Low-Thrust Real-Time Guidance Algorithm for Proximity Operations about an Asteroid

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This paper proposes a real-time implementable Lyapunov-based guidance algorithm for small-body proximity operations. The guidance solution accounts for the secular effects coming from the oblateness of the main body and for the non-impact condition between the latter one and the spacecraft. The first perturbation with respect to the two-body problem is implemented via an update of the Keplerian elements based on the secular precessing ellipse model, while the second one through a check on the radial distance at periapsis. Advantages are taken of the fact that real-time guidance algorithms require only the knowledge of the current state of the spacecraft and of the target. The proposed mission scenario implements an approximate model of 433 Eros as the main body and a low-thrust point mass spacecraft executing orbital transfers about it. It is shown that, for transfers lasting tens of days, ignoring the oblateness of the main body results in errors on the order of ten degrees for the right ascension of the ascending node, the mean anomaly, and the argument of periapsis. These errors are reduced by two orders of magnitude when the proposed guidance algorithm is applied.

Nomenclature

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<td>(f)</td>
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The interest in small-bodies is a topic in which space industries and research interests go hand in hand, supporting each other. The first definition of these celestial bodies was given during the 26th General Assembly of the International Astronomical Union. Specifically, in Resolution 5A, it is stated that all bodies in the Solar System can be divided into planets, dwarf planets, satellites, and small-bodies, the last ones being defined as "all other objects" not part of the previous categories.

Small-bodies are seen as intermediate steps towards interplanetary manned missions, providing incremental capabilities at a low risk. Due to their distance from the Earth, their vicinity environments are often not well-known before arrival and communication delays of several minutes are unavoidable. Moreover, their gravitational parameter is tiny w.r.t. planets, making any manoeuvre particularly sensitive to any premature or delayed command execution. As a result, there is a major interest in advancing the performances of autonomous guidance and navigation subsystems.

Previous work about real-time guidance algorithms has been based on the two-body problem assumption, i.e., considering the main body as a point mass. As a consequence, two main problems arise: the spacecraft could collide with the small-body without even realising it and, if any mass irregularity w.r.t. a perfect sphere is taken into account, the gravitational potential will differ from the one of a point mass. This issue is considered as one of the main perturbations affecting a spacecraft orbiting a small-body, together with solar radiation pressure, and third-body effects. However, among these three perturbations, the non-sphericity of the main body is the only one for which the magnitude increases when the spacecraft is closer to the surface. A partial solution has been offered by Kechichian, who presented a method to optimise the direction of the thrust vector to provide a minimum-time solution able to match any elliptic orbit with the two-body dynamics perturbed only by the main body's oblateness, by Kluever and Oleson, who introduced a simulator able to produce results close to optimality with improvements in robustness, and by Perez and Varga, who modified and optimised the Q-law to include the effects of $J_2$. However, in all the previous cases, the computational time is considered excessive for the requirements of a real-time guidance algorithm: the solutions available in the literature to the oblateness problem lack in simplicity and real-time implementability.

The current paper will build on those results, taking advantage of the flexibility of Lyapunov-based algorithms. In particular, the one proposed by Hernandez and Akella, generally valid for any main body-spacecraft combination in a two-body problem configuration and any propulsion system, is tailored towards the specific needs of small-body proximity operations. Nevertheless, it is highlighted that any improvement leading to the distortion of the simplicity and the low computational effort of the initial algorithm would be counterproductive. To reach this goal, the main core of the guidance solution is still based on the two-body assumption, adding at the end of each time step a contribution taking into account the secular effects of $J_2$.
for the orbital elements of specific interest, i.e., the right ascension of the ascending node, the argument of periapsis and the mean anomaly, indicated by Ω, ω, and M, respectively, while leaving the others unaltered. On the other hand, to avoid an impact between the spacecraft and the asteroid, considered a likely event during proximity operations about irregular bodies, a check on the radial distance at periapsis is implemented: if \( r_p \) is smaller than a certain pre-specified limit, the spacecraft would start a circularisation procedure before proceeding with the semi-major axis matching. Using this approach, some opportune changes are made to an already well-developed and tested algorithm from Ref. 1, implementing new features to deal with small-body needs, rather than starting from scratch.

The layout of this paper, divided into four sections, including this introduction, is presented in the following. Section II is focused on the description of the guidance algorithm proposed by Hernandez and Akella,\(^1\) with an emphasis on the execution of a semi-major axis matching and a full transfer between two orbits. Then, Sec. III presents some modifications to the latter algorithm, based on the two-body problem, to include the oblateness of the main body through the implementation of the Secularly Precessing Ellipse model and the non-impact condition between the celestial body and the spacecraft. Finally, conclusions are drawn in Sec. IV.

II. Lyapunov-based guidance in KS variables

This section is based on the work by Hernandez and Akella,\(^1\) who proposed a Lyapunov-based guidance algorithm able to execute orbital manoeuvres within the two-body assumption. Due to the presence of a thrust acceleration \( f \), the resulting equation of motion is:

\[
\frac{d^2 r}{dt^2} = -\frac{\mu}{r^3} r + f
\]  

(1)

where \( r \) is the distance between the spacecraft and the main body, \( \mu \) represents its standard gravitational parameter and \( t \) is the time. It is highlighted that, since an asteroid-fixed asteroid-centered reference frame is used, the main body is a point mass placed at the origin of the reference frame.

