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1 Ill-posedness in modelling 2D morphodynamic 2 problems: Effects of bed slope and secondary flow

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9 **Abstract**

10 A two-dimensional model describing river morphodynamic processes under mixed-size sed-
11 iment conditions is analysed with respect to its well-posedness. Well-posedness guarantees the
12 existence of a unique solution continuously depending on the problem data. When a model
13 becomes ill-posed, infinitesimal perturbations to a solution grow infinitely fast. Apart from the
14 fact that this behaviour cannot represent a physical process, numerical simulations of an ill-
15 posed model continue to change as the grid is refined. For this reason, ill-posed models cannot
16 be used as predictive tools. One source of ill-posedness is due to the simplified description of
17 the processes related to vertical mixing of sediment. The current analysis reveals the existence
18 of two additional mechanisms that lead to model ill-posedness: secondary flow due to the flow
19 curvature and the gravitational pull on the sediment transport direction. When parametris-
20 ing secondary flow, accounting for diffusion in the transport of secondary flow intensity is a
21 requirement for obtaining a well-posed model. When considering the theoretical amount of
22 diffusion, the model predicts instability of perturbations that are incompatible with the shal-
23 low water assumption. The gravitational pull is a necessary mechanism to yield a well-posed
24 model, but not all closure relations to account for this mechanism are valid under mixed-size
25 sediment conditions. Numerical simulations of idealised situations confirm the results of the
26 stability analysis and highlight the consequences of ill-posedness.

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1 Introduction

Modelling of fluvial morphodynamic processes is a powerful tool not only to predict the future state of a river after, for instance, an intervention or a change in the discharge regime (*Blom et al.*, 2017), but also as a source of understanding of the natural processes responsible for patterns such as dunes, meanders, and bars (*Callander*, 1969; *Seminara*, 2006; *Colombini and Stocchino*, 2012). A framework for modelling the morphodynamic development of alluvial rivers is composed of a system of partial differential equations for modelling the flow, change in bed elevation, and change in the bed surface texture. The *Saint-Venant* (1871) equations account for conservation of water mass and momentum and enable modelling processes with a characteristic length scale significantly longer than the flow depth in one-dimensional cases. The Shallow Water Equations describe the depth-averaged flow in two-dimensional cases. Conservation of unisize bed sediment is typically modelled using the *Exner* (1920) equation and, under mixed-size sediment conditions, the active layer model (*Hirano*, 1971) accounts for mass conservation of bed sediment of each grain size.

Although widely successful in predicting river morphodynamics, a fundamental problem arises when using the above framework. Under certain conditions the description of the natural phenomena is not captured by the system of equations, which manifests as an ill-posed model. Models describe a simplified version of reality, which allows us to understand the key elements playing a major role in the dynamics of the system one studies (*Paola and Leeder*, 2011). Major simplifications such as reducing streamwise morphodynamic processes to a diffusion equation allow for insight on the creation of stratigraphic records and evolution on large spatial scales (*Paola et al.*, 1992; *Paola*, 2000; *Paola and Leeder*, 2011). There is a difference between greatly simplified models and models that do not capture the physical processes. A simplified model reproduces a reduced-complexity version of reality (*Murray*, 2007) and it is mathematically well-posed, as a unique solution exists that depends continuously on the data (*Hadamard*, 1923; *Joseph and Saut*, 1990). An ill-posed model lacks crucial physical processes that cause the model to be unsuitable to capture the dynamics of the system (*Fowler*, 1997). An ill-posed model is unrepresentative of a physical phenomenon, as the growth rate of infinitesimal perturbations to a solution (i.e., negligible noise from a physical perspective) tends to infinity (*Kabanikhin*, 2008). This is different from chaotic systems, in which noise similarly causes the solution to diverge but not infinitely fast (*Devaney*, 1989; *Banks et al.*, 1992).

An example of an ill-posed model is the one describing the dynamics of granular flow. The continuum formulation of such a problem depends on deriving a model for the granular viscosity. *Jop et al.* (2005, 2006) relate viscosity to a dimensionless shear rate. The model captures the dynamics of granular flows if the dimensionless shear rate is within a certain range, but otherwise the model is ill-posed and loses its predictive capabilities (*Barker et al.*, 2015). A better representation of the physical processes guaranteeing that viscosity tends to 0 when the dimensionless shear rate tends

63 to 0 extends the domain of well-posedness (*Barker and Gray, 2017*).

64 Under unisize sediment and one-dimensional flow conditions, the Saint-Venant-Exner model
65 may be ill-posed when the Froude number is larger than 6 (*Cordier et al., 2011*). As most flows
66 of interest are well below this limit, we can consider modelling of fluvial problems under unisize
67 sediment conditions to be well-posed. This is not the case when considering mixed-size sediment.
68 Using the active layer model we assume that the bed can be discretised into two layers: the active
69 layer and the substrate. The sediment transport rate depends on the grain size distribution of
70 the active layer. A vertical flux of sediment occurs between the active layer and the substrate if
71 the elevation of the interface between the active layer and the substrate changes. The active layer
72 is well-mixed, whereas the substrate can be stratified. The above simplification of the physical
73 processes responsible for vertical mixing causes the active layer model to be ill-posed (*Ribberink,*
74 *1987; Stecca et al., 2014; Chavarrías et al., 2018*). In particular, the active layer is prone to be ill-
75 posed under degradational conditions into a substrate finer than the active layer (i.e., an armoured
76 bed (*Parker and Sutherland, 1990*)) for any value of the Froude number.

77 Previous analyses of river morphodynamic models regarding their well-posedness have been
78 focused on conditions of one-dimensional flow (*Ribberink, 1987; Cordier et al., 2011; Stecca et al.,*
79 *2014; Chavarrías et al., 2018*). Our objective is to extend these analyses to conditions of two-
80 dimensional flow. More specifically we include the secondary flow and the bed slope effect in the
81 analysis of the well-posedness of the system of equations.

82 As the flow is intrinsically three-dimensional, the depth-averaging procedure eliminates an
83 important flow component: the secondary flow (*Van Bendegom, 1947; Rozovskii, 1957*). The
84 secondary flow causes, for instance, an increase in the amplitude of meanders (*Kitanidis and*
85 *Kennedy, 1984*) and plays an important role in bar development (*Olesen, 1982*). To understand the
86 morphology of two-dimensional features, it is necessary to account for the fact that the sediment
87 transport direction is affected by the gravitational pull when the bed slope in the transverse
88 direction is significant (*Dietrich and Smith, 1984; Seminara, 2006*). This is usually done using
89 a closure relation that sets the angle between the flow and the sediment transport directions as
90 a function of the flow and sediment parameters (*Van Bendegom, 1947; Engelund, 1974; Talmon*
91 *et al., 1995; Seminara et al., 2002; Parker et al., 2003; Francalanci and Solari, 2007, 2008; Baar*
92 *et al., 2018*).

93 In this paper we show that combining these two effects, secondary flow and sediment deflection
94 by the bed slope, leads in some cases to an ill-posed system of equations. The paper is organised
95 as follows. In Section 2 we present the model equations describing the primary and secondary flow,
96 as well as changes in bed elevation and surface texture. In Section 3 we extend the explanation of
97 ill-posedness and relate it to growth of perturbations. We subsequently conduct a stability analysis
98 of the equations, which indicates the conditions under which the secondary flow model and the
99 closure relation for the bed slope effect yield an ill-posed model (Section 4). In Section 5 we run

100 numerical simulations of idealised cases to test the validity of the analytical results and study the
 101 consequences of ill-posedness.

102 2 Mathematical Model

103 In this section we present the two-dimensional mathematical model of flow, accounting for sec-
 104 ondary flow, coupled to a morphodynamic model for mixed-size sediment. We subsequently intro-
 105 duce the equations describing the primary flow (Section 2.1), the secondary flow (Section 2.2), and
 106 morphodynamic change (Section 2.3). In Section 2.4 we linearise the system of equations to study
 107 the stability of perturbations.

108 2.1 Primary Flow Equations

109 The primary flow is described using the depth-averaged Shallow Water Equations (e.g. *Vreugdenhil*,
 110 1994):

$$111 \quad \frac{\partial h}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0, \quad (1)$$

$$112 \quad \begin{aligned} & \frac{\partial q_x}{\partial t} + \frac{\partial(q_x^2/h + gh^2/2)}{\partial x} + \frac{\partial(\frac{q_x q_y}{h})}{\partial y} + gh \frac{\partial \eta}{\partial x} - F_{sx} = \\ 113 \quad & = 2 \frac{\partial}{\partial x} \left(\nu h \frac{\partial(\frac{q_x}{h})}{\partial x} \right) + \frac{\partial}{\partial y} \left(\nu h \left(\frac{\partial(\frac{q_x}{h})}{\partial y} + \frac{\partial(\frac{q_y}{h})}{\partial x} \right) \right) - gh S_{fx}, \end{aligned} \quad (2)$$

$$114 \quad \begin{aligned} & \frac{\partial q_y}{\partial t} + \frac{\partial(q_y^2/h + gh^2/2)}{\partial y} + \frac{\partial(\frac{q_x q_y}{h})}{\partial x} + gh \frac{\partial \eta}{\partial y} - F_{sy} = \\ 115 \quad & 2 \frac{\partial}{\partial y} \left(\nu h \frac{\partial(\frac{q_y}{h})}{\partial y} \right) + \frac{\partial}{\partial x} \left(\nu h \left(\frac{\partial(\frac{q_y}{h})}{\partial x} + \frac{\partial(\frac{q_x}{h})}{\partial y} \right) \right) - gh S_{fy}, \end{aligned} \quad (3)$$

116 where (x, y) [m] are Cartesian coordinates and t [s] is the time coordinate. The variables $(q_x, q_y) =$
 117 (uh, vh) [m²/s] are the specific water discharges in the x and y direction, respectively, where h
 118 [m] is the flow depth and u [m/s] and v [m/s] are the depth-averaged flow velocities. The variable
 119 η [m] is the bed elevation and g [m/s²] the acceleration due to gravity. The friction slopes are
 120 (S_{fx}, S_{fy}) [-] and the diffusion coefficient ν [m²/s] is the horizontal eddy viscosity. The depth-
 121 averaging procedure of the equations of motion introduces terms that originate from the difference
 122 between the actual velocity at a certain elevation in the water column and the depth-averaged
 123 velocity. We separate the contributions due to turbulent motion and secondary flow caused by
 124 the flow curvature. The contribution due to turbulent motion is accounted for by the diffusion
 125 coefficient. *Elder* (1959) derived an expression for the diffusion coefficient that accounts for the
 126 effect of turbulent motion on the depth-averaged flow assuming a logarithmic profile for the primary
 127 flow and negligible effect of molecular viscosity:

$$128 \quad \nu_E = \frac{1}{6} \kappa h u^*, \quad (4)$$

129 where $\kappa = 0.41 [-]$ is the Von Kármán constant and $u^* = \sqrt{C_f}Q/h$ [m/s] is the friction velocity.
 130 Parameter $C_f [-]$ is a nondimensional friction coefficient, which we assume to be constant (*Iked*
 131 *et al.*, 1981; *Schielen et al.*, 1993) and $Q = \sqrt{q_x^2 + q_y^2}$ [m²/s] is the module of the specific water
 132 discharge. In the numerical simulations we will assume the eddy viscosity to be a constant equal
 133 to the value given by ν_E in a reference state (e.g. *Falconer*, 1980; *Lien et al.*, 1999). Appendix A
 134 presents the limitations of the coefficient derived by *Elder* (1959).

135 The terms (F_{sx}, F_{sy}) [m²/s²] account for the effect of secondary flow. These terms are respon-
 136 sible for a transfer of momentum that shifts the maximum velocity to the outer bend (*Kalkwijk*
 137 *and De Vriend*, 1980), as well as for a sink of energy in the secondary circulation (*Flokstra*, 1977;
 138 *Begnudelli et al.*, 2010). We deal with these terms in Section 2.2.

139 We assume a Chézy-type friction:

$$140 \quad S_{fx} = \frac{C_f q_x Q}{gh^3}, \quad S_{fy} = \frac{C_f q_y Q}{gh^3}. \quad (5)$$

141 One underlying assumption of the system of equations presented above is that the vertical length
 142 and velocity scales are negligible with respect to the horizontal ones. Another assumption is the
 143 fact that the concentration of sediment (the ratio between the solid and liquid discharge) is small
 144 (below 6×10^{-3} (*Garegnani et al.*, 2011, 2013)), such that we apply the clear water approximation.

145 2.2 Secondary Flow Equations

146 This section describes the equations that model secondary flow (i.e., formulations for F_{sx} and F_{sy}
 147 in equations (2) and (3)). The secondary flow velocity profile u^s [m/s] (i.e., the vertical profile
 148 of the velocity component perpendicular to the primary flow) is assumed to have a universal
 149 shape as a function of the relative elevation in the water column $\zeta = (z - \eta)/h [-]$, where z
 150 [m] is the vertical Cartesian coordinate perpendicular to x and y increasing in upward direction
 151 (*Rozovskii*, 1957; *Engelund*, 1974; *De Vriend*, 1977, 1981; *Booij and Pennekamp*, 1984). Worded
 152 differently, the vertical profile of the secondary flow is parametrised by a single value representing
 153 the intensity of the secondary flow I [m/s], such that $u^s = f(\zeta)I$. The secondary flow intensity I is
 154 the integral of the absolute value of the secondary flow velocity profile (*De Vriend*, 1981). Among
 155 others, *Rozovskii* (1957), *Engelund* (1974), and *De Vriend* (1977), derive equilibrium profiles of
 156 the secondary flow that differ in the description of the eddy viscosity, vertical profile of the primary
 157 flow, and the boundary condition of the flow at the bed. Following *De Vriend* (1977), we assume
 158 a logarithmic profile for the primary flow (i.e., a parabolic distribution of the eddy viscosity) and
 159 vanishing velocity close to the bed at $\zeta = \exp(-1 - 1/\alpha)$ where $\alpha = \frac{\sqrt{C_f}}{\kappa} < 0.5$.

160 The depth-averaging procedure yields the integral value (along z) of the force per unit mass
 161 that the secondary flow exerts on the primary flow (*De Vriend*, 1977; *Kalkwijk and De Vriend*,

162 1980):

$$163 \quad F_{sx} = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y}, \quad (6)$$

$$164 \quad F_{sy} = \frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y}, \quad (7)$$

166 where T_{lm} [m^3/s^2] is the integral shear stress per unit mass in the direction l - m . Assuming a large
167 width-to-depth ratio (i.e., $B/h \gg 1$, where B [m] is the characteristic channel width) and a mild
168 curvature (i.e., $h/R_s \ll 1$, where R_s [m] is the radius of curvature of the streamlines), the shear
169 stress terms are:

$$170 \quad T_{xx} = -2 \frac{\beta^* I}{Q} q_x q_y, \quad (8)$$

$$171 \quad T_{xy} = T_{yx} = \frac{\beta^* I}{Q} (q_x^2 - q_y^2), \quad (9)$$

$$172 \quad T_{yy} = T_{yx} = 2 \frac{\beta^* I}{Q} q_x q_y, \quad (10)$$

173 where $\beta^* = 5\alpha - 15.6\alpha^2 + 37.5\alpha^3$.

