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The UMD Property for Musielak–Orlicz Spaces

Nick Lindemulder, Mark Veraar, and Ivan Yaroslavtsev

Dedicated to Ben de Pagter on the occasion of his 65th birthday

Abstract In this paper we show that Musielak–Orlicz spaces are UMD spaces under the so-called Δ_2 condition on the generalized Young function and its complemented function. We also prove that if the measure space is divisible, then a Musielak–Orlicz space has the UMD property if and only if it is reflexive. As a consequence we show that reflexive variable Lebesgue spaces $L^{p(\cdot)}$ are UMD spaces.

Keywords UMD · Musielak–Orlicz spaces · Variable L^p-spaces · Young functions · Vector-valued martingales · Variable Lebesgue spaces

1 Introduction

Theclass of Banach spaces X with the UMD (unconditional martingale differences) property is probably the most important one for vector-valued analysis. Harmonic and stochastic analysis in UMD spaces can be found in [6, 8, 23, 34] and references therein. Among other things the UMD property of X implies the following results in X-valued harmonic analysis:

- Marcinkiewicz/Mihlin Fourier multiplier theorems (see [8], [23, Theorem 5.5.10] and [24, Theorem 8.3.9]);
- the T_b-theorem (see [21]);
- the L^p-boundedness of the lattice maximal function (see [34, Theorem 3]);

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and in X-valued stochastic analysis:

- the L^p-boundedness of martingale transforms (see [6, 23]);
- the continuous time Burkholder–Davis–Gundy inequalities (see [36–38]);
- the lattice Doob maximal L^p-inequality (see [34, 37]).

Most of the classical reflexive spaces are UMD spaces. A list of known spaces with UMD can be found in [23, p. 356]. On the other hand, a relatively simple to state space without UMD is given by $X = L^p(L^q(L^p(L^q(\ldots)\ldots)))$ with $1 < p \neq q < \infty$ (see [33]). The latter space is not only reflexive, but also uniformly convex.

In [17] and [28] it has been shown that an Orlicz space L^Φ has the UMD property if and only if it is reflexive. In [17] the proof is based on an interpolation argument and in [28] a more direct argument is given which uses known Φ-analogues of the defining estimates in UMD. The Musielak-Orlicz spaces of course include all Orlicz spaces but also the important class of variable Lebesgue spaces $L^{p(\cdot)}$.

It seems that a study of the UMD property of Musielak-Orlicz spaces L^Φ and even $L^{p(\cdot)}$ is not available in the literature yet. In the present paper we show that under natural conditions on Φ, the Musielak-Orlicz space L^Φ has UMD. We did not see how to prove this by interpolation arguments and instead we use an idea from [28]. Even in the Orlicz case our proof is simpler, and at the same time it provides more information on the UMD constant.

Theorem 1.1 *Assume that $\Phi, \Psi : T \times [0, \infty) \to [0, \infty]$ are complementary Young functions which both satisfy the Δ_2 condition. Then the Musielak-Orlicz space $L^\Phi(T)$ is a UMD space.*

This theorem is a special case of Theorem 3.1 below, in which we also have an estimate for the UMD constant in terms of the constants appearing in the Δ_2 condition for Φ, Ψ. The result implies the following new result for the variable Lebesgue spaces.

Corollary 1.2 *Assume $1 < p_0 < p_1 < \infty$ and $p : T \to [p_0, p_1]$ is measurable. Then $L^{p(\cdot)}(T)$ is a UMD space.*

In the case the measure space is divisible, one can actually characterize the UMD property in terms of Δ_2 and even in terms of reflexivity (see Corollary 3.3 below).

In the Orlicz setting (i.e. Φ does not dependent on T) the noncommutative analogue of [17, 28] was obtained in [15, Corollary 1.8]. It would be interesting to obtain the noncommutative analogues of our results as well. Details on noncommutative analysis and interpolation theory can be found in the forthcoming book [16].

Notation For a number $p \in [1, \infty]$ we write $p' \in [1, \infty]$ for its Hölder conjugate which satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. For a random variable f, $\mathbb{E}(f)$ denotes the expectation of f.

2 Preliminaries

2.1 Musielak–Orlicz Spaces

For details on Orlicz spaces we refer to [27, 35] and references therein. Details on Musielak–Orlicz spaces can be found in [14, 25, 26, 30, 40].

Let X be a Banach space and let (T, Σ, μ) be a σ-finite measure space. We say that a measurable function $\Phi : T \times [0, \infty) \to [0, \infty]$ is a *Young function* if for each $t \in T$,

1. $\Phi(t, 0) = 0$, $\exists x_1, x_2 > 0$ s.t. $\Phi(t, x_1) > 0$ and $\Phi(t, x_2) < \infty$;
2. $\Phi(t, \cdot)$ is increasing, convex and left-continuous.

As a consequence of the above $\lim_{x\to\infty} \Phi(t, x) = \infty$.

