Critical behavior of Ising spin systems
Phase transition, metastability and ergodicity

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Critical behavior of Ising spin systems

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Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof. dr. ir. T.H.J.J. van der Hagen,
voorzitter van het College voor Promoties,
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door

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ヒデキ
がんばれ

JANEIRO, 2010 DITIAN E BATIAN

When I became the first person in my family
who could study at a public university,
my grandparents put a lot of effort
into writing this message for me.
It says: "Hideki, do your best".

Contents

Summary

Physical phenomena commonly observed in nature such as phase transitions, critical phenomena and metastability when studied from a mathematical point of view may give arise to a rich variety of behavior whose study becomes interesting in itself.

In Chapter 1 we illustrate the phase transition phenomenon at low temperatures for one-dimensional long range Ising models with inhomogeneous external fields. More precisely, we consider Ising spins arranged on the one-dimensional integer lattice where such spins interact via ferromagnetic pairwise interactions whose strength is inversely proportional to their distance to the power α; furthermore, the system is put under the influence of an external magnetic field that vanishes with polynomial power δ as the distance between the spin and the origin increases. In that case we show that a phase transition manifests itself in the form of the existence of two distinct infinite-volume Gibbs states, obtained by means of the application of the thermodynamic limit considering “plus” and “minus” boundary conditions respectively, whenever the system is subject at low temperatures and an inequality involving α and δ holds. The proof of this result is done by means of the Peierls’ contour argument adapted to one-dimensional long range Ising models, first introduced by J. Fröhlich and T. Spencer in 1982 and later modified by M. Cassandro, P.A. Ferrari, I. Merola and E. Presutti in 2005. Our results improve the one obtained by the latter authors since we managed to avoid the assumption of large nearest-neighbor interactions and added the influence of an external field, showing an interplay between the constants α and δ in order to guarantee the manifestation of the phase transition.

In Chapter 2 we apply standard techniques presented by F. Manzo, F.R. Nardi, E. Olivieri and E. Scoppola in 2003 in order to approach the problem of metastability of Ising spin systems. The problem addressed in this chapter, differently from the one in the previous chapter, has a dynamical nature, where we explore metastable features of one-dimensional ferromagnetic Ising systems with long range pairwise interactions in the presence of a uniform external field defined in a finite volume with free boundary condition. We characterize the asymptotic behavior of the tunneling time between metastable configurations and stable configurations as the temperature approaches zero. Moreover, the critical configurations are determined in the general case as well as in some particular situations, such as in the cases where the strength of the pair interactions decays polynomially or exponentially.

The main concern of Chapter 3 is to explore ergodic properties of probabilistic cellular automata (PCAs) on infinite rooted trees. We start by establishing a partial relationship between ergodicity/non-ergodicity of PCAs and uniqueness/phase transition for a related equilibrium statistical mechanical model defined on space-time configurations, where we construct a correspondence between stationary measures for the PCA dynamic and time-invariant Gibbs states for its correspondent space-time model. Such a result is an extension of the one obtained by S. Goldstein, R. Kuik, J. Lebowitz and C.

Maes in 1989 which was done for PCAs on the d-dimensional cubic lattice. After that we develop some necessary and sufficient conditions that guarantee the ergodicity for PCAs on d-ary trees obtaining explicit computations for their critical parameters and stationary measures.

Samenvatting

Fysische verschijnselen die vaak in de natuur worden waargenomen, zoals faseovergangen, kritische verschijnselen en metastabiliteit wanneer ze vanuit wiskundig oogpunt worden bestudeerd, kunnen een rijke verscheidenheid aan gedrag veroorzaken waarvan de studie op zich interessant wordt.

In Hoofdstuk 1 illustreren we het fenomeen van faseovergangen bij lage temperaturen voor eendimensionale Ising-modellen met oneindige dracht met inhomogene externe velden. Om precies te zijn, beschouwen we Ising-spins gerangschikt in het eendimensionale, heeltallige rooster waar dergelijke spins wisselwerken via ferromagnetische paarsgewijze interacties waarvan de sterkte omgekeerd evenredig is met hun afstand tot de macht α; bovendien wordt het systeem onder invloed gebracht van een extern magnetisch veld dat verdwijnt als de afstand van de spin tot de oorsprong tot de macht δ. In dat geval laten we zien dat de faseovergang zich manifesteert in de vorm van het bestaan van twee verschillende Gibbs-toestanden van het oneindige volume, verkregen door de toepassing van de thermodynamische limiet, rekening houdend met respectievelijk “plus” en “minus” randvoorwaarden, telkens wanneer het systeem onderhevig is aan lage temperaturen. Het bewijs van dit resultaat wordt geleverd door het contourargument van Peierls, aangepast aan eendimensionale langedrachts Ising-modellen, voor het eerst geïntroduceerd door J. Frohlich en T. Spencer in 1982 en later veralgemeend door M. Cassandro, PA Ferrari, I. Merola en E. Presutti in 2005. Onze resultaten verbeteren het resultaat van de latere auteurs omdat we de aanname van grote naastebureninteracties hebben kunnen vermijden en de invloed van een extern veld hebben toegevoegd, met een ongelijkheid tissen de constanten. α en δ om de het optreden van de fase-overgang te garanderen.

In Hoofdstuk 2 passen we standaardtechnieken toe, gepresenteerd door F. Manzo, F.R. Nardi, E. Olivieri en E. Scoppola in 2003 om het probleem van metastabiliteit van Ising-spinsystemen te aan te pakken. Het probleem dat in dit hoofdstuk wordt aangepakt, anders dan in het vorige hoofdstuk, heeft een dynamisch karakter, waarbij we metastabiele kenmerken van eendimensionale ferromagnetische systemen onderzoeken met lang-bereik paarsgewijze interacties in de aanwezigheid van een uniform extern veld dat is gedefinieerd in een eindig volume met vrije randvoorwaarde. We karakteriseren het asymptotische gedrag van de tunnelingstijd tussen metastabiele configuraties en stabiele configuraties wanneer de temperatuur tot nul nadert. Bovendien worden de kritische configuraties zowel in het algemene geval als in sommige specifieke situaties bepaald, zoals in de gevallen waarin de sterkte van de paar interacties polynomiaal of exponentieel vervalt.

De hoofdvraag van Hoofdstuk 3 is om ergodische eigenschappen van probabilistische cellulaire automaten (PCA’s) op oneindige grote bomen te onderzoeken. We beginnen met het vaststellen van een gedeeltelijke relatie tussen ergodiciteit / niet-ergodiciteit van PCA’s en uniciteit / fase-overgang voor een gerelateerd statistisch-mechanisch evenwichtsmodel gebaseerd op ruimte-tijdconfiguraties, waarbij we een verband leggen tussen stationaire maten voor de PCA-dynamiek en tijd-invariante Gibbs maten voor een corresponderend ruimte-tijdmodel. Een dergelijk resultaat is een uitbreiding van dat verkregen door S. Goldstein, R. Kuik, J. Lebowitz en C. Maes in 1989, dat werd gedaan voor PCA's op het d -dimensionale kubische rooster. Daarna ontwikkelen we enkele noodzakelijke en voldoende voorwaarden die de ergodiciteit garanderen voor PCA's op d -ary-bomen gebaseerd op expliciete berekeningen voor hun kritische parameters en stationaire maten.

1

LONG RANGE ISING MODEL

1.1. INTRODUCTION

The rigorous study of phase transitions for one-dimensional Ising models with long range slowly decaying interactions (Dyson models) is a classical subject in one-dimensional statistical mechanics. One of the earliest highlights, almost 50 years ago, was Dyson's proof of a phase transition [1–3] proving a conjecture due to Kac and Thompson [4]. Long range Ising models with slow polynomial decay, as well as the somewhat related hierarchical models, have been called "Dyson models" in the literature. We will mostly refer to our polynomially decaying models as "long range Ising models" but sometimes refer to them as "Dyson models".

The formal Hamiltonian of these models is given by

$$H(\sigma) = -\sum_{x \neq y} J_{x,y}\sigma_x\sigma_y - \sum_x h_x\sigma_x. \tag{1.1}$$

Here the sites x, y lie in the integer lattice $\mathbb{Z}$, and the σ_x's are Ising spins. More precise definitions are given in the next section. We first mention what is known for the zero-field case, i.e. when $h_x = 0$ for all x.

If we consider ferromagnetic interactions $J_{x,y} \geq 0$ given by $J_{x,y} = |x-y|^{-2+\alpha}$ with $\alpha < 1$, then it is well known that for $\alpha < 0$ there is no phase transition, and Dyson showed in [1] via comparison with a hierarchical model that, for $\alpha \in (0,1)$, such a system undergoes phase transition at low temperature.

Afterwards different proofs were developed to show the appearance of such a phenomenon. One of them relied on Reflection Positivity [5]. The method of infrared bounds offers an alternative way to obtaining bounds on contour probabilities. In fact, the authors of [5] remark that they can cover a general class of long range one-dimensional pair interactions, including the ones treated in [1].

Shortly after, Fröhlich and Spencer [6] showed the existence of a phase transition for $\alpha = 0$. The proof of these authors was done by a contour argument; they invented a notion of one-dimensional contours on $\mathbb{Z}$ in order to prove the phase transition. Their

1

strategy more or less followed the classical Peierls contour argument used for the standard nearest-neighbor Ising model, but with a substantially more sophisticated definition of contours. Phase transitions for larger $\alpha \in (0,1)$ can then be deduced by Griffiths inequalities for low enough temperature.

Yet another way to derive the transition was a comparison with independent long range percolation via Fortuin inequalities and Griffiths inequalities for the $\alpha = 0$ case, as discussed in [7]. In that paper it was also shown that the transition for $\alpha = 0$ is a hybrid one, in the sense that the magnetization is discontinuous and at the same time the energy is continuous as a function of temperature, the so-called Thouless effect. Moreover, for $\alpha = 0$ it is known that there is a temperature interval below the transition temperature where the system is critical, in the sense that the covariance is nonsummable, and at the same time the system is magnetized.

Cassandro et al. in [8] rigorously formalized the contour argument of [6] in the parameter regime $0 \le \alpha < \alpha_+$, where $\alpha_+ := \log 3/\log 2 - 1 \approx 0.5849$. The construction allows a more precise description of various properties of the model. It has been used in various follow-up papers [9–15]. We should emphasize that, although the use of contour arguments may look somewhat unwieldy in comparison with other approaches, it is much more robust. Indeed it has been used to analyze Dyson models in random [12, 13] and periodic fields [16], for interface behavior and phase separation [10, 11], for entropic repulsion [9], and here for the model in decaying magnetic fields, all problems where alternative methods appear to break down. See also [17] for another, somewhat related approach.

However, the adaptation proposed by Cassandro et al. in [8] needed the following technical assumptions: (A1): $\alpha \in [0, \alpha^+)$ and (A2): $J(1) \gg 1$. Even the case of $\alpha = 0$, previously obtained by Fröhlich and Spencer, needs $J(1) \gg 1$ in the adaptation proposed by them. The intuition behind the condition is more or less clear; it makes the model closer to a nearest-neighbor interaction model where, in principle, contour arguments might work more easily. Despite the condition being rather artificial and proof-generated, the constraint asking for $J(1) \gg 1$ is present in many later papers about Dyson models and the proof presented in [8] depends strongly on this hypothesis.

As regards the restriction on α, Littin in his thesis [14], and then Littin and Picco [15], showed that, using quasi-additive properties of the Hamiltonian of the corresponding contour model and applying the results from [8], one can modify the contour argument so that it implies the phase transition for all $\alpha \in [0,1)$. Due to the fact that the authors in [15] use energetic lower bounds from [8] which assume large nearest-neighbor interaction $J(1)$, they still use assumption (A2) in their arguments.

Our motivation for the present work is two-fold: first we want to present an argument to remove assumption (A2) for the zero-field case and secondly we want to show persistence of a phase transition for one-dimensional long range models in the presence of external fields decaying to zero at infinity with a power δ, in particular, for fields given by $h_x = h^*(1+|x|)^{-\delta}$ and $1-\alpha < \delta$. More precisely, our results combined with existing results imply that there is a trade-off between restricting the parameter range of δ to $\delta > \max\{1-\alpha, 1-\alpha^*\}$ and $J(1) = 1$ and assuming $J(1) \gg 1$ and choosing $\delta > \max\{1-\alpha, 1-\alpha_+\}$ where $\alpha^* < \alpha_+$ will be specified later. Note that our results apply to the latter case as well.

Before describing the rest of this work, we will discuss briefly the context of these results with respect to the hypotheses and technicalities of the proof. Let us mention that a short announcement of some of our results, but without rigorous proofs, is contained in [18]. Furthermore, all the results presented in this chapter were rigorously reported in [19].

Considering the first result in the zero-field case, although proofs for the existence of a phase transition were known, our estimates allow firstly to drop the (A2) assumption, and then, by using monotonicity of the Hamiltonian with respect to α, we are also able to remove the first assumption (A1).

As regarding the decaying-field case we know that phase transitions for non-zero uniform fields are forbidden due to the Lee-Yang circle theorem [20].

The heuristics behind the inequality $1-\alpha<\delta$ can be obtained as follows. We observe that the contribution of the interaction of a finite interval Λ with its complement is of order $O(|\Lambda|^{\alpha})$, whereas the contribution from the external field is of order $O(|\Lambda|^{1-\delta})$.

We now compare the exponents. If the interaction energy dominates the field energy for large Λ, a contour argument has a chance of working. This intuition is also what is underlying Imry-Ma arguments for analyzing the stability of phase transitions in the presence of random fields. It has been confirmed for decaying fields in higher-dimensional nearest-neighbor models, see below.

It can also be applied to a decaying field the strength of which decays with power δ but which has random signs. In this case the field energy behaves like $O(|\Lambda|^{\frac{1}{2}-\delta})$. This case has also been considered before by J. Littin (private communication) [21]. We note that the case $\delta=0$ reduces to the known Imry-Ma analysis as presented in [12, 13].

Note that the analogous question of the persistence of phase transitions in decaying fields already was studied before in some short-range models, see [22–25].

This chapter is organized as follows. In Section 1.2 we introduce the some theoretical background, define our model of interest and fix some notation. In the next section, Section 1.3, we construct the first main block that constitutes the Peierls' argument through the introduction of a graphical representation for one-dimensional spin configurations by means of triangle configurations, and define the notion of the contours. A detailed exposition of the entropy estimates suitable for such a kind of contours can be found in Section 1.5. Finally, Sections 1.4 and 1.6 contain the proofs of the main theorems including the Peierls' argument.

1.2. THE LONG RANGE ISING MODEL

Let us consider a ferromagnetic one-dimensional long range Ising model together with a nonuniform external magnetic field. As usual, we describe the set of all possible configurations of a system constituted by $+1$ and -1 spins arranged on the one-dimensional integer lattice $\mathbb{Z}$ by means of the configuration space Ω given by $\Omega=\{-1,+1\}^{\mathbb{Z}}$. Fixed a real number α in the interval $[0,1)$, let $J_\alpha:\mathbb{N}\to\mathbb{R}$ be a function defined by

$$J_\alpha(n)=\frac{1}{n^{2-\alpha}}. \tag{1.2}$$

For any pair x,y of distinct spin locations, we interpret the number $J_\alpha(|x-y|)$ as the coupling constant related to the ferromagnetic pair interaction between the spins located at

1

these sites. Furthermore, we add the effect of an external field by means of a family of real numbers $\mathbf{h} = (h_x)_{x\in\mathbb{Z}}$, where h_x is interpreted as the strength of this field at x.

We define our model by means of the interaction potential $\Phi^{\alpha,\mathbf{h}} = (\Phi^{\alpha,\mathbf{h}}_A)_{A\in\mathscr{S}}$, where $\mathscr{S}$ denotes the collection of all nonempty finite subsets of $\mathbb{Z}$, as follows. At each point ω in Ω, let us define $\Phi^{\alpha,\mathbf{h}}_A(\omega)$ by

$$\Phi^{\alpha,\mathbf{h}}_A(\omega) = \begin{cases} -J_\alpha(|x-y|)\omega_x\omega_y & \text{if } A=\{x,y\}, \text{ where } x,y \text{ are distinct elements of } \mathbb{Z}, \\ -h_x\omega_x & \text{if } A=\{x\}, \text{ where } x \text{ belongs to } \mathbb{Z}, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Observe that this interaction is absolutely summable, because the sum

$$\sum_{A\in\mathscr{S},\, A\ni x} \|\Phi^{\alpha,\mathbf{h}}_A\|_\infty = |h_x| + 2\sum_{n=1}^{\infty} J_\alpha(n)$$

is finite for all x. Furthermore, for each nonempty finite subset Λ of $\mathbb{Z}$, it is easy to check that the expression for the Hamiltonian $H^{\alpha,\mathbf{h}}_\Lambda$ is given by

$$H^{\alpha,\mathbf{h}}_\Lambda(\omega) = -\sum_{\substack{\{x,y\}\subseteq\Lambda \\ x\neq y}} J_\alpha(|x-y|)\,\omega_x\omega_y - \sum_{x\in\Lambda}\sum_{y\in\mathbb{Z}\setminus\Lambda} J_\alpha(|x-y|)\,\omega_x\omega_y - \sum_{x\in\Lambda} h_x\,\omega_x \tag{1.4}$$

at each point ω in Ω.

Now, let $+$ denote the configuration of Ω that assigns the value $+1$ at each point of $\mathbb{Z}$. If we restrict ourselves only to configurations with “plus” boundary condition $\tau = +$, then, for every such configuration σ in Ω of the form $\sigma = \omega_\Lambda\tau_{\mathbb{Z}\setminus\Lambda}$, we have

$$H^{\alpha,\mathbf{h}}_\Lambda(\sigma) = 2\left(\frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J_\alpha(|x-y|)\mathbb{1}_{\{\sigma_x\neq\sigma_y\}} + \sum_{x\in\mathbb{Z}} h_x\mathbb{1}_{\{\sigma_x=-1\}}\right) + H^{\alpha,\mathbf{h}}_\Lambda(+). \tag{1.5}$$

For convenience, we also introduce a new energy function defined for any spin configuration with “plus” boundary condition, denoted by $h^{\alpha,\mathbf{h}}$, whose expression is given by

$$h^{\alpha,\mathbf{h}}(\sigma) = \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J_\alpha(|x-y|)\mathbb{1}_{\{\sigma_x\neq\sigma_y\}} + \sum_{x\in\mathbb{Z}} h_x\mathbb{1}_{\{\sigma_x=-1\}}. \tag{1.6}$$

1.3. THE GRAPHICAL REPRESENTATION

The ideas that represent the core of the technique we develop in this chapter, the so-called Peierls contour argument, were first published in 1936 by R. Peierls in his work [26] whose objective was to prove the existence of the phase transition phenomenon at low temperatures for the two-dimensional Ising model when considered with ferromagnetic nearest-neighbor pair interactions in the absence of an external magnetic field. The argument on which Peierls’ result was based relies on a graphical representation that can provide us with a way of visualizing each Ising spin configuration on $\mathbb{Z}^2$ with homogeneous boundary condition as a collection of closed curves on the plane, also known as the Peierls contours. One of the advantages of such a geometrical approach is that it makes possible the establishment of energy bounds that imply on a relationship between the typical configurations of the system at a given temperature with the total length of their corresponding contours. More specifically, one can show that once the system is subject to low temperatures the typical configurations of a system with "plus" boundary condition consist of small islands containing spins with value -1 surrounded by a large ocean of spins with value $+1$. Finally, the last ingredient consists of finding entropy bounds, in the sense that, it is merely the employment of a combinatorial argument that provide the possibility to count the number of contours surrounding the origin. The combination of all these ingredients together reveals the existence of a competition between energetic and entropic terms where the first dominates the second whenever the temperature is sufficiently small. Such an effect is expressed in probabilistic terms in the form of the existence of two distinct Gibbs states μ_β^+ and μ_β^- whenever the parameter $\beta = \frac{1}{T}$ is large enough, which represents the manifestation the phase transition phenomenon in the sense of the one described in [27].

In the following sections, we proceed towards the construction of the phase transition argument for the model defined in Section 1.2 aiming at implementing ideas similar to those we briefly discussed above. We start this section by introducing a modified graphical representation that consists of a representation for one-dimensional Ising spin configurations (again, with homogeneous boundary condition) developed by Cassandro et al. [8] which was derived from the one employed by Fröhlich and Spencer [6]. Differently from the traditional technique, the contours that we will be dealing with consist of collections of triangles grouped together according to suitable separation properties that will be shown to be crucial for obtaining results whose roles are analogous to those present in the original case, allowing us to extend the highly acclaimed Peierls' argument to the one-dimensional case.

In order to show that such a model defined by the interaction potential $\Phi^{\alpha,\mathbf{h}}$ given by equation (1.3) exhibits the phase transition phenomenon at low temperatures via Peierls contour argument, let us consider only spin configurations in $\Omega = \{-1,+1\}^{\mathbb{Z}}$ with "plus" boundary condition, since the analogous results considering "minus" boundary condition follow by means of a simple spin-flip argument. Thus, let Ω_+ be defined as the set of all spin configurations in Ω whose spin values are equal to $+1$ up to a finite numbers of sites, more precisely, we define

$$\Omega_+ = \{\omega \in \Omega : \omega_x = +1 \text{ holds for } |x| \text{ sufficiently large}\}. \tag{1.7}$$

1.3.1. INTERFACE POINTS

Before we dive into the construction of the contours that best suits the one-dimensional case, it is reasonable to start by defining the concept of interface points and to make clear how fundamental is the role played by them. Recall that for the two-dimensional case (see [26–28]), for each configuration with "plus" boundary condition an interface is placed perpendicularly to the midpoint of the edge that joins two sites whenever the values of their spins differ, see Figure 1.1. In that way, we end up with a collection of interfaces that fully characterizes that spin configuration, in the sense that, since we know the fact that the spins sufficiently far from the origin have the value $+1$ and the values of the spins are flipped whenever an interface is crossed, then the original spin configuration can be reconstructed from the knowledge of its corresponding interfaces. The reason for choosing the interface locations exactly in between two sites was to allow the establishment of a direct relationship between the energy of a given configuration and the total sum of the lengths of its associated interfaces. However, for the one-dimensional model that we discuss in this chapter, due to its long range nature, such a kind of relationship does not hold anymore and needs to be adapted, because of that, in our case, the choice of the interface locations is merely arbitrary.

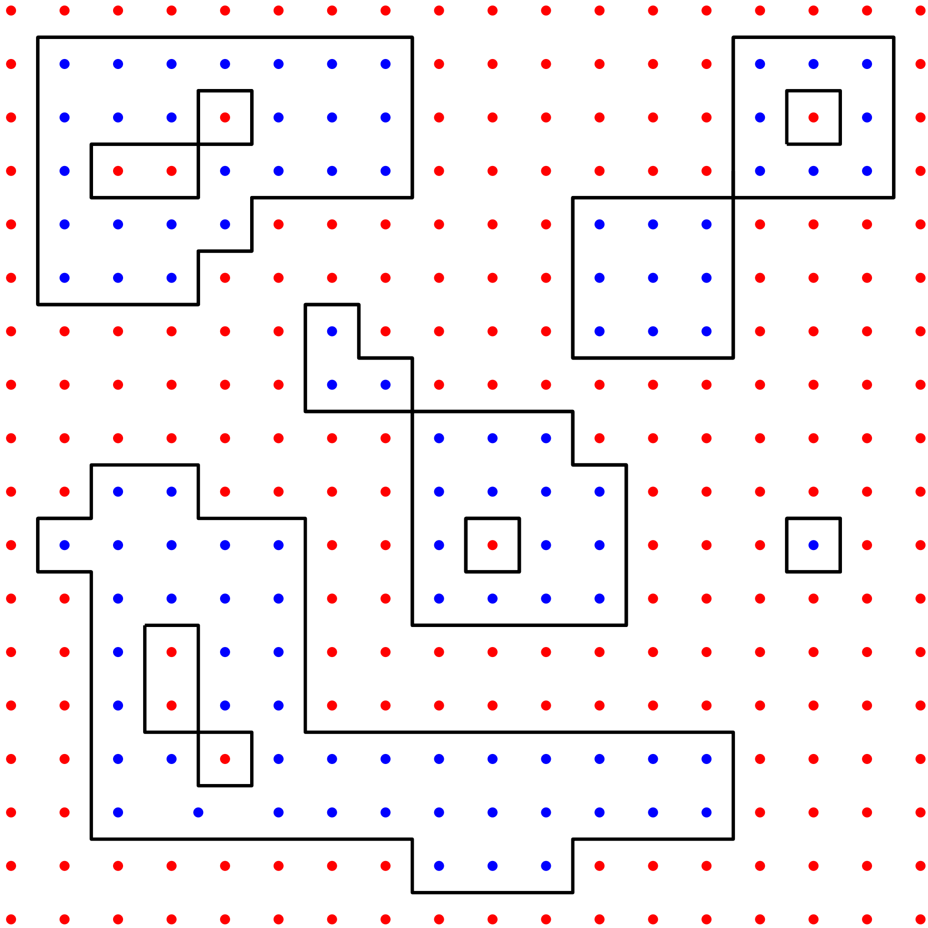


Figure 1.1: Illustration of the use of interfaces to indicate the spin-flip locations associated to a two-dimensional configuration with "plus" boundary condition. The red dots stand for the sites of $\mathbb{Z}^2$ whose spins have value $+1$, while the blue dots represent the sites whose spins have value -1.

Suppose that we are given a spin configuration ω in Ω_+, then, there must be a finite number of sites x that are associated with a change of phase, that is, sites x for which $\omega_x\omega_{x+1} = -1$. Following a reasoning similar to the one applied in the traditional case, we will place an interface perpendicularly to the real line at the point r_x situated between x and $x+1$ whenever we face a change of phase at x, see Figure 1.2.

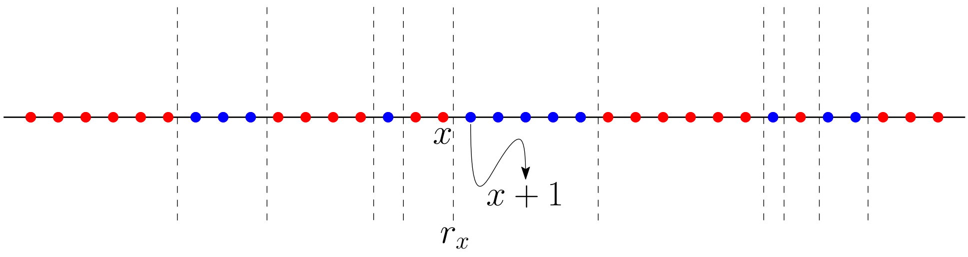


Figure 1.2: Illustration of the use of interfaces to indicate the spin-flip locations associated to a one-dimensional configuration with "plus" boundary condition. Differently from the two-dimensional case, the interfaces are not placed in the midpoint of the line segment determined by neighbors with opposite spins.

As was mentioned before, we can take advantage from the fact that the choice of the fixed location of each interface can be made freely, so, let us assume that their positions were previously arranged in such a way that they are described by a family $(r_x)_{x\in\mathbb{Z}}$ of real numbers such that the distances corresponding to any two pairs of r_x's are distinct. The following lemma shows that such a choice is always possible.

Lemma 1.1. *For each $\delta_0 \in (0, \frac{1}{4})$, there is a family $(r_x)_{x\in\mathbb{Z}}$ of real numbers such that each r_x belongs to the interval $(x+\frac{1}{2}-\delta_0, x+\frac{1}{2}+\delta_0)$ and the relation*

$$|r_{x_1} - r_{x_2}| \neq |r_{y_1} - r_{y_2}| \tag{1.8}$$

holds whenever x_1, x_2 and y_1, y_2 are distinct pairs satisfying $x_1 \neq x_2$ and $y_1 \neq y_2$.

Proof. Let A be the set given by

$$A = \bigcup_{n\geq 0} \big\{f : [-n,n]\cap\mathbb{Z} \to \mathbb{R} : f(x) \in (x+1/2-\delta_0, x+1/2+\delta_0) \text{ for each point } x\text{, and } |f(x_1)-f(x_2)| \neq |f(y_1)-f(y_2)| \text{ whenever } x_1, x_2 \text{ and } y_1, y_2 \text{ are distinct pairs satisfying } x_1 \neq x_2 \text{ and } y_1 \neq y_2\big\},$$

that is, let A be the set consisting of all functions defined on symmetric bounded intervals of $\mathbb{Z}$ that satisfy the required properties. Then, corresponding to each function $f \in A$, let us define the set X_f by

$$X_f = \big\{g \in A : \mathrm{dom}(g) \supsetneq \mathrm{dom}(f) \text{ and } g \text{ extends } f\big\}.$$

The reader can easily verify that X_f is nonempty. It follows from the axiom of choice that there is a function F from A into $\cup_{f\in A} X_f$ that associates to each element f of A an extension $F(f)$ in X_f. So, given a function $f_0 \in A$, by means of the recursive formula

$$f_{n+1} = F(f_n),$$

1

we obtain a sequence $(f_n)_{n\geq 0}$ consisting of compatible functions such that $\cup_{n\geq 0}\mathrm{dom}(f_n) = \mathbb{Z}$. Therefore, if we let $r : \mathbb{Z} \to \mathbb{R}$ be the unique extension of this family of functions, then it is straightforward to check that r satisfies the required conditions. ■

Since the existence of the interfaces that fulfill our required properties is guaranteed by Lemma 1.1, let us choose a real number δ_0 lying in the interval $(0, \frac{1}{4})$ and fix a family of interface locations $(r_x)_{x\in\mathbb{Z}}$ for the remaining of this chapter.

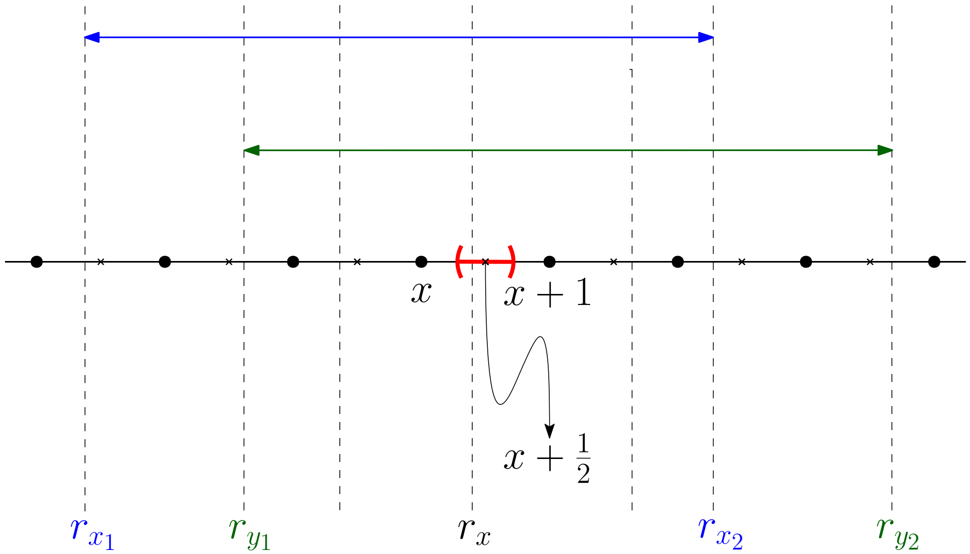


Figure 1.3: The one-dimensional integer lattice $\mathbb{Z}$ together with its fixed interfaces. The distances corresponding to the pairs of interfaces r_{x_1}, r_{x_2} and r_{y_1}, r_{y_2} are distinct and indicated by the blue and green arrows, respectively.

1.3.2. TRIANGLE CONFIGURATIONS

In the following, we show that each one-dimensional spin configuration with "plus" boundary condition can be regarded as a collection of triangles obtained from its spin-flip interfaces. First, note that given an element ω of Ω_+, as we briefly mentioned in the previous section, the set

$$\{x \in \mathbb{Z} : \omega_x\omega_{x+1} = -1\}, \tag{1.9}$$

that consists of all points that correspond to a change of sign in ω, is finite, moreover, it contains an even number of elements. Indeed, let N be a positive integer such that $\omega_x = +1$ holds for all x satisfying $|x| \geq N$. It follows that $\{x \in \mathbb{Z} : |x| > N\} \subseteq \{x \in \mathbb{Z} : \omega_x\omega_{x+1} = 1\}$, hence, we have $\{x \in \mathbb{Z} : \omega_x\omega_{x+1} = -1\} \subseteq \{x \in \mathbb{Z} : |x| \leq N\}$. Now, in order to prove the second part of our claim, we just need to use the fact that

$$\omega_{-n}\omega_{n+1} = \prod_{x=-n}^{n} \omega_x\omega_{x+1} = (-1)^{\#\{x\in\mathbb{Z}:\omega_x\omega_{x+1}=-1\}\cap[-n,n]}$$

holds for every nonnegative integer n, in particular, if we consider n sufficiently large (for instance, $n = N$), we conclude that

$$(-1)^{\#\{x\in\mathbb{Z}:\omega_x\omega_{x+1}=-1\}} = 1.$$

This remark shows that the number of interfaces associated to a given configuration in Ω_+ is even, therefore, we can group them in pairs according to a specific rule, namely the rule of minimal pairwise distance, in such a way that to each pair of such interfaces we attach the endpoints of the base of a triangle. So, in the end, the resulting picture consists of a collection of triangles, a so-called triangle configuration. In the following, we give a precise description of this construction.

Let us denote by $\Delta(a,b)$ the closed interval in $\mathbb{R}$ whose endpoints are a and b, where $a < b$ and both belong to the set $\{r_x : x \in \mathbb{Z}\}$ that consists of all possible interface locations. For graphical purposes such intervals will often be regarded as triangles since, as will be seen later, it is more convenient to visualize them as the diagonals of isosceles right triangles whose endpoints are attached to a pair of interface points. So, for that reason, instead of referring to such an object of the form $\Delta(a,b)$ as the base of the triangle we may refer to it as the triangle by itself, moreover, we refer to its endpoints a and b as the roots of that triangle. Given a configuration ω in Ω_+, let us define its set of spin-flip interfaces by

$$I_1(\omega) = \{r_x : x \text{ is an integer such that } \omega_x\omega_{x+1} = -1\}. \tag{1.10}$$

Let us consider the function $\mathfrak{m}$ that maps each subset I of $\{r_x : x \in \mathbb{Z}\}$ containing an even number of elements to the set

$$\mathfrak{m}(I) = \begin{cases} \emptyset & \text{if } I = \emptyset, \text{ and} \\ \{a,b\} & \text{otherwise, where } a \text{ and } b \text{ belong to } I,\ a < b, \text{ and} \\ & |a-b| = \min\{|a'-b'| : a',b' \in I, a' \neq b'\}. \end{cases} \tag{1.11}$$

Note that the property (1.8) from Lemma 1.1 guarantees that the minimal distance taken in equation (1.11) is attained by a unique pair of interfaces, so, $\mathfrak{m}$ is indeed well defined. Then, the set $I_{n+1}(\omega)$ can be recursively defined by using $I_1(\omega)$ and the relation

$$I_{n+1}(\omega) = I_n(\omega)\setminus\mathfrak{m}(I_n(\omega)) \tag{1.12}$$

for each positive integer n.

Proceeding with the construction we just described, we end up with a sequence of sets where each $I_{n+1}(\omega)$ is obtained by removing from $I_n(\omega)$ its pair of interfaces that minimizes the distance among any other pairs; moreover, this sequence satisfies $I_n(\omega) \neq \emptyset$ whenever $n \leq \#I_1(\omega)/2$, and $I_n(\omega) = \emptyset$ otherwise. Therefore, let us consider all the pairs of minimizing interfaces, say

$$\mathfrak{m}(I_n(\omega)) = \{a_n, b_n\} \tag{1.13}$$

for each n such that $1 \leq n \leq \#I_1(\omega)/2$, and define the triangle configuration associated to ω by letting

$$\Psi(\omega) = \{\Delta(a_n,b_n) : 1 \leq n \leq \#I_1(\omega)/2\}. \tag{1.14}$$

The step-by-step constructions of triangle configurations are illustrated in detail at the end of this section, see Examples 1.9 and 1.10.

1

Proposition 1.2. *The function Ψ defined above is one-to-one.*

Proof. Let ω and η be elements of Ω_+ such that the equality $\Psi(\omega) = \Psi(\eta)$ holds. It is straightforward to check that $I_1(\omega) = I_1(\eta)$. It follows from this identity that

$$\{x \in \mathbb{Z} : \omega_x \omega_{x+1} = -1\} = \{x \in \mathbb{Z} : \eta_x \eta_{x+1} = -1\}. \tag{1.15}$$

For each site x in $\mathbb{Z}$, equation (1.15) implies that

$$\omega_x \eta_x = (\omega_x \omega_{x+1})(\omega_{x+1}\eta_{x+1})(\eta_x \eta_{x+1}) = \omega_{x+1}\eta_{x+1}.$$

By using an induction argument, we conclude that

$$\omega_x \eta_x = \omega_y \eta_y$$

holds for every x and y in $\mathbb{Z}$. Therefore, by choosing y with $|y|$ large enough in such a way that the condition $\omega_y = \eta_y = 1$ is satisfied, we obtain $\omega_x \eta_x = 1$ for all x, that is, $\omega = \eta$. ∎

According to the construction developed so far, every spin configuration with "plus" boundary condition can be unambiguously represented in a graphical form as a collection of isosceles right triangles. Note that the configuration consisting of only $+1$ spins is identified with the empty collection of triangles. From now on, let us denote the range of Ψ by $\mathscr{T}$ and refer to its elements as triangle configurations.

In the remaining of this section we explore some geometric features of such a representation. First of all, it is important to mention that it is not true that every finite collection of triangles corresponds to some spin configuration, that is, despite the fact that the map Ψ is one-to-one, it is not a function from Ω_+ onto the set of all possible finite collection of triangles. In the following, we derive a necessary and sufficient condition that must be satisfied by a collection of triangles to belong to $\mathscr{T}$. Given two triangles T and T', say $T = \Delta(a,b)$ and $T' = \Delta(a',b')$, let us associate to them the length $\ell(T,T')$ defined by

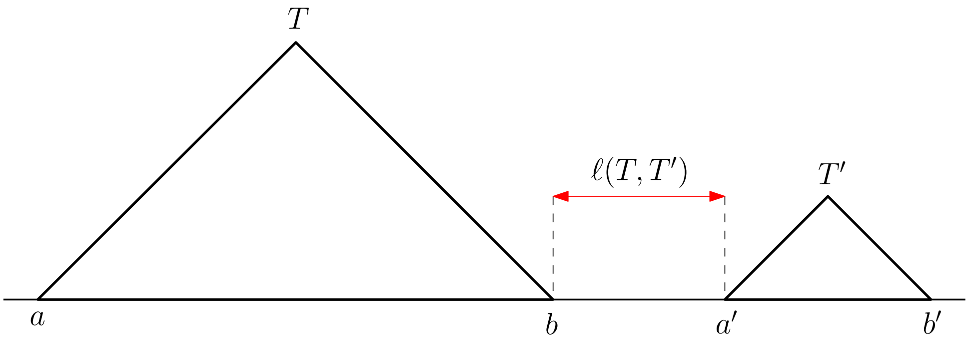
$$\ell(T,T') = \min\{|a-a'|, |a-b'|, |b-a'|, |b-b'|\}, \tag{1.16}$$

that is, such a quantity is defined as the minimal distance between the roots of T and the roots of T'. Note that $\ell(T,T') = 0$ if and only if $\{a,b\} \cap \{a',b'\} \neq \emptyset$, in other words, the quantity $\ell(T,T')$ vanishes if and only if T and T' share at least one common root. In case T and T' do not share a common root, we can split equation (1.16) into six remaining cases and express it as

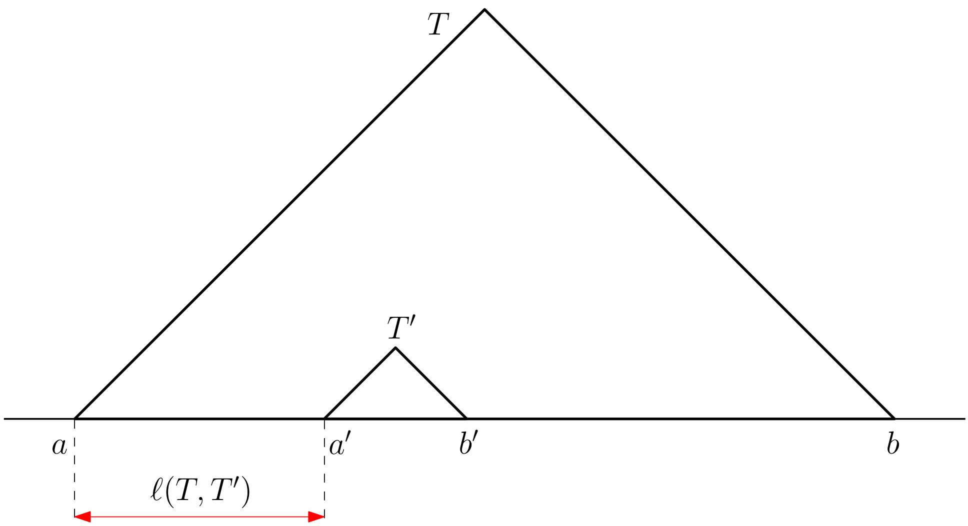
$$\ell(T,T') = \begin{cases} a'-b & \text{if } a<b<a'<b', \\ a-b' & \text{if } a'<b'<a<b, \\ (a'-a)\wedge(b-b') & \text{if } a<a'<b'<b, \\ (a-a')\wedge(b'-b) & \text{if } a'<a<b<b', \\ (a'-a)\wedge(b-a')\wedge(b'-b) & \text{if } a<a'<b<b', \text{and} \\ (a-a')\wedge(b'-a)\wedge(b-b') & \text{if } a'<a<b'<b. \end{cases} \tag{1.17}$$

From now on, let us use $\ell(T)$ to denote the quantity $\ell(T) = b-a$ which is equal to the length of the base of the triangle corresponding to T.

Remark 1.3. Note that for the first four cases in equation (1.17), which correspond to the cases where T and T' are either disjoint or one of them includes the other, the quantity $\ell(T,T')$ coincides exactly with the length of the smallest interval determined by these two triangles. As will be shown by Proposition 1.4, the triangle configurations are built in such a special way that any pair of their triangles necessarily falls into one of these former cases, thus, since we will be dealing mostly with triangle configurations, the last two cases from (1.17) will be irrelevant to us.

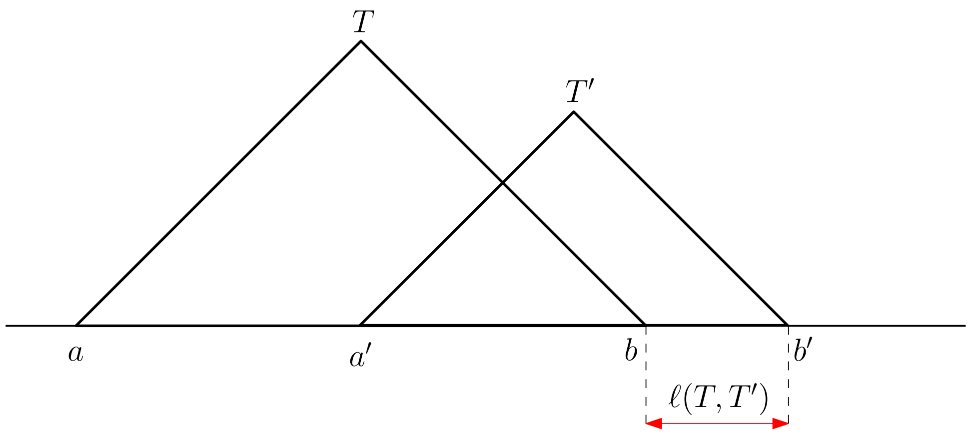


(a) This figure represents the first case from (1.17), where T is on the left of T' and $\ell(T,T')$ coincides with the distance between the right root of T and the left root of T'.

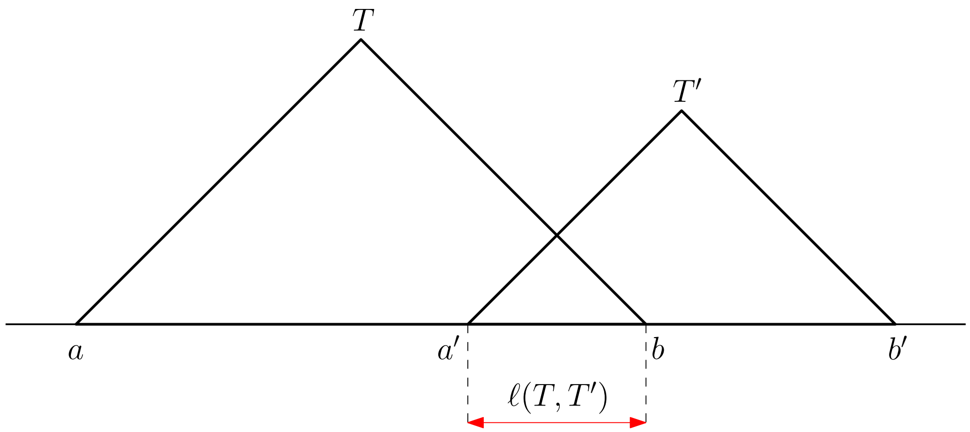


(b) This figure represents the third case from (1.17), where T includes T' and $\ell(T,T')$ is the least of the distance between the left roots of T and T' and the distance between the right roots of T and T'.

Figure 1.4: Some possible scenarios regarding the relative positions between T and T'.



(c) This figure represents the fifth case from (1.17), where T and T' have a nonempty intersection but no one includes the other. In the present case, $\ell(T,T')$ is equal to the distance between the right roots of T and T'.



(d) This figure also represents the fifth case from (1.17). In the present case, $\ell(T,T')$ is equal to the length of the intersection of the bases of T and T'.

Figure 1.4: Some possible scenarios regarding the relative positions between T and T'.

