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ENTROPY AND KOLMOGOROV COMPLEXITY

Entropy and Kolmogorov Complexity

Proefschrift

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Part I

Introduction

What is Entropy?

The general concept of *entropy* is elusive and hard to define rigorously, even though its instances occur a lot in sciences and applications. Depending on the context, the word 'entropy' might actually have different meanings. For example, entropy is seen in statistical thermodynamics, in particular, it appears in the somewhat infamous Second Law of Thermodynamics saying that

The total entropy of any isolated thermodynamic system tends to increase over time, approaching a maximum value.

Very often 'entropy' is intuitively linked to some form of *complexity*. For instance, entropy - more precisely, combinatorial entropy - can be used to describe the complexity of large strings of symbols. That is, if the string of bits forming a digital file has large entropy, then it is impossible to achieve a good lossless compression ratio for such a file.

In this thesis we study entropy and its relation to complexity in the context of dynamical systems.

Entropy and Dynamical Systems

One way to make the term 'entropy' mathematically precise is to introduce it as a certain invariant for studying dynamical systems of a topological or probabilistic origin. For a given group Γ, a Γ-dynamical system is a pair (X, π), where X is a 'space' and π is a representation of Γ in the group $\mathrm{Aut}(\mathrm{X})$ of automorphisms of X. Of course, the group $\mathrm{Aut}(\mathrm{X})$ of automorphisms of X depends on the underlying structure of the space X. When X is a topological space, we call the pair (X, π) a topological dynamical system, and when X is a probability space we call (X, π) a measure-preserving system. The collection of all topological dynamical systems for a fixed group Γ forms a category, and, similarly, we have the category of measure-preserving dynamical systems. The crucial consequence of any meaningful definition of entropy of a dynamical system is that entropy is an invariant, i.e., it remains constant on isomorphism classes of systems in a given category. In the category of topological dynamical systems, mathematicians are mainly interested in the topological entropy. In the category of measure-preserving systems, the Kolmogorov-Sinai entropy is used instead.

Originally, the study of dynamical systems was focused on the special case in which Γ is the group $\mathbb{Z}$ of integers, and the original definitions of entropy were given for the case $\Gamma = \mathbb{Z}$ accordingly. Entropy of a measure-preserving $\mathbb{Z}$-system was defined by A. Kolmogorov, and later it was modified by Ya. Sinai, leading to what we know today as the Kolmogorov-Sinai entropy. This concept proved to be useful immediately by giving a negative answer to an open problem of isomorphism of Bernoulli shifts. The entropy of a topological $\mathbb{Z}$-dynamical system was defined by R. L. Adler, and later an equivalent definition was given by R. Bowen. It was later shown in [Pal76] by G. Palm that both topological and Kolmogorov-Sinai entropies of $\mathbb{Z}$-systems are in fact instances of a more general entropy defined on what he called abstract dynamical lattices.

Later, the original definitions of the topological and the Kolmogorov-Sinai entropies were extended for amenable group actions using the lemma of D. S. Ornstein and B. Weiss. The corresponding generalization of the work of G. Palm to representations of amenable groups is discussed later in the thesis.

Entropy and Kolmogorov Complexity

Entropy is often seen as a certain measure of complexity. The appropriate mathematical definition of complexity was suggested by A. Kolmogorov in [Kol65], and it is known today as Kolmogorov complexity. Informally speaking, a decompressor is a computer program that takes finite binary words as the input and produces finite words as the output. The Kolmogorov complexity of a finite word ω with respect to a fixed decompressor A is defined as the length of the shortest binary program that serves as the input to A such that the word ω is printed as the output. It turns out that there exist optimal decompressors, i.e., decompressors that allow for (essentially) shorter descriptions of words as compared to any other decompressor. When such an optimal decompressor A^* is fixed, we simply talk about Kolmogorov complexity without referring to A^* explicitly. The 'optimality' of A^* has many consequences - so, for instance, a long periodic word would have small Kolmogorov complexity relative to its size.

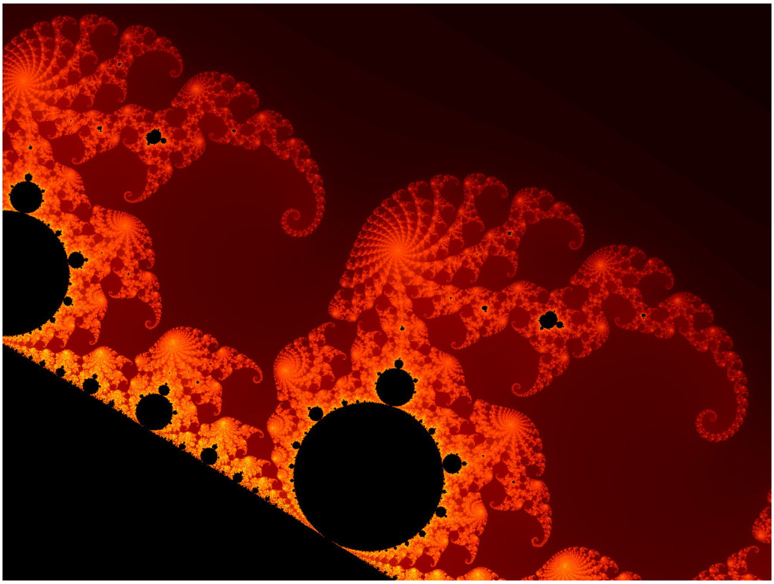

FIGURE 1. A fragment of the Mandelbrot set

For another example, consider the image in Figure 1 depicting a fragment of the Mandelbrot set. This image would take approximately 2 million bits if stored as plain data, but there is a much shorter computer program that can generate this picture using the mathematical definition of the Mandelbrot set. It follows that the Kolmogorov complexity of the image above is much lower than 2 million. On the other hand, words without any regularities are expected to have large Kolmogorov complexity and are considered to be most random.

The bridge between the entropy of dynamical systems and the Kolmogorov complexity of the trajectories was built by A. A. Brudno in a series of papers [Bru74] and [Bru82], where he showed that for every $\mathbb{N}$-dynamical system its Kolmogorov-Sinai entropy equals the Kolmogorov complexity of the orbit of almost every point. However, Brudno obtained his results long before the entropy theories for amenable group actions were developed, and some important tools needed to generalize these results became available only recently. Most importantly, when the acting group is amenable, we need a generalization of the classical pointwise ergodic theorem. E. Lindenstrauss proved in [Lin01] that the pointwise ergodic theorem and the Shannon-McMillan-Breiman theorem do indeed hold under some mild restrictions on the Følner sequence. A 'weighted' version of the pointwise ergodic theorem for general amenable group actions, which will be used as well, was developed by P. Zorin-Kranich in [ZK14].

Our second major aim in this thesis is to present these generalizations, using the tools that became available recently. We prove in [Mor15b] that the original results of A. Brudno can be extended under certain assumptions to a large class of computable groups, and in [Mor15c] we prove that his later results can also be extended, but at the cost of introducing more restrictions.

