = \text{down}) \le 0.6$, and $0.64 \le P(Z = \text{down}) \le 0.8$.

In Fig. 9, given the evidence in Z, i.e. $m_z$ (down) = 1.0, the updated belief masses of X and Y can be used in the same way to calculate their posterior probability intervals of down states as: $0.63 \le P(X = \text{down} \mid Z = \text{down}) \le 0.69$ and $0.63 \le P(Y = \text{down} \mid Z = \text{down}) \le 0.76$.

#### 5.3.2. Result of Monte Carlo simulation

Since X and Y are the root nodes of the BN (or EN), they are conditionally independent (as long as the state of Z is unknown, due to the d-separation rule (Pearl, 1988)). As Z is connected to X and Y by OR gate, the probability of Z in down state can be calculated as:

$$P(Z = down) = P(X = down) + P(Y = down) - P(X = down)P(Y = down) \tag{19}$$

Furthermore, using the Bayes'rule, the updated probability of X (or Y) being in the down state given Z in the down state can be calculated as:

[6] Misuri et al. (2018) used GL2U (http://people.idsia.ch/~sun/gl2u.html) and JavaBayes (http://www.cs.cmu.edu/~javabayes/) just to find out these two packages would result in different posterior probabilities, with the latter resulting in more logical posteriors according to the evidence.

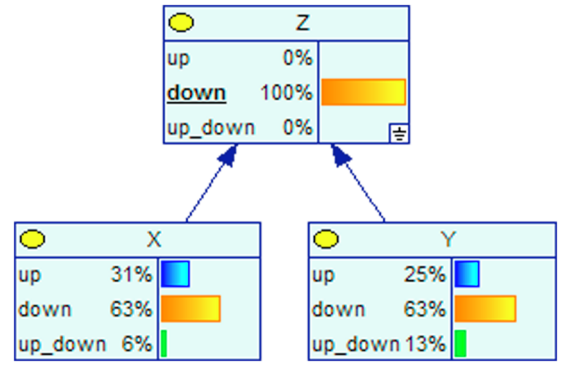


**Fig. 9.** EN of Fig. 6 which has been updated given the evidence $m_Z(\text{down}) = 1.0$.

$$P(X = down|Z = down) = \frac{P(X = down)P(Z = down|X = down)}{P(Z = down)} = \frac{P(X = down)}{P(Z = down)} \tag{20}$$

To perform the Monte Carlo simulation, 1000 samples were generated for P(X = down) and P(Y = down) based on their prior probability intervals $0.4 \le P(X = \text{down}) \le 0.5$ and $0.4 \le P(Y = \text{down}) \le 0.6$ assuming uniform distributions (the first two columns of Table 8).

For each pair of P(X = down) and P(Y = down), the probability of P (Z = down) can then be calculated using Eq. (19) as in the 3rd column of Table 8 while the conditional probabilities of P(X = down|Z = down) and P(Y = down|Z = down) can be calculated using Eq. (20) as in the 4th and 5th columns of Table 8. Sorting the probabilities listed in the 3rd, 4th, and 5th columns of Table 8 from the lowest to the highest, the rounded-up probability intervals were identified as: $0.65 \le P(Z = \text{down}) \le 0.79$, $0.53 \le P(X = \text{down}|Z = \text{down}) \le 0.71$, and $0.57 \le P(Y = \text{down}|Z = \text{down}) \le 0.79$.

The results of the EN analysis and Monte Carlo simulation are summarized in Table 9, showing a good agreement between the predicted and updated probabilities calculated using these two methods.

## 6. Safety assessment with imprecise probabilities

### 6.1. Truss under tensile stress

To demonstrate an application of EN to safety assessment under epistemic uncertainty, consider a truss consisting of two[7] axial members AB and AC, each with respective uniform square section areas of $a_{AB} = 2 \times 10^{-4}\,m^2$ and $a_{AC} = 4 \times 10^{-4}\,m^2$, under a concentrated load F (kN) with a trunctated normal distribution as F ~ Normal ($\mu = 180$, $\sigma = 25$) as shown in Fig. 10. By solving the equilibrium equation of joint A, the amounts of tensile forces in the bars AB and AC are determined, respectively, as $F_{AB} = F_1 = 0.5\,F$ and $F_{AC} = F_2 = 0.866\,F$.