Proposition 1.4. *A finite collection $\underline{T}$ of triangles is a triangle configuration if and only if*

$$\ell(T,T') > \ell(T) \wedge \ell(T) \quad \text{holds for every pair } T, T' \text{ of distinct elements of } \underline{T}\,. \tag{1.18}$$

Proof. Let T and T' be two distinct elements of a triangle configuration $\underline{T}$, say $T = \Delta(a,b)$, $T' = \Delta(a',b')$, and $\underline{T} = \Psi(\omega)$ for some spin configuration ω in Ω_+. As we have seen, there are distinct integers n and m such that $\{a,b\} = \mathfrak{m}(I_n(\omega))$ and $\{a',b'\} = \mathfrak{m}(I_m(\omega))$. Without loss of generality, we can assume that $n < m$. It follows from the fact that $I_m(\omega)$ is included in $I_{n+1}(\omega) = I_n(\omega)\setminus\{a,b\}$ that $|a-b| < |a'-b'|$, therefore, using once again the fact that the pair a, b is the one that minimizes the distance between any pair of elements of $I_n(\omega)$, by means of equation (1.16), we conclude that

$$\ell(T,T') > |a-b| = \ell(T) \wedge \ell(T').$$

Now, let us prove the converse statement. Note that in the cases where $\underline{T} = \emptyset$ or $\underline{T}$ consists of a unique triangle, the condition (1.18) is immediately fulfilled and $\underline{T}$ is a legit triangle configuration. Let us suppose that $\underline{T}$ is a triangle configuration whenever it satisfies (1.18) and $\underline{T}$ contains n triangles, where $n \geq 1$. Now, let $\underline{T}$ be a collection of triangles with $n+1$ elements such that condition (1.18) holds, say $\underline{T} = \{T_0, T_1, \dots, T_n\}$, $T_i = \Delta(a_i, b_i)$, and $\ell(T_k) < \ell(T_{k+1})$ for each $k = 0, \dots, n-1$. Then, according to our induction hypothesis, there is a spin configuration σ in Ω_+ such that $\Psi(\sigma) = \{T_1, \dots, T_n\}$. Note that

$$I_1(\sigma) = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\},$$

and for any pair a, b of distinct elements of $I_1(\sigma)$ we have $|a-b| \geq |a_1 - b_1| > |a_0 - b_0|$. Furthermore, since the triangles T_0 and T_i satisfy the inequality from (1.18) for each i such that $1 \leq i \leq n$, then, by means of equation (1.16), the lengths $|a_0 - a_i|, |a_0 - b_i|, |b_0 - a_i|$, and $|b_0 - b_i|$ are greater than $|a_0 - b_0|$. It follows that every pair a, b of distinct elements of the set

$$\{a_i : 0 \leq i \leq n\} \cup \{b_i : 0 \leq i \leq n\}$$

satisfies $|a - b| \geq |a_0 - b_0|$, where the minimum is reached only for the pair a_0, b_0. If we let ω be the spin configuration in Ω_+ whose set of spin-flip interfaces is given by

$$I_1(\omega) = \{a_i : 0 \leq i \leq n\} \cup \{b_i : 0 \leq i \leq n\},$$

then, we have

$$\mathfrak{m}(I_1(\omega)) = \{a_0, b_0\}, \tag{1.19}$$

and

$$I_{k+1}(\omega) = I_k(\sigma) \tag{1.20}$$

for every positive integer k. Hence, by means of equations (1.19) and (1.20), we conclude that $\Psi(\omega) = \{T_0, T_1, \dots, T_n\}$. ■

Corollary 1.5. *Every subset of a triangle configuration is still a triangle configuration.*

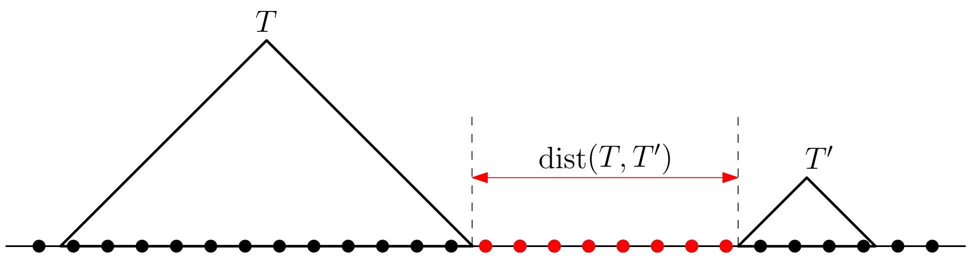
Remark 1.6. As the reader can easily verify, it follows from equation (1.17) and Proposition 1.4 that given an arbitrary triangle configuration $\underline{T}$, for any two distinct triangles T and T' that belong to it, we can only have $T \cap T' = \emptyset$, $T \subsetneq T'$ or $T' \subsetneq T$, in other words, the triangles are arranged in such a way that they are either disjoint or one of them is strictly included inside the other.

We will see in the forthcoming sections that, in the same way as in the classical two-dimensional Peierls' argument, there is the necessity to establish a link between the graphical representation and certain physical quantities originated by model in order to express the energy and entropy bounds in a proper way. That requirement is fulfilled by introducing the notion of the size of contours in terms of which we write those bounds. While in the two-dimensional case the size of a contour is measured based on its total length, in the one-dimensional case we will be dealing with a slightly different quantity, the mass of the contour. In the following, we start by defining the notion of the mass of a triangle in such a way that after we derive the definition of contours in the next section this idea can be naturally extended for such objects; furthermore, we also precise the idea of the distance between triangles that provide us with a concrete way of connecting their geometry with their masses.

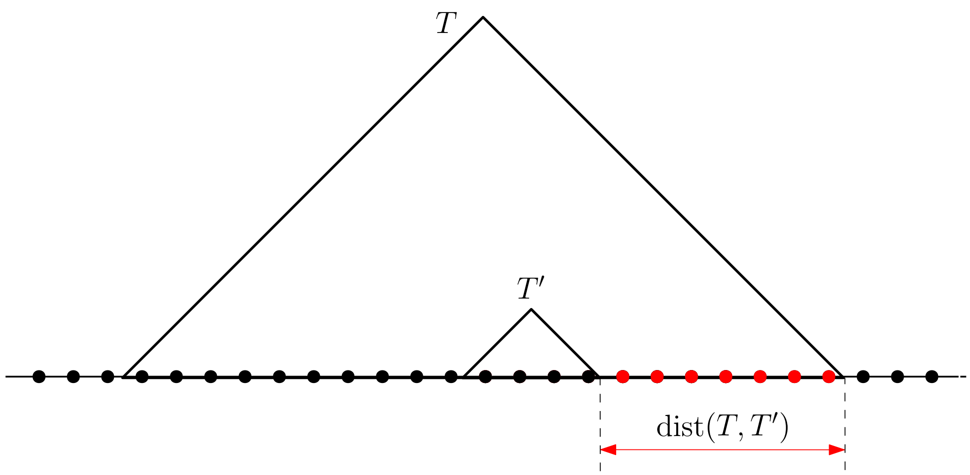
For any triangle T, let us define its mass $|T|$ as the number of integer points contained inside of it, that is, we define

$$|T| = \# T \cap \mathbb{Z}. \tag{1.21}$$

Given any pair T, T' of triangles, we define their distance $\mathrm{dist}(T, T')$ as the number of integers between the interface points that attains the minimum from equation (1.16). Note that $\mathrm{dist}(T, T') = 0$ if and only if T and T' have at least one root in common.



(a) If T and T' are disjoint, their distance is given by the number of integers that lie between them.



(b) If one includes the other, their distance is given by the minimum of the number of integers between their left roots and the number of integers between their right roots.

Figure 1.5: Illustration of the distance between T and T', where each dot stands for an integer number in the real line.

Corollary 1.7. *Let $\underline{T}$ be a triangle configuration. Then, for any pair T, T' of distinct triangles in $\underline{T}$ we have*

$$\operatorname{dist}(T, T') \geq |T| \wedge |T'|. \tag{1.22}$$

Before we follow to proof of the statement above, let us show the existing interplay between the length and the mass of a triangle. Let r_x and r_y be two distinct interfaces such that $r_x < r_y$. Since r_x and r_y respectively belong to the intervals $(x+\frac{1}{2}-\delta_0, x+\frac{1}{2}+\delta_0)$ and $(y+\frac{1}{2}-\delta_0, y+\frac{1}{2}+\delta_0)$, then we have

$$(y-x)-2\delta_0 < r_y - r_x < (y-x)+2\delta_0.$$

Using the fact that $\delta_0 < \frac{1}{4}$, we obtain the inequalities

$$(y-x)-\frac{1}{2} < r_y - r_x < (y-x)+\frac{1}{2} \tag{1.23}$$

that express the relationship between the separation distance of the interfaces r_x and r_y and number of integers between them. Note that for any triangle T, the relation (1.23) implies

$$|T| - \frac{1}{2} < \ell(T) < |T| + \frac{1}{2}. \tag{1.24}$$

Proof of Corollary 1.7. It is straightforward to check that it follows directly from our definition of distance, Proposition 1.4, and equations (1.23) and (1.24) that

$$\operatorname{dist}(T, T') + \frac{1}{2} > \ell(T, T') > \ell(T) \wedge \ell(T') > (|T| \wedge |T'|) - \frac{1}{2},$$

thus, equation (1.22) holds. ∎

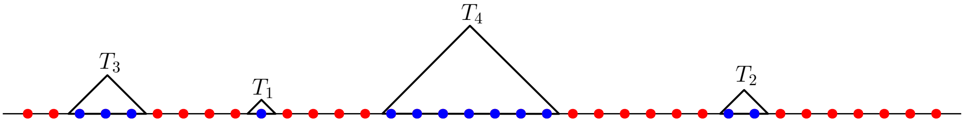
Remark 1.8. As the final remark of this section, the reader can easily verify that equation (1.24) implies that given two triangles T and T' the inequality $|T| \leq |T'|$ holds whenever $\ell(T) < \ell(T')$, more generally, the number of integers between two interfaces is monotonic (non-decreasing) with respect to their separation distance. The main consequence of this fact is that the distance $\operatorname{dist}(T, T')$ between the triangles T and T', say $T = \Delta(r_x, r_y)$ and $T' = \Delta(r_{x'}, r_{y'})$, can be written explicitly as

$$\operatorname{dist}(T, T') = \min\left\{|x-x'|, |x-y'|, |y-x'|, |y-y'|\right\}. \tag{1.25}$$

The reader may notice that equation (1.25) coincides with the definition provided in [15].

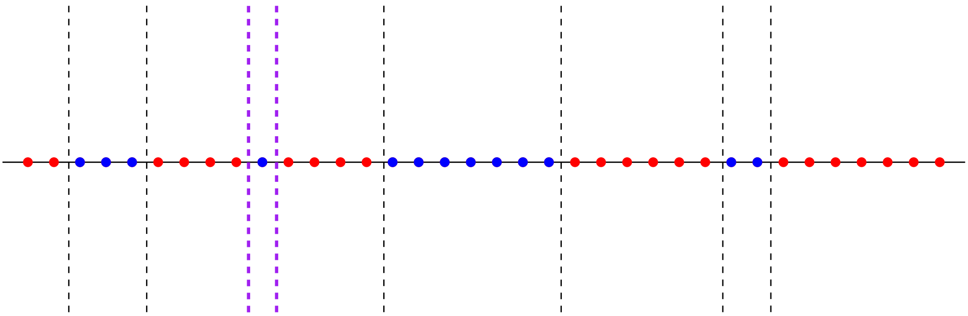
Example 1.9. Let us consider the spin configuration ω in Ω_+ illustrated in Figure 1.6. In the following, we provide a step-by-step construction of the triangle configuration $\Psi(\omega)$ corresponding to ω.

(a) Spin configuration ω.

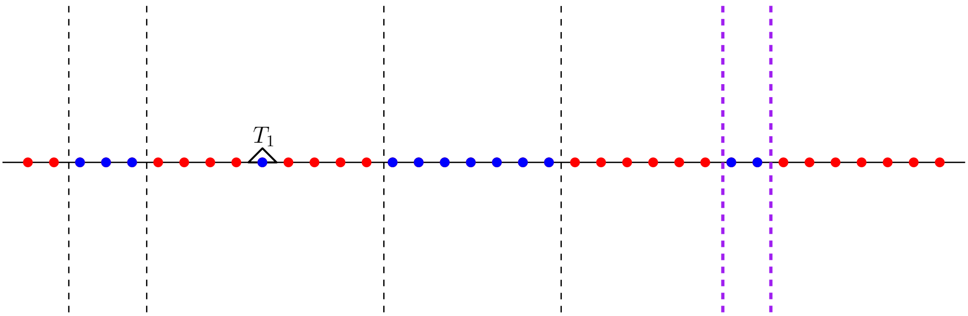


(b) Triangle configuration $\Psi(\omega) = \{T_1, T_2, T_3, T_4\}$.

Figure 1.6: The spin configuration with "plus" boundary condition and its set of triangles. The red dots stand for the sites whose spins have value $+1$, while the blue dots represent the sites whose spins have value -1.

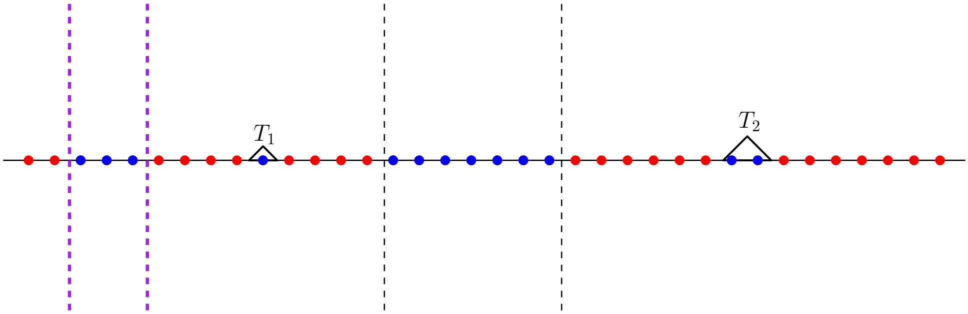


(a) First, we place the interfaces, represented above by the dashed lines, to indicate the location of the elements of $I_1(\omega)$. Then, let us pick the pair of interfaces that has the minimal distance among the other pairs and highlight them in purple. Note that the purple dashed lines indicate the location of the elements of $\mathfrak{m}(I_1(\omega))$.

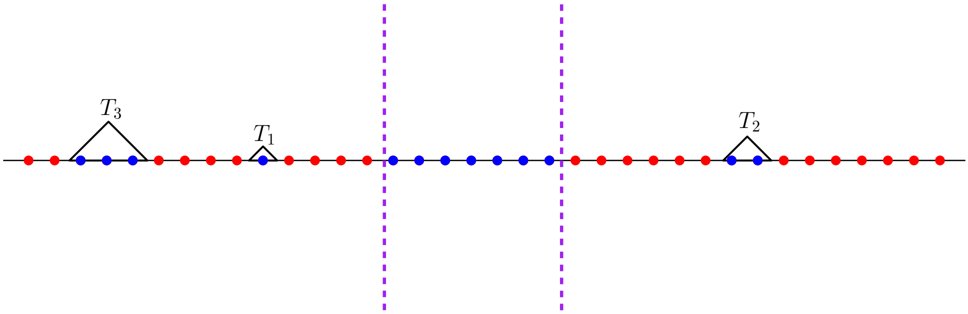


(b) After erasing the interfaces corresponding to $\mathfrak{m}(I_1(\omega))$ we attach to their former positions an isosceles right triangle T_1. So, the remaining interfaces indicate the elements of $I_2(\omega)$. Again, we highlight in purple the pair of interfaces that minimizes the distance among the remaining pairs, indicating the location of the elements of $\mathfrak{m}(I_2(\omega))$.

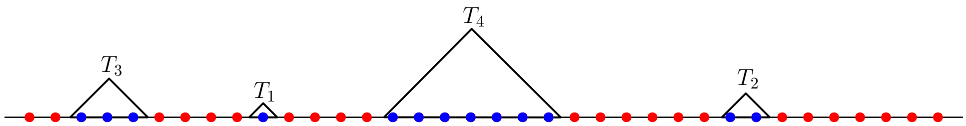
Figure 1.7: Step-by-step construction of the triangle configuration $\Psi(\omega)$.



(c) We repeat the same procedure to $I_2(\omega)$. After erasing the interfaces corresponding to $\mathfrak{m}(I_2(\omega))$ we attach to their former positions an isosceles right triangle T_2. So, the remaining interfaces indicate the elements of $I_3(\omega)$. Again, we highlight in purple the pair of interfaces that minimizes the distance among the remaining pairs, indicating the location of the elements of $\mathfrak{m}(I_3(\omega))$.



(d) After removing the interfaces corresponding to $\mathfrak{m}(I_3(\omega))$ and introducing the triangle T_3, we end up with a unique pair of interfaces.

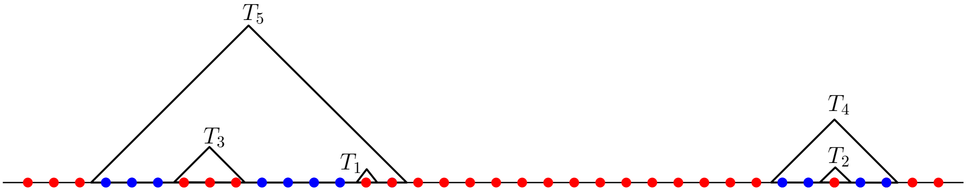


(e) Replacing the last interfaces by the triangle T_4, we finish the construction of the triangle configuration associated to ω.

Figure 1.7: Step-by-step construction of the triangle configuration $\Psi(\omega)$.

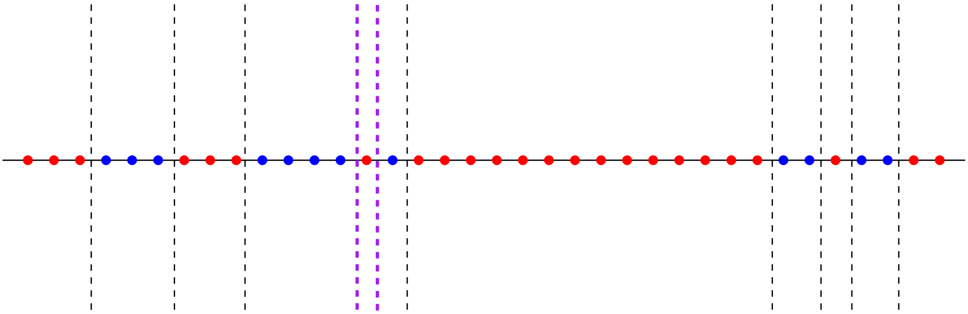
Example 1.10. Let us consider the spin configuration σ in Ω_+ illustrated in Figure 1.8. In the following, we provide a step-by-step construction of the triangle configuration $\Psi(\sigma)$ corresponding to σ.

(a) Spin configuration σ.

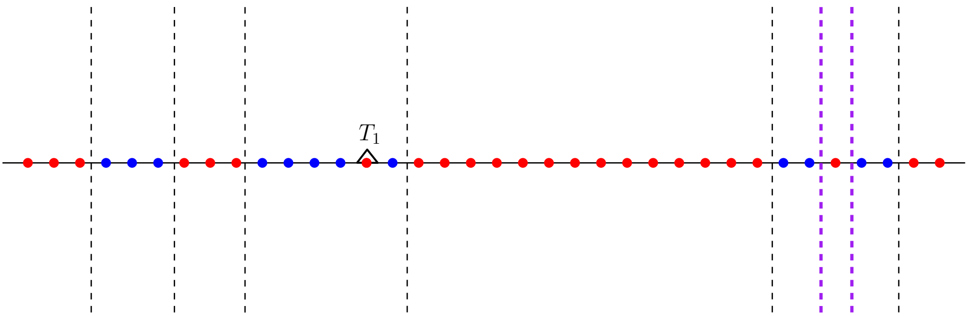


(b) Triangle configuration $\Psi(\sigma) = \{T_1, T_2, T_3, T_4, T_5\}$.

Figure 1.8: The spin configuration with "plus" boundary condition and its set of triangles. The red dots stand for the sites whose spins have value $+1$, while the blue dots represent the sites whose spins have value -1.

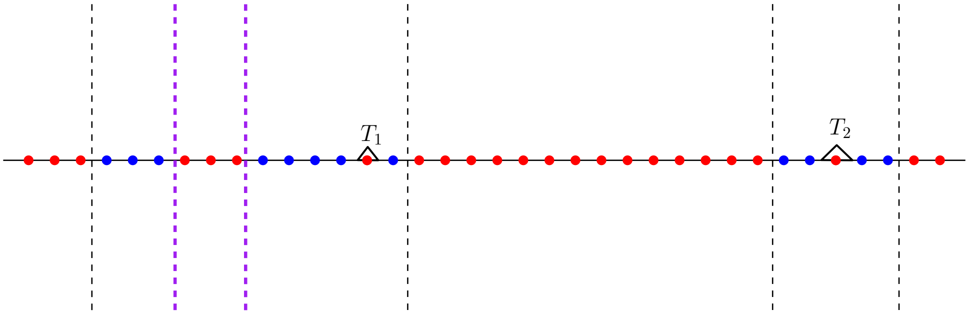


(a) Like in the previous example, we place the interfaces to indicate the location of the elements of $I_1(\sigma)$. Then, let us pick the pair of interfaces that has the minimal distance among the other pairs and highlight them in purple. Note that the purple dashed lines indicate the location of the elements of $\mathfrak{m}(I_1(\sigma))$.

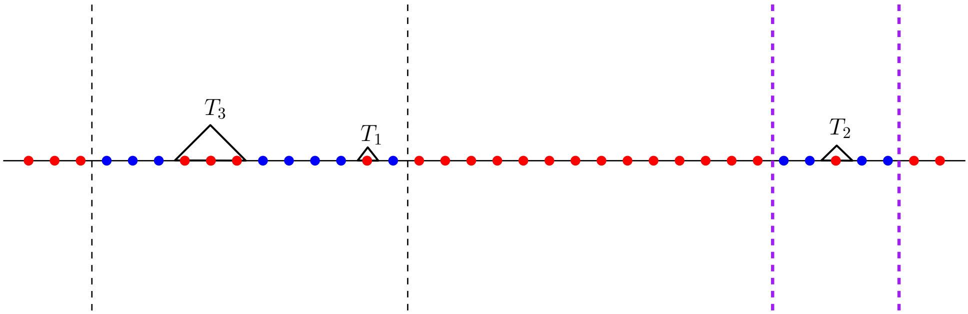


(b) After erasing the interfaces corresponding to $\mathfrak{m}(I_1(\sigma))$ we attach to their former positions an isosceles right triangle T_1. So, the remaining interfaces indicate the elements of $I_2(\sigma)$. Again, we highlight in purple the pair of interfaces that minimizes the distance among the remaining pairs, indicating the location of the elements of $\mathfrak{m}(I_2(\sigma))$.

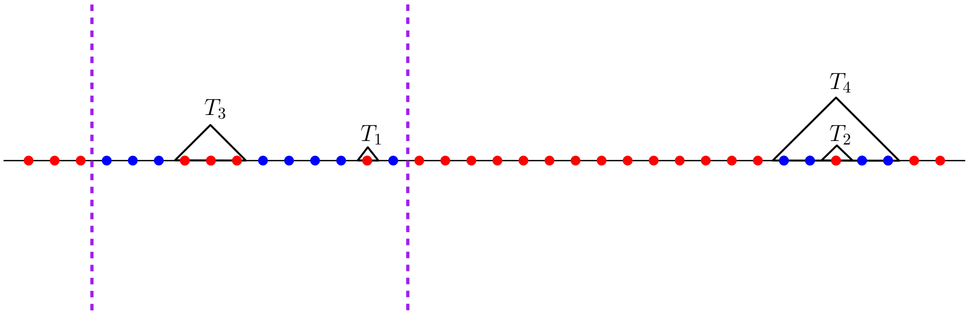
Figure 1.9: Step-by-step construction of the triangle configuration $\Psi(\sigma)$.



(c) We repeat the same procedure to $I_2(\sigma)$. After erasing the interfaces corresponding to $\mathfrak{m}(I_2(\sigma))$ we attach to their former positions an isosceles right triangle T_2. So, the remaining interfaces indicate the elements of $I_3(\sigma)$. Again, we highlight in purple the pair of interfaces that minimizes the distance among the remaining pairs, indicating the location of the elements of $\mathfrak{m}(I_3(\sigma))$.

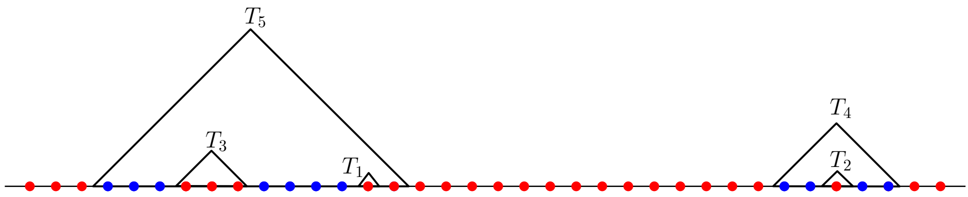


(d) Likewise, we erase $\mathfrak{m}(I_3(\sigma))$, introduce the triangle T_3 and identify $\mathfrak{m}(I_4(\sigma))$.



(e) After removing the interfaces corresponding to $\mathfrak{m}(I_4(\sigma))$ and introducing the triangle T_4, we end up with a unique pair of interfaces.

Figure 1.9: Step-by-step construction of the triangle configuration $\Psi(\sigma)$.



(f) Replacing the last interfaces by the triangle T_5, we finish the construction of the triangle configuration associated to σ.

Figure 1.9: Step-by-step construction of the triangle configuration $\Psi(\sigma)$.

1.3.3. CONTOURS

In this section we finally introduce the main concept of this whole chapter, the notion of contours for one-dimensional Ising models. Recall that for the well-known two-dimensional case, the construction of contours is essentially based on spin-flip interfaces (such as those we have shown in Figure 1.1) associated to some configuration with homogeneous boundary condition. This construction consists of considering the interfaces obtained from such a configuration ω and deforming them according to a certain rule in such a way that we end up with a finite collection $\Gamma(\omega) = \{\gamma_1, \dots, \gamma_n\}$ of non-overlapping closed curves on the plane, where we refer to each one of the γ_i's as a contour of ω, see Figure 1.10.

Now, with respect to one-dimensional Ising models, in order to define the contours of an element ω in Ω_+ a slightly different approach is required. In the present case the fact of having the interfaces at our disposal does not provide us with an immediate way of determining their corresponding contours. We overcome this problem by adopting the procedure that consists of associating to ω its triangle configuration $\underline{T} = \Psi(\omega)$, and then, after that, we split it through a partition $\Gamma(\underline{T}) = \{\gamma_1, \dots, \gamma_n\}$, where each of its elements is a triangle configuration defined in such a way that, in some sense, the triangles that belong to the same γ_i are "close to each other" while the triangles from different γ_i's are "well-separated". In this setting, we may refer to each γ_i interchangeably as a contour of ω or even a contour of $\underline{T}$. At a first glance this notion of contours may seem artificial and counter-intuitive, in fact, it is, however, despite the fact that it has no obvious physical insight behind of it, along the next sections we strive to make analogies with the classical case in order to convince the reader that its properties are of extreme relevance. Such a construction requires a higher degree of abstraction and we describe it precisely as follows.

As we discussed above, given a triangle configuration $\underline{T}$ the set of contours associated to it will be defined as a collection $\Gamma(\underline{T}) = \{\gamma_1, \dots, \gamma_n\}$ consisting of a finite partition of $\underline{T}$ into triangle configurations that satisfy suitable separation properties. Before we proceed to the proof of the existence and uniqueness of such a function Γ, let us introduce some notation and clarify what is the meaning of the expression "well-separated". Given an arbitrary triangle configuration γ, let us define the mass of γ as the sum of the masses of all triangles that belong to it, explicitly, its mass $|\gamma|$ is given by

$$|\gamma| = \sum_{T \in \gamma} |T|. \tag{1.26}$$

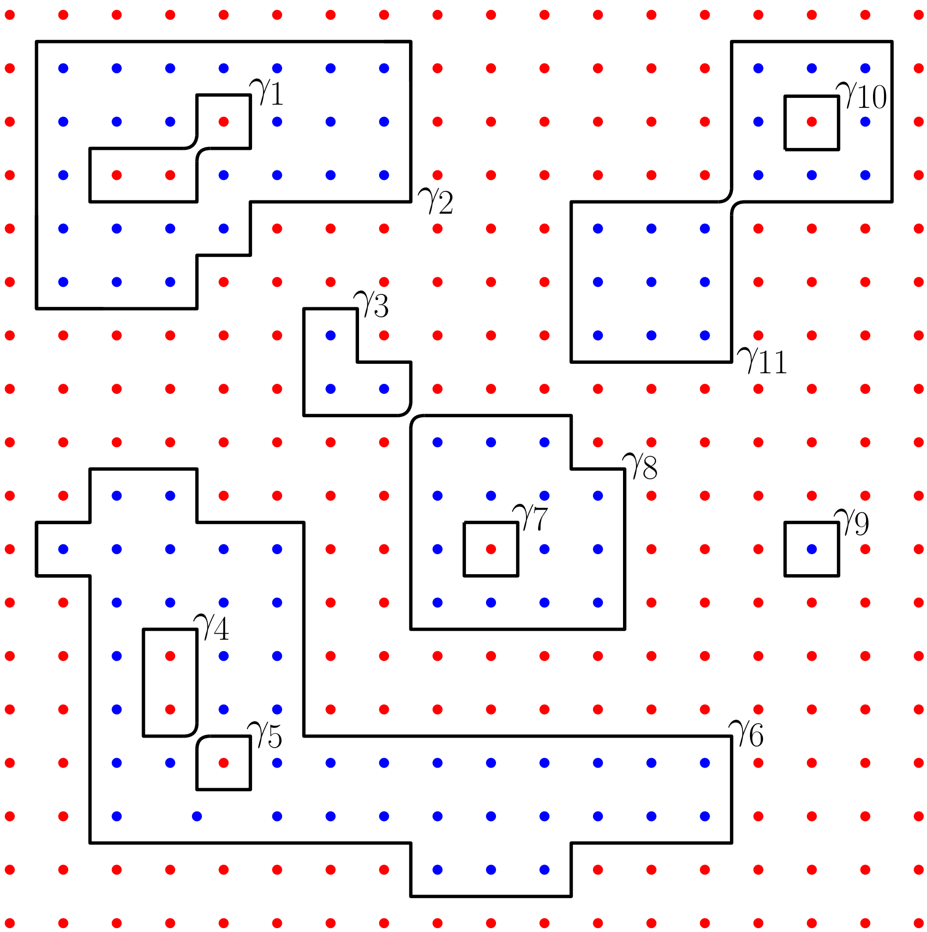


Figure 1.10: The contours for a two-dimensional spin configuration with "plus" boundary condition. The red dots stand for the sites of $\mathbb{Z}^2$ whose spins have value $+1$, while the blue dots represent the sites whose spins have value -1.

Now, for any pair γ, γ' of nonempty triangle configurations, let us define their distance $\operatorname{dist}(\gamma, \gamma')$ as the smallest distance between any pair of triangles where one of them belongs to γ and the other belongs to γ', that is,

$$\operatorname{dist}(\gamma, \gamma') = \min_{T \in \gamma, T' \in \gamma'} \operatorname{dist}(T, T'), \tag{1.27}$$

moreover, we also use $T(\gamma)$ to denote the smallest triangle that contains all the triangles in γ.

Theorem 1.11. *Fixed a positive real number c, there exists a unique function Γ defined on $\mathscr{T}$ that satisfies the following properties.*

1

(P0) We have

$$\Gamma(\underline{T}) = \{\gamma_1, \ldots, \gamma_n\} \tag{1.28}$$

for some positive integer n, where each γ_i is a triangle configuration such that $\underline{T} = \cup_{i=1}^{n} \gamma_i$.

(P1) For every pair γ, γ' of distinct elements in $\Gamma(\underline{T})$, one of the following alternatives holds.

(a) In case the triangles $T(\gamma)$ and $T(\gamma')$ are disjoint, we have the inequality

$$\operatorname{dist}(\gamma, \gamma') > c \cdot |\gamma|^3 \wedge |\gamma'|^3. \tag{1.29}$$

(b) In case the triangles $T(\gamma)$ and $T(\gamma')$ have nonempty intersection, we must have either $T(\gamma)$ included in $T(\gamma')$ or vice versa. If the first inclusion is verified, then for every triangle T' in γ' either $T(\gamma) \subseteq T'$ or $T(\gamma) \cap T' = \emptyset$, moreover,

$$\operatorname{dist}(\gamma, \gamma') > c|\gamma|^3. \tag{1.30}$$

Now, if the second inclusion holds, then we have analogous properties obtained by interchanging the roles of γ and γ' above.

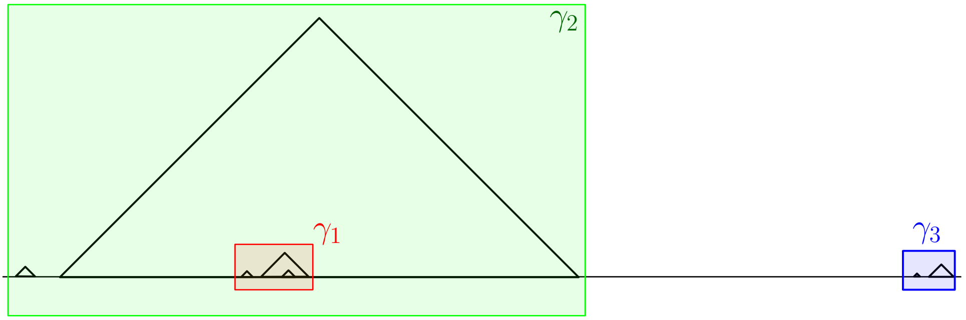
(P2) If $\underline{T}$ is a triangle configuration that can be decomposed as $\underline{T} = \cup_{i=1}^{n} \underline{T}^{(i)}$, where any pair γ, γ' of distinct elements of $\cup_{i=1}^{n} \Gamma(\underline{T}^{(i)})$ satisfies conditions (P1)(a) and (P1)(b), then $\Gamma(\underline{T})$ can be expressed as

$$\Gamma(\underline{T}) = \cup_{i=1}^{n} \Gamma(\underline{T}^{(i)}). \tag{1.31}$$

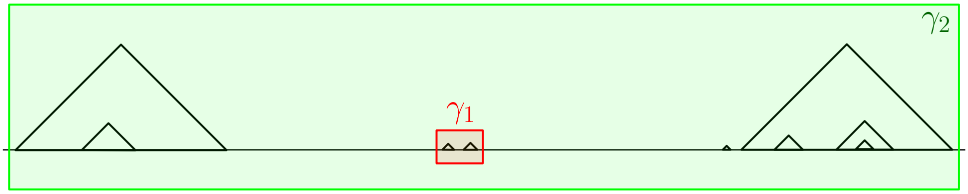
From now on, following the terminology introduced in [8], we may refer to any pair γ, γ' of triangle configurations that satisfies conditions (P1)(a) and (P1)(b) as well-separated. Note that (P1) implies that any two elements of $\Gamma(\underline{T})$ are disjoint, since otherwise equations (1.29) and (1.30) would be contradicted. Therefore, it follows from properties (P0) and (P1) that $\Gamma(\underline{T})$ defines, in fact, a partition of $\underline{T}$ that consists of well-separated triangle configurations. Furthermore, analogously to the traditional two-dimensional contours representation (see Figure 1.10), such contours γ from (P1)(b) that satisfy $T(\gamma) \subseteq T(\gamma')$ may be referred to as inner contours. Figure 1.11 synthesizes such concepts we briefly discussed above.

Again, adopting the same nomenclature as in [8], we may call (P2) the independence property of contours. This property essentially states that if we are given a triangle configuration $\underline{T}$ that can be decomposed into a finite number of triangle configurations $\underline{T}^{(1)}, \ldots, \underline{T}^{(n)}$, once we determine their contour sets $\Gamma(\underline{T}^{(1)}), \ldots, \Gamma(\underline{T}^{(n)})$ and show that all the contours involved are "well-separated", then all such contours a those that correspond to the whole configuration $\underline{T}$. One of the next results shows that property (P2) is crucial to ensure the uniqueness of Γ, moreover, we also provide in the end of this section practical examples where the application of such a property is extremely helpful for determining of the contours associated to a given triangle configuration.

Lemma 1.12. *Let γ_1, γ_2 be a pair of well-separated triangle configurations , and let γ and γ' be nonempty subsets of γ_1 and γ_2, respectively. Then, γ and γ' are also well-separated.*



(a) Note that $T(\gamma_1) \cap T(\gamma_3) = \emptyset$ and $T(\gamma_2) \cap T(\gamma_3) = \emptyset$. Furthermore, $T(\gamma_1) \subseteq T(\gamma_2)$ and for every triangle T in γ_2 we have either $T(\gamma_1) \subseteq T$ or $T(\gamma_1) \cap T = \emptyset$.



(b) In this case, $T(\gamma_1) \subseteq T(\gamma_2)$ and every triangle T in γ_2 satisfies $T(\gamma_1) \cap T = \emptyset$.

Figure 1.11: Illustration of property (P1) from Theorem 1.11.

Proof. If we assume that $T(\gamma_1)$ and $T(\gamma_2)$ are disjoint, then it follows that $T(\gamma)$ and $T(\gamma')$ are also disjoint, moreover, the inequalities

$$\operatorname{dist}(\gamma, \gamma') \geq \operatorname{dist}(\gamma_1, \gamma_2) > c \cdot |\gamma_1|^3 \wedge |\gamma_2|^3 \geq c \cdot |\gamma|^3 \wedge |\gamma'|^3$$

hold. Now, if $T(\gamma_1)$ and $T(\gamma_2)$ have nonempty intersection, then, without loss of generality, we can suppose that $T(\gamma_1)$ is included in $T(\gamma_2)$. It follows that for each triangle T in γ', we have either $T(\gamma) \subseteq T(\gamma_1) \subseteq T$ or $T(\gamma) \cap T = \emptyset$. This fact implies that necessarily either $T(\gamma)$ and $T(\gamma')$ are disjoint, or $T(\gamma)$ is included in $T(\gamma')$ with the property that for every T in γ' either $T(\gamma) \subseteq T$ or $T(\gamma) \cap T = \emptyset$. If the first alternative holds, then we have the inequalities

$$\operatorname{dist}(\gamma, \gamma') \geq \operatorname{dist}(\gamma_1, \gamma_2) > c \cdot |\gamma_1|^3 \geq c \cdot |\gamma|^3 \geq c \cdot |\gamma|^3 \wedge |\gamma'|^3,$$

while if the second one holds, we have

$$\operatorname{dist}(\gamma, \gamma') \geq \operatorname{dist}(\gamma_1, \gamma_2) > c \cdot |\gamma_1|^3 \geq c \cdot |\gamma|^3.$$

■

Proof of the Existence. Let us start by proving the existence of such a function Γ. If $\underline{T} = \emptyset$, it is immediate to check that by defining $\Gamma(\underline{T}) = \{\emptyset\}$ such a value we associate to $\underline{T}$ satisfies the conditions (P0) and (P1). Then, let us suppose that $\underline{T}$ is a nonempty triangle configuration. Let $\mathscr{C}(\underline{T})$ be defined as the set of all partitions $\mathscr{P}$ of $\underline{T}$ into nonempty triangle configurations such that any pair γ, γ' of distinct elements of $\mathscr{P}$ is well-separated.

1

Now, let us endow $\mathscr{C}(\underline{T})$ with the following partial order. We say that $\mathscr{P}$ is finer than $\mathscr{P}'$, denoting by $\mathscr{P} \succeq \mathscr{P}'$, if for every $\gamma \in \mathscr{P}$ there exists an element $\gamma' \in \mathscr{P}'$ such that $\gamma \subseteq \gamma'$. In the following, we show that for every $\mathscr{P}$ and $\mathscr{P}'$ that belong to $\mathscr{C}(\underline{T})$, the partition

$$\mathscr{P} \vee \mathscr{P}' = \{\gamma \cap \gamma' : \gamma \in \mathscr{P}, \gamma' \in \mathscr{P}', \text{and } \gamma \cap \gamma' \text{ is nonempty}\} \tag{1.32}$$

also belongs to it. Let us verify that each pair of distinct elements of $\mathscr{P} \vee \mathscr{P}'$ is well-separated. Let $\gamma_1 \cap \gamma_1'$ and $\gamma_2 \cap \gamma_2'$ be distinct elements of $\mathscr{P} \vee \mathscr{P}'$, where $\gamma_1, \gamma_2 \in \mathscr{P}$ and $\gamma_1', \gamma_2' \in \mathscr{P}'$. It follows that $\gamma_1 \neq \gamma_2$ or $\gamma_1' \neq \gamma_2'$, so, let us concentrate only on the first case since the treatment of the second one is similar. Note that γ_1 and γ_2 are well-separated, then, by using Lemma 1.12, we conclude that so do $\gamma_1 \cap \gamma_1'$ and $\gamma_2 \cap \gamma_2'$. Hence, in fact, $\mathscr{P} \vee \mathscr{P}'$ belongs to $\mathscr{C}(\underline{T})$. Since $\mathscr{C}(\underline{T})$ is nonempty, it follows that it admits a greatest element with respect to the partial order described above, thus, let us define $\Gamma(\underline{T})$ as the finest partition of $\underline{T}$ into nonempty triangle configurations that satisfies condition (P1).

Since the function Γ defined above fulfills conditions (P0) and (P1), it only remains to show that condition (P2) is also satisfied. The reader can check that (P2) is easily verified for the case where $\underline{T} = \emptyset$. Then, let $\underline{T}$ be a nonempty triangle configuration that can be written as $\underline{T} = \cup_{i=1}^n \underline{T}^{(i)}$, where we assume that any pair γ, γ' of distinct elements of $\cup_{i=1}^n \Gamma(\underline{T}^{(i)})$ satisfy conditions (P1)(a) and (P1)(b). The fact that $\cup_{i=1}^n \Gamma(\underline{T}^{(i)})$ belongs to $\mathscr{C}(\underline{T})$ implies that $\Gamma(\underline{T})$ is finer than $\cup_{i=1}^n \Gamma(\underline{T}^{(i)})$, so, it is straightforward to show that the identity

$$\underline{T}^{(i)} = \bigcup \{\gamma \in \Gamma(\underline{T}) : \gamma \subseteq \underline{T}^{(i)}\}$$

holds, moreover, the partition $\{\gamma \in \Gamma(\underline{T}) : \gamma \subseteq \underline{T}^{(i)}\}$ belongs to $\mathscr{C}(\underline{T}^{(i)})$. It follows that $\Gamma(\underline{T}^{(i)}) \succeq \{\gamma \in \Gamma(\underline{T}) : \gamma \subseteq \underline{T}^{(i)}\}$. Reciprocally, by means of a similar argument using again the fact that $\Gamma(\underline{T})$ is finer than $\cup_{i=1}^n \Gamma(\underline{T}^{(i)})$, we obtain $\{\gamma \in \Gamma(\underline{T}) : \gamma \subseteq \underline{T}^{(i)}\} \succeq \Gamma(\underline{T}^{(i)})$, hence

$$\Gamma(\underline{T}^{(i)}) = \{\gamma \in \Gamma(\underline{T}) : \gamma \subseteq \underline{T}^{(i)}\}. \tag{1.33}$$

Thus, by using equation (1.33), we finally conclude that $\Gamma(\underline{T}) = \cup_{i=1}^n \Gamma(\underline{T}^{(i)})$. ■

Lemma 1.13. *Let Γ be a function defined on $\mathscr{T}$ that satisfies conditions (P0), (P1), and (P2). Then, given any contour γ in $\Gamma(\underline{T})$, we have $\Gamma(\gamma) = \{\gamma\}$.*

Proof. Note that the result can be easily verified if $\underline{T} = \emptyset$, then, let us suppose that $\underline{T}$ is a nonempty triangle configuration. Under this assumption, the associated set of contours $\Gamma(\underline{T})$ is given by

$$\Gamma(\underline{T}) = \{\gamma_1, \ldots, \gamma_n\}$$

for some positive integer n, moreover, for each contour γ_i, its corresponding $\Gamma(\gamma_i)$ can be written as

$$\Gamma(\gamma_i) = \{\gamma_{i,1}, \ldots, \gamma_{i,m_i}\}$$

where m_i is a positive integer. Similarly as in the proof of the existence of Γ, it is straightforward to verify that each pair of distinct elements of $\cup_{i=1}^n \Gamma(\gamma_i) = \{\gamma_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m_i\}$ is well-separated. Therefore, by using property (P2), we conclude that $\Gamma(\underline{T}) = \cup_{i=1}^n \Gamma(\gamma_i)$, so, each m_i is equal to 1 and $\gamma_i = \gamma_{i,1}$. ■

Proof of the Uniqueness. Let us suppose that there exist two functions $\Gamma^{(1)}$ and $\Gamma^{(2)}$ satisfying properties (P0), (P1), and (P2). It follows directly from the fact that such functions satisfy property (P0) that $\Gamma^{(1)}(\emptyset) = \Gamma^{(2)}(\emptyset)$, so, let us assume once again that $\underline{T}$ is a nonempty triangle configuration. Note that, according to the properties (P0) and (P1), the sets of contours with respect to $\Gamma^{(1)}$ and $\Gamma^{(2)}$, expressed in the form $\Gamma^{(k)}(\underline{T}) = \{\gamma_i^{(k)} : 1 \leq i \leq n_k\}$, are partitions of $\underline{T}$ into nonempty triangle configurations. In the following we prove that these two partitions coincide. If we consider the collection

$$\{\gamma_i^{(1)} \cap \gamma_j^{(2)} : 1 \leq i \leq n_1, 1 \leq j \leq n_2 \text{ and } \gamma_i^{(1)} \cap \gamma_j^{(2)} \text{ is nonempty}\}, \tag{1.34}$$

then, for each of its elements the associated set of contours with respect to $\Gamma^{(1)}$ can be written as

$$\Gamma^{(1)}(\gamma_i^{(1)} \cap \gamma_j^{(2)}) = \{\gamma_{i,j,k} : 1 \leq k \leq m_{i,j}\}, \tag{1.35}$$

for some positive integer $m_{i,j}$. Analogously as before, it is straightforward to show that corresponding to each i, any pair of distinct elements of the collection

$$\{\gamma_{i,j,k} : 1 \leq j \leq n_2, 1 \leq k \leq m_{i,j} \text{ and } \gamma_i^{(1)} \cap \gamma_j^{(2)} \text{ is nonempty}\}$$

is well-separated, thus, according to Lemma 1.13 and property (P2), we have

$$\Gamma^{(1)}(\gamma_i^{(1)}) = \{\gamma_i^{(1)}\} = \{\gamma_{i,j,k} : 1 \leq j \leq n_2, 1 \leq k \leq m_{i,j} \text{ and } \gamma_i^{(1)} \cap \gamma_j^{(2)} \text{ is nonempty}\}.$$

It follows that $\gamma_i^{(1)} = \gamma_i^{(1)} \cap \gamma_j^{(2)}$ for some j, hence the partition $\Gamma^{(1)}(\underline{T})$ is finer than $\Gamma^{(2)}(\underline{T})$. Conversely, by means of an analogous argument, we can also prove that $\Gamma^{(2)}(\underline{T})$ is finer that $\Gamma^{(1)}(\underline{T})$, therefore, we conclude the proof of the uniqueness. ■

We dedicate the remaining of this section to enlighten what have been discussed so far by illustrating and providing concrete examples where contours are determined from a given triangle configuration. In order to do so, we need the following result, called the monotonic property of contours, which essentially says that a contour cannot be split into more pieces by adding new triangles to a given configuration. In such case, at most what would happen would be the merger of contours into a larger one.