Overview

The thesis is structured as follows. We devote Part II to the general preliminaries on amenable groups, dynamical systems and computability theory. We begin by defining amenable groups in Chapter 1, giving some examples and basic properties. We discuss topological dynamical systems in Chapter 2, and measure-preserving systems in Chapter 3. Chapter 4 is devoted to the notions of computability and Kolmogorov complexity. We will define, among others, computable groups and computable Følner monotilings of computable amenable groups. The results of this chapter will be of importance in the last part of the thesis.

Part III concerns the theory of entropy of amenable group actions. In Chapter 5 we will define the Kolmogorov-Sinai and the topological entropy of dynamical systems for amenable group actions using the lemma of Ornstein and Weiss from Chapter 1, provide some examples and prove some basic properties. We close the third part with a chapter based upon [Mor15a], where we present the first major result of this thesis, namely the generalization of the work of G. Palm for amenable group actions in the language of measurement functors.

We establish the link between entropy theory and complexity in Part IV by proving two theorems of Brudno in Chapter 7. Here we rely on the tools from the previous chapters such as the Shannon-McMillan-Breiman theorem and the theorem on (weighted) pointwise convergence of ergodic averages. This chapter is based on [Mor15b] and [Mor15c].

Almost every chapter is closed with a 'Remarks' section, where some additional comments, explanations and references are provided. We do not use this material in the main part of the text.

Part II

Preliminaries

CHAPTER 1

Amenable Groups

As we have already mentioned in the introduction, the theory of dynamical systems was originally focused on studying dynamical systems with a single transformation, i.e., $\mathbb{Z}$-systems. So, for instance, the key theorems such as the Birkhoff pointwise convergence theorem and the Shannon-McMillan-Breiman theorem were first proved in this setting. It turns out that one can generalize many of these classical results to the much more general case of amenable group actions. The class of amenable groups includes the standard examples, such as the groups $\mathbb{Z}^d$ for $d \geq 1$, together with many others.

This chapter is structured as follows. We begin with the definitions in Section 1.1, give some examples and state some basic properties in Section 1.2. We devote Section 1.3 to the notion of a Følner monotiling of an amenable group which is originally due to B. Weiss, that plays a crucial role later in the thesis. In general, one does not know whether an arbitrary amenable group has a Følner monotiling, but the existence of a quasi-tiling is always guaranteed by the results of D. S. Ornstein and B. Weiss. The presence of quasi-tilings in an arbitrary amenable group is extremely useful, e.g. this fact is used in the proof of the Ornstein-Weiss lemma. We will discuss this result in Section 1.4, and we will rely on it later in the definitions of topological (Section 5.3), Kolmogorov-Sinai (Section 5.1) and Palm (Section 6.3) entropies. We close the chapter with Section 1.5, containing some additional comments and remarks.

1.1. Definition

There are a few equivalent ways of defining an amenable group. In this subsection we give the definition using Følner sequences. We stress that all the groups that we consider when talking about amenability are at most countably infinite and discrete.

Let Γ be a group with the counting measure $|\cdot|$. A sequence of finite sets $(F_n)_{n\geq 1}$ is called

1) a **left (right) weak Følner sequence** if for every finite set $K \subseteq \Gamma$ one has

$$\frac{|F_n \triangle K F_n|}{|F_n|} \to 0 \quad \left(\text{resp. } \frac{|F_n \triangle F_n K|}{|F_n|} \to 0\right);$$

2) a **left (right) strong Følner sequence** if for every finite set $K \subseteq \Gamma$ one has

$$\frac{\left|\partial^{\mathrm{l}}_K(F_n)\right|}{|F_n|} \to 0 \quad \left(\text{resp. } \frac{|\partial^{\mathrm{r}}_K(F_n)|}{|F_n|} \to 0\right),$$

where

$$\partial^{\mathrm{l}}_K(F) := K^{-1}F \cap K^{-1}F^{\mathrm{c}} \quad \left(\text{resp. } \partial^{\mathrm{r}}_K(F) := FK^{-1} \cap F^{\mathrm{c}}K^{-1}\right)$$

is the **left (right) K-boundary** of F;

3) a **(C-)tempered sequence** if there is a constant C such that for every j one has

$$\left|\bigcup_{i<j} F_i^{-1}F_j\right| < C\,|F_j|\,.$$

One can show that a sequence of sets $(F_n)_{n\geq 1}$ is a weak left Følner sequence if and only if it is a strong left Følner sequence (see [CSC10, Section 5.4]), hence we will simply call it a left Følner sequence. The same holds for right Følner sequences. If we call a sequence of sets a '**Følner sequence**' without qualifying it as 'left' or 'right', we always mean a left Følner sequence. A sequence of sets $(F_n)_{n\geq 1}$ which is simultaneously a left and a right Følner sequence is called a **two-sided Følner sequence**. A group Γ is called **amenable** if it admits a left Følner sequence. If Γ is infinite, for every Følner sequence $(F_n)_{n\geq 1}$ we have $|F_n| \to \infty$ as $n \to \infty$. If, on the other hand, Γ is finite, then for every Følner sequence $(F_n)_{n\geq 1}$ we have $F_n = \Gamma$ for all sufficiently large n.

For finite sets $F, K \subseteq \Gamma$ the sets

$$\mathrm{int}^{\mathrm{l}}_K(F) := F \setminus \partial^{\mathrm{l}}_K(F) \quad (\text{resp. } \mathrm{int}^{\mathrm{r}}_K(F) := F \setminus \partial^{\mathrm{r}}_K(F))$$

are called the **left (right) K-interior** of F respectively. It is clear that if a sequence of finite sets $(F_n)_{n\geq 1}$ is a left (right) Følner sequence, then for every finite $K \subseteq \Gamma$ one has

$$\left|\mathrm{int}^{\mathrm{l}}_K(F_n)\right| / |F_n| \to 1 \quad (\text{resp. } |\mathrm{int}^{\mathrm{r}}_K(F_n)| / |F_n| \to 1)\,.$$

1.2. Examples

We begin with the most basic examples. It is clear that all finite groups are amenable.

Example 1.2.1. Consider the group $\mathbb{Z}^d$ for some $d \geq 1$. Consider the sequence $(F_n)_{n\geq 1}$ in $\mathbb{Z}^d$ given by

$$F_n := [0, 1, 2, \ldots, n-1]^d.$$

It is easy to see that $(F_n)_{n\geq 1}$ is a tempered two-sided Følner sequence.

The simplest non-abelian example of an infinite amenable group is the discrete Heisenberg group, which we discuss next.