Assume that the analyst is not sure whether the bars are of the same type and made of steel (Steel), alluminium alloy (Alloy), or ductile iron (Iron), with respective ultimate tensile stresses (UTS) of 500 MPa, 480 MPa, and 410 MPa. The analyst thus asks two experts to express their degree of belief about the type of the bars, in the form of the lower and upper bound probabilities as listed in Table 10. Given the heuristic technique in Section 3, the epistemic uncertainty about the materials type can be expressed via belief masses in Table 11.

### 6.2. Failure assessment

If the tensile stress ($\tau$) in a bar exceeds the respective UTS, the bar

[7] Under the current loading condition, the member BC is not subject to any axial load, and thus not contributing to the structure's safety.

**Table 8**
Part of the results generated by Monte Carlo simulation.

| P (X = down) | P(Y = down) | P(Z = down) | P(X = down\|Z = down) | P(Y = down\|Z = down) |
|---|---|---|---|---|
| 0.496 | 0.456 | 0.726 | 0.683 | 0.629 |
| 0.420 | 0.518 | 0.721 | 0.583 | 0.719 |
| 0.438 | 0.579 | 0.763 | 0.573 | 0.759 |
| 0.458 | 0.445 | 0.699 | 0.655 | 0.636 |
| 0.488 | 0.428 | 0.707 | 0.690 | 0.605 |

**Table 9**
Comparison between the results of evidential network (EN) and Monte Carlo (MC) simulation.

| | Forward analysis | Backward analysis | |
|---|---|---|---|
| Technique | P(Z = down) | P(X = down\|Z = down) | P(Y = down\|Z = down) |
| EN | 0.64 ≤ P ≤ 0.80 | 0.63 ≤ P ≤ 0.69 | 0.63 ≤ P ≤ 0.76 |
| MC | 0.65 ≤ P ≤ 0.79 | 0.53 ≤ P ≤ 0.71 | 0.57 ≤ P ≤ 0.79 |

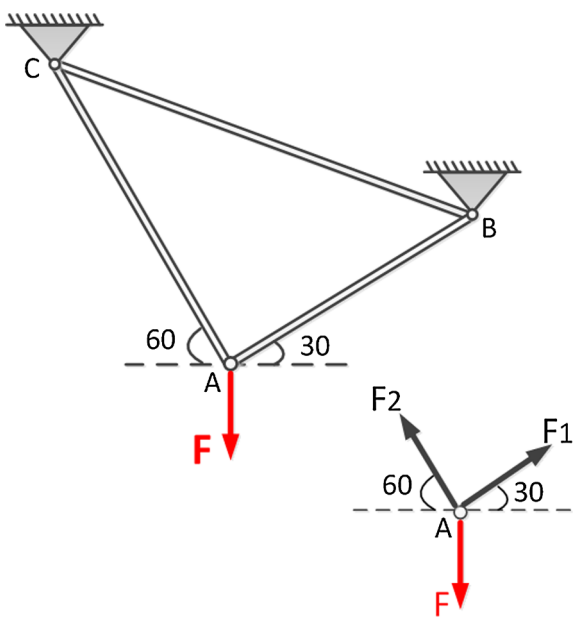


**Fig. 10.** Truss under a concentrated load F.

**Table 10**
Prior point probabilities estimated by the experts for the type of the bars in Fig. 10.

| Type | Bar AB | | Bar AC | |
|---|---|---|---|---|
| | Expert 1 | Expert 2 | Expert 1 | Expert 2 |
| Steel | 0.2 | 0.3 | 0.4 | 0.5 |
| Iron | 0.3 | 0.4 | 0.2 | 0.1 |
| Alloy | 0.5 | 0.3 | 0.4 | 0.4 |