Corollary 1.14. *Let $\underline{T}$ and $\underline{T}'$ be triangle configurations such that $\underline{T}$ is a subset of $\underline{T}'$. Then, given a contour $\gamma \in \Gamma(\underline{T})$ there exists a contour $\gamma' \in \Gamma(\underline{T}')$ that includes γ.*

Proof. The result follows immediately in the case where $\underline{T}$ is the empty triangle configuration, so, let us suppose that $\underline{T}$ is nonempty. Let the contour set $\Gamma(\underline{T}')$ be given by $\Gamma(\underline{T}') = \{\gamma_1, \ldots, \gamma_n\}$ for some positive integer n, and let us consider the collection

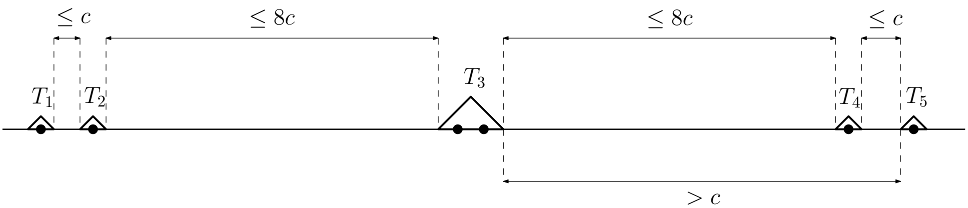
$$\{\gamma \cap \gamma_i : 1 \leq i \leq n \text{ and } \gamma \cap \gamma_i \text{ is nonempty}\}. \tag{1.36}$$

Note that Lemma 1.12 implies that the collection from equation (1.36) is a partition of γ into nonempty triangle configurations such that each of its pairs of distinct elements is well-separated, therefore, it follows from the construction of Γ and Lemma 1.13 that

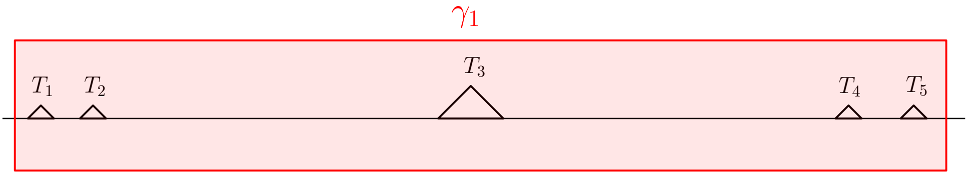
$$\{\gamma\} = \Gamma(\gamma) = \{\gamma \cap \gamma_i : 1 \leq i \leq n \text{ and } \gamma \cap \gamma_i \text{ is nonempty}\},$$

and our claim is proved. ■

Example 1.15. Let $\underline{T}$ be the triangle configuration illustrated in Figure 1.12a. Note that the configuration consisting only of T_1 and T_2 generates a single contour. Indeed, if T_1 and T_2 belong to distinct contours, say γ_1 and γ_2, respectively, then we would have $\mathrm{dist}(\gamma_1, \gamma_2) = \mathrm{dist}(T_1, T_2) \leq c$, a contradiction. Now, if we consider the triangle configuration consisting of T_1, T_2 and T_3, it follows from Corollary 1.14 that T_1 and T_2 must belong to the same contour. Using an argument similar to the one used before, we conclude that the configuration consisting of T_1, T_2 and T_3 generates a single contour. An analogous argument guarantees that the same result holds if we consider the configuration consisting of T_3, T_4 and T_5,this, by means of Corollary 1.14, the whole configuration $\underline{T}$ generates a unique contour, see Figure 1.12b.



(a) Triangle configuration $\underline{T}$ with arrows indicating lower bounds or upper bounds for the number of integers between their corresponding triangles and dots indicating the integer numbers contained inside of each triangle.

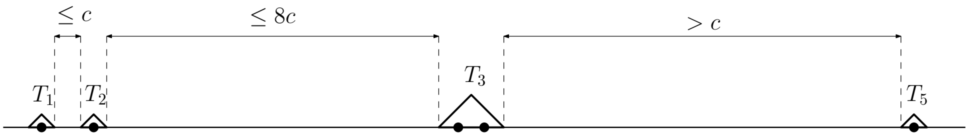


(b) In this case $\Gamma(\underline{T})$ is a singleton.

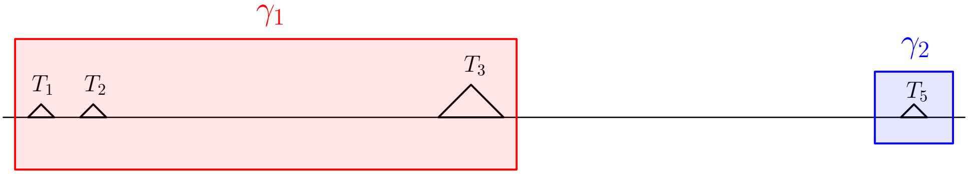
Figure 1.12: Triangle configuration and its set of contours corresponding to Example 1.15.

Example 1.16. Let us consider the triangle configuration $\underline{T}$ obtained by removing the triangle T_4 from the configuration from Example 1.15, see Figure 1.13a. If we let $\underline{T}^{(1)}$ and $\underline{T}^{(2)}$ be two triangle configurations respectively given by $\underline{T}^{(1)} = \{T_1, T_2, T_3\}$ and $\underline{T}^{(2)} = \{T_5\}$. Note that, by repeating the same argument we used in the previous example, $\Gamma(\underline{T}^{(1)})$ is a singleton as well as $\Gamma(\underline{T}^{(2)})$, say $\Gamma(\underline{T}^{(1)}) = \{\gamma_1\}$ and $\Gamma(\underline{T}^{(2)}) = \{\gamma_2\}$. Since γ_1 and γ_2 are well-separated (see Figure 1.13b), it follows from property (P2) that $\Gamma(\underline{T}) = \{\gamma_1, \gamma_2\}$.

Example 1.17. Let $\underline{T}$ be the triangle configuration from Figure 1.14a. Let us split it intro three configurations $\underline{T}^{(1)}$, $\underline{T}^{(2)}$ and $\underline{T}^{(3)}$ respectively given by $\underline{T}^{(1)} = \{T_1, T_2, T_3\}$, $\underline{T}^{(2)} = \{T_4\}$ and $\underline{T}^{(3)} = \{T_5\}$. By using the same argument as applied in Example 1.15, it is straightforward to prove that $\Gamma(\underline{T}^{(1)})$ is a singleton. Thus, if we write $\Gamma(\underline{T}^{(1)}) = \{\gamma_1\}$, $\Gamma(\underline{T}^{(2)}) = \{\gamma_2\}$ and $\Gamma(\underline{T}^{(3)}) = \{\gamma_3\}$, by considering the distances in Figure 1.14b and using property (P2), we conclude that $\Gamma(\underline{T}) = \{\gamma_1, \gamma_2, \gamma_3\}$.

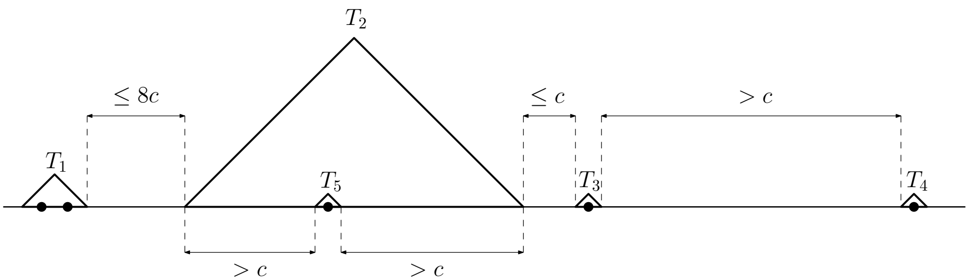


(a) Triangle configuration $\underline{T}$ with arrows indicating lower bounds or upper bounds for the number of integers between their corresponding triangles and dots indicating the integer numbers contained inside of each triangle.

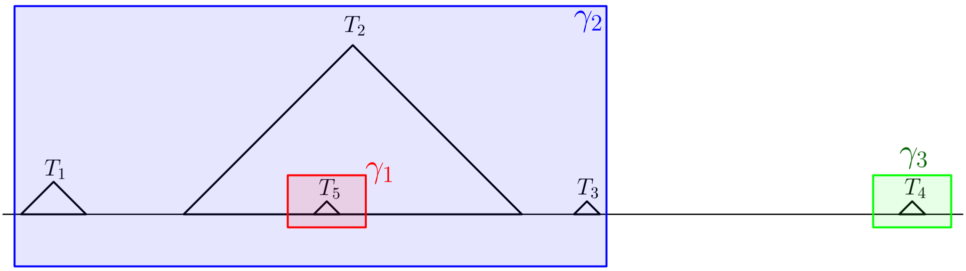


(b) If we remove triangle T_4 from the configuration from Example 1.15, then, the contour from Figure 1.12b is split into two.

Figure 1.13: Triangle configuration and its set of contours corresponding to Example 1.16.



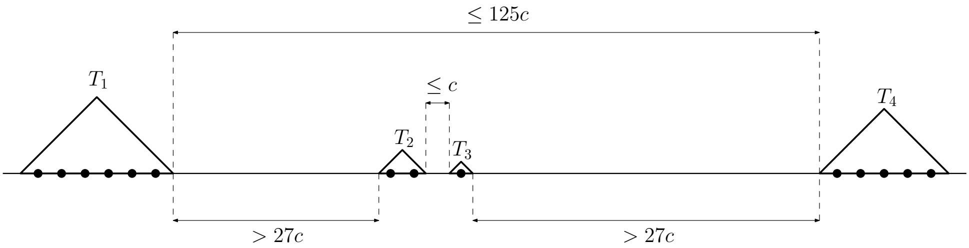
(a) Triangle configuration $\underline{T}$ with arrows indicating lower bounds or upper bounds for the number of integers between their corresponding triangles and dots indicating the integer numbers contained inside of each triangle.



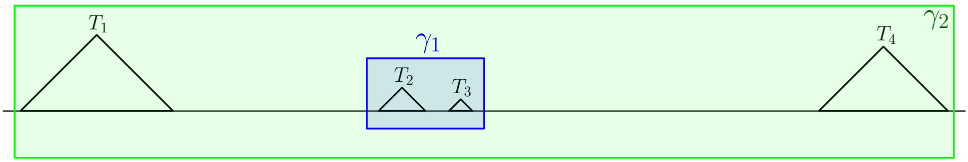
(b) In this case $\Gamma(\underline{T})$ is given by $\Gamma(\underline{T}) = \{\gamma_1, \gamma_2, \gamma_3\}$.

Figure 1.14: Triangle configuration and its set of contours corresponding to Example 1.17.

Example 1.18. Let us consider the triangle configuration $\underline{T}$ illustrated in Figure 1.15a. Let us we split $\underline{T}$ into two triangle configurations $\underline{T}^{(1)}$ and $\underline{T}^{(2)}$ respectively given by $\underline{T}^{(1)} = \{T_1, T_4\}$ and $\underline{T}^{(2)} = \{T_2, T_3\}$. If we keep in mind the distance between the triangles shown in Figure 1.15a, then, it is straightforward to show that γ_1 and γ_2 are well-separated. It follows from property (P2) that $\Gamma(\underline{T}) = \{\gamma_1, \gamma_2\}$.



(a) Triangle configuration $\underline{T}$ with arrows indicating lower bounds or upper bounds for the number of integers between their corresponding triangles and dots indicating the integer numbers contained inside of each triangle.



(b) In this case $\Gamma(\underline{T})$ is given by $\Gamma(\underline{T}) = \{\gamma_1, \gamma_2\}$.

Figure 1.15: Triangle configuration and its set of contours corresponding to Example 1.18.

1.4. ENERGY BOUNDS

In this section we derive the Peierls estimates for the one-dimensional long range Ising model introduced in Section 1.2. Recall that for the two-dimensional nearest-neighbor ferromagnetic Ising model with zero external field it is possible to show that the probability of observing a contour γ according to a Gibbs distribution in the square $\Lambda_n = [-n, n]^2 \cap \mathbb{Z}^2$ with "plus" boundary condition at inverse temperature $\beta > 0$ can be bounded above in the form

$$\mu^+_{\Lambda_n,\beta}(\gamma \in \Gamma) \leq e^{-2\beta|\gamma|}. \tag{1.37}$$

The inequality above suggests that if we consider such a system subject to low temperatures it is more likely to observe the appearance contours with smaller lengths, in other words, the typical configurations may be recognized as being small perturbations of the ground state, which is the configuration consisting of only $+1$ spins.

Our goal in this section is to reproduce an inequality similar to the one from equation (1.37) for the one-dimensional long range Ising model. In order to do so, we approach this problem from the point of view of the graphical representation introduced in Section 1.3 and establish some estimates that states the minimum amount of energy required to add a contour to a given triangle configuration. Let us start by assigning to each triangle configuration $\underline{T}$ its energy $\mathscr{H}^{\alpha,\mathbf{h}}(\underline{T})$ defined by

$$\mathscr{H}^{\alpha,\mathbf{h}}(\underline{T}) = h^{\alpha,\mathbf{h}}(\omega) \tag{1.38}$$

where ω is the element of Ω_+ such that $\Psi(\omega) = \underline{T}$.

The first step to fulfill our goal will be providing some estimates of the minimum amount of energy needed to add a triangle to a given triangle configuration. The next result shows the effect on the spins of a given configuration due to the removal of one of its triangles, more precisely, it states that all the spins inside of a triangle are flipped as soon as it is removed.

Lemma 1.19. *Let $\underline{T}$ be a nonempty triangle configuration, and let ω be the spin configuration such that $\Psi(\omega) = \underline{T}$. Given a triangle T in $\underline{T}$, the spin configuration σ such that $\Psi(\sigma) = \underline{T}\backslash\{T\}$ is given by*

$$\sigma_x = \begin{cases} -\omega_x & \text{if } x \in T, \text{ and} \\ \omega_x & \text{if } x \notin T. \end{cases} \tag{1.39}$$

Proof. The reader can easily verify that for every subset A of $\{r_x : x \in \mathbb{Z}\}$ with an even number of elements, the spin configuration $\eta \in \{-1,+1\}^{\mathbb{Z}}$ defined by

$$\eta_x = (-1)^{\# A\cap(-\infty,x)}$$

at each site x, belongs to Ω_+ and $I_1(\eta) = A$. Since the elements of Ω_+ are fully characterized by their set of spin-flip interfaces, it follows that ω and σ can be expressed as

$$\omega_x = (-1)^{\# I_1(\omega)\cap(-\infty,x)} \tag{1.40}$$

and

$$\sigma_x = (-1)^{\# I_1(\sigma)\cap(-\infty,x)} \tag{1.41}$$

for every x in $\mathbb{Z}$. Note that $I_1(\sigma) = I_1(\omega)\backslash\{a,b\}$, where a and b are the roots of T. Therefore, we have

$$\begin{aligned} \omega_x &= (-1)^{\# I_1(\sigma)\cap(-\infty,x)}(-1)^{\#\{a,b\}\cap(-\infty,x)} \\ &= \sigma_x \cdot (-1)^{\mathbb{1}_{\{a<x<b\}}}, \end{aligned}$$

and the result follows. ∎

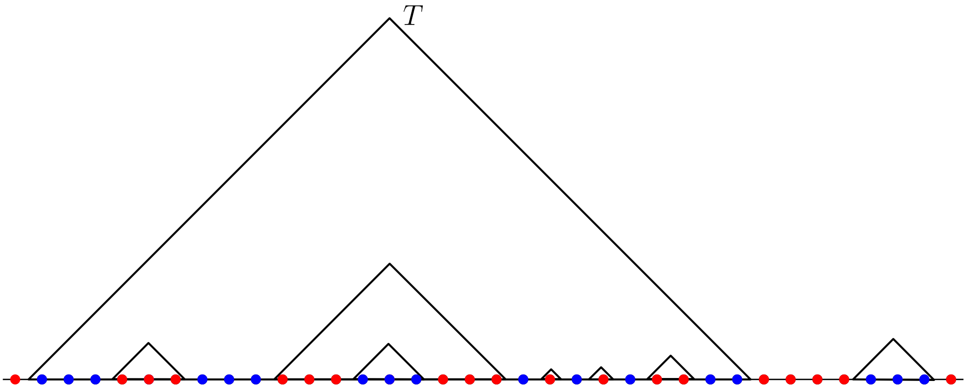
Proposition 1.20. *Let $\underline{T} = \{T_1,\dots,T_n\}$ be a triangle configuration such that $|T_k| \le |T_{k+1}|$ for each $k = 1,\dots,n-1$. Then, we have*

$$\mathscr{H}^{\alpha,\mathbf{h}}(\underline{T}\backslash\{T_1,\dots,T_{i-1}\}) - \mathscr{H}^{\alpha,\mathbf{h}}(\underline{T}\backslash\{T_1,\dots,T_i\}) \ge W_\alpha(|T_i|) - \sum_{x\in T_i\cap\mathbb{Z}} |h_x| \tag{1.42}$$

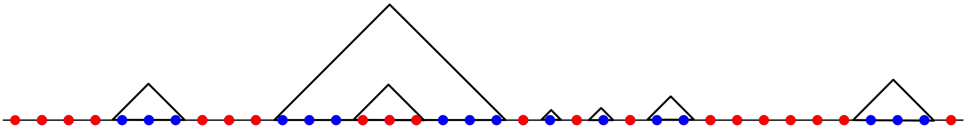
for every $i = 1,\dots,n$, where $W_\alpha(L)$ is the quantity defined by equation (A.1) that can also be expressed as

$$W_\alpha(L) = 2\left(\sum_{x=1}^{L}\sum_{y=L+1}^{2L} J_\alpha(|x-y|) - \sum_{x=1}^{L}\sum_{y=2L+1}^{\infty} J_\alpha(|x-y|)\right) \tag{1.43}$$

for each positive integer L.



(a) Configuration ω.



(b) Configuration σ.

Figure 1.16: The effect on the sign of the spins after the removal a triangle.

Proof. In order to simplify our notation, let us denote the coupling constant $J_\alpha(|x-y|)$ simply by $J_{x,y}$. The energy cost to add the triangle T_i to the configuration $\underline{T}\backslash\{T_1,\dots,T_i\}$ can be written as

$$\mathscr{H}^{\alpha,\mathbf{h}}(\underline{T}\backslash\{T_1,\dots,T_{i-1}\}) - \mathscr{H}^{\alpha,\mathbf{h}}(\underline{T}\backslash\{T_1,\dots,T_i\}) = h^{\alpha,\mathbf{h}}(\omega) - h^{\alpha,\mathbf{h}}(\sigma),$$

where ω and σ are elements of Ω_+ such that $\Psi(\omega) = \underline{T}\backslash\{T_1,\dots,T_{i-1}\}$ and $\Psi(\sigma) = \underline{T}\backslash\{T_1,\dots,T_i\}$, respectively. According to Lemma 1.19, we have

$$\begin{aligned} h^{\alpha,\mathbf{h}}(\omega) \quad &= \quad \frac{1}{2}\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in T_i\cap\mathbb{Z}} J_{x,y}\mathbb{1}_{\{\omega_x\neq\omega_y\}} + \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\backslash T_i} J_{x,y}\mathbb{1}_{\{\omega_x\neq\omega_y\}} + \sum_{x\in T_i\cap\mathbb{Z}} h_x\mathbb{1}_{\{\omega_x=-1\}} \\ &\quad + \frac{1}{2}\sum_{x\in\mathbb{Z}\backslash T_i}\sum_{y\in\mathbb{Z}\backslash T_i} J_{x,y}\mathbb{1}_{\{\omega_x\neq\omega_y\}} + \sum_{x\in\mathbb{Z}\backslash T_i} h_x\mathbb{1}_{\{\omega_x=-1\}}, \end{aligned}$$

and

$$\begin{aligned} h^{\alpha,\mathbf{h}}(\sigma) \quad &= \quad \frac{1}{2}\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in T_i\cap\mathbb{Z}} J_{x,y}\mathbb{1}_{\{\omega_x\neq\omega_y\}} + \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\backslash T_i} J_{x,y}\mathbb{1}_{\{\omega_x=\omega_y\}} + \sum_{x\in T_i\cap\mathbb{Z}} h_x\mathbb{1}_{\{\omega_x=+1\}} \\ &\quad + \frac{1}{2}\sum_{x\in\mathbb{Z}\backslash T_i}\sum_{y\in\mathbb{Z}\backslash T_i} J_{x,y}\mathbb{1}_{\{\omega_x\neq\omega_y\}} + \sum_{x\in\mathbb{Z}\backslash T_i} h_x\mathbb{1}_{\{\omega_x=-1\}}. \end{aligned}$$

Let us consider consider the interval of integers I_i^+ and I_i^- respectively defined by $I_i^+ = (T_i\cap\mathbb{Z}) + |T_i|$ and $I_i^- = (T_i\cap\mathbb{Z}) - |T_i|$, in other words, let I_i^+ (resp. I_i^-) be the interval of integers to the right (resp. left) of T_i with L elements. It follows that

$$\begin{aligned}
h^{\alpha,\mathbf{h}}(\omega) - h^{\alpha,\mathbf{h}}(\sigma) &= \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\setminus T_i} J_{x,y}\mathbb{1}_{\{\omega_x\neq\omega_y\}} - \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\setminus T_i} J_{x,y}\mathbb{1}_{\{\omega_x=\omega_y\}}\\
&\quad + \sum_{x\in T_i\cap\mathbb{Z}} h_x\mathbb{1}_{\{\omega_x=-1\}} - \sum_{x\in T_i\cap\mathbb{Z}} h_x\mathbb{1}_{\{\omega_x=+1\}}\\
&= \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\setminus T_i} J_{x,y} - 2\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\setminus T_i} J_{x,y}\mathbb{1}_{\{\omega_x=\omega_y\}} + \sum_{x\in T_i\cap\mathbb{Z}} h_x\mathbb{1}_{\{\omega_x=-1\}}\\
&\quad - \sum_{x\in T_i\cap\mathbb{Z}} h_x\mathbb{1}_{\{\omega_x=+1\}}\\
&= \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in I_i^{\pm}} J_{x,y} - \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\setminus(T_i\cup I_i^{\pm})} J_{x,y}\\
&\quad +2\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\setminus(T_i\cup I_i^{\pm})} J_{x,y}(1-\mathbb{1}_{\{\omega_x=\omega_y\}}) - 2\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in I_i^{\pm}} J_{x,y}\mathbb{1}_{\{\omega_x=\omega_y\}}\\
&\quad + \sum_{x\in T_i\cap\mathbb{Z}} h_x\mathbb{1}_{\{\omega_x=-1\}} - \sum_{x\in T_i\cap\mathbb{Z}} h_x\mathbb{1}_{\{\omega_x=+1\}}.
\end{aligned}$$

Hence, by using the fact that

$$W_\alpha(|T_i|) = \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in I_i^{\pm}} J_{x,y} - \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\setminus(T_i\cup I_i^{\pm})} J_{x,y}$$

and

$$\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in\mathbb{Z}\setminus(T_i\cup I_i^{\pm})} J_{x,y}(1-\mathbb{1}_{\{\omega_x=\omega_y\}}) \geq 0,$$

we conclude that

$$h^{\alpha,\mathbf{h}}(\omega) - h^{\alpha,\mathbf{h}}(\sigma) \geq W_\alpha(|T_i|) - 2\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in I_i^{\pm}} J_{x,y}\mathbb{1}_{\{\omega_x=\omega_y\}} - \sum_{x\in T_i\cap\mathbb{Z}}|h_x|. \tag{1.44}$$

Since $\mathrm{dist}(T_i,T_j)\geq|T_i|$ holds for all $j>i$, then the spins inside of T_i and $I_i^{\pm}$ have opposite signs. Therefore, the result follows. ■

It follows from Proposition 1.20 that the energy cost to add the triangles $T_1,\ldots,T_k$ to $\underline{T}\setminus\{T_1,\ldots,T_k\}$ in order to obtain $\underline{T}$ can be bounded by

$$\mathscr{H}^{\alpha,\mathbf{h}}(\underline{T}) - \mathscr{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\{T_1,\ldots,T_k\}) \geq \sum_{i=1}^{k}\left(W_\alpha(|T_i|) - \sum_{x\in T_i\cap\mathbb{Z}}|h_x|\right). \tag{1.45}$$

However, recall that we are targeting to obtain such a quantity associated to the addition of a contour, and, in general, the inequality above is no longer valid since it is not necessarily true that the triangles from that contour are those with smaller masses.

We show in Theorem 1.21 that in order to overcome this obstacle it is necessary to restrict the range of the interaction power decay. In Appendix A we prove that for α in the interval $[0,\alpha^*)$, where α^* is the number that satisfies $0<\alpha^*<1$ and $\sum_{k=1}^{\infty}\frac{1}{k^{2-\alpha^*}}=2$, there is a positive constant ζ_α such that

$$W_\alpha(L) \geq \zeta_\alpha \mathcal{X}_\alpha(L) \tag{1.46}$$

1

holds for every positive integer L, where the function $\mathcal{X}_\alpha$ is given by

$$\mathcal{X}_\alpha(L) = \begin{cases} L^\alpha & \text{if } \alpha > 0, \text{ and} \\ \log(L) + 4 & \text{if } \alpha = 0. \end{cases} \tag{1.47}$$

Theorem 1.21. *Let $\alpha \in [0, \alpha^*)$, and let the constant c from property (P1) be large enough. Then, given a triangle configuration $\underline{T}$, for any $\gamma_0 \in \Gamma(\underline{T})$ we have*

$$\mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}) - \mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\gamma_0) \geq \sum_{T\in\gamma_0}\left(\frac{1}{2}W_\alpha(|T|) - \sum_{x\in T\cap\mathbb{Z}}|h_x|\right). \tag{1.48}$$

Proof. If $\underline{T}$ is the empty triangle configuration, then the result follows immediately. So, let us suppose that $\underline{T}$ is nonempty. In this case, let us write $\Gamma(\underline{T}) = \{\gamma_0, \dots, \gamma_n\}$ and $\gamma_0 = \{T_1, \dots, T_k\}$ where $n \geq 0$, $k \geq 1$ and $|T_i| \leq |T_{i+1}|$ for $i = 1, \dots, k-1$. The left-hand side of equation (1.48) can be expressed in terms of the telescoping sum

$$\mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}) - \mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\gamma_0) = \sum_{i=1}^{k}\left(\mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\{T_1,\dots,T_{i-1}\}) - \mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\{T_1,\dots,T_i\})\right). \tag{1.49}$$

As in Proposition 1.20, let us write

$$\mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\{T_1,\dots,T_{i-1}\}) - \mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\{T_1,\dots,T_i\}) = h^{\alpha,\mathbf{h}}(\omega) - h^{\alpha,\mathbf{h}}(\sigma),$$

where ω and σ are elements of Ω_+ such that $\Psi(\omega) = \underline{T}\setminus\{T_1,\dots,T_{i-1}\}$ and $\Psi(\sigma) = \underline{T}\setminus\{T_1,\dots,T_i\}$, respectively. Using exactly the same computations as before, we obtain

$$h^{\alpha,\mathbf{h}}(\omega) - h^{\alpha,\mathbf{h}}(\sigma) \geq W_\alpha(|T_i|) - 2\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in I_i^\pm} J_{x,y}\mathbb{1}_{\{\omega_x=\omega_y\}} - \sum_{x\in T_i\cap\mathbb{Z}}|h_x|. \tag{1.50}$$

Before we proceed further in the computations, let us point out some remarks.

Lemma 1.22. *Under the hypotheses stated above, we have the following conditions.*

(a) We have $(I_i^- \cup I_i^+)\cap T_j = \emptyset$ whenever j is an integer such that $j > i$ and $T_j \not\supseteq T_i$,

(b) for all $j \geq 1$ such that $|\gamma_j| \geq |\gamma_0|$,

$$(I_i^- \cup I_i^+)\cap T = \emptyset$$

holds whenever T is a triangle that belongs to γ_j and satisfies $T \not\supseteq T_i$, and

(c) for all $j \geq 1$ such that $|\gamma_j| < |\gamma_0|$, the inequality

$$\mathrm{dist}(T, T_i) > c|\gamma_j|^3$$

holds for all T in γ_j.

Let us return to the proof of the theorem. Note that for every x in $T_i \cap \mathbb{Z}$ and $y \in I_i^- \cup I_i^+$ such that $\omega_x = \omega_y$ there exists a triangle T in $\underline{T}\backslash\{T_1,\dots,T_{i-1}\}$ that does not contain T_i such that either T contains x but not y or T contains y but not x. Indeed, suppose that this assertion does not hold. If y belongs to I_i^+, let y_0 be the largest integer inside of T_i. Since y_0 belongs to the interval $[x, y-1]$, then

$$
\begin{aligned}
\omega_x\omega_y &= \prod_{j=x}^{y-1} \omega_j\omega_{j+1} = (-1)^{\#\{j\in\mathbb{Z}: x\le j\le y-1 \text{ and } \omega_j\omega_{j+1}=-1\}} \\
&= -(-1)^{\#\{j\in\mathbb{Z}: x\le j\le y-1, j\neq y_0 \text{ and } \omega_j\omega_{j+1}=-1\}}.
\end{aligned}
$$

We say that any two elements x' and y' of $\{j \in \mathbb{Z} : x \le j \le y-1, j \neq y_0 \text{ and } \omega_j\omega_{j+1} = -1\}$ are neighbors if either we have $x' < y'$ with $\Delta(r_{x'}, r_{y'}) \in \underline{T}\backslash\{T_1,\dots,T_{i-1}\}$ or $y' < x'$ with $\Delta(r_{y'}, r_{x'}) \in \underline{T}\backslash\{T_1,\dots,T_{i-1}\}$. It is easy to see that this set together with this graph structure is a graph with degree 1, thus it contains an even number of elements. It follows that $\omega_x\omega_y = -1$, contradicting the assumption that $\omega_x = \omega_y$. Analogously, if y belongs to I_i^-, we consider the smallest integer x_0 inside of T_i and the proof follows similarly. Thus, our assertion is proved. Furthermore, note that the triangle T from the assertion above does not belong to γ_0, otherwise, we would have $T = T_j$ for some $j > i$, and according to Lemma 1.22(a), this would result in a contradiction. Therefore, we derive

$$
\begin{aligned}
h^{\alpha,\mathbf{h}}(\omega) - h^{\alpha,\mathbf{h}}(\sigma) \quad \ge \quad & W_\alpha(|T_i|) - 2\sum_{M=1}^{\infty}\sum_{j=1}^{n} \mathbb{1}_{|\gamma_j|=M} \sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in I_i^\pm} J_{x,y}(\mathbb{1}_{y\in A(T_i,\gamma_j)} + \mathbb{1}_{x\in A(T_i,\gamma_j)}) \\
& - \sum_{x\in T_i\cap\mathbb{Z}} |h_x| \qquad (1.51)
\end{aligned}
$$

where

$$
A(T_i,\gamma_j) = \bigcup_{T\in\gamma_j, T\not\supseteq T_i} T\cap\mathbb{Z}.
$$

Now, let us prove the inequality

$$
\sum_{j=1}^{n} \mathbb{1}_{|\gamma_j|=M} \sum_{y\in I_i^+} J_{x,y}\mathbb{1}_{y\in A(T_i,\gamma_j)} \le \frac{M}{\lfloor cM^3\rfloor} \sum_{y\in I_i^+} J_{x,y}. \qquad (1.52)
$$

Let us consider the set

$$
\{j \in \{1,\dots,n\} : |\gamma_j| = M \text{ and } I_i^+ \cap A(T_i,\gamma_j) \neq \emptyset\} \qquad (1.53)
$$

and define $y_l = y_{l-1} + \lfloor cM^3 \rfloor$ for each positive integer l (recall that y_0 is the rightmost integer inside of T_i). If the set above is empty, then equation (1.52) is trivial. So, let us suppose that the set above has N elements. In the following, we prove that the elements of (1.53) can be written as $j_1,\dots,j_N$ in such a way that every triangle from γ_{j_l} that does not contain T_i and intersects I_i^+ is on the right of y_l. Considering the contours γ_j where j belongs to (1.53), according to Lemma 1.22(b),(c), every triangle in γ_j that does not contain T_i and intersects I_i^+ is on the right of y_1. If $N = 1$, our assertion follows immediately. Otherwise, if $N > 1$, we define the contour γ_{j_1} as the contour that

1

contains the closest triangle that does not contain T_i and intersects I_i^+ on the right of y_1 among all such triangles from γ_j, where j belongs to (1.53). Suppose that we constructed $j_1,\dots,j_m$, $1 \le m < N$, in such a way that every triangle from γ_j, j belonging (1.53) distinct from $j_1 \dots, j_{l-1}$, that does not contain T_i and intersects I_i^+ is to the right of y_l; moreover, each γ_{j_l} contains the closest such triangle on the right of y_l among all such triangles from these γ_j's. Considering a contour γ_j, j belonging to (1.53) distinct from $j_1,\dots,j_m$, we have

$$\mathrm{dist}(\gamma_j,\gamma_{j_m}) > cM^3.$$

Thus, every triangle in γ_j that does not contain T_i and intersects I_i^+ is on the right of y_{m+1}, and define $\gamma_{j_{m+1}}$ as the contour that contains the closest such triangle on the right of y_{m+1} among all such triangles from these γ_j's. So, we prove our assertion.

Therefore, we have

$$\sum_{j=1}^{n} \mathbb{1}_{|\gamma_j|=M} \sum_{y\in I_i^+} J_{x,y}\mathbb{1}_{y\in A(T_i,\gamma_j)} \le M\sum_{l=1}^{m} J_{x,y_l} \le M\sum_{l=1}^{m}\frac{1}{\lfloor cM^3\rfloor}\sum_{y\in(y_{j-1},y_l)} J_{x,y},$$

thus

$$\sum_{j=1}^{n} \mathbb{1}_{|\gamma_j|=M} \sum_{y\in I_i^+} J_{x,y}\mathbb{1}_{y\in A(T_i,\gamma_j)} \le \frac{M}{\lfloor cM^3\rfloor}\sum_{y\in I_i^+} J_{x,y}. \tag{1.54}$$

Repeating the same argument to the other sums in (1.51), we obtain

$$h^{\alpha,\mathbf{h}}(\omega)-h^{\alpha,\mathbf{h}}(\sigma) \ge W_\alpha(|T_i|) - \left(\sum_{M=1}^{\infty}\frac{4M}{\lfloor cM^3\rfloor}\right)\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in I_i^\pm} J_{x,y} - \sum_{x\in T_i\cap\mathbb{Z}}|h_x| \tag{1.55}$$

The reader can easily verify that there is a positive constant k_α such that

$$\sum_{x\in T_i\cap\mathbb{Z}}\ \sum_{y\in\mathbb{Z}\setminus(T_i\cup I_i^\pm)} J_{x,y} \le k_\alpha \mathcal{X}_\alpha(|T_i|),$$

where $\mathcal{X}_\alpha$ is defined by equation (A.11) from Appendix A.1.2. It follows that

$$\sum_{x\in T_i\cap\mathbb{Z}}\sum_{y\in I_i^\pm} J_{x,y} \le W_\alpha(|T_i|) + k_\alpha\mathcal{X}_\alpha(|T_i|) \le W(|T_i|)\left(1+\frac{k_\alpha}{\zeta_\alpha}\right),$$

thus, we obtain

$$h^{\alpha,\mathbf{h}}(\omega)-h^{\alpha,\mathbf{h}}(\sigma) \ge W_\alpha(|T_i|)\left[1-\left(\sum_{M=1}^{\infty}\frac{4M}{\lfloor cM^3\rfloor}\right)\left(1+\frac{k_\alpha}{\zeta_\alpha}\right)\right] - \sum_{x\in T_i\cap\mathbb{Z}}|h_x|. \tag{1.56}$$

If we take c large enough in such a way that

$$\left(\sum_{M=1}^{\infty}\frac{4M}{\lfloor cM^3\rfloor}\right)\left(1+\frac{k_\alpha}{\zeta_\alpha}\right) \le \frac{1}{2}$$

holds, we conclude that

$$\mathscr{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\{T_1,\dots,T_{i-1}\}) - \mathscr{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\{T_1,\dots,T_i\}) \ge \frac{1}{2}W_\alpha(|T_i|) - \sum_{x\in T_i\cap\mathbb{Z}}|h_x|. \tag{1.57}$$

Therefore, the result follows by using equation (1.49). ■

In order to apply the Peierls contour argument it is sufficient to show that for suitable interaction power decay $2-\alpha$ and external field power decay δ, there exists a positive constant ξ (possibly depending on these parameters) such that given a triangle configuration $\underline{T}$, for each contour $\gamma \in \Gamma(\underline{T})$ the inequality

$$\mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}) - \mathcal{H}^{\alpha,\mathbf{h}}(\underline{T}\setminus\gamma) \geq \xi \sum_{T\in\gamma} \chi_\alpha(|T|) \tag{1.58}$$

holds. Indeed, let us consider the probability of occurrence of a contour γ according to the Gibbs distribution in $\Lambda_n = [-n,n]\cap\mathbb{Z}$ with "plus" boundary condition and external field $\mathbf{h}$ at inverse temperature $\beta > 0$, given by

$$\mu^+_{\Lambda_n,\beta,\mathbf{h}}(\gamma\in\Gamma) := \mu^+_{\Lambda_n,\beta,\mathbf{h}}\big(\omega\in\Omega^+_{\Lambda_n} : \gamma\in\Gamma(\omega)\big), \tag{1.59}$$

where $\Omega^+_{\Lambda_n}$ is the set of all Ising spin configurations whose spin values are equal to $+1$ outside of Λ_n. Then,

$$\begin{aligned}
\mu^+_{\Lambda_n,\beta,\mathbf{h}}(\gamma\in\Gamma) &= \frac{\sum\limits_{\omega\in\Omega^+_{\Lambda_n}:\gamma\in\Gamma(\omega)} e^{-\beta H^{\alpha,\mathbf{h}}_{\Lambda_n}(\omega)}}{\sum\limits_{\eta\in\Omega^+_{\Lambda_n}} e^{-\beta H^{\alpha,\mathbf{h}}_{\Lambda_n}(\eta)}} \\
&= \frac{\sum\limits_{\omega\in\Omega^+_{\Lambda_n}:\gamma\in\Gamma(\omega)} e^{-2\beta\mathcal{H}^{\alpha,\mathbf{h}}(\Psi(\omega))}}{\sum\limits_{\eta\in\Omega^+_{\Lambda_n}} e^{-2\beta\mathcal{H}^{\alpha,\mathbf{h}}(\Psi(\eta))}} \\
&\leq e^{-2\beta\xi\sum\limits_{T\in\gamma}\chi_\alpha(|T|)} \times \frac{\sum\limits_{\omega\in\Omega^+_{\Lambda_n}:\gamma\in\Gamma(\omega)} e^{-2\beta\mathcal{H}^{\alpha,\mathbf{h}}(\Psi(\omega)\setminus\gamma)}}{\sum\limits_{\eta\in\Omega^+_{\Lambda_n}} e^{-2\beta\mathcal{H}^{\alpha,\mathbf{h}}(\Psi(\eta))}}.
\end{aligned}$$

By using the fact that the last quotient above is smaller or equal than 1 since it describes the probability of an event, we obtain the inequality

$$\mu^+_{\Lambda_n,\beta,\mathbf{h}}(\gamma\in\Gamma) \leq e^{-2\beta\xi\sum\limits_{T\in\gamma}\chi_\alpha(|T|)}, \tag{1.60}$$

establishing the relationship between the probability of the appearance of γ and the mass of its triangles. Thus, under the assumption of validity of energy bounds like (1.58), we obtained equation (1.60), suggesting that the typical contours we observe at low temperatures are those with smaller masses. Furthermore, since we are dealing with configurations with "plus" boundary condition, in order to have a spin with value -1 at the origin there must exist a contour with a triangle containing it. So, let us use $\gamma\odot 0$ to denote the fact that there is a triangle that belongs to the contour γ that contains the origin. Then, as usual, we bound the probability of the event $\{\sigma_0=-1\}$ in the form

$$\begin{aligned}
\mu^+_{\Lambda_n,\beta,\mathbf{h}}(\sigma_0=-1) &\leq \mu^+_{\Lambda_n,\beta,\mathbf{h}}(\text{There is } \gamma\in\Gamma \text{ such that } \gamma\odot 0) \\
&\leq \sum_{\gamma:\gamma\odot 0} \mu^+_{\Lambda_n,\beta,\mathbf{h}}(\gamma\in\Gamma),
\end{aligned}$$

1

and, by means of equation (1.60), it follows that

$$\mu^{+}_{\Lambda_n,\beta,\mathbf{h}}(\sigma_0=-1) \leq \sum_{m=1}^{\infty}\left(\sum_{\substack{\gamma:\gamma\odot 0\\ |\gamma|=m}} e^{-2\beta\xi\sum_{T\in\gamma}\chi_\alpha(|T|)}\right). \tag{1.61}$$

The solution of the combinatorial problem that consists of finding an upper bound for the summation included inside the brackets above is given in the next section (see Theorem 1.23), however, due to the technicality of its exposition, a more impatient reader may skip it at a first reading without any problem. Therefore, by using Theorem 1.23, we conclude that for β sufficiently large we have

$$\mu^{+}_{\Lambda_n,\beta,\mathbf{h}}(\sigma_0=-1) \leq 2\sum_{m=1}^{\infty} m e^{-2\beta\xi\chi_\alpha(m)}, \tag{1.62}$$

where the right-hand side converges to zero as β approaches infinity, so, the system undergoes phase transition at low temperatures.

In view of the comments above, we dedicate Section 1.5 to give the proof of the entropy estimates and Section 1.6 to find sufficient conditions on α and δ so that such a condition like (1.58) holds, in order to ensure the phase transition phenomenon at low temperatures.

1.5. ENTROPY

We dedicate this whole section to establish the entropy bounds required to control the right-hand side of equation (1.61) and then conclude the final step of the Peierls' argument. Our goal is to prove the following result.

Theorem 1.23. *Let $\alpha \in [0,1)$, and let β be a positive real number. If β is sufficiently large, then, the inequality*

$$\sum_{\substack{\gamma:\gamma\odot 0\\ |\gamma|=m}} w^{\alpha}_{\beta}(\gamma) \leq 2me^{-\beta\chi_\alpha(m)} \tag{1.63}$$

holds for every positive integer m, where $w^{\alpha}_{\beta}(\gamma)$ is the weight we associate to the contour γ whose expression is given by

$$w^{\alpha}_{\beta}(\gamma) = \prod_{T\in\gamma} e^{-\beta\chi_\alpha(|T|)}. \tag{1.64}$$

First, let us introduce some notation. Given a triangle T, say $T = \Delta(r_x, r_y)$, let us denote the sites associated to its left and right endpoints by $x_l(T)$ and $x_r(T)$, respectively, in other words, we define $x_l(T)$ and $x_r(T)$ by letting $x_l(T) = x$ and $x_r(T) = y$. Now, if γ is a nonempty triangle configuration and $T(\gamma)$ the smallest triangle that contains all its elements, then, we extend the previous definition by defining the points $x_l(\gamma)$ and $x_r(\gamma)$ by $x_l(\gamma) = x_l(T(\gamma))$ and $x_r(\gamma) = x_r(T(\gamma))$.