Example 1.2.2. Consider the group $\mathsf{UT}_3(\mathbb{Z})$, i.e., the discrete Heisenberg group H_3. By definition,

$$\mathsf{UT}_3(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

To simplify the notation, we will denote a matrix

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{UT}_3(\mathbb{Z})$$

by the corresponding triple (a, b, c) of its entries. Then the products and inverses in $\mathsf{UT}_3(\mathbb{Z})$ can be computed by the formulas

$$\begin{aligned} (a, b, c)(x, y, z) &= (a + x, b + y, c + z + ya), \\ (a, b, c)^{-1} &= (-a, -b, ba - c). \end{aligned}$$

The sequence $(F_n)_{n\geq 1}$ defined by

$$F_n := \{(a, b, c) \in \mathsf{UT}_3(\mathbb{Z}) : 0 \leq a, b < n, 0 \leq c < n^2\}$$

for all $n \geq 1$ is a two-sided Følner sequence (this follows from a straightforward computation, see [LSV11] for the details). In order to check the temperedness of $(F_n)_{n\geq 1}$, note that for every $n > 1$

$$\bigcup_{i<n} F_i^{-1} F_n \subseteq F_n^{-1} F_n,$$

where

$$F_n^{-1} \subseteq \{(a, b, c) : -n < a, b \leq 0, -n^2 < c < n^2\}.$$

It is easy to see that for every $n > 1$

$$F_n^{-1} F_n \subseteq \{(a, b, c) : -n < a, b < n, -3n^2 < c < 3n^2\}.$$

Since $|F_n| = n^4$ for every n, the sequence $(F_n)_{n\geq 1}$ is tempered.

Amenable groups enjoy some useful properties, which we state without proofs below. For the proofs we refer to [CSC10, Chapter 4].

Proposition 1.2.3. *Suppose that G, K are amenable groups and that the sequence*

$$1 \to G \xrightarrow{\imath} F \xrightarrow{\pi} K \to 1$$

is exact. Then the group F is amenable as well.

We will discuss a related result in the context of Følner monotilings later in Section 1.5.3. It follows from Proposition 1.2.3 that the group $\mathsf{UT}_d(\mathbb{Z})$ is amenable for all $d \geq 2$. We will return to the question about 'nice' Følner sequences in $\mathsf{UT}_d(\mathbb{Z})$ later in Section 7.2.

Proposition 1.2.4. *Every group which is the limit of an inductive system of amenable groups is amenable.*

Hence the group $\mathbb{Q}$ is amenable. In fact, it follows from Proposition 1.2.4 that all abelian groups are amenable.

1.3. Følner Monotilings

The purpose of this section is to discuss the notion of a *Følner monotiling*, that was introduced by B. Weiss in [Wei01]. However, we have to introduce both 'left' and 'right' monotilings, while the original notion introduced by Weiss is a 'left' monotiling.

A **left monotiling** $[F, \mathcal{Z}]$ in a discrete group Γ is a pair of a finite set $F \subseteq \Gamma$, which we call **a tile**, and a set $\mathcal{Z} \subseteq \Gamma$, which we call a set of **centers**, such that $\{Fz : z \in \mathcal{Z}\}$ is a covering of Γ by disjoint translates of F. Similarly, given a **right monotiling** $[\mathcal{Z}, F]$ we require that $\{zF : z \in \mathcal{Z}\}$ is a covering of Γ by disjoint translates of F. A **left (right) Følner monotiling** is a sequence of monotilings $([F_n, \mathcal{Z}_n])_{n\geq 1}$ (resp. $([\mathcal{Z}_n, F_n])_{n\geq 1}$) such that $(F_n)_{n\geq 1}$ is a left (resp. right) Følner sequence in Γ. A left Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is called **symmetric** if for every $k \geq 1$ the set of centers $\mathcal{Z}_k$ is symmetric, i.e. $\mathcal{Z}_k^{-1} = \mathcal{Z}_k$. It is clear that if $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is a symmetric Følner monotiling, then $([\mathcal{Z}_n, F_n^{-1}])_{n\geq 1}$ is a right Følner monotiling.

We begin with a basic example.

Example 1.3.1. Consider the group $\mathbb{Z}^d$ for some $d \geq 1$ and the Følner sequence $(F_n)_{n\geq 1}$ in $\mathbb{Z}^d$ given by

$$F_n := [0, 1, 2, \ldots, n-1]^d.$$

Furthermore, for every n let

$$\mathcal{Z}_n := n\mathbb{Z}^d.$$

Here $n\mathbb{Z}^d$ stands for the subgroup of $\mathbb{Z}^d$, consisting of d-tuples of integers which are divisible by n. It is easy to see that $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is a symmetric Følner monotiling of $\mathbb{Z}^d$.

A less trivial example is given by Følner monotilings of the discrete Heisenberg group $\mathsf{UT}_3(\mathbb{Z})$. We will return to the Følner monotilings of $\mathsf{UT}_d(\mathbb{Z})$ for $d > 3$ later in Section 7.2.

Example 1.3.2. Consider the group $\mathsf{UT}_3(\mathbb{Z})$, i.e., the discrete Heisenberg group H_3. By definition,

$$\mathsf{UT}_3(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

For every $n \geq 1$, consider the subgroup

$$\mathcal{Z}_n := \{(a, b, c) \in \mathsf{UT}_3(\mathbb{Z}) : a, b \in n\mathbb{Z}, c \in n^2\mathbb{Z}\}.$$

This is a finite index subgroup, and it is easy to see that for every n the finite set

$$F_n := \{(a, b, c) \in \mathsf{UT}_3(\mathbb{Z}) : 0 \leq a, b < n, 0 \leq c < n^2\}$$

is a fundamental domain for $\mathcal{Z}_n$. As we have already mentioned, $(F_n)_{n\geq 1}$ is a two-sided Følner sequence. We conclude that $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is a symmetric Følner monotiling.

In what follows we will need the following simple proposition, which tells us that the sets of centers of a Følner monotiling have positive density.

PROPOSITION 1.3.3. *Let $([F_n, \mathcal{Z}_n])_{n\geq 1}$ be a left Følner monotiling of Γ such that $\mathrm{e} \in F_n$ for every n, where $\mathrm{e} \in \Gamma$ is the neutral element. Then for every fixed k*

$$\frac{\left|\mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \cap \mathcal{Z}_k\right|}{|F_n|} \to \frac{1}{|F_k|} \tag{1.3.1}$$

and

$$\frac{|F_n \cap \mathcal{Z}_k|}{|F_n|} \to \frac{1}{|F_k|} \tag{1.3.2}$$

as $n \to \infty$. If, additionally, $(F_n)_{n\geq 1}$ is a two-sided Følner sequence, then for every fixed k

$$\frac{\left|\mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{r}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k\right|}{|F_n|} \to \frac{1}{|F_k|} \tag{1.3.3}$$

as $n \to \infty$.