A function Φ with the above properties is a.e. differentiable, the right-derivative $\varphi := \partial_x \Phi$ is increasing and

$$\Phi(t, x) = \int_0^x \varphi(t, \lambda) d\lambda, \quad t \in T, \ x \in \mathbb{R}_+.$$

Note that the function $\varphi(t, \cdot)$ has a right-continuous version since any increasing function has at most countably many discontinuities, so $\varphi(t, \lambda) = \lim_{\varepsilon\to 0} \varphi(t, \lambda+\varepsilon)$ for each $t \in T$ for a.e. $\lambda \in [0, \infty)$.

For a strongly measurable function $f : T \to X$ we say that $f \in L^\Phi(T; X)$ if there exists a $\lambda > 0$ such that

$$\int_T \Phi(t, \|f(t)\|_X/\lambda)\, \mathrm{d}\mu(t) < \infty.$$

The space $L^\Phi(T; X)$ equipped with the norm

$$\|f\|_{L^\Phi(T;X)} := \inf\left\{\lambda > 0 : \int_T \Phi(t, \|f(t)\|_X/\lambda)\, \mathrm{d}\mu(t) \le 1\right\} \tag{2.1}$$

is a Banach space. Here as usual we identify functions which are almost everywhere identical. The space $L^\Phi(T; X)$ is called the X-valued *Musielak-Orlicz space* associated with Φ.

The following norm will also be useful in the sequel.

$$\|f\|_{X,\Phi} := \inf_{\lambda>0} \frac{1}{\lambda}\Big[1 + \int_T \Phi(t, \lambda\|f(t)\|_X)\, \mathrm{d}\mu(t)\Big]. \tag{2.2}$$

It is simple to check that this gives an equivalent norm (see [22, Lemma 2.1])

$$\|f\|_{L^\Phi(T;X)} \le \|f\|_{X,\Phi} \le 2\|f\|_{L^\Phi(T;X)}. \tag{2.3}$$

In case $X = \mathbb{R}$ or $X = \mathbb{C}$, we write $L^\Phi(T)$ for the above space.

Example 2.1 Let $p : T \to [1, \infty)$ be a measurable function and let $\Phi(t, \lambda) = |\lambda|^{p(t)}$. Then $L^{\Phi}(T)$ coincides with the variable Lebesgue space $L^{p(\cdot)}$.

Next we recall condition Δ_2 from [30, Theorem 8.13]. There it was used to study the dual space of the Musielak–Orlicz space and to prove uniform convexity and in particular reflexivity (see [30, Section 11]). Let $L^1_+(T) \subset L^1(T)$ be the set of all nonnegative integrable functions.

Definition 2.2 A Young function $\Phi : T \times [0, \infty) \to [0, \infty]$ is said to be in Δ_2 if there exists a $K > 1$ and an $h \in L^1_+(T)$ such that for a.a. $t \in T$

$$\Phi(t, 2\lambda) \leq K\Phi(t, \lambda) + h(t), \quad \lambda \in [0, \infty).$$

Note that $\Phi \in \Delta_2$ implies that $\Phi(t, \lambda) < \infty$ for almost all $t \in T$ and all $\lambda \in [0, \infty)$. Unlike is standard for Young's function independent of T, the condition Δ_2 depends on the measure space; namely, if one has that $\mu(T) = \infty$ and h does not depend on $t \in T$, then $h = 0$.

If $\Phi : T \times [0, \infty) \to [0, \infty]$ is a Young function, we define its *complemented function* $\Psi : T \times [0, \infty) \to [0, \infty]$ by the Legendre transform

$$\Psi(t, x) = \sup_{y \geq 0}(xy - \Phi(t, y)).$$

Then Ψ is a Young function as well. Moreover, one can check that the complemented function of $\Psi(t, \cdot)$ equals $\Phi(t, \cdot)$.

Example 2.3 Let the notations be as in Example 2.1. Then the following statements hold.

(i) Φ is in Δ_2 if and only if $p \in L^{\infty}(T)$, in which case Φ satisfies the Δ_2-condition with $K = 2^{\|p\|_{\infty}}$ and $h = 0$.
(ii) The complemented function Ψ to Φ is given by

$$\Psi(t, x) = x^{p'(t)}\mathbf{1}_{\{p>1\}\times[0,\infty)}(t, x) + \infty \cdot \mathbf{1}_{\{p=1\}\times(1,\infty)}(t, x),$$

where $p'(t) = p(t)'$ is the Hölder conjugate.

In particular, Φ and Ψ are both in Δ_2 if and only if $1 < \operatorname{ess\,inf} p \leq \operatorname{ess\,sup} p < \infty$, in which case $\Psi(t, x) = x^{p'(t)}$ for a.a. $t \in T$ and all $x \in [0, \infty)$.