176 The simplest strategy to account for secondary flow assumes that the secondary flow is fully
177 developed. This is equivalent to saying that the secondary flow intensity is equal to the equilib-
178 rium value $I_e = Q/R_s$ [m/s] found in an infinitely long bend (*Rozovskii*, 1957; *Engelund*, 1974;
179 *De Vriend*, 1977, 1981; *Booij and Pennekamp*, 1983). A change in channel curvature leads to the
180 streamwise adaptation of secondary flow to the equilibrium value (*De Vriend*, 1981; *Ikeda and*
181 *Nishimura*, 1986; *Johannesson and Parker*, 1989; *Seminara and Tubino*, 1989). *Booij and Pen-*
182 *nekamp* (1984) and *Kalkwijk and Booij* (1986) not only account for the spatial adaptation but also
183 the temporal adaptation of the secondary flow associated with a variable discharge or tides. Here
184 we adopt the latter strategy, which has been applied, for instance, in modelling the morphody-
185 namics of braided rivers (*Javernick et al.*, 2016; *Williams et al.*, 2016; *Javernick et al.*, 2018). The
186 spatial and temporal adaptation of secondary flow is expressed by (*Jagers*, 2003):

$$187 \quad \frac{\partial I}{\partial t} + \frac{q_x}{h} \frac{\partial I}{\partial x} + \frac{q_y}{h} \frac{\partial I}{\partial y} - \frac{\partial}{\partial x} \left(\nu \frac{\partial I}{\partial x} \right) - \frac{\partial}{\partial y} \left(\nu \frac{\partial I}{\partial y} \right) = S_s, \quad (11)$$

188 where S_s [m/s^2] is a source term which depends on the difference between the local secondary flow
189 intensity and its equilibrium value:

$$190 \quad S_s = -\frac{I - I_e}{T_I}, \quad (12)$$

191 where T_I [s] is the adaptation time scale of the secondary flow:

$$192 \quad T_I = \frac{L_I h}{Q}, \quad (13)$$

193 where $L_I = L_I^* h$ [m] is the adaptation length scale of the secondary flow, which depends on the
194 nondimensional length scale $L_I^* = \frac{1-2\alpha}{2\kappa^2\alpha}$ (*Kalkwijk and Booij*, 1986).

195 The radius of curvature of the streamlines is defined as (e.g. *Legleiter and Kyriakidis, 2006*):

$$196 \quad \frac{1}{R_s} = \frac{\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right)^{3/2}}, \quad (14)$$

197 assuming steady flow and in terms of water discharge we obtain:

$$198 \quad \frac{1}{R_s} = \frac{-q_x q_y \frac{\partial q_x}{\partial x} + q_x^2 \frac{\partial q_y}{\partial x} - q_y^2 \frac{\partial q_x}{\partial y} + q_x q_y \frac{\partial q_y}{\partial y}}{(q_x^2 + q_y^2)^{3/2}}. \quad (15)$$

199 The secondary flow model described in this section closes the primary flow model described in
200 Section 2.1 given a certain bed elevation. In the following section we describe the model equations
201 that describe changes in bed elevation as a function of the primary and secondary flow.

202 2.3 Morphodynamic Equations

203 We consider an alluvial bed composed of an arbitrary number N of non-cohesive sediment fractions
204 characterised by a grain size d_k [m], where the subscript k denotes the grain size fraction in
205 increasing order (i.e., $d_1 < d_2 < \dots < d_N$). Bed elevation change depends on the divergence of the
206 sediment transport rate (*Exner, 1920*):

$$207 \quad \frac{\partial \eta}{\partial t} + \frac{\partial q_{bx}}{\partial x} + \frac{\partial q_{by}}{\partial y} = 0, \quad (16)$$

208 where $q_{bx} = \sum_{k=1}^N q_{bxk}$ [m²/s] and $q_{by} = \sum_{k=1}^N q_{byk}$ [m²/s] are the total specific (i.e., per unit of
209 differential length) sediment transport rates including pores in the x and y direction, respectively.
210 The variables q_{bxk} [m²/s] and q_{byk} [m²/s] are the specific sediment transport rates of size fraction
211 k including pores. For simplicity we assume a constant porosity and density of the bed sediment.
212 The sediment transport rate is assumed to be locally at capacity, which implies that we do not
213 model the temporal and spatial adaptation of the sediment transport rate to capacity conditions
214 (*Bell and Sutherland, 1983; Phillips and Sutherland, 1989; Jain, 1992*).

215 Changes in the bed surface grain size distribution are accounted for using the active layer model
216 (*Hirano, 1971*). For simplicity, we assume a constant active layer thickness L_a [m]. Conservation
217 of sediment mass of size fraction k in the active layer reads:

$$218 \quad \frac{\partial M_{ak}}{\partial t} + f_k^I \frac{\partial \eta}{\partial t} + \frac{\partial q_{bxk}}{\partial x} + \frac{\partial q_{byk}}{\partial y} = 0 \quad k \in \{1, N-1\}, \quad (17)$$

219 and in the substrate (*Chavarrías et al., 2018*):

$$220 \quad \frac{\partial M_{sk}}{\partial t} - f_k^I \frac{\partial \eta}{\partial t} = 0 \quad k \in \{1, N-1\}, \quad (18)$$

221 where $M_{ak} = F_{ak}L_a$ [m] and $M_{sk} = \int_{\eta_0}^{\eta_0+\eta-L_a} f_{sk}(z)dz$ [m] are the volume of sediment of size
 222 fraction k per unit of bed area in the active layer and the substrate, respectively. Parameter η_0 [m]
 223 is a datum for bed elevation. Parameters $F_{ak} \in [0, 1]$, $f_{sk} \in [0, 1]$, and $f_k^I \in [0, 1]$ are the volume
 224 fraction content of sediment of size fraction k in the active layer, substrate, and at the interface
 225 between the active layer and the substrate, respectively. By definition, the sum of the volume
 226 fraction content over all size fractions equals 1:

$$227 \quad \sum_{k=1}^N F_{ak} = 1, \quad \sum_{k=1}^N f_{sk}(z) = 1, \quad \sum_{k=1}^N f_k^I = 1. \quad (19)$$

228 Under degradational conditions, the volume fraction content of size fraction k at the interface
 229 between the active layer and the substrate is equal to that at the top part of the substrate ($f_k^I =$
 230 $f_{sk}(z = \eta - L_a)$ for $\partial\eta/\partial t < 0$). This allows for modelling of arbitrarily abrupt changes in grain
 231 size due to erosion of previous deposits. Under aggradational conditions the sediment transferred
 232 to the substrate is a weighted mixture of the sediment in the active layer and the bed load (*Parker,*
 233 *1991; Hoey and Ferguson, 1994; Toro-Escobar et al., 1996*). Here we simplify the analysis and we
 234 assume that the contribution of the bed load to the depositional flux is negligible (i.e., $f_k^I = F_{ak}$
 235 for $\partial\eta/\partial t > 0$) (*Hirano, 1971*).

236 The magnitude of the sediment transport rate is assumed to be a function of the local bed shear
 237 stress. We apply the sediment transport relation by *Engelund and Hansen (1967)* in a fractional
 238 manner (*Blom et al., 2016, 2017*) as well as the one by *Ashida and Michiue (1971)* (Appendix B).

239 The direction of the sediment transport (φ_{sk} [rad]) is affected by the secondary flow and the
 240 bed slope (*Van Bendegom, 1947*):

$$241 \quad \tan \varphi_{sk} = \frac{\sin \varphi_\tau - \frac{1}{g_{sk}} \frac{\partial \eta}{\partial y}}{\cos \varphi_\tau - \frac{1}{g_{sk}} \frac{\partial \eta}{\partial x}} \quad k \in \{1, N\}, \quad (20)$$

242 where $g_{sk} [-]$ is a function that accounts for the influence of the bed slope on the sediment transport
 243 direction and φ_τ [rad] is the direction of the sediment transport accounting for the secondary flow
 244 only:

$$245 \quad \tan \varphi_\tau = \frac{q_y - h\alpha_I \frac{q_x I}{Q}}{q_x - h\alpha_I \frac{q_y I}{Q}}. \quad (21)$$

246 Assuming a mild curvature, uniform flow conditions, and a logarithmic profile of the primary flow,
 247 the constant $\alpha_I [-]$ is (*De Vriend, 1977*):

$$248 \quad \alpha_I = \frac{2}{\kappa^2} (1 - \alpha). \quad (22)$$

249 The effect of the bed slope on the sediment transport direction depends on the grain size (*Parker*

250 and Andrews, 1985). We account for this effect setting:

$$251 \quad g_{sk} = A_s \theta_k^{B_s} \quad k \in \{1, N\}, \quad (23)$$

252 where A_s [-] and B_s [-] are nondimensional parameters and θ_k [-] is the *Shields* (1936) stress
 253 (Appendix B). Different values of the coefficients A_s and B_s have been proposed (for a recent
 254 review, see *Baar et al.* (2018)). We consider two possibilities: (1) $A_s = 1$, $B_s = 0$ (*Schielen et al.*,
 255 1993) and (2) $A_s = 1.70$ and $B_s = 0.5$ (*Talmon et al.*, 1995). In the first and simpler case, the bed
 256 slope effect is independent from the bed shear stress (*Engelund and Skovgaard*, 1973; *Engelund*,
 257 1975). In the second, more complex, case, the bed slope effect is assumed to be dependent on
 258 the fluid drag force on the grains, which is assumed to depend on the Shields stress (*Koch and*
 259 *Flokstra*, 1981).

260 2.4 Linearised System of Equations

261 The system of equations describing the flow, change of bed level, and change of the bed surface
 262 texture is highly non-linear. Here we linearise the system of equations to provide insight on the
 263 fundamental properties of the model and to study the stability of perturbations. To this end we
 264 consider a reference state that is a solution to the system of equations. The reference state is a
 265 steady uniform straight flow in the x direction over an inclined plane bed composed of an arbitrary
 266 number of size fractions. Mathematically: $h_0 = \text{ct.}$, $q_{x0} = \text{ct.}$, $q_{y0} = 0$, $I_0 = 0$, $\frac{\partial \eta}{\partial x} = \text{ct.} = \frac{-C_\tau q_{x0}^2}{gh_0^3}$,
 267 $\frac{\partial \eta}{\partial y} = 0$, $M_{ak0} = \text{ct.} \forall k \in \{1, N-1\}$, where ct. denotes a constant different from 0 and subscript 0
 268 indicates the reference solution.

269 We add a small perturbation to the reference solution denoted by $'$ and we linearise the resulting
 270 system of equations. After substituting the reference solution we obtain a system of equations of
 271 the perturbed variables:

$$272 \quad \frac{\partial \mathbf{Q}'}{\partial t} + \mathbf{D}_{x0} \frac{\partial^2 \mathbf{Q}'}{\partial x^2} + \mathbf{D}_{y0} \frac{\partial^2 \mathbf{Q}'}{\partial y^2} + \mathbf{A}_{x0} \frac{\partial \mathbf{Q}'}{\partial x} + \mathbf{A}_{y0} \frac{\partial \mathbf{Q}'}{\partial y} + \mathbf{B}_0 \mathbf{Q}' = 0, \quad (24)$$

273 where the vector of dependent variables is:

$$274 \quad \mathbf{Q}' = [h', q'_x, q'_y, I', \eta', [M'_{ak}]]^\top, \quad (25)$$

275 where the square bracket indicates the vector character.

276 The diffusive matrix in x direction is:

$$277 \quad \mathbf{D}_{x0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2\nu \frac{q_{x0}}{h_0} & -2\nu & -\nu & 0 & 0 & 0 \\ 0 & 0 & -\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (26)$$

278 where $\mathbb{0}$ denotes the zero matrix. The diffusive matrix in y direction is:

$$279 \quad \mathbf{D}_{y0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \nu \frac{q_{x0}}{h_0} & -\nu & 0 & 0 & 0 & 0 \\ \nu \frac{q_{x0}}{h_0} & -\nu & -2\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial q_{by}}{\partial \frac{\partial \eta}{\partial y}} \Big|_0 & 0 \\ \hline 0 & 0 & 0 & 0 & \left[\frac{\partial q_{byk}}{\partial \frac{\partial \eta}{\partial y}} \Big|_0 - f_{k0}^I \frac{\partial q_{by}}{\partial \frac{\partial \eta}{\partial y}} \Big|_0 \right] & 0 \end{bmatrix}. \quad (27)$$

280 The advective matrix in x direction is:

$$281 \quad \mathbf{A}_{x0} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ gh_0 - \left(\frac{q_{x0}}{h_0} \right)^2 & 2 \frac{q_{x0}}{h_0} & 0 & 0 & gh_0 & 0 \\ 0 & 0 & \frac{q_{x0}}{h_0} & -\beta^* q_{x0} & 0 & 0 \\ 0 & 0 & -\frac{q_{x0}}{h_0^2 L_1^*} & \frac{q_{x0}}{h_0} & 0 & 0 \\ \hline \frac{-q_{x0}}{h_0} \frac{\partial q_{bx}}{\partial q_x} \Big|_0 & \frac{\partial q_{bx}}{\partial q_x} \Big|_0 & 0 & 0 & 0 & \left[\frac{\partial q_{bx}}{\partial M_{al}} \Big|_0 \right] \\ \left[\frac{-q_{x0}}{h_0} \frac{\partial q_{bk}}{\partial q_x} \Big|_0 + f_{k0}^I \frac{q_{x0}}{h_0} \frac{\partial q_b}{\partial q_x} \Big|_0 \right] & \left[\frac{\partial q_{bxk}}{\partial q_x} \Big|_0 - f_{k0}^I \frac{\partial q_{bx}}{\partial q_x} \Big|_0 \right] & 0 & 0 & 0 & \left[\frac{\partial q_{bxk}}{\partial M_{al}} \Big|_0 - f_{k0}^I \frac{\partial q_{bx}}{\partial M_{al}} \Big|_0 \right] \end{bmatrix}. \quad (28)$$

282 The advective matrix in y direction is:

$$283 \quad \mathbf{A}_{y0} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{q_{x0}}{h_0} & -\beta^* q_{x0} & 0 & 0 \\ gh_0 & 0 & 0 & 0 & gh_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial q_{by}}{\partial q_y} \Big|_0 & \frac{\partial q_{by}}{\partial I} \Big|_0 & 0 & 0 \\ \hline 0 & 0 & \left[\frac{\partial q_{byk}}{\partial q_y} \Big|_0 - f_{k0}^I \frac{\partial q_{by}}{\partial q_y} \Big|_0 \right] & \left[\frac{\partial q_{byk}}{\partial I} \Big|_0 - f_{k0}^I \frac{\partial q_{by}}{\partial I} \Big|_0 \right] & 0 & 0 \end{bmatrix}. \quad (29)$$

284 The matrix of linear terms is:

$$\mathbf{B}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-3C_f q_{x0}^2}{h_0^3} & \frac{2C_f q_{x0}}{h_0^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{C_f q_{x0}}{h_0^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q_{x0}}{h_0^2 L_1^*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \quad (30)$$

285
286 We assume that the perturbations can be represented as a Fourier series, which implies that
287 they are piecewise smooth and bounded for $x = \pm\infty$. Using this assumption the solution of the
288 perturbed system is expressed in the form of normal modes:

$$289 \quad \mathbf{Q}' = \text{Re} \left(\mathbf{V} e^{i(k_{wx} + k_{wy} - \omega t)} \right) , \quad (31)$$

290 where i is the imaginary unit, k_{wx} [rad/m] and k_{wy} [rad/m] are the real wave numbers in x and
291 y direction, respectively, $\omega = \omega_r + i\omega_i$ [rad/s] is the complex angular frequency, \mathbf{V} is the complex
292 amplitude vector, and Re denotes the real part of the solution (which we will omit in the subsequent
293 steps). The variable ω_r is the angular frequency and ω_i the attenuation coefficient. A value of $\omega_i > 0$
294 implies growth of perturbations and $\omega_i < 0$ decay. Substitution of equation (31) in equation (24)
295 yields:

$$296 \quad [\mathbf{M}_0 - \omega \mathbb{1}] \mathbf{V} = 0 , \quad (32)$$

297 where:

$$298 \quad \mathbf{M}_0 = \mathbf{D}_{x0} k_{wx}^2 \mathbf{i} + \mathbf{D}_{y0} k_{wy}^2 \mathbf{i} + \mathbf{A}_{x0} k_{wx} + \mathbf{A}_{y0} k_{wy} - \mathbf{B}_0 \mathbf{i} , \quad (33)$$

299 and $\mathbb{1}$ denotes the unit matrix. Equation (32) is an eigenvalue problem in which the eigenvalues
300 of \mathbf{M}_0 (as a function of the wave number) are the values of ω satisfying equation (32).