PROOF. Observe first that, under the initial assumptions of the theorem, for every set $A \subseteq \Gamma$, $k \geq 1$ and $g \in \Gamma$ we have

$$g \in \mathrm{int}^{\mathrm{l}}_{F_k}(A) \Leftrightarrow F_k g \subseteq A$$

and

$$g \in \mathrm{int}^{\mathrm{r}}_{F_k^{-1}}(A) \Leftrightarrow g F_k^{-1} \subseteq A.$$

Let $k \geq 1$ be fixed. For every $n \geq 1$, consider the finite set $A_{n,k} := \{g \in \mathcal{Z}_k : F_k g \cap \mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \neq \varnothing\}$. Then the translates $\{F_k z : z \in A_{n,k}\}$ form a disjoint cover of the set $\mathrm{int}^{\mathrm{l}}_{F_k}(F_n)$. It is easy to see that

$$\Gamma = \mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \sqcup \partial^{\mathrm{l}}_{F_k}(F_n) \sqcup \mathrm{int}^{\mathrm{l}}_{F_k}(F_n^{\mathrm{c}}).$$

Since $A_{n,k} \cap \mathrm{int}^{\mathrm{l}}_{F_k}(F_n^{\mathrm{c}}) = \varnothing$, we can decompose the set of centers $A_{n,k}$ as follows:

$$A_{n,k} = (A_{n,k} \cap \mathrm{int}^{\mathrm{l}}_{F_k}(F_n)) \sqcup (A_{n,k} \cap \partial^{\mathrm{l}}_{F_k}(F_n)).$$

Since $(F_n)_{n\geq 1}$ is a Følner sequence,

$$\frac{\left|F_k(A_{n,k} \cap \partial^{\mathrm{l}}_{F_k}(F_n))\right|}{|F_n|} = \frac{|F_k| \cdot \left|A_{n,k} \cap \partial^{\mathrm{l}}_{F_k}(F_n)\right|}{|F_n|} \to 0$$

and $\left|\mathrm{int}^{\mathrm{l}}_{F_k}(F_n)\right| / |F_n| \to 1$ as $n \to \infty$. Then from the inequalities

$$\begin{aligned}\frac{\left|\mathrm{int}^{\mathrm{l}}_{F_k}(F_n)\right|}{|F_n|} &\leq \frac{\left|F_k(A_{n,k} \cap \partial^{\mathrm{l}}_{F_k}(F_n))\right|}{|F_n|} + \frac{\left|F_k(A_{n,k} \cap \mathrm{int}^{\mathrm{l}}_{F_k}(F_n))\right|}{|F_n|} \\ &\leq \frac{\left|F_k(A_{n,k} \cap \partial^{\mathrm{l}}_{F_k}(F_n))\right|}{|F_n|} + 1\end{aligned}$$

we deduce that

$$\frac{|F_k| \cdot \left|A_{n,k} \cap \mathrm{int}^{\mathrm{l}}_{F_k}(F_n)\right|}{|F_n|} \to 1 \tag{1.3.4}$$

as $n \to \infty$. It remains to note that $A_{n,k} \cap \mathrm{int}^{\mathrm{l}}_{F_k}(F_n) = \mathcal{Z}_k \cap \mathrm{int}^{\mathrm{l}}_{F_k}(F_n)$ and the first statement follows. The second statement follows trivially from the first one. To obtain the last statement, observe that $\left|\mathrm{int}^{\mathrm{r}}_{F_k^{-1}}(F_n)\right| / |F_n| \to 1$ as $n \to \infty$ since $(F_n)_{n\geq 1}$ is a right Følner sequence, thus

$$\lim_{n\to\infty} \frac{\left|\mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \cap F_n \cap \mathcal{Z}_k\right|}{|F_n|} = \lim_{n\to\infty} \frac{\left|\mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{r}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k\right|}{|F_n|} = \frac{1}{|F_k|}.$$

□

1.4. Lemma of Ornstein and Weiss

In the classical definitions of the topological and the Kolmogorov-Sinai entropies the following elementary lemma is used.

LEMMA 1.4.1 (Subadditivity lemma). *Let $(a_n)_{n\geq 1}$ be a sequence of non-negative numbers such that, for every $n, m \geq 1$, $a_{m+n} \leq a_m + a_n$. Then the limit*

$$\lim_{n\to\infty} \frac{a_n}{n}$$

exists and equals $\inf\limits_{n\geq 1} \frac{a_n}{n}$.

PROOF. Let $a := \inf\limits_{n\geq 1} \frac{a_n}{n}$. Given $\varepsilon > 0$, there is N such that $\frac{a_N}{N} < a + \varepsilon$. Now, for all $n \geq N$, we have $n = sN + r$, where the integers s, r are nonnegative and $r < N$. Using subadditivity, we see that

$$a_n \leq a_{sN} + a_r \leq s a_N + a_r.$$

Dividing both sides by n, we see that

$$\frac{a_n}{n} \leq \frac{sN}{n}\frac{a_N}{N} + \frac{a_r}{n} \leq (a + \varepsilon) + \frac{a_r}{n}.$$

Since ε is arbitrary, this shows that $\limsup\limits_{n\geq 1} \frac{a_n}{n} \leq a$, hence the limit $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists and equals a. □

For general amenable groups this result is replaced by the following lemma. For a set X we let

$$\mathcal{P}_0(X) := \{A \subseteq X : A \text{ finite}\}$$

be the set of all finite subsets of X.

PROPOSITION 1.4.2 (Ornstein-Weiss lemma). *Let $f : \mathcal{P}_0(\Gamma) \to \mathbb{R}_{\geq 0}$ be a function from the set of finite subsets of an amenable group Γ to the set of non-negative reals satisfying the following conditions*

a) *f is monotone, i.e. $f(F_1) \leq f(F_2)$ holds for any two finite subsets $F_1 \subseteq F_2 \subseteq \Gamma$;*
b) *f is subadditive, i.e. $f(F_1 \cup F_2) \leq f(F_1) + f(F_2)$ holds for any two finite subsets $F_1, F_2 \subseteq \Gamma$;*
c) *f is right-invariant, i.e. $f(Fg) = f(F)$ holds for all finite $F \subseteq \Gamma$ and $g \in \Gamma$.*

Then for every Følner sequence $(F_n)_{n\in\mathbb{N}}$ of Γ the limit

$$\lim_{n\to\infty} \frac{f(F_n)}{|F_n|}$$

in $\mathbb{R}_{\geq 0}$ exists and is independent of the choice of Følner sequence.

For the proofs we refer to [LW00, Theorem 6.1], [Kri07] or [Gro99, Section 1.3].

1.5. Remarks

1.5.1. Equivalent Definitions of Amenability. A few equivalent definitions of amenability can be given. Let Γ be a group, which we assume to be at most countable and discrete. A **finite mean** is a finitely supported function $\lambda : \Gamma \to \mathbb{R}_{\geq 0}$. Of course, finite means belong to $\ell^1(\Gamma;\mathbb{R})$. A non-negative linear functional $\lambda : \ell^\infty(\Gamma;\mathbb{R}) \to \mathbb{R}$ is called a **mean** if $\lambda(\mathbf{1}) = 1$. A mean $\lambda : \ell^\infty(\Gamma;\mathbb{R}) \to \mathbb{R}$ is called **left-invariant** if for all $x \in \Gamma$ and $f \in \ell^\infty(\Gamma;\mathbb{R})$ we have $\lambda(L_x f) = \lambda(f)$. Here $L.$ denotes the left-regular representation, i.e. for all $x, y \in \Gamma$ we have

$$(L_x f)(y) = f(x^{-1}y).$$

We summarize some useful equivalent definitions of amenability in the following proposition. For the proofs we refer to [CSC10, Chapter 4].