Proof Let us only give the proof of Example (2.3)(i). If $p \in L^{\infty}(T)$, then, for a.a. $t \in T$,

$$\Phi(t, 2\lambda) = 2^{p(t)}\Phi(t, \lambda) \leq 2^{\|p\|_{\infty}}\Phi(t, \lambda), \quad \lambda \in [0, \infty).$$

Conversely assume that Φ is in Δ_2. Let K and h be as in the Δ_2 condition for Φ. Then, for a.a. $t \in T$ and all $\lambda \in [0, \infty)$,

$$2^{p(t)}\Phi(t, \lambda) = \Phi(t, 2\lambda) \leq K\Phi(t, \lambda) + h(t)$$

and thus

$$(2^{p(t)} - K)\Phi(t, \lambda) \leq h(t).$$

As $\lim_{\lambda\to\infty} \Phi(t, \lambda) = \infty$, this implies that $2^{p(t)} \leq K$ for a.a. $t \in T$. Hence, $p \in L^\infty(T)$. □

By the properties of the functions Φ and Ψ one can check that for $\varphi = \partial_x \Phi$ and $\psi = \partial_x \Psi$ (where φ and ψ are taken right-continuous in x), we have $\varphi^{-1}(t, \cdot) = \psi(t, \cdot)$, where

$$\varphi^{-1}(t, y) = \sup\{x : \varphi(t, x) \leq y\} \quad y \geq 0. \tag{2.4}$$

Note that $\psi(t, \varphi(t, x)) \geq x$ and $\varphi(t, \psi(t, x)) \geq x$ because of the above choices.

Recall Young's inequality (see [27, Section I.2] or [35, Proposition 15.1.2]) for a.a. $t \in T$,

$$xy \leq \Phi(t, x) + \Psi(t, y), \quad x, y \geq 0 \tag{2.5}$$

with equality if and only if $y = \varphi(t, x)$ or $x = \psi(t, y)$.

Lemma 2.4 *Let $\Phi : T \times [0, \infty) \to [0, \infty)$ be a Young function and let Ψ be its complemented function. If $\Phi \in \Delta_2$ with constant $K > 1$ and $h \in L^1_+(T)$, then for almost all $t \in T$,*

$$\Psi(t, \lambda) \leq \frac{K}{K-1}\lambda\psi(t, \lambda) + \frac{1}{K}h(t), \quad \lambda \geq 0.$$

Proof We use a variation of the argument in [27, Section 1.4]. By the Δ_2 condition there exist $K > 1$ and $h \in L^1_+(T)$ such that for almost all $t \in T$ and all $\lambda \geq 0$

$$K\Phi(t, \lambda) + h(t) \geq \Phi(t, 2\lambda) = \int_0^{2\lambda} \varphi(t, x)dx \geq \int_\lambda^{2\lambda} \varphi(t, x)dx \geq \lambda\varphi(t, \lambda),$$

where we used the fact that $\varphi(t, \cdot)$ is increasing. Using the identity case of (2.5) we obtain

$$K\lambda\varphi(t, \lambda) - K\Psi(t, \varphi(t, \lambda)) + h(t) \geq \lambda\varphi(t, \lambda)$$

Therefore,

$$\frac{K\Psi(t, \varphi(t, \lambda))}{\lambda\varphi(t, \lambda)} \leq K - 1 + \frac{h(t)}{\lambda\varphi(t, \lambda)}.$$

Taking $\lambda = \psi(t,x)$ and using the estimates below (2.4) and the fact that $y \mapsto \frac{\Psi(t,y)}{y}$ is increasing (see [27, (1.18)]) we obtain

$$\begin{aligned}\frac{K\Psi(t,x)}{x\psi(t,x)} &\leq \frac{K\Psi(t,\varphi(t,\psi(t,x)))}{\psi(t,x)\varphi(t,\psi(t,x))} \\ &\leq K-1+\frac{h(t)}{\psi(t,x)\varphi(t,\psi(t,x))} \leq K-1+\frac{h(t)}{x\psi(t,x)}.\end{aligned}$$

We may conclude that

$$\Psi(t,x) \leq \frac{K-1}{K} x\psi(t,x) + \frac{1}{K}h(t).$$

□

2.2 UMD Spaces

For details on UMD spaces the reader is referred to [6, 34] and the monographs [23, 24]. Let $(\Omega, \mathcal{A}, (\mathscr{F}_n)_{n\geq 0}, \mathbb{P})$ denote a filtered probability space which is rich enough in the sense that it supports an i.i.d. sequence $(\varepsilon_n)_{n\geq 0}$ such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = \frac{1}{2}$ for each $n \geq 0$. Such a sequence is called a Rademacher sequence.

For a sequence of random variables $f = (f_n)_{n\geq 0}$ with values in X, we write $f_n^* = \sup_{k\leq n} \|f_k\|_X$ and $f^* = \sup_{k\geq 0} \|f_k\|_X$. Moreover, if $\epsilon = (\epsilon_n)_{n\geq 0}$ is a sequence of signs, we write $(\epsilon * f)_n = \sum_{k=0}^n \epsilon_k (f_k - f_{k-1})$, where $f_{-1} = 0$.