301 The solution of the linear model provides information regarding the development of small
302 amplitude oscillations only, but for an arbitrary wave number. For this reason the linear model
303 is convenient for studying the well-posedness of the model, which we will assess in the following
304 section.

3 Instability, Hyperbolicity, and Ill-Posedness

Ill-posedness has been related to the system of governing equations losing its hyperbolic character. Stability analysis investigates growth and decay of perturbations of a base state. The two mathematical problems may seem unrelated but in fact they are strongly linked. In this section we clarify the terms unstable, hyperbolic, and ill-posed, and present the mathematical framework that we use to study the well-posedness of the system of equations.

A system is stable if perturbations to an equilibrium state decay and the solution returns to its original state. This is equivalent to saying that all possible combinations of wave numbers in the x and y directions yield a negative growth rate (ω_i , equation (31)). An example of a stable system in hydrodynamics is the inviscid Shallow Water Equations (iSWE) for a Froude number smaller than 2 (*Jeffreys, 1925; Balmforth and Mandre, 2004; Colombini and Stocchino, 2005*). In figure 1a we show the maximum growth rate of perturbations to a reference solution (Case I1, tables 1 and 2) of the iSWE on an inclined plane (i.e., the first 3 equations of the complete system, equation (24), with neither secondary flow nor diffusion). The growth rate is obtained numerically by computing the eigenvalues of the reduced matrix \mathbf{M}_0 (the first 3 rows and columns in equation (33)) for wave numbers between 0 and 250 rad/m, which is equivalent to wavelengths ($l_{wx} = 2\pi/k_{wx}$ and equivalently for y) down to 1 cm. Figure 1b presents the same information as figure 1a in terms of wavelength rather than wave number to better illustrate the behaviour for large wavelengths. The growth rate is negative for all wave numbers, which confirms that the iSWE for $Fr < 2$ yield a stable solution.

u [m/s]	v [m/s]	h [m]	C_f [-]
1	0	1	0.007

Table 1: Reference state.

Case	model	Fr	stability	mathematical character
I1	iSWE	0.32	stable	well-posed
B1	iSWE+Exner	0.32	unstable	well-posed
I2	iSWE	2.01	unstable	ill-posed

Table 2: Cases of a stable well-posed model (I1), an unstable well-posed model (B1), and an ill-posed model (I2). Case I2 has the same parameter values as Case I1 but for the mean flow velocity which is equal to 6.30 m/s.

A system is unstable when perturbations to an equilibrium state grow and the solution diverges from the initial equilibrium state. The growth of river bars is an example of an unstable system in river morphodynamics. A straight alluvial channel is stable if the width-to-depth ratio is sufficiently small and, above a certain threshold value, the channel becomes unstable and free alternate bars grow (*Engelund and Skovgaard, 1973; Fredsøe, 1978; Colombini et al., 1987; Schielen et al., 1993*).

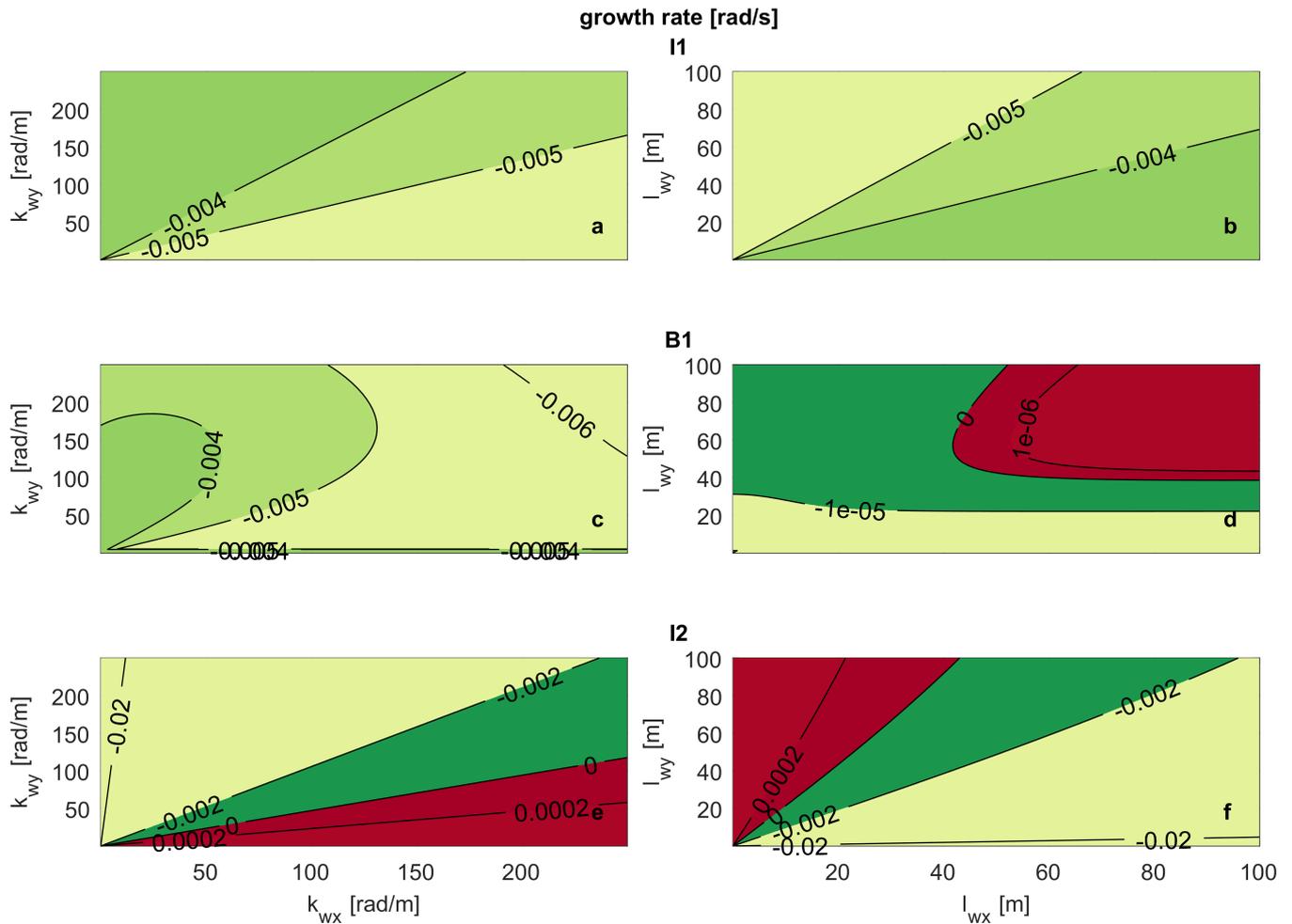


Figure 1: Growth rate of perturbations added to the reference case (tables 1 and 2) as a function of the wave number and the wavelength: (a)-(b) iSWE, $Fr < 2$ (Case I1, well-posed), (c)-(d) iSHE+Exner (Case B1, well-posed), and (e)-(f) iSWE, $Fr > 2$ (Case I2, ill-posed). The subplots in the two columns show the same information but highlight the behaviour for large wave numbers (left column) and for large wavelengths (right column). Red and green indicates growth and decay of perturbations, respectively.

330 Mathematically, an unstable system has a region, a domain in the wave number space, in which the
 331 growth rate of perturbations is positive. In figure 1c-d we present the growth rate of perturbations
 332 to a reference solution consisting of uniform flow (table 1) on an alluvial bed composed of unisize
 333 sediment with a characteristic grain size equal to 0.001 m (Case B1, table 2). The sediment
 334 transport rate is computed using the relation by *Engelund and Hansen* (1967) (equation (42)) and
 335 the effect of the bed slope on the sediment transport direction is accounted for using the simplest
 336 formulation, $g_s = 1$. Figure 1d confirms the classical result of linear bar theory: there exists a
 337 critical transverse wavelength (l_{wyc}) below which all perturbations decay. In our particular case
 338 $l_{wyc} = 40.2$ m. Impermeable boundary conditions at the river banks limit the possible wavelengths
 339 to fractions of the channel width B [m] such that $l_{wy} = 2B/m$ for $m = 1, 2, \dots$ (*Callander*, 1969). As
 340 the most unstable mode is the first one (i.e., $m = 1$, alternate bars) (*Colombini et al.*, 1987; *Schielen*
 341 *et al.*, 1993), the minimum channel width above which perturbations grow is $B_c = l_{wyc}/2 = 20.1$ m,

342 which confirms the results of *Schielen et al.* (1993). Figure 1c highlights, as for case I1, the decay
343 of short waves.

344 A particular case of instability is that in which the domain of positive growth rate extends
345 to infinitely large wave numbers (i.e., short waves). Under this condition there is no cutoff wave
346 number above which we can neglect the contribution of ever shorter waves with non-zero growth
347 rates. For any unstable perturbation a shorter one can be found which is even more unstable.
348 This implies that the growth rate of an infinitesimal perturbation (i.e., noise) tends to infinity.
349 Such a system cannot represent a physical phenomenon, as the growth rate of any physical process
350 in nature is bounded. A system in which the growth rate of infinitesimal perturbations tends
351 to infinity does not have a unique solution depending continuously on the initial and boundary
352 conditions, which implies that the system is ill-posed (*Hadamard, 1923; Joseph and Saut, 1990*).
353 An example of an ill-posed hydrodynamic model is the iSWE for flow with a Froude number larger
354 than 2. In figure 1e-f we show the growth rate of perturbations to the reference solution of a
355 case in which the Froude number is slightly larger than 2 (Case I2, table 2). The growth rate
356 extends to infinitely large wave numbers, which confirms that this case is ill-posed. A model being
357 ill-posed is an indication that there is a relevant physical mechanism that has been neglected in the
358 model derivation (*Fowler, 1997*). Viscous forces regularise the iSWE (i.e., make the model well-
359 posed) and rather than ill-posed, the viscous Shallow Water Equations become simply unstable
360 for a Froude number larger than 2, predicting the formation of roll-waves (*Balmforth and Mandre,*
361 *2004; Balmforth and Vakil, 2012; Rodrigues and Zumbrun, 2016; Barker et al., 2017a,b*).

362 Chaotic models, just as ill-posed models, are sensitive to the initial and boundary conditions
363 and lose their predictive capabilities in a deterministic sense (*Lorenz, 1963*). Yet, there are two
364 essential differences. First, chaotic systems lose their predictive capabilities after a certain time
365 (*Devaney, 1989; Banks et al., 1992*), yet there exists a finite time in which the dynamics are
366 predictable. In ill-posed models infinitesimal perturbations to the initial condition cause a finite
367 divergence in the solution in an arbitrarily (but fixed) short time. Second, while the dynamics
368 of a chaotic model are not predictable in deterministic terms after a certain time, these continue
369 to be predictable in statistical terms. For this reason, although being sensitive to the initial and
370 boundary conditions, a model presenting chaotic properties can be used, for instance, to capture
371 the essential dynamics and spatio-temporal features of river braiding (*Murray and Paola, 1994,*
372 *1997*). On the contrary, the dynamics of an ill-posed model cannot be analysed in statistical terms.

373 The numerical solution of an ill-posed problem continues to change as the grid is refined because
374 a smaller grid size resolves larger wave numbers with faster growth rates (*Joseph and Saut, 1990;*
375 *Kabanikhin, 2008; Barker et al., 2015; Woodhouse et al., 2012*). In other words, the numerical
376 solution of an ill-posed problem does not converge when the grid cell size is reduced. This property
377 emphasizes the unrealistic nature of ill-posed problems and shows that ill-posed models cannot be
378 applied in practice.

379 We present an example of grid dependence specifically related to river morphodynamics under
 380 conditions with mixed-size sediment. We consider a case of degradation into a substrate finer
 381 than the active layer, as this is a situation in which the active layer model is prone to be ill-posed
 382 (Section 1). The reference state is the same as in Case B1, yet the sediment is a mixture of two sizes
 383 equal to 0.001 m and 0.010 m. The bed surface is composed of 10 % of fine sediment. The active
 384 layer thickness is equal to 0.05 m, which in this case is representative of small dunes covering the
 385 bed (e.g. *Deigaard and Fredsøe, 1978; Armanini and Di Silvio, 1988; Blom, 2008*). Depending on
 386 the substrate composition, this situation yields an ill-posed model (*Chavarrías et al., 2018*). When
 387 the substrate is composed of 50 % of fine sediment (Case H1, table 3), the problem is well-posed
 388 and it is ill-posed when the substrate is composed of 90 % of fine sediment (Case H2, table 3).

389 We use the software package Delft3D (*Lesser et al., 2004*) to solve the system of equations.
 390 We stress that the problem of ill-posedness is inherent to the system of equations and independent
 391 from the numerical solver. We have implemented a subroutine that assesses the well-posedness of
 392 the system of equations at each node and time step. The domain is 100 m long and 10 m wide.
 393 The downstream water level is lowered at a rate of 0.01 m/h to induce degradational conditions.
 394 The upstream sediment load is constant and equal to the equilibrium value of the reference state
 395 (*Blom et al., 2017*). The cells are square and we consider three different sizes (table 3). The time
 396 step varies between simulations to maintain a constant value of the CFL number.

Case	f_1^I [-]	Δx [m]	mathematical character
H1a	0.5	0.50	well-posed
H1b	0.5	0.25	well-posed
H1c	0.5	0.10	well-posed
H2a	0.9	0.50	ill-posed
H2b	0.9	0.25	ill-posed
H2c	0.9	0.10	ill-posed

Table 3: Cases showing the effect of grid cell size on the numerical solution of well-posed and ill-posed models.

397 Figure 2 presents the bed elevation after 10 h. The result of the well-posed case (H1, left
 398 column) is grid independent. The result of the ill-posed case (H2, right column) changes as the
 399 grid is refined and presents an oscillatory pattern characteristic of ill-posed simulations (*Joseph and*
 400 *Saut, 1990; Woodhouse et al., 2012; Barker et al., 2015; Chavarrías et al., 2018*). The bed seems to
 401 be flat in the ill-posed simulation with a coarser grid (figure 2b). This is because oscillations grow
 402 slowly on a coarse grid and require more time to be perceptible. The waviness of the bed is seen
 403 in the result of the check routine, as it predicts ill-posedness only at those locations where the bed
 404 degrades (the stoss face of the oscillations). The fact that the model is well-posed in almost the
 405 entire domain in the ill-posed case solved using a cell sizes equal to 0.25 m (H2b, figure 2d) and
 406 0.10 m (H2c, figure 2f) does not mean that the results are realistic. Non-physical oscillations have
 407 grown and vertically mixed the sediment such that the situation is well-posed after 10 h (*Chavarrías*

408 *et al.*, 2018). We provide a movie of figure 2 in the online supplementary material.