PROPOSITION 1.5.1. *Let Γ be an at most countable group. The following assertions are equivalent:*

(i) *There exists a left-invariant mean $\lambda : \ell^\infty(\Gamma;\mathbb{R}) \to \mathbb{R}$;*
(ii) *For every finite subset $S \subseteq \Gamma$ and every $\varepsilon > 0$ there exists a finite mean λ such that $\|\lambda - L_x\lambda\|_{\ell^1(\Gamma;\mathbb{R})} < \varepsilon$ for all $x \in S$;*
(iii) *For every finite subset $S \subseteq \Gamma$ and every $\varepsilon > 0$ there exists a non-empty finite set $A \subseteq \Gamma$ such that, for all $x \in S$,*

$$\frac{|xA \triangle A|}{|A|} < \varepsilon;$$

(iv) *There exists a Følner sequence $(F_n)_{n\geq 1}$ in Γ, i.e. Γ is amenable.*
(v) *Every affine action of Γ on a nonempty convex compact subset of a Hausdorff topological vector space admits a fixed point.*

1.5.2. Two-sided Følner Sequences. We have made a clear distinction between the *left* and the *right* Følner sequences in our definitions, while in the definition of a regular Følner monotiling (Section 7.2) we will require that the Følner sequence is two-sided. However, *every* amenable group admits a two-sided Følner sequence. For the proof we refer to [BCRZ14, Section 2.2] and [OW87, Chapter I. §1, Proposition 2]. It is not clear, on the other hand, if one can construct a two-sided Følner sequence that would tile the group from a left/right Følner sequence tiling the group.

1.5.3. Existence of Følner Monotilings. One of the main results in [Wei01] is the following:

PROPOSITION 1.5.2. *Suppose that G, K are amenable groups that both admit Følner monotilings and that the sequence*

$$1 \to G \xrightarrow{\imath} F \xrightarrow{\pi} K \to 1$$

is exact. Then the group F admits a Følner monotiling as well.

We discuss a 'computable' version of this result later in Theorem 4.5.2. It is clear that all finitely generated abelian groups admit Følner monotilings. Furthermore, all countable abelian groups admit Følner monotilings as well (any such group Γ is an increasing union $\bigcup_{n\geq 1} \Gamma_n$ of finitely generated subgroups obtained by adding one extra generator at each step, so, given a sufficiently invariant set $F \subseteq \Gamma_n$ which tiles Γ_n, one can cover the whole group Γ by disjoint translates of F). In particular, the group $\mathbb{Q}$ admits a Følner monotiling, even though we do not have a nice formula for it. This result, together with Proposition 1.5.2, yields the following:

PROPOSITION 1.5.3. *Any countable solvable group admits a Følner monotiling.*

1.5.4. Non-amenable Groups and Sofic Groups. Not all countable groups are amenable, even if we restricted to the finitely generated ones. A basic example of a non-amenable group is the free group F_2 on two generators. However, this group is an example of a *sofic group*, which have recently become important in the sofic entropy theory. We refer to [CSC10, Section 7.5] for the definition of a sofic group, and to [Bow10] for the introduction to the sofic entropy theory.

CHAPTER 2

Topological Dynamical Systems

A large part of the theory of dynamical systems is devoted to studying topological dynamical systems, i.e. the systems coming from continuous actions of groups on topological spaces. In this setting one studies, among others, the problem of isomorphism of topological dynamical systems; topological recurrence; the structure theory of topological dynamical systems, such as the Furstenberg distal structure theorem, and so on.

We will not go deep into topological dynamics in this thesis since our main object of interest is entropy. In this chapter we will present the basic prerequisites that are needed to talk about entropy for topological dynamical systems. Also, we will not impose restrictions on the groups acting on topological spaces unless it is stated otherwise; but later we will restrict ourselves to amenable group actions when defining the topological entropy.

In Section 2.1 we will discuss the definitions and prove some basic properties, among others the notion of a *factor*. We say a bit more about factors and related category-theoretic questions in Section 2.3. Section 2.2 is devoted to the basic examples. We will follow a 'categorical' view on the subject by using the language of category theory.

2.1. Definition

Let Top be the category of compact Hausdorff topological spaces with surjective continuous maps as morphisms. Then, clearly, for every topological space $\mathrm{X} \in \mathsf{Top}$ the group $\mathrm{Aut}(\mathrm{X})$ is the group of homeomorphisms from X to X.

Let Γ be a discrete group, $\mathrm{X} \in \mathsf{Top}$ and $\pi : \Gamma \to \mathrm{Aut}(\mathrm{X})$ be a group homomorphism. This defines a left action of Γ on X by setting

$$\gamma \cdot x := \pi_\gamma(x).$$

The pair

$$\mathbf{X} = (\mathrm{X}, \pi)$$

is called a **topological dynamical system**. When we want to stress that the acting group is Γ, we will sometimes say that this is a **topological Γ-system**. When we work with a topological Γ-system $\mathbf{X} = (\mathrm{X}, \pi)$ and the representation π is fixed, we will often write γ instead of π_γ to denote the morphism $\pi_\gamma : \mathrm{X} \to \mathrm{X}$ for $\gamma \in \Gamma$. This coincides with the standard notation in the theory of dynamical systems, and typically does not cause any confusion.

Let us define the category Top_Γ. The objects are, by definition, pairs (X, π), with X being a compact Hausdorff topological space and $\pi : \Gamma \to \mathrm{Aut}(\mathrm{X})$ being a group homomorphism. Let $\mathbf{X} = (\mathrm{X}, \pi), \mathbf{Y} = (\mathrm{Y}, \rho)$ be topological dynamical systems. We define $\mathrm{Hom}(\mathbf{X}, \mathbf{Y})$ as the set of all morphisms $\phi : \mathrm{X} \to \mathrm{Y}$ such that $\rho_\gamma \circ \phi = \phi \circ \pi_\gamma$ for all γ. That is, we require that the diagram

$$\begin{tikzcd} \mathrm{X} \arrow[r, "\phi"] & \mathrm{Y} \\ \mathrm{X} \arrow[u, "\pi_\gamma"] \arrow[r, "\phi"'] & \mathrm{Y} \arrow[u, "\rho_\gamma"'] \end{tikzcd}$$

commutes for all γ. Then $\mathbf{Y}$ is called a **factor** of $\mathbf{X}$, and ϕ is called a (topological) **factor map**.

A **subsystem** of a topological dynamical system (X, π) is a nonempty closed subset $\mathrm{Y} \subseteq \mathrm{X}$ which is **Γ-invariant**, i.e.

$$\pi_\gamma(x) \in \mathrm{Y}$$

holds for all $x \in \mathrm{Y}, \gamma \in \Gamma$. Clearly, any subsystem $\mathrm{Y} \subseteq \mathrm{X}$ becomes a topological dynamical system (Y, π) by restricting the action of Γ to Y. A topological dynamical system $\mathbf{X} = (\mathrm{X}, \pi)$ is called **minimal** if it does not have proper subsystems. A simple argument using Zorn's lemma gives the following proposition.