Remark 1.24. Note that, according to the notation introduced above, it follows from equation (1.25) that the distance between the triangles T and T' can be rewritten in the form

$$\operatorname{dist}(T,T') = \min\left\{|x_l(T)-x_l(T')|, |x_l(T)-x_r(T')|, |x_r(T)-x_l(T')|, |x_r(T)-x_r(T')|\right\}.$$

In this section we will be dealing with nonempty triangle configurations $\underline{T}$ for which $\Gamma(\underline{T})$ consists of a unique contour. The proof of Theorem 1.23 relies on the fact that one of the following alternatives holds. In the first case, $\underline{T}$ can be decomposed as a maximal triangle T and triangle configurations $\gamma_1, \ldots, \gamma_k$, where k is a nonnegative integer, the $T(\gamma_i)$'s are pairwise disjoint triangles included in T and each $\Gamma(\gamma_i)$ is a singleton; moreover, the relations

$$
\begin{aligned}
&1 \le x_-(\gamma_1) - x_-(T) \le c|\gamma_1|^3, && && (1.65)\\
&1 \le x_-(\gamma_j) - x_+(\gamma_{j-1}) \le c|\gamma_j|^3 && \text{for each } 2 \le j \le p, && (1.66)\\
&1 \le x_-(\gamma_{j+1}) - x_+(\gamma_j) \le c|\gamma_j|^3 && \text{for each } p+1 \le j \le k-1, \text{ and} && (1.67)\\
&1 \le x_+(T) - x_+(\gamma_k) \le c|\gamma_k|^3 && && (1.68)
\end{aligned}
$$

hold for some integer p satisfying $0 \le p \le k$. Otherwise, $\underline{T}$ can be split into triangle configurations $\gamma_1, \ldots, \gamma_n, \gamma_1^{(1)}, \ldots, \gamma_{k_1}^{(1)}, \ldots, \gamma_1^{(n-1)}, \ldots, \gamma_{k_{n-1}}^{(n-1)}$, where n is an integer number greater or equal than 2, each k_i is a nonnegative integer, all the triangles $T(\gamma)$'s are pairwise disjoint and each $\Gamma(\gamma)$ is a singleton. In addition, for each i such that $1 \le i \le n-1$, the triangle $T(\gamma_i)$ is on the left of $T(\gamma_{i+1})$ and their distance satisfies

$$\operatorname{dist}(\gamma_i, \gamma_{i+1}) \le c \cdot |\gamma_i|^3 \wedge |\gamma_{i+1}|^3; \tag{1.69}$$

furthermore, the triangles $T(\gamma_1^{(i)}), \ldots, T(\gamma_{k_i}^{(i)})$ are arranged in between $T(\gamma_i)$ and $T(\gamma_{i+1})$ in such a way that the relations

$$
\begin{aligned}
&1 \le x_-(\gamma_1^{(i)}) - x_+(\gamma_i) \le c|\gamma_1^{(i)}|^3, && && (1.70)\\
&1 \le x_-(\gamma_j^{(i)}) - x_+(\gamma_{j-1}^{(i)}) \le c|\gamma_j^{(i)}|^3 && \text{for each } 2 \le j \le p_i, && (1.71)\\
&1 \le x_-(\gamma_{j+1}^{(i)}) - x_+(\gamma_j^{(i)}) \le c|\gamma_j^{(i)}|^3 && \text{for each } p_i+1 \le j \le k_i-1, \text{ and} && (1.72)\\
&1 \le x_-(\gamma_{i+1}) - x_+(\gamma_{k_i}^{(i)}) \le c|\gamma_{k_i}^{(i)}|^3 && && (1.73)
\end{aligned}
$$

hold for some integer p_i satisfying $0 \le p_i \le k_i$.

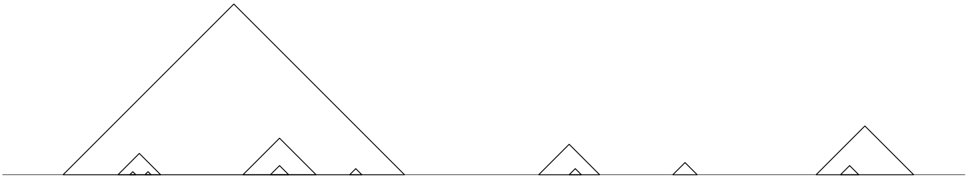
1.5.1. SQUARE CONFIGURATIONS AT TIME $t = 0$

Let $\underline{T}$ be a triangle configuration. A triangle T in $\underline{T}$ is said to be maximal if there is no other triangle from that configuration that includes T. Note that any pair of distinct maximal triangles of $\underline{T}$ necessarily must be disjoint. For the sake of clarity, instead of visualizing a maximal triangle as we normally do, let us replace its usual graphical representation with a square whose one of its sides coincides with the base of that triangle. For that reason, such maximal triangles we just described will often be referred to as squares (at time $t = 0$) and typically denoted by S. See Figure 1.17.

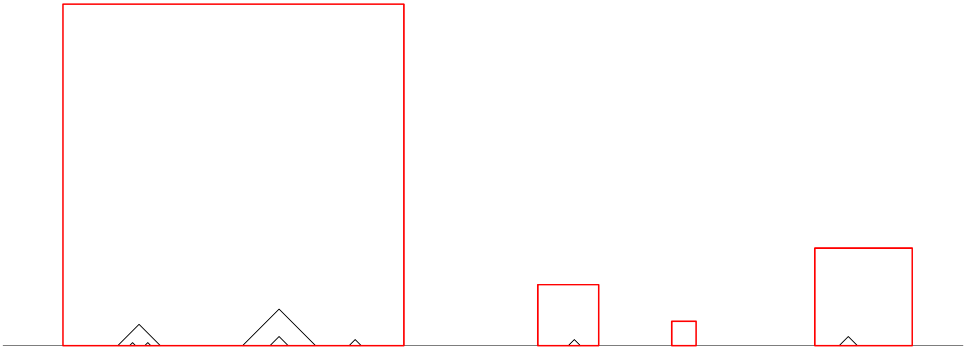
In the following, we provide the construction of the square configurations corresponding to time $t = 0$ and derive their basic properties. Given a square S in $\underline{T}$, let us consider the cluster of triangles $[\underline{T}]_S$ defined by

$$[\underline{T}]_S = \{T \in \underline{T} : T \text{ is a subset of } S\}, \tag{1.74}$$

in other words, let $[\underline{T}]_S$ be defined as the set consisting of the square S and all the triangles in $\underline{T}$ (strictly) included in S.



(a) Triangle configuration.



(b) Triangle configuration with its maximal triangles replaced by squares.

Figure 1.17: Representation of maximal triangles as squares.

From now on, we will mostly be dealing with nonempty triangle configurations $\underline{T}$ such that $\Gamma(\underline{T})$ consists of a unique contour. In such cases, we assign to every such a configuration $\underline{T}$ its corresponding square configuration $\underline{S}$ at time $t = 0$ by defining $\underline{S}$ as the set whose elements are all the squares of $\underline{T}$, in other words, we define the configuration $\underline{S}$ as the set of all maximal triangles of $\underline{T}$. As the reader can easily verify, the collection of clusters $\{[\underline{T}]_S : S \in \underline{S}\}$ defines a partition of $\underline{T}$. In part (a) of Theorem 1.25 we prove that each cluster $[\underline{T}]_S$ generates a single contour, while in part (b) we show precisely how the triangles inside of each square S are organized according to their relative positions described by relations (1.75) and (1.76).

Theorem 1.25. *Let $\underline{T}$ be a nonempty triangle configuration such that $\Gamma(\underline{T})$ is a singleton, and let S be a square in $\underline{T}$. Then, the following properties hold.*

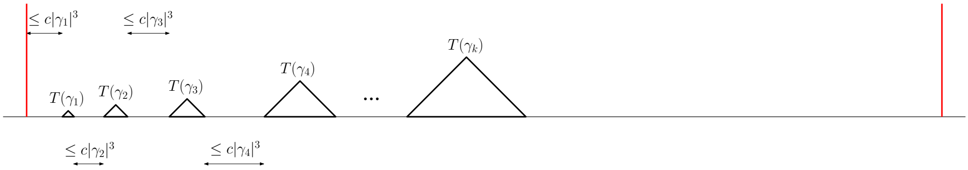
(a) $\Gamma([\underline{T}]_S)$ *is a singleton.*

(b) If $[\underline{T}]_S \backslash \{S\}$ is nonempty, then $\Gamma([\underline{T}]_S \backslash \{S\})$ consists of contours $\gamma_1, \ldots, \gamma_k$ labeled in such a way that $T(\gamma_1), \ldots, T(\gamma_k)$ is a sequence of disjoint triangles ordered from the left to the right for which there exists p satisfying $0 \le p \le k$ such that

$$1 \le a_i - b_{i-1} \le c|\gamma_i|^3 \quad \text{holds if } 1 \le i \le p, \text{ and} \tag{1.75}$$

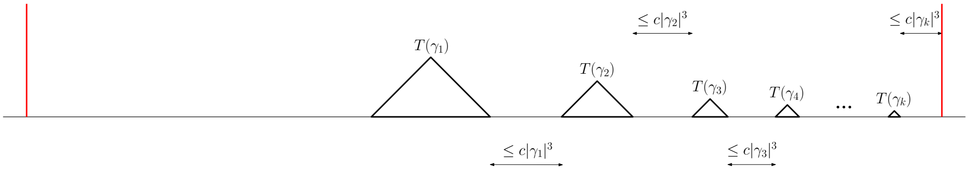
$$1 \le a_{i+1} - b_i \le c|\gamma_i|^3 \quad \text{holds if } p+1 \le i \le k, \tag{1.76}$$

where $b_0 = x_l(S)$, $a_i = x_l(\gamma_i)$, $b_i = x_r(\gamma_i)$, and $a_{k+1} = x_r(S)$.



(a) Case $p = k$.

(b) Case $0 < p < k$.



(c) Case $p = 0$.

Figure 1.18: Contours associated to $[\underline{T}]_S \setminus \{S\}$, where the red lines stand for the sides of the square S.

Let $\underline{T}$ be an arbitrary triangle configuration and let $\underline{T}''$ be a subset of $\underline{T}$. We say that a closed interval $[a,b]$ of the real line is $\underline{T}''$-compatible with respect to $\underline{T}$ if each of its endpoints is a root of some triangle in $\underline{T}''$, and the intersection $T \cap (a,b)$ is equal to $\emptyset$, (a,b), or T for each triangle T in $\underline{T}$, moreover, only the first two possibilities can occur in case T belongs to $\underline{T}''$. The following lemma will be essential for the proof of Theorem 1.25.

Lemma 1.26. *Let $\underline{T}''$ be a subset of an arbitrary triangle configuration $\underline{T}$ such that $\Gamma(\underline{T}'')$ consists of a single contour, let $[a,b]$ be a $\underline{T}''$-compatible interval with respect to $\underline{T}$, and let $\underline{T}'$ be the collection of all the triangles in $\underline{T}$ whose bases lie inside of (a,b). If $\Gamma(\underline{T}' \cup \underline{T}'')$ is not a singleton, then $\Gamma(\underline{T})$ is not a singleton either.*

Proof. Since $\Gamma(\underline{T}'')$ is a singleton, it follows from Corollary 1.14 that there is a contour γ_0 in $\Gamma(\underline{T}' \cup \underline{T}'')$ that includes $\underline{T}''$. Then, let us express $\Gamma(\underline{T}' \cup \underline{T}'')$ as

$$\Gamma(\underline{T}' \cup \underline{T}'') = \{\gamma_0, \dots, \gamma_n\} \tag{1.77}$$

where n is a positive integer. Note that $\underline{T}' \cap (\underline{T} \setminus \gamma_0) = \gamma_1 \cup \dots \cup \gamma_n$, so, for that reason, the bases of the triangles $T(\gamma_1), \dots, T(\gamma_n)$ are included in the open interval (a,b). Now, let us consider

$$\Gamma((\underline{T} \setminus \underline{T}') \cup \gamma_0) = \{\gamma'_1, \dots, \gamma'_k\} \tag{1.78}$$

1

where k is a positive integer. In the following, we prove that the identity

$$\Gamma(\underline{T}) = \{\gamma_1, \dots, \gamma_n, \gamma'_1, \dots, \gamma'_k\} \tag{1.79}$$

holds, therefore, the result follows. Such identity can be demonstrated through the application of property (P2) once we show that each pair of distinct elements of the set on the right-hand side of equation (1.79) satisfies properties (P1)(a) and (P1)(b). In order to do so, it is sufficient to verify that any pair of the form γ_i, γ'_j is well-separated.

Without loss, we may assume that γ'_1 is the contour from the right-hand side of equation (1.78) that includes γ_0. Let us show that $\gamma = \gamma_i$ and $\gamma' = \gamma'_1$ satisfy condition (P1)(b) for each $i \geq 1$. Since the base of $T(\gamma_i)$ lies inside of the interval (a, b) and γ'_1 includes γ_0, we have $T(\gamma_i) \subseteq T(\gamma_0) \subseteq T(\gamma'_1)$. In addition to that, using the fact that γ'_1 is a subset of $(\underline{T} \backslash \underline{T}') \cup \gamma_0$, it is straightforward to prove that each triangle T in γ'_1 satisfies either $T(\gamma_i) \subseteq T$ or $T(\gamma_i) \cap T = \emptyset$. Furthermore, the inequality

$$\operatorname{dist}(\gamma_i, \gamma'_1) = \operatorname{dist}(\gamma_i, \gamma_0) > c|\gamma_i|^3$$

holds. Now, let us prove that $\gamma = \gamma_i$ and $\gamma' = \gamma'_j$ satisfy conditions (P1)(a) and (P1)(b) whenever $i \geq 1$ and $j \geq 2$. The proof of the first part of these conditions follows directly from the fact that γ'_j is included in $\underline{T} \backslash \underline{T}'$ while the base of $T(\gamma_i)$ is included in (a, b). Therefore, both in the case where $T(\gamma_i) \subseteq T(\gamma'_j)$ and in the case where $T(\gamma_i) \cap T(\gamma'_j) = \emptyset$, we have

$$\operatorname{dist}(\gamma_i, \gamma'_j) \geq \operatorname{dist}(\gamma_i, \gamma_0) > c|\gamma_i|^3 \geq c \cdot |\gamma_i|^3 \wedge |\gamma'_j|^3.$$

■

The main lesson we extract from Lemma 1.26 is that a compatible interval $[a, b]$ plays the role of a protection for the contours of $\underline{T}' \cup \underline{T}''$ which are "surrounded" by the open interval (a, b), in the sense that, under the same assumptions as those mentioned above, if $\Gamma(\underline{T}' \cup \underline{T}'')$ can be written as in equation (1.77), then, according to the proven identity (1.79), the contours γ's of $\underline{T}' \cup \underline{T}''$ for which the base of $T(\gamma)$ lies inside of (a, b), namely $\gamma_1, \dots, \gamma_n$, will remain preserved among the other contours associated to the full triangle configuration $\underline{T}$.

Proof of Theorem 1.25(a). Let a and b be the left and right endpoints of S, respectively. Let us consider $\underline{T}'' = \{S\}$ and define $\underline{T}'$ as the set of all triangles in $\underline{T}$ whose bases lie inside of the interval (a, b), then, it is straightforward to verify that $[a, b]$ is $\underline{T}''$-compatible with respect to $\underline{T}$ and $[\underline{T}]_S = \underline{T}' \cup \underline{T}''$. Since $\Gamma(\underline{T})$ is a singleton, then by using Lemma 1.26, we conclude that $\Gamma([\underline{T}]_S)$ also is a singleton. ■

Proof of Theorem 1.25(b). Note that if $k = 1$, we must have $\operatorname{dist}(S, \gamma_1) \leq c|\gamma_1|^3$, otherwise we would have a contradiction with the fact that $\Gamma([\underline{T}]_S)$ consists of a single contour. It follows that $1 \leq x_l(\gamma_1) - x_l(S) \leq c|\gamma_1|^3$ or $1 \leq x_r(S) - x_r(\gamma_1) \leq c|\gamma_1|^3$, thus, in case the first (resp. second) inequality holds, then equations (1.75) and (1.76) hold for $p = 1$ (resp. $p = 0$).

Now, let us suppose that $k \geq 2$. Let us start by proving that the contours can be arranged in such a way that $T(\gamma_1), \dots, T(\gamma_k)$ is a sequence of disjoint triangles placed from the left to the right. Indeed, suppose that there is a pair γ_i, γ_j of distinct contours such that $T(\gamma_i) \subseteq T(\gamma_j)$. Note that such contours can be chosen in such a way that $T(\gamma_i)$ is minimal among all the triangles $T(\gamma_1), \ldots, T(\gamma_k)$, and $T(\gamma_j)$ is the smallest one that strictly includes $T(\gamma_i)$. Let us define a (resp. b) as the largest (resp. smallest) endpoint of a triangle from γ_j located to the left (resp. right) of $T(\gamma_i)$, then it is straightforward to show that $[a,b]$ is a γ_j-compatible interval with respect to $[\underline{T}]_S$. So, if we let $\underline{T}'$ be the set of all triangles in $[\underline{T}]_S$ included inside of (a,b), then, $\Gamma(\underline{T}' \cup \gamma_j)$ would consist of γ_j and all the other γ's for which $T(\gamma)$ is included in (a,b). Thus, according to Lemma 1.26, $\Gamma([\underline{T}]_S)$ would not be a singleton, a contradiction. Therefore, we may assume that $T(\gamma_1), \ldots, T(\gamma_k)$ are indexed from the left to the right.

Claim 1.27. *If we assume that $k \geq 2$, then the following properties hold.*

(a) For each i such that $1 \leq i \leq k$,

$$a_i - b_{i-1} > c|\gamma_i|^3 \textit{ implies } a_{i+1} - b_i \leq c|\gamma_i|^3 \tag{1.80}$$

and

$$a_{i+1} - b_i > c|\gamma_i|^3 \textit{ implies } a_i - b_{i-1} \leq c|\gamma_i|^3. \tag{1.81}$$

(b) For every i that satisfies $2 \leq i \leq k-1$,

$$a_i - b_{i-1} \leq c|\gamma_{i-1}|^3 \textit{ implies } a_{i+1} - b_i \leq c|\gamma_i|^3 \tag{1.82}$$

and

$$a_{i+1} - b_i \leq c|\gamma_{i+1}|^3 \textit{ implies } a_i - b_{i-1} \leq c|\gamma_i|^3. \tag{1.83}$$

Proof of Claim 1.27. Let us start by proving implication (1.80). In order to do that, let us show that if inequalities $a_i - b_{i-1} > c|\gamma_i|^3$ and $a_{i+1} - b_i > c|\gamma_i|^3$ hold, then we would have a contradiction with the fact that $\Gamma([\underline{T}]_S)$ consists of a single contour. In the case where $i = 1$, we would have

$$x_l(\gamma_1) - x_l(S) > c|\gamma_1|^3$$

and

$$x_l(\gamma_2) - x_r(\gamma_1) > c|\gamma_1|^3,$$

thus, the inequality $\mathrm{dist}(\gamma_1, [\underline{T}]_S \backslash \{\gamma_1\}) > c|\gamma_1|^3$ follows, contradicting the fact that $\Gamma([\underline{T}]_S)$ is the finest partition of $[\underline{T}]_S$ into well-separated triangle configurations. Similarly, in case $i = k$, we would have

$$x_l(\gamma_k) - x_r(\gamma_{k-1}) > c|\gamma_k|^3$$

and

$$x_r(S) - x_r(\gamma_k) > c|\gamma_k|^3,$$

that is, $\mathrm{dist}(\gamma_k, [\underline{T}]_S \backslash \{\gamma_k\}) > c|\gamma_k|^3$ holds, and by means of an analogous argument, we also derive a contradiction. Now, if $1 < i < k$, then we would have inequalities

$$x_l(\gamma_i) - x_r(\gamma_{i-1}) > c|\gamma_i|^3$$

and

$$x_l(\gamma_{i+1}) - x_r(\gamma_i) > c|\gamma_i|^3,$$

which would imply that $\operatorname{dist}(\gamma_i, [\underline{T}]_S \setminus \{\gamma_i\}) > c|\gamma_i|^3$, again resulting in the same kind of contradiction. By using the fact that implication (1.81) follows immediately from (1.80), we conclude the proof of part (a).

Now, let us prove implication (1.82). Suppose that $x_l(\gamma_i) - x_r(\gamma_{i-1}) \le c|\gamma_{i-1}|^3$, in other words, let us assume that $\operatorname{dist}(\gamma_{i-1}, \gamma_i) \le c|\gamma_{i-1}|^3$. Since $\operatorname{dist}(\gamma_{i-1}, \gamma_i) > c \cdot |\gamma_{i-1}|^3 \wedge |\gamma_i|^3$, it follows that $\operatorname{dist}(\gamma_{i-1}, \gamma_i) > c|\gamma_i|^3$, that is,

$$x_l(\gamma_i) - x_r(\gamma_{i-1}) > c|\gamma_i|^3.$$

Thus, by using implication (1.80), we have $x_l(\gamma_{i+1}) - x_r(\gamma_i) \le c|\gamma_i|^3$. The reader can easily check that by means of an analogous argument implication (1.83) also follows. □

Let us consider the case where inequalities $a_1 - b_0 \le c|\gamma_1|^3$ and $a_{k+1} - b_k \le c|\gamma_k|^3$ hold simultaneously. Let p be defined as the largest number satisfying $1 \le p \le k$ such that $a_i - b_{i-1} \le c|\gamma_i|^3$ holds whenever $1 \le i \le p$. In case $p = k$, inequalities (1.75) and (1.76) follow immediately. Now, if $p < k$, then $a_{p+1} - b_p > c|\gamma_{p+1}|^3$, and by using Claim 1.27(a), we have $a_{p+2} - b_{p+1} \le c|\gamma_{p+1}|^3$. Through the application of Claim 1.27(b), we obtain inequalities (1.75) and (1.76).

Now, let us consider the cases where $a_1 - b_0 > c|\gamma_1|^3$ or $a_{k+1} - b_k > c|\gamma_k|^3$. If the first inequality holds, then by means of implications (1.80) and (1.82), we have $a_{i+1} - b_i \le c|\gamma_i|^3$ for each i such that $1 \le i \le k-1$. Analogously, by using implications (1.81) and (1.83), the second inequality implies that $a_i - b_{i-1} \le c|\gamma_i|^3$ holds for each i such that $2 \le i \le k$. Note that conditions $a_1 - b_0 > c|\gamma_1|^3$ and $a_{k+1} - b_k > c|\gamma_k|^3$ cannot hold simultaneously, since otherwise that would imply that $\operatorname{dist}(\gamma_1, \gamma_2) \le c \cdot |\gamma_1|^3 \wedge |\gamma_2|^3$, a contradiction. So, necessarily either $a_1 - b_0 > c|\gamma_1|^3$ and $a_{k+1} - b_k \le c|\gamma_k|^3$ hold or $a_1 - b_0 \le c|\gamma_1|^3$ and $a_{k+1} - b_k > c|\gamma_k|^3$ hold. If the first condition is satisfied, then inequalities (1.75) and (1.76) are fulfilled by letting $p = 0$, while if the second condition is verified, then such inequalities are satisfied for $p = k$. ■

1.5.2. SQUARE CONFIGURATIONS AT TIME $t+1$

Let us assume that once a nonempty triangle configuration $\underline{T}$ such that $\Gamma(\underline{T})$ consists of a unique contour has been set from the beginning, we generated its square configuration $\underline{S}$ corresponding to time t whose elements are pairwise disjoint squares (at time t) constructed in such a way that the endpoints of their bases coincide with roots of triangles of $\underline{T}$, moreover, we also assume that the collection of the clusters

$$[\underline{T}]_S = \{T \in \underline{T} : T \text{ is a subset of } S\} \tag{1.84}$$

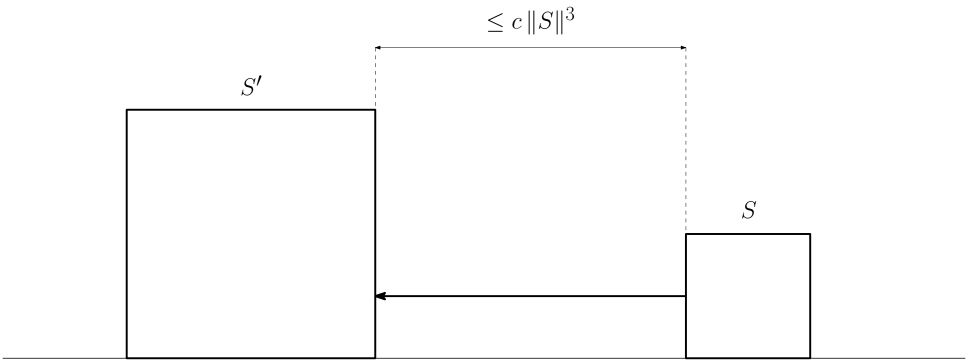
indexed by $\underline{S}$ forms a partition of $\underline{T}$ and each $\Gamma([\underline{T}]_S)$ is a singleton.

Our goal in this section is to generate a new configuration of squares (which will be associated to time $t+1$) by using those corresponding to time t in such a manner that, analogously as in the previous step, the new collection of clusters indexed by the squares at time $t+1$ defines a partition of $\underline{T}$, and each new cluster generates a single contour. In order to construct such a new configuration, let us start by introducing an intermediate step where we establish a procedure that allow us to identify the objects that would be eligible to be the squares at time $t+1$, the so-called protosquares.

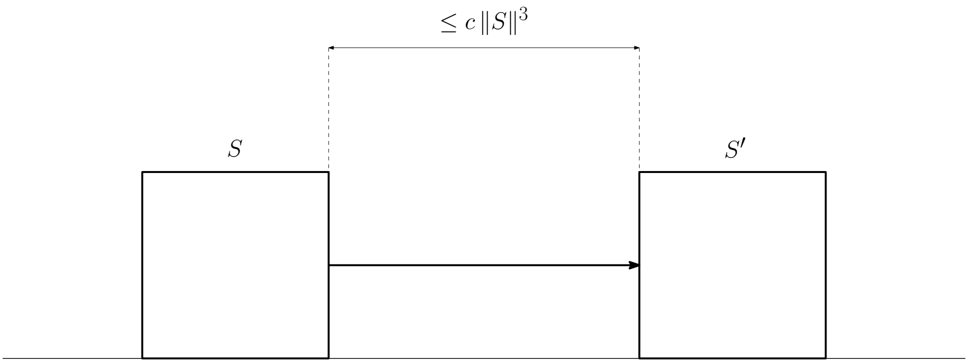
First, let us make clear what we mean when we refer to the mass of a square (at time t) and the distance between two such objects. Since the endpoints of the base of a square (at time t) are supposed to be attached to interface points, then we can naturally extend the notion of distance between triangles to such objects. In particular, given a pair S, S' of distinct squares in $\underline{S}$, their distance $\mathrm{dist}(S, S')$ will be simply given by the number of integers between them. Keeping in mind that the initial triangle configuration $\underline{T}$ is fixed from the beginning, let us define the mass of a square S (at time t) by the expression

$$\|S\| = \sum_{T \in \underline{T}, T \subseteq S} |T|, \tag{1.85}$$

in other words, the mass $\|S\|$ is defined as the sum of the masses of all triangles of $\underline{T}$ included in S. Note that we are using different symbols $|\cdot|$ and $\|\cdot\|$ to discriminate masses of triangles and squares, since the second one does not represent an intrinsic quantity, in the sense that, it describes the total mass of a cluster of triangles and its expression may vary depending on the fixed initial triangle configuration $\underline{T}$.



(a) Case $\|S\| < \|S'\|$ and $\mathrm{dist}(S, S') \leq c\|S\|^3$.



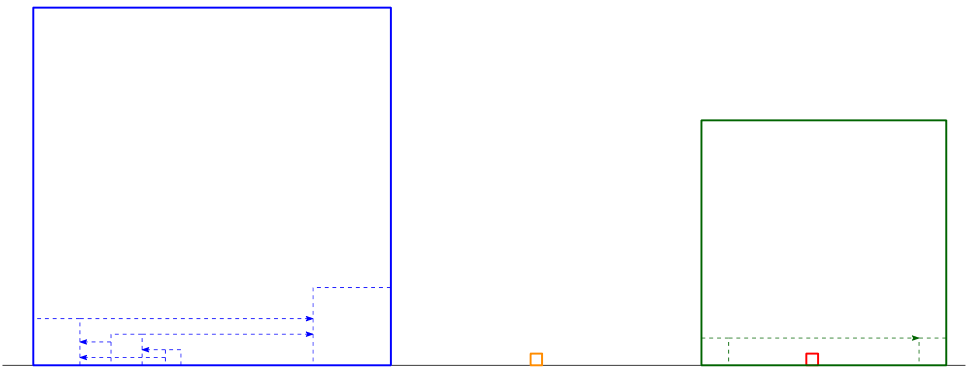
(b) Case $\|S\| = \|S'\|$ and $\mathrm{dist}(S, S') \leq c\|S\|^3$.

Figure 1.19: Squares S and S' connected by an arrow.

Now, let us describe the construction of protosquares. Let $\underline{S}$ be the square configuration at time t, then let us join a pair S, S' of distinct squares in $\underline{S}$ by connecting them through an arrow (S, S') oriented from S to S' whenever $\|S\| \le \|S'\|$ and $\mathrm{dist}(S, S') \le c\|S\|^3$, where in case the equality $\|S\| = \|S'\|$ holds we keep only the arrow oriented from the left to the right, see Figure 1.19. In addition, we define the shadow of the arrow (S, S') as the (bounded) closed interval determined by these two squares. After that, we say that two squares S and S' are arrow-connected if there exists a sequence $S_0, \dots, S_k$ of elements of $\underline{S}$ such that $S = S_0$ and $S' = S_k$ where the consecutive squares S_i and S_{i+1} are joined by an arrow (with any orientation). So, corresponding to each arrow-connected component, we define its associated protosquare as the square whose base is the smallest closed interval that includes the bases of all squares of that component, see Figure 1.20.

(a) Squares at time t.

(b) Connected components generated by squares connected by arrows.



(c) Protosquares.

Figure 1.20: Construction of protosquares.

Remark 1.28. According to the rules of establishing arrows we defined above, the reader can easily verify that if $\underline{S}$ contains at least two squares, then, there exists at least one pair of squares that can be joined by an arrow, otherwise we would contradict the fact that $\underline{T}$ generates a unique contour. The main consequence of this fact is that the number of protosquares (therefore, the number of squares at time $t+1$) will be strictly smaller than the number of squares at time t, so, after a finite number of iterations of this mechanism of generating square configurations, this process must stabilize after we reach the point where all the squares collapse into a single square.

Now, our next step is to show that the protosquares are either disjoint or one is included in the other, so, for that reason, the criterion that will be chosen to decide whether a protosquare is eligible or not to be a square at time $t+1$ will be based on its maximality. In order to prove the claim we just posed, it is convenient to erase some arrows in such a proper way that the new notion of arrow-connectivity keeps all the properties and objects we have obtained so far unchanged. For each square S in $\underline{S}$, let us remove all the arrows emanating from it oriented to the right (resp. to the left) keeping only the arrow whose shadow has the smallest length. Such remaining arrows will be referred to as new arrows, and analogously as defined before, we say that a pair of distinct squares of $\underline{S}$ are new arrow-connected if there is a path connecting them such that any two consecutive squares from this path are joined by a new arrow (with any orientation). The next claims synthesizes the properties of such objects.

Claim 1.29. *Two squares are new arrow-connected if and only if they are arrow-connected.*

Proof. Note that if two squares are new arrow-connected, then it immediately follows that they are arrow-connected. On the other hand, let us suppose that there is a pair of squares that are arrow-connected, but are not new arrow-connected. It is straightforward to verify that such a assumption implies that there must exist a pair of squares that can be joined by an arrow, but are not new arrow-connected. Following the same terminology as the one used in [8], we may refer to such a kind of pair as an odd pair. The proof of our claim is completed by proving that for each odd pair S, S' there is a square S'' between S and S' such that either S, S'' is odd or S'', S' is odd, which implies on the existence of an arbitrarily large number of odd pairs, a contradiction . Without loss of generality, we may assume that S is to the left of S' and they are connected by an arrow that goes from S to S'. Note that the arrow (S, S') is not a new arrow, so, it follows that there is a square S'' between S and S' such that (S, S'') is the new arrow emanating from S which is oriented to the right. In case $\|S''\| \leq \|S'\|$, we obtain inequalities

$$\mathrm{dis}(S'', S') < \mathrm{dis}(S, S') \leq c\|S\|^3 \leq c\|S''\|^3,$$

which means that there is an arrow from S'' to S'. Note that such squares are not new arrow-connected, otherwise we would contradict the fact that S, S' is an odd pair, therefore, we conclude that S'', S' is odd. Now, in case $\|S'\| < \|S''\|$, we have

$$\mathrm{dis}(S', S'') < \mathrm{dis}(S, S') \leq c\|S\|^3 \leq c\|S'\|^3,$$

thus, by means of an analogous argument, we conclude that S'' and S' define an odd pair. ∎

Claim 1.30. *The shadows of two new arrows are either disjoint or one is included in the other.*

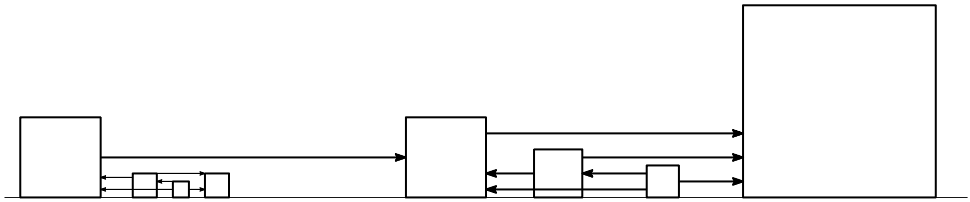
Proof. Let S_1, S_2, S_3, and S_4 be squares indexed from the left to the right, where S_1 and S_3 are connected by a new arrow as well as S_2 and S_4. Let us consider the case where (S_1, S_3)

and (S_2, S_4) are new arrows. Note that $\mathrm{dist}(S_1, S_2) < \mathrm{dist}(S_1, S_3) \le c\|S_1\|^3$ and (S_1, S_2) is cannot be an arrow, so, it follows that $\|S_1\| > \|S_2\|$. Analogously, using the fact that $\mathrm{dist}(S_2, S_3) < \mathrm{dist}(S_2, S_4) \le c\|S_2\|^3$ and (S_2, S_3) is not an arrow, we also obtain $\|S_2\| > \|S_3\|$. Hence, in that case we would have $\|S_1\| > \|S_3\|$, contradicting our assumption that there is an arrow from S_1 to S_3. Now, let us suppose that (S_1, S_3) and (S_4, S_2) are new arrows. As before, we have $\|S_1\| > \|S_2\|$, moreover, since $\mathrm{dist}(S_4, S_3) < \mathrm{dist}(S_4, S_2) \le c\|S_4\|^3$ and (S_4, S_3) is not an arrow, it follows that $\|S_4\| > \|S_3\|$. Therefore, taking into account that $\|S_1\| \le \|S_3\|$ and $\|S_4\| < \|S_2\|$, we derive a contradiction. The treatment of the remaining two cases is done in a similar way, thus, we conclude the proof of our claim. ■

Recall that to each (new) arrow-connected component we assigned the smallest square containing it, a so-called protosquare, so, as the reader can easily check, Claim 1.30 implies that whenever we have two distinct protosquares, they are either disjoint or one is included in the other. Therefore, let us define the configuration of squares associated to time $t+1$ as the set of all maximal protosquares generated from squares at time t. Note that the maximality of such objects implies that they are pairwise disjoint and the new collection of clusters of the form (1.84) indexed by the squares at time $t+1$ defines a partition of $\underline{T}$. Now, it only remains to show that these new clusters originate single contours. In order to so, let us investigate further how the squares at time t are arranged inside their corresponding square at time $t+1$.

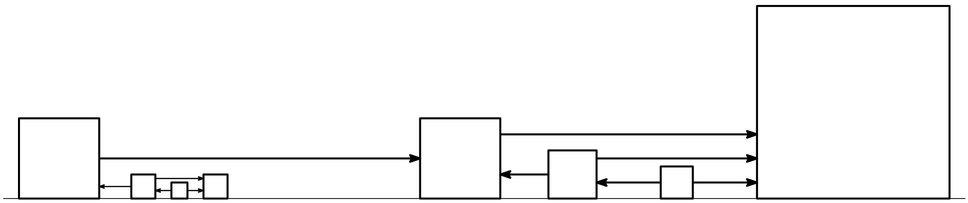
Keeping in mind the claims we just proved above, let us refer to the new arrows with maximal shadows as primary arrows, where we classify their endpoints as primary squares. Furthermore, let us classify the new arrows which are not primary as secondary arrows. The reader can easily verify that a new arrow is secondary if and only if its shadow is strictly included in the shadow of a primary arrow. So, for that reason, we also categorize the squares whose bases lie in the shadow of a primary arrow as secondary squares. See Figure 1.21.

Let S_1 and S_2 be two squares joined by a primary arrow, where S_1 is located to the left of S_2. Then, let us denote by $\underline{T}'$ the set of all triangles in $\underline{T}$ whose bases lie in the shadow of the primary arrow connecting S_1 and S_2, and define the configuration $\underline{T}''$ by letting $\underline{T}'' = [\underline{T}]_{S_1} \cup [\underline{T}]_{S_2}$. In part (a) of Theorem 1.31 we show that the set $\underline{T}' \cup \underline{T}''$, whose elements are all the triangles in $\underline{T}$ located between S_1 and S_2, or included in the base of S_1 or S_2, generates a unique contour, while in part (b) we provide a quantitative description of the relative positions of the triangles placed in between these primary squares.

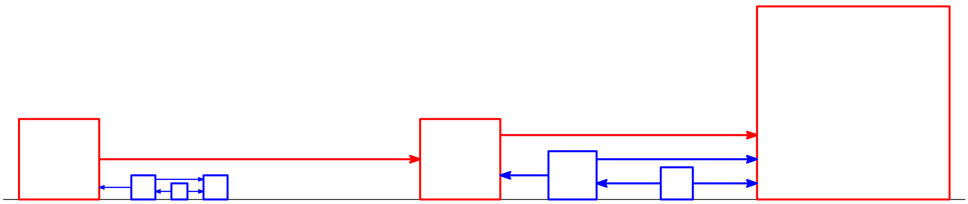


(a) Arrow-connected component with "old" arrows.

Figure 1.21: Determination of primary (resp. secondary) arrows and squares.



(b) Arrow-connected component with new arrows.



(c) Primary arrows and squares (in red) and secondary arrows and squares (in blue).

Figure 1.21: Determination of primary (resp. secondary) arrows and squares.

Theorem 1.31. *Let $\underline{T}$ be a nonempty triangle configuration such that $\Gamma(\underline{T})$ is a singleton, and let S_1 and S_2 be squares at time t joined by a primary arrow, where S_1 is located to the left of S_2. Using the notation introduced above, we have the following properties.*

(a) $\Gamma(\underline{T}' \cup \underline{T}'')$ *is a singleton.*

(b) If $\underline{T}'$ is nonempty, then $\Gamma(\underline{T}')$ consists of contours $\gamma_1, \dots, \gamma_k$ labeled in such a way that $T(\gamma_1), \dots, T(\gamma_k)$ is a sequence of disjoint triangles ordered from the left to the right for which there exists p satisfying $0 \le p \le k$ such that

$$1 \le a_i - b_{i-1} \le c|\gamma_i|^3 \quad \text{holds if } 1 \le i \le p, \text{ and} \tag{1.86}$$

$$1 \le a_{i+1} - b_i \le c|\gamma_i|^3 \quad \text{holds if } p+1 \le i \le k, \tag{1.87}$$

where $b_0 = x_r(S_1)$, $a_i = x_l(\gamma_i)$, $b_i = x_r(\gamma_i)$, and $a_{k+1} = x_l(S_2)$.

Proof. Let us start by proving part (a). Recall that S_1 and S_2 are squares constructed at time t in such a way that $\Gamma([\underline{T}]_{S_1})$ and $\Gamma([\underline{T}]_{S_2})$ are singletons, moreover, since they are connected by an arrow, it follows that their distance satisfies $\mathrm{dist}(S_1, S_2) \le c \cdot \|S_1\|^3 \wedge \|S_2\|^3$. By applying Corollary 1.14, we conclude that $\Gamma(\underline{T}'')$ is a singleton. If we define a and b as the rightmost endpoint of S_1 and the leftmost endpoint of S_2 respectively, then, it is straightforward to show that the interval $[a, b]$ is $\underline{T}''$-compatible with respect to $\underline{T}$. Thus, Lemma 1.26 implies that $\Gamma(\underline{T}' \cup \underline{T}'')$ is a singleton. The proof of part (b) can be omitted since it is very similar to the proof of Theorem 1.25(b). ■

Finally, let us show that for every square S associated to time $t+1$ its corresponding cluster of triangles $[\underline{T}]_S$ originates a single contour. Let S' and S'' be respectively

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the leftmost and the rightmost squares at time t which are included in S. Note that if $S' = S''$, our assertion follows immediately, so, let us suppose that S' and S'' are distinct. By using Claim 1.30 and the fact that S is a maximal smallest square that includes a (new) arrow-connected component, it is straightforward to show that there is a chain of squares $S_1, \ldots, S_n$ from $S' = S_1$ to $S'' = S_n$ where the consecutive squares S_i and S_{i+1} are joined by a primary arrow. Let us consider the sequence of triangle configurations $\underline{T}^{(1)}, \ldots, \underline{T}^{(n-1)}$, where each $\underline{T}^{(i)}$ is the set of all triangles in $\underline{T}$ which are located between S_i and S_{i+1}, or that belong to $[\underline{T}]_{S_i} \cup [\underline{T}]_{S_{i+1}}$. Note that the union of all such $\underline{T}^{(i)}$'s coincides with $[\underline{T}]_S$, moreover, according to Theorem 1.31(a), each $\Gamma(\underline{T}^{(i)})$ is a singleton. Therefore, by using the monotonic property of contours from Corollary 1.14, we prove that $\Gamma([\underline{T}]_S)$ also is a singleton.

1.5.3. CONSTRUCTION OF TREES

Let $\underline{T}$ be a nonempty triangle configuration such that $\Gamma(\underline{T})$ is a singleton. Once such a configuration is fixed, by repeatedly applying the process we described in the previous sections, we obtain a sequence $\underline{S}^{(0)}, \underline{S}^{(1)}, \underline{S}^{(2)}, \ldots$ of configurations of squares, where $\underline{S}^{(t)}$ stands for the square configuration corresponding to time t. According to Remark 1.28, the first time t for which $\underline{S}^{(t)}$ consists of a unique square is well defined, so, let us denote it by $\tau(\underline{T})$, or simply by τ in case the initial triangle configuration is understood from the context.

For every such a configuration $\underline{T}$ we will associate a tree whose vertices are classified as heavy triangles or spheres. This nomenclature is adopted so that the reader does not confuse the nodes of the tree with the triangles from proper triangles configurations. The heavy triangles can be colored black or white. Black triangles are the only nodes that can generate an offspring, where the members of such an offspring fit into any of these categories we just mentioned, moreover, the spheres can only be located between two heavy triangles generated by the same parent or inside of a white triangle.

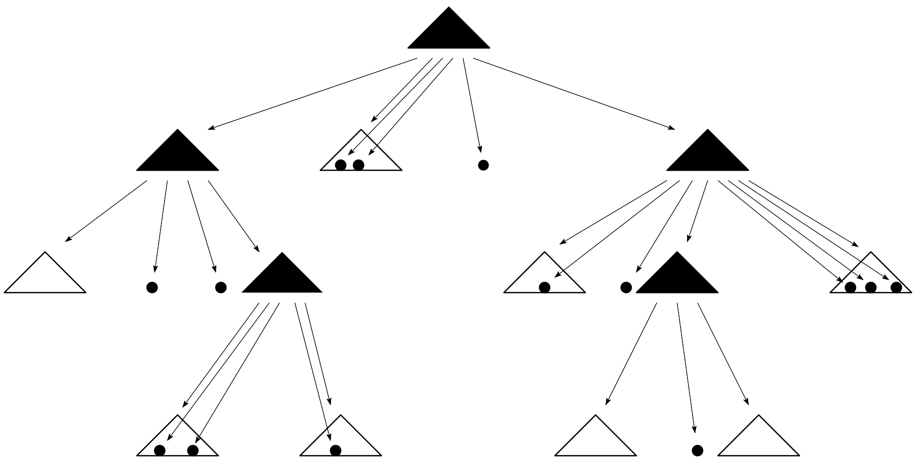
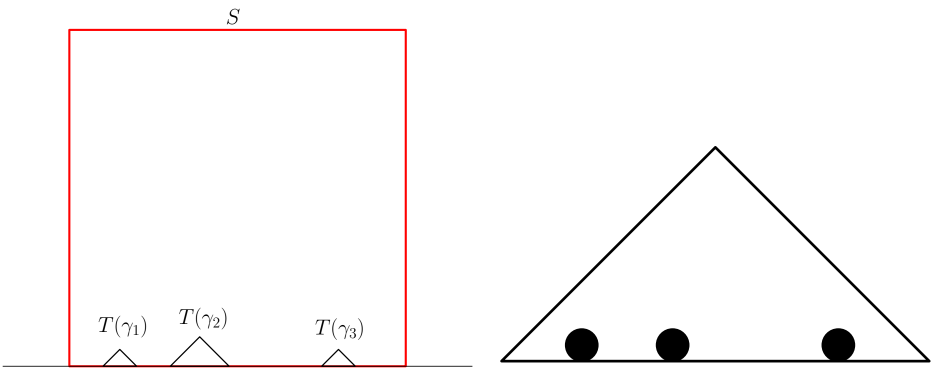


Figure 1.22: General picture of a tree associated to $\underline{T}$.

Let us suppose that $\tau = 0$. It follows that the square configuration $\underline{S}^{(0)}$ has only one element, this means that $\underline{T}$ consists of a unique maximal triangle T possibly with some triangles inside of it. Part (b) of Theorem 1.25 implies that $\underline{T}$ can be split into the maximal triangle T and triangle configurations $\gamma_1, \ldots, \gamma_k$, where k is a nonnegative integer and each $\Gamma(\gamma_i)$ is a singleton, moreover, their distances must satisfy relations (1.65)-(1.68) for some integer p such that $0 \le p \le k$. Let us represent the maximal triangle T by a white heavy triangle and the contours γ_i's by spheres. So, the resulting picture we associate to $\underline{T}$ is a tree consisting of a single root, which is a white heavy triangle with k spheres attached to it.

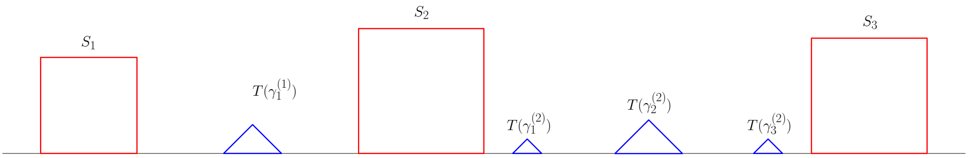


(a) The unique square at $t = 0$ associated to $\underline{T}$ and the contours generated by $[\underline{T}]_S \setminus \{S\}$.