The structure of this section, which provides the theoretical basis for Sec. III, i.e., the most innovative part of this paper, is presented in the following. Firstly, the Lyapunov Direct Method of Stability is introduced, which ensures the convergence of the spacecraft to the intended final solution. Secondly, the coordinate system on which the core loop of the algorithm is carried out, i.e, the KS model by Kustaanheimo and Stiefel,\(^8\) is briefly explained. Finally, after these introductory definitions, the guidance strategy proposed by Hernandez and Akella\(^1\) is treated in detail.

A. Lyapunov Direct Method

The Lyapunov Direct Method is a tool to find the control parameter(s) to bring an autonomous system, in which the derivative of the state does not depend explicitly on the independent variable, to an equilibrium point. This method is based on the research of a candidate potential \( V \) which satisfies two requirements: it is positive except at its equilibrium point, and its time-derivative is negative semi-definite. As a result, if \( \dot{V} = 0 \) when the current orbit and the one to be matched coincide, the control parameter(s) derived by the application of this method will bring the spacecraft to the intended final solution. Moreover, applying this theorem, it is not needed to solve a non-linear differential system for every initial condition. Finally, the theorem is enunciated in the following:

- Let \( x_{eq} = 0 \) be an equilibrium point, contained in the domain \( D \in \mathbb{R}^N \), of the system \( \dot{x} = f(x) \).
- Let \( V : D \rightarrow \mathbb{R} \) be a continuously differentiable function, named Lyapunov function or potential, such that \( V(x_{eq}) = 0 \) and \( V(x) > 0 \in Dn\{x_{eq}\} \).

Then:

- if \( \dot{V} \leq 0 \in D \), the equilibrium point \( x_{eq} \) is stable
- if \( \dot{V} < 0 \in Dn\{x_{eq}\} \), the equilibrium point \( x_{eq} \) is asymptotically stable.
B. KS transformation

The KS model, introduced by Kustaanheimo and Stiefel,\(^8\) provides a regularisation for the three-dimensional system presented in Eq. (1) with the following mapping tool: \( \mathbf{r} = (x, y, z) \in \mathbb{R}^3 \rightarrow \mathbf{u}_r = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \). Basically, the term regularisation refers to the removal of the singularity when the position vector \( r \rightarrow 0 \) through the introduction of a new coordinate system. Due to the dimension mismatch between \( r \) and \( \mathbf{u}_r \), a class of augmented vectors is introduced:

\[
\mathbf{y} = (y_1, y_2, y_3)^T \rightarrow \mathbf{y}_a = (y_1, y_2, 0)^T
\]

where \( \mathbf{y} \) is a generic 3D vector and \( \mathbf{y}_a \) is the corresponding augmented one.

The KS model is obtained through two steps: the use of a fictitious time \( \tau \) instead of the real one \( t \) and the already introduced position transformation. In mathematical terms, the relationship between \( t \)- and \( \tau \)-derivatives are indicated by \( () \) and \( ()' \), respectively, is:

\[
\frac{dr}{dt} = \frac{1}{r} \frac{dr}{d\tau} \Leftrightarrow \dot{r} = \frac{1}{r} \dot{r}' \tag{3}
\]

On the other hand, the formula linking the two four-dimensional position vectors \( \mathbf{r}_a \) and \( \mathbf{u}_r \) is:

\[
\mathbf{r}_a = \mathbf{L}(\mathbf{u}_r)\mathbf{u}_r =
\begin{bmatrix}
  u_1 & -u_2 & -u_3 & u_4 \\
  u_2 & u_1 & -u_4 & -u_3 \\
  u_3 & u_4 & u_1 & u_2 \\
  u_4 & -u_3 & u_2 & -u_1
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4
\end{bmatrix}
\tag{4}
\]

The inverse transformation is presented in the following:\(^{11}\)

\[
x_0 \geq 0 : \quad u_{1,0} = \sqrt{\frac{T_0 + x_0}{2}} \quad u_{2,0} = \frac{y_0}{2u_{1,0}} \quad u_{3,0} = \frac{z_0}{2u_{1,0}} \quad u_{4,0} = 0
\]

\[
x_0 < 0 : \quad u_{1,0} = \frac{y_0}{2u_{2,0}} \quad u_{2,0} = \sqrt{\frac{T_0 - x_0}{2}} \quad u_{3,0} = 0 \quad u_{4,0} = \frac{z_0}{2u_{2,0}}
\tag{5}
\]

C. Guidance strategy

The equations of motion in KS coordinates are presented in the following:

\[
\begin{align*}
\mathbf{u}_r'' &= -\frac{\mu_\alpha}{4} \mathbf{u}_r + \frac{1}{2} \frac{T}{r} \mathbf{L}^T(\mathbf{u}_r)\mathbf{f}_a \\
\alpha' &= -\frac{4}{\mu} \mathbf{u}_r \cdot \mathbf{L}^T(\mathbf{u}_r)\mathbf{f}_a \\
m' &= -\frac{T}{c} \\
t &= \mathbf{u}_r \cdot \mathbf{u}_r
\end{align*}
\tag{6}
\]

where \( \alpha \) is the inverse of the semi-major axis \( a \), \( m \) is the mass of the spacecraft, \( T \) is the thrust magnitude and \( c \) is the velocity of the exhaust gas. The thrust acceleration vector \( \mathbf{f} \) is expressed in a body-fixed reference frame defined by three unit vectors, i.e., \( \mathbf{r}', \mathbf{h} \) and \( \mathbf{p} \). In particular, the first two unit vectors are the velocity and the angular momentum, respectively, while the third one \( \mathbf{p} = \mathbf{r}' \times \mathbf{h} \) completes the right-handed tern. Then, the thrust acceleration, function of two control angles, i.e., the azimuth \( \delta \) and the elevation \( \beta \), is presented in the following:

\[
\mathbf{f} = \frac{T}{m} \left[ \sin \beta \mathbf{r}' + \cos \beta \cos \delta \mathbf{p} + \cos \beta \sin \delta \mathbf{h} \right]
\tag{7}
\]

where the thrust magnitude remains constant during the whole orbital transfer. In the following, the formulae to match the orbital energy and to execute a full transfer, i.e., the targeting of the complete Keplerian state excluding the true anomaly \( \theta \), are introduced. However, it is highlighted that the algorithm reported here is also able to match different subsets.
1. Matching the semi-major axis

The selected candidate potential for the $a$ matching is function of $e_\alpha = \alpha - \alpha^*$, which is the difference between the current $\alpha$ and the targeted one. This quadratic potential is defined as:

$$V_a = \frac{1}{2} e_\alpha^2$$

(8)

and it is positive by definition, except at the equilibrium point where $e_\alpha = 0$. Then, the latter potential is $\tau$-derived:

$$V'_a = (\alpha - \alpha^*) \alpha' = -2 e_\alpha \mu \frac{T_0}{m} r'_a \cdot \left[ \sin \beta r'_a + \cos \beta \cos \delta \hat{p}_a + \cos \beta \sin \delta \hat{h}_a \right]$$

(9)

where $\alpha'$ from Eq. (6) has been introduced. Substituting Eq. (7) in the last equation, it is obtained:

$$V'_a = -2 e_\alpha \mu \frac{T_0}{m} r'_a \cdot \sin \beta e_\alpha$$

(10)

Finally, since the unit vectors in the bracket of the $\tau$-derivative of the potential form a basis, they are mutually orthogonal, i.e., $r_a \cdot \hat{p}_a = 0$ and $r_a \cdot \hat{h}_a = 0$. As a result, it is found:

$$V'_a = -2 e_\alpha \mu \frac{T_0}{m} r'_a \sin \beta e_\alpha$$

(11)

with the potential not dependent on $\delta$, i.e., the orbital energy can only be changed thrusting in the direction of the velocity. Then, expressing the elevation angle as $\beta = \arcsin(K e_\alpha)$ and choosing a positive control gain, the non-increasing nature of the $\tau$-derivative of the potential is certified:

$$V'_a = -2 T_0 \mu \frac{r'_a}{m} K e_\alpha^2$$

(12)

Due to the fact that $V_a$ fulfils both requirements of the Lyapunov Direct Method of Stability, applying this strategy a spacecraft starting from $a_0$ will converge to any $a^*$.

2. Full transfer

This section deals with the implementation of a full three-dimensional transfer, equivalent to converge simultaneously to the angular momentum vector and the first two components of the eccentricity one. These two vectors are expressed in KS coordinates in the following:

$$h = \left( \frac{2}{r} u_r^T A u_r', \frac{2}{r} u_r^T B u_r', \frac{2}{r} u_r^T C u_r' \right)^T$$

(13)

$$e = \frac{1}{r \mu} r'_a^T \left[ Y_A, Y_B, Y_C \right]^T h_a - \hat{r}$$

(14)

where $A$, $B$ and $C$ are skew symmetric matrices only function of $u_r$, defined as:

$$A = \mathcal{L}(u_r)^T Y_A \mathcal{L}(u_r), \quad B = \mathcal{L}(u_r)^T Y_B \mathcal{L}(u_r), \quad C = \mathcal{L}(u_r)^T Y_C \mathcal{L}(u_r)$$

(15)

with $Y_A$, $Y_B$ and $Y_C$ being three constant matrices, reported in the following:

$$Y_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Y_B = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Y_C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(16)

Then, the $\tau$-derivative of the angular momentum and the eccentricity vectors are needed:

$$h' = - \left( \mathcal{L}(u_r) [A, B, C]^T u_r \right) \cdot f_a$$

(17)
\[ e' = \left\{ \frac{r}{\mu} \left[ Y_A, Y_B, Y_C \right]^T h_a - \frac{\mathcal{L}(u)}{r\mu} \left[ Y_A, Y_B, Y_C \right]^T u \right\} \cdot f_a \]  

Finally, the targeted value of the latter two vectors are provided in the following:

\[ h^* = \sqrt{\mu a^* (1 - e^*^2)} \left( \sin i^* \sin \Omega^*, \sin i^* \cos \Omega^*, \cos i^* \right)^T \]

\[ e^* = e^* \left( \cos \omega^* \cos \Omega^* - \sin \omega^* \sin \Omega^* \cos i^*, \sin \omega^* \cos \Omega^* \cos i^* + \cos \omega^* \sin \Omega^*, \text{free} \right)^T \]

After this brief introduction, in the following, a potential is defined as the sum of two quadratic differences, i.e., between \( h \) and \( h^* \), and between the first two components of \( e \) and \( e^* \):

\[ V_o = \frac{1}{2} (h - h^*)^T (h - h^*) + \frac{1}{2} (e_1 - e_1^*)^2 + \frac{1}{2} (e_2 - e_2^*)^2 = \frac{1}{2} e_h + \frac{1}{2} e_1 + \frac{1}{2} e_2 \]