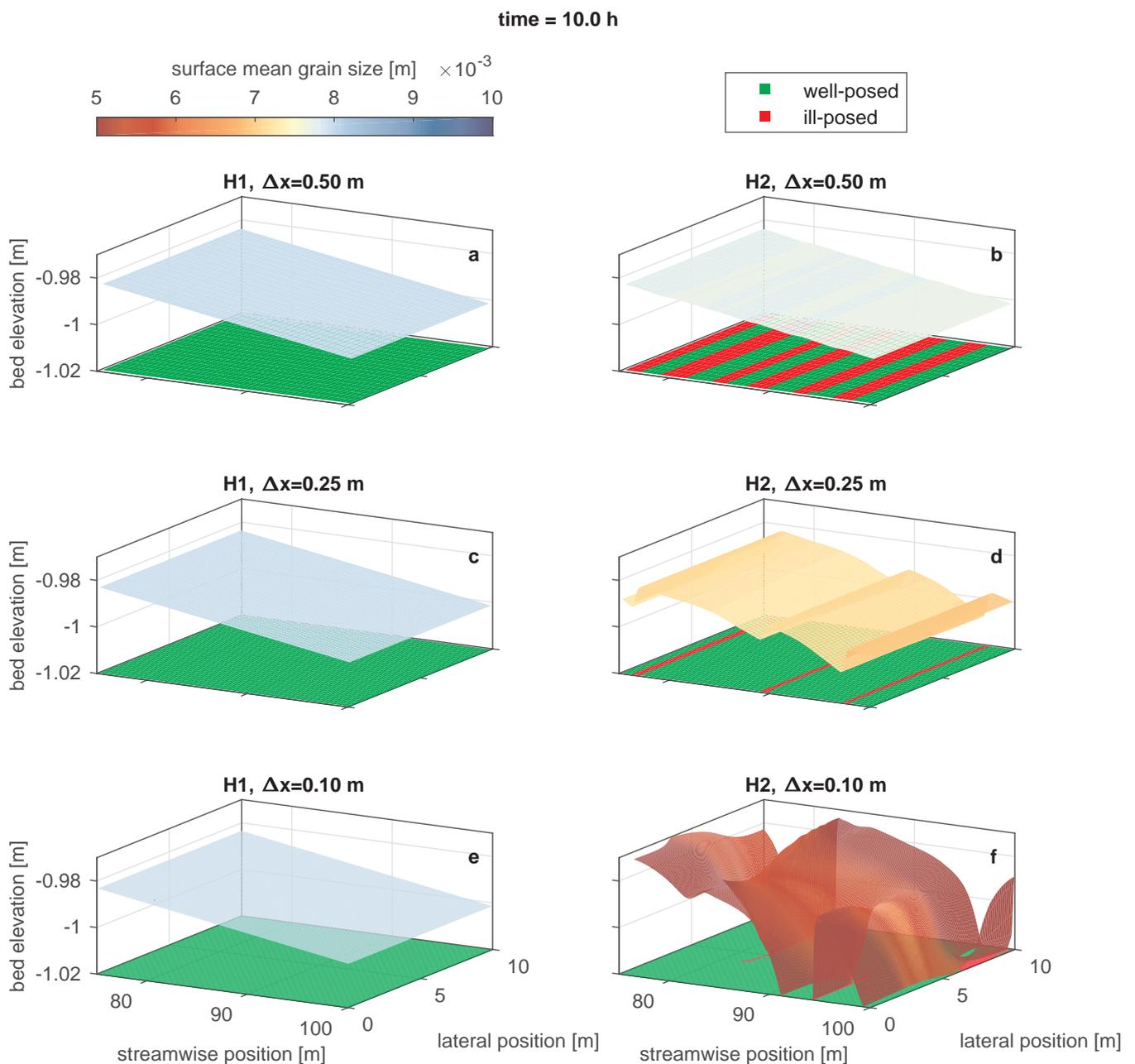


Figure 2: Simulated bed elevation (surface) and mean grain size at the bed surface (colour) of a well-posed case (left column, H1, table 3) and an ill-posed case (right column, H2, table 3). In each row we present the results for varying cell size. The colour of the $x - y$ plane shows the result of the routine that checks whether the conditions at each node yield a well-posed (green) or an ill-posed (red) model.

409 In the above idealised situations it is evident that the oscillations are non-physical and it is
 410 straightforward to do a converge test to clarify that the solution is grid dependent. In complex
 411 domains in which several processes play a role, it is more difficult to associate oscillations to
 412 ill-posedness. Moreover, in long term applications the growth rate of perturbations may be fast
 413 compared to the frequency at which model results are assessed, which may hide the consequences

414 of ill-posedness. If one studies a process that covers months or years (and consequently analyses
 415 the results on a monthly basis) but perturbations due to ill-posedness grow on an hourly scale, it
 416 may be difficult to identify that the problem is ill-posed. Using poor numerical techniques to solve
 417 the system of equations also contributes to hiding the consequences of ill-posedness as numerical
 418 diffusion dampens perturbations. These factors may explain why the problem of ill-posedness in
 419 mixed-sediment river morphodynamics is not widely acknowledged.

420 In the river morphodynamics community, the term ellipticity has been used to refer to ill-
 421 posedness of the system of equations in contrast to hyperbolicity, which is associated to well-
 422 posedness (*Ribberink*, 1987; *Mosselman*, 2005; *Stecca et al.*, 2014; *Siviglia et al.*, 2017; *Chavarrías*
 423 *et al.*, 2018). In general the terms are equivalent, but not always. We consider a unit vector
 424 \hat{n} in the direction (x, y) , $\hat{n} = (\hat{n}_x, \hat{n}_y)$. The system of equations (24) is hyperbolic if matrix
 425 $\mathbf{A} = \mathbf{A}_{x0}\hat{n}_x + \mathbf{A}_{y0}\hat{n}_y$ diagonalises with real eigenvalues $\forall \hat{n}$ (e.g. *LeVeque*, 2004; *Castro et al.*,
 426 2009). Neglecting friction and diffusive processes (i.e., $\mathbf{B}_0 = \mathbf{D}_{x0} = \mathbf{D}_{y0} = \mathbf{0}$), hyperbolicity
 427 implies that the eigenvalues of \mathbf{M}_0 (equation (33)) are real. In this case, as the growth rate of
 428 perturbations (i.e., the imaginary part of the eigenvalues of \mathbf{M}_0) is equal to 0 regardless of the
 429 wave number, the system of equations is well-posed. As the coefficients of \mathbf{A} are real, complex
 430 eigenvalues appear in conjugate pairs. This means that if \mathbf{A} has a complex eigenvalue (i.e., the
 431 problem is not hyperbolic), at least one wave will have a positive growth rate. Neglecting friction
 432 and diffusive processes, non-hyperbolicity implies that infinitely large wave numbers have a positive
 433 growth rate. We conclude that, in the absence of diffusion and friction, lack of hyperbolicity implies
 434 ill-posedness. Note that ellipticity (i.e., the eigenvalues of \mathbf{A} are all complex) is not required for
 435 the problem to be ill-posed, as it suffices that the problem is not hyperbolic. When considering
 436 diffusion and friction even when \mathbf{A} has complex eigenvalues, the imaginary part of the eigenvalues
 437 of \mathbf{M}_0 may all be negative and the problem well-posed.

438 Finally, well-posedness and hyperbolicity are similar terms when dealing with problems arising
 439 from conservation laws and changes with time, as hyperbolicity guarantees the existence of wave
 440 solutions (*Lax*, 1980; *Courant and Hilbert*, 1989; *Strikwerda*, 2004; *Toro*, 2009; *Dafermos*, 2010;
 441 *Bressan*, 2011; *Dafermos*, 2016). In communities such as materials science, it is the term hyper-
 442 bolicity that is associated to ill-posedness, as a smooth solution of, for instance the stress, requires
 443 that the system is elliptic (*Knowles and Sternberg*, 1975, 1976; *Veprek et al.*, 2007).

444 4 Stability Analysis

445 In this section we study the applicability of the system of equations to model two-dimensional river
 446 morphodynamics by means of a stability analysis of perturbations. We study the effects of the
 447 secondary flow model (Sections 4.1) and the bed slope (Section 4.2) on model ill-posedness.

4.1 Ill-Posedness Due to Secondary Flow

In this section we study how the stability of the system of equations is affected by the secondary flow model. To gain insight we compare three cases. In the first case we omit secondary flow. In the second and third cases we include the secondary flow model with and without considering diffusion (table 4).

Case	secondary flow	ν	stability	mathematical character
S1	no	ν_E	stable	well-posed
S2	yes	ν_E	unstable	well-posed
S3	yes	0	unstable	ill-posed

Table 4: Variations to the reference state (table 1) and results of the linear analysis with respect to secondary flow.

The first case is equivalent to I1 (table 2), yet the eddy viscosity is equal to the value derived by *Elder* (equation (4), $\nu = \nu_E = 0.0057 \text{ m}^2/\text{s}$). In figure 3a-b we plot the maximum growth rate of perturbations as a function of the wave number and the wavelength, respectively. Diffusion appears to significantly dampen perturbations (compare figure 1a in which diffusion is neglected to figure 3a).

In the second case we repeat the analysis including the equation for advection and diffusion of the secondary flow intensity (i.e., the first 4 rows and columns of matrix \mathbf{M}_0 in equation (33), Case S2, table 4). We observe that accounting for secondary flow introduces an instability mechanism (figure 3d). For the specific conditions of the case, a growth domain appears for wavelengths between 0.7 m and 39 m long and between 0.4 m and 19 m wide. The maximum growth corresponds to a wavelength in the x and y direction equal to 1.29 m and 0.74 m, respectively. This situation is well-posed, as for large wave numbers perturbations decay (figure 3c). Yet, the model is unsuitable for reproducing such instability, as it predicts growth of perturbations with a length scale of the order of the flow depth and shorter, for which the SWE model is not suited. Given the fact that we consider a depth-averaged formulation of the primary flow, processes that scale with the flow depth are not resolved by the model and consequently perturbations at that scale must decay to yield physically realistic results. Otherwise, scales of the order of the flow depth become relevant, which contradicts the assumptions of the depth-averaged formulation. To model processes that scale with the flow depth such as dune growth, it is necessary to account for non-depth-averaged flow formulations that consider, for instance, rotational flow (*Colombini and Stocchino*, 2011, 2012), or non-hydrostatic pressure (*Giri and Shimizu*, 2006; *Shimizu et al.*, 2009).

In the third case we test the secondary flow model without accounting for diffusion in the system of equations ($\nu = 0$, Case S3, table 4). We observe that the instability domain extends to infinitely large wave numbers (figure 3e), which implies that this model is ill-posed (Section 3). We now aim to prove that the Shallow Water Equations in combination with the secondary flow model without

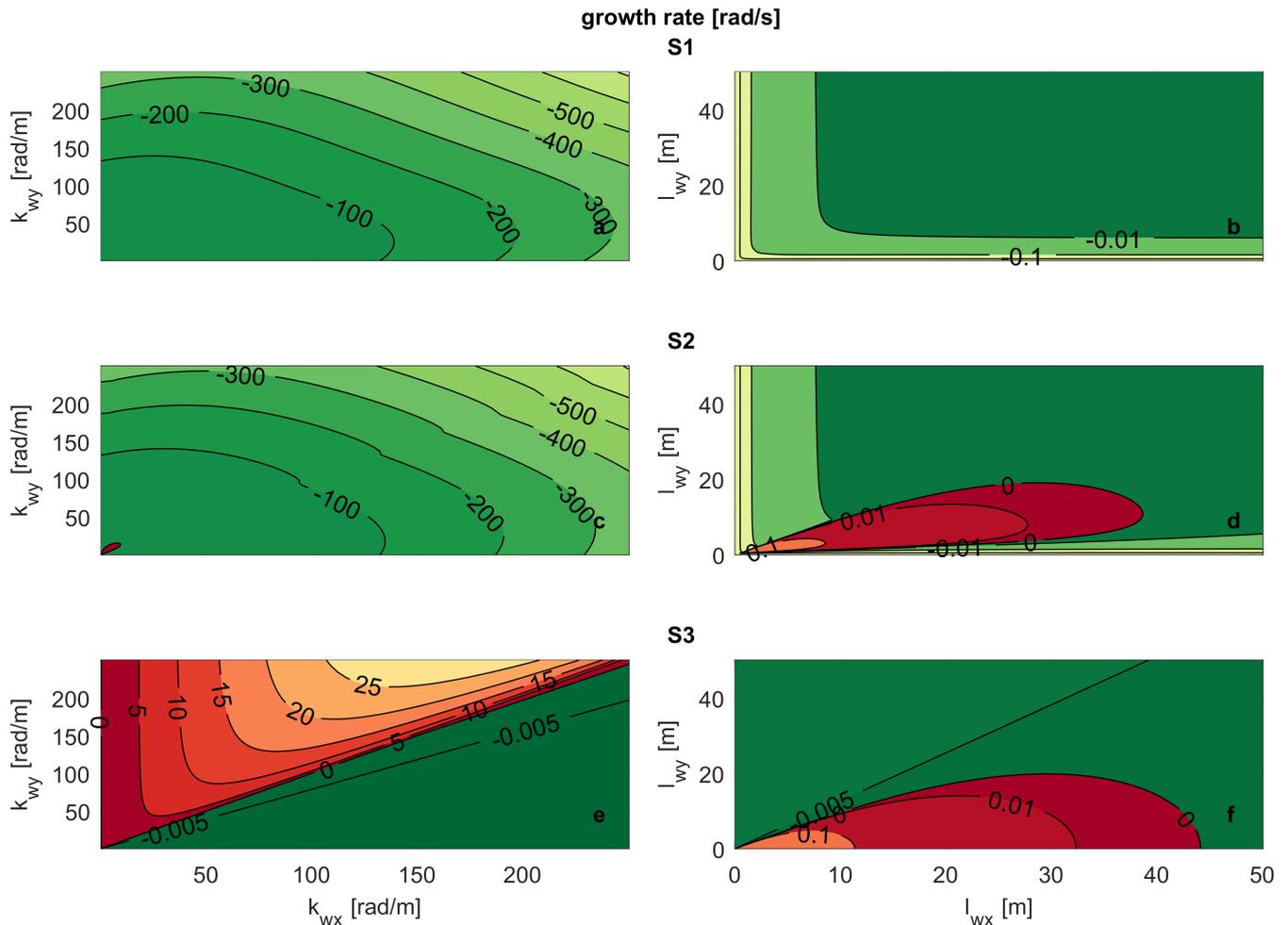


Figure 3: Growth rate of perturbations added to the reference case (tables 1 and 4) as a function of the wave number and the wavelength: (a)-(b) without secondary flow (Case S1, well-posed), (c)-(d) accounting for secondary flow with diffusion (Case S2, well-posed), and (e)-(f) accounting for secondary flow without diffusion (Case S3, ill-posed). The subplots in the two columns show the same information but highlight the behaviour for large wave numbers (left column) and for large wavelengths (right column). Red and green indicates growth and decay of perturbations, respectively.

478 diffusion always yields an ill-posed model. To this end we obtain the characteristic polynomial
 479 of matrix \mathbf{M}_0 (equation (33)). We compute the discriminant of the fourth order characteristic
 480 polynomial and we find that for $k_{wx} < k_{wy}$ the growth rate of perturbations is positive (Appendix
 481 C). The model is ill-posed, as there always exists a domain of growth extending to infinitely large
 482 wave numbers in the transverse direction.

483 We assess how the length scale of the instability related to the secondary flow model depends
 484 on the flow parameters. For this purpose we compute the shortest wave with positive growth
 485 for a varying diffusion coefficient and flow conditions (figure 4). We observe that, independently
 486 from the flow conditions, the theoretical value of the diffusion coefficient derived by *Elder* (1959)
 487 (equation (4)) is insufficient for dampening oscillations scaling with the flow depth. We conclude
 488 that if the diffusion coefficient is realistic, the treatment of the secondary flow yields an unrealistic
 489 model. It is necessary to use an unrealistically large value of the diffusion coefficient to obtain a

490 realistic depth-averaged model in which perturbations scaling with the flow depth decay.