**Table 11**
Belief masses of the types of the bars based on the values in Table 10.

| Focal set | Bar AB | Bar AC |
|---|---|---|
| {Steel} | 0.2 | 0.4 |
| {Iron} | 0.3 | 0.1 |
| {Alloy} | 0.3 | 0.4 |
| {Steel, Iron} | 0.0 | 0.1 |
| {Steel, Alloy} | 0.1 | 0.0 |
| {Iron, Alloy} | 0.1 | 0.0 |
| {Steel, Iron, Alloy} | 0.0 | 0.0 |

fails. The truss thus loses its structural integrity (the system fails) if either AB or AC fails. The failure of each of the bars under tensile stress can be modeled as:

$$P(AB\ \ fails) = P(\tau_{AB} \geq UTS_{AB}) = P\left(\frac{F_{AB}}{a_{AB}} \geq UTS_{AB}\right) = P\left(\frac{0.5\ \ F}{2 \times 10^{-4}} \geq UTS_{AB}\right) = P(F \geq 4 \times 10^{-4}\ \ UTS_{AB}) \tag{21}$$

$$P(AC\ \ fails) = P(\tau_{AC} \geq UTS_{AC}) = P\left(\frac{F_{AC}}{a_{AC}} \geq UTS_{AC}\right) = P\left(\frac{0.866\ \ F}{4 \times 10^{-4}} \geq UTS_{AC}\right) = P(F \geq 4.6 \times 10^{-4}\ \ UTS_{AC}) \tag{22}$$

As such, AB and AC fail if the amount of F exceeds the amounts listed in Table 12 considering different possible types of materials. The EN for assessing the failure probability of the truss has been displayed in Fig. 11.

In Fig. 11, the continuous variable F has been discretized into intervals due to the fact that 99.7% of the density lies in μ ± 3σ. However, since an F less than 164 kN would not cause failure in neither of the bars (see Table 14), the first two intervals, i.e., μ − 2σ and μ − 3σ, have been merged together as the first state of node 'F'. The states of the nodes "Type of AB" and "Type of AC" and the respective probabilities (belief masses) have already been identified in Table 11. Having the intervals of F, the type of bars, and thus the corresponding minimum amounts of F needed for the failure of each type (Table 12), the state probabilities of nodes "AB fails" and "AC fails" can readily be calculated. For instance, consider a case where 'F = Btw_180_205' (i.e., 180 ≤ F <2 0 5):

- For "Type of AB = Steel", since the minimum F required for the failure of an AB made of steel is 200 kN (see Table 12), the failure probability of AB can be calculated as $P(AB\ \ fails|\ Type\ \ of\ \ AB = Steel) = P(F \geq 200)$. As F is already between 180 and 205 kN, this probability could be modified as $P(AB\ \ fails|\ Type\ \ of\ \ AB = Steel, 180 \leq F < 205) = P(F \geq 200\ |180 \leq F < 205)$. $= P(200 \leq F < 205)$

**Table 12**
Amounts of F associated with the failure of truss members in Fig. 10.

| | | Minimum F required for failure (KN) | |
|---|---|---|---|
| Type | UTS (MPa) | AB | AC |
| Steel | 500 | 200 | 230 |
| Iron | 410 | 164 | 189 |
| Alloy | 480 | 192 | 221 |