PROPOSITION 2.1.1. *Let* (X, π) *be a topological dynamical system. Then there exists a minimal subsystem* $\mathrm{Y} \subseteq \mathrm{X}$.

Many interesting examples of dynamical systems are $\mathbb{Z}$-systems. In this case we adapt the notation slightly and write $(\mathrm{X}; \varphi)$ instead of (X, π), where $\varphi := \pi_1$ is a homeomorphism $\mathrm{X} \to \mathrm{X}$.

2.2. Examples

In this section we collect some elementary examples of topological dynamical systems. Later we will show how to compute the topological entropy for some of these examples.

EXAMPLE 2.2.1. Let $\mathbb{R}/\mathbb{Z}$ be the unit torus written additively, and let $\alpha \in \mathbb{R}/\mathbb{Z}$ be fixed. Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the transformation

$$x \mapsto x + \alpha \quad \text{for all } x \in \mathbb{R}/\mathbb{Z}.$$

The dynamical system $(\mathbb{R}/\mathbb{Z}; R_\alpha)$ is called a **torus rotation**. It is easy to show using Dirichlet's principle that that $(\mathbb{R}/\mathbb{Z}; R_\alpha)$ is minimal if and only if $\alpha \in \mathbb{R}/\mathbb{Z}$ is irrational.

EXAMPLE 2.2.2. Let G be a compact abelian group and $g \in G$ be a fixed element. Let $R_g : G \to G$ be the transformation

$$x \mapsto x + g \quad \text{for all } x \in G.$$

The dynamical system $(G; R_g)$ is called a **compact group rotation**. One can show [EFHN15, Theorem 3.4] that the following assertions are equivalent:

(i) $(G; R_g)$ is minimal;

(ii) $\{g^n\}_{n\geq 1}$ is dense in G;

(iii) $\{g^n\}_{n\in\mathbb{Z}}$ is dense in G.

EXAMPLE 2.2.3. Let $(\mathbb{R}/\mathbb{Z})^2$ be the two-dimensional torus and $\alpha \in \mathbb{R}/\mathbb{Z}$ be fixed. Consider the transformation $\varphi : (\mathbb{R}/\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2$ given by

$$(x, y) \mapsto (x+\alpha, x+y) \quad \text{for all } (x, y) \in (\mathbb{R}/\mathbb{Z})^2.$$

The topological dynamical system $((\mathbb{R}/\mathbb{Z})^2; \varphi)$ is called the **skew-shift**. One can show [Fur81, Lemma 1.25] that the skew-shift is minimal.

EXAMPLE 2.2.4. Let Λ be a finite alphabet and Γ be a discrete, at most countably infinite group. We define a compact Hausdorff space

$$\mathrm{X} := \Lambda^{\Gamma},$$

carrying the product topology. Consider the representation π of Γ in $\mathrm{Aut}(\mathrm{X})$ given by

$$(g\cdot\omega)(x) := \omega(xg) \quad \text{for all } x, g \in \Gamma, \omega \in \mathrm{X}.$$

The dynamical system (X, π) is called the **right shift** over Γ with alphabet Λ. Any subsystem $\mathrm{Y} \subseteq \mathrm{X}$ is called a **subshift**.

2.3. Factors

2.3.1. Coproducts. There exists a 'natural' structure of category on the collection of factors of a fixed system $\mathbf{X} \in \mathsf{Top}_\Gamma$. First, we need to remind the reader of the notion of a **coproduct** from category theory. It is a 'dual' notion to the notion of product. Let C be a category, and let $\{\mathrm{X}_\lambda\}_{\lambda\in\Lambda}$ be a family of objects of this category indexed by a set Λ. The coproduct of this family is a pair of an object $\mathrm{X} \in \mathsf{C}$ and a collection of morphisms $\{\pi_\lambda\}_{\lambda\in\Lambda}$, $\pi_\lambda : \mathrm{X}_\lambda \to \mathrm{X}$ such that for any object $\mathrm{Y} \in \mathsf{C}$ and a collection of morphisms $\{\rho_\lambda\}_{\lambda\in\Lambda}$, $\rho_\lambda : \mathrm{X}_\lambda \to \mathrm{Y}$ there exists a unique morphism $\phi : \mathrm{X} \to \mathrm{Y}$ such that $\phi \circ \pi_\lambda = \rho_\lambda$ for all indices $\lambda \in \Lambda$. A standard argument then shows that coproducts are unique up to a unique isomorphism when they exist. We write $\coprod\limits_{\lambda\in\Lambda} \mathrm{X}_\lambda$ to denote the coproduct of the family $\{\mathrm{X}_\lambda\}_{\lambda\in\Lambda}$.

We give a couple of examples of coproducts. Consider the category Set of sets with maps between sets being the morphisms. Then the disjoint union $\bigsqcup\limits_{\lambda\in\Lambda} \mathrm{X}_\lambda$ of these sets together with the maps $\pi_\lambda : \mathrm{X}_\lambda \to \bigsqcup\limits_{\lambda\in\Lambda} \mathrm{X}_\lambda$, which are the canonical set inclusions, is, up to a unique isomorphism, the coproduct of a family $\{\mathrm{X}_\lambda\}_{\lambda\in\Lambda}$ of sets. Verifying the universal property is straightforward. It is also not difficult to show that in the categories Ab of abelian groups and Vect_k of vector spaces over a fixed field k the notion of a coproduct coincides with that of a direct sum.

2.3.2. Poset of Factors. For the moment, let D be a category and let A be a fixed object of D. We define the category $\mathsf{Fac}(\mathrm{A})$ as follows. Let S be the collection of all epimorphisms $\phi : \mathrm{A} \to \mathrm{B}$ in D. Given arrows $\phi : \mathrm{A} \to \mathrm{B}$ and $\psi : \mathrm{A} \to \mathrm{C}$ in S, we say ϕ and ψ are *equivalent as factor maps* if there is an isomorphism $\zeta : \mathrm{B} \to \mathrm{C}$ such that $\zeta \circ \phi = \psi$. We define $\mathrm{Obj}(\mathsf{Fac}(\mathrm{A}))$ as the collection of isomorphism classes of elements of S modulo this equivalence relation. We define a preorder on $\mathrm{Obj}(\mathsf{Fac}(\mathrm{A}))$ as follows. For $[\phi], [\psi] \in \mathrm{Obj}(\mathsf{Fac}(\mathrm{A}))$ we say that $[\phi] \geq [\psi]$ if there exists a morphism ζ such that $\psi = \zeta \circ \phi$. This definition is independent of the choice of the representatives ϕ and ψ. Given ψ and ϕ, such ζ is unique because ϕ is an epimorphism. Furthermore, ζ is an epimorphism because ψ is an epimorphism as well. Given $[\phi], [\psi] \in \mathrm{Obj}(\mathsf{Fac}(\mathrm{A}))$, we let

$$\mathrm{Hom}([\phi], [\psi]) := \begin{cases} \{\geq\} & : [\phi] \geq [\psi] \\ \varnothing & : \text{otherwise.} \end{cases}$$

This makes $\mathsf{Fac}(\mathrm{A})$ a *poset category*, i.e. a category satisfying the following additional assertions for all objects $[\phi], [\psi] \in \mathrm{Obj}(\mathsf{Fac}(\mathrm{A}))$:

a) there exists at most one morphism from $[\phi]$ to $[\psi]$;

b) if $\mathrm{Hom}([\phi], [\psi])$ and $\mathrm{Hom}([\psi], [\phi])$ are nonempty, then $[\phi] = [\psi]$.