We say that X is a UMD space if there exists a $p \in (1,\infty)$ and $\beta \in [1,\infty)$ such that for all L^p-martingales $f = (f_n)_{n\geq 0}$ and all sequences of signs $\epsilon = (\epsilon_n)_{n\geq 0}$ we have that

$$\|(\epsilon * f)_n\|_{L^p(\Omega;X)} \leq \beta \|f_n\|_{L^p(\Omega;X)}, \quad n \geq 0,$$

where the least admissible constant β is denoted by $\beta_{p,X}$ and is called *the UMD constant*. If the above holds for some $p \in (1,\infty)$, then it holds for all $p \in (1,\infty)$. Examples and counterexamples of UMD spaces have been mentioned in the introduction. Every UMD space is (super-)reflexive (see [23, Theorem 4.3.8]).

We say that $f = (f)_{n\geq 0}$ is a *Paley–Walsh martingale* if f is a martingale with respect to the filtration $(\mathscr{F}_n)_{n\geq 0}$ with $\mathscr{F}_0 = \{\varnothing, \Omega\}$ and $\mathscr{F}_n = \sigma\{\varepsilon_k : 1 \leq k \leq n\}$ for some Rademacher sequence $(\varepsilon_k)_{k\geq 0}$ and if $f_0 = 0$.

The following result follows from [4, Theorems 1.1 and 3.2].

Proposition 2.5 *Let X be a Banach space. Then X is a UMD space if and only if for all Paley–Walsh martingales f and all sequences of signs ϵ we have*

$$\sup_{n\geq 0} \|f_n\|_{L^\infty(\Omega;X)} < \infty \implies \mathbb{P}(\sup_{n\geq 0} \|(\epsilon * f)_n\|_X < \infty) > 0.$$

We will also need the following lemma which allows to estimate the ϵ-transform for different functions than $\Phi(x) = |x|^p$. This lemma is a straightforward extension [3, p. 1001] where the case $b = 0$ was considered. Moreover, since we only state it for Paley–Walsh martingales it follows from [6, Proof of (10)].

Lemma 2.6 *Assume X is a UMD space. Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function and assume that there exist constants $K > 1$ and $b \geq 0$ such that*

$$\Phi(2\lambda) \leq K\Phi(\lambda) + b, \quad \lambda \geq 0. \tag{2.6}$$

Let $f = (f_n)_{n\geq 0}$ be a Paley–Walsh martingale, $\epsilon = (\epsilon_n)_{n\geq 0}$ be a sequence of signs, and set $g := \epsilon * f$. Then there exists a constant $C_{K,X} \geq 0$ only depending on K and (the UMD constant of) X such that

$$\mathbb{E}\Phi(g^*) \leq C_{K,X}(\mathbb{E}\Phi(f^*) + b).$$

Remark 2.7 To obtain Lemma 2.6 in the case of general martingales (as it is done in [3, p. 1001]), one can use the Davis decomposition to reduce to a bad part and a good part of f. To estimate the bad part of the Davis decomposition one can use [7, Theorem 3.2 and the proof of Theorem 2.1] (see [31, Proposition A-3-5] and [29, Theorem 53] for a simpler proof).

Recall that X is a UMD space if and only if it is *ζ-convex*, i.e. there exists a biconvex function $\zeta : X \times X \to \mathbb{R}$ such that $\zeta(0,0) > 0$ and $\zeta(x,y) \leq \|x+y\|$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ (see [4, 5, 23]). By the ζ-function we will usually mean the *optimal ζ-function* which can be defined as the supremum over all admissible ζ's, and this obviously satisfies the required conditions.

The following theorem can be found in [6, equation (20)].

Theorem 2.8 (Burkholder) *Let X be a UMD Banach space and let $\zeta : X \times X \to \mathbb{R}$ be an optimal ζ-function (i.e. $\zeta(0,0)$ is maximal). For any $1 < p < \infty$ one then has that*

$$\frac{1}{\zeta(0,0)} \leq \beta_{p,X} \leq \frac{72}{\zeta(0,0)} \frac{(p+1)^2}{p-1}. \tag{2.7}$$

The following lemma follows from [4, p. 49].

Lemma 2.9 *Let X be a UMD Banach space and let $\zeta : X \times X \to \mathbb{R}$ be an optimal ζ-function (i.e. $\zeta(0,0)$ is maximal). Then for any $\varepsilon > 0$ there exists an X-valued Paley–Walsh martingale $f = (f_n)_{n\geq 1}$ which starts in zero and a sequence of signs*

$\epsilon = (\epsilon_n)_{n\geq 1}$ *such that* $\mathbb{P}(g^* > 1) = 1$ *and* $\sup_{n\geq 1}\mathbb{E}\|f_n\| \leq \frac{\zeta(0,0)}{2} + \varepsilon$, *where* $g := \epsilon * f$.