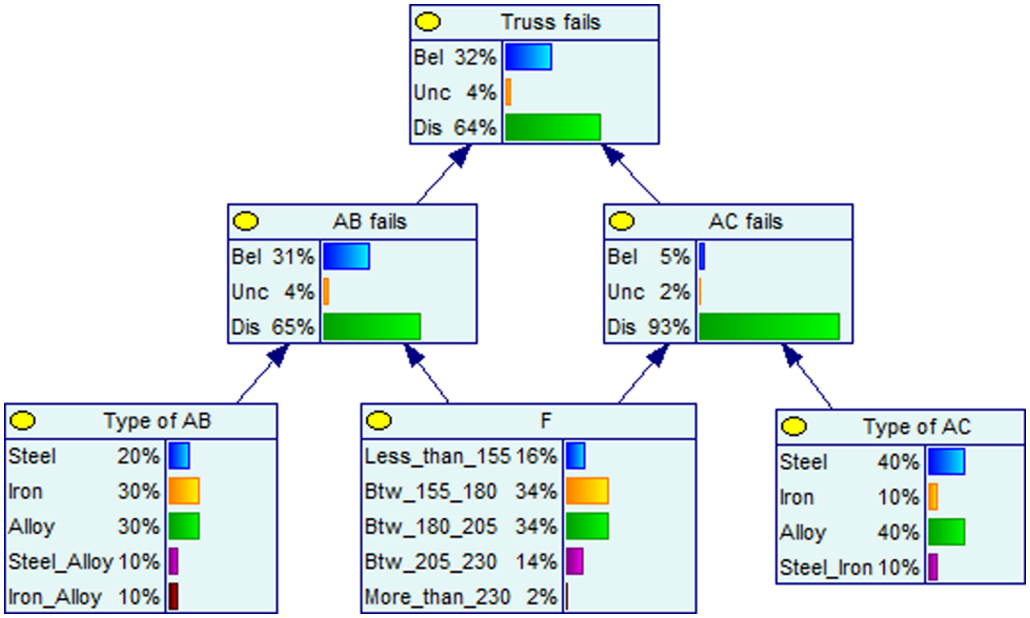


**Fig. 11.** EN for failure modeling of the truss under epistemic uncertainty.

Because F ~ Normal (μ = 180, σ = 25), this probability can be calculated as $P(200 \le F < 205) = \Phi\left(\frac{205-180}{25}\right) - \Phi\left(\frac{200-180}{25}\right) = 0.053$, where Φ(.) is the cumulative density function of standard normal distribution, available in tabular forms or calculable from the error function.

Since the type of AB is known for this certain state, Bel(AB fails) = Pls(AB fails) = 0.053, resulting in Unc(AB fails) = 0.0 and thus Dis(AB fails) = 1 − 0.053 = 0.947.

Similarly:

- For "Type of AB = Iron": $P(AB\ fails|\ Type\ of\ AB = Iron, 180 \le F < 205) = P(F \ge 164\ |180 \le F < 205) = 1.0$; thus: Bel(AB fails) = Pls(AB fails) = 1.0, Unc(AB fails) = 0.0, Dis(AB fails) = 0.0; and
- For "Type of AB = Alloy": $P(AB\ fails|\ Type\ of\ AB = Alloy, 180 \le F < 205) = P(F \ge 192\ |180 \le F < 205) = P(192 \le F < 205) = 0.157$; thus: Bel(AB fails) = Pls(AB fails) = 0.157, Unc(AB fails) = 0.0, Dis(AB fails) = 0.843.

Having the conditional belief mass functions (and probabilities) calculated for certain types of AB, the ones for uncertain types of AB can readily be determined:

- For "Type of AB = Steel_Alloy", which refers to the uncertainty about the type of AB, the smaller failure probability of 0.053 (which is attributed to 'Type of AB = Steel") can be taken as Bel(AB fails) whereas the larger failure probability of 0.157 (which is attributed to "Type of AB = Alloy") can be taken as Pls(AB fails). As such, Unc(AB fails) = 0.157 − 0.053 = 0.104, and Dis(AB fails) = 1 − 0.053 − 0.104 = 0.843. Likewise:
- For "Type of AB = Iron_Alloy", the smaller failure probability of 0.157 (which is attributed to "Type of AB = Alloy") can be taken as Bel(AB fails) whereas the larger failure probability of 1.0 (which is attributed to "Type of AB = Iron") can be taken as Pls(AB fails). As such, Unc(AB fails) = 0.843, and Dis(AB fails) = 0.0.

Having the conditional probabilities of nodes "AB fails" and "AC fails" determined this way, the conditional probabilities of node 'Truss fails" can be defined as an OR gate as presented in Table 13. For the sake of clarity, the conditional probabilities for an AND gate have also been included in Table 13. According to the belief masses in Fig. 11, the probability intervals of the truss and its members can readily be calculated as: 0.32 ≤ P(Truss fails) ≤ 0.36; 0.31 ≤ P(AB fails) ≤ 0.35, and 0.05 ≤ P(AC fails) ≤ 0.07, implying a relatively higher contribution of AB's failure to the failure of truss.