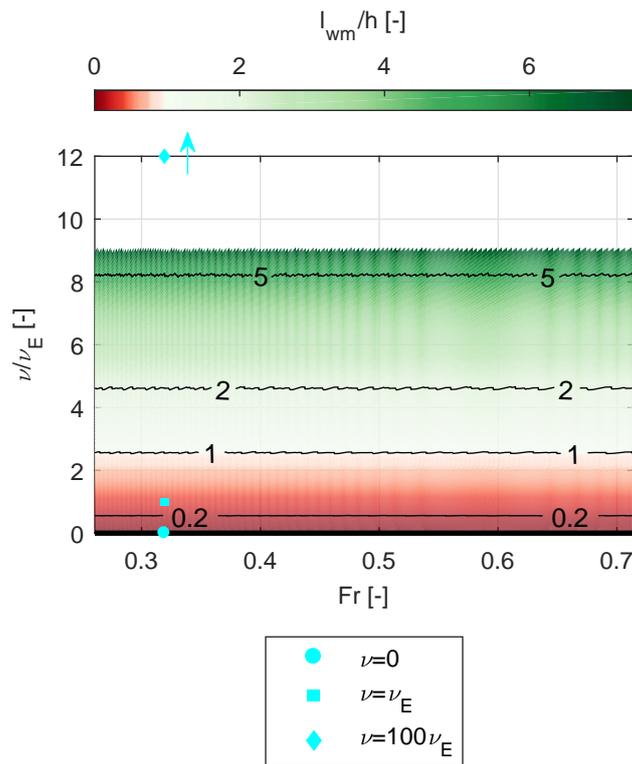


Figure 4: Wavelength of the shortest perturbation with positive growth rate (l_{wm}) relative to the flow depth (h) as a function of the Froude number (Fr) and the diffusion coefficient (ν) relative to the diffusion coefficient according to *Elder* (1959) (ν_E). Different flow conditions are studied varying the flow depth between 0.2 m and 1.5 m from the reference case (table 1). The cyan markers indicate the conditions of three numerical simulations with different values of the diffusion coefficient (Section 5.1). The arrow next to the diamond marker indicates that the value lies outside the figure. Red (green) colours indicate that the shortest wave length with positive growth rate are smaller (larger) than the flow depth.

491 4.2 Ill-Posedness Due to Bed Slope Effect

492 In this section we study the influence of considering the effect of the bed slope on model well-
 493 posedness. To gain insight we compare 5 cases in which we consider unisize and mixed-size sedi-
 494 ment, various sediment transport relations, and various bed slope functions (table 5). We neglect
 495 secondary flow and diffusion to reduce the complexity of the problem (*Parker*, 1976; *Fredsøe*, 1978;
 496 *Colombini et al.*, 1987; *Schielen et al.*, 1993).

497 Our reference case is B1 (Section 3) which considers unisize sediment conditions, and the
 498 effect of the bed slope on the sediment transport direction is accounted for using the simplest
 499 formulation, $g_s = 1$. We have shown that this case is well-posed. Neglecting the effect of the bed
 500 slope on the sediment transport direction (Case B2, table 5) makes the problem ill-posed (figure
 501 5a). This illustrates that accounting for the effect of the bed slope is required for obtaining not
 502 only physically realistic but also mathematically well-posed results. We prove that the Shallow

503 Water Equations in combination with the *Exner* (1920) equation without considering the effect of
 504 the bed slope always yields an ill-posed model by studying the growth rate of perturbations in the
 505 limit for the wave number k_{wy} tending to infinity (Appendix D).

Case	sediment	d_2 [m]	sed. trans.	bed slope	mathematical character
B1	unisize	-	EH	$g_s = 1$	well-posed
B2	unisize	-	EH	No	ill-posed
B3	mixed-size	0.004	AM	$g_{sk} = 1$	well-posed
B4	mixed-size	0.004	AM	$g_{sk} = 1.7\theta_k^{0.5}$	ill-posed
B5	mixed-size	0.012	AM	$g_{sk} = 1$	ill-posed

Table 5: Variations to the reference state (table 1) and results of the linear analysis with respect to the effect of the bed slope on the sediment transport direction. EH and AM refer to the sediment transport relations by *Engelund and Hansen* (1967) and *Ashida and Michiue* (1971), respectively.

506 The fact that the bed slope effect dampens perturbations under unisize conditions is expected
 507 from the fact that the only diffusive term in the system of equations is $\partial q_{by}/\partial s_y$ (equation (27)),
 508 where $s_y = \partial\eta/\partial y$. This term is negative and approximately equal to $-q_b/g_s$ for a small streamwise
 509 slope. When we consider more than one grain size, diffusive terms appear in each active layer
 510 equation. We find that these diffusive terms may be positive, which hints at the possibility of an
 511 antidiffusive behaviour, which may lead to ill-posedness. To study this possibility we compute the
 512 growth rate of perturbations of a simplified case consisting of two sediment size fractions. In the
 513 limit for the wave numbers tending to infinity, the maximum growth rate is:

$$514 \quad \omega_i^{\text{lim}} = \alpha_1 (r_{y1} - d_{x1,1})^2 + \alpha_2 (r_{y1} - d_{x1,1}) + \alpha_3, \quad (34)$$

515 where α_i for $i = 1, 2, 3$ are parameters relating the flow and the sediment transport rate (Appendix
 516 E). The parameter r_{y1} explains how the sediment transport rate of the fine fraction is affected by
 517 changes in the transverse bed slope:

$$518 \quad r_{y1} = \frac{\partial q_{by1}/\partial s_y}{\partial q_{by}/\partial s_y}, \quad (35)$$

519 and the parameter $d_{x1,1}$ relates changes in the sediment transport rate to changes in the volume
 520 of sediment in the active layer:

$$521 \quad d_{x1,1} = \frac{\partial q_{bx1}/\partial M_{a1}}{\partial q_{bx}/\partial M_{a1}}. \quad (36)$$

522 As $\alpha_1 > 0$ (Appendix E), there exist an interval $(r_{y1} - d_{x1,1})^- < (r_{y1} - d_{x1,1}) < (r_{y1} - d_{x1,1})^+$ in
 523 which $\omega_i^{\text{lim}} < 0$ and the model is well-posed. Outside the interval, $\omega_i^{\text{lim}} > 0$ and the problem is
 524 ill-posed.

525 The physical interpretation of the limit values for obtaining a well-posed model is as follows.
 526 The effect of the transverse bed slope (r_{y1}) needs to be balanced with respect to the effect of
 527 changes in surface texture ($d_{x1,1}$) to obtain a well-posed model. For a given $d_{x1,1}$, if parameter

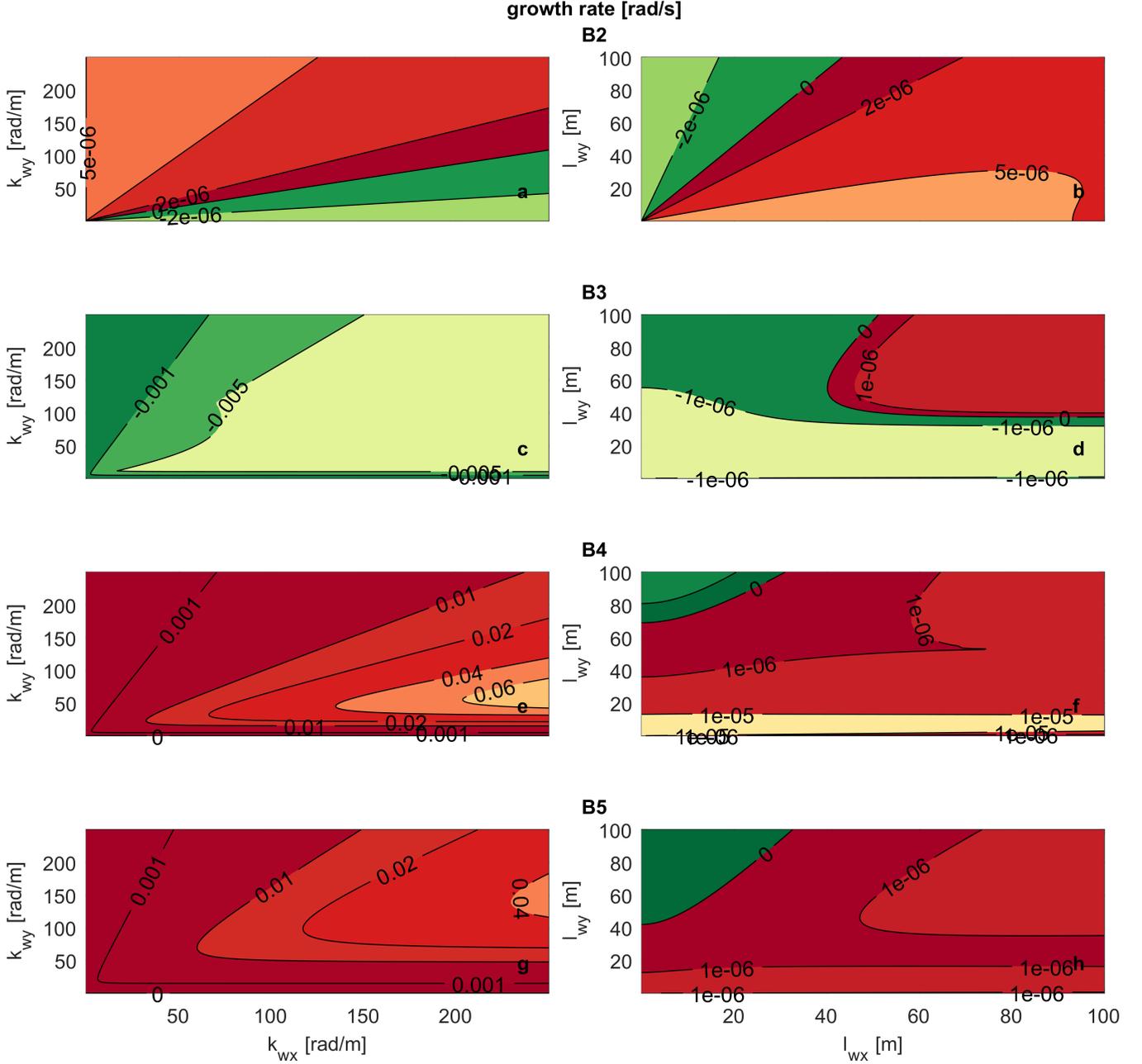


Figure 5: Growth rate of perturbations added to the reference case (tables 1 and 5) as a function of the wave number and the wavelength: (a)-(b) Case B2 (ill-posed), (c)-(d) Case B3 (well-posed), (e)-(f) Case B4 (ill-posed), and (g)-(h) Case B5 (ill-posed). The subplots in the two columns show the same information but highlight the behaviour for large wave numbers (left column) and for large wavelengths (right column). Red and green indicates growth and decay of perturbations, respectively.

528 r_{y1} is too small (i.e., the bed slope effect is not sufficiently strong) perturbations in the transverse
 529 direction are not dampened and the model is ill-posed. On the other hand, for a given r_{y1} , if
 530 parameter $d_{x1,1}$ is too small (e.g. due to relatively strong hiding or in conditions close to incipient
 531 motion) perturbations in the streamwise direction do not decay and the model is also ill-posed.
 532 The critical values r_{y1}^{\pm} that allow for a well-posed model are shown in Appendix E.

533 In Cases B3-B5 we illustrate the possibility of ill-posedness due to the bed slope closure relation

(table 5). In Case B3 the sediment mixture consists of two grain size fractions with characteristic grain sizes equal to 0.001 m and 0.004 m. The volume fraction content of the fine sediment in the active layer and at the interface between the active layer and the substrate is equal to 0.5. The sediment transport rate is computed using the relation developed by *Ashida and Michiue* (1971). The other parameters are equal to the reference case (table 1). The system is well-posed when the effect of the bed slope does not depend on the bed shear stress (figure 5c). The situation is ill-posed if the effect of the bed slope depends on the bed shear stress (Case B4, table 5, figure 5e). The cause of ill-posedness is not found in the closure relation for the bed slope effect only but in the combination of the closure relation with the flow and bed surface conditions. A case equal to B3 except for the fact that the coarse grain size is equal to 0.012 m is ill-posed (Case B5, table 5, figure 5g).

We assess how the domain of ill-posedness due to the bed slope effect depends on the model parameters. For given sediment sizes, flow, and bed surface texture, parameter B_s (equation (23)) controls the effect of the bed slope by modifying r_{y1} only. The parameter A_s (equation (23)) cancels in r_{y1} and does not play a role. For this reason we study how g_{s1}/A_s [-] affects the domain of ill-posedness for varying sediment properties, flow, and bed surface grain size distribution (figure 6). We consider Case B3 and we vary B_s between 0 and 0.5 to vary the bed slope effect. The sediment size of the coarse fraction varies between d_1 and 0.020 m. The mean flow velocity varies between 1 m/s and 2.8 m/s. The volume fraction content of fine sediment at the bed surface varies between 0 and 1. We aim to isolate the problem of ill-posedness due to bed slope effect from the problem of ill-posedness due to a different grain size distribution at the bed surface and at the interface between the bed surface and the substrate (*Chavarrías et al.*, 2018). For this reason, in this analysis the volume fraction content of fine sediment at the interface is equal to the one at the bed surface. Under this condition the problem can be ill-posed due to the effect of the bed slope only.

As we have shown analytically, under unisize conditions (i.e., $d_1/d_2 = 1$ or $F_{a1} = 1$ or $F_{a1} = 0$) the model is well-posed (figure 6a and 6c). For sufficiently different grain sizes ($d_1/d_2 \lesssim 0.15$) the model is well-posed regardless of the bed slope effect but for a small range of values ($0.08 \lesssim d_1/d_2 \lesssim 0.1$). This small range of ill-posed values is associated with r_{y1} increasing for decreasing values of d_1/d_2 and acquiring a value larger than r_{y1}^+ such that the model becomes ill-posed for all values of the bed slope effect. A further decrease in d_1/d_2 increases the limit value r_{y1}^+ faster than r_{y1} such that the model becomes well-posed for all values of the bed slope effect.

An increase in the Froude number decreases the domain of well-posedness, which is explained from the fact that an increase in Froude number decreases r_{y1} while it does not modify r_{y1}^- . We have assumed steady flow in deriving ω_1^{lim} to reduce the complexity of the model such that we can find an analytical solution (Appendix E). This causes a physically unrealistic resonance phenomenon for $\text{Fr} \rightarrow 1$ (*Colombini and Stocchino*, 2005). In reality we do not expect that for $\text{Fr}=1$ the model is

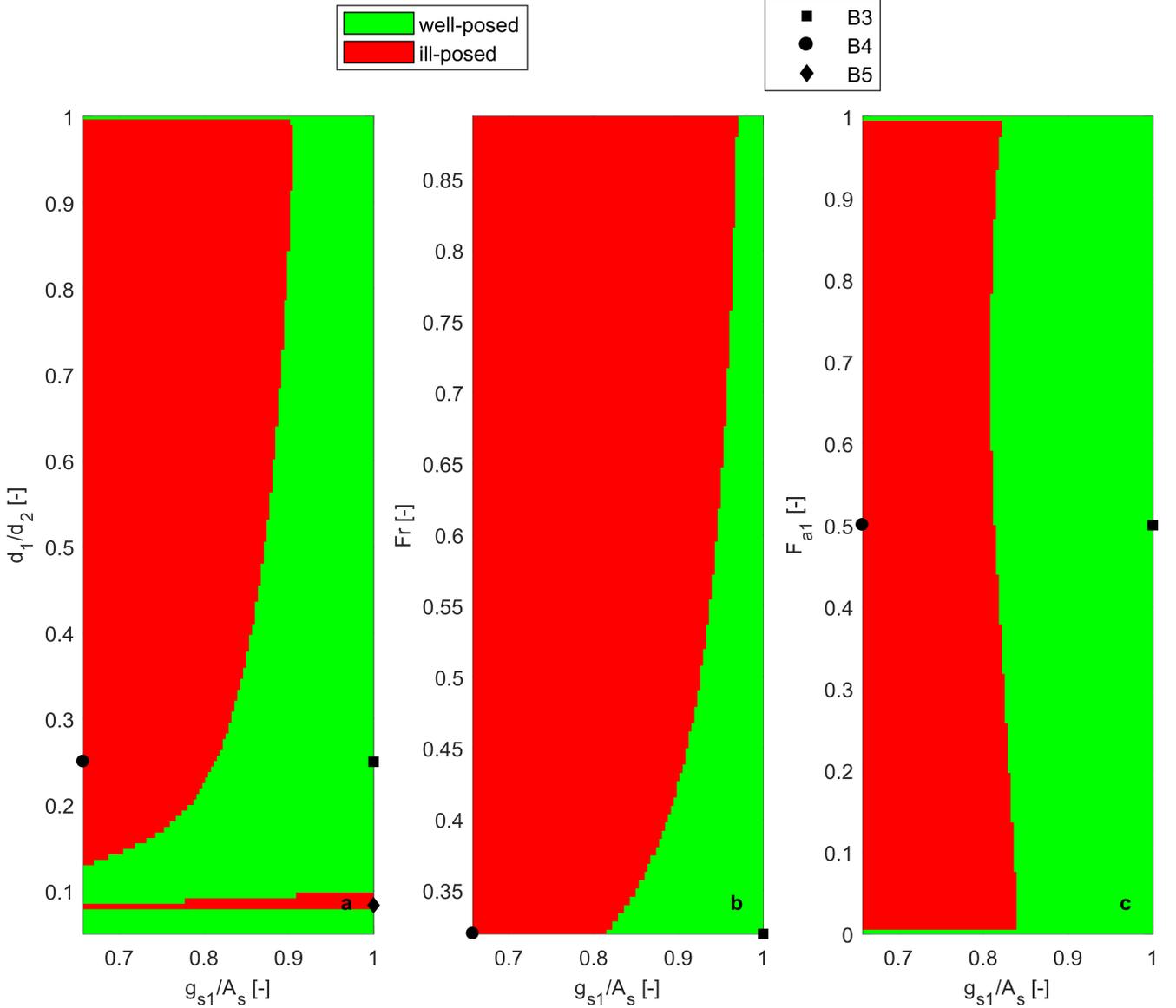


Figure 6: Domain of ill-posedness due to the bed slope effect under mixed-size sediment conditions: as a function of the ratio between fine and coarse sediment (**a**), the Froude number (**b**), and the volume fraction content of fine sediment in the active layer (**c**). The bed slope effect is measured by g_{s1}/A_s and the range of parameters is obtained by varying B_s (equation (23)). The range of values of d_1/d_2 is obtained by varying d_2 . The range of values of the Froude number is obtained by varying u . The volume fraction content of fine sediment at the interface between the active layer and the substrate is kept equal to the volume fraction content of fine sediment in the active layer. The conditions represent unisize sediment when $d_1/d_2 = 1$, $F_{a1} = 0$, or $F_{a1} = 1$.