Since the latter potential is positive by definition, the last step to fulfill the Lyapunov Direct Method of Stability is to demonstrate that its \( \tau \)-derivative is negative semi-definite. In particular, \( V_o' \) can be split into two parts, i.e., the thrust acceleration contribution and the vector \( \mathbf{g} \), which includes all the components of the \( \tau \)-derivative of the potential excluding the thrust:

\[ V_o' = -\frac{T_0}{m} \left[ \mathbf{g} \cdot (\cos \delta \hat{p}_a + \sin \delta \hat{h}_a) \right] \]

with

\[ \mathbf{g} = (h_1 - h_1^*) [\mathcal{L}(u_r)A u_r] + (h_2 - h_2^*) [\mathcal{L}(u_r)B u_r] + (h_3 - h_3^*) [\mathcal{L}(u_r)C u_r] + \\
- e_{e_1} \left[ r a Y_A h_a - \frac{1}{r} \mathcal{L}(u_r) (r_2' C - r_3' B) u_r \right] - e_{e_2} \left[ r a Y_B h_a - \frac{1}{r} \mathcal{L}(u_r) (r_2' A - r_1' C) u_r \right] \]

Finally, to make the derivative of the candidate potential negative semi-definite, and then to satisfy the Lyapunov Direct Method of Stability, the vector \( \mathbf{g} \) must be set parallel to the parenthesis in Eq. (22). This is possible by selecting the azimuth angle as it follows:

\[ \delta = \arctan \left( \frac{g \cdot \hat{h}_a}{g \cdot \hat{p}_a} \right) \]

### III. Low-thrust guidance in KS variable with \( J_2 \)

The layout of this section, divided into three parts, is presented in the following. Firstly, the Secularly Precessing Ellipse model, which allows to approximate the secular effects of \( J_2 \) through three analytically integrable equations, is introduced. Then, it is explained how to introduce this model into the algorithm by Hernandez and Akella, signaling that this operation is appropriate only for low-thrust engines. Finally, a conditional statement on the radius at periapsis is introduced, forcing the spacecraft to a circularisation procedure if it is too close to the asteroid surface.

#### A. Orbit Propagation with \( J_2 \)

In the two-body problem, the gravitational potential is a function of only the distance from the spacecraft to the main body and the gravitational parameter of the latter one:

\[ U_{2BP} = -\frac{\mu}{r} \]

However, when one or more of the assumptions of the two-body problem are not considered strong enough w.r.t. the level of accuracy desired, the latter equation loses its reliability. Nevertheless, it is possible to split the gravitational potential into two different contributions: \( U_{2BP} \), associated with the two-body problem, and \( U' \), which includes all the perturbations w.r.t. the two-body potential. As a result, the gravitational potential may be expressed as a sum of two terms:

\[ U = U_{2BP} + U' \]
There are three main perturbations affecting a spacecraft orbiting a small-body: solar radiation pressure, mass distribution and third-body effects. Among these three phenomena, the second one is considered the most important and difficult to implement for small-body proximity manoeuvres: in fact, the effects of the solar radiation pressure may be included, to first approximation, in a sort of augmented gravitational parameter, allowing to treat the dynamics as a two-body problem, while the closer the spacecraft is to the main body, the smaller the contribution by a third one on the vehicle. On the other hand, the magnitude of the second perturbation, which takes into account the violation of the Keplerian assumption on the spherical shape and homogeneous density of the main body, is larger when the spacecraft is closer to the surface.

In particular, the potential associated to the irregularities in the mass distribution can be modelled through an infinite series of Legendre polynomials as it follows:

\[
U_{MD} = \frac{\mu}{r} \left[ \sum_{n=2}^{\infty} \frac{J_n}{n} \left( \frac{R}{r} \right)^n P_n(\sin \phi) + \sum_{n=2}^{\infty} \sum_{m=1}^{n} \frac{J_{n,m}}{n} \left( \frac{R}{r} \right)^n P_{n,m}(\sin \phi) \cos(m(\Lambda - \Lambda_{n,m})) \right]
\]  

(27)

where the first term is the two-body acceleration, \( n \) and \( m \) are called degree and order, respectively, \( J_n \), \( J_{n,m} \) and \( \Lambda_{n,m} \) are model parameters, \( \phi \) and \( \Lambda \) complete with \( r \) the set of spherical coordinates, \( P_{n,m}(\cdot) \) is a Legendre polynomial and \( R \) is the mean radius of the main body. Naturally, the accuracy of Eq. (27) is dependent on the number of terms considered: the more, the better.

Nevertheless, it is clear that Eq. (27) has to be truncated: an excessive number of parameters decreases the computational efficiency of the on-board algorithm considerably, and the knowledge of \( J_n \) and \( J_{n,m} \) is strongly limited by the data available on the specific small-body. Additionally, extensive work has been done on modelling the effects of the main body oblateness, for planets and small-bodies: as a result, \( J_2 \) is considered the only non-spherical contribution in the following. Moreover, the effects by the oblateness can be classified into secular variations and periodic ones: while the former ones are unbounded, being proportional to a certain power \( k \) of time, the latter ones average out at the end of their period and are complex to model. Then, to maintain the simplicity of the original algorithm, the periodic effects are ignored in the following, obtaining:

\[
U_{MD} = \frac{n^2 R^2 J_2}{4(1 - e^2)^{3/2}} \left[ 3 \sin^2 i - 2 \right]
\]  

(28)