**Table 13**
Conditional probability table of node "Truss fails" in the case of OR/AND gate. OR gate has been used in Fig. 11.

| | | Truss fails (OR gate) | | | Truss fails (AND gate) | | |
|---|---|---|---|---|---|---|---|
| AB fails | AC fails | Bel | Unc | Dis | Bel | Unc | Dis |
| Bel | Bel | 1 | 0 | 0 | 1 | 0 | 0 |
| Bel | Unc | 1 | 0 | 0 | 0 | 1 | 0 |
| Bel | Dis | 1 | 0 | 0 | 0 | 0 | 1 |
| Unc | Bel | 1 | 0 | 0 | 0 | 1 | 0 |
| Unc | Unc | 0 | 1 | 0 | 0 | 1 | 0 |
| Unc | Dis | 0 | 1 | 0 | 0 | 0 | 1 |
| Dis | Bel | 1 | 0 | 0 | 0 | 0 | 1 |
| Dis | Unc | 0 | 1 | 0 | 0 | 0 | 1 |
| Dis | Dis | 0 | 0 | 1 | 0 | 0 | 1 |

### 6.3. Probability updating

As demonstrated in Section 5.3, the BN-based EN can also be used for belief updating or diagnostic (backward) analysis. For this purpose, evidence in the form of new observations can be used to update prior beliefs about the system and its components. For instance, assume that the analysist observes the truss not fail under F = 190 kN. This observation can be considered as two pieces of evidence and thus being implemented in the EN by instantiating the states of nodes "Truss fails = Dis" and "F = Btw_180_205" as shown in Fig. 12.

Accordingly, using the updated belief mass functions for the types of AB and AC, the updated lower and upper bound probabilities can be calculated. For instance, the updated belief and plausibility functions of "Type of AB = Alloy" can be calculated using Eqs. (4) and (5) as $Bel_{AB}(Alloy) = m_{AB}(Alloy) = 0.48$ and $Pls_{AB}(Alloy) = m_{AB}(Alloy) + m_{AB}(Steel\_Alloy) + m_{AB}(Iron\_Alloy) = 0.48 + 0.16 + 0.0 = 0.64$. Accordingly updated probability of AB being made of aluminum alloy can be presented as 0.48 ≤ P(Type of AB = Alloy|F = 190, Truss does not