571 always ill-posed regardless of the bed slope effect. Apart from the limit values in which the problem
 572 becomes unisize, the surface volume fraction content does not significantly affect the domain of
 573 ill-posedness (figure 6c) as it rescales in more or less a similar way r_{y1}^{\pm} and r_{y1} .

574 While Case B4 is ill-posed because the effect of the bed slope (r_{y1}) is small, Case B5 is ill-posed
 575 because parameter $d_{x1,1}$ is small. The different origin of ill-posedness does not cause a significant
 576 difference in the growth rate of perturbations as a function of the wave number (figure 5e-g).
 577 However, we will find out that the pattern resulting from the perturbations depends on the origin

578 of ill-posedness.

579 5 Application

580 The results of the linear stability analysis (Section 4) neglect second order terms and non-linear
581 interactions. In this section we study the effects of the terms neglected in the linear analysis
582 and the development of perturbations by means of numerical simulations. We use the software
583 package Delft3D (*Lesser et al.*, 2004). This exercise provides information on the consequences of ill-
584 posedness in numerical simulations and clarifies the limitations of the current modelling approach.
585 We study the effect of secondary flow (Section 5.1) and the bed slope effect (Section 5.2).

586 5.1 Secondary Flow

587 We run 5 numerical simulations with a fixed bed (i.e., only the flow is computed) to study the
588 role of the secondary flow model and the diffusion coefficient on the ill-posedness of the system
589 of equations. The first 3 simulations reproduce the conditions of Cases S1, S2, and S3 (table 4).
590 The domain is rectangular, 100 m long and 10 m wide. We use square cells with size equal to
591 0.1 m. The time step is equal to 0.01 s and we simulate 10 minutes of flow. We set a constant
592 water discharge and the downstream water level remains constant with time. The initial condition
593 represents normal flow for the values in table 1 (i.e., equilibrium conditions).

594 The simulation not accounting for secondary flow does not present growth of perturbations as
595 predicted by the linear analysis and remains stable with time (figure 7a). We observe growth of
596 perturbations when we account for secondary flow with the diffusion coefficient derived by *Elder*
597 (1959) (figure 7b). The results are physically unrealistic as the flow depth presents variations of
598 up to 7 % of the normal flow depth and the length scale of perturbations is smaller than the
599 flow depth. We have not introduced any perturbation in the initial or boundary conditions which
600 implies that perturbations grow from numerical truncation errors. This supports the fact that the
601 simulation is physically unrealistic. The case with a diffusion coefficient equal to 0 is ill-posed and
602 the solution presents unreasonably large oscillations (figure 7c). These numerical results confirm
603 the results of the linear stability analysis.

604 In the fourth simulation we set a diffusion coefficient 100 times larger than the one derived by
605 *Elder* (1959) (figure 7d). Under this condition the linear analysis predicts all short waves to decay
606 (diamond in figure 4). These numerical results confirm the linear theory. The last simulation is
607 equal to Case S2 except for the fact that we use a coarser grid ($\Delta x = \Delta y = 1$ m). In this case the
608 numerical grid is not sufficiently detailed for resolving the perturbations due to secondary flow and
609 the simulation is stable (figure 7e). This last case highlights an important limitation of a physically
610 unrealistic model. Although Case S2 is mathematically well-posed, the solution presents similarities
611 with ill-posed cases in the sense that a refinement of the grid causes non-physical oscillations to

612 appear.

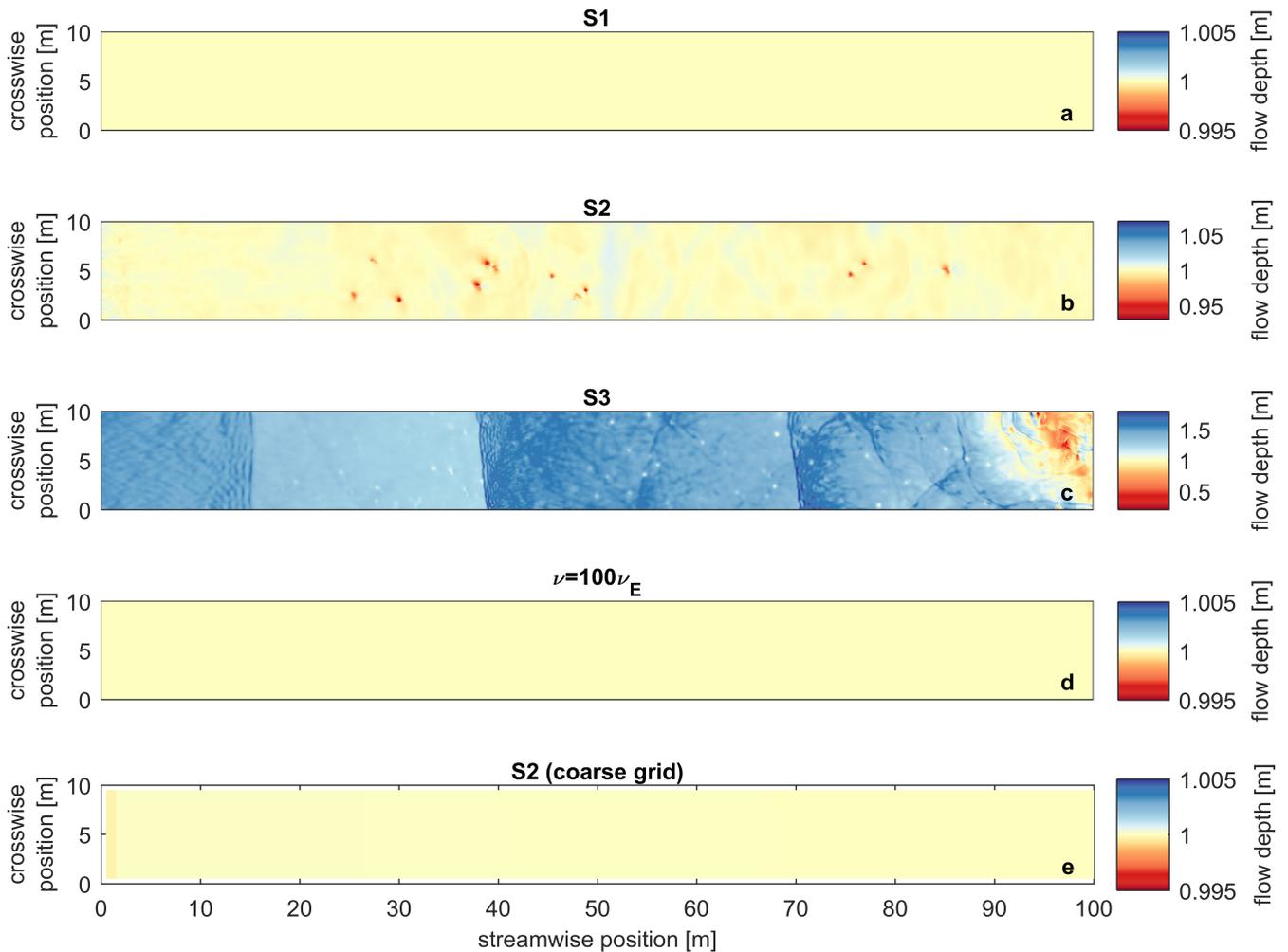


Figure 7: Flow depth at the end of the simulations: (a) without accounting for secondary flow (Case S1), (b) setting $\nu = \nu_E$ (Case S2), (c) setting $\nu = 0$ (Case S3), (d) setting $\nu = 100\nu_E$, and (e) setting $\nu = \nu_E$ using a coarser numerical grid (Case S2). The colour map is adjusted for each case and centred around the initial and equilibrium value ($h = 1$ m).

613 5.2 Bed Slope Effect

614 In this section we focus on the consequences of accounting for the bed slope effect on the mathemat-
 615 ical character of the model. To this end we run 5 more numerical simulations without accounting
 616 for secondary flow and updating the bed (i.e., accounting for morphodynamic change). The simu-
 617 lations reproduce Cases B1-B5 (table 5). We simulate 8 h using a time step $\Delta t = 0.1$ s.

618 We have proved that accounting for the effect of the bed slope makes a unisize simulation well-
 619 posed (Section 4.2 and figure 1c). The numerical solution of this case (B1, table 2) is stable and
 620 perturbations do not grow (figure 8a). Moreover, no alternate bars appear as the channel width
 621 is below the critical value (Section 3). Perturbations grow when the effect of the bed slope is not
 622 taken into account (Case B2, figure 8b), which confirms that this case is ill-posed.

623 The mixed-size sediment conditions of Case B3 yield a well-posed model (figure 5e) and the

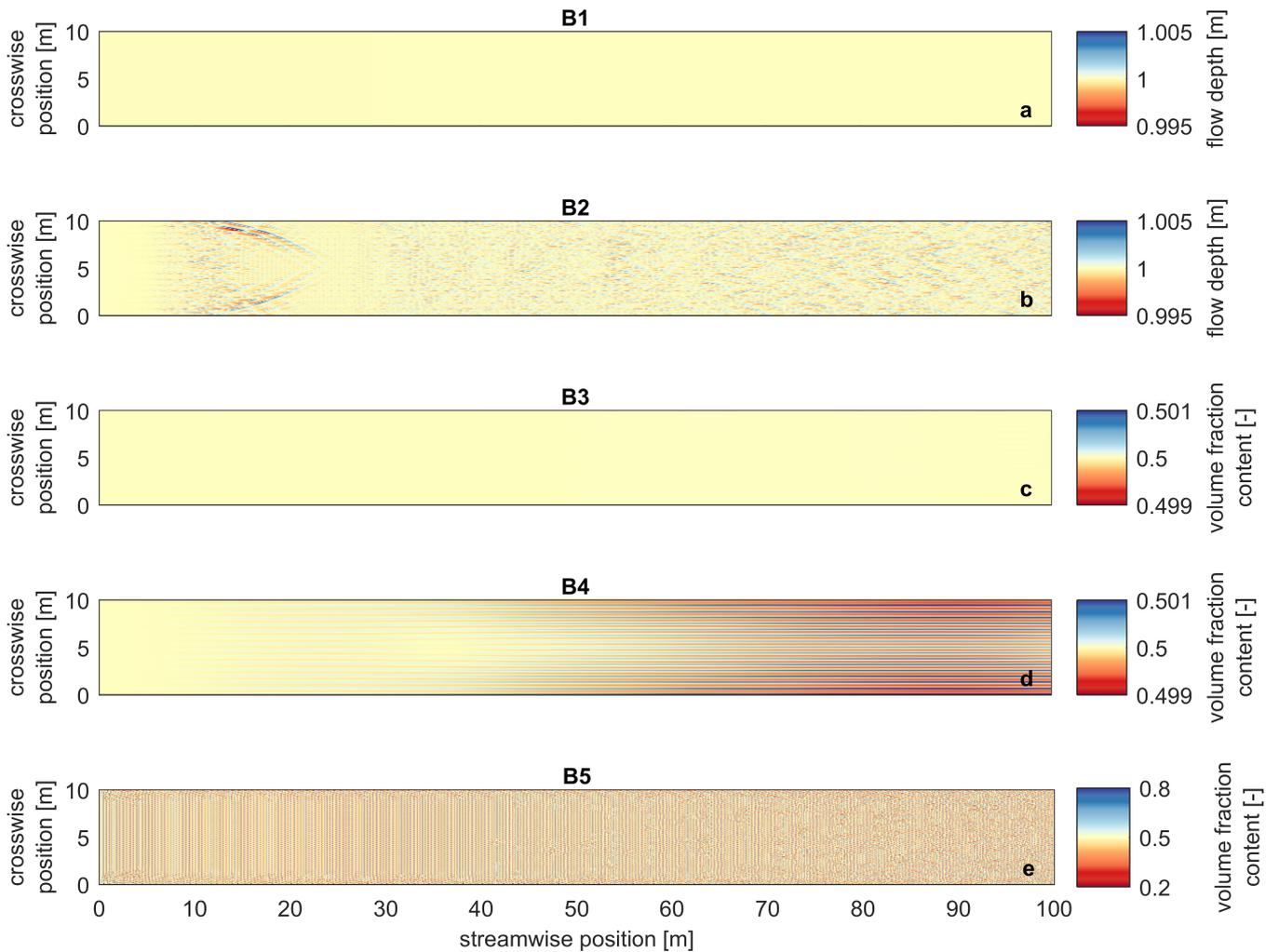


Figure 8: Flow depth at the end of the simulations of: (a) Case B1, (b) Case B2; and volume fraction content of fine sediment in the active layer: (c) Case B3, (d) Case B4, (e) Case B5. The colour map is adjusted for each case and centred around the initial and equilibrium value.

624 simulation is stable (figure 8c). On the other hand, the ill-posed cases B4 and B5 present growth
 625 of unrealistic and non-physical perturbations (figure 8d-e). While the growth of perturbations in
 626 Case B5 seems random, in Case B4 we observe a clear pattern. Moreover, oscillations have grown
 627 significantly faster in Case B5 than in Case B4. While after 8 h the changes in volume fraction
 628 content at the bed surface are of the order of 10^{-3} in Case B4, these are of order 1 in Case B5.

629 The fact that oscillations grow faster in Case B5 than in Case B4 is related to the different origin
 630 of ill-posedness. While Case B4 is ill-posed because the effect of the bed slope is not sufficiently
 631 strong (i.e., $r_{y1} < r_{y1}^-$), Case B5 is ill-posed because changes in the sediment transport rate due
 632 to changes in the volume of fine sediment in the active layer are too small (i.e., $r_{y1} > r_{y1}^+$). The
 633 first case is closely linked to the lateral direction, in which sediment transport is initially zero.
 634 The fact that initially the lateral sediment transport rate is zero limits the rate at which lateral
 635 changes occur. In the second case perturbations are linked to the streamwise direction, in which
 636 the sediment transport rate initially is non-zero, which enhances the rate at which changes develop.