The latter potential is introduced in the Lagrange Planetary Equations, and, while \( a \), \( e \), and \( i \) are not affected by \( U_{MD} \), for the other three elements the following set is obtained:

\[
\frac{d\omega}{dt} = -\frac{3}{2} J_2 a \left( \frac{R}{a} \right)^2 \frac{\cos i}{(1 - e^2)^2}
\]

\[
\frac{d\Omega}{dt} = \frac{3}{4} J_2 a \left( \frac{R}{a} \right)^2 \frac{5 \cos^2 i - 1}{(1 - e^2)^2}
\]

\[
\frac{dM}{dt} = n_{avg} - \frac{3}{4} J_2 a \left( \frac{R}{a} \right)^2 \frac{3 \cos^2 i - 1}{(1 - e^2)^{3/2}}
\]  

(29)

where \( a \), \( e \) and \( i \) represent the mean elements. However, since the former equations are part of a first-order analysis and the difference between the osculating elements and the initial ones is on the order of \( J_2 \), for sake of simplicity, the latter ones may also be used. In fact, despite not mathematically rigorous, the error caused by this approach is on the order of \( J_2^2 \), with \( M \) as the only exception: due to its fast nature, the first \( n \) in the last equation of Eq. (29) is substituted by \( n_{avg} \), calculated with \( n_{avg} \) to avoid substantial errors w.r.t. the full model.

The set in Eq. (29) can be integrated analytically, and it forms the basis of a linear model called Secularly Precessing Ellipse. To properly understand its effects, in Fig. 1, the history of the orbital elements \( \Omega, \omega \) and \( M \) is presented for three different formulations: two-body problem, full dynamics perturbed only by \( J_2 \) and linearised formulation. While the first two models require the numerical integration of six differential equations, in the last case the integration is analytical and straightforward. Furthermore, the angles in Fig. 1 are not wrapped to 360 deg, to show in an unambiguous way the linear trend driving their evolution over time. Then, two considerations arise:
• After 25 days, the error made when applying the two-body problem formulation w.r.t. the full dynamics ranges from about 25 degrees in the case of $\Omega$ to more than 60 degrees for $M$, with $e_\omega$ lying in the middle; on the other hand, the error coming from the application of the linearised formulation is about two orders of magnitude smaller.

• The variation of $M$ over time is almost totally dominated by the term $nt$: the lines representing the two-body problem and the full dynamics are clearly distinguishable for $\Omega$ and $\omega$, but not for $M$. Nevertheless, looking at the errors, they all have the same magnitude: due to this behaviour, the mean anomaly will require special attention.

Finally, it is highlighted that the oblateness factor implemented in the last example does not represent the real parameter for 433 Eros, but it is chosen, for sake of simplicity, as five times the ones of the Earth, due to the fact that these small-bodies are usually way more irregular than planets. Nonetheless, from the results of the NEAR Shoemaker mission, extensive data on the shape and gravity models of 433 Eros are available. 

Figure 1: $\Omega$, $\omega$ and $M$ propagation around 433 Eros: two-body problem, perturbed by $J_2 = 0.005$ and linearised formulation. Orbit0: $a_0 = 35$ km, $e_0 = 0.3$, $i_0 = 25^\circ$, $\Omega_0 = 60^\circ$, $\omega_0 = 15^\circ$, $\theta_0 = 0^\circ$.

B. Simulator architecture

The simulator architecture for the new algorithm is identical to the one introduced in Section II, with the only changes being in the introduction of the radius of the main body and its $J_2$ as inputs. In this way, the simulator is able to take into account the non-spherical shape of the main body about which the spacecraft orbits, if these irregularities are limited to its oblateness: the celestial body is treated as an ellipsoid of revolution with all the zonal harmonics set to zero, except $J_2$.

In the following, the main steps to implement the improved algorithm and the key differences w.r.t. the original one are treated in details. The starting point is the knowledge of three physical data about the celestial body, i.e., its standard gravitational parameter $\mu$, its first zonal harmonic $J_2$ and its mean radius $R$, and the initial orbit in Keplerian Elements, i.e., $\mathbf{x}_{Kep}(t_0) = (a_0, e_0, i_0, \Omega_0, \omega_0, \theta_0)^T$. It is highlighted that, in the first simulator, $R$ was not necessary: within the two-body problem assumption, the main body is considered as a point mass. This is not true with the current algorithm, and $R$ is a key contributor to all the equations from the last section. Then, the data of the final orbit in Keplerian elements, i.e.,
\( \mathbf{x}_{\text{Kep}} = (a^*, e^*, i^*, \Omega^*, \omega^*, \theta)^T \), are introduced, with the final true anomaly \( \theta^* \) always unspecified. As a result, the number of parameters to match may vary from a minimum of one to a maximum of five: based on this number, an equal set of tolerance \( \epsilon_\gamma \) is created, i.e., the maximum difference between the actual orbital element(s) \( \gamma \) and the desired one(s) \( \gamma^* \) to consider the targeting process finished and successful. At this point, the initial Keplerian elements are transformed into a set of Cartesian coordinates centered on the main body’s center, i.e., \( \mathbf{x}_0 \), which is used to initialise the KS state \( \mathbf{u}_0 \). Finally, setting \( i = 1 \), the core loop, based on the propagation of the KS variables, can start.