Remark 2.10 Let us compute an upper bound for $C_{K,X}$ in Lemma 2.6. Let $M \geq 1$ be the least integer such that $2^{-M} \leq \frac{\zeta(0,0)}{48K}$. Fix $\beta := 2$ and $\delta := 2^{-M}$. Then by formula [3, (1.8)] one has that for f and g from Lemma 2.6

$$\mathbb{P}(g^* > 2\lambda, f^* \leq 2^{-M}\lambda) \leq \varepsilon\mathbb{P}(g^* > \lambda), \quad \lambda > 0,$$

where $\varepsilon = 3c\delta/(\beta - \delta - 1) \leq 1/(2K)$, and where we used the fact that the constant c from [3, (1.2)] can be bounded from above by $4/\zeta(0,0)$ by [32, Theorem 3.26 and Lemma 3.23] (see also [39]). Note that by (2.6)

$$\Phi(\beta\lambda) \leq K\Phi(\lambda) + b, \quad \Phi(\delta^{-1}\lambda) \leq K^M\Phi(\lambda) + bMK^M, \quad \lambda > 0,$$

where one needs to iterate (2.6) M times in order to get the latter inequality. Therefore by exploiting [2, proof of Lemma 7.1] one has the following analogue of the formula [2, (7.6)]

$$\mathbb{E}\Phi(2^{-1}g^*) \leq \varepsilon\mathbb{E}\Phi(g^*) + K^M\mathbb{E}\Phi(f^*) + bMK^M,$$

and by using the fact that $\mathbb{E}\Phi(g^*) \leq K\mathbb{E}\Phi(2^{-1}g^*) + b$ and the fact that $\varepsilon K \leq 1/2$ one has that

$$\mathbb{E}\Phi(g^*) \leq 2K^{M+1}\mathbb{E}\Phi(f^*) + 2b(1 + MK^{M+1}) \leq 2(MK^{M+1} + 1)(\mathbb{E}\Phi(f^*) + b),$$

so $C_{K,X} \leq 2(MK^{M+1} + 1)$, where M can be taken $[\log_2 \frac{48K}{\zeta(0,0)}] + 1$. Of course this bound is not optimal.

3 Musielak-Orlicz Spaces Are UMD Spaces

The main result of this paper is the following.

Theorem 3.1 *Assume X is a UMD space. Let $\Phi, \Psi : T \times [0,\infty) \to [0,\infty)$ be complemented Young functions which both satisfy Δ_2. Then the Musielak-Orlicz space $L^\Phi(T; X)$ is a UMD space.*

Moreover, if $\Phi \in \Delta_2$ with constant K_Φ and $h_\Phi \in L^1_+(T)$ and $\Psi \in \Delta_2$ with constant K_Ψ and $h_\Psi \in L^1_+(T)$, then for the optimal ζ-function $\zeta : L^\Phi(T; X) \times L^\Phi(T; X) \to \mathbb{R}$ (see the discussion preceding Theorem 2.8) one has that

$$\zeta(0,0) \geq \frac{1}{6K_\Psi C_{K_\Phi,X} C_h}, \tag{3.8}$$

and

$$\beta_{p,L^\Phi(T;X)} \le 432 K_\Psi C_{K_\Phi,X} C_h \frac{(p+1)^2}{p-1}, \tag{3.9}$$

where $C_{K_\Phi,X}$ *is as in Lemma 2.6 and* $C_h := 2 + \|h_\Phi\|_{L^1(T)} + \frac{1}{K_\Psi}\|h_\Psi\|_{L^1(T)}$.

This result is well-known in the case of $\Phi(x) = |x|^p$, and then it is a simple consequence of Fubini's theorem which allows to write $L^p(\Omega; L^p(T; X)) = L^p(T; L^p(\Omega; X))$ and to apply the UMD property of X pointwise a.e. in T (see [23, Proposition 4.2.15]). Such a Fubini argument is necessarily limited to L^p-spaces. Indeed, the Kolmogorov–Nagumo theorem says that for Banach function spaces E and F one has $E(F) = F(E)$ isomorphically, if and only if E and F are weighted L^p-spaces (see [1, Theorem 3.1]).

To prove Theorem 3.1 we will use several results from the preliminaries. Moreover, we will need the following scalar-valued result which is a well-known version of Doob's maximal inequality for a certain class of Young functions.

Proposition 3.2 *Suppose that* $\Phi : [0,\infty) \to [0,\infty]$ *is a Young function with a right-continuous derivative* $\varphi : [0,\infty) \to [0,\infty)$ *and that there exists a* $q \in (1,\infty)$ *and* $c \in [0,\infty)$ *such that*

$$\Phi(\lambda) \le \frac{1}{q}\lambda\varphi(\lambda) + c, \quad \lambda \ge 0.$$

Then for all nonnegative submartingales $(f_n)_{n\ge 0}$

$$\mathbb{E}\Phi(f_n^*) \le \mathbb{E}\Phi(q' f_n) + c, \quad n \ge 0.$$

In particular, $\|f_n^*\|_\Phi \le q'(1+c)\|f_n\|_\Phi$.