6 Discussion

The origin of the instability due to secondary flow is found in the source term (S_s in equation (11)). As the source term depends on the flow curvature, the source term is 0 in a straight flow. A small perturbation in the flow causes the flow to curve. The flow curvature causes a source of secondary flow intensity, which further increases the flow curvature, causing a positive feedback. The flow curvature is largest for the smallest perturbations, which explains why the model is ill-posed if a dampening mechanism (i.e., diffusion) is not taken into account. This destabilizing mechanism may seem plausible to explain secondary flow circulation observed in straight channels (*Nikuradse*, 1930; *Brundrett and Baines*, 1964; *Nezu and Nakagawa*, 1984; *Gavrilakis*, 1992). However, secondary flow in a straight channel can only be caused by anisotropy of turbulence (*Einstein and Li*, 1958; *Gessner and Jones*, 1965; *Bradshaw*, 1987; *Colombini*, 1993), which is not included in the model for secondary flow. For this reason, the secondary flow model must predict decay of secondary flow intensity in case of straight flow. Diffusion of secondary flow intensity causes decay of perturbations, but the theoretical diffusion coefficient derived by *Elder* (1959) appears to be insufficient to dampen perturbations.

The advection equation of the secondary flow intensity was initially derived for steady decaying secondary flow on a straight reach after a bend neglecting the effect of diffusion (*De Vriend*, 1981). It is assumed that the same advective behaviour is valid for a varying curvature (*De Vriend*, 1981; *Olesen*, 1982) and in an unsteady situation (*Booij and Pennkamp*, 1984). These assumptions have, to our knowledge, not been tested. Moreover, secondary flow affects the vertical profile of the primary flow. This feedback mechanism, which limits the development of secondary flow (*Blanckaert and De Vriend*, 2004; *Blanckaert*, 2009), is not included in the model. *Blanckaert and de Vriend* (2003), *Blanckaert and Graf* (2004) and *Ottevanger et al.* (2013) propose non-linear models that take into consideration this mechanism. We expect that accounting for the feedback mechanism yields a well-posed model.

The feedback mechanism between the secondary and the primary flow may be seen as an increase of diffusivity, as it causes an enhanced momentum redistribution. For a situation in which the non-linear model for the secondary flow appears to be excessively expensive in computational terms, a diffusion coefficient depending on the secondary flow intensity would (partially) account for the enhanced momentum redistribution and provide a well-posed and realistic model.

We have assumed that the diffusion coefficient is constant and equal in all directions, which is a crude approximation, as in the streamwise direction diffusion is larger than in the transverse direction (Appendix A). It would be interesting to study the effect of anisotropic diffusion, however, we do not expect that this will significantly alter our results. This is because a larger diffusion coefficient in the streamwise direction will not alter the most unstable wavelength in the lateral direction. For this reason the shortest unstable waves remain to be of the order of the flow depth.

673 The non-linear relation between the flow and the sediment transport rate causes the growth of
674 perturbations in bed elevation. Worded differently, a deep flow attracts the flow and vice versa,
675 which enhances the growth of perturbations. This mechanism is counteracted by the bed slope
676 effect, which causes deep parts to fill in. In this sense, it seems logical that it is necessary to
677 account for bed slope effects to obtain a well-posed model. This may be confirmed by the facts
678 that *Parker* (1976), by not considering the bed slope effect, found that all streams tend to form
679 bars and, similarly, *Olesen* (1982) concluded that “the stream will develop an infinite number
680 of submerged bars”. From our point of view the fact that all streams seem to be unstable and
681 develop an infinite number of submerged bars is a consequence of the model being ill-posed. Our
682 analysis shows that the bed slope effect is a crucial physical process in analysing two-dimensional
683 morphodynamic processes.

684 Nevertheless, the numerical simulations by *Qian et al.* (2016) of bar development without ac-
685 counting for the bed slope effect do not show unrealistic oscillatory behaviour as is characteristic
686 of ill-posedness. Yet, there is an essential difference between their model and the one we analyse
687 here. We do not model the interaction between the sediment in the bed and the sediment in
688 transport as we assume that the sediment transport rate adapts instantaneously to changes in the
689 flow (i.e., the sediment transport rate depends on the flow variables only). Essentially, sediment
690 in transport is not conserved and bed elevation and surface texture changes depend on the diver-
691 gence of the sediment transport rate at capacity conditions. *Qian et al.* (2016) account for the
692 conservation of mass of the sediment in transport and use an entrainment-deposition formulation
693 for modelling bed elevation and surface texture changes (*Parker et al.*, 2000). In this formulation
694 changes depend on the difference between the rate at which sediment is entrained from the bed
695 and at which it is deposited on the bed. The fact that their model does not show symptoms of
696 ill-posedness, while the effect of the bed slope is not taken into consideration, raises the question
697 whether the entrainment-deposition formulation in combination with mass conservation of the sed-
698 iment in transport is responsible, like the bed slope effect, for a mechanism that counteracts growth
699 of perturbations in bed elevation. If the model used by *Qian et al.* (2016) is indeed well-posed,
700 the effect of the bed slope may be a crucial process only when mass conservation of the sediment
701 in transport is not considered.

702 *Lanzoni and Tubino* (1999) investigated the development of alternate bars under mixed-size
703 sediment conditions using a model similar to the one we apply here. They assumed secondary
704 flow to be negligible and considered a different set of closure relations for friction, the sediment
705 transport rate, and the effect of the bed slope. Under the conditions they studied, they found
706 that, similarly to the unisize case, growth of perturbations occurs if the width-to-depth ratio is
707 above a critical value. This implies that they found that their model is well-posed, as short wave
708 length perturbations decay. Given that the essence of the closure relations they considered is the
709 same as the ones considered here and there is no fundamental difference, we suppose that their

710 model may become ill-posed if different conditions are studied (i.e., different flow or sediment
711 parameters). This is because well-posedness is not related to the model equations only, but also
712 to the conditions in which the model is applied.

713 The bed slope effect (represented by the parameter r_{y1}) needs to be balanced with respect
714 to the effect of changes in the bed surface grain size distribution (represented by $d_{x1,1}$) to yield
715 a well-posed model. The balance depends on the flow and bed conditions. For this reason, a
716 particular closure relation may yield an ill-posed model in some subdomain of a simulation and a
717 well-posed model in some other subdomain. It is necessary to further study the development of
718 the transverse bed slope under mixed-size sediment conditions (e.g. *Baar et al.*, 2018) to obtain a
719 universally applicable closure relation.

720 Overall, there are three causes of ill-posedness of the model: (1) the secondary flow parametri-
721 sation, (2) the closure relation for the bed slope effect, and (3) the representation of the vertical
722 mixing processes when using the active layer model (*Ribberink*, 1987; *Chavarrías et al.*, 2018). We
723 summarise all the conditions in which the model may become ill-posed in figure 9.

724 Only in idealised simulations it is straightforward to relate instability of the system of equations
725 to ill-posedness. We advocate for an *a priori* test of whether the system of equations is well-posed
726 or ill-posed, especially when dealing with mixed-size sediment cases. If at some time a location
727 in the model becomes ill-posed, the model results should be carefully evaluated. The fact that
728 some domain area has always been well-posed does not guarantee a unique solution, as oscillations
729 caused by upstream or downstream ill-posed areas propagate through the domain. Similarly, the
730 fact that the entire domain is well-posed at some time is no guarantee of a unique solution, as past
731 oscillations due to ill-posedness affect the present solution.

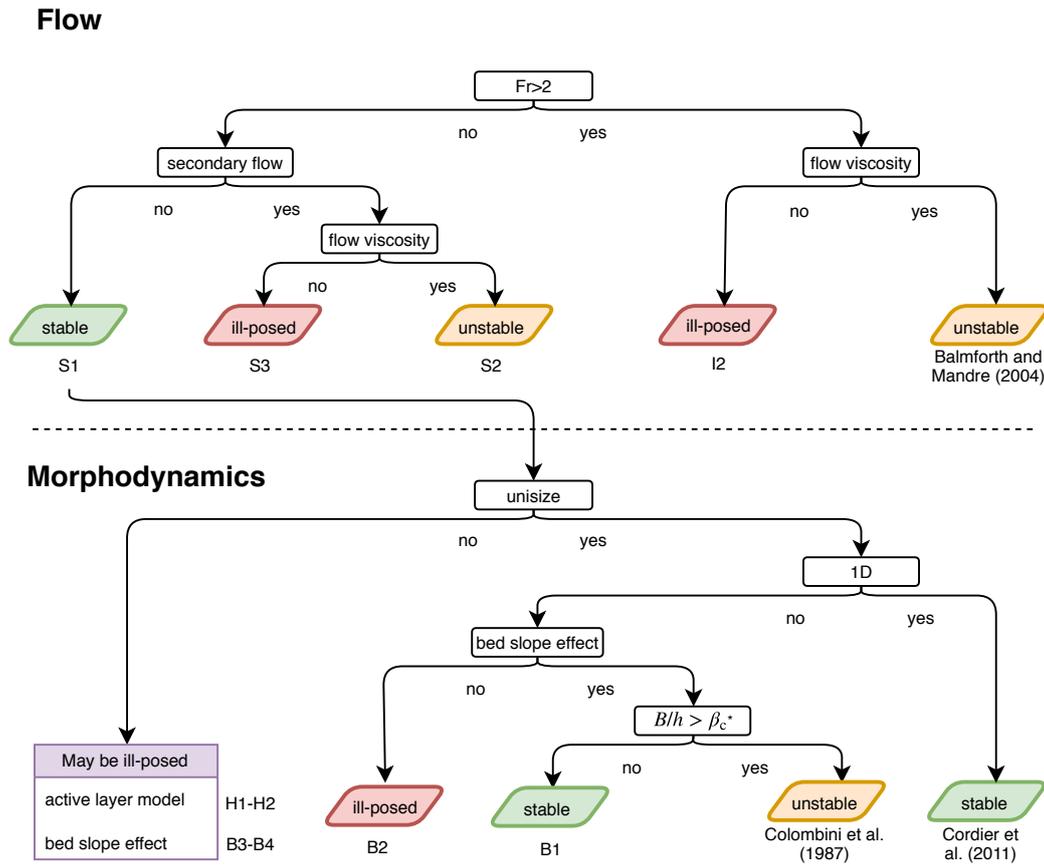


Figure 9: Conditions in which the flow model (top) and the morphodynamic model (bottom) is stable, unstable, or ill-posed. The code below the model type (e.g., S1) indicates an example case of such a situation. See tables 2, 3, 4, and 5 for an explanation of the cases S1-3, B1-4, H1-2, and I2. * Parameter β_c denotes the critical width-to-depth ratio (*Engelund and Skovgaard, 1973; Colombini et al., 1987; Schielen et al., 1993*).

732 **7 Conclusions**

733 We have studied a two-dimensional system of equations used to model mixed-size river morpho-
 734 dynamics as regards to its well-posedness. The model is based on the depth-averaged Shallow
 735 Water Equations in combination with the *Exner* (1920) and active layer (*Hirano, 1971*) equations
 736 to model bed elevation and surface texture changes, respectively. In particular we have focused on
 737 modelling of the secondary flow induced by flow curvature and the effect of the bed slope on the
 738 sediment transport direction, which causes particles to deviate from the direction of the bed shear
 739 stress.

740 By means of a linear stability analysis of the system of equations we find that:

- 741 • The parametrisation accounting for secondary flow yields an ill-posed model if diffusion is
 742 not accounted for.
- 743 • The theoretical amount of diffusion due to depth-averaging the vertical profile of the primary
 744 flow (*Elder, 1959*) yields a well-posed model but it predicts growth of perturbations that are
 745 incompatible with the shallow water assumption.

- 746 • The effect of the bed slope on the direction of the sediment transport is a necessary mech-
747 anism for the model being well-posed. Yet, a different modelling strategy accounting for
748 conservation of the sediment in transport and an entrainment-deposition formulation may
749 yield a well-posed model without accounting for the effect of the bed slope.
- 750 • Not all closure relations accounting for the bed slope effect are valid under mixed-size sed-
751 iment conditions. There needs to be a balance between the effect of the bed slope and the
752 effect of the streamwise variation of grain size distribution on the sediment transport rate.

753 Numerical simulations of idealised cases confirm the above results of the linear stability analysis.

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758 the manuscript.

759 A Eddy Viscosity

760 In general terms, given the anisotropy of the flow field, the diffusion tensor has non-diagonal terms
761 and the diagonal terms are not equal (i.e., the diffusion coefficient in the streamwise direction ν_s is
762 different than in the transverse direction ν_n). The non-diagonal terms become significant close to
763 corners (*Fischer*, 1973) but far from corners the diagonal terms dominate. *Elder* (1959) derived
764 an eddy viscosity coefficient in the streamwise and lateral direction assuming a logarithmic profile
765 for the primary flow:

$$766 \nu_s = \left(\frac{0.4041}{\kappa^3} + \frac{1}{6}\kappa \right) hu^*, \quad (37)$$

$$767 \nu_n = \frac{1}{6}\kappa hu^*. \quad (38)$$

769 *Elder* neglected the effect of the viscous sublayer, which causes his analytical expression to be a
770 lower limit of the diffusion coefficient (*Fischer*, 1967).

771 Several researchers (e.g. *Erdogan and Chatwin*, 1967; *Simons and Albertson*, 1963; *Fischer*,
772 1969; *Holley*, 1971; *Fischer*, 1973; *Kyong and Il*, 2016) propose values for the diffusion coefficient
773 that are significantly larger than the one derived by *Elder* (1959). These values are used, for
774 instance, by *Parker* (1978); *Ikeda and Nishimura* (1985) and *Van Prooijen and Uijtewaal* (2002).
775 These values of the diffusion coefficient are derived from experimental measurements and implicitly
776 account for the enhanced momentum redistribution due to secondary flow that we account for by
777 means of the dispersive stresses.

778 In numerical simulations resolving the secondary flow, the diffusion coefficients derived by *Elder*
 779 (1959) are valid if the grid is of the order of magnitude of the flow depth (assuming that the relevant
 780 turbulent processes scale with the flow depth). Otherwise the numerical grid filters out significant
 781 two-dimensional turbulent motions that need to be accounted for in the closure model (*Talstra,*
 782 2011). In our numerical runs the grid cell size is always smaller than the flow depth.

783 B Magnitude of the Sediment Transport Rate

784 The module of the specific sediment transport rate of size fraction k , q_{bk} [m²/s], has a direction
 785 given by the angle φ_{sk} [rad]:

$$786 \quad (q_{bxk}, q_{byk}) = q_{bk}(\cos \varphi_{sk}, \sin \varphi_{sk}) . \quad (39)$$

787 The magnitude of the sediment transport rate is equal to:

$$788 \quad q_{bk} = F_{ak} \sqrt{gRd_k^3} (1 - p) q_{bk}^* , \quad (40)$$

789 where p is the porosity and q_{bk}^* [-] is a nondimensional sediment transport rate (*Einstein, 1950*)
 790 dependent on the *Shields* (1936) stress:

$$791 \quad \theta_k = \frac{C_f \left(\frac{Q}{h}\right)^2}{gRd_k} . \quad (41)$$

792 The parameter $R = \rho_s/\rho_w - 1$ [-] is the submerged sediment density, $\rho_s = 2650$ kg/m³ is the
 793 sediment density and $\rho_w = 1000$ kg/m³ is the water density. To compute the nondimensional
 794 sediment transport rate we use a fractional form (*Blom et al., 2016, 2017*) of the relation proposed
 795 by *Engelund and Hansen* (1967) neglecting form drag:

$$796 \quad q_{bk}^* = \frac{0.05}{C_f} \theta_k^{5/2} , \quad (42)$$

797 and the relation including a nondimensional critical shear stress θ_c [-] proposed by *Ashida and*
 798 *Michiue* (1971):

$$799 \quad q_{bk}^* = 17 (\theta_k - \xi_k \theta_c) \left(\sqrt{\theta_k} - \sqrt{\xi_k \theta_c} \right) . \quad (43)$$

800 The parameter ξ_k [-] is the hiding factor that accounts for the fact that fine sediment in a mixture
 801 hides behind larger grains and a coarse sediment in a mixture is more exposed than in unisize

802 coarse sediment (*Einstein*, 1950). *Ashida and Michiue* (1971) proposes $\theta_c = 0.05$ and the relation:

$$803 \quad \xi_k = \begin{cases} 0.843 \left(\frac{d_k}{D_m} \right)^{-1} & \text{for } \frac{d_k}{D_m} \leq 0.4 \\ \left(\frac{\log_{10}(19)}{\log_{10}(19 \frac{d_k}{D_m})} \right)^2 & \text{for } \frac{d_k}{D_m} > 0.4 \end{cases}, \quad (44)$$

804 where D_m is a characteristic mean grain size of the sediment mixture.