The derivatives of the KS state \( \mathbf{u}_i \), the inverse of the semi-major axis \( \alpha \), the spacecraft mass \( m \) and actual time \( t \) w.r.t. the fictitious time \( \tau \), named \( \mathbf{u}' \), \( \alpha' \), \( m' \), \( t' \), respectively, are calculated assuming the main body to be a perfect homogeneous sphere. Through integration of their derivatives, these variables are found at the successive step, i.e., \( u_{i+1}, \alpha_{i+1}, m_{i+1} \) and \( t_{i+1} \). Then, the updated Keplerian elements are calculated through the well known formula to transform the KS variables into Keplerian ones. At this point, the orbital elements are corrected to take into account the oblateness of the main body, passing from a generic orbital element \( \gamma_{i+1} \) to \( \gamma_{i+1}^* \), by adding a \( \Delta\gamma_{i+1}^* \).

In mathematical terms, integrating Eq. (29), it is found:

\[
\Delta\gamma_{i+1}^* = \int_{t_i}^{t_{i+1}} \left( \frac{d\gamma}{dt} \right)^2 \Delta t
\]

in case \( \gamma \) corresponds to \( \Omega, \omega \) and \( M \), while \( \Delta\gamma_{i+1}^* = 0 \) for \( \gamma \) equal to \( a, e \) and \( i \). In particular, in the last section, the validity of Eq. (30) was demonstrated for an orbit propagation, i.e., no applied thrust. In that case, the last equation can be integrated analytically, resulting in:

\[
\Delta\gamma_{i+1}^* = \left( \frac{d\gamma}{dt} \right)^2 \Delta t
\]

with \( \Delta t = t_{i+1} - t_i \). This is due to the fact that the derivative in the last equation is only dependent on \( \mu, R, a, e \) and \( i \): while the first two parameters are constant by definition, the latter three ones do not change over time when short- and long-periodic perturbation effects by \( J_2 \) are ignored. This statement is not true when a thrust is applied, i.e., during an orbital manoeuvre: for in-plane transfers \( a \) and \( e \) may vary dramatically, and the same is possible with \( i \) during out-of-plane manoeuvres. To give an idea of the additional complexity, Eq. (30) is rewritten for \( \omega \) during an orbit matching:

\[
\Delta\omega_{i+1}^* = \frac{3}{2} J_2 R^2 \sqrt{\mu} \int_{t_i}^{t_{i+1}} \frac{\cos i(t)}{a(t)^{5/2}(1 - e(t)^2)^2} dt
\]

From a quick glance at Eq. (32), it is clear why it is not analytically integrable: there is no known relationship expressing \( a, e \) and \( i \) as function of the time \( t \). Moreover, proceeding numerically, the main advantages of simplicity and computational efficiency of the formulation presented in the last section would be lost or at least seriously compromised.

Nevertheless, if the integration interval is small enough and the magnitude of the applied thrust is not too large, it is possible to consider \( a, e \) and \( i \) constant during a \( \Delta t \). Naturally, this approximation is particularly accurate for low-thrust systems, with \( T_0 \) on the order of \( mN \), while it totally loses its validity in case of chemical propulsion. Then, the following set is obtained:

\[
\Delta\omega_{i+1}^* = \left[ \frac{3}{2} J_2 n_i \left( \frac{R}{a_i} \right)^2 \frac{\cos i}{(1 - e_i^2)^2} \right] \Delta t
\]

\[
\Delta\Omega_{i+1}^* = \left[ \frac{3}{4} J_2 n_i \left( \frac{R}{a_i} \right)^2 \frac{5 \cos^2 i - 1}{(1 - e_i^2)^2} \right] \Delta t
\]

\[
\Delta M_{i+1}^* = \left[ \frac{3}{4} J_2 n_i \left( \frac{R}{a_i} \right)^2 \frac{3 \cos^2 i - 1}{(1 - e_i^2)^{3/2}} \right] \Delta t
\]

It is highlighted that the term in parenthesis in the last equation is slightly different than Eq. (29), due to the absence of \( n_{\text{avg}} \), which dominates the propagation of \( M \) (cf. Fig. 1). However, this absence is due
to the fact that $n_{avg}$ is not a consequence of the secular perturbation by $J_2$, but it is just the essential part of the fast nature of the mean anomaly. Then, after this clarification, adding the $\Delta \gamma^{J_2}$ from Eq. (33) to the Keplerian elements coming from the two-problem assumption and keeping in mind that no secular contribution from $J_2$ is applied to $a$, $e$ and $i$, an updated Keplerian state is obtained:

$$
\begin{pmatrix}
\frac{a^{J_2}_{i+1}}{e^{J_2}_{i+1}}
\frac{i^{J_2}_{i+1}}{\omega^{J_2}_{i+1}}
\frac{\Omega^{J_2}_{i+1}}{M^{J_2}_{i+1}}
\end{pmatrix} =
\begin{pmatrix}
a_{i+1}
e_{i+1}i_{i+1}
\omega_{i+1}+\Delta \omega^{J_2}_{i+1}
\Omega_{i+1}+\Delta \Omega^{J_2}_{i+1}
\end{pmatrix} +
\begin{pmatrix}
\Delta a^{J_2}_{i+1}
\Delta e^{J_2}_{i+1}
\Delta i^{J_2}_{i+1}
\end{pmatrix}
$$

(34)

Although this approach makes it simple to add an important perturbation to the algorithm introduced by Hernandez and Akella,¹ it is noted that the latter one is applicable to both high- and low-thrust propulsion systems: then, the assumption of small $T_0$ made here constitutes a limitation to its original capabilities. Nevertheless, enhancing the algorithm capabilities on small-bodies while renouncing to some features not necessary for them, perfectly fits with the original goal.