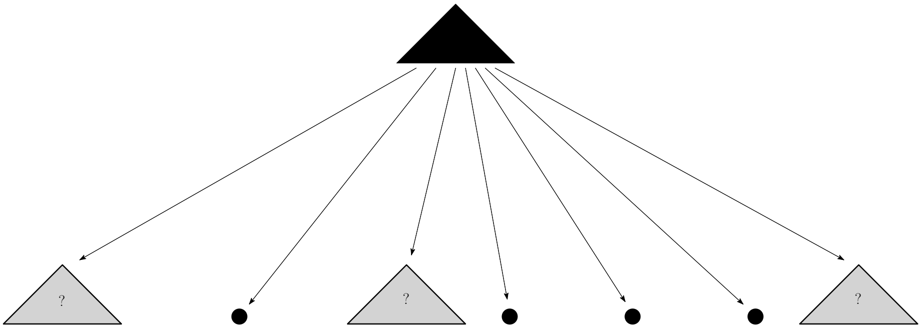
(b) White triangle with spheres attached to it.

Figure 1.23: Construction of trees in the case $\tau = 0$.

Now, if we suppose that $\tau > 0$, then the square configuration at time $\tau - 1$ consists of $n \ge 2$ primary squares $S_1, \ldots, S_n$, which are assumed to be labeled from the left to the right and the squares S_i and S_{i+1} are joined by a primary arrow, with secondary squares lying in the shadows of primary arrows. Recall that the cluster $\gamma_i = [\underline{T}]_{S_i}$ corresponding to the primary square S_i generates a single contour and the distance between two consecutive clusters γ_i and γ_{i+1} satisfy the inequality (1.69). Furthermore, it follows from part (b) of Theorem 1.31 that the set of triangles of $\underline{T}$ that are located in between S_i and S_{i+1} generate $k_i \ge 0$ contours, say $\gamma_1^{(i)}, \ldots, \gamma_{k_i}^{(i)}$, whose relative distances satisfy relations (1.70)-(1.73) for some integer p_i such that $0 \le p_i \le k_i$. In the tree representation, each γ_i will be associated to a heavy triangle (black or white), while the contours $\gamma_1^{(i)}, \ldots, \gamma_{k_i}^{(i)}$ will give rise to the spheres placed between the heavy triangles corresponding to γ_i and γ_{i+1} . So, the tree that represents $\underline{T}$ is drawn by starting from its root which we agree will be a black triangle. Its offspring consists of n heavy triangles, whose colors cannot yet be determined, and k_i spheres between the i-th and $(i+1)$-th heavy triangles. In order to determine whether such heavy triangles are black or white, we apply the same process to each cluster $\gamma_i = [\underline{T}]_{S_i}$ and exhaustively construct the remaining parts of the tree until all its leaves are either spheres or white triangles.



(a) First we identify (in red) the primary squares at time $\tau - 1$, and then recognize the contours (in blue) that will give rise to the spheres.



(b) Each cluster $\gamma_i = [\underline{T}]_{S_i}$ is represented by a heavy triangle and the remaining contours of the form $\gamma_j^{(i)}$ will be represented by spheres that lie between the heavy triangles. Note that the colors of the heavy triangles are still unknown.

Figure 1.24: First step of the construction of the tree corresponding to a configuration $\underline{T}$ with $\tau > 0$.

1.5.4. COUNTING CONTOURS

In order to prove Theorem 1.23 it is convenient to embed the set of all possible contours in a larger set $\mathfrak{C}$ which is simpler to deal with. The main feature of $\mathfrak{C}$ is that it is invariant under translations by integer numbers, moreover, all the intrinsic properties of contours (such as their masses, weights, and tree structures) can be naturally extended to the elements of $\mathfrak{C}$. After that, such an embedding will be used to reduce the proof of inequality (1.63) to the computation of an upper bound for a simpler quantity.

Let us start by introducing some notation. For each triangle configuration γ such that $\Gamma(\gamma)$ is a singleton, let us define Z_γ as the collection whose elements are the sets of integer points that belong to each triangle in γ, that is,

$$Z_\gamma = \{T \cap \mathbb{Z} : T \in \gamma\}, \tag{1.88}$$

moreover, we associate to each set of the form (1.88) its translate by an integer i whose expression is given by

$$Z_\gamma + i = \{(T \cap \mathbb{Z}) + i : T \in \gamma\}. \tag{1.89}$$

Note that if the identity $Z_\gamma + i = Z_{\gamma'} + j$ holds, then, each triangle T in γ, say $T = \Delta(r_x, r_y)$, can be associated to a triangle $T' = \Delta(r_{x+i-j}, r_{y+i-j})$ in γ'. It immediately follows that such a map $T \mapsto T'$ is a one-to-one correspondence from γ onto γ' that preserves the masses of the triangles and their relative distances, therefore, the trees derived from to γ and γ', whose constructions were described in the previous sections, have the same structure.

As we mentioned in the beginning of this section, let us show that the set of all possible contours (in other words, the set of all γ's for which $\Gamma(\gamma) = \{\gamma\}$) can be embedded in the set $\mathfrak{C}$ defined by

$$\mathfrak{C} = \{\Upsilon : \Upsilon = Z_\gamma + i \text{ for some } \gamma \text{ such that } \Gamma(\gamma) \text{ is a singleton and some integer } i\}. \tag{1.90}$$

It follows from the first part of the remark above that the map $\gamma \mapsto Z_\gamma$ is one-to-one. By using the same remark, the masses and weights w_β^α's associated to contours can be unambiguously extended to the elements of $\mathfrak{C}$ in the way we describe as follows. Let Υ be an element of $\mathfrak{C}$ that can be written as $\Upsilon = Z_\gamma + i$. Then, let us define its mass $|\Upsilon|$ and its weight $w_\beta^\alpha(\Upsilon)$ by letting $|\Upsilon| = |\gamma|$ and $w_\beta^\alpha(\Upsilon) = w_\beta^\alpha(\gamma)$. Furthermore, we may also say that the root of Υ is white (resp. black) if the root of γ is white (resp. black).

By using the fact that the map $\gamma \mapsto Z_\gamma$ is one-to-one, the contours' weights summed over all possible contours surrounding the origin with mass m can be expressed as

$$\sum_{\substack{\gamma:\,\gamma\odot 0\\ |\gamma|=m}} w_\beta^\alpha(\gamma) = \sum_{\substack{\gamma:\,\bigcup Z_\gamma \text{ contains } 0\\ |Z_\gamma|=m}} w_\beta^\alpha(Z_\gamma). \tag{1.91}$$

Note that the right-hand side of equation (1.91) can be bounded from above by the sum of the weights indexed by all the elements Υ' of $\mathfrak{C}$ with mass m such that one of its elements contains the origin. It is straightforward to show that every such element Υ' can be uniquely expressed in the form $\Upsilon' = \Upsilon + i$ for some integer i and an element Υ of $\mathfrak{C}_0(m)$, where $\mathfrak{C}_0(m)$ is given by

$$\mathfrak{C}_0(m) = \{\Upsilon \in \mathfrak{C} : |\Upsilon| = m \text{ and } \min(\bigcup \Upsilon) = 0\},$$

that is, $\mathfrak{C}_0(m)$ is defined as the set of all elements of $\mathfrak{C}$ with mass m whose leftmost site surrounded by it is the origin. It follows from the comments above that

$$\sum_{\substack{\gamma:\,\gamma\odot 0\\ |\gamma|=m}} w_\beta^\alpha(\gamma) \le \sum_{\substack{\Upsilon' \in \mathfrak{C}:\,\bigcup \Upsilon' \text{ contains } 0\\ |\Upsilon'|=m}} w_\beta^\alpha(\Upsilon') = \sum_{\Upsilon \in \mathfrak{C}_0(m)} \sum_{\substack{i\in\mathbb{Z}\\ \bigcup(\Upsilon+i) \text{ contains } 0}} w_\beta^\alpha(\Upsilon + i).$$

Since w_β^α remains invariant under translation, $\bigcup(\Upsilon + i)$ contains 0 if and only if $\bigcup \Upsilon$ contains $-i$, and $\bigcup \Upsilon$ contains at most m points, we finally conclude that

$$\sum_{\substack{\gamma:\,\gamma\odot 0\\ |\gamma|=m}} w_\beta^\alpha(\gamma) \le m \sum_{\Upsilon \in \mathfrak{C}_0(m)} w_\beta^\alpha(\Upsilon). \tag{1.92}$$

Therefore, according to equation (1.92), the proof of Theorem 1.23 will be concluded once we show that the inequality

$$\sum_{\Upsilon \in \mathfrak{C}_0(m)} w_\beta^\alpha(\Upsilon) \le 2e^{-\beta\chi_\alpha(m)}. \tag{1.93}$$

holds for each positive integer m whenever β is sufficiently large.

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Proof of Theorem 1.23. Let us suppose that inequality (1.93) holds whenever m satisfies $m \leq M-1$. By means of a translation argument, our assumption implies that

$$\sum_{\Upsilon\in\mathfrak{C}_i(m)} w_\beta^\alpha(\Upsilon) = \sum_{\Upsilon\in\mathfrak{C}_0(m)} w_\beta^\alpha(\Upsilon) \leq 2e^{-\beta\chi_\alpha(m)}. \tag{1.94}$$

holds for each integer i and each positive integer m such that $m \leq M-1$, where $\mathfrak{C}_i(m)$ is the translation of $\mathfrak{C}_0(m)$ by i (in other words, $\mathfrak{C}_i(m)$ is the set of all elements of $\mathfrak{C}$ with mass m whose leftmost site surrounded by it is i). Let us split the left-hand side of inequality (1.93) into the sum of two terms $\Theta_\beta^\alpha(M)$ and $\Xi_\beta^\alpha(M)$ given by

$$\Theta_\beta^\alpha(M) = \sum_{\substack{\Upsilon\in\mathfrak{C}_0(M)\\ \text{the root of }\Upsilon\text{ is a}\\ \text{white triangle}}} w_\beta^\alpha(\Upsilon) \tag{1.95}$$

and

$$\Xi_\beta^\alpha(M) = \sum_{\substack{\Upsilon\in\mathfrak{C}_0(M)\\ \text{the root of }\Upsilon\text{ is a}\\ \text{black triangle}}} w_\beta^\alpha(\Upsilon). \tag{1.96}$$

Let us start by finding an upper bound for $\Theta_\beta^\alpha(M)$. Let Υ be an element of $\mathfrak{C}_0(M)$, say $\Upsilon = Z_\gamma + i'$ for some integer i' and some contour γ whose root is a white triangle. Recall that γ can be decomposed as a maximal triangle S together with contours $\gamma_1,\ldots,\gamma_k$, say the mass of S is m_0 and the mass of each γ_i is m_i, such that the γ_i's are arranged sequentially from the left to the right inside of S in such a way that the points $b_0 = x_l(S)$, $a_1 = x_l(\gamma_1)$, $b_1 = x_r(\gamma_1),\ldots,a_k = x_l(\gamma_k)$, $b_k = x_r(\gamma_k)$, and $a_{k+1} = x_r(S)$ satisfy the conditions

$$1 \leq a_i - b_{i-1} \leq cm_i^3 \quad \text{if } 1\leq i\leq p\text{, and} \tag{1.97}$$
$$1 \leq a_{i+1} - b_i \leq cm_i^3 \quad \text{if } p+1\leq i\leq k, \tag{1.98}$$

for some p satisfying $0 \leq p \leq k$. Furthermore, each weight $w_\beta^\alpha(\Upsilon)$ can we written as

$$w_\beta^\alpha(\Upsilon) = e^{-\beta\chi_\alpha(m_0)} w_\beta^\alpha(Z_{\gamma_1})\cdots w_\beta^\alpha(Z_{\gamma_k}). \tag{1.99}$$

If we assume that the masses m_0, m_1, $\ldots$, m_k are fixed as well as the points b_0, a_1, b_1, $\ldots$, a_k, b_k and a_{k+1}, then, the sum of the terms on the right-hand side of (1.99) over the corresponding γ_i's can be bounded by

$$e^{-\beta\chi_\alpha(m_0)}\left(\sum_{\Upsilon_1\in\mathfrak{C}_{a_1+1}(m_1)} w_\beta^\alpha(\Upsilon_1)\right)\cdots\left(\sum_{\Upsilon_k\in\mathfrak{C}_{a_k+1}(m_k)} w_\beta^\alpha(\Upsilon_k)\right) \leq e^{-\beta\chi_\alpha(m_0)}\prod_{i=1}^k 2e^{-\beta\chi_\alpha(m_i)},$$

moreover, the number of possibilities the locations of the a_i's and b_i's can be arranged so that conditions (1.97) and (1.98) are fulfilled is bounded above by

$$(k+1)\left[(cm_1^3)\wedge m_0\right]\cdots\left[(cm_k^3)\wedge m_0\right] = (k+1)\prod_{i=1}^k\left[(cm_i^3)\wedge m_0\right].$$

Therefore, it follows that

$$\Theta_\beta^\alpha(M) \le \sum_{k\ge 0}(k+1) \sum_{\substack{m_0,m_1,\dots,m_k\ge 1\\ m_0+m_1+\cdots+m_k=M}} e^{-\beta\chi_\alpha(m_0)} \prod_{i=1}^{k} 2\left[(cm_i^3)\wedge m_0\right] e^{-\beta\chi_\alpha(m_i)}. \tag{1.100}$$

In the cases where m_0 is the largest number among the m_i's, the inequality

$$e^{-\beta\chi_\alpha(m_0)} \prod_{i=1}^{k} 2\left[(cm_i^3)\wedge m_0\right] e^{-\beta\chi_\alpha(m_i)} \le e^{-\beta\chi_\alpha(m_0)} \prod_{i=1}^{k} 2cm_i^3 e^{-\beta\chi_\alpha(m_i)} \tag{1.101}$$

follows, while if the maximum is attained by m_j for some $j>0$, by using the fact that $(cm_j^3)\wedge m_0 \le cm_0^3$ and $(cm_i^3)\wedge m_0 \le cm_i^3$ holds for each $i \ne j$, we have

$$e^{-\beta\chi_\alpha(m_0)} \prod_{i=1}^{k} 2\left[(cm_i^3)\wedge m_0\right] e^{-\beta\chi_\alpha(m_i)} \le e^{-\beta\chi_\alpha(m_j)} \prod_{\substack{i=0\\ i\ne j}}^{k} 2cm_i^3 e^{-\beta\chi_\alpha(m_i)}. \tag{1.102}$$

Then, by splitting the summation over the m_i's on the right-hand side of (1.100) according to the cases described above and applying (1.101) and (1.102), the inequality

$$\begin{aligned}
\Theta_\beta^\alpha(M) &\le \sum_{k\ge 0}(k+1)^2 \sum_{\substack{m_0,m_1,\dots,m_k\ge 1\\ m_0+m_1+\cdots+m_k=M\\ m_1,\dots,m_k\le m_0}} e^{-\beta\chi_\alpha(m_0)} \prod_{i=1}^{k} 2cm_i^3 e^{-\beta\chi_\alpha(m_i)} \\
&= \sum_{k\ge 0}(k+1)^2 \sum_{\substack{m_0,m_1,\dots,m_k\ge 1\\ m_0+m_1+\cdots+m_k=M\\ m_1,\dots,m_k\le m_0}} e^{-\beta\chi_\alpha(m_0)-(\beta-a)\sum_{i=1}^k \chi_\alpha(m_i)} \prod_{i=1}^{k} 2cm_i^3 e^{-a\chi_\alpha(m_i)}
\end{aligned}$$

holds for any arbitrary positive parameter a. Let us write $\beta = ab$, where b is a positive real number that depends only on α chosen in such a way that

$$e^{-\beta\chi_\alpha(y)-(\beta-a)\sum_{i=1}^k \chi_\alpha(x_i)} \le e^{-\beta\chi_\alpha(x_1+\cdots+x_k+y)} \tag{1.103}$$

holds whenever $x_1,\dots,x_k$, and y are positive integers such that $x_1+\cdots+x_k \le y$. It follows that

$$\begin{aligned}
\Theta_\beta^\alpha(M) &\le e^{-\beta\chi_\alpha(M)} \sum_{k\ge 0}(k+1)^2 \sum_{\substack{m_0,m_1,\dots,m_k\ge 1\\ m_0+m_1+\cdots+m_k=M\\ m_1,\dots,m_k\le m_0}} \prod_{i=1}^{k} 2cm_i^3 e^{-a\chi_\alpha(m_i)} \\
&\le e^{-\beta\chi_\alpha(M)} \left(1+\sum_{k\ge 1}(k+1)^2 \sum_{\substack{m_1,\dots,m_k\ge 1\\ m_1,\dots,m_k\le M-1}} \prod_{i=1}^{k} 2cm_i^3 e^{-a\chi_\alpha(m_i)}\right) \\
&\le e^{-\beta\chi_\alpha(M)} \left(1+\sum_{k\ge 1}(k+1)^2 \sum_{m_1=1}^{M-1}\cdots\sum_{m_k=1}^{M-1} \prod_{i=1}^{k} 2cm_i^3 e^{-a\chi_\alpha(m_i)}\right) \\
&\le e^{-\beta\chi_\alpha(M)} \left[1+\sum_{k\ge 1}(k+1)^2 \left(2c\sum_{m=1}^{\infty} m^3 e^{-a\chi_\alpha(m)}\right)^k\right],
\end{aligned}$$

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where the quantity $\epsilon = \epsilon(a)$ given by

$$\epsilon(a) = 2c \sum_{m=1}^{\infty} m^3 e^{-a\chi_\alpha(m)} \tag{1.104}$$

converges to 0 as the parameter a approaches infinity. Therefore, by choosing a sufficiently large, we conclude that

$$\Theta_\beta^\alpha(M) \leq \frac{3}{2} e^{-\beta\chi_\alpha(M)}. \tag{1.105}$$

Now, let us find an upper bound for $\Xi_\beta^\alpha(M)$. Similarly as before, let Υ be an element of $\mathfrak{C}_0(M)$, say $\Upsilon = Z_\gamma + i'$ for some integer i' and some contour γ whose root is a black triangle. Note that the root of the tree associated to γ generates $n \geq 2$ heavy triangles (black or white), where $k_i \geq 0$ spheres are placed between the i-th and $(i+1)$-th heavy triangle. More precisely, the contour γ can be split into contours $\gamma_1, \ldots, \gamma_n$, $\gamma_1^{(1)}, \ldots, \gamma_{k_1}^{(1)}$, $\ldots$, $\gamma_1^{(n-1)}, \ldots, \gamma_{k_{n-1}}^{(n-1)}$, say each γ_i has mass m_i and each $\gamma_j^{(i)}$ has mass $m_j^{(i)}$, such that the γ_i's correspond to the heavy triangles generated by the root (placed sequentially from the left to the right) whose relative distances are described by

$$1 \leq a_{i+1} - b_i \leq c(m_{i,i+1})^3 \quad \text{for every } 1 \leq i \leq n-1, \tag{1.106}$$

where $a_{i+1} = x_l(\gamma_{i+1})$, $b_i = x_r(\gamma_i)$ and $m_{i,i+1} = m_i \wedge m_{i+1}$; moreover, for each i, the $\gamma_j^{(i)}$'s correspond to the spheres sequentially arranged from the left to the right between $T(\gamma_i)$ and $T(\gamma_{i+1})$ such that for some integer p_i satisfying $0 \leq p_i \leq k_i$ their relative distances are

$$1 \leq a_j^{(i)} - b_{j-1}^{(i)} \leq c(m_j^{(i)})^3 \quad \text{if } 1 \leq j \leq p_i, \text{ and} \tag{1.107}$$

$$1 \leq a_{j+1}^{(i)} - b_j^{(i)} \leq c(m_j^{(i)})^3 \quad \text{if } p_i + 1 \leq j \leq k_i, \tag{1.108}$$

where $b_0^{(i)} = x_r(\gamma_i)$, $a_1^{(i)} = x_l(\gamma_1^{(i)})$, $b_1^{(i)} = x_r(\gamma_1^{(i)})$, $\ldots$, $a_{k_i}^{(i)} = x_l(\gamma_{k_i}^{(i)})$, $b_{k_i}^{(i)} = x_r(\gamma_{k_i}^{(i)})$ and $a_{k_i+1}^{(i)} = x_l(\gamma_{i+1})$. Since the weight $w_\beta^\alpha(\Upsilon)$ can be written as

$$w_\beta^\alpha(\Upsilon) = \left(\prod_{i=1}^{n} w_\beta^\alpha(Z_{\gamma_i})\right)\left(\prod_{j=1}^{k_1} w_\beta^\alpha(Z_{\gamma_j^{(1)}})\right)\cdots\left(\prod_{j=1}^{k_{n-1}} w_\beta^\alpha(Z_{\gamma_j^{(n-1)}})\right),$$

then, once we fix the masses m_i's and $m_j^{(i)}$'s and the positions a_i's, b_i's, $a_j^{(i)}$'s and $b_j^{(i)}$'s, the sum of $w_\beta^\alpha(\Upsilon)$ over all the corresponding contours γ_i's and $\gamma_j^{(i)}$'s can be bounded

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above by

$$\left(\prod_{i=1}^{n}\sum_{\Upsilon_i\in\mathfrak{C}_{a_i+1}(m_1)} w_\beta^\alpha(\Upsilon_i)\right)\left(\prod_{j=1}^{k_1}\sum_{\Upsilon_j^{(1)}\in\mathfrak{C}_{a_j^{(1)}+1}(m_j^{(1)})} w_\beta^\alpha(\Upsilon_j^{(1)})\right)\cdots$$
$$\left(\prod_{j=1}^{k_{n-1}}\sum_{\Upsilon_j^{(n-1)}\in\mathfrak{C}_{a_j^{(n-1)}+1}(m_j^{(n-1)})} w_\beta^\alpha(\Upsilon_j^{(n-1)})\right)\le\left(\prod_{i=1}^{n}2e^{-\beta\chi_\alpha(m_i)}\right)\left(\prod_{j=1}^{k_1}2e^{-\beta\chi_\alpha(m_j^{(1)})}\right)\cdots$$
$$\left(\prod_{j=1}^{k_{n-1}}2e^{-\beta\chi_\alpha(m_j^{(n-1)})}\right).$$

Note that the number of combinations the points a_i's and b_i's can be arranged in such a way that their relative distances satisfy (1.106) is bounded by $\prod_{i=1}^{n-1}c(m_{i,i+1})^3$, moreover, the number of possibilities the points $a_j^{(i)}$'s and $b_j^{(i)}$'s can be arranged in such a way that conditions (1.107) and (1.108) are fulfilled for some p_i such that $0\le p_i\le k_i$ is bounded by $(k_i+1)\prod_{j=1}^{k_i}c[(m_{i,i+1})^3\wedge(m_j^{(i)})^3]$. If we recall that the n heavy triangles generated by the root can be painted as black or white at most in 2^n different ways, then, we conclude that

$$\begin{aligned}\Xi_\beta^\alpha(M)\le\;&\sum_{n\ge 2}2^n\sum_{k_1\ge 0}\cdots\sum_{k_{n-1}\ge 0}\sum_{m_1,\dots,m_n>0}\sum_{m_1^{(1)},\dots,m_{k_i}^{(1)}>0}\cdots\sum_{m_1^{(n-1)},\dots,m_{k_{n-1}}^{(n-1)}>0}\\&\left(\prod_{i=1}^{n-1}c(m_{i,i+1})^3\right)\left(\prod_{i=1}^{n}2e^{-\beta\chi_\alpha(m_i)}\right)\\&\left((k_1+1)\prod_{j=1}^{k_1}c[(m_{1,2})^3\wedge(m_j^{(1)})^3]\right)\left(\prod_{j=1}^{k_1}2e^{-\beta\chi_\alpha(m_j^{(1)})}\right)\cdots\\&\left((k_{n-1}+1)\prod_{j=1}^{k_{n-1}}c[(m_{n-1,n})^3\wedge(m_j^{(n-1)})^3]\right)\left(\prod_{j=1}^{k_{n-1}}2e^{-\beta\chi_\alpha(m_j^{(n-1)})}\right)\\&\mathbb{1}_{\{\sum_{i=1}^n m_i+\sum_{i=1}^{n-1}\sum_{j=1}^{k_i}m_j^{(i)}=M\}}.\end{aligned}\tag{1.109}$$

Similarly as in the previous case, fixed the numbers n, k_1, …, k_{n-1}, let us bound the remaining sums by the sum over the cases where one of the m_i's or one of the $m_j^{(i)}$'s is the largest mass among the others. In case the maximum is reached by m_ℓ, let us apply the inequality

$$\left(\prod_{i=1}^{n-1}c(m_{i,i+1})^3\right)\left(\prod_{i=1}^{n}2e^{-\beta\chi_\alpha(m_i)}\right)\le 2e^{-\beta\chi_\alpha(m_\ell)}\left(\prod_{\substack{i=1\\ i\ne\ell}}^{n}2cm_i^3e^{-\beta\chi_\alpha(m_i)}\right)\tag{1.110}$$

and use the fact that $c[(m_{i,i+1})^3\wedge(m_j^{(i)})^3]\le c(m_j^{(i)})^3$ holds in the remaining terms; and if the maximum is reached by $m_k^{(\ell)}$, then we apply inequalities (1.110),

$$\begin{aligned}
&\left((k_\ell+1)\prod_{j=1}^{k_\ell} c[(m_{\ell,\ell+1})^3 \wedge (m_j^{(\ell)})^3]\right)\left(\prod_{j=1}^{k_\ell} 2e^{-\beta\chi_\alpha(m_j^{(\ell)})}\right) \leq \\
&2cm_\ell^3 e^{-\beta\chi_\alpha(m_k^{(\ell)})}\left((k_\ell+1)\prod_{\substack{j=1\\ j\neq k}}^{k_\ell} 2c(m_j^{(\ell)})^3 e^{-\beta\chi_\alpha(m_j^{(\ell)})}\right),
\end{aligned} \tag{1.111}$$

and again $c[(m_{i,i+1})^3 \wedge (m_j^{(i)})^3] \leq c(m_j^{(i)})^3$ for the remaining terms. After that, we obtain

$$\begin{aligned}
\Xi_\beta^\alpha(M) \quad \leq \quad & \sum_{n\geq 2} 2^n \sum_{k_1\geq 0} \cdots \sum_{k_{n-1}\geq 0} (n+k_1+\cdots+k_{n-1}) \sum_{\substack{m_0,m_1,\ldots,\\ m_{n-1+k_1+\cdots+k_{n-1}}\geq 1\\ \text{such that } m_i\leq m_0}} (k_1+1)\cdots(k_{n-1}+1) \\
& 2e^{-\beta\chi_\alpha(m_0)}\left(\prod_{i=1}^{n-1+k_1+\cdots+k_{n-1}} 2cm_i^3 e^{-\beta\chi_\alpha(m_i)}\right)\mathbb{1}_{\{\sum_{i=0}^{n-1+k_1+\cdots+k_{n-1}} m_i = M\}}.
\end{aligned}$$

Proceeding analogously as we did for Θ_β^α, we have

$$\begin{aligned}
\Xi_\beta^\alpha(M) \quad \leq \quad & e^{-\beta\chi_\alpha(M)}\sum_{n\geq 2} 2^{n+1} \sum_{k_1\geq 0} \cdots \sum_{k_{n-1}\geq 0} (n+k_1+\cdots+k_{n-1})(k_1+1)\cdots(k_{n-1}+1) \\
& \sum_{\substack{m_0,m_1,\ldots,\\ m_{n-1+k_1+\cdots+k_{n-1}}\geq 1\\ \text{such that } m_i\leq m_0}} \left(\prod_{i=1}^{n-1+k_1+\cdots+k_{n-1}} 2cm_i^3 e^{-a\chi_\alpha(m_i)}\right)\mathbb{1}_{\{\sum_{i=0}^{n-1+k_1+\cdots+k_{n-1}} m_i = M\}} \\
\leq \quad & e^{-\beta\chi_\alpha(M)}\sum_{n\geq 2} 2^{n+1} \sum_{k_1\geq 0} \cdots \sum_{k_{n-1}\geq 0} (n+k_1+\cdots+k_{n-1})(k_1+1)\cdots(k_{n-1}+1) \\
& \left(2c\sum_{m=1}^{\infty} m^3 e^{-a\chi_\alpha(m)}\right)^{n-1+k_1+\cdots+k_{n-1}}.
\end{aligned}$$

Note that the constant $\epsilon = \epsilon(a)$ defined by equation (1.104) is present in the last term of the summations above, so,

$$\begin{aligned}
\Xi_\beta^\alpha(M) \quad \leq \quad & e^{-\beta\chi_\alpha(M)}\sum_{n\geq 2} 2^{n+1}\epsilon^{n-1} \sum_{k_1\geq 0}(k_1+1)\epsilon^{k_1} \cdots \sum_{k_{n-1}\geq 0}(k_{n-1}+1)\epsilon^{k_{n-1}}[(k_1+1)+\cdots+ \\
& (k_{n-1}+1)+1] \\
\leq \quad & e^{-\beta\chi_\alpha(M)}\sum_{n\geq 2} 2^{n+1}\epsilon^{n-1} n\left(\sum_{k\geq 0}(k+1)^2\delta^k\right)^{n-1}.
\end{aligned}$$

If we consider the parameter a large enough so that inequalities

$$\sum_{k\geq 0}(k+1)^2\epsilon^k \leq 2$$

and

$$\sum_{n\geq 2} 2^{2n} n\epsilon^{n-1} \leq \frac{1}{2}$$

hold, then

$$\Xi_\beta^\alpha(M) \le \frac{1}{2} e^{-\beta \chi_\alpha(M)}. \tag{1.112}$$

Therefore, by means of equations (1.105) and (1.112), we conclude that

$$\sum_{\Upsilon \in \mathfrak{C}_0(M)} w_\beta^\alpha(\Upsilon) \le \Theta_\beta^\alpha(M) + \Xi_\beta^\alpha(M) \le 2e^{-\beta \chi_\alpha(M)}. \tag{1.113}$$

■

1.6. PHASE TRANSITION AT LOW TEMPERATURE

Theorem 1.32. *Let $\alpha \in [0, \alpha^*)$, and let $\mathbf{h} = (h_x)_{x\in\mathbb{Z}}$ be an external field satisfying $|h_x| \cdot |x|^{1-\alpha} \to 0$ as $|x|$ approaches infinity. Then, the system undergoes phase transition as the inverse temperature β approaches infinity.*

Proof. It follows from our assumption made about the field $\mathbf{h} = (h_x)_{x\in\mathbb{Z}}$ that given a positive real number ε there exists a nonnegative integer $L = L(\varepsilon)$ such that

$$|h_x| \le \frac{\varepsilon}{(|x|+1)^{1-\alpha}} \tag{1.114}$$

holds whenever $|x| > L$. It is convenient to define a field $\widetilde{\mathbf{h}} = (\widetilde{h}_x)_{x\in\mathbb{Z}}$ whose expression is given by the right-hand side of equation (1.114), that is, we define $\widetilde{\mathbf{h}}$ by letting

$$\widetilde{h}_x = \frac{\varepsilon}{(|x|+1)^{1-\alpha}}$$

at each site x in $\mathbb{Z}$. Let us modify the fields $\mathbf{h}$ and $\widetilde{\mathbf{h}}$ in the following way. Let $\mathbf{h}_L = (h_{L,x})_{x\in\mathbb{Z}}$ and $\widetilde{\mathbf{h}}_L = (\widetilde{h}_{L,x})_{x\in\mathbb{Z}}$ be respectively given by

$$h_{L,x} = \begin{cases} \frac{\varepsilon}{(L+1)^{1-\alpha}} & \text{if } |x| \le L \\ h_x & \text{otherwise,} \end{cases}$$

and

$$\widetilde{h}_{L,x} = \begin{cases} \frac{\varepsilon}{(L+1)^{1-\alpha}} & \text{if } |x| \le L \\ \frac{\varepsilon}{(|x|+1)^{1-\alpha}} & \text{otherwise.} \end{cases}$$

The modified external fields $\mathbf{h}_L$ and $\widetilde{\mathbf{h}}_L$ satisfy the inequality $|h_{L,x}| \le |\widetilde{h}_{L,x}|$ at each point x in $\mathbb{Z}$. As we proved in Theorem 1.21, if the constant c from property (P1) is sufficiently large, then, for any contour γ from a triangle configuration $\underline{T}$ we have

$$\begin{aligned} \mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}) - \mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}\setminus\gamma) &\ge \sum_{T\in\gamma}\left(\frac{1}{2}W_\alpha(|T|) - \sum_{x\in T\cap\mathbb{Z}}|h_{L,x}|\right) & (1.115)\\ &\ge \sum_{T\in\gamma}\left(\frac{1}{2}W_\alpha(|T|) - \sum_{x\in T\cap\mathbb{Z}}|\widetilde{h}_{L,x}|\right). & (1.116)\end{aligned}$$

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First, let us consider the case $\alpha > 0$. Suppose that we chose ε sufficiently small so that the constant ξ_α defined by

$$\xi_\alpha = \frac{\zeta_\alpha}{2} - \frac{2^{1-\alpha}}{\alpha}\varepsilon$$

is positive. It follows from Propositions A.1 and A.2(a) that

$$\mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}) - \mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}\setminus\gamma) \geq \xi_\alpha \sum_{T\in\gamma} |T|^\alpha. \tag{1.117}$$

For $\alpha = 0$, analogously as before, let ε be small enough in such way that the constant ξ_0 defined by

$$\xi_0 = \frac{\zeta_0}{2} - 2\varepsilon$$

is positive. Using Propositions A.1 and A.2(b), we obtain

$$\mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}) - \mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}\setminus\gamma) \geq \xi_\alpha \sum_{T\in\gamma} [\log(|T|)+4]. \tag{1.118}$$

According to our discussion in the end of Section 1.4, equations (1.117) and (1.118) are sufficient to show that for $\alpha \in [0,\alpha^*)$ the Gibbs states $\mu^+_{\beta,\mathbf{h}_L}$ and $\mu^-_{\beta,\mathbf{h}_L}$ differs as the inverse temperature β approaches infinity. Since the fields $\mathbf{h}$ and $\mathbf{h}_L$ coincide up to a finite number of sites, then, it follows from the macroscopic equivalence of Gibbs simplices (see Theorem 7.33 from [27]) that the non-uniqueness of Gibbs measures for the system with field $\mathbf{h}_L$ is equivalent to the non-uniqueness of Gibbs measures for the system with field $\mathbf{h}$, therefore, the result follows.

■

Corollary 1.33. *Let $\alpha \in [0,1)$, and let $\mathbf{h} = (h_x)_{x\in\mathbb{Z}}$ be the external field given by*

$$h_x = \frac{h^*}{(|x|+1)^\delta}, \tag{1.119}$$

where h^* is an arbitrary real number and $\delta > \max\{1-\alpha, 1-\alpha^*\}$. Then, the system undergoes phase transition as the inverse temperature β approaches infinity.

Proof. In case $0 \leq \alpha < \alpha^*$, the result follows directly from Theorem 1.32. Now, let us suppose that $\alpha^* \leq \alpha < 1$ and show that there is a nonnegative integer L for which the system subject to the modified external field $\mathbf{h}_L = (h_{L,x})_{x\in\mathbb{Z}}$ (see Appendix A.2) exhibits phase transition at low temperatures. Let us fix a number α' in the interval $0 < \alpha' < \alpha^*$ such that $\delta > 1-\alpha'$.

Note that given a triangle configuration $\underline{T}$ and a contour γ of $\underline{T}$, the inequality

$$\mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}) - \mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}\setminus\gamma) \geq \mathscr{H}^{\alpha,0}(\underline{T}) - \mathscr{H}^{\alpha,0}(\underline{T}\setminus\gamma) - \sum_{T\in\gamma}\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}|$$

holds. According to the quasi-additivity of the Hamiltonian (in the absence of external field) with respect to contours proved by Littin and Picco in [15], if we choose the constant c from property (P1) sufficiently large, then there exists a positive constant $K_c(\alpha)$ such that for every contour γ of a triangle configuration $\underline{T}$ we have

$$\mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}) - \mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}\setminus\gamma) \geq K_c(\alpha)\mathscr{H}^{\alpha,0}(\gamma) - \sum_{T\in\gamma}\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}|. \tag{1.120}$$

Since the inequality

$$\mathscr{H}^{\alpha,0}(\gamma) \geq \mathscr{H}^{\alpha',0}(\gamma)$$

and Propositions 1.20 and A.1 imply

$$\mathscr{H}^{\alpha,0}(\gamma) \geq \sum_{T\in\gamma} W_{\alpha'}(|T|) \geq \zeta_{\alpha'} \sum_{T\in\gamma} \chi_{\alpha'}(|T|), \tag{1.121}$$

then, it follows that

$$\mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}) - \mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}\setminus\gamma) \geq K_c(\alpha)\zeta_{\alpha'} \sum_{T\in\gamma} |T|^{\alpha'} - \sum_{T\in\gamma}\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}|. \tag{1.122}$$

Let us define the external field $\widetilde{\mathbf{h}} = (\widetilde{h}_x)_{x\in\mathbb{Z}}$ by letting

$$\widetilde{h}_x = \frac{\varepsilon}{(|x|+1)^{1-\alpha'}} \tag{1.123}$$

at each site x in $\mathbb{Z}$, where ε is a positive number chosen sufficiently small in such a way that the constant ξ_α given by

$$\xi_\alpha = K_c(\alpha)\zeta_{\alpha'} - \frac{2^{1-\alpha'}}{\alpha'}\varepsilon$$

is positive. Then, by considering the length L sufficiently large so that the modified field $\widetilde{\mathbf{h}}_L = (\widetilde{h}_{L,x})_{x\in\mathbb{Z}}$ given by

$$\widetilde{h}_{L,x} = \begin{cases} \frac{\varepsilon}{(L+1)^{1-\alpha'}} & \text{if } |x| \leq L \\ \frac{\varepsilon}{(|x|+1)^{1-\alpha'}} & \text{otherwise} \end{cases}$$

satisfies $|h_{L,x}| \leq |\widetilde{h}_{L,x}|$ for every integer x, and by using the fact that

$$\sum_{x\in T\cap\mathbb{Z}} |\widetilde{h}_{L,x}| \leq \frac{2^{1-\alpha'}\varepsilon}{\alpha'}|T|^{\alpha'} \tag{1.124}$$

(see Proposition A.2(a)), we conclude that

$$\mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}) - \mathscr{H}^{\alpha,\mathbf{h}_L}(\underline{T}\setminus\gamma) \geq \xi_\alpha \sum_{T\in\gamma} |T|^{\alpha'}. \tag{1.125}$$

Therefore, since the system subject to the external field $\mathbf{h}_L$ undergoes phase transition at low temperatures, so does the system subject to the original field $\mathbf{h}$. ■

Remark 1.34. Note that if $\alpha \in [0,\alpha^*)$ and the constant δ from (1.119) satisfies $\delta = 1-\alpha$, then the phase transition also holds at low temperatures provided h^* is sufficiently small.

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1.7. CONCLUSION

In this chapter we extended the famous results regarding the existence of phase transition for ferromagnetic long range Ising models in one dimension by using the so-called Peierls contour argument adapted to one-dimensional systems, which was introduced by Fröhlich and Spencer [6] and Cassandro et al. [8]. The assumption imposed by [8] that the coupling constant $J(1)$ associated to the nearest-neighbor interactions has to be sufficiently large was avoided, moreover, the phase transition phenomenon persists even with the presence of a spatially inhomogeneous external field. The first step to achieve this result was to use the techniques introduced by Cassandro et al. and extend their results considering $J(1) = 1$ with α restricted to a small interval $[0, \alpha^*)$, where $\alpha^* \approx 0.2713$. Such a range of α is analogous to the one obtained by these authors where α had to belong to the interval $[0, \alpha_+)$, in that case, $\alpha_+ = \frac{\log 3}{\log 2} - 1 \approx 0.5849$. The restriction we obtained is the price that must be paid from dropping the assumption that the nearest-neighbor interactions have to be large, however, the result could be extended to the whole interval $[0, 1)$ by using the quasi-additive property of the Hamiltonian with respect to contours, derived by Littin and Picco [15].

The main result (see Corollary 1.33) states that if the pair interaction is inversely proportional to the distance between the spins to the power $2-\alpha$ and the external field vanishes polynomialy with power δ of the distance to the origin, where $0 \le \alpha < 1$ and $\delta > \max\{1-\alpha, 1-\alpha^*\}$, then, the system undergoes phase transition as the temperature approaches zero. The interplay between δ and α that involves α^* emerges from the technique and should be improved in the future by adopting a different approach other than the Peierls' argument. It is expected that the phase transition should hold for every $\alpha \in [0, 1)$ and $\delta > 1 - \alpha$. Questions regarding the uniqueness of Gibbs measures should also be considered in the future.

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2

METASTABILITY

2.1. INTRODUCTION

Metastability is a dynamical phenomenon observed in many different contexts, such as physics, chemistry, biology, climatology, economics. Despite the variety of scientific areas, the common feature of all these situations is the existence of multiple, well-separated time scales. On short time scales the system is in a quasi-equilibrium within a single region, while on long time scales it undergoes rapid transitions between quasi-equilibria in different regions. A rigorous description of metastability in the setting of stochastic dynamics is relatively recent, dating back to the pioneering paper [1], and has experienced substantial progress in the last decades. See [2–5] for reviews and for a list of the most important papers on this subject.

One of the big challenges in rigorous study of metastability is understanding the dependence of the metastable behaviour and of the nucleation process of the stable phase on the dynamics. The nucleation process of the critical droplet, i.e. the configuration triggering the crossover, has been indeed studied in different dynamical regimes: serial ([6, 7]) vs. parallel dynamics ([8–10]); non-conservative ([6, 7]) vs. conservative dynamics ([11–13]); finite ([14]) vs. infinite volumes ([15]); competition ([16–19]) vs. non-competition of metastable phases ([20, 21]). All previous studies assumed that the microscopic interaction is of short-range type.

In this chapter we push further this investigation, studying the dependence of the metastability scenario on the range of the interaction of the model. Long range Ising models in low dimensions are known to behave like higher-dimensional short-range models. For instance in [22, 23] (and later generalized by [24, 25]) it was shown that long range Ising models undergo a phase transition already in one dimension, and this transition persists in fast enough decaying fields. Furthermore, Dobrushin interfaces are rigid already in two dimensions for anisotropic long range Ising models, see [26].

We consider the question: does indeed a long range interaction change substantially the nucleation process? Are we able to define in this framework a critical configuration triggering the crossover towards the stable phase? In ([27]) the author already considered the Dyson-like long range models, i.e. the one-dimensional lattice model of Ising spins with interaction decaying with a power α, in a external magnetic field. Despite the long range potential, the author showed, by instanton arguments, that the system has a finite-sized critical droplet. In the present work we want to make rigorous this claim for a general long range interaction, showing as well that the long range interaction completely changes the metastability scenario: in the short–range one-dimensional Ising model a droplet of size one, already nucleates indeed the stable phase. The results present in this chapter were reported in [28].

We show that for given h and $J(n)$, we can define a nucleation droplet which for infinite-range $J(n)$ gets larger for smaller h, in contrast to the nearest-neighbor case. An interval of minuses of length l which grows to $l+1$ gains energy 2h, but loses $E_l = \sum_{n=l}^{\infty} J(n)$. Such a quantity E_l converges to zero as $l \to \infty$, but the smaller h is, the larger the size of the critical droplet. Moreover, taking h volume-dependent, going to zero with N as $N^{-\delta}$, can make the nucleation interval mesoscopic $O(N^{\delta})$ or macroscopic $O(N)$, but this also happens in higher dimension. This happens if at a phase transition one phase gets more stable due to a boundary term or an infinitesimal field, you could call it dynamically metastable, but we have a thermodynamically stable phase.

This chapter is organized as follows. In Section 2.2 we describe the lattice model and we give the main definitions; in Section 2.3 the main results are stated, while in Section 2.4 and 2.5 the proofs of the model-dependent results are given.

2.2. THE MODEL AND MAIN DEFINITIONS

Let Λ be a finite interval of $\mathbb{Z}$, and let us denote by h a positive external field. Given a configuration σ in $\Omega_\Lambda = \{-1,1\}^\Lambda$, we define the Hamiltonian with with respect to free boundary condition by

$$H_{\Lambda,h}(\sigma) = -\sum_{\{i,j\}\subseteq\Lambda} J(|i-j|)\sigma_i\sigma_j - \sum_{i\in\Lambda} h\sigma_i, \tag{2.1}$$

where $J:\mathbb{N}\to\mathbb{R}$, the pair interaction, is assumed to be positive and decreasing. The class of interactions that we want to include in the present analysis are of long range type, for instance,

1. exponential decay: $J(|i-j|) = J\cdot\lambda^{-|i-j|}$ with constants $J>0$ and $\lambda>1$;
2. polynomial decay: $J(|i-j|) = J\cdot|i-j|^{-\alpha}$, where $\alpha>0$ is a parameter.