Proof The result for $c = 0$ is proved in [13, estimate (104.5)], and the case $c > 0$ follows by a simple modification of that argument. The final assertion follows from the obtained estimate since for any $\lambda > 0$ we have

$$\begin{aligned}\|f_n^*\|_\Phi &\le \lambda^{-1}(1 + \mathbb{E}\Phi(\lambda f_n^*))\\ &\le \lambda^{-1}(c + 1 + \mathbb{E}\Phi(\lambda q' f_n))\\ &\le (c+1)\lambda^{-1}(1 + \mathbb{E}\Phi(\lambda q' f_n)).\end{aligned}$$

Taking the infimum over all $\lambda > 0$ yields the required conclusion. □

Proof of Theorem 3.1 Let $Y := L^{\Phi}(T; X)$. In order to prove the theorem we will use Proposition 2.5. Let $f = (f_n)_{n\geq 0}$ be a Paley–Walsh martingale with values in Y. Let $\epsilon = (\epsilon_n)_{n\geq 0}$ be a sequence of signs and $g := \epsilon * f$. We will show that $g^* < \infty$ a.s. For this it is enough to show that

$$\mathbb{E} \sup_{n\geq 0} \|g_n\|_Y \leq K_\Psi C_{K_\Phi, X} C_h \sup_{n\geq 0} \|f_n\|_{L^\infty(\Omega;Y)}, \tag{3.10}$$

where $C_{K_\Phi,X}$ is as in Lemma 2.6 and $C_h = 2 + \|h_\Phi\|_{L^1(T)} + \frac{1}{K_\Psi}\|h_\Psi\|_{L^1(T)}$. By homogeneity we can assume $\sup_{n\geq 0} \|f_n\|_{L^\infty(\Omega;Y)} = 1$.

We know that $K_\Phi > 1$ and a function $h_\Phi \in L^1_+(T)$ satisfy the following inequality

$$\Phi(t, 2\lambda) \leq K_\Phi \Phi(t, \lambda) + h_\Phi(t), \quad \lambda \in [0, \infty), \ t \in T. \tag{3.11}$$

Since Ψ satisfies Δ_2 with constant $K_\Psi > 1$ and $h_\Psi \in L^1_+(T)$ it follows from Lemma 2.4 that

$$\Phi(t, \lambda) \leq \frac{K_\Psi - 1}{K_\Psi} \lambda \varphi(t, \lambda) + \frac{1}{K_\Psi} h_\Psi(t), \quad \lambda \in [0, \infty), \quad t \in T. \tag{3.12}$$

One can check that for a.e. $t \in T$, $f(t)$ is an X-valued martingale and $g_n(t) = (\epsilon * (f(t)))_n$ (use that f is a Paley–Walsh martingale). Therefore, first applying (3.11) and Lemma 2.6 and then (3.12) and Proposition 3.2 to the submartingale $(\|f_k(t)\|_X)_{k\geq 0}$ gives that for almost all $t \in T$,

$$\begin{aligned}
\mathbb{E}\Phi(t, \sup_{k\leq n} \|g_k(t)\|_X) &\leq C_{K_\Phi,X} \left[\mathbb{E}\Phi(t, \sup_{k\leq n} \|f_k(t)\|_X) + h_\Phi(t) \right] \\
&\leq C_{K_\Phi,X} \left[\mathbb{E}\Phi(t, K_\Psi \|f_n(t)\|_X) + h_\Phi(t) + \tfrac{1}{K_\Psi} h_\Psi(t) \right].
\end{aligned}$$

The same holds with (f, g) replaced by $(\lambda f, \lambda g)$ for any $\lambda > 0$. Integrating over $t \in T$ (and using (2.2)) we find that

$$\begin{aligned}
\mathbb{E} \sup_{k\leq n} \|g_k\|_{X,\Phi} &\leq \mathbb{E} \sup_{k\leq n} \frac{1}{\lambda} \Big(1 + \int_T \Phi(t, \lambda \|g_k(t)\|_X) d\mu(t) \Big) \\
&\overset{(*)}{\leq} \mathbb{E} \frac{1}{\lambda} \Big(1 + \int_T \Phi(t, \sup_{k\leq n} \lambda \|g_k(t)\|_X) d\mu(t) \Big) \\
&\leq C_{K_\Phi,X} \mathbb{E} \frac{1}{\lambda} \Big(1 + \int_T \Phi(t, K_\Psi \lambda \|f_n(t)\|_X) d\mu(t) + \|h_\Phi\|_{L^1(T)} + \|\tfrac{1}{K_\Psi} h_\Psi\|_{L^1(T)} \Big),
\end{aligned}$$

where $(*)$ follows form the fact that $\sup \int \leq \int \sup$ and the fact that the map $\lambda \mapsto \Phi(t, \lambda)$ is increasing in $\lambda \geq 0$. Since $\int_T \Phi(t, \|f_n(t)\|_X) d\mu(t) \leq 1$ a.s. by (2.1) and

the assumption $\|f_n\|_{L^\infty(\Omega;Y)} \le 1$, it follows by setting $\lambda = 1/K_\Psi$ that

$$\mathbb{E}\sup_{k\le n}\|g_k\|_{X,\Phi} \le K_\Psi C_{K_\Phi,X} C_h.$$

Now the required estimate (3.10) follows from (2.3).