Afterwards, the Cartesian state $x_{i+1}$ is found through the formula to pass from Keplerian to Cartesian coordinates. Then, the errors $(e_\gamma)_{i+1}$ and their relations with the pre-set tolerances $\epsilon_\gamma$ are defined: they are calculated as the absolute difference between the updated Keplerian elements $\gamma^{J_2}_{i+1}$ and the targeted ones $\gamma^*$. At this point, if $(e_\gamma)_{i+1} \leq \epsilon_\gamma$ for all the targeted orbital elements, the simulation is considered concluded, while, if the error associated with one or more targeted elements exceed the pre-set tolerance, the algorithm calculates a way to get the spacecraft closer to them. In the second case, the control angles $\beta_{i+1}$ and $\delta_{i+1}$ are found as shown in Section II.

Once the control variables are defined, the second main difference between the new and the old version of the algorithm is presented: the calculation of $u^{J_2}_{i+1}$. In fact, in the initial algorithm, the core loop is fully in KS variables, and the usefulness of the Keplerian and the Cartesian coordinates is limited to the evaluation of the errors $e_\gamma$ and to plotting, respectively. Now, instead, without the update from $u_{i+1}$ to $u^{J_2}_{i+1}$, the contribution of $J_2$ would be still visible in $x_{i+1}$ and $x_{kep,i+1}$, but lost in $u_{i+2}$, i.e., at the end of each iteration: as a result, a huge piece of information would be lost.

Basically, the main complication that stems from regularising a set of coordinates, from Cartesian to KS ones, is the difference in dimensions: no mapping like $r \in \mathbb{R}^3 \rightarrow u_r \in \mathbb{R}^3$ is available with the current state-of-art. Using the KS model, an additional degree of freedom is added, and for all the infinite points that lie on a certain circle about the origin in the $u$-space, there is a single point in the physical $x$-space. However, it is possible to implement Eq. (5), and the results of its application are presented in Fig. 2, where, starting from the knowledge of the position vector $u_r$, a KS transformation is performed twice. In this way, two KS position vectors and two Cartesian ones are obtained: the "original" $u_r$ and the Cartesian vector obtained from it are the "propagated position vectors"; instead, the $u_r$ obtained through transformation of the propagated Cartesian vector, and the subsequent Cartesian vector obtained from it are called the "transformed position vectors". After these clarifications, it is noted that, although the propagated $u_r$ and the transformed one do not coincide, their conversions to Cartesian coordinates are identical: as shown in Fig. 2b, the difference between the propagation error $10^{-11}$ m for the position errors and $10^{-15}$ m/s for the velocity ones. This is due to the fact that the floating-point relative accuracy for a certain variable in MATLAB R2017a is equal to $\epsilon = 2.2204 \times 10^{-16}$ times the variable itself: while $x$, $y$ and $z$ are on the order of ten kilometres, $v_x$, $v_y$ and $v_z$ are on the order of meters per second.

Finally, after calculating $u^{J_2}_{i+1}$, $J_2$ is removed from the name of all the variables containing it, i.e., $\gamma^{J_2}_{i+1}$ and $u^{J_2}_{i+1}$: the latter two variables are allocated as $\gamma_{i+1}$ and $u_{i+1}$. Then, the last step is to update the time step, i.e., $i = i + 1$, and to iterate the procedure up to the point where the answer to the decision box in the simulator architecture is affirmative. It is highlighted again that this does not happen only when a full orbit is matched, since it is possible to target also a Keplerian subset.

The real-time guidance algorithm described in this section is implemented in Fig. 3, where the matching of the semi-major axis for an ascent transfer about 433 Eros is shown. The spacecraft is equipped with
Figure 2: Comparison of the propagated spacecraft state and the one from Eq. (5), during $a$, $e$, $i$, $\Omega$ and $\omega$ matching around 433 Eros. Orbit$_0$: $a_0 = 30$ km, $e_0 = 0.2$, $i_0 = 15^\circ$, $\Omega_0 = 30$, $\omega_0 = 45$, $M_0 = 15^\circ$. Target: $\epsilon_a = 10^{-3}$ DU$^{-1}$, $\epsilon_e = 10^{-2}$, $\epsilon_i = \epsilon_\Omega = \epsilon_\omega = 0.25^\circ$, $a_T = 50$ km, $e_T = 0.1$, $i_T = 5^\circ$, $\Omega_T = 0^\circ$, $\omega_T = 0^\circ$. SC: $T = 5$ mN, $I_{sp} = 3000$ s, $m_0 = 2500$ kg. Gain: $K_a = 50$, $K_e = 50$. 
a low-thrust engine, with $T = 5 \, \text{mN}$ and $I_{sp} = 3000 \, \text{s}$, and it starts in a quasi-circular orbit about three kilometres higher than the average radius of the asteroid. During the manoeuvre, it is noted that the right ascension of the ascending node does not present the expected linear trend, which characterises the orbit propagation, due to the fact that the formula in Eq. (29) decreases proportionally to $a^{3.5}$: as a result, in the first five days, $\Omega$ declines almost linearly, while afterwards the steep of the curve becomes remarkably less evident. The time of flight is the same for both the two-body and the perturbed formulation, due to the implementation of the Secularly Precessing Ellipse, while the effects on the out-of-plane elements $\Omega$ and $\omega$ are appreciable, highlighting the importance of adding the main body’s oblateness.