The finite-volume Gibbs measure will be denoted by

$$\mu_\Lambda(\sigma) = \frac{1}{Z_\Lambda}\exp\left(-\beta H_{\Lambda,h}(\sigma)\right), \tag{2.2}$$

where $\beta>0$ is proportional to the inverse temperature and Z_Λ is a normalizing constant. The set of ground states $\mathscr{X}^s$ is defined as $\mathscr{X}^s := \operatorname{argmin}_{\sigma\in\Omega_\Lambda} H_{\Lambda,h}(\sigma)$. Note that for the class of interactions considered $\mathscr{X}^s = \{+\mathbf{1}\}$, where $+\mathbf{1}$ stands for the configuration with all spins equal to $+1$.
Given an integer $k\in\{0,\dots,\#\Lambda\}$, we consider the manifold $\mathscr{M}_k := \{\sigma\in\Omega_\Lambda : \#\{i:\sigma_i=1\} =$

$k\}$ consisting of configurations in Ω_Λ with k positive spins, and we define the configurations $L^{(k)}$ and $R^{(k)}$ as follows. Let

$$L_i^{(k)} = \begin{cases} +1 & \text{if } 1 \le i \le k, \text{ and} \\ -1 & \text{otherwise,} \end{cases} \tag{2.3}$$

and

$$R_i^{(k)} = \begin{cases} -1 & \text{if } 1 \le i \le \#\Lambda - k, \text{ and} \\ +1 & \text{otherwise,} \end{cases} \tag{2.4}$$

i.e., the configurations respectively with k positive spins on left side of the interval and on the right one. We will show that $L^{(k)}$ and $R^{(k)}$ are the minimizers of the energy function $H_{\Lambda,h}$ on $\mathscr{M}_k$ (see Proposition 2.12). Let us denote by $\mathscr{P}^{(k)}$ the set $\mathscr{P}^{(k)} := \{L^{(k)}, R^{(k)}\}$ consisting of the minimizers of the energy on $\mathscr{M}_k$. With abuse of notation we will indicate with $H_{\Lambda,h}(\mathscr{P}^{(k)})$ the energy of the elements of the set, that is, $H_{\Lambda,h}(\mathscr{P}^{(k)}) := H_{\Lambda,h}(L^{(k)}) = H_{\Lambda,h}(R^{(k)})$.

We choose the evolution of the system to be described by a discrete-time Markov chain $X = (X(t))_{t\ge 0}$, in particular, we consider the discrete-time serial Glauber dynamics given by the Metropolis weights, i.e., the transition matrix of such dynamics is given by

$$p(\sigma,\eta) := c(\sigma,\eta) e^{-\beta[H_{\Lambda,h}(\eta) - H_{\Lambda,h}(\sigma)]_+},$$

where $[\cdot]_+$ denotes the positive part, and $c(\cdot,\cdot)$ is its connectivity matrix that is equal to $1/|\Lambda|$ in case the two configurations σ and η coincide up to the value of a single spin, and zero otherwise. Notice that such dynamics is reversible with respect to the Gibbs measure defined in (2.2). Let us define the hitting time τ_η^σ of a configuration η of the chain X started at σ as

$$\tau_\eta^\sigma := \inf\{t > 0 : X(t) = \eta\}. \tag{2.5}$$

For any positive integer n, a sequence $\gamma = (\sigma^{(1)}, \dots, \sigma^{(n)})$ such that $\sigma^{(i)} \in \Omega_\Lambda$ and $c(\sigma^{(i)}, \sigma^{(i+1)}) > 0$ for all $i = 1, \dots, n-1$ is called a path joining $\sigma^{(1)}$ to $\sigma^{(n)}$; we also say that n is the length of the path. For any path γ of length n, we let

$$\Phi_\gamma := \max_{i=1,\dots,n} H_{\Lambda,h}(\sigma^{(i)}) \tag{2.6}$$

be the height of the path. We also define the communication height between σ and η by

$$\Phi(\sigma,\eta) := \min_{\gamma\in\Omega(\sigma,\eta)} \Phi_\gamma, \tag{2.7}$$

where the minimum is restricted to the set $\Omega(\sigma,\eta)$ of all paths joining σ to η. By reversibility, it easily follows that

$$\Phi(\sigma,\eta) = \Phi(\eta,\sigma) \tag{2.8}$$

for all $\sigma,\eta \in \Omega_\Lambda$. We extend the previous definition for sets $\mathscr{A},\mathscr{B} \subseteq \Omega_\Lambda$ by letting

$$\Phi(\mathscr{A},\mathscr{B}) := \min_{\gamma\in\Omega(\mathscr{A},\mathscr{B})} \Phi_\gamma = \min_{\sigma\in\mathscr{A},\eta\in\mathscr{B}} \Phi(\sigma,\eta), \tag{2.9}$$

where $\Omega(\mathscr{A},\mathscr{B})$ denotes the set of paths joining a state in $\mathscr{A}$ to a state in $\mathscr{B}$. The communication cost of passing from σ to η is given by the quantity $\Phi(\sigma,\eta) - H_{\Lambda,h}(\sigma)$. Moreover, if we define $\mathscr{I}_\sigma$ as the set of all states η in Ω_Λ such that $H_{\Lambda,h}(\eta) < H_{\Lambda,h}(\sigma)$, then the stability level of any $\sigma \in \Omega_\Lambda \setminus \mathscr{X}^s$ is given by

$$V_\sigma := \Phi(\sigma, \mathscr{I}_\sigma) - H_{\Lambda,h}(\sigma) \geq 0. \tag{2.10}$$

Following [29], we now introduce the notion of maximal stability level. Assuming that $\Omega_\Lambda \setminus \mathscr{X}^s \neq \emptyset$, we let the maximal stability level be

$$\Gamma_{\mathrm{m}} := \sup_{\sigma\in\Omega_\Lambda\setminus\mathscr{X}^s} V_\sigma. \tag{2.11}$$

We give the following definition.

Definition 2.1. We call metastable set $\mathscr{X}^m$, the set

$$\mathscr{X}^m := \{\sigma \in \Omega_\Lambda \setminus \mathscr{X}^s : V_\sigma = \Gamma_{\mathrm{m}}\}. \tag{2.12}$$

Following [29], we shall call $\mathscr{X}^m$ the set of metastable states of the system and refer to each of its elements as metastable. We denote by Γ the quantity

$$\Gamma := \max_{k=0,\dots,\#\Lambda} H_{\Lambda,h}(\mathscr{P}^{(k)}) - H_{\Lambda,h}(-\mathbf{1}). \tag{2.13}$$

We will show in Corollary 2.4 that under certain assumptions $\Gamma = \Gamma_{\mathrm{m}}$.

2.3. MAIN RESULTS

2.3.1. MEAN EXIT TIME

In this section we will study the first hitting time of the configuration $+\mathbf{1}$ when the system is prepared in $-\mathbf{1}$, in the limit $\beta \to \infty$. We will restrict our analysis to the case given by the following condition.

Condition 2.2. Let N be an integer such that $N \geq 2$. We consider $\Lambda = \{1,\dots,N\}$ and h such that

$$0 < h < \sum_{n=1}^{N-1} J(n). \tag{2.14}$$

By using the general theory developed in [29], we need first to solve two model-dependent problems: the calculation of the minimax between $-\mathbf{1}$ and $+\mathbf{1}$ (item 1 of Theorem 2.3) and the proof of a recurrence property in the energy landscape (item 3 of Theorem 2.3).

Theorem 2.3. *Assume that Condition 2.2 is satisfied.Then, we have*

1. $\Phi(-\mathbf{1},+\mathbf{1}) = \Gamma + H_{\Lambda,h}(-\mathbf{1})$,
2. $V_{-\mathbf{1}} = \Gamma > 0$, *and*
3. $V_\sigma < \Gamma$ *for any* $\sigma \in \Omega_\Lambda \setminus \{-\mathbf{1},+\mathbf{1}\}$.

As a corollary we have that $-\mathbf{1}$ is the only metastable state for this model.

Corollary 2.4. *Assume that Condition 2.2 is satisfied. It follows that*

$$\Gamma = \Gamma_m, \tag{2.15}$$

and

$$\mathscr{X}^m = \{-\mathbf{1}\}. \tag{2.16}$$

Therefore, the asymptotic of the exit time for the system started at the metastable states is given by the following theorem.

Theorem 2.5. *Assume that Condition 2.2 is satisfied. It follows that*

1. *for any* $\epsilon > 0$
$$\lim_{\beta\to\infty} \mathbb{P}\left(e^{\beta(\Gamma-\epsilon)} < \tau^{-\mathbf{1}}_{+\mathbf{1}} < e^{\beta(\Gamma+\epsilon)}\right) = 1,$$

2. *the limit*
$$\lim_{\beta\to\infty} \frac{1}{\beta}\log\left(\mathbb{E}\left(\tau^{-\mathbf{1}}_{+\mathbf{1}}\right)\right) = \Gamma$$

   holds.

Once the model-dependent results in Theorem 2.3 have been proven, the proof of Theorem 2.5 easily follows from the general theory present in [29]: item 1 follows from Theorem 4.1 in [29] and item 2 from Theorem 4.9 in [29].

2.3.2. MESOSCOPIC VS. MACROSCOPIC NUCLEATION

We are going to show that for small enough external magnetic field, the size of the critical droplet is a macroscopic fraction of the system (i.e., macroscopic nucleation), while for h sufficiently large, the critical configuration will be a mesoscopic fraction of the system.

Let us define $L := \lfloor \frac{N}{2} \rfloor$, and let $h_k^{(N)}$ be

$$h_k^{(N)} := \sum_{n=1}^{N-k-1} J(n) - \sum_{n=1}^{k} J(n) \tag{2.17}$$

for each $k = 0, \dots, L-1$. One can easily verify that

$$0 < h_{L-1}^{(N)} < \cdots < h_1^{(N)} < h_0^{(N)} = \sum_{n=1}^{N-1} J(n) \tag{2.18}$$

Proposition 2.6. *Under the assumption that Condition (2.2) is satisfied, one of the following conditions holds.*

1. *Case* $h < h_{L-1}^{(N)}$*, we have*
$$H_{\Lambda,h}(\mathscr{P}^{(L)}) > \max_{\substack{0\le k\le N\\ k\neq L}} H_{\Lambda,h}(\mathscr{P}^{(k)}).$$

2. *Case* $h_k^{(N)} < h < h_{k-1}^{(N)}$ *for some* $k \in \{1, \dots, L-1\}$, *we have*

$$H_{\Lambda,h}(\mathscr{P}^{(k)}) > \max_{\substack{0 \le i \le N \\ i \ne k}} H_{\Lambda,h}(\mathscr{P}^{(i)}).$$

3. *Case* $h = h_k^{(N)}$ *for some* $k \in \{1, \dots, L-1\}$, *we have*

$$H_{\Lambda,h}(\mathscr{P}^{(k)}) = H_{\Lambda,h}(\mathscr{P}^{(k+1)}) > \max_{\substack{0 \le i \le N \\ i \ne k, i \ne k+1}} H_{\Lambda,h}(\mathscr{P}^{(i)}).$$

The first point of Proposition 2.6 describes the less interesting and, in a way, artificial, situation of very low external magnetic fields: in this regime the bulk term is negligible so that the energy of the droplet increases until the positive spins are the majority (i.e. $k = L$, see Figure 2.3). Therefore, the second point contains the most interesting situation, where there is an interplay between the bulk and the surface term. The following Corollary is a consequence of Proposition 2.6 when N is large enough and gives a characterisation of the critical size k_c of the critical droplet.

Corollary 2.7. *If we assume that* $\sum_{n=1}^{\infty} J(n)$ *converges and*

$$0 < h < \sum_{n=1}^{\infty} J(n), \tag{2.19}$$

then, the size of the critical droplet will be given by

$$k_c = \min\left\{ k \in \mathbb{N} : \sum_{n=k+1}^{\infty} J(n) \le h \right\} \tag{2.20}$$

whenever N *is sufficiently large.*

As a consequence of Corollary 2.7, the set of critical configurations $\mathscr{P}_c$ is given by

$$\mathscr{P}_c := \{L^{(k_c)}, R^{(k_c)}\} \tag{2.21}$$

for N large enough. The following result shows the reason why configurations in $\mathscr{P}_c$ are referred to as critical configurations: they indeed trigger the transition towards the stable phase.

Lemma 2.8. *Under the conditions stated above, we have*

1. *any path* $\gamma \in \Omega(-\mathbf{1}, +\mathbf{1})$ *such that* $\Phi_\gamma - H_{\Lambda,h}(-\mathbf{1}) = \Gamma$ *visits* $\mathscr{P}_c$, *and*

2. *the limit*

$$\lim_{\beta \to \infty} \mathbb{P}(\tau_{\mathscr{P}_c}^{-\mathbf{1}} < \tau_{+\mathbf{1}}^{-\mathbf{1}}) = 1$$

holds.

The proof of the previous Theorem is a straightforward consequence of Theorem 5.4 in [29].

2.3.3. EXAMPLES

Let us give two interesting examples of the general theory so far developed.

EXAMPLE 1: EXPONENTIALLY DECAYING COUPLING

We consider

$$J(n) = \frac{J}{\lambda^{n-1}},$$

where J and λ are positive real numbers with $\lambda > 1$.

Proposition 2.9. *Under the same hypotheses as Corollary 2.7, we have that the critical droplet length k_c is equal to*

$$k_c = \left\lceil \log_\lambda \left(\frac{J}{h(1-\lambda^{-1})} \right) \right\rceil \tag{2.22}$$

whenever N is sufficiently large.

Proof. By Corollary 2.7, we have

$$J \sum_{n=k_c+1}^{\infty} \lambda^{-(n-1)} \le h < J \sum_{n=k_c}^{\infty} \lambda^{-(n-1)}$$

that implies

$$\frac{\lambda^{-k_c}}{1-\lambda^{-1}} \le \frac{h}{J} < \frac{\lambda^{-(k_c-1)}}{1-\lambda^{-1}}$$

Thus

$$k_c - 1 < -\frac{\log\left(\frac{h(1-\lambda^{-1})}{J}\right)}{\log \lambda} \le k_c. \tag{2.23}$$

∎

As a remark we notice that in case of exponential decay of the interaction, the system behaves essentially as the one-dimensional Ising model with nearest-neighbor interactions. Note that

$$\lim_{\lambda\to\infty} J(n) = \begin{cases} J & \text{if } n = 1, \text{ and} \\ 0 & \text{otherwise;} \end{cases} \tag{2.24}$$

moreover, if $h < J = \lim_{\lambda\to\infty} \sum_{n=1}^{\infty} J(n)$, then $k_c = 1$ whenever λ is large enough. So, we conclude that typically a single plus spin in the lattice will trigger the nucleation of the stable phase. As you can see in Figure 2.1 the energy exitations $H_{\Lambda,h}(\mathscr{P}^{(k)}) - H_{\Lambda,h}(-\mathbf{1})$ are strictly descreasing in k, as expected.

EXAMPLE 2: POLYNOMIALLY DECAYING COUPLING

Let the coupling constants be given by

$$J(n) = J \cdot n^{-\alpha},$$

where J and α are positive real numbers with $\alpha > 1$. As it is shown in Figures 2.2 and 2.3, for the polynomially decaying coupling model, we have that, for h small enough the critical droplet is essentially the half interval, while for large enough magnetic external

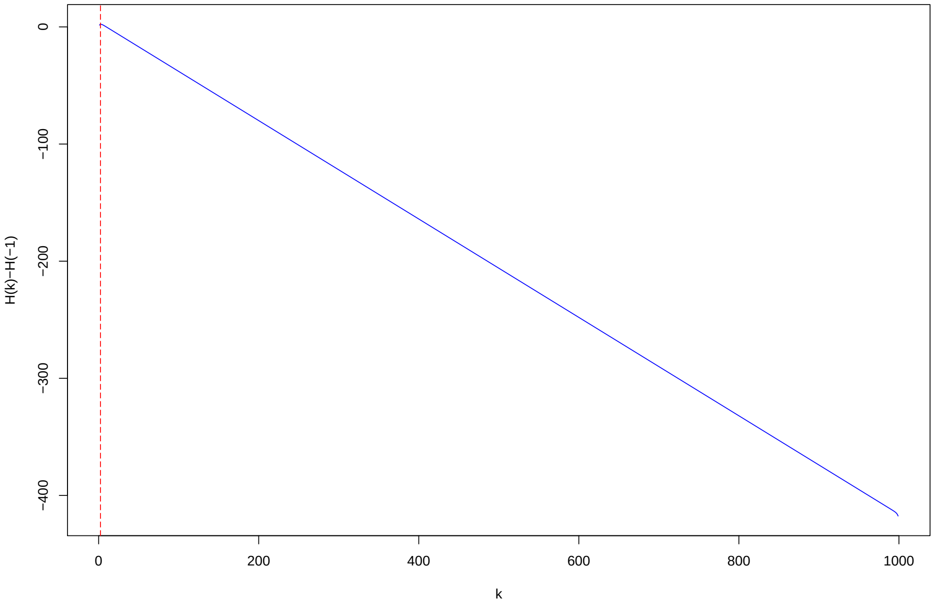

Figure 2.1: The blue line is the excitation energy $H_{\Lambda,h}(\mathscr{P}^{(k)}) - H_{\Lambda,h}(-\mathbf{1})$ for $N = 1000$, $\lambda = 2, h = 0.21, J = 1$; while the red line indicates the size of the critical droplet.

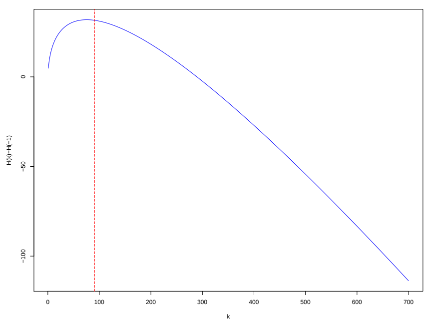

Figure 2.2: Blue line is the excitation energy $H_{\Lambda,h}(\mathscr{P}^{(k)}) - H_{\Lambda,h}(-\mathbf{1})$ for $N = 10000$, $\alpha = 3/2, h = 0.21, J = 1$; the red line represents the critical length $k_c \approx 91$.

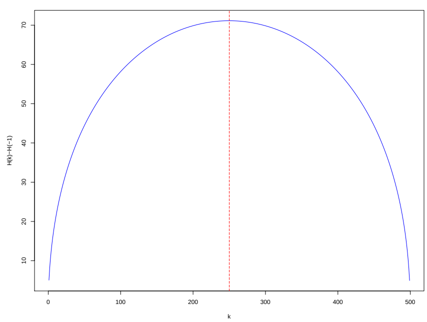

Figure 2.3: Blue line is the excitation energy $H_{\Lambda,h}(\mathscr{P}^{(k)}) - H_{\Lambda,h}(-\mathbf{1})$ for $N = 500$, $\alpha = 3/2, h = 0.0001, J = 1$; the red line represents the critical length $k_c = 250$.

magnetic field, the critical droplet is the configuration with k_c plus spins at the sides, with $k_c \approx \left(\frac{J}{h(\alpha-1)}\right)^{\frac{1}{\alpha-1}}$.

We can prove indeed the following proposition.

Proposition 2.10. *Under the same hypotheses as Corollary 2.7, we have that k_c satisfies*

$$\left| k_c - \left(\frac{J}{h(\alpha-1)}\right)^{\frac{1}{\alpha-1}} \right| < 1 \tag{2.25}$$

whenever N is large enough.

Proof. By Corollary 2.7, it follows that

$$J \sum_{n=k_c+1}^{\infty} n^{-\alpha} \leq h < J \sum_{n=k_c}^{\infty} n^{-\alpha}.$$

Moreover, note that

$$\int_{k_c+1}^{\infty} \frac{1}{x^\alpha} dx < \sum_{n=k_c+1}^{\infty} n^{-\alpha}$$

and

$$\sum_{n=k_c}^{\infty} n^{-\alpha} < \int_{k_c-1}^{\infty} \frac{1}{x^\alpha} dx$$

so that

$$\frac{(k_c+1)^{1-\alpha}}{\alpha-1} < \frac{h}{J} < \frac{(k_c-1)^{1-\alpha}}{\alpha-1}.$$

Hence,

$$(k_c-1)^{\alpha-1} < \frac{J}{h(\alpha-1)} < (k_c+1)^{\alpha-1}. \tag{2.26}$$

■

2.4. PROOF THEOREM 2.3

We start the proof of the main theorem giving some general results about the control of the energy of a general configuration. First of all we note that equation (2.1) can be written as

$$\begin{aligned}
H_{\Lambda,h}(\sigma) &= -\frac{1}{2}\sum_{i\in\Lambda}\sum_{j\in\Lambda} J(|i-j|)\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i \\
&= \sum_{i\in\Lambda}\sum_{j\in\Lambda} J(|i-j|)\left(\frac{1-\sigma_i\sigma_j}{2}\right) - h\sum_{i\in\Lambda}\sigma_i - \frac{1}{2}\sum_{i\in\Lambda}\sum_{j\in\Lambda} J(|i-j|) \\
&= \sum_{i\in\Lambda}\sum_{j\in\Lambda} J(|i-j|)\mathbb{1}_{\{\sigma_i\neq\sigma_j\}} - h\sum_{i\in\Lambda}\sigma_i - \frac{1}{2}\sum_{i\in\Lambda}\sum_{j\in\Lambda} J(|i-j|).
\end{aligned}$$

Moreover, given an integer $k \in \{0,\dots,N\}$, if $\sigma \in \mathcal{M}_k$, then

$$H_{\Lambda,h}(\sigma) = \sum_{i\in\Lambda}\sum_{j\in\Lambda} J(|i-j|)\mathbb{1}_{\{\sigma_i\neq\sigma_j\}} + h(N-2k) - \frac{1}{2}\sum_{i\in\Lambda}\sum_{j\in\Lambda} J(|i-j|). \tag{2.27}$$

2

Therefore, restricting ourselves to configurations that contains only k spins with the value 1, in order to find such configurations with minimal energy, it is sufficient to minimize the first term of the right-hand side of equation (2.27).

Proposition 2.11. *Let N be a positive integer and $k \in \{0,\dots,N\}$, if we restrict to all $\sigma \in \mathcal{M}_k$, then*

$$\sum_{i=1}^{N}\sum_{j=1}^{N} J(|i-j|)\mathbb{1}_{\{\sigma_i \neq \sigma_j\}} \geq 2\sum_{i=1}^{k}\sum_{j=k+1}^{N} J(|i-j|). \tag{2.28}$$

Under this restriction, the equality in the equation above holds if and only if $\sigma = L^{(k)}$ or $\sigma = R^{(k)}$.

Proof. Let us prove the result by induction. Let $\mathcal{H}_N$ be defined by

$$\mathcal{H}_N(\sigma_1,\dots,\sigma_N) = \sum_{i=1}^{N}\sum_{j=1}^{N} J(|i-j|)\mathbb{1}_{\{\sigma_i\neq\sigma_j\}} = 2\sum_{i:\sigma_i=1}\sum_{j:\sigma_j=-1} J(|i-j|). \tag{2.29}$$

Note that the result is trivial if $N = 1$. Assuming that it holds for $N \geq 1$, let us prove that it also holds for $N+1$. In case $\sigma_1 = 1$, applying our induction hypothesis and Lemma B.1, we have

$$\begin{aligned}
\mathcal{H}_{N+1}(1,\sigma_2,\dots,\sigma_{N+1}) &= 2\sum_{j=1}^{N} J(j)\mathbb{1}_{\{\sigma_{j+1}=-1\}} + \mathcal{H}_N(\sigma_2,\dots,\sigma_{N+1}) && (2.30)\\
&\geq 2\sum_{j=k}^{N} J(j) + 2\sum_{i=1}^{k-1}\sum_{j=k}^{N} J(|i-j|) && (2.31)\\
&= 2\sum_{i=1}^{k}\sum_{j=k+1}^{N+1} J(|i-j|). && (2.32)
\end{aligned}$$

Replacing the inequality sign in equation (2.31) by an equality, it follows that

$$0 \leq \mathcal{H}_N(\sigma_2,\dots,\sigma_{N+1}) - 2\sum_{i=1}^{k-1}\sum_{j=k}^{N} J(|i-j|) = 2\sum_{j=k}^{N} J(j) - 2\sum_{j=1}^{N} J(j)\mathbb{1}_{\{\sigma_{j+1}=-1\}} \leq 0, \tag{2.33}$$

hence,

$$\sum_{j=1}^{k-1} J(j) - \sum_{j=1}^{N} J(j)\mathbb{1}_{\{\sigma_{j+1}=1\}} = 0. \tag{2.34}$$

Using Lemma B.1 again, we conclude that $\sigma_j = 1$ whenever $1 \leq j \leq k$, and $\sigma_j = -1$ whenever $k+1 \leq j \leq N+1$. Now, in case $\sigma_1 = -1$, we write $\mathcal{H}_{N+1}(-1,\sigma_2,\dots,\sigma_{N+1})$ as

$$\mathcal{H}_{N+1}(-1,\sigma_2,\dots,\sigma_{N+1}) = \mathcal{H}_{N+1}(1,-\sigma_2,\dots,-\sigma_{N+1}) \tag{2.35}$$

and apply our previous result in order to obtain

$$\mathcal{H}_{N+1}(-1,\sigma_2,\dots,\sigma_{N+1}) \geq 2\sum_{i=1}^{N+1-k}\sum_{j=N+2-k}^{N+1} J(|i-j|) = 2\sum_{i=1}^{k}\sum_{j=k+1}^{N+1} J(|i-j|), \tag{2.36}$$

where the equality holds only if $\sigma_j = -1$ whenever $1 \leq j \leq N+1-k$, and $\sigma_j = 1$ whenever $N+2-k \leq j \leq N+1$. ■

As an immediate consequence of Proposition 2.11 the next results follows.

Theorem 2.12. *Given an integer $k \in \{0,\dots,N\}$, if we restrict to all $\sigma \in \mathscr{M}_k$, then*

$$H_{\Lambda,h}(\sigma) \geq 2\sum_{i=1}^{k}\sum_{j=k+1}^{N} J(|i-j|) + h(N-2k) - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} J(|i-j|). \tag{2.37}$$

Under this restriction, the equality in the equation above holds if and only if $\sigma = R^{(k)}$ or $\sigma = L^{(k)}$.

2

2.4.1. Proof of Theorem 2.3.1 (minimax)

Proof of Theorem 2.3.1. Define $f : \{0,\dots,N\} \to \mathbb{R}$ as

$$f(k) = H_{\Lambda,h}(\mathscr{P}^{(k)}). \tag{2.38}$$

It follows that

$$\begin{aligned}
\Delta f(k) &= f(k+1) - f(k) \\
&= 2\left(\sum_{i=1}^{k+1}\sum_{j=k+2}^{N} J(|i-j|) - \sum_{i=1}^{k}\sum_{j=k+1}^{N} J(|i-j|) - h\right) \\
&= 2\left(\sum_{j=k+2}^{N} J(|k+1-j|) + \sum_{i=1}^{k}\sum_{j=k+2}^{N} J(|i-j|) - \sum_{i=1}^{k}\sum_{j=k+1}^{N} J(|i-j|) - h\right) \\
&= 2\left(\sum_{j=k+2}^{N} J(|k+1-j|) - \sum_{i=1}^{k} J(|i-(k+1)|) - h\right) \\
&= 2\left(\sum_{i=1}^{N-k-1} J(i) - \sum_{i=1}^{k} J(i) - h\right)
\end{aligned}$$

holds for all k such that $0 \leq k \leq N-1$, and

$$\begin{aligned}
\Delta^2 f(k) &= \Delta f(k+1) - \Delta f(k) \\
&= 2\left(\sum_{i=1}^{N-k-2} J(i) - \sum_{i=1}^{N-k-1} J(i) - \sum_{i=1}^{k+1} J(i) + \sum_{i=1}^{k} J(i)\right) \\
&= -2(J(N-k-1) + J(k+1))
\end{aligned}$$

holds whenever $0 \leq k \leq N-2$.

Note that

$$\Delta f(0) = 2\left(\sum_{i=1}^{N-1} J(i) - h\right) > 0, \tag{2.39}$$

$1 \leq \left\lfloor \frac{N}{2} \right\rfloor \leq N-1$, and

$$\Delta f\left(\left\lfloor \frac{N}{2} \right\rfloor\right) < 0. \tag{2.40}$$

It follows from $\Delta^2 f < 0$ and equations (2.39) and (2.40) that f satisfies

$$f(0) < f(1) \tag{2.41}$$

2

and

$$f\left(\left\lfloor \frac{N}{2} \right\rfloor\right) > \cdots > f(N), \tag{2.42}$$

therefore, $f(k_0) = \max_{0\le k\le N} f(k)$ for some $k_0 \in \{1, \dots, \lfloor \frac{N}{2} \rfloor\}$.

Defining the path $\gamma : -\mathbf{1} \to +\mathbf{1}$ by $\gamma = (L^{(0)}, L^{(1)}, \dots, L^{(N)})$, it is easy to see that

$$\Phi(-\mathbf{1}, +\mathbf{1}) = \max_{\sigma\in\gamma} H_{\Lambda,h}(\sigma) = \max_{0\le k\le N} H_{\Lambda,h}(\mathscr{P}^{(k)}) = \Gamma + H_{\Lambda,h}(-\mathbf{1}). \tag{2.43}$$

■

2.4.2. Proof of Theorem 2.3.2 and 2.3.3

Before giving the proof of the second point of the main theorem, we give some results about the control of the energy of a spin-flipped configuration. Given a configuration σ and $k \in \Lambda$, the spin-flipped configuration $\theta_k\sigma$ is defined as:

$$(\theta_k\sigma)_i = \begin{cases} -\sigma_k & \text{if } i = k \text{, and} \\ \sigma_i & \text{otherwise.} \end{cases} \tag{2.44}$$

Note that the energetic cost to flip the spin at position k from the configuration σ is given by

$$\begin{aligned} H_{\Lambda,h}(\theta_k\sigma) - H_{\Lambda,h}(\sigma) &= \sum_{\{i,j\}\subseteq\Lambda} J(|i-j|)(\sigma_i\sigma_j - (\theta_k\sigma)_i(\theta_k\sigma)_j) + h\sum_{i\in\Lambda}(\sigma_i - (\theta_k\sigma)_i) \\ &= \left(\sum_{j\in\Lambda} J(|k-j|)2\sigma_k\sigma_j + 2h\sigma_k\right) \\ &= 2\sigma_k\left(\sum_{j\in\Lambda} J(|k-j|)\sigma_j + h\right). \end{aligned}$$

Proposition 2.13. *Under Condition 2.2, given a configuration σ such that*

$$H_{\Lambda,h}(\theta_k\sigma) - H_{\Lambda,h}(\sigma) \ge 0 \tag{2.45}$$

for every $k \in \{1, \dots, N\}$, then either $\sigma = -\mathbf{1}$ or $\sigma = +\mathbf{1}$.

Proof. Let $k \in \{1, \dots, N-1\}$, and let σ be a configuration such that $\sigma_i = +1$ whenever $1 \le i \le k$ and $\sigma_{k+1} = -1$. In the following, we show that every such σ cannot satisfy property (2.45). If property (2.45) is satisfied, then

$$\begin{cases} H_{\Lambda,h}(\theta_k\sigma) - H_{\Lambda,h}(\sigma) \ge 0 \\ H_{\Lambda,h}(\theta_{k+1}\sigma) - H_{\Lambda,h}(\sigma) \ge 0 \end{cases} \tag{2.46}$$

that is,

$$\begin{cases} \sum_{i=1}^{k-1} J(|k-i|) - J(1) + \sum_{i=k+2}^{N} J(|k-i|)\sigma_i + h \ge 0 \\ -\left(\sum_{i=1}^{k} J(|k+1-i|) + \sum_{i=k+2}^{N} J(|k+1-i|)\sigma_i + h\right) \ge 0. \end{cases} \tag{2.47}$$

Summing both equations above, we have

$$\begin{aligned} 0 &\leq -J(k) - J(1) + \sum_{i=k+2}^{N} (J(i-k) - J(i-k-1))\sigma_i \\ &\leq -J(k) - J(1) + \sum_{i=k+2}^{N} (J(i-k-1) - J(i-k)) \\ &= -J(k) - J(1) + \sum_{i=1}^{N-k-1} (J(i) - J(i+1)) \\ &= -J(k) - J(N-k) \end{aligned}$$

that is a contradiction. Analogously, every configuration σ such that such that $\sigma_i = -1$ whenever $1 \leq i \leq k$ and $\sigma_{k+1} = 1$ for some $k \in \{1, \dots, N-1\}$, property (2.45) cannot be satisfied. Therefore, we conclude that for every σ different from $-\mathbf{1}$ and $+\mathbf{1}$, property (2.45) does not hold.

The proof of the converse statement is straightforward. ∎

As an immediate consequence of the result above, the next result follows.

Corollary 2.14. *Under Condition 2.2, for every configuration σ different from $-\mathbf{1}$ and $+\mathbf{1}$, there is a path $\gamma = (\sigma^{(1)}, \dots, \sigma^{(n)})$, where $\sigma^{(1)} = \sigma$ and $\sigma^{(n)} \in \{-\mathbf{1}, +\mathbf{1}\}$, such that $H_{\Lambda,h}(\sigma^{(i+1)}) < H_{\Lambda,h}(\sigma^{(i)})$.*

We have now all the element for proving item 2 and 3 of Theorem 2.3.

Proof of Theorem 2.3.2. First, note that it follows from inequality (2.41) that $\Gamma > 0$. Now, let us show that $V_{-\mathbf{1}}$ satisfies

$$V_{-\mathbf{1}} = \Phi(-\mathbf{1}, +\mathbf{1}) - H_{\Lambda,h}(-\mathbf{1}). \tag{2.48}$$

Since $+\mathbf{1} \in \mathscr{I}_{-\mathbf{1}}$, we have

$$V_{-\mathbf{1}} \leq \Phi(-\mathbf{1}, +\mathbf{1}) - H_{\Lambda,h}(-\mathbf{1}). \tag{2.49}$$

So, we conclude the proof if we show that

$$\Phi(-\mathbf{1}, +\mathbf{1}) \leq \Phi(-\mathbf{1}, \eta) \tag{2.50}$$

holds for every $\eta \in \mathscr{I}_{-\mathbf{1}}$. Let $\gamma_1 : -\mathbf{1} \to \eta$ be a path from $-\mathbf{1}$ to η given by $\gamma_1 = (\sigma^{(1)}, \dots, \sigma^{(n)})$, then, according to Corollary 2.14, there is a path $\gamma_2 : \eta \to +\mathbf{1}$, say $\gamma_2 = (\eta^{(1)}, \dots, \eta^{(m)})$, along which the energy decreases. Hence, the path $\gamma : -\mathbf{1} \to +\mathbf{1}$ given by

$$\gamma = (\sigma^{(1)}, \dots, \sigma^{(n-1)}, \eta^{(1)}, \dots, \eta^{(m)}) \tag{2.51}$$

satisfies

$$\Phi_\gamma(-\mathbf{1}, +\mathbf{1}) = \Phi_{\gamma_1}(-\mathbf{1}, \eta) \vee \Phi_{\gamma_2}(\eta, +\mathbf{1})) = \Phi_{\gamma_1}(-\mathbf{1}, \eta). \tag{2.52}$$

Hence, the inequality

$$\Phi(-\mathbf{1}, +\mathbf{1}) \leq \Phi_{\gamma_1}(-\mathbf{1}, \eta) \tag{2.53}$$

holds for every path $\gamma_1 : -\mathbf{1} \to \eta$, and equation (2.50) follows. ∎

2

Proof of Theorem 2.3.3. Given $\sigma \notin \{-\mathbf{1},+\mathbf{1}\}$, let us show now that

$$\Phi(\sigma,\eta) - H_{\Lambda,h}(\sigma) < V_{-\mathbf{1}} \tag{2.54}$$

holds for any $\eta \in \mathscr{I}_\sigma$. Let us consider the following cases.

1. Case $\eta = +\mathbf{1}$. According to Corollary (2.14), there is a path $\gamma = (\sigma^{(1)},\dots,\sigma^{(n)})$ from $\sigma^{(1)} = \sigma$ to $\sigma^{(n)} \in \{-\mathbf{1},+\mathbf{1}\}$ along which the energy decreases.
   (a) If $\sigma^{(n)} = -\mathbf{1}$, then the path $\gamma_0 : \sigma \to \eta$ given by $\gamma_0 = (\sigma^{(1)},\dots,\sigma^{(n-1)}, L^{(0)},\dots,L^{(N)})$ satisfies

$$\begin{aligned}
\Phi(\sigma,\eta) - H_{\Lambda,h}(\sigma) &\le \max_{\zeta\in\gamma_0} H_{\Lambda,h}(\zeta) - H_{\Lambda,h}(\sigma)\\
&\le \left(\max_{\zeta\in\gamma} H_{\Lambda,h}(\zeta)\right) \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)})\right) - H_{\Lambda,h}(\sigma)\\
&= 0 \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)}) - H_{\Lambda,h}(\sigma)\right)\\
&< \max_{0\le k\le N} H_{\Lambda,h}(L^{(k)}) - H_{\Lambda,h}(-\mathbf{1})\\
&= V_{-\mathbf{1}}.
\end{aligned}$$

   (b) Otherwise, if $\sigma^{(n)} = +\mathbf{1}$, then

$$\begin{aligned}
\Phi(\sigma,\eta) - H_{\Lambda,h}(\sigma) &\le \max_{\zeta\in\gamma} H_{\Lambda,h}(\zeta) - H_{\Lambda,h}(\sigma)\\
&= 0\\
&< V_{-\mathbf{1}}.
\end{aligned}$$

2. Case $\eta = -\mathbf{1}$. According to Corollary (2.14), there is a path $\gamma = (\sigma^{(1)},\dots,\sigma^{(n)})$ from $\sigma^{(1)} = \sigma$ to $\sigma^{(n)} \in \{-\mathbf{1},+\mathbf{1}\}$ along which the energy decreases.
   (a) If $\sigma^{(n)} = +\mathbf{1}$, then the path $\gamma_0 : \sigma \to \eta$ given by $\gamma_0 = (\sigma^{(1)},\dots,\sigma^{(n-1)}, L^{(N)},\dots,L^{(0)})$ satisfies

$$\begin{aligned}
\Phi(\sigma,\eta) - H_{\Lambda,h}(\sigma) &\le \max_{\zeta\in\gamma_0} H_{\Lambda,h}(\zeta) - H_{\Lambda,h}(\sigma)\\
&\le \left(\max_{\zeta\in\gamma} H_{\Lambda,h}(\zeta)\right) \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)})\right) - H_{\Lambda,h}(\sigma)\\
&= 0 \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)}) - H_{\Lambda,h}(\sigma)\right)\\
&< \max_{0\le k\le N} H_{\Lambda,h}(L^{(k)}) - H_{\Lambda,h}(-\mathbf{1})\\
&= V_{-\mathbf{1}}.
\end{aligned}$$

   (b) Otherwise, if $\sigma^{(n)} = -\mathbf{1}$, then

$$\begin{aligned}
\Phi(\sigma,\eta) - H_{\Lambda,h}(\sigma) &\le \max_{\zeta\in\gamma} H_{\Lambda,h}(\zeta) - H_{\Lambda,h}(\sigma)\\
&= 0\\
&< V_{-\mathbf{1}}.
\end{aligned}$$

3. Case $\eta \notin \{-\mathbf{1},+\mathbf{1}\}$. Let $\gamma_1 = (\sigma^{(1)},\dots,\sigma^{(n)})$ and $\gamma_2 = (\eta^{(1)},\dots,\eta^{(m)})$ be paths from $\sigma^{(1)} = \sigma$ to $\sigma^{(n)} \in \{-\mathbf{1},+\mathbf{1}\}$ and from $\eta^{(1)} = \eta$ to $\eta^{(m)} \in \{-\mathbf{1},+\mathbf{1}\}$, respectively, along which the energy decreases.

   (a) If $\sigma^{(n)} = \eta^{(m)}$, define the path $\gamma:\sigma \to \eta$ given by $\gamma_0 = (\sigma^{(1)},\dots,\sigma^{(n-1)},\eta^{(m)},\dots,\eta^{(1)})$ in order to obtain

$$\begin{aligned}
\Phi(\sigma,\eta) - H_{\Lambda,h}(\sigma) &\le \max_{\zeta\in\gamma_0} H_{\Lambda,h}(\zeta) - H_{\Lambda,h}(\sigma)\\
&= \left(\max_{\zeta\in\gamma_1} H_{\Lambda,h}(\zeta)\right) \vee \left(\max_{\zeta\in\gamma_2} H_{\Lambda,h}(\zeta)\right) - H_{\Lambda,h}(\sigma)\\
&= H_{\Lambda,h}(\sigma) \vee H_{\Lambda,h}(\eta) - H_{\Lambda,h}(\sigma)\\
&= 0\\
&< V_{-\mathbf{1}}.
\end{aligned}$$

   (b) If $\sigma^{(n)} = -\mathbf{1}$ and $\eta^{(m)} = +\mathbf{1}$, let us define the path $\gamma_0:\sigma \to \eta$ given by

$$\gamma_0 = (\sigma^{(1)},\dots,\sigma^{(n-1)},L^{(0)},\dots,L^{(N)},\eta^{(m-1)},\dots,\eta^{(1)}) \tag{2.55}$$

   satisfies

$$\begin{aligned}
\Phi(\sigma,\eta) - H_{\Lambda,h}(\sigma) &\le \max_{\zeta\in\gamma_0} H_{\Lambda,h}(\zeta) - H_{\Lambda,h}(\sigma)\\
&= \left(\max_{\zeta\in\gamma_1} H_{\Lambda,h}(\zeta)\right) \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)})\right) \vee \left(\max_{\zeta\in\gamma_2} H_{\Lambda,h}(\zeta)\right) - H_{\Lambda,h}(\sigma)\\
&= H_{\Lambda,h}(\sigma) \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)})\right) \vee H_{\Lambda,h}(\eta) - H_{\Lambda,h}(\sigma)\\
&= 0 \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)}) - H_{\Lambda,h}(\sigma)\right)\\
&< \max_{0\le k\le N} H_{\Lambda,h}(L^{(k)}) - H_{\Lambda,h}(-\mathbf{1})\\
&= V_{-\mathbf{1}}.
\end{aligned}$$

   (c) If $\sigma^{(n)} = +\mathbf{1}$ and $\eta^{(m)} = -\mathbf{1}$, let us define the path $\gamma_0:\sigma \to \eta$ given by

$$\gamma_0 = (\sigma^{(1)},\dots,\sigma^{(n-1)},L^{(N)},\dots,L^{(0)},\eta^{(m-1)},\dots,\eta^{(1)}) \tag{2.56}$$

   satisfies

$$\begin{aligned}
\Phi(\sigma,\eta) - H_{\Lambda,h}(\sigma) &\le \max_{\zeta\in\gamma_0} H_{\Lambda,h}(\zeta) - H_{\Lambda,h}(\sigma)\\
&= \left(\max_{\zeta\in\gamma_1} H_{\Lambda,h}(\zeta)\right) \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)})\right) \vee \left(\max_{\zeta\in\gamma_2} H_{\Lambda,h}(\zeta)\right) - H_{\Lambda,h}(\sigma)\\
&= H_{\Lambda,h}(\sigma) \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)})\right) \vee H_{\Lambda,h}(\eta) - H_{\Lambda,h}(\sigma)\\
&= 0 \vee \left(\max_{0\le k\le N} H_{\Lambda,h}(L^{(k)}) - H_{\Lambda,h}(\sigma)\right)\\
&< \max_{0\le k\le N} H_{\Lambda,h}(L^{(k)}) - H_{\Lambda,h}(-\mathbf{1})\\
&= V_{-\mathbf{1}}.
\end{aligned}$$

We conclude that for every $\sigma \notin \{-\mathbf{1}, +\mathbf{1}\}$, we have $V_\sigma < V_{-\mathbf{1}}$. ■

2.5. Proofs of the critical droplets results

Proof of Proposition 2.6. As in the proof of Theorem 2.3, let us define $f : \{0, \dots, N\} \to \mathbb{R}$ as

$$f(i) = H_{\Lambda,h}(L^{(i)}), \tag{2.57}$$

and recall that

$$\Delta f(i) = 2\left(\sum_{n=1}^{N-i-1} J(n) - \sum_{n=1}^{i} J(n) - h\right). \tag{2.58}$$

In the first case, we have $\Delta f(L-1) = 2(h_{L-1}^{(N)} - h) > 0$, thus, since f decreases for all i greater than L, and since $\Delta^2 f < 0$, we conclude that f attains a unique strict global maximum at L. In the second case, we have $\Delta f(k-1) = 2(h_{k-1}^{(N)} - h) > 0$ and $\Delta f(k) = 2(h_k^{(N)} - h) < 0$, so, f attains a unique strict global maximum at k. Finally, in the third case, we have $\Delta f(k) = 0$, that is, $f(k) = f(k+1)$. Using the fact that $\Delta f(k+1) < 0 < \Delta f(k-1)$, we conclude that the global maximum of f can we only be reached at k and $k+1$. ■

Proof of Corollary 2.7. Since $\sum_{n=1}^{\infty} J(n)$ converges, it follows that the set in equation (2.20) is nonempty, thus k_c is well defined. Then, we have

$$\sum_{n=k_c+1}^{\infty} J(n) \le h < \sum_{n=k_c}^{\infty} J(n). \tag{2.59}$$

For all N sufficiently large such that $\lfloor \frac{N}{2} \rfloor > k_c$ and

$$\sum_{n=N-k_c+1}^{\infty} J(n) < \sum_{n=k_c}^{\infty} J(n) - h, \tag{2.60}$$

we have

$$h < \sum_{n=k_c}^{\infty} J(n) - \sum_{n=N-k_c+1}^{\infty} J(n) = h_{k_c-1}^{(N)} \tag{2.61}$$

and

$$h_{k_c}^{(N)} = \sum_{n=k_c+1}^{\infty} J(n) - \sum_{n=N-k_c}^{\infty} J(n) < h. \tag{2.62}$$

Therefore, by means of Proposition 2.6, we conclude that for N large enough, k_c satisfies

$$H_{\Lambda,h}(\mathscr{P}^{(k_c)}) > \max_{\substack{0 \le i \le N \\ i \ne k_c}} H_{\Lambda,h}(\mathscr{P}^{(i)}). \tag{2.63}$$

■

2.6. CONCLUSION

In this work we managed to provide the answers for the questions posed at the beginning of this chapter. Although the proposed problem was relatively simple compared to those which inspired this work, our results outweigh this fact by revealing a rich and unexpected behavior of long range Ising systems. By adopting a pathwise approach, we studied the energy landscape of such systems in one dimension in a very general setting and derived its metastability features, including descriptions related to tunneling time, nucleation and critical droplets.