For proving (3.8) and (3.9) we will use Lemma 2.9. By the first part of the proof, $Y = L^\Phi(T;X)$ is UMD. Fix $\varepsilon > 0$. Then by Lemma 2.9 there exists a Y-valued Paley–Walsh martingale $f = (f_n)_{n\ge 0}$ which starts in zero and a sequence of signs $\epsilon = (\epsilon_n)_{n\ge 0}$ such that $\mathbb{P}(g^* > 1) = 1$ and $\sup_{n\ge 0}\mathbb{E}\|f_n\|_Y \le \frac{\zeta(0,0)}{2} + \varepsilon$, where $g := \epsilon * f$. By Burkholder [4, Lemma 3.1] there exist discrete Y-valued Paley–Walsh martingales $F = (F_n)_{n\ge 0}$ and $G = (G_n)_{n\ge 0}$ such that $G = \epsilon * F$, $\mathbb{P}(G^* > 1) \le 1/2$, and

$$\sup_{n\ge 0}\|F_n\|_{L^\infty(\Omega;Y)} \le 6\sup_{n\ge 0}\mathbb{E}\|f_n\|_Y.$$

Therefore, by (3.10),

$$\begin{aligned}\frac{1}{2} \le \mathbb{E}G^* &\le K_\Psi C_{K_\Phi,X} C_h \sup_{n\ge 0}\|F_n\|_{L^\infty(\Omega;Y)} \\ &\le 6K_\Psi C_{K_\Phi,X} C_h \sup_{n\ge 0}\mathbb{E}\|f_n\|_Y \le 3K_\Psi C_{K_\Phi,X} C_h(\zeta(0,0) + 2\varepsilon),\end{aligned}$$

so letting $\varepsilon \to 0$ gives (3.8). Equation (3.9) follows from (3.8) and (2.7). □

We recover the following result of [17] and [28]. Recall that a measure space (T, Σ, μ) is divisible if for every $A \in \Sigma$ and $t \in (0,1)$ there exist sets $B, C \in \Sigma$ such that $B, C \subseteq A$, $\mu(B) = t\mu(A)$ and $\mu(C) = (1-t)\mu(A)$. The divisibility condition is only needed in the implication (ii)⇒(iii).

Corollary 3.3 *Let $X \ne \{0\}$ be a Banach space and assume that T is divisible and σ-finite. Suppose $\Phi, \Psi : T \times [0,\infty) \to [0,\infty]$ are complementary Young functions. Then the following are equivalent:*

(i) $L^\Phi(T;X)$ is a UMD space;
(ii) $L^\Phi(T)$ is reflexive and X is a UMD space;
(iii) Φ and Ψ both satisfy Δ_2 and X is a UMD space.

For the proof we will need the following lemma which follows from [26, Theorem 2.2] and [25, Theorem 4.7].

Lemma 3.4 *Let $\Phi, \Psi : T \times [0,\infty) \to [0,\infty]$ be complementary Young functions. Then there exists a decomposition $L^\Phi(T)^* = L^\Psi(T) \oplus \Lambda$ of the dual of $L^\Phi(T)$ into a direct sum of two Banach spaces, where $g \in L^\Psi(T)$ acts on $L^\Phi(T)$ in the*

following way:

$$\langle f, g \rangle = \int_T fg \, \mathrm{d}\mu, \quad f \in L^\Phi(T).$$

Proof of Corollary 3.3 (i)⇒(ii): Fix $h \in L^\Phi(T)$ and $x \in X$ of norm one. Then $L^\Phi(T)$ and X can be identified with the closed subspaces $L^\Phi(T) \otimes x$ and $h \otimes X$ of the UMD space $L^\Phi(T; X)$, respectively, and therefore have UMD themselves. In particular, $L^\Phi(T)$ is reflexive.

(ii)⇒(iii): We show that Φ satisfies Δ_2. The proof for Ψ is similar. By Lemma 3.4, $L^\Phi(T)^* = L^\Psi(T) \oplus \Lambda$, so

$$L^\Phi(T)^{**} = L^\Psi(T)^* \oplus \Lambda^* \supseteq L^\Phi(T) \oplus \Lambda^*,$$

where the latter inclusion follows from Lemma 3.4 and means that $L^\Phi(T) \oplus \Lambda^*$ is a closed subspace of $L^\Psi(T)^* \oplus \Lambda^*$, and hence of $L^\Phi(T)^{**}$. Thus $\Lambda = 0$ due to the reflexivity of $L^\Phi(T)$, and since T is divisible the desired statement follows from [26, Corollary 1.7.4].