805 C Proof of Ill-posedness Due to Secondary Flow without 806 Diffusion

807 In this section we prove that the model based on the Shallow Water Equations accounting for
808 secondary flow without diffusion is ill-posed.

809 The system of equations is composed of the first four rows and columns of the full system of
810 equations in equation (24). Neglecting diffusive processes matrices $\mathbf{D}_{\mathbf{x}\mathbf{0}}$ and $\mathbf{D}_{\mathbf{y}\mathbf{0}}$ are equal to $\mathbf{0}$.
811 As we are interested in the short-wave domain, friction can be neglected. The resulting matrix
812 \mathbf{M}_0 of the linearised eigenvalue problem (equation (33)) is:

$$813 \quad \mathbf{M}_0 = \mathbf{A}_{\mathbf{x}\mathbf{0}} k_{\text{wx}} + \mathbf{A}_{\mathbf{y}\mathbf{0}} k_{\text{wy}}. \quad (45)$$

814 We compute the fourth order characteristic polynomial of matrix \mathbf{M}_0 . The roots of the char-
815 acteristic polynomial are the eigenvalues (i.e., the angular frequencies ω in equation (31)). The
816 discriminant of a fourth order polynomial $p(\omega) = p_4\omega^4 + p_3\omega^3 + p_2\omega^2 + p_1\omega + p_0 = 0$ is equal to
817 (*Beeler et al.*, 1972):

$$818 \quad \begin{aligned} \Delta_4 = & (p_1^2 p_2^2 p_3^2 - 4p_1^3 p_3^3 - 4p_1^2 p_2^3 p_4 + 18p_1^3 p_2 p_3 p_4 - 27p_1^4 p_4^2 + 256p_0^3 p_4^3) \\ & + p_0(-4p_2^3 p_3^2 + 18p_1 p_2 p_3^3 + 16p_2^4 p_4 - 80p_1 p_2^2 p_3 p_4 - 6p_1^2 p_3^2 p_4 + 144p_1^2 p_2 p_4^2) \\ & + p_0^2(-27p_3^4 + 144p_2 p_3^2 p_4 - 128p_2^2 p_4^2 - 192p_1 p_3 p_4^2). \end{aligned} \quad (46)$$

819 We find that the discriminant of the characteristic polynomial is:

$$820 \quad \Delta_4 = \frac{16gh^2 T^2 \beta_u}{L_I} k_{\text{wx}}^2 (k_{\text{wx}}^2 - k_{\text{wy}}^2), \quad (47)$$

821 where $\beta_u = \beta^* q_x^2 / h^2$ and:

$$822 \quad T = L_I g \left[L_I g (k_{\text{wx}}^2 + k_{\text{wy}}^2)^2 + \beta_u (6k_{\text{wx}}^2 k_{\text{wy}}^2 - 2k_{\text{wx}}^4) \right] + \beta_u^2 k_{\text{wx}}^4. \quad (48)$$

823 As the coefficients of the characteristic polynomial $p(\omega)$ are all real, a positive discriminant indicates
824 that either all the roots are real or all the roots are complex. A negative discriminant indicates that

825 there are two real and two complex roots. The complex roots come in pairs of complex conjugates.
 826 For this reason, if the discriminant is negative there exist an eigenvalue with a positive imaginary
 827 component. As the discriminant is negative for $k_{wx} < k_{wy}$ independently from the wave number,
 828 there exists always a region of growth. This implies that the model is ill-posed.

829 **D Proof of Ill-posedness Due to Lack of Bed Slope Effect** 830 **under Unsize Conditions**

831 In this section we prove that the model based on the Shallow Water Equations without accounting
 832 for the effect of secondary flow in combination with the *Exner* (1920) equation to model bed
 833 elevation changes is ill-posed if the effect of the bed slope on the direction of the sediment transport
 834 is not taken into consideration.

835 The system of equations is composed of the first three and the fifth rows and columns of the
 836 system of equations in equation (24). Neglecting diffusive processes in the momentum equations
 837 and the effect of the bed slope, matrices \mathbf{D}_{x0} and \mathbf{D}_{y0} are equal to $\mathbb{0}$. The system of equations
 838 has 4 unknowns (h , q_x , q_y , and η). The unknowns are coupled meaning that a change in bed
 839 elevation influences the flow and vice versa. The celerity of perturbations associated with the flow
 840 variables (i.e., h , q_x , and q_y) are orders of magnitude larger than the celerity of perturbations
 841 in bed elevation if the Froude number is sufficiently small ($Fr \lesssim 0.7$ (*De Vries*, 1965, 1973; *Lyn*
 842 *and Altinakar*, 2002)). Under this condition we can decouple the system and consider steady
 843 flow to study the propagation of perturbations in bed elevation (i.e., quasi-steady flow assumption
 844 (*De Vries*, 1965; *Cao and Carling*, 2002; *Colombini and Stocchino*, 2005)). In this manner we
 845 reduce the number of unknowns to one (η), which means that there is only one eigenvalue (ω). We
 846 obtain ω equating to 0 the determinant of matrix:

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix} - \mathbf{M}_0 \quad (49)$$

848 The growth rate (the imaginary part of ω) is:

$$\omega_i = \frac{q_b C_f k_{wx}^2}{k_{wx}^2 w_2^2 + w_1^2} (w_3 + (n-1) k_{wy}^4) \quad (50)$$

850 where w_1 , w_2 , and w_3 are second degree polynomials on k_{wy} :

$$w_1 = C_f [(1 - 4Fr^2) k_{wx}^2 + 2k_{wy}^2] \quad (51)$$

$$w_2 = b_1 + (1 - \text{Fr}^2) k_{\text{wx}}^2 + k_{\text{wy}}^2, \quad (52)$$

$$w_3 = -3\text{Fr}^2 n k_{\text{wx}}^4 - b_1 n k_{\text{wx}}^2 + [n(2 - \text{Fr}^2) - (2 + \text{Fr}^2)] k_{\text{wx}}^2 k_{\text{wy}}^2 + b_1 (n - 3) k_{\text{wy}}^2, \quad (53)$$

where b_1 is:

$$b_1 = \frac{3C_f^2 \text{Fr}^2}{h^2}. \quad (54)$$

Parameter n is the degree of non-linearity of the sediment transport relation (*Mosselman et al., 2008*):

$$n = \frac{Q}{q_b} \frac{\partial q_b}{\partial Q}, \quad (55)$$

which is larger than 1. For instance, $n = 5$ in the relation developed by *Engelund and Hansen (1967)* and $n > 3$ in the one by *Meyer-Peter and Müller (1948)*. In general $n > 3$ for the sediment transport relation to be physically realistic (*Mosselman, 2005*).

For k_{wy} tending to infinity, parameter w_3 becomes negligible with respect to $(n - 1)k_{\text{wy}}^4$. As all other terms in equation (50) are positive, for a large wave number the growth rate is positive which implies that the model is ill-posed.

E Well-Posed Domain under Mixed-Size Sediment Conditions

In this section we show that the Shallow Water Equations in combination with the active layer model (*Hirano, 1971*) used to account for mixed-size sediment morphodynamics may yield an ill-posed model depending on the closure relation used to account for the effect of the bed slope on the sediment transport direction.

We consider a model with two sediment size fractions. The system of equations is composed of the first three, the fifth and the sixth rows and columns of the full system of equations in equation (24). We neglect diffusive processes in the momentum equations. The system of equations has 5 unknowns (h , q_x , q_y , η , and M_{a1}). We consider that the Froude number is sufficiently small such that the quasi-steady approximation is valid (Appendix D) and we assume that the celerity associated with changes in the grain size distribution of the bed surface are of the same order of magnitude as the celerity of bed elevation changes (*Ribberink, 1987; Sieben, 1997; Stecca et al., 2016*). Under these conditions it is valid to decouple the system and consider steady flow to study the propagation of perturbations in bed elevation and bed surface grain size distribution. In this manner we reduce the number of unknowns to two (η and M_{a1}), which means that there are two

882 angular frequencies to find. We obtain ω equating to 0 the determinant of matrix:

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & \omega \end{bmatrix} - \mathbf{M}_0 \quad . \quad (56)$$

884 We define a set of physically meaningful parameters useful to simplify the expression of the
885 growth rate. Subscripts k and l refer to the grain size fraction while the subscript j refers to the
886 direction (i.e., x and y). The parameters are a generalization of the parameters used by *Stecca*
887 *et al.* (2014) and *Chavarrías et al.* (2018) to the x and y direction.

888 Parameter ψ_j [–] represents the sediment transport intensity (e.g. *De Vries*, 1965; *Lyn and*
889 *Altinakar*, 2002; *Stecca et al.*, 2014) and ranges between 0 (no sediment transport) and $\mathcal{O}(10^{-2})$
890 (high sediment discharge):

$$\psi_j = \frac{\partial q_{bj}}{\partial q_j} \quad . \quad (57)$$

892 Parameter $c_{jk} \in [0, 1]$ [–] represents the sediment transport intensity of fraction k relative to the
893 total sediment transport intensity:

$$c_{jk} = \frac{1}{\psi_j} \frac{\partial q_{bjk}}{\partial q_j} \quad . \quad (58)$$

895 Parameter γ_{jk} [–] represents the sediment transport intensity of fraction k relative to the fraction
896 content of sediment of fraction k at the interface between the active layer and the substrate:

$$\gamma_{jk} = c_{jk} - f_k^I \quad , \quad (59)$$

898 Parameter χ_{jk} [–] represents the nondimensional rate of change of the total sediment transport
899 rate with respect to the change of volume of sediment of size fraction k in the active layer:

$$\chi_{jk} = \frac{1}{u_j} \frac{\partial q_{bj}}{\partial M_{ak}} \quad . \quad (60)$$

901 Parameter $d_{jk,l}$ [–] represents the nondimensional rate of change of the sediment transport rate of
902 size fraction l with respect to the volume of sediment of size fraction k in the active layer:

$$d_{jk,l} = \frac{1}{u_j \chi_{jk}} \frac{\partial q_{bjl}}{\partial M_{ak}} \quad . \quad (61)$$

904 Parameter $\mu_{jk,l}$ [–] represents the rate of change of the sediment transport rate with respect to
905 the volume of sediment in the active layer relative to the fraction content of sediment of fraction

906 k at the interface between the active layer and the substrate:

$$907 \quad \mu_{jk,l} = d_{jk,l} - f_k^I. \quad (62)$$

908 Parameter $R_j < 0$ [m²/s] represents the effect of the bed slope on the direction of the sediment
909 transport rate:

$$910 \quad R_j = \frac{\partial q_{bj}}{\partial s_j}, \quad (63)$$

911 where $s_j = \partial\eta/\partial j$. Parameter r_{jk} [-] represents the effect of the bed slope on the direction of the
912 sediment transport rate of fraction k relative to the total effect:

$$913 \quad r_{jk} = \frac{1}{R_j} \frac{\partial q_{bjk}}{\partial s_j}. \quad (64)$$

914 Parameter l_{jk} [-] represents the effect of the bed slope on the direction of the sediment transport
915 rate of fraction k relative to the fraction content of sediment at the interface between the active
916 layer and the substrate:

$$917 \quad l_{jk} = r_{jk} - f_k^I. \quad (65)$$

918 The largest of the two growth rates (i.e., the largest imaginary part of the two eigenvalues ω
919 of the system) is:

$$920 \quad \omega_i = \frac{1}{2} \left(\frac{\sqrt{2}}{2} \sqrt{f_1} - \sqrt{f_2} \right), \quad (66)$$

921 where:

$$922 \quad f_1 = \sqrt{m_1^2 + m_2} - m_1, \quad (67)$$

923 and:

$$924 \quad f_2 = R_y^2 k_{wy}^4. \quad (68)$$

925 When parameter f_1 is larger than $2f_2$, $\omega_i > 0$ and perturbations grow. Parameter f_1 becomes
926 large with respect to f_2 when parameter m_2 becomes large with respect to m_1 where:

$$927 \quad m_1 = k_{wx}^2 u^2 a_3 - f_2, \quad (69)$$

928 and:

$$929 \quad m_2 = 4k_{wx}^2 u^2 f_2 o^2. \quad (70)$$

930 Focusing on the bed slope effect, for a given value of f_2 (i.e., a given value of R_y), parameter m_2
931 becomes large with respect to m_1 when parameter o becomes large, where:

$$932 \quad o = a_1 + 2\chi_{x1} (r_{y1} - d_{x1,1}). \quad (71)$$

933 Thus, the growth rate of perturbations is prone to be positive when the absolute value of $r_{y1} - d_{x1,1}$
 934 increases. The parameters a_m for $m = 1, 2, 3$ are:

$$935 \quad a_1 = e_x + e_y + \chi_{x1}\mu_{x1,1}, \quad (72)$$

$$936 \quad a_2 = \gamma_{x1}e_x + \gamma_{y1}e_y - \mu_{x1,1}e_x - \mu_{x1,1}e_y, \quad (73)$$

$$937 \quad a_3 = a_1^2 + 4\chi_{x1}a_2. \quad (74)$$

938
 939
 940 The parameters e_j for $j = x, y$ are:

$$941 \quad e_x = \psi_x \frac{k_{wx}^2}{(1 - \text{Fr}^2)k_{wx}^2 + k_{wy}^2}, \quad (75)$$

$$942 \quad e_y = \psi_y \frac{k_{wy}^2}{(1 - \text{Fr}^2)k_{wx}^2 + k_{wy}^2}. \quad (76)$$

943
 944 We compute the limit of the growth rate (equation (66)) for k_{wx} and k_{wy} tending to infinity:

$$945 \quad \omega_1^{\text{lim}} = \alpha_1 (r_{y1} - d_{x1,1})^2 + \alpha_2 (r_{y1} - d_{x1,1}) + \alpha_3, \quad (77)$$

946 where:

$$947 \quad \alpha_1 = \frac{-u^2\chi_{x1}}{R_y}\chi_{x1}, \quad \alpha_2 = \frac{-u^2\chi_{x1}}{R_y}a_1^{\text{lim}}, \quad \alpha_3 = \frac{u^2\chi_{x1}}{R_y}a_2^{\text{lim}}, \quad (78)$$

948 where the superscript lim indicates that these are the limit values and:

$$949 \quad e_x^{\text{lim}} = \frac{\psi_x}{2 - \text{Fr}^2}, \quad (79)$$

$$950 \quad e_y^{\text{lim}} = \frac{\psi_y}{2 - \text{Fr}^2}. \quad (80)$$

951
 952 As $R_y < 0$ and $\chi_{x1} > 0$, the mathematical character of the system of equations is given by the
 953 sign of the second degree polynomial with variable $(r_{y1} - d_{x1,1})$. The fact that $\alpha_1 > 0$ (the factor
 954 of the squared term) indicates that the model is well-posed when $r_{y1}^- < r_{y1} < r_{y1}^+$ where:

$$955 \quad r_{y1}^{\pm} = \frac{1}{2\chi_{x1}} \left(-a_1^{\text{lim}} \pm \sqrt{a_1^{\text{lim}^2} + 4\chi_{x1}a_2^{\text{lim}}} \right) + d_{x1,1}. \quad (81)$$

956

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