We call this category the category of *factors* of A. This is a 'dual' notion to the standard notion of the category of *subobjects* (see [Gol84] for the details).

2.3.3. Factors of Topological Dynamical Systems. Now, let Top_Γ be the category of topological Γ-systems. By definition, all morphisms in this category are epimorphisms. Let $\mathbf{X} = (\mathrm{X}, \pi) \in \mathsf{Top}_\Gamma$ be a fixed topological dynamical systems, and let $\mathsf{Fac}(\mathbf{X})$ be the associated category of factors. We want to prove that $\mathsf{Fac}(\mathbf{X})$ can be identified with the set of all Γ-invariant subtopologies on X, and that the '$\geq$' relation of factors is simply the set-theoretic relation '$\supseteq$' of topologies.

If $\phi : \mathbf{X} \to \mathbf{Y}, \psi : \mathbf{X} \to \mathbf{Z}$ are equivalent as factor maps, then the corresponding subtopologies on X coincide. Conversely, if $\phi : \mathbf{X} \to \mathbf{Y}, \psi : \mathbf{X} \to \mathbf{Z}$ are factor maps such that the corresponding subtopologies on X coincide, then ϕ and ψ are equivalent as factor maps. Indeed, it is clear that the fibers $\phi^{-1}(y), y \in \mathrm{Y}$ of points are precisely the minimal proper closed sets in X with respect to the subtopology of Y. The same holds for fibers $\psi^{-1}(z), z \in \mathrm{Z}$. Since the subtopologies coincide, each fiber $\phi^{-1}(y)$ is in fact a fiber $\psi^{-1}(z)$ for some uniquely determined $z \in \mathrm{Z}$. This defines a bijection $\zeta : \mathrm{Y} \to \mathrm{Z}$, and it is easy to see that ζ is a homeomorphism and Γ-intertwining.

Given a Γ-invariant subtopology $\mathcal{V}$ on X, we define an equivalence relation $\sim$ on X by saying that two points $x, y \in \mathrm{X}$ are equivalent if and only if for all open sets $V \in \mathcal{V}$ we have $x, y \in V$ or $x, y \notin V$. Then the quotient space $\mathrm{X}/\!\sim$, endowed with the induced action of Γ, is a factor of $\mathbf{X}$. Finally, given $[\phi], [\psi] \in \mathrm{Obj}(\mathsf{Fac}(\mathbf{X}))$, it is clear that $[\phi] \geq [\psi]$ if and only if the subtopology corresponding to $[\phi]$ is finer than the subtopology corresponding to $[\psi]$.

In what follows we will need to understand the coproducts in $\mathsf{Fac}(\mathbf{X})$. We describe a coproduct of two factors first. So let $\phi : \mathbf{X} \to \mathbf{Y}$ and $\psi : \mathbf{X} \to \mathbf{Z}$ be representatives of $[\phi], [\psi] \in \mathsf{Fac}(\mathbf{X})$. Consider the factor $\mathbf{W}$ of $\mathbf{X}$ obtained by

1) lifting the topologies of $\mathbf{Y}$ and $\mathbf{Z}$ to subtopologies on X via maps ϕ and ψ;
2) taking the intersection of these subtopologies, obtaining a new compact topology on X;
3) gluing points that are not separated by this topology to define a quotient map $\chi : \mathrm{X} \to \mathrm{W}$ and taking the induced action of Γ on W.

Then $\mathbf{W} \in \mathsf{Top}_\Gamma$ is a topological dynamical system and $[\chi] \in \mathsf{Fac}(\mathbf{X})$ is the coproduct of $[\psi]$ and $[\phi]$. A similar construction applies to infinite coproducts.

CHAPTER 3

Measure-preserving Dynamical Systems

Another classical part of the theory of dynamical systems is studying measure-preserving dynamical systems, i.e. measure-preserving actions of groups on probability spaces, which are often assumed to be standard. Just like in topological dynamics, one studies the problem of isomorphism, (measure-theoretic) recurrence and the structure theory. Of course, there are interesting new problems, which are specific to measure-preserving dynamics, as well. One of the main examples is given by ergodic theorems, which say that we can 'average' measure-preserving actions of sufficiently nice groups. Another example is the Shannon-McMillan-Breiman theorem, which connects the Kolmogorov-Sinai entropy of an ergodic measure-preserving system to the amount of information in the sense of Shannon that one obtains by observing the time evolution of a system.

This chapter is structured as follows. Our first goal is to define the category Prob of standard probability spaces. We do so in several steps. In Section 3.1 we introduce measure-preserving maps between probability spaces and measure algebras of probability spaces. Measure-preserving maps will essentially play the role of the morphisms in Prob. The measure algebra $\Sigma(\mathrm{X})$ of a probability space $(X, \mathcal{B}, \mu)$ is the quotient of $\mathcal{B}$ modulo null sets. The main reason to restrict ourselves to *standard* probability spaces is that this restriction allows to define morphisms of probability spaces by defining morphisms of the associated measure algebras.

We begin Section 3.2 by introducing abstract measure algebras and morphisms between them. With this terminology, we complete the definition of the category Prob of standard probability spaces at the end of Section 3.2. We introduce the category Prob_Γ of measure-preserving dynamical systems on standard probability spaces in Section 3.3. At the end of this section we show that the category $\mathsf{Fac}(\mathbf{X})$ of factors of a measure-preserving system $\mathbf{X}$ is isomorphic to the category of Γ-invariant σ-complete subalgebras of the measure algebra $\Sigma(\mathrm{X})$ of X. This fact will be used later in Section 6.3.

We discuss ergodic theorems for amenable groups actions and their weighted versions in Section 3.4. Finally, some additional remarks are provided in Section 3.5. The connection of measure-preserving and topological dynamics is discussed in Section 3.5.1, where we state the Krylov-Bogolyubov theorem, asserting that every topological dynamical system over an amenable group can be endowed with an invariant probability measure, making it a measure-preserving system.

3.1. Probability Spaces

3.1.1. Measure-preserving Maps. Let $\mathrm{X} := (X, \mathcal{B}, \mu)$ and $\mathrm{Y} = (Y, \mathcal{C}, \nu)$ be probability spaces and $\varphi : X \to Y$ be a measurable map. We say that φ is **measure-preserving** if for every measurable set $C \in \mathcal{C}$ we have

$$\mu(\varphi^{-1}(C)) = \nu(C).$$

If $\varphi : X \to X$ is measure-preserving, we say that the measure μ is **invariant** under φ. Similarly, if we are given a group Γ of measure-preserving transformations of X, we say that μ is invariant under Γ.