(iii)⇒(i): This follows from Theorem 3.1. □

As a consequence of the above results many other spaces are UMD as well. Indeed, it suffices to be isomorphic to a closed subspace (or quotient space) of an $L^\Phi(T; X)$ space with UMD. This applies to the Musielak–Orlicz variants of Sobolev, Besov, and Triebel–Lizorkin spaces.

Remark 3.5 A result of Rubio de Francia (see [34, p. 214]) states that for a Banach function space E and a Banach space X one has that $E(X)$ is a UMD space if and only if E and X are both UMD spaces. Therefore, it actually suffices to consider $X = \mathbb{R}$ in the proof of Theorem 3.1. Since our argument works in the vector-valued case without difficulty, we consider that setting from the start.

For the variable Lebesgue spaces we obtain the following consequence. For a measurable mapping $p : T \to [1, \infty]$ we will write $p_+ = \|p\|_{L^\infty(T)}$ and $p_- = \|1/p\|^{-1}_{L^\infty(T)}$.

Corollary 3.6 *Let $X \neq \{0\}$ be a Banach space and assume T is divisible and σ-finite. Assume $p : T \to [1, \infty]$ is measurable. Then the following assertions are equivalent.*

(i) $L^{p(\cdot)}(T; X)$ *is a UMD space;*
(ii) $L^{p(\cdot)}(T)$ *is reflexive and X is a UMD space;*
(iii) $p_- > 1$ *and* $p_+ < \infty$ *and X is a UMD space.*

The result that $L^{p(\cdot)}(T)$ is reflexive if and only if $p_- > 1$ and $p_+ < \infty$ can also be found in [12, Proposition 2.79&Corollary 2.81] and [14, Remark 3.4.8].

Proof This is an immediate consequence of Corollary 3.3 and Example 2.3. □

Remark 3.7 Let $Y := L^{p(\cdot)}(T; X)$. Let us bound $\zeta(0,0)$ from below using (3.8). Note that by Example 2.3 one has that $K_\Phi = p_+$, $K_\Psi = p'_-$, and $h_\Phi = h_\Psi = 0$, so

$$\zeta(0,0) \geq \frac{1}{6K_\Psi C_{K_\Phi,X} C_h} = \frac{1}{3p'_- C_{p_+,X}},$$

where an upper bound for $C_{p_+,X}$ can be found using Remark 2.10. From this one can obtain an upper bound for the UMD constant using (2.7).

In [20] the analytic Radon–Nikodym (ARNP) and analytic UMD (AUMD) properties are shown to hold for Musielak-Orlicz spaces $L^\Phi(T)$ where Φ satisfies a condition which is slightly more restrictive than Δ_2. To end the paper we want to state a related conjecture about spaces satisfying a randomized version of UMD. In order to introduce it let $(\Omega', \mathscr{A}', \mathbb{P}')$ be a second probability space with a Rademacher sequence $\varepsilon' = (\varepsilon'_n)_{n\geq 1}$. A Banach space X is said to be a $\mathrm{UMD}^-_{\mathrm{PW}}$ space if there is a $p \in [1,\infty)$ and a constant $C \geq 0$ such that for all Paley–Walsh martingales f,

$$\|f\|_{L^p(\Omega;X)} \leq \|\varepsilon' * f\|_{L^p(\Omega\times\Omega';X)}.$$

This property turns out to be p-independent, and it gives a more general class of Banach spaces than the UMD spaces (see [9–11, 18]). For instance, L^1 is a $\mathrm{UMD}^-_{\mathrm{PW}}$ space.

Conjecture 3.8 Assume $\Phi : T \times [0,\infty) \to [0,\infty)$ is a Young function such that $\Phi \in \Delta_2$. Then $L^\Phi(T)$ is $\mathrm{UMD}^-_{\mathrm{PW}}$.

The conjecture is open also in the case Φ is not dependent on T. If $\Phi : [0,\infty) \to [0,\infty)$ is merely continuous, increasing to infinity and $\Phi(0) = 0$ and satisfies Δ_2, then the same question can be asked. However, in this case $L^\Phi(T)$ is not a Banach space, but only a quasi-Banach space. Some evidence for the conjecture can be found in [11, Theorem 4.1] and [19, Theorem 1.1] where analogues of Lemma 2.6 can be found (only $\Phi \in \Delta_2$ is needed in the proof). Doob's inequality plays a less prominent role for UMD^- because of [11, Lemma 2.2]. Similar questions can be asked for the possibly more restrictive "decoupling property" of a quasi-Banach space X introduced in [10, 11].

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