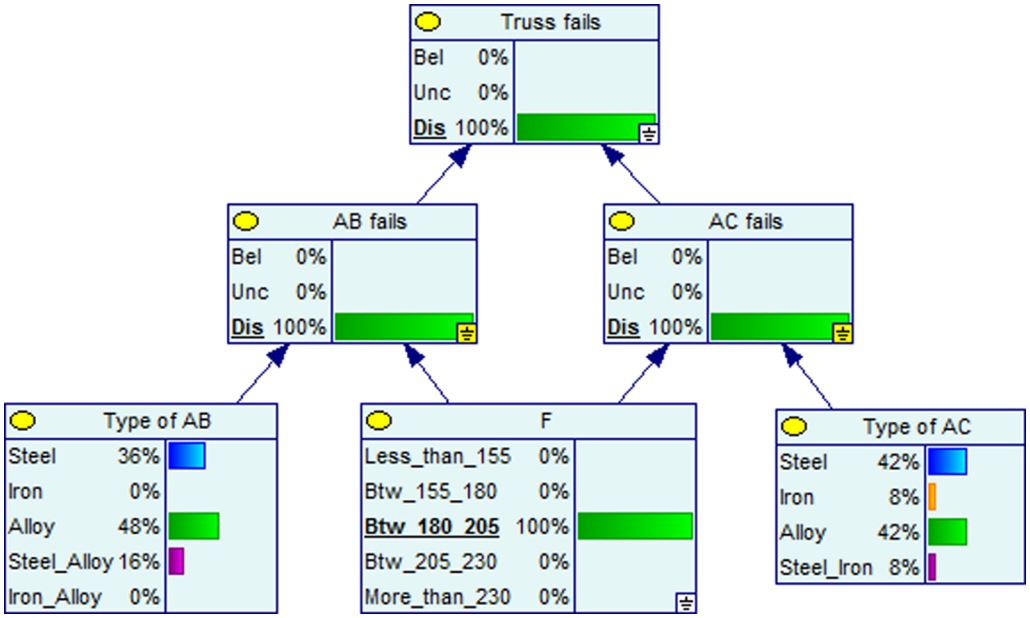


**Fig. 12.** Backward analysis in EN knowing that the truss has survived a load of 190 kN.

**Table 14**
Prior and posterior probability bounds for the type of AB. Updated probabilities have been calculated given that the truss does not fail under F = 190 kN.

| Type of AB | Posterior probability (from Fig. 12) | Prior probability (from Table 10) | VR |
|---|---|---|---|
| Steel | $0.36 \le P \le 0.52$ | $0.2 \le P \le 0.3$ | 0.76 |
| Alloy | $0.48 \le P \le 0.64$ | $0.3 \le P \le 0.5$ | 0.40 |
| Iron | $P = 0$ | $0.3 \le P \le 0.4$ | −1.00 |

fail) ≤ 0.64, which compared to the prior probability 0.3 ≤ P(Type of AB = Alloy) ≤ 0.5 (see Table 10) has notably increased. Using a similar approach, the updated lower and upper bound probabilities for the type of AB and AC can be calculated as reported in Table 14.

Selecting the most probable type of AB merely based on either prior or posterior probabilities is likely to lead to incorrect results. To identify the most likely type of AB given the above-mentioned observation, a variation ratio[8] (VR) can be calculated for each state as:

$$VR(x) = \frac{P(x|E) - P(x)}{P(x)} \tag{23}$$

where P(x|E) is the updated probability of X = x given evidence E (posterior probability of x), and P(x) is the prior probability of X = x. The results have been presented in the last column of Table 14. It is worth mentioning that the mean value of the posterior and prior probability intervals have been used in Eq. (23) to calculate VR. As can be seen, despite a higher posterior probability interval for "Type of AB = Alloy" than "Type of AB = Steel", it is Steel that has been identified as the most likely type of AB, according to its higher VR.

## 7. Conclusions

In the present study, we developed a methodology for using imprecise probabilities in Bayesian network for system safety assessment under uncertainty. In a nutshell, the developed methodology consists of three steps: (i) identifying belief masses from imprecise probabilities, (ii) propagating the belief masses in Bayesian network, and (iii) converting the predicted and/or updated belief masses back into imprecise probabilities:

- As for the first step, we in Section 3 developed an innovative heuristic approach for identifying joint belief masses of multi-state events from their imprecise probabilities with no need for Dempster's rule of combination. The heuristic approach is particularly useful in the case of events with four or more states, where the application of Dempster-Shafer Theory could lead to an undetermined and insoluble system of equations for belief masses.
- As for the second step, we in Section 5.2 modified the Bayesian network approach originally proposed by Simon et al. (2008) so that the combination and propagation of belief masses could be performed more intuitively and with less complexity. This modification was demonstrated to facilitate the modeling of complex systems through a simpler Bayesian network.
- As for the third step, in Section 5.3 we demonstrated that the Bayesian network can be used for belief mass updating the same way it can be used for probability mass updating. This achievement especially enables the modeler to update the imprecise probabilities with no need for resorting to other techniques such as Credal network.

Nevertheless, it should be noted that the predicted and updated imprecise probabilities calculated using the developed Bayesian network are credible as long as the prior belief masses assigned to the root nodes of the Bayesian network are not specified through Dempster's rule of combination. This is because Dempster's rule of combination tends to produce counterintuitive and inconsistent results when used to combine the belief masses, let alone when used for their updating.

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[8] This score is different than the variation ratio as a simple measure of statistical dispersion in nominal distributions (Freeman, 1965).

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