C. Non-impact condition

When only the semi-major axis is targeted, there is an important difference on the path followed by the spacecraft, based on the relation between $a_0$ and its final value $a^*$: if the purpose of the manoeuvre is to match a higher orbit, i.e., $a_0 < a^*$, there is no risk of impact; on the other hand, if the spacecraft is directed towards a lower one, i.e., $a_0 > a^*$, some complications arise. In fact, assuming that the vehicle starts from an orbit in which the combination of $a_0$ and $e_0$ does not allow to have an impact, i.e., the radius at periapsis $r_p = a(1 - e)$ is larger than $R$, decreasing the height results in a negative effect on $r_p$. As an example, in Fig. 4a, a transfer towards a lower orbit is shown, where the targeted semi-major axis is only 7% larger than the asteroid radius, i.e., $a^* = 18 \, \text{km}$. At the beginning of the orbital manoeuvre, the spacecraft state is defined by $a_0 = 30 \, \text{km}$, $e_0 = 0.25$ and $\omega = 0^\circ$, corresponding to an altitude one-third larger than the asteroid radius. The total transfer lasts 1.35 days and only 300 grams of propellant are burnt, thanks to a tiny thrust magnitude and a considerable specific impulse, i.e., $T = 25 \, \text{mN}$ and $I_{sp} = 1000 \, \text{s}$, but, after about half of the total time of flight, the spacecraft impacts on the main body surface. However, due to the two-body problem assumption on which the algorithm by Hernandez and Akella$^1$ is based, 433 Eros is considered as a point mass and the algorithm is not able to address this problem.

The easiest solution is to maintain the original structure and to implement an additional check on the evolution of the periapsis radius: if this quantity is smaller than a certain limit, named $r_{pi,m}$, the spacecraft would stop the $a$ matching and start a circularisation, decreasing $e$ to take $r_p$ to a safer value. This approach is presented in Fig. 4b, where the same initial and final data of the last example are used, but a limit on $r_p$ is imposed, i.e., $r_{pi,m} = 17.5 \, \text{km}$, equivalent to the asteroid mean radius plus a 4% margin. Comparing the two cases treated in Fig. 4, it is noted that avoiding the impact has a huge cost: the fuel mass increases by 50% and the time of flight almost triples, from 300 to 470 grams and from 1.35 to 3.14 days, respectively. Moreover, the thrust is not applied continuously any more, and this explains why the increase in the transfer duration is larger than the one for the spacecraft mass: eight coast arcs are needed, during which, naturally, no fuel is wasted. In this way, targeting the semi-major axis corresponds to a sort of double matching of $a$ and $e$, due to the fact that, if $a^*$ is only slightly larger than $R$, non-circular orbits tend to result in impacts.

IV. Conclusions

In this paper the development of a real-time guidance algorithm able to execute manoeuvres in the proximity of a small-body was investigated. In particular, the starting point has been the algorithm presented by Hernandez and Akella,$^1$ founded on the two-body problem, to which new features have been added to allow for more flexibility.

The main perturbations affecting the two-body problem when applied to these environments regard the perfect sphere and point mass assumptions on the main body. To limit these effects, two different paths are followed. Firstly, to avoid an impact between the spacecraft and the small-body, a check on the periapsis radius is implemented. Secondly, the effects of the mass irregularities are limited to the secular contribution of $J_2$, and implemented via the Secularly Precessing Ellipse. In both cases, following this strategy, the architecture of the initial algorithm is maintained.

The proposed mission scenario implements an approximate model of 433 Eros as the main body and a low-thrust point mass spacecraft executing orbital transfers about it. It was shown that ignoring $J_2$ results in errors on the order of tens of degrees for the $\Omega$, $\omega$, and $M$, for transfers lasting tens of days. They are reduced by up to two orders of magnitude when the proposed simulator is applied. Moreover, avoiding an impact via a circularisation procedure produces a fuel burnt increase of 50 while the time of flight almost triples, due to the amplification by possible coasting arcs.
(a) Cartesian coordinates

Figure 3: a matching around 433 Eros with $J_2 = 0.005$. Orbit$_0$: $a_0 = 20$ km, $e_0 = 0.05$, $i_0 = 5^\circ$, $\Omega_0 = 60^\circ$, $\omega_0 = 15^\circ$, $\theta_0 = 0^\circ$. Target: $\epsilon_\alpha = 10^{-3}$ DU$^{-1}$, $a^* = 100$ km. SC: $T = 5$ mN, $I_{sp} = 3000$ s, $m_0 = 2500$ kg. Gain: $K_a = 50$, $K_e = 50$. 

(b) $a$, $e$, $i$, $\Omega$, $\omega$ and $\Delta m$ history
Figure 4: a matching around 433 Eros. Orbit0: $a_0 = 30$ km, $e_0 = 0.25$, $i_0 = \Omega_0 = \omega_0 = 0^\circ$, $\theta_0 = 30^\circ$. Target: $\epsilon_a = 1$ m, $a^* = 18$ km. SC: $T = 25$ mN, $I_{sp} = 1000$ s, $m_0 = 2500$ kg. Gain: $K_a = 1$. $r_{p,lim} = 17.5$ km.
Unfortunately, the proposed methodology is not feasible for chemical rockets, limiting the range of applicability of the initial algorithm to low-thrust engines; however, these are the propulsion systems implemented for the intended mission scenarios, i.e., small-body proximity missions. Due to their simplicity and the scarcity of real-time guidance algorithms able to perform orbital manoeuvres around oblate bodies and to avoid possible impacts, the results are promising and innovative.

References