Let us note that our results refer to one-dimensional long range Ising models with free boundary condition. So, it is expected that the same results should also hold if we consider periodic boundary conditions. In fact, such results can be straightforwardly extended to that context (to be published) by computing the ground states corresponding to the new Hamiltonian restricted to each manifold $\mathcal{M}_k$ and using techniques similar to those we applied here. In the forthcoming works we are going to direct our efforts towards the generalization of the results considering such a class of systems in higher dimension. We expect that a richer variety of behavior would emerge due to the fact that in higher dimension the nucleation process may generate droplets with more complex shapes which strongly depend on the chosen long range pair interaction.

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3

Probabilistic Cellular Automata

3.1. Introduction

Cellular Automata (CAs) are discrete-time dynamical systems on a spatially extended discrete space. They are well known for being easy to implement and for exhibiting a rich and complex nonlinear behaviour as emphasized for instance in [1–4]; furthermore, they can give rise to multiple levels of organization [5]. Probabilistic Cellular Automata (PCAs) whose the updating rule is now considered to be stochastic, see [6], are a straightforward generalization of CAs and are employed as modeling tools in a wide range of applications, e.g. HIV infection [7], biological immune system [8], weather forecast [9], heart pacemaker tissue [10], and opinion forming [11]. Moreover, a natural context in which the PCAs main ideas are of interest is that of evolutionary games [12–14].

Strong relations exist as well between PCAs and the general equilibrium statistical mechanics framework [15–24]. A central question is the characterization of the equilibrium behavior of a general PCA dynamics. For instance, one primary interest is the study of its ergodic properties, e.g. the long-term behavior of the PCA and its dependence on the initial probability distribution. Regarding the ergodicity for PCAs on infinite lattices, see for instance [23] for details and references. Moreover, conditions for ergodicity for general PCAs can be found in the following papers: [17, 25–28]. Furthermore, in case of a translation-invariant PCA on $\mathbb{Z}^d$ with positive rates, it has been shown in [20] that the law of the trajectories, starting from any stationary distribution, is given by a Gibbs state for some space-time associated potential (in $\mathbb{Z}^{d+1}$). Moreover, it has also been proven that the converse is true: all the translation-invariant Gibbs states for such potential correspond to statistical space-time histories for the PCA. Therefore, phase transition for the space-time potential is closely related to the PCA ergodicity in the sense that non-uniqueness of translation invariant Gibbs states is equivalent to non-uniqueness of stationary measures for the PCA. The main ingredient for proving this result is the use of the local variational principle for the entropy density of the Gibbs measure. However, as it has been proved in [29], the variational principle for Gibbs states fails for nearest-neighbor finite state statistical mechanics systems on 3-ary trees. Hence, a first result to this chapter is to extend the results presented by [20] for a class of PCAs on infinite rooted trees. In particular, the PCAs considered in this work have positive rate shift-invariant local transition probabilities such that each local probabilistic rule depends only on the spins of the children of the node. This class of PCAs has generally a Bernoulli product measure as an invariant measure, and they are the natural generalization on trees of the models considered in [30].

A second type of results in this work is to give conditions for ergodicity in case of d-ary trees, with $d \in \{1,2,3\}$. Our positive rate PCAs satisfy indeed such conditions (i.e. (3.8) and (3.14)) that, when iterating the dynamics from the Bernoulli product measure, the resulting space-time diagram defines non-trivial random fields with very weak dependences. This fact allows us to give a detailed analysis of the ergodicity problem and, for two relevant examples of PCA dynamics, we are able to find the critical parameters. All the results in this chapter were published in [31].

This chapter is organized as follows. In Section 3.2 we extend the results of [20] in case of infinite rooted d-ary trees. We first define the PCA on a countably infinite set and in this general framework we show how stationary measures for a PCA can be naturally associated to Gibbs measures (Theorem 3.3). In order to state the converse result, we first restrict ourselves to the case of infinite rooted trees and to PCAs with non-degenerate shift-invariant local transition probabilities that depends only on the spins of the children of the node. For this class of PCAs, we state that all the time-invariant Gibbs states for the potential correspond to statistical space-time histories for the PCA (Theorem 3.6). In Section 3.3 we give results concerning conditions for the ergodicity of the PCA on d-ary trees. First we characterize Bernoulli product stationary measures via Lemma 3.8. In Theorem 3.9 we show that for $d = 1$ the PCA is always ergodic, and the same occurs for $d = 2$ with the additional assumption of spin-flip symmetry of the local transition probabilities. In Theorem 3.10 the case of $d = 3$ is studied. We give two examples (in Section 3.3.1) where the critical parameters can be computed. Section 3.4 and the Appendices are devoted to the proofs of the main results.

3.2. FROM PCAS TO GIBBS MEASURES AND BACK

3.2.1. PCAS ON COUNTABLY INFINITE SETS

Let the single spin space be a nonempty finite set S and let V denote a countably infinite set (for example, the d-dimensional cubic lattice $\mathbb{Z}^d$ or, more generally, the vertex set of a countably infinite graph). In the following we introduce a special class of discrete-time Markov chains on the state space $\Omega_0 = S^{\mathsf{V}}$ whose main feature is the fact that given the previous configuration, for the next one all spins are simultaneously updated accoding to independent local transition probabilities (parallel updating), the so-called probabilistic cellular automata.

We define the probabilistic cellular automaton as follows.

Definition 3.1. A PCA is a discrete-time Markov chain on Ω_0 with the following properties. At each site i in V

(a) corresponding to each configuration $x \in \Omega_0$ we associate a probability measure $p_i(\cdot|x)$ on S, and

(b) assume that for every spin s, the map

$$x \longmapsto p_i(s|x)$$

is a local function. So, there is a finite subset $U(i)$ of V such that the equality $p_i(s|x) = p_i(s|y)$ holds for every s whenever x and y satisfy $x_j = y_j$ for each j in $U(i)$.

In this setting, we associate to each point x in Ω_0 the product measure

$$P(dy|x) = \bigotimes_{i \in \mathsf{V}} p_i(dy_i|x), \tag{3.1}$$

and introduce the probabilistic cellular automaton dynamics on our state space Ω_0 by considering the Markov kernel P given by the expression

$$P(x, B) = P(B|x) \tag{3.2}$$

where B is a Borel set of Ω_0.

Now, we recall the definition of a stationary measure for the dynamics P.

Definition 3.2. A probability measure ν on Ω_0 is called stationary for the dynamics P defined above if

$$\int P(x, B)\nu(dx) = \nu(B)$$

holds for every Borel set B of Ω_0.

3.2.2. FROM PCA TO GIBBS MEASURES...

In this section we will show how stationary measures for a PCA can be naturally associated to Gibbs measures for a corresponding equilibrium statistical mechanical model. Let us consider the set of sites given by the countably infinite set $\mathbb{Z} \times \mathsf{V}$, the collection $\mathscr{S}$ consisting of all nonempty finite subsets of $\mathbb{Z} \times \mathsf{V}$. We also consider the configuration space $\Omega = S^{\mathbb{Z} \times \mathsf{V}}$ together with its product σ-algebra $\mathscr{F}$. Given an arbitrary space-time spin configuration ω in Ω, for each site x in $\mathbb{Z} \times \mathsf{V}$, say $\mathsf{x} = (n, i)$, let $\omega_{n,i}$ denote the value ω_{x} of the spin at this site, just for simplicity. Furthermore, for each integer n and each configuration ω, we define the configuration at time n as the element ω_n of Ω_0 given by $\omega_n = (\omega_{n,i})_{i \in \mathsf{V}}$.

Now, let us consider again the setting from the previous section. We will assume that the PCA dynamics is nondegenerate, that is, the local transition probabilities have positive rates: $p_i(s|x) > 0$ holds for all $i \in \mathsf{V}$, $s \in S$ and $x \in \Omega_0$. Furthermore, we also suppose that for each site i, the set

$$\{j \in \mathsf{V} : i \in U(j)\} \tag{3.3}$$

is finite, which means that at each step in the dynamics of the PCA, each spin can have influence only on the future state of a finite number of spins. Given a stationary measure ν

for P, it is possible to construct a probability measure μ_ν on $(\Omega, \mathscr{F})$ uniquely determined by the identity

$$\mu_\nu(\omega_t \in B_0, \omega_{t+1} \in B_1, \dots, \omega_{t+n} \in B_n) = \int_{B_0} \nu(dx_0) \int_{B_1} P(x_0, dx_1) \cdots \int_{B_n} P(x_{n-1}, dx_n), \tag{3.4}$$

where t is an integer, n a positive integer, and $B_0, B_1, \dots, B_n$ are Borel sets of Ω_0. In the following, given a site x in $\mathbb{Z} \times \mathsf{V}$, say $\mathsf{x} = (n, i)$, we will use $U(\mathsf{x})$ to denote the set

$$U(\mathsf{x}) = \{(n-1, j) : j \in U(i)\}.$$

Observe that our assumption (3.3) is equivalent to say that for each point x, the set

$$\{\mathsf{y} \in \mathbb{Z} \times \mathsf{V} : \mathsf{x} \in U(\mathsf{y})\}$$

is finite. This remark is very useful for proving the next theorem, whose proof is given in Appendix C.

Theorem 3.3. *The space-time measure μ_ν obtained from a stationary measure ν for the PCA is a Gibbs measure for the interaction $\Phi = (\Phi_A)_{A \in \mathscr{S}}$, where each $\Phi_A : \Omega \to \mathbb{R}$ is given by*

$$\Phi_A(\omega) = \begin{cases} -\log p_i(\omega_{\mathsf{x}} | \omega_{n-1}) & \textit{if } A = \{\mathsf{x}\} \cup U(\mathsf{x}) \textit{ for some } \mathsf{x} = (n, i), \\ 0 & \textit{otherwise.} \end{cases} \tag{3.5}$$

3.2.3. PCA ON INFINITE ROOTED TREES

We specify now the class of PCAs that will be considered in this work. We introduce indeed probabilistic cellular automata on d-ary trees $\mathsf{V} = \mathbb{T}^d$ with root $\mathbf{o}$ and degree $\deg(x) = d+1$ for all vertices $x \neq \mathbf{o}$ and $\deg(\mathbf{o}) = d$. Without loss of generality, the d-ary tree $\mathbb{T}^d$ can be regarded as the set

$$\bigcup_{n \geq 0} \{0, \dots, d-1\}^n$$

consisting of all finite sequences of integers from 0 to $d-1$. Given finite sequences i in $\{0, \dots, d-1\}^n$ and j in $\{0, \dots, d-1\}^m$, say $i = (i_k)_{k=0}^{n-1}$ and $j = (j_k)_{k=0}^{m-1}$, we naturally define their sum $i + j$ as the concatenation of these sequences, i.e., the sum is the element of $\{0, \dots, d-1\}^{m+n}$ given by

$$(i+j)_k = \begin{cases} i_k & \text{if } k \in \{0, \dots, n-1\}, \\ j_{k-n} & \text{if } k \in \{n, \dots, m+n-1\}. \end{cases}$$

Once defined the translation on $\mathbb{T}^d$, then we are allowed to associate to each site i in $\mathbb{T}^d$ the shift map $\Theta_i : S^{\mathbb{T}^d} \to S^{\mathbb{T}^d}$ defined by

$$\Theta_i x = (x_{i+j})_{j \in \mathbb{T}^d} \tag{3.6}$$

at each point $x = (x_j)_{j \in \mathbb{T}_d}$. Furthermore, for each $k \in \{0, \dots, d-1\}$, we denote by e_k the sequence $e_k = (k)$ consisting only of the number k, therefore, the e_k's are the neighbors of the root $\mathbf{o}$ of $\mathbb{T}^d$.

From now on, we consider the single spin space $S = \{-1,+1\}$, so, the state space Ω_0 is described as $\Omega_0 = \{-1,+1\}^{\mathbb{T}^d}$. Following [32, 33], we give the definitions of attractive dynamics and of repulsive dynamics. In order to do that we introduce the notation $x \le y$ to indicate that x and y are elements of Ω_0 that satisfy $x_i \le y_i$ for all $i \in \mathbb{T}^d$.

Definition 3.4. We call the dynamics P attractive if for every positive integer n, for all configurations x, y such that $x \le y$ and each nondecreasing local function f, we have

$$P^n(x, f) \le P^n(y, f). \tag{3.7}$$

3

Definition 3.5. We call the dynamics P repulsive if for every positive integer n, for all configurations x, y such that $x \le y$ and each nondecreasing local function f, we have

$$P^n(x, f) \ge P^n(y, f). \tag{3.8}$$

By [32, 33] it follows that the dynamics is attractive if and only if for all configurations x, y such that $x \le y$ we have $p_{\mathbf{o}}(+1|x) \le p_{\mathbf{o}}(+1|y)$; furthermore, it is repulsive if and only if for all configurations x, y such that $x \le y$ we have $p_{\mathbf{o}}(+1|x) \ge p_{\mathbf{o}}(+1|y)$.

The PCAs considered in this work have nondegenerate shift-invariant local transition probabilities such that each probabilistic rule $p_i(\cdot|x)$ depends only on the spins of the children of i. More precisely, we will state the following assumptions on the transition kernel.

Assumptions:

(A1) each $p_{\mathbf{o}}(\cdot|x)$ is a probability measure such that $p_{\mathbf{o}}(s|x) > 0$ holds for all $s \in \{-1,+1\}$,

(A2) the map $x \mapsto p_{\mathbf{o}}(s|x)$ depends only on the values of x on $U(\mathbf{o}) = \{e_0, \dots, e_{d-1}\}$, and

(A3) for each i in $\mathbb{T}^d \setminus \{\mathbf{o}\}$, the local transition probability $p_i(\cdot|x)$ satisfies

$$p_i(s|x) = p_{\mathbf{o}}(s|\Theta_i x). \tag{3.9}$$

Note that Assumption (A1) is the so-called nondegeneracy property, while Assumption (A3) is the invariance of the PCA dynamics under tree shifts. We remark as well that, it follows from $(A2)$ and $(A3)$ that the map $x \mapsto p_i(s|x)$ depends only on the values assumed by the spins of x on $U(i) = i + \{e_0, \dots, e_{d-1}\}$. One of the crucial features of this dynamics P is that under Assumptions (A2) and (A3) the relation

$$P^n(x, \{y_F = \xi\}) = \prod_{i \in F} P^n(\Theta_i x, \{y_{\mathbf{o}} = \xi_i\}) \tag{3.10}$$

holds for every configuration x, finite volume configuration $(\xi_i)_{i \in F}$ for some $F \subseteq \mathbb{T}^d$, and positive integer n.

3.2.4. ...AND BACK

According to Theorem 3.3, every stationary measure for the PCA defined above can be associated to a Gibbs measure for the corresponding statistical mechanical model Φ defined by (3.5). Next, we show that for the class of PCAs on trees we are dealing with, under suitable conditions, the converse is also valid.

3

Theorem 3.6. *Under the Assumption (A1)-(A3), let μ be a Gibbs measure for the interaction Φ defined by (3.5), such that it is invariant under time translations, i.e., μ is a Gibbs measure that satisfies*

$$\mu(\omega_m \in B) = \mu(\omega_{m-1} \in B)$$

for each integer m and each Borel subset B of Ω_0. Then, there is a stationary measure ν for the corresponding PCA such that $\mu = \mu_\nu$.

Therefore, thanks to Theorem 3.6 the study of the ergodicity of the PCA can be closely related to the study the uniqueness of the Gibbs measure on space-time associated to it.

Remark 3.7. In Appendix C, we give a more general proof for Theorem 3.6. It actually holds for any PCA on $\Omega_0 = S^{\mathsf{V}}$, where S is a nonempty finite set and V is a (locally finite) infinite rooted tree, satisfying $(A1)$ and

(A2') Let $d : \mathsf{V} \times \mathsf{V} \to \mathbb{R}$ be the distance function that assigns to each pair (i, j) of vertices the length of the unique path connecting them. Corresponding to each point i that belongs to V the set $U(i)$ is a finite set such that

$$U(i) \subseteq \{j \in \mathsf{V} : d(\mathbf{o}, i) < d(\mathbf{o}, j)\}. \tag{3.11}$$

3.3. CONDITIONS FOR ERGODICITY FOR PCAS ON TREES

In this section we will present some results regarding sufficient conditions for ergodicity for the class of PCAs described previously. Note that equation (3.10) implies that the probability distributions of the spins at time n are independent, so, this suggests that the typical stationary measures we have to look for are product measures. This remark leads us to state a lemma regarding the characterization of stationary Bernoulli product measures, whose proof is given in Appendix C.3.

Lemma 3.8. *A Bernoulli product measure $\nu = \mathsf{Bern}(\mathsf{p})^{\otimes \mathbb{T}^d}$ with parameter $\mathsf{p} \in [0,1]$, is a stationary measure for P if and only if*

$$\int p_{\mathbf{o}}(+1|x)\nu(dx) = \mathsf{p} \tag{3.12}$$

i.e. if and only if

$$\sum_{l=0}^{d} (-1)^l \left[\sum_{\substack{I \subseteq \{0,\dots,d-1\} \\ |I| = l}} \left(\sum_{\substack{\xi \in \{-1,+1\}^d \\ \xi_k = -1 \text{ for all } k \in I}} (-1)^{\#\{m : \xi_m = -1\}} p_{\mathbf{o}}(+1|\xi) \right) \right] \mathsf{p}^{d-l} = \mathsf{p}. \tag{3.13}$$

Moreover, the probability to find the spin $+1$ at the root of $\mathbb{T}^d$ after $n+1$ steps of this dynamics starting from the configuration x can be written as

$$P^{n+1}(x,\{y_{\mathbf{o}}=+1\}) = \tag{3.14}$$

$$\sum_{l=0}^{d}(-1)^l\left[\sum_{\substack{I\subseteq\{0,\dots,d-1\}\\|I|=l}}\left(\sum_{\substack{\xi\in\{-1,+1\}^d\\ \xi_k=-1\text{ for all }k\in I}}(-1)^{\#\{m:\xi_m=-1\}}p_{\mathbf{o}}(+1|\xi)\right)\prod_{k\in\{0,\dots,d-1\}\setminus I}P^n(\Theta_{e_k}x,\{y_{\mathbf{o}}=+1\})\right].$$

3.3.1. ERGODICITY AND EXAMPLES

From now on, we will abbreviate $+1$ by $+$ (resp. -1 by $-$). In the first theorem we prove ergodicity results for the line and the binary trees, while in the second theorem we prove ergodicity and non-ergodicity results for the 3-ary trees.

Theorem 3.9. *Let us consider a PCA with transition probabilities satisfying (A1)-(A3). Then, we have the following results.*

(a) If $d=1$, then the PCA dynamics is ergodic. The unique stationary measure is a Bernoulli product measure with parameter

$$\mathsf{p}=\frac{p_{\mathbf{o}}(+|-)}{p_{\mathbf{o}}(-|+)+p_{\mathbf{o}}(+|-)}. \tag{3.15}$$

(b) Let $d=2$ and the transition probabilities being symmetric under spin-flip, i.e., the equality $p_{\mathbf{o}}(s|x)=p_{\mathbf{o}}(-s|-x)$ holds for every spin s and each configuration x. Then the PCA dynamics is ergodic, where its unique stationary measure is $\mathrm{Bern}\left(\frac{1}{2}\right)^{\otimes\mathbb{T}^2}$.

Theorem 3.10. *Let $d=3$ and let the transition probabilities be symmetric under spin-flip. Denote by $\alpha:=p_{\mathbf{o}}(+|+++)$ and $\gamma:=p_{\mathbf{o}}(+|-++)+p_{\mathbf{o}}(+|+-+)+p_{\mathbf{o}}(+|++-)$. Then the PCA transition rule is*

(a) ergodic, if α and γ satisfy

(i) $1+\alpha-\gamma=0$, or

(ii) the PCA dynamics is attractive and $1+\alpha-\gamma\neq 0$ and $3\alpha+\gamma\le 5$, or

(iii) the PCA dynamics is repulsive and $1+\alpha-\gamma\neq 0$ and $3\alpha+\gamma\ge 1$.

In this case the unique stationary measure is given by $\mathrm{Bern}\left(\frac{1}{2}\right)^{\otimes\mathbb{T}^3}$.

(b) non-ergodic, if α and γ satisfy

(i) $1+\alpha-\gamma\neq 0$ and $3\alpha+\gamma>5$. In this case, we have several stationary Bernoulli product measures with parameter

$$\mathsf{p}\in\left\{\frac{1}{2},\frac{1+\sqrt{1+\frac{4(1-\alpha)}{1+\alpha-\gamma}}}{2},\frac{\sqrt{1-\frac{4(1-\alpha)}{1+\alpha-\gamma}}}{2}\right\},$$

or

(ii) the PCA dynamics is repulsive and $1+\alpha-\gamma\neq 0$ *and* $3\alpha+\gamma<1$.

Remark 3.11. In the last case (Theorem 3.10 (b)-(ii)), we can actually prove that the PCA oscillates between two Bernoulli product measures with distinct parameters p. Further details are presented in Section 3.4.2.

Before we pass to the proofs of the theorems we will discuss some examples.

3

EXAMPLE 1

For $d=3$ and $\beta>0$, let us consider the PCA with transition probabilities given by

$$p_i(s|x) = \frac{1}{2}\left(1+s\tanh\left(\beta\sum_{k=0}^{2} J_k x_{i+e_k}\right)\right) \tag{3.16}$$

where J_0, J_1 and $J_2 \in \mathbb{R}$. Hence, for suitable values of the constants, there exists a critical $\beta_c \in (0,\infty)$ such that the PCA is ergodic for $\beta \leq \beta_c$ and non-ergodic otherwise. In fact the following result holds.

Proposition 3.12. *Suppose that one of the following conditions on the coupling constants* J_0, J_1, J_2 *is fulfilled.*

(C1) $J_0, J_1, J_2 > 0$ *and* $J_0 \leq J_1 + J_2$, $J_1 \leq J_0 + J_2$, *and* $J_2 \leq J_0 + J_1$.

(C2) $J_0, J_1, J_2 < 0$ *and* $J_0 \geq J_1 + J_2$, $J_1 \geq J_0 + J_2$, *and* $J_2 \geq J_0 + J_1$.

Let α, γ *be defined as in Theorem 3.10, and let function* $f : \mathbb{R}_+ \to \mathbb{R}$ *be defined as*

$$f(\beta) = 3\alpha + \gamma.$$

Then, there exists $\beta_c \in (0,\infty)$ *depending on the constants* J_0, J_1, J_2 *such that for*

(a) $\beta \leq \beta_c$ *the PCA dynamics associated to the local transition probabilities given by* (3.16) *is ergodic, and*

(b) $\beta > \beta_c$ *the dynamics is non-ergodic.*

Remark 3.13. Note that, thanks to the spin-flip symmetry of the probabilities (3.16), we can apply Theorem 3.10. Moreover, we remark that the lattice model equivalent to (3.16) has been extensively studied in [34].

Remark 3.14. If condition $(C1)$ holds, then $\beta_c = f^{-1}(5)$. Otherwise, if $(C2)$ holds, then $\beta_c = f^{-1}(1)$. In particular, if $J_0 = J_1 = J_2 = J \in \mathbb{R}\backslash\{0\}$, it follows that $\beta_c = \frac{1}{2|J|}\log(1+2^{2/3})$. In [26] a similar ferromagnetic PCA has been studied on $\mathbb{Z}^d$ where in the particular case $d=2$ the value of β_c is given by $\beta_c = \frac{1}{2J}\log(1+\sqrt{2})$.

EXAMPLE 2

Let us consider the PCA on the 3-ary tree defined as follows. Suppose that at each step every spin assume the value corresponding to the majority among their children. After that each spin make an error with a probability $\epsilon \in (0,1)$ independently of each other, that is, if the spin at the site i assumed the value $+1$ (resp. -1), then it will change to -1 (resp. $+1$) with probability ϵ and keep the value $+1$ (resp. -1) with probability $1-\epsilon$. Note that such a system follows a CA dynamics, namely the majority rule, with the addition of a noise. For a more detailed study of this kind of PCAs, see [35].

In the example described above, we have

$$p_{\mathbf{o}}(+|+++) = p_{\mathbf{o}}(+|++-) = p_{\mathbf{o}}(+|+-+) = p_{\mathbf{o}}(+|-++) = 1-\epsilon.$$

This PCA has been first studied in [32], where non-ergodicity has been proven only for sufficiently small ϵ. In the next proposition we fully characterize its behavior for the whole range of ϵ.

Proposition 3.15. *There exist two critical values $\epsilon_c^{(1)} = \frac{1}{6}$ and $\epsilon_c^{(2)} = \frac{5}{6}$ such that for every* $\epsilon \in (0,1)$

(a) *the PCA dynamics is ergodic if $\epsilon_c^{(1)} \le \epsilon \le \epsilon_c^{(2)}$, and*

(b) *non-ergodic for $\epsilon \notin [\epsilon_c^{(1)}, \epsilon_c^{(2)}]$.*

3.4. PROOFS OF ERGODICITY RESULTS

3.4.1. PROOF OF THEOREM 3.9

CASE (A)

Proof. Note that a PCA on $\mathbb{T}^1$ is equivalent to a PCA model on $\mathbb{Z}_+$. In order to simplify the computations, let us use a and b to denote $p_{\mathbf{o}}(+|+)$ and $p_{\mathbf{o}}(+|-)$, respectively. Since the local transition probabilities have positive rates, then, we have $|a-b| < 1$. It follows that for each point x in Ω_0, we have

$$\begin{aligned} P^{n+1}(x,\{y_{\mathbf{o}}=+1\}) &= \int P(z,\{y_{\mathbf{o}}=+1\})P^n(x,dz) \\ &= a\cdot P^n(x,\{y_{e_0}=+1\}) + b\cdot P^n(x,\{y_{e_0}=-1\}) \\ &= (a-b)\cdot P^n(x,\{y_{e_0}=+1\}) + b \\ &= (a-b)\cdot P^n(\Theta_{e_0}x,\{y_{\mathbf{o}}=+1\}) + b \end{aligned}$$

for each positive integer n. Note that the relation above can also be obtained by means of equation (3.14). Thus, the quantity above can be expressed as

$$P^n(x,\{y_{\mathbf{o}}=+1\}) = (a-b)^{n-1}\cdot p_{\mathbf{o}}(+1|\Theta_{\underbrace{e_0+\cdots+e_0}_{n-1\,\text{times}}}x) + b\cdot\sum_{k=0}^{n-2}(a-b)^k.$$

It follows that for any initial configuration x, the probability $P^n(x,\{y_{\mathbf{o}}=+1\})$ converges to $\mathsf{p} = \frac{b}{1-(a-b)}$ as n approaches infinity. Therefore, using equation (3.10), we conclude that this PCA is ergodic, where its unique attractive stationary measure is $\mathsf{Bern}(\mathsf{p})^{\otimes\mathbb{T}^1}$. ■

CASE (B)

Proof. Let $a, b \in (0,1)$ defined by $a = p_{\mathbf{o}}(+|--) = 1 - p_{\mathbf{o}}(+|++)$ and $b = p_{\mathbf{o}}(+|-+) = 1 - p_{\mathbf{o}}(+|+-)$, respectively. Let us show that $\mathrm{Bern}(\frac{1}{2})^{\otimes\mathbb{T}^2}$, in fact, is the unique attractive stationary measure, that is, for every initial configuration x we have $P^n(x,\cdot) \to \mathrm{Bern}(\frac{1}{2})^{\otimes\mathbb{T}^2}$ as n approaches infinity. According to equation (3.14), we have

$$P^{n+1}(x,\{y_{\mathbf{o}}=+1\}) = (1-b-a)P^n(\Theta_{e_0}x,\{y_{\mathbf{o}}=+1\}) + (b-a)P^n(\Theta_{e_1}x,\{y_{\mathbf{o}}=+1\}) + a.$$

By induction, we can show that

$$\begin{aligned} P^n(x,\{y_{\mathbf{o}}=+1\}) \quad &= \sum_{i\in\{0,1\}^{n-1}} (1-b-a)^{\#\{k:i_k=0\}}(b-a)^{\#\{k:i_k=1\}}P(\Theta_i x,\{y_{\mathbf{o}}=+1\}) \\ &\quad + a\sum_{l=0}^{n-2}\sum_{i\in\{0,1\}^l}(1-b-a)^{\#\{k:i_k=0\}}(b-a)^{\#\{k:i_k=1\}}. \end{aligned}$$

Using the fact that for any real numbers p and q, the relation

$$\sum_{i\in\{0,1\}^l} p^{\#\{k:i_k=0\}}q^{\#\{k:i_k=1\}} = (p+q)^l$$

holds for every nonnegative integer l, it follows that

$$P^n(x,\{y_{\mathbf{o}}=+1\}) = \sum_{i\in\{0,1\}^{n-1}} (1-b-a)^{\#\{k:i_k=0\}}(b-a)^{\#\{k:i_k=1\}}P(\Theta_i x,\{y_{\mathbf{o}}=+1\}) + a\sum_{l=0}^{n-2}(1-2a)^l. \tag{3.17}$$

Since the absolute value of the first term of equation (3.17) is bounded by

$$\sum_{i\in\{0,1\}^{n-1}} |1-b-a|^{\#\{k:i_k=0\}}|b-a|^{\#\{k:i_k=1\}} = (|1-b-a|+|b-a|)^{n-1},$$

then

$$\lim_{n\to\infty} P^n(x,\{y_{\mathbf{o}}=+1\}) = a\sum_{l=0}^{\infty}(1-2a)^l = \frac{1}{2}. \tag{3.18}$$

Therefore, by means of equation (3.10), we conclude that $\mathrm{Bern}(\frac{1}{2})^{\otimes\mathbb{T}^2}$ is the unique attractive stationary measure of the PCA. ■

3.4.2. PROOF OF THEOREM 3.10

CASE (A)-(I) AND (B)-(I)

Proof. Recall we abbreviated $\alpha = p_{\mathbf{o}}(+|+++)$ and $\gamma = p_{\mathbf{o}}(+|-++) + p_{\mathbf{o}}(+|+-+) + p_{\mathbf{o}}(+|++-)$. From Lemma 3.8 we know that a stationary product measure has to satisfy the condition

$$\int p_{\mathbf{o}}(+1|x)\nu(dx) = \mathrm{p} \tag{3.19}$$

which was equivalent to solving equation (3.13), i.e.

$$2(1+\alpha-\gamma)\mathrm{p}^3 - 3(1+\alpha-\gamma)\mathrm{p}^2 + (3\alpha-\gamma-1)\mathrm{p} + (1-\alpha) = 0. \tag{3.20}$$

Since $\mathsf{p} = \frac{1}{2}$ is a solution for the equation above, then, it can be written as

$$2\left(\mathsf{p}-\frac{1}{2}\right)\left[(1+\alpha-\gamma)\mathsf{p}^2-(1+\alpha-\gamma)\mathsf{p}-(1-\alpha)\right]=0. \tag{3.21}$$

Suppose that $1+\alpha-\gamma=0$. Then, analogously as in the previous case, we have

$$\begin{aligned}&P^n(x,\{y_{\mathbf{o}}=+1\})\\ &=\sum_{i\in\{0,1,2\}^{n-1}}(\alpha-p_{\mathbf{o}}(+|++-))^{\#\{k:i_k=0\}}(\alpha-p_{\mathbf{o}}(+|+-+))^{\#\{k:i_k=1\}}(\alpha-p_{\mathbf{o}}(+|-++))^{\#\{k:i_k=2\}}P(\Theta_i x,\{y_{\mathbf{o}}=+1\})\\ &\quad+(1-\alpha)\sum_{l=0}^{n-2}\sum_{i\in\{0,1,2\}^l}(\alpha-p_{\mathbf{o}}(+|++-))^{\#\{k:i_k=0\}}(\alpha-p_{\mathbf{o}}(+|+-+))^{\#\{k:i_k=1\}}(\alpha-p_{\mathbf{o}}(+|-++))^{\#\{k:i_k=2\}}.\end{aligned}$$

The equation above implies that $P^n(x,\{y_{\mathbf{o}}=+1\})\to\frac{1}{2}$ as n approaches infinity, therefore, by means of the same argument as used in Section 3.4.1, we conclude that the dynamics is ergodic.

Now, if $1+\alpha-\gamma\neq 0$, we have two other solutions

$$\mathsf{p}_+ = \frac{1+\sqrt{1+\frac{4(1-\alpha)}{1+\alpha-\gamma}}}{2} \tag{3.22}$$

and

$$\mathsf{p}_- = \frac{1-\sqrt{1+\frac{4(1-\alpha)}{1+\alpha-\gamma}}}{2}. \tag{3.23}$$

Therefore, both p_- and p_+ are inside the interval $(0,1)$ and are different from $\frac{1}{2}$ if and only if $3\alpha+\gamma>5$. ∎

CASE (A)-(II)

Proof. Let us consider a PCA with attractive dynamics. Again, by using Lemma 3.8, we can find a map $F:[0,1]\to\mathbb{R}$

$$F(\mathsf{p}) = 2(1+\alpha-\gamma)\mathsf{p}^3-3(1+\alpha-\gamma)\mathsf{p}^2+(3\alpha-\gamma)\mathsf{p}+(1-\alpha) \tag{3.24}$$

such that its fixed points correspond to the parameters of the stationary Bernoulli product measures. We will show that F has a unique attractive fixed point at $\mathsf{p}=\frac{1}{2}$, that is, such fixed point satisfies $F^n(\mathsf{q})\to\mathsf{p}$ as n approaches infinity for any point $\mathsf{q}\in[0,1]$. Let us prove that F is an increasing function that satisfies

$$\begin{cases} F(\mathsf{p})>\mathsf{p} & \text{for all } \mathsf{p}<\frac{1}{2},\\ F(\frac{1}{2})=\frac{1}{2} & \text{and}\\ F(\mathsf{p})<\mathsf{p} & \text{for all } \mathsf{p}>\frac{1}{2}.\end{cases} \tag{3.25}$$

Suppose that $1+\alpha-\gamma<0$. Due to the attractiveness of the dynamics, it follows that $3\alpha\geq\gamma$ and the minimum value of F' given by $F'(0)=F'(1)=3\alpha-\gamma$ is nonnegative. Therefore, F is increasing. Moreover, the property (3.25) follows from the identity

$$F(\mathsf{p})-\mathsf{p}=2\left(\mathsf{p}-\frac{1}{2}\right)\left[(1+\alpha-\gamma)\mathsf{p}^2-(1+\alpha-\gamma)\mathsf{p}-(1-\alpha)\right] \tag{3.26}$$

where $(1+\alpha-\gamma)\mathsf{p}^2-(1+\alpha-\gamma)\mathsf{p}-(1-\alpha)<0$ for all $\mathsf{p}\neq\frac{1}{2}$. Now, let us consider the case where $1+\alpha-\gamma>0$. The attractiveness of the dynamics implies that $\gamma\geq 3(1-\alpha)$, so, the minimum value of F' is $F'(\frac{1}{2})=(-3+3\alpha+\gamma)/2\geq 0$. Again, we prove that F is increasing. Furthermore, we have (3.25) by means of the equation

$$F(\mathsf{p})-\mathsf{p}=2\left(\mathsf{p}-\frac{1}{2}\right)(1+\alpha-\gamma)(\mathsf{p}-\mathsf{p}_-)(\mathsf{p}-\mathsf{p}_+) \tag{3.27}$$

where $\mathsf{p}_-<0$ and $\mathsf{p}_+>1$ are given by equation (3.23) and (3.22), respectively. Since F is increasing, $F(0)=1-\alpha<\frac{1}{2}$ and $F(1)=\alpha>\frac{1}{2}$, then $F(\mathsf{p})$ belongs to $\left[1-\alpha,\frac{1}{2}\right)\subseteq\left[0,\frac{1}{2}\right)$ for all p in $\left[0,\frac{1}{2}\right)$ and $F(\mathsf{p})$ belongs to $\left(\frac{1}{2},\alpha\right]\subseteq\left(\frac{1}{2},1\right]$ for all p in $\left(\frac{1}{2},1\right]$. Using the continuity of F, we easily conclude that $\lim_{n\to\infty}F^n(\mathsf{q})=\frac{1}{2}$ for every point q that belongs to the interval $[0,1]$, therefore, $\mathsf{p}=\frac{1}{2}$ is the unique attractive fixed point for F.

It follows from equation (3.14) that

$$P^{n+1}(x_-,\{y_{\mathbf{o}}=+1\})=F(P^n(x_-,\{y_{\mathbf{o}}=+1\}))$$

and

$$P^{n+1}(x_+,\{y_{\mathbf{o}}=+1\})=F(P^n(x_+,\{y_{\mathbf{o}}=+1\})),$$

where x_- and x_+ are respectively the configurations with all spins -1 and $+1$ on $\mathbb{T}^3$. The conclusion above implies that both $P^n(x_-,\{y_{\mathbf{o}}=+1\})$ and $P^n(x_+,\{y_{\mathbf{o}}=+1\})$ converge to $\frac{1}{2}$ as n approaches infinity. Therefore, since the inequality $x_-\leq x\leq x_+$ holds for every configuration x, it follows from Definition 3.4 that

$$P^n(x_-,\{y_{\mathbf{o}}=+1\})\leq P^n(x,\{y_{\mathbf{o}}=+1\})\leq P^n(x_+,\{y_{\mathbf{o}}=+1\}), \tag{3.28}$$

therefore,

$$\lim_{n\to\infty}P^n(x,\{y_{\mathbf{o}}=+1\})=\frac{1}{2}. \tag{3.29}$$

Finally, we conclude that the probability $P^n(x,\cdot)$ converges to $\mathsf{Bern}(\frac{1}{2})^{\otimes\mathbb{T}^3}$ as n approaches infinity, independently on the initial configuration x, hence, the PCA dynamics is ergodic. ∎

CASE (A)-(III) AND (B)-(II)

Proof. Let us consider a new PCA described by a probability kernel Q defined by

$$Q(dy|x)=\bigotimes_{i\in\mathbb{T}^3}q_i(dy_i|x), \tag{3.30}$$

where each probability q_i is given by

$$q_i(\cdot|x)=p_i(\cdot|-x). \tag{3.31}$$

It is easy to see that this PCA satisfies the spin-flip condition. In the case where we have $3\alpha+\gamma\geq 1$, if we consider α' and γ' respectively defined by $\alpha'=q_{\mathbf{o}}(+|+++)$ and $\gamma'=q_{\mathbf{o}}(+|++-)+q_{\mathbf{o}}(+|+-+)+q_{\mathbf{o}}(+|-++)$, then we have

$$1+\alpha'-\gamma'=-(1+\alpha-\gamma)\neq 0,$$

and

$$3\alpha' + \gamma' = 6 - (3\alpha + \gamma) \leq 5.$$

Therefore, in this case the PCA dynamics described by Q is ergodic. It is easy to check that $P^n(x,\cdot) = Q^n((-1)^n x,\cdot)$ holds for every positive integer n and each configuration x. Therefore, the ergodicity of P follows.

In order to prove the non-ergodicity for the case $3\alpha + \gamma < 1$, let us consider again the function $F\colon [0,1] \to \mathbb{R}$ given by equation (3.24). It is straightforward to show that

$$F(\mathsf{p}) - (1-\mathsf{p}) = 2(1-\alpha-\gamma)\left(\mathsf{p} - \frac{1}{2}\right)(1-\mathsf{q}_-)(1-\mathsf{q}_+), \tag{3.32}$$

where q_- and q_+ are the elements in the interval $(0,1)$ given by

$$\mathsf{q}_- = \frac{1+\sqrt{1-\frac{4\alpha}{1+\alpha-\gamma}}}{2} \tag{3.33}$$

and

$$\mathsf{q}_+ = \frac{1-\sqrt{1-\frac{4\alpha}{1+\alpha-\gamma}}}{2}, \tag{3.34}$$

respectively. It follows that

$$\begin{cases} F(\mathsf{p}) < 1-\mathsf{p} & \text{if } \mathsf{p} \in [0,\mathsf{q}_-), \\ F(\mathsf{p}) > 1-\mathsf{p} & \text{if } \mathsf{p} \in (\mathsf{q}_-,\frac{1}{2}), \\ F(\mathsf{p}) < 1-\mathsf{p} & \text{if } \mathsf{p} \in (\frac{1}{2},\mathsf{q}_+), \text{ and} \\ F(\mathsf{p}) > 1-\mathsf{p} & \text{if } \mathsf{p} \in (\mathsf{q}_+,1]. \end{cases} \tag{3.35}$$

Because of the repulsiveness of the dynamics, we have $3\alpha - \gamma \leq 0$ and $F'(\frac{1}{2}) = \frac{1}{2}(-3+3\alpha+\gamma) < -1$, thus, F is a decreasing function. In addition, we have $F(\mathsf{p}) = 1 - F(1-\mathsf{p})$ for every p in $[0,1]$. So, we obtain

$$\begin{cases} \mathsf{p} < F^2(\mathsf{p}) < \mathsf{q}_- & \text{if } \mathsf{p} \in [0,\mathsf{q}_-), \\ \mathsf{q}_- < F^2(\mathsf{p}) < \mathsf{p} & \text{if } \mathsf{p} \in (\mathsf{q}_-,\frac{1}{2}), \\ \mathsf{p} < F^2(\mathsf{p}) < \mathsf{q}_+ & \text{if } \mathsf{p} \in (\frac{1}{2},\mathsf{q}_+), \text{ and} \\ \mathsf{q}_+ < F^2(\mathsf{p}) < \mathsf{p} & \text{if } \mathsf{p} \in (\mathsf{q}_+,1]. \end{cases} \tag{3.36}$$

Therefore, we conclude that

$$\lim_{n\to\infty} F^{2n}(\mathsf{p}) = \begin{cases} \mathsf{q}_- & \text{if } \mathsf{p} \in [0,\frac{1}{2}), \text{ and} \\ \mathsf{q}_+ & \text{if } \mathsf{p} \in (\frac{1}{2},1]; \end{cases} \tag{3.37}$$

similarly, we also have

$$\lim_{n\to\infty} F^{2n+1}(\mathsf{p}) = \begin{cases} \mathsf{q}_+ & \text{if } \mathsf{p} \in [0,\frac{1}{2}), \text{ and} \\ \mathsf{q}_- & \text{if } \mathsf{p} \in (\frac{1}{2},1]. \end{cases} \tag{3.38}$$

Thus, we finally conclude that, by means of equations (3.14), (3.37) and (3.38), the probabilities $P^{2n+1}(x_+,\cdot)$ and $P^{2n}(x_+,\cdot)$ converge to $\mathrm{Bern}(\mathsf{q}_-)^{\otimes\mathbb{T}^3}$ and $\mathrm{Bern}(\mathsf{q}_+)^{\otimes\mathbb{T}^3}$, respectively, as n approaches infinity. So, the PCA dynamics is not ergodic. ■

3.4.3. PROOF OF PROPOSITION 3.12

Proof. The PCA is fully described by the numbers

$$\begin{aligned} p_{\mathbf{o}}(+|+++) &= \frac{1}{2}\big(1+\tanh\beta(J_0+J_1+J_2)\big), \\ p_{\mathbf{o}}(+|++-) &= \frac{1}{2}\big(1+\tanh\beta(J_0+J_1-J_2)\big), \\ p_{\mathbf{o}}(+|+-+) &= \frac{1}{2}\big(1+\tanh\beta(J_0-J_1+J_2)\big), \end{aligned}$$

and

$$p_{\mathbf{o}}(+|-++) = \frac{1}{2}\big(1+\tanh\beta(-J_0+J_1+J_2)\big).$$

Note that assumption (C1) from Example 1 implies that $J_0+J_1-J_2 < J_0+J_1+J_2$, $J_0-J_1+J_2 < J_0+J_1+J_2$, and $-J_0+J_1+J_2 < J_0+J_1+J_2$; and at most one of the quatities $J_0+J_1-J_2$, $J_0-J_1+J_2$ and $-J_0+J_1+J_2$ can be equal zero. Therefore, the map $g:\mathbb{R}\to\mathbb{R}$ given by

$$\begin{aligned} g(\beta) &= 1+\alpha-\gamma \\ &= \frac{1}{2}(\tanh\beta(J_0+J_1+J_2)-\tanh\beta(J_0+J_1-J_2)-\tanh\beta(J_0-J_1+J_2) \\ &\quad -\tanh\beta(-J_0+J_1+J_2)) \end{aligned}$$

satisfies $g(0)=0$ and

$$\begin{aligned} g'(\beta) &= \frac{1}{2}\bigg(\frac{J_0+J_1+J_2}{\cosh^2\beta(J_0+J_1+J_2)}-\frac{J_0+J_1-J_2}{\cosh^2\beta(J_0+J_1-J_2)}-\frac{J_0-J_1+J_2}{\cosh^2\beta(J_0-J_1+J_2)} \\ &\quad -\frac{-J_0+J_1+J_2}{\cosh^2\beta(-J_0+J_1+J_2)}\bigg) \\ &< \frac{1}{2\cosh^2\beta(J_0+J_1+J_2)}((J_0+J_1+J_2)-(J_0+J_1-J_2)-(J_0-J_1+J_2)-(-J_0+J_1+J_2)) \\ &= 0. \end{aligned}$$

It follows that $g(\beta)=1+\alpha-\gamma<0$ for all $\beta>0$. Moreover, note that the function $f:\mathbb{R}\to\mathbb{R}$ given by

$$\begin{aligned} f(\beta) &= 3\alpha+\gamma \\ &= 3+\frac{3}{2}\tanh\beta(J_0+J_1+J_2)+\frac{1}{2}(\tanh\beta(J_0+J_1-J_2)+\tanh\beta(J_0-J_1+J_2) \\ &\quad +\tanh\beta(-J_0+J_1+J_2)) \end{aligned}$$

is increasing, $f(0)=3$, and $\lim_{\beta\to\infty} f(\beta)\geq 5+\frac{1}{2}$. It follows that there is a unique positive real number β_c that satisfies $f(\beta_c)=5$. Since this PCA dynamics satisfies the spin-flip property and is attractive, according to Theorem 3.10, the PCA is ergodic for $\beta\leq\beta_c$ and non-ergodic for $\beta>\beta_c$.

Since we proved the result considering the case where condition $(C1)$ holds, the proof for the case $(C2)$ is straightforward. ■

3.4.4. PROOF OF PROPOSITION 3.15

Proof. Clearly the PCA satisfies the spin-flip property. Note that in both cases we have $1+\alpha-\gamma = 2\epsilon-1$. It follows that the PCA is ergodic for $\epsilon = \frac{1}{2}$. Furthermore, note that the PCA is attractive for $0 < \epsilon < \frac{1}{2}$, repulsive for $\frac{1}{2} < \epsilon < 1$, and in both cases we have $1+\alpha-\gamma \neq 0$.

Let us suppose that $\epsilon \in (0, \frac{1}{2})$. Since $3\alpha+\gamma = 6(1-\epsilon)$, it follows from Theorem 3.10 that the PCA is non-ergodic for $\epsilon < \frac{1}{6}$ and ergodic for $\frac{1}{6} \le \epsilon < \frac{1}{2}$. Now, if $\epsilon \in (\frac{1}{2}, 1)$, then again by Theorem 3.10, the PCA is ergodic for $\frac{1}{2} < \epsilon \le \frac{5}{6}$ and non-ergodic for $\frac{5}{6} < \epsilon < 1$. ■

3.5. CONCLUSION

In this work we proved the correspondence between stationary measures for PCAs on infinite rooted trees and time-invariant Gibbs measures for a corresponding statistical mechanical model. As mentioned before, the proof of such correspondence is very general and can be applied for any PCA on a (locally finite) infinite rooted tree with finite single spin space S. The main implication of this fact is once we establish conditions for uniqueness of Gibbs measures for such a system, we guarantee the uniqueness of stationary distributions for the associated PCA. On the other hand, the existence of multiple stationary measures implies on the phase transition in the statistical mechanical model. In this way we provide a partial relationship between ergodicity and phase transition extending the results from [20].

Restricting to the study of PCAs on a d-ary tree $\mathbb{T}^d$ with translation-invariant local transition probabilities with single spin space $S = \{-1,+1\}$, we were able to find ergodicity properties for such class of PCAs. The assumption that the choice of a local transition probability at a site i only depends upon the values of the spins of the children of i allowed us to derive several important properties, for instance, equations (3.10) and (3.14). Equation (3.10) shows us that the probability distributions of the spins at time n are independent, such fact lead us to characterize the stationary measures of such a system whose form are product measures. In this way, we naturally obtained a polynomial function F defined on the interval $[0,1]$ whose expression is given by

$$F(\mathsf{p}) = \sum_{l=0}^{d} (-1)^l \left[\sum_{\substack{I \subseteq \{0,\dots,d-1\} \\ |I| = l}} \left(\sum_{\substack{\xi \in \{-1,+1\}^d \\ \xi_k = -1 \text{ for all } k \in I}} (-1)^{\#\{m: \xi_m = -1\}} p_{\mathbf{o}}(+1|\xi) \right) \right] \mathsf{p}^{d-l} \tag{3.39}$$

such that, according to equation (3.13), a Bernoulli product measure with parameter p is a stationary measure for the PCA dynamics if and only if p is a fixed point of F. Furthermore, based on equation (3.14), the convergence of $P^n(x, \{y_{\mathbf{o}} = +1\})$ for a shift-invariant configuration x (that is, for x that satisfies $\theta_i x = x$ for all i) can be studied in terms on the behavior of the iterations F^n, since the identity $P^{n+1}(x, \{y_{\mathbf{o}} = +1\}) = F(P^n(x, \{y_{\mathbf{o}} = +1\}))$ holds.

We applied the techniques described above in the cases where $d = 1, 2$, and 3. For $d = 1$, we the PCA dynamics is ergodic and the unique stationary measure is a Bernoulli product measure with parameter p given by equation (3.15). Note that this case is equivalent to the study of a PCA on $\mathbb{N}$ where the choice of the value of the spin located at

3

i at time $n+1$ depends only on the value of the spin at $i+1$ at time n. Extensions of this result where $U(i) = \{i, i+1\}$ were extensively studied in [32], moreover, more recent generalizations that considers one-dimensional PCAs with general finite alphabets and characterizations of Markov stationary measures can be found in [36, 37]. For the cases $d = 2$ and $d = 3$ we assumed the invariance of the local transition probabilities under spin-flip in order to guarantee the existence of a stationary Bernoulli product measure (which has parameter $\frac{1}{2}$). Under this restriction, we obtained a full characterization the dynamics of PCAs with $d = 2$, and $d = 3$ with the additional hypothesis of attractiveness (resp. repulsiveness).

For further generalizations, in order to drop the assumption of spin-flip symmetry and extend the results for any d, it is necessary to investigate the general properties of the polynomial function F regarding its fixed points and the behavior of its iterates F^n. It is also worth investigating generalizations of the PCAs from Examples 1 and 2 (the generalization of Example 2 has been found but not published yet). Note that Theorem 3.3 together with Dobrushin's uniqueness theorem implies that for a PCA on $\mathbb{T}^d$ whose local transition probabilities are given by

$$p_i(s|x) = \frac{1}{2}\left(1 + s\tanh\left(\beta \sum_{k=0}^{d-1} J_k x_{i+e_k}\right)\right) \tag{3.40}$$

there is a unique stationary measure given by $\mathsf{Bern}(\frac{1}{2})^{\otimes \mathbb{T}^d}$ for β small enough, suggesting the ergodicity at high temperatures.

Another kind direction that should be considered in the future is the possibility of inclusion of finite alphabets other that $S = \{-1, +1\}$ and the possibility of influence of the state at the vertex i at time n on its state at time $n+1$, more precisely, the possibility of considering $U(i) = \{i, i+e_{\mathbf{o}}, \ldots, i+e_{d-1}\}$. Such assumptions require a new approach once equations (3.10), (3.13) and (3.14) would no longer be valid, so, one possible direction that should be chosen would be towards an extension of the results from [36, 37].

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3

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Appendix

A.1. Energetic lower bounds

Let us consider the function W_α defined by

$$W_\alpha(L) = \sum_{x=1}^{L} \left[\sum_{\substack{y \in [L+1, 2L] \cap \mathbb{Z} \\ y \in [-L+1, 0] \cap \mathbb{Z}}} J_\alpha(|x-y|) - \sum_{\substack{y \in [2L+1, \infty) \cap \mathbb{Z} \\ y \in (-\infty, -L] \cap \mathbb{Z}}} J_\alpha(|x-y|) \right]. \tag{A.1}$$

Note that $W_\alpha(L)$ can be written as

$$\begin{aligned} W_\alpha(L) &= \sum_{x=1}^{L} \sum_{y=L+1}^{2L} J_{x,y} + \sum_{x=1}^{L} \sum_{y=-L+1}^{0} J_{x,y} - \sum_{x=1}^{L} \sum_{y=-\infty}^{-L} J_{x,y} - \sum_{x=1}^{L} \sum_{y=2L+1}^{\infty} J_{x,y} \\ &= \sum_{x=1}^{L} \sum_{y=L+1}^{2L} J_{x,y} + \sum_{x=L+1}^{2L} \sum_{y=1}^{L} J_{x-L,y-L} - \sum_{x=1}^{L} \sum_{y=-\infty}^{-L} J_{L+1-x,L+1-y} - \sum_{x=1}^{L} \sum_{y=2L+1}^{\infty} J_{x,y} \\ &= 2\sum_{x=1}^{L} \sum_{y=L+1}^{2L} J_{x,y} - 2\sum_{x=1}^{L} \sum_{y=2L+1}^{\infty} J_{x,y}, \end{aligned}$$

and, by applying the explicit formula for the coupling constants $J_{x,y} = J_\alpha(|x-y|)$, we conclude that the identity

$$W_\alpha(L) = 2\sum_{x=1}^{L} \sum_{y=L+1-x}^{2L-x} \frac{1}{y^{2-\alpha}} - 2\sum_{x=1}^{L} \sum_{y=2L+1-x}^{\infty} \frac{1}{y^{2-\alpha}} \tag{A.2}$$

holds for every positive integer L. By splitting the first term of the equation above and changing the order of the sums, we find

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$$
\begin{aligned}
W_\alpha(L) &= 2\sum_{x=1}^{L}\sum_{y=L+1-x}^{L}\frac{1}{y^{2-\alpha}} + 2\sum_{x=1}^{L}\sum_{y=L+1}^{2L-x}\frac{1}{y^{2-\alpha}} - 2\sum_{x=1}^{L}\sum_{y=2L+1-x}^{\infty}\frac{1}{y^{2-\alpha}} \\
&= 2\sum_{x=1}^{L}\sum_{y=L+1-x}^{L}\frac{1}{y^{2-\alpha}} + 4\sum_{x=1}^{L}\sum_{y=L+1}^{2L-x}\frac{1}{y^{2-\alpha}} - 2\sum_{x=1}^{L}\sum_{y=L+1}^{\infty}\frac{1}{y^{2-\alpha}} \\
&= 2\sum_{y=1}^{L} y\frac{1}{y^{2-\alpha}} + 4\sum_{y=L+1}^{2L-1}(2L-y)\frac{1}{y^{2-\alpha}} - 2L\sum_{y=L+1}^{\infty}\frac{1}{y^{2-\alpha}},
\end{aligned}
$$

thus,

$$
W_\alpha(L) = 2\sum_{y=1}^{L}\frac{1}{y^{1-\alpha}} - 4\sum_{y=L+1}^{2L-1}\frac{1}{y^{1-\alpha}} + 8L\sum_{y=L+1}^{2L-1}\frac{1}{y^{2-\alpha}} - 2L\sum_{y=L+1}^{\infty}\frac{1}{y^{2-\alpha}}. \tag{A.3}
$$

For the next sections will be convenient to introduce the so-called generalized harmonic numbers. Given a real number k and a positive integer n, let us define the n-th generalized harmonic number $H_n^{(k)}$ of order k by

$$
H_n^{(k)} = \sum_{y=1}^{n}\frac{1}{y^k}. \tag{A.4}
$$

In particular, if $k=1$, we denote $H_n^{(1)}$ simply by H_n.

A.1.1. Lower bound for L large enough

In this section we show that for α in the inteval $[0,\alpha_+)$, where $\alpha_+ = \frac{\log 3}{\log 2} - 1$, there is a positive real number ζ_α^* for which the inequality

$$
W_\alpha(L) \geq \zeta_\alpha^* \chi_\alpha(L) \tag{A.5}
$$

holds whenever L is sufficiently large, where the expression of $\chi_\alpha(L)$ is given by (A.11).

It follows from expression (A.3) that

$$
\begin{aligned}
W_\alpha(L) &= \left(6H_L^{(1-\alpha)} - 4H_{2L-1}^{(1-\alpha)}\right) + 8L\sum_{y=L}^{2L-1}\frac{1}{y^{2-\alpha}} - 8L\frac{1}{L^{2-\alpha}} - 2L\sum_{y=L+1}^{\infty}\frac{1}{y^{2-\alpha}} \\
&= 2\left(3H_L^{(1-\alpha)} - 2H_{2L-1}^{(1-\alpha)} - \frac{4}{L^{1-\alpha}}\right) + 8L\sum_{y=L}^{2L-1}\frac{1}{y^{2-\alpha}} - 2L\sum_{y=L+1}^{\infty}\frac{1}{y^{2-\alpha}} \\
&\geq 2\left(3\frac{1}{L^\alpha}H_L^{(1-\alpha)} - 2\frac{1}{L^\alpha}H_{2L-1}^{(1-\alpha)} - \frac{4}{L}\right)L^\alpha + 8L\int_L^{2L}\frac{1}{z^{2-\alpha}}dz - 2L\int_L^{\infty}\frac{1}{z^{2-\alpha}}dz \\
&= 2\left(3\frac{1}{L^\alpha}H_L^{(1-\alpha)} - 2\frac{1}{L^\alpha}H_{2L-1}^{(1-\alpha)} - \frac{4}{L}\right)L^\alpha + 8L\left(\frac{L^{-1+\alpha}}{1-\alpha} - \frac{(2L)^{-1+\alpha}}{1-\alpha}\right) - 2L\frac{L^{-1+\alpha}}{1-\alpha} \\
&= 2\left(3\frac{1}{L^\alpha}H_L^{(1-\alpha)} - 2\frac{1}{L^\alpha}H_{2L-1}^{(1-\alpha)} - \frac{4}{L}\right)L^\alpha + \frac{2}{1-\alpha}\left(3-2^{1+\alpha}\right)L^\alpha.
\end{aligned}
$$

Let us suppose that α belongs to the interval $[0,\alpha_+)$. The condition $\alpha \in (0,\alpha_+)$ implies that

$$\lim_{L\to\infty}\left(3\frac{1}{L^\alpha}H_L^{(1-\alpha)} - 2\frac{1}{L^\alpha}H_{2L-1}^{(1-\alpha)} - \frac{4}{L}\right) = \frac{1}{\alpha}(3-2^{1+\alpha}) > 0. \tag{A.6}$$

Therefore, it follows that for L large enough we have

$$W_\alpha(L) \geq \zeta_\alpha^* L^\alpha, \tag{A.7}$$

where $\zeta_\alpha^* = \frac{2}{1-\alpha}\left(3-2^{1+\alpha}\right)$. Now, if $\alpha = 0$, the quantity $W_0(L)$ satisfies

$$\begin{aligned} W_0(L) &\geq 2\left(3H_L - 2H_{2L-1} - \frac{4}{L} + 1\right) \\ &= 2\left(3\frac{1}{\log(L)+4}H_L - 2\frac{1}{\log(L)+4}H_{2L-1} - \frac{4}{L(\log(L)+4)} + \frac{1}{\log(L)+4}\right)(\log(L)+4). \end{aligned}$$

Using the fact that $\log(L+1) \leq H_L \leq 1+\log(L)$, we prove that

$$\lim_{L\to\infty}\left(3\frac{1}{\log(L)+4}H_L - 2\frac{1}{\log(L)+4}H_{2L-1} - \frac{4}{L(\log(L)+4)} + \frac{1}{\log(L)+4}\right) = 1.$$

Thus, we conclude that

$$W_0(L) \geq \zeta_0^*[\log(L)+4] \tag{A.8}$$

holds for L sufficiently large, where $\zeta_0^* = 1$.

A.1.2. LOWER BOUND FOR ALL L

Note that in order to obtain lower bounds for $W_\alpha(L)$ like in equations (A.7) and (A.8) that hold for all L, it suffices to show that $W_\alpha(L)$ is positive for each L. It is straightforward to show that

$$W_\alpha(1) = 2\left(2 - \sum_{y=1}^{\infty}\frac{1}{y^{2-\alpha}}\right). \tag{A.9}$$

Restricting the range of α to a smaller interval $[0,\alpha^*) \subseteq [0,\alpha_+)$, where α^* is the number that belongs to the interval $(0,1)$ that satisfies

$$\sum_{y=1}^{\infty}\frac{1}{y^{2-\alpha^*}} = 2,$$

we have $W_\alpha(1) > 0$.

Proposition A.1. *Let $\alpha \in [0,\alpha^*)$, then there is a constant $\zeta_\alpha > 0$ such that*

$$W_\alpha(L) \geq \zeta_\alpha \chi_\alpha(L) \tag{A.10}$$

holds for all $L \geq 1$, where

$$\chi_\alpha(L) = \begin{cases} L^\alpha & \textit{if } \alpha > 0, \textit{ and} \\ \log(L)+4 & \textit{if } \alpha = 0. \end{cases} \tag{A.11}$$

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Proof. Based on the remark above, in order to prove that $W_\alpha(L) > 0$ holds for all L, let us show that W_α is an increasing function with respect to L. Note that $W_\alpha(L)$ can be expressed as

$$W_\alpha(L) = 6H_L^{(1-\alpha)} - 4H_{2L-1}^{(1-\alpha)} + 8LH_{2L-1}^{(2-\alpha)} - 6LH_L^{(2-\alpha)} - 2L\sum_{y=1}^{\infty}\frac{1}{y^{2-\alpha}}. \tag{A.12}$$

Therefore, the quantity $\Delta W_\alpha(L)$ given by $\Delta W_\alpha(L) = W_\alpha(L+1) - W_\alpha(L)$ can be written as

$$\begin{aligned}
\Delta W_\alpha(L) &= 6\left(H_{L+1}^{(1-\alpha)} - H_L^{(1-\alpha)}\right) - 4\left(H_{2L+1}^{(1-\alpha)} - H_{2L-1}^{(1-\alpha)}\right) + 8(L+1)H_{2L+1}^{(2-\alpha)} - 8LH_{2L-1}^{(2-\alpha)} \\
&\quad -6(L+1)H_{L+1}^{(2-\alpha)} + 6LH_L^{(2-\alpha)} - 2\sum_{y=1}^{\infty}\frac{1}{y^{2-\alpha}} \\
&= 6\left(H_{L+1}^{(1-\alpha)} - H_L^{(1-\alpha)}\right) - 4\left(H_{2L+1}^{(1-\alpha)} - H_{2L-1}^{(1-\alpha)}\right) + 8(L+1)H_{2L-1}^{(2-\alpha)} \\
&\quad +8(L+1)\left(H_{2L+1}^{(2-\alpha)} - H_{2L-1}^{(2-\alpha)}\right) - 8LH_{2L-1}^{(2-\alpha)} - 6(L+1)H_L^{(2-\alpha)} \\
&\quad -6(L+1)\left(H_{L+1}^{(2-\alpha)} - H_L^{(2-\alpha)}\right) + 6LH_L^{(2-\alpha)} - 2\sum_{y=1}^{\infty}\frac{1}{y^{2-\alpha}} \\
&= \frac{8}{(2L)^{2-\alpha}} + \frac{4}{(2L+1)^{2-\alpha}} + 8H_{2L-1}^{(2-\alpha)} - 6H_L^{(2-\alpha)} - 2\sum_{y=1}^{\infty}\frac{1}{y^{2-\alpha}} \\
&= \frac{8}{(2L)^{2-\alpha}} + \frac{4}{(2L+1)^{2-\alpha}} + 6\left(H_{2L-1}^{(2-\alpha)} - H_L^{(2-\alpha)}\right) - 2\sum_{y=2L}^{\infty}\frac{1}{y^{2-\alpha}} \\
&= \frac{6}{(2L)^{2-\alpha}} + \frac{4}{(2L+1)^{2-\alpha}} + 6\sum_{y=L+1}^{2L-1}\frac{1}{y^{2-\alpha}} - 2\sum_{y=2L+1}^{\infty}\frac{1}{y^{2-\alpha}}.
\end{aligned}$$

The reader can easily verify that $\Delta W_\alpha(1) \geq W_\alpha(1) > 0$ and $\Delta W_\alpha(2) \geq W_\alpha(1) > 0$. Now, for $L \geq 3$, we have

$$\begin{aligned}
\Delta W_\alpha(L) &> 6\int_{L+1}^{2L}\frac{1}{z^{2-\alpha}}dz - 2\int_{2L}^{\infty}\frac{1}{z^{2-\alpha}}dz = 6\left[\frac{z^{-1+\alpha}}{-1+\alpha}\right]_{L+1}^{2L} - 2\left[\frac{z^{-1+\alpha}}{-1+\alpha}\right]_{2L}^{\infty} \\
&= \frac{2}{1-\alpha}\left[3(L+1)^{-1+\alpha} - 3\cdot 2^{-1+\alpha}L^{-1+\alpha} - 2^{-1+\alpha}L^{-1+\alpha}\right] \\
&= \frac{2}{1-\alpha}\left[3\left(\frac{L+1}{L}\right)^{-1+\alpha} - 2^{1+\alpha}\right]L^{-1+\alpha}.
\end{aligned}$$

Thus,

$$\Delta W_\alpha(L) > \frac{2}{1-\alpha}\left[3\left(\frac{4}{3}\right)^{-1+\alpha} - 2^{1+\alpha}\right]L^{-1+\alpha} > 0 \tag{A.13}$$

for all $L \geq 3$. ■

A.2. EXTERNAL FIELD ESTIMATES

Given $\delta > 0$ and $h^* \in \mathbb{R}$ let us consider the external field $\mathbf{h} = (h_x)_{x\in\mathbb{Z}}$ given by

$$h_x = \frac{h^*}{(|x|+1)^\delta}. \tag{A.14}$$

For convenience, we will modify this field as described as follows. Given a nonnegative integer L, let us consider the field $\mathbf{h}_L = (h_{L,x})_{x\in\mathbb{Z}}$ truncated inside the box of length $2L+1$ centered around the origin defined by

$$h_{L,x} = \begin{cases} \frac{h^*}{(L+1)^\delta} & \text{if } |x| \le L \\ \frac{h^*}{(|x|+1)^\delta} & \text{otherwise.} \end{cases} \tag{A.15}$$

Proposition A.2. *We have the following statements.*

(a) If $\alpha \in (0,1)$ and δ satisfies $1-\alpha \le \delta < 1$, then, for every triangle T, the inequality

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le \frac{2^{1-\alpha}}{1-\delta}\frac{|h^*|}{(L+1)^{\delta-(1-\alpha)}}|T|^\alpha \tag{A.16}$$

holds. In particular, if $\delta = 1-\alpha$, we have

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le \frac{2^{1-\alpha}|h^*|}{\alpha}|T|^\alpha. \tag{A.17}$$

(b) If $\delta = 1$, then

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le 2|h^*|\left[\log(|T|)+4\right] \tag{A.18}$$

holds for every triangle T.

Proof. Let us prove part (a). Let p be the nonnegative real number defined by $p = \delta - (1-\alpha)$. In the case where $|T| \le 2L+1$, we have

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le \frac{|h^*|}{(L+1)^\delta}|T| = \frac{|h^*|}{(L+1)^\delta}|T|^{\delta-p}|T|^{1-\delta+p} \le \frac{|h^*|}{(L+1)^\delta}(2L+1)^{\delta-p}|T|^\alpha,$$

thus

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le \left(\frac{2L+1}{L+1}\right)^\delta \frac{|h^*|}{(2L+1)^p}|T|^\alpha \le \frac{2^\delta}{1-\delta}\frac{|h^*|}{(2L+1)^p}|T|^\alpha. \tag{A.19}$$

Now, let us suppose that $|T| > 2L+1$. If $|T|-(2L+1)$ is even, that is, $|T|-(2L+1) = 2k$ for some positive integer k, we have

$$\begin{aligned}
\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| &\le \frac{|h^*|}{(L+1)^\delta}(2L+1) + 2|h^*|\sum_{x=L+1}^{L+k}\frac{1}{(x+1)^\delta} \\
&\le |h^*|\frac{2L+1}{(L+1)^\delta} + 2|h^*|\int_{L+1}^{L+k+1}\frac{1}{z^\delta}\,dz \\
&= |h^*|\frac{2L+1}{(L+1)^\delta} + \frac{2|h^*|}{1-\delta}\left[(L+k+1)^{1-\delta} - (L+1)^{1-\delta}\right] \\
&= |h^*|\left[\frac{2L+1}{(L+1)^\delta} - \frac{2}{1-\delta}(L+1)^{1-\delta} + \frac{2}{1-\delta}(L+k+1)^{1-\delta}\right].
\end{aligned}$$

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So,

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le |h^*| \left[\frac{2L+1}{(L+1)^\delta} - \frac{2}{1-\delta}(L+1)^{1-\delta} + \frac{2^\delta}{1-\delta}(2L+2k+1)^{1-\delta} \right]. \tag{A.20}$$

Similarly, in the case where $|T|-(2L+1)$ is odd, that is, $|T|-(2L+1)=2k+1$ for some nonnegative integer k, we have

$$\begin{aligned}
\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| &\le \frac{|h^*|}{(L+1)^\delta}(2L+1) + |h^*| \sum_{x=L+1}^{L+k} \frac{1}{(x+1)^\delta} + |h^*| \sum_{x=L+1}^{L+k+1} \frac{1}{(x+1)^\delta} \\
&\le |h^*| \frac{2L+1}{(L+1)^\delta} + |h^*| \int_{L+1}^{L+k+1} \frac{1}{z^\delta} dz + |h^*| \int_{L+1}^{L+k+2} \frac{1}{z^\delta} dz \\
&= |h^*| \frac{2L+1}{(L+1)^\delta} + \frac{|h^*|}{1-\delta} \left[(L+k+1)^{1-\delta} + (L+k+2)^{1-\delta} - 2(L+1)^{1-\delta} \right] \\
&= |h^*| \left[\frac{2L+1}{(L+1)^\delta} - \frac{2}{1-\delta}(L+1)^{1-\delta} + \frac{1}{1-\delta} \left((L+k+1)^{1-\delta} + (L+k+2)^{1-\delta} \right) \right] \\
&= |h^*| \left[\frac{2L+1}{(L+1)^\delta} - \frac{2}{1-\delta}(L+1)^{1-\delta} + \frac{2}{1-\delta} \left(\frac{1}{2}(L+k+1)^{1-\delta} + \frac{1}{2}(L+k+2)^{1-\delta} \right) \right] \\
&= |h^*| \left[\frac{2L+1}{(L+1)^\delta} - \frac{2}{1-\delta}(L+1)^{1-\delta} + \frac{2}{1-\delta} \left(\frac{2L+2k+3}{2} \right)^{1-\delta} \right].
\end{aligned}$$

Then,

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le |h^*| \left[\frac{2L+1}{(L+1)^\delta} - \frac{2}{1-\delta}(L+1)^{1-\delta} + \frac{2^\delta}{1-\delta}(2L+2k+3)^{1-\delta} \right]. \tag{A.21}$$

Thus, if we express inequalities (A.20) and (A.21) in terms of $|T|$, we conclude that

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le |h^*| \left[\frac{2L+1}{(L+1)^\delta} - \frac{2}{1-\delta}(L+1)^{1-\delta} + \frac{2^\delta}{1-\delta}(|T|+1)^{1-\delta} \right] \tag{A.22}$$

holds for every triangle T such that $|T|>2L+1$. It is straightforward to check that the term on the right-hand side of equation (A.22) divided by $|T|^{1-\delta}$ is nondecreasing with respect to $|T|$ for all $|T|>2L+1$, moreover, it converges to $\frac{2^\delta |h^*|}{1-\delta}$ as $|T|$ approaches infinity. Therefore,

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le \frac{2^\delta |h^*|}{1-\delta}|T|^{1-\delta} \le \frac{2^\delta}{1-\delta}\frac{|h^*|}{(2L+2)^p}|T|^{1-\delta+p} = \frac{2^{1-\alpha}}{1-\delta}\frac{|h^*|}{(L+1)^p}|T|^\alpha, \tag{A.23}$$

concluding the proof of part (a).

Now, let us prove part (b). Analogously as before, let us consider the case where $|T|\le 2L+1$. In this case, we have

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le \frac{|h^*|}{(L+1)}|T| \le |h^*| \left(\frac{2L+1}{L+1} \right) \le 2|h^*|,$$

hence

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le 2|h^*|\left[\log(|T|)+4\right]. \tag{A.24}$$

Given a triangle T such that $|T|-(2L+1)>0$, if we have $|T|-(2L+1)=2k$ for some positive integer k, then

$$\begin{aligned}
\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| &\le \frac{|h^*|}{L+1}(2L+1)+2|h^*|\sum_{x=L+1}^{L+k}\frac{1}{x+1}\\
&\le |h^*|\left(\frac{2L+1}{L+1}\right)+2|h^*|\int_{L+1}^{L+k+1}\frac{1}{z}dz\\
&= |h^*|\left(\frac{2L+1}{L+1}\right)+2|h^*|[\log(L+k+1)-\log(L+1)]\\
&= |h^*|\left(\frac{2L+1}{L+1}\right)+2|h^*|[\log(2L+2k+2)-\log(2L+2)]\\
&= |h^*|\left[\left(\frac{2L+1}{L+1}\right)-2\log(2L+2)\right]+2|h^*|\log(2L+2k+2).
\end{aligned}$$

Using the fact that the term inside the brackets in the equation above is negative, we obtain

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le 2|h^*|\log(2L+2k+2). \tag{A.25}$$

Finally, if we have $|T|-(2L+1)=2k+1$ for some nonnegative integer k, then

$$\begin{aligned}
\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| &\le \frac{|h^*|}{L+1}(2L+1)+|h^*|\sum_{x=L+1}^{L+k}\frac{1}{x+1}+|h^*|\sum_{x=L+1}^{L+k+1}\frac{1}{x+1}\\
&\le |h^*|\left(\frac{2L+1}{L+1}\right)+|h^*|\int_{L+1}^{L+k+1}\frac{1}{z}dz+|h^*|\int_{L+1}^{L+k+2}\frac{1}{z}dz\\
&= |h^*|\left(\frac{2L+1}{L+1}\right)+|h^*|[\log(L+k+1)+\log(L+k+2)-2\log(L+1)]\\
&= |h^*|\left(\frac{2L+1}{L+1}\right)+|h^*|[\log(2L+2k+2)+\log(2L+2k+4)-2\log(2L+2)]\\
&= |h^*|\left[\left(\frac{2L+1}{L+1}\right)-2\log(2L+2)\right]+|h^*|\left[\log(2L+2k+2)+\log(2L+2k+4)\right]\\
&\le 2|h^*|\left[\frac{1}{2}\log(2L+2k+2)+\frac{1}{2}\log(2L+2k+4)\right].
\end{aligned}$$

So,

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le 2|h^*|\log(2L+2k+3). \tag{A.26}$$

It follows from inequalities (A.25) and (A.26) that for every triangle T that satisfies $|T|>2L+1$, we have

$$\sum_{x\in T\cap\mathbb{Z}} |h_{L,x}| \le 2|h^*|\log(|T|+1). \tag{A.27}$$

Using the fact that $\frac{\log(|T|+1)}{\log(|T|)+4}\le 1$, we conclude the proof. ■

B

Appendix

Lemma B.1. *Let* Λ *be a finite subset of* $\mathbb{N}$*, then*

$$\sum_{i\in\Lambda} J(i) \leq \sum_{i=1}^{\#\Lambda} J(i), \tag{B.1}$$

moreover, the equality holds if and only if $\Lambda = \{1,\dots,\#\Lambda\}$.

Proof. Let k be the number of elements of Λ. Note that for $k = 0$ the result holds, so, suppose that it holds whenever Λ has k elements. Given a subset Λ of $\mathbb{N}$ containing $k+1$ elements, let k_0 be its the maximal element, then, using our induction hypothesis and the fact that $k_0 \geq k+1$, we have

$$\sum_{i\in\Lambda} J(i) = J(k_0) + \sum_{i\in\Lambda\backslash\{k_0\}} J(i) \leq J(k+1) + \sum_{i=1}^{k} J(i) = \sum_{i=1}^{k+1} J(i). \tag{B.2}$$

In case we have an equality in equation (B.2), we have

$$0 \leq \sum_{i=1}^{k} J(i) - \sum_{i\in\Lambda\backslash\{k_0\}} J(i) = J(k_0) - J(k+1) \leq 0, \tag{B.3}$$

thus, $\Lambda\backslash\{k_0\} = \{1,\dots,k\}$ and $k_0 = k+1$. ∎

C

Appendix

C.1. Proof of Theorem 3.3

Before we follow to the proof of Theorem 3.3 it will be convenient to construct a special sequence $(\Delta_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{Z}\times\mathsf{V}$. Given a positive integer n and a nonempty finite subset F of V, let us define a subset $\Delta(n,F)$ of $\mathbb{Z}\times\mathsf{V}$ as follows. Let Λ_n be the set given by

$$\Lambda_n = \{(n,i) : i \in F\},$$

and for each integer $m < n$ let

$$\Lambda_m = \bigcup_{\mathsf{x}\in\Lambda_{m+1}} U(\mathsf{x}) \cup \{(m,i) : i \in F\}$$

Then, we define $\Delta(n,F)$ by

$$\Delta(n,F) = \bigcup_{m=-n}^{n} \Lambda_m.$$

Remark C.1. Observe that

(a) $\Delta(n,F)$ is a finite subset of $\mathbb{Z}\times\mathsf{V}$,

(b) we have $\{-n,\dots,0,\dots,n\}\times F \subseteq \Delta(n,F) \subseteq \{-n,\dots,0,\dots,n\}\times\mathsf{V}$, and

(c) for every point x in $\Delta(n,F)$, if $\pi_{\mathbb{Z}}(\mathsf{x}) \neq -n$, then $U(\mathsf{x}) \subseteq \Delta(n,F)$.

Now, if φ is a one-to-one function from $\mathbb{N}$ onto V, then let

$$\Delta_1 = \Delta(1,\{\varphi(1)\}), \tag{AC.1}$$

and

$$\Delta_{n+1} = \Delta(n+1, \pi_{\mathsf{V}}(\Delta_n) \cup \{\varphi(n+1)\}) \tag{AC.2}$$

for each positive integer n. Observe that $(\Delta_n)_{n\in\mathbb{N}}$ is an increasing sequence of elements of $\mathscr{S}$ such that $\mathbb{Z}\times\mathsf{V} = \bigcup_{n\in\mathbb{N}} \Delta_n$.

Lemma C.2. *Let $\Delta = \Delta_m$ for some $m \in \mathbb{N}$, and let $\overline{\Delta}$ be an element of $\mathscr{S}$ defined by*

$$\overline{\Delta} = \bigcup_{\substack{\mathsf{x}\in\Delta \\ \pi_{\mathbb{Z}}(\mathsf{x})=-m}} U(\mathsf{x}).$$

Given a finite volume configuration ξ in S^{Δ}, the measure λ^{ξ} on $(\Omega, \mathscr{F}_{\overline{\Delta}})$ defined by

$$\lambda^{\xi}(B) = \int_B \prod_{\mathsf{x}=(n,i)\in\Delta} p_i(\xi_{\mathsf{x}}|(\xi\omega_{\Delta^c})_{n-1})\,\mu_\nu(d\omega) \tag{AC.3}$$

can be expressed as

$$\lambda^{\xi}(B) = \int_B \mathbb{1}_{[\xi]}(\omega)\mu_\nu(d\omega). \tag{AC.4}$$

Proof of Lemma C.2. It suffices to show the identity for cylinder sets of the form $[\zeta]$, where each ζ belongs to $S^{\overline{\Delta}}$. The result follows by using the fact that the map

$$\omega \mapsto \prod_{\mathsf{x}=(n,i)\in\Delta} p_i(\xi_{\mathsf{x}}|(\xi\omega_{\Delta^c})_{n-1})$$

depends only on the values of ω assumed on $\overline{\Delta}$. ■

Proof of Theorem 3.3. Let us fix a set $\Lambda \in \mathscr{S}$ and a finite volume configuration σ in S^{Λ}. Let $\Delta = \Delta_m$ for some positive integer m such that

$$\{\mathsf{x} \in \mathbb{Z}\times\mathsf{V} : (\{\mathsf{x}\}\cup U(\mathsf{x}))\cap\Lambda \neq \emptyset\} \subseteq \Delta_m.$$

Then, for each ω in Ω, we have

$$\begin{aligned} e^{-H^{\Phi}_{\Lambda}(\sigma\omega_{\Lambda^c})} &= \prod_{\substack{\mathsf{x}=(n,i) \\ (\{\mathsf{x}\}\cup U(\mathsf{x}))\cap\Lambda\neq\emptyset}} p_i((\sigma\omega_{\Lambda^c})_{\mathsf{x}}|(\sigma\omega_{\Lambda^c})_{n-1}) \\ &= \frac{\prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\sigma\omega_{\Lambda^c})_{\mathsf{x}}|(\sigma\omega_{\Lambda^c})_{n-1})}{\prod_{\substack{\mathsf{x}=(n,i)\in\Delta \\ (\{\mathsf{x}\}\cup U(\mathsf{x}))\cap\Lambda=\emptyset}} p_i(\omega_{\mathsf{x}}|\omega_{n-1})}, \end{aligned}$$

thus

$$\frac{e^{-H^{\Phi}_{\Lambda}(\sigma\omega_{\Lambda^c})}}{\sum_{\sigma'\in S^{\Lambda}} e^{-H^{\Phi}_{\Lambda}(\sigma'\omega_{\Lambda^c})}} = \frac{\prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\sigma\omega_{\Lambda^c})_{\mathsf{x}}|(\sigma\omega_{\Lambda^c})_{n-1})}{\sum_{\sigma'\in S^{\Lambda}} \prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\sigma'\omega_{\Lambda^c})_{\mathsf{x}}|(\sigma'\omega_{\Lambda^c})_{n-1})}. \tag{AC.5}$$

Now, given a finite volume configuration η in $S^{\Delta\setminus\Lambda}$, using equation (AC.5), we obtain

$$\begin{aligned}
\int_{[\eta]} \mathbb{1}_{[\sigma]}(\omega)\mu_\nu(d\omega) &= \lambda^{\sigma\eta}(\Omega) = \int \prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\sigma\eta)_{\mathsf{x}}|(\sigma\eta\omega_{\Delta^c})_{n-1})\,\mu_\nu(d\omega) \\
&= \sum_{\zeta\in S^\Lambda}\int \frac{e^{-H_\Lambda^\Phi(\sigma\eta\omega_{\Delta^c})}}{\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\eta\omega_{\Delta^c})}} \prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\zeta\eta)_{\mathsf{x}}|(\zeta\eta\omega_{\Delta^c})_{n-1})\,\mu_\nu(d\omega) \\
&= \sum_{\zeta\in S^\Lambda}\int \frac{e^{-H_\Lambda^\Phi(\sigma\eta\omega_{\Delta^c})}}{\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\eta\omega_{\Delta^c})}} \lambda^{\zeta\eta}(d\omega) \\
&= \sum_{\zeta\in S^\Lambda}\int \frac{e^{-H_\Lambda^\Phi(\sigma\eta\omega_{\Delta^c})}}{\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\eta\omega_{\Delta^c})}} \mathbb{1}_{[\zeta\eta]}\mu_\nu(d\omega) \\
&= \int_{[\eta]} \frac{e^{-H_\Lambda^\Phi(\sigma\omega_{\Lambda^c})}}{\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\omega_{\Lambda^c})}} \mu_\nu(d\omega).
\end{aligned}$$

Since $(\Delta_n)_{n\in\mathbb{N}}$ is an increasing sequence of elements of $\mathscr{S}$ such that $\mathbb{Z}\times\mathsf{V} = \bigcup_{n\in\mathbb{N}} \Delta_n$, it follows that the equality

$$\mu_\nu([\sigma]|\mathscr{F}_{\Lambda^c})(\omega) = \frac{e^{-H_\Lambda^\Phi(\sigma\omega_{\Lambda^c})}}{\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\omega_{\Lambda^c})}} \tag{AC.6}$$

holds for μ_ν-almost every point ω in Ω. ■

C.2. PROOF OF THEOREM 3.6

Let m and N be integers, where $N \geq 0$, and let us consider the set

$$\Delta = \{m\}\times\{j\in\mathsf{V} : d(\mathbf{o}, j)\leq N\}. \tag{AC.7}$$

If we consider the nonempty finite subset Λ of $\mathbb{Z}\times\mathsf{V}$ given by

$$\Lambda = \bigcup_{l=0}^{N}\{m+l\}\times\{j\in\mathsf{V} : d(\mathbf{o}, j)\leq N-l\}, \tag{AC.8}$$

it follows that

$$\begin{aligned}
e^{-H_\Lambda^\Phi(\xi\omega_{\Lambda^c})} &= \prod_{\mathsf{x}=(n,i)\in\Lambda} p_i(\xi_{\mathsf{x}}|(\xi\omega_{\Lambda^c})_{n-1})\cdot \prod_{\substack{\mathsf{x}=(n,i)\notin\Lambda \\ U(\mathsf{x})\cap\Lambda\neq\emptyset}} p_i(\omega_{\mathsf{x}}|(\xi\omega_{\Lambda^c})_{n-1}) \\
&= \prod_{\mathsf{x}=(n,i)\in\Lambda} p_i(\xi_{\mathsf{x}}|(\xi\omega_{\Lambda^c})_{n-1})\cdot \prod_{\substack{\mathsf{x}=(n,i)\notin\Lambda \\ U(\mathsf{x})\cap\Lambda\neq\emptyset}} p_i(\omega_{\mathsf{x}}|\omega_{n-1})
\end{aligned}$$

holds for all finite volume configuration ξ in S^Λ and for every ω in Ω. Since μ is a Gibbs measure, then for μ-almost every point ω in Ω we have

$$\mu([\xi]|\mathscr{F}_{\Lambda^c})(\omega) = \frac{\prod_{x=(n,i)\in\Lambda} p_i(\xi_x|(\xi\omega_{\Lambda^c})_{n-1})}{\sum_{\eta\in S^\Lambda}\prod_{x=(n,i)\in\Lambda} p_i(\eta_x|(\eta\omega_{\Lambda^c})_{n-1})} = \prod_{x=(n,i)\in\Lambda} p_i(\xi_x|(\xi\omega_{\Lambda^c})_{n-1}),$$

and summing over all possible spins inside the volume $\Lambda\setminus\Delta$, we conclude that

$$\mu([\xi_\Delta]|\mathscr{F}_{\Lambda^c})(\omega) = \prod_{x=(m,i)\in\Delta} p_i(\xi_x|\omega_{m-1}). \tag{AC.9}$$

If we define the σ-algebra $\mathscr{F}_{<m}$ as the σ-algebra $\mathscr{F}_{\Gamma(m)}$ of subsets of Ω, where $\Gamma(m) = \{x \in S : \pi_{\mathbb{Z}}(x) < m\}$, it follows from (AC.9) that

$$\mu(\{\omega' \in \Omega : \omega'_m \in B\}|\mathscr{F}_{<m})(\omega) = P(B|\omega_{m-1}) \tag{AC.10}$$

holds for μ-almost every ω in Ω and for any measurable subset B of Ω_0.

Since μ is invariant under time translations, it follows that the measure ν on $(\Omega_0, \mathscr{B}(\Omega_0))$ defined by

$$\nu(B) = \mu(\{\omega' \in \Omega : \omega'_m \in B\}) \tag{AC.11}$$

does not depends on the choice of the integer m, moreover, it is easy to show that ν is a stationary measure for the PCA. Using equation (AC.10) and Kolmogorov consistency theorem, we finally conclude that $\mu = \mu_\nu$.

C.3. PROOF OF LEMMA 3.8

Proof. Let us proof that given a function $a : \{-1,+1\}^d \to \mathbb{R}$ and a probability measure μ on $\{-1,+1\}^{\mathbb{T}^d}$, we have

$$\begin{aligned}
&\sum_{\xi\in\{-1,+1\}^d} a(\xi) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k} = \xi_k) \\
&= \sum_{l=0}^{d} (-1)^l \left[\sum_{\substack{I\subseteq\{0,\dots,d-1\}\\ |I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi) \right) \prod_{k\in\{0,\dots,d-1\}\setminus I} \mu(x_{e_k} = +1) \right]
\end{aligned} \tag{BC.1}$$

We prove the equation above by induction. For the case where $d = 1$, we proof is straightforward. If we suppose that the result is proven for d, then

$$\begin{aligned}
\sum_{\xi\in\{-1,+1\}^{d+1}} a(\xi) \prod_{k\in\{0,\dots,d\}} \mu(x_{e_k} = \xi_k) &= \sum_{\xi\in\{-1,+1\}^d} a(\xi,+1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k} = \xi_k)\cdot\mu(x_{e_d} = +1) \\
&+ \sum_{\xi\in\{-1,+1\}^d} a(\xi,-1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k} = \xi_k)\cdot\mu(x_{e_d} = -1)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\xi\in\{-1,+1\}^d} a(\xi,+1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k}=\xi_k)\cdot\mu(x_{e_d}=+1) \\
&\quad - \sum_{\xi\in\{-1,+1\}^d} a(\xi,-1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k}=\xi_k)\cdot\mu(x_{e_d}=+1) \\
&\quad + \sum_{\xi\in\{-1,+1\}^d} a(\xi,-1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k}=\xi_k) \\
&= \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\dots,d-1\}\\ |I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi,+1) \right) \prod_{k\in\{0,\dots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad - \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\dots,d-1\}\\ |I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi,-1) \right) \prod_{k\in\{0,\dots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad + \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\dots,d-1\}\\ |I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi,-1) \right) \prod_{k\in\{0,\dots,d-1\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&= \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\dots,d\}\\ |I|=l, d\notin I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi,+1) \right) \prod_{k\in\{0,\dots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad - \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\dots,d\}\\ |I|=l, d\notin I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi,-1) \right) \prod_{k\in\{0,\dots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad \sum_{l=0}^{d}(-1)^{l+1} \left[\sum_{\substack{I\subseteq\{0,\dots,d\}\\ |I|=l+1, d\in I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^{d+1}\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi) \right) \prod_{k\in\{0,\dots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&= \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\dots,d\}\\ |I|=l, d\notin I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^{d+1}\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi) \right) \prod_{k\in\{0,\dots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad \sum_{l=0}^{d}(-1)^{l+1} \left[\sum_{\substack{I\subseteq\{0,\dots,d\}\\ |I|=l+1, d\in I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^{d+1}\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi) \right) \prod_{k\in\{0,\dots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&= \sum_{l=0}^{d+1}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\dots,d\}\\ |I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^{d+1}\\ \xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi) \right) \prod_{k\in\{0,\dots,d\}\setminus I} \mu(x_{e_k}=+1) \right].
\end{aligned}
$$

Therefore the result follows.

If we consider the the particular case where $a(\xi) = p_{\mathbf{o}}(+1|\xi)$ and $\mu = \mathsf{Bern}(\mathsf{p})^{\otimes \mathbb{T}^d}$ that satisfies (3.12), then equation (3.13) follows. Now, if we let $a(\xi) = p_{\mathbf{o}}(+1|\xi)$ and $\mu = P^n(x,\cdot)$, then equations (3.10) and (BC.1) imply equation (3.14). ■

C

Curriculum Vitæ

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List of Publications

4. A.C.D. van Enter, B. Kimura, W.M. Ruszel, and C. Spitoni. Nucleation for One-Dimensional Long-Range Ising Models. J Stat Phys (2019) 174: 1327. https://doi.org/10.1007/s10955-019-02238-y

3. B. Kimura, W.M. Ruszel, and C. Spitoni. Ergodicity versus non-ergodicity for Probabilistic Cellular Automata on rooted trees. Markov Processes Relat. Fields (2019) 25, 189–215.

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