We give now some basic examples. First of all, recall the torus rotation from Example 2.2.1.

EXAMPLE 3.1.1. Let $\mathbb{R}/\mathbb{Z}$ be the unit torus written additively, and let $\alpha \in \mathbb{R}/\mathbb{Z}$ be fixed. Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the transformation

$$x \mapsto x + \alpha \quad \text{for all } x \in \mathbb{R}/\mathbb{Z}.$$

Since the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ is translation-invariant, R_α is a measure-preserving map.

EXAMPLE 3.1.2. Let G be a compact abelian group and $g \in G$ be a fixed element. Let $R_g : G \to G$ be the transformation

$$x \mapsto x + g, \quad \text{for all } x \in G.$$

Then, similar to the torus rotation case, R_g is a measure-preserving transformation.

EXAMPLE 3.1.3. Let $(\mathbb{R}/\mathbb{Z})^2$ be a two-dimensional torus and $\alpha \in \mathbb{R}/\mathbb{Z}$ be fixed. Consider the transformation $\varphi : (\mathbb{R}/\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2$ given by

$$(x, y) \mapsto (x + \alpha, x + y) \text{ for all } (x, y) \in (\mathbb{R}/\mathbb{Z})^2.$$

It is easy to see that φ is, in fact, an affine transformation on the compact abelian group $(\mathbb{R}/\mathbb{Z})^2$, and hence it is measure-preserving.

Our last example is the 'Bernoulli shift' transformation.

EXAMPLE 3.1.4. Let $\Lambda = \{1, 2, \ldots, k\}$ be a finite alphabet and let $p = (p_1, p_2, \ldots, p_k)$ be a probability vector. Let

$$\mathrm{X} := \Lambda^{\mathbb{Z}}$$

be the measurable space carrying the Borel structure coming from the product topology. We define a probability measure μ on X via defining it on cylinder sets as follows:

$$\mu(\{\omega : \omega_{i_1} = p_{j_1}, \omega_{i_2} = p_{j_2}, \ldots, \omega_{i_k} = p_{j_k}\}) := \prod_{l=1}^{k} p_{j_l}$$

for all indices $i_1 < i_2 < \cdots < i_k$ and all $k \in \mathbb{N}$. It is easy to see that the shift transformation $\varphi : \mathrm{X} \to \mathrm{X}$ defined by

$$(\varphi\omega)(k) := \omega(k+1) \quad \text{for all } \omega \in \mathrm{X}, k \in \mathbb{Z}$$

is measure-preserving.

3.1.2. Measure Algebras and Maps. Let $\mathrm{X} = (X, \mathcal{B}, \mu)$ and $\mathrm{Y} = (Y, \mathcal{C}, \nu)$ be probability spaces. If $\varphi, \psi : X \to Y$ are measure-preserving maps, we say that φ and ψ are **equivalent** if for μ-a.e. $x \in X$ we have $\varphi(x) = \psi(x)$. We denote the equivalence class of a measure-preserving map $\varphi : X \to Y$ in the set of all measure-preserving maps from X to Y by $[\varphi]$. If $\varphi : X \to Y$ and $\psi : Y \to X$ are measure-preserving maps, we say that ψ is an **essential inverse** of φ if

$$\psi \circ \varphi = \mathrm{id}_X \ \ \mu\text{-a.e.}$$

and

$$\varphi \circ \psi = \mathrm{id}_Y \ \ \nu\text{-a.e.}$$

A measure-preserving map φ is called **essentially invertible** if it admits an essential inverse.

We claim that the 'almost everywhere' equivalence class of a measure-preserving map φ is essentially determined by the map $\varphi^{-1} : \mathcal{C} \to \mathcal{B}$ between the corresponding σ-algebras. To make this precise, we need to introduce the *measure algebras* of the underlying probability spaces. Let $\mathrm{X} := (X, \mathcal{B}, \mu)$ be a probability space. We want to identify those measurable sets in $\mathcal{B}$ which are 'essentially the same'. Let

$$\mathcal{N}(\mathrm{X}) := \{A \in \mathcal{B} : \mu(A) = 0\}$$

be the σ-ideal of null sets of X. The **measure algebra of** X is the pair

$$\Sigma(\mathrm{X}) := (\mathcal{B}/\mathcal{N}(\mathrm{X}), \widetilde{\mu})$$

of the quotient Boolean algebra $\mathcal{B}/\mathcal{N}(\mathrm{X})$ and the function $\widetilde{\mu} : \mathcal{B}/\mathcal{N}(\mathrm{X}) \to [0,1]$, which is induced by the measure μ on $\mathcal{B}$. The quotient Boolean algebra $\mathcal{B}/\mathcal{N}(\mathrm{X})$ is the set of equivalence classes of sets in $\mathcal{B}$ modulo the equivalence relation

$$A \sim B \Leftrightarrow \mu(A \triangle B) = 0.$$

The equivalence class of a set $A \in \mathcal{B}$ in $\mathcal{B}/\mathcal{N}(\mathrm{X})$ is denoted by $[A]$, but we will often ignore this distinction and write A to denote the equivalence class of A in $\mathcal{B}/\mathcal{N}(\mathrm{X})$. The Boolean algebra operations $\vee$, $\wedge$ and $\cdot^c$ on $\mathcal{B}/\mathcal{N}(\mathrm{X})$ are induced by taking union, intersection and complement of sets in $\mathcal{B}$ respectively. Additionally, the Boolean algebra $\mathcal{B}/\mathcal{N}(\mathrm{X})$ is **σ-complete** in the sense that any countable subset of $\mathcal{B}/\mathcal{N}(\mathrm{X})$ has the least upper and the greatest lower bound. More precisely, we have

$$\bigvee_{i \geq 1} [A_i] = \left[\bigcup_{i \geq 1} A_i \right]$$

and

$$\bigwedge_{i\geq 1}[A_i] = \left[\bigcap_{i\geq 1} A_i\right].$$

One can show that the algebra ${}^{\mathcal{B}}/_{\mathcal{N}(\mathrm{X})}$ is complete as well (see [EFHN15, Corollary 7.8] for the proof), but we will not need this fact. The function $\widetilde{\mu} : {}^{\mathcal{B}}/_{\mathcal{N}(\mathrm{X})} \to [0,1]$ is induced by μ, that is

$$\widetilde{\mu}([A]) := \mu(A) \quad \text{ for all } A \in \mathcal{B},$$

and this definition is clearly independent of the particular choice of a representative in $[A]$. It is easy to see that $\widetilde{\mu}$ is σ-additive in the sense that for every sequence $([A_n])_{n\geq 1}$ of elements of ${}^{\mathcal{B}}/_{\mathcal{N}(\mathrm{X})}$ such that $\mu(A_n \cap A_m) = 0$ whenever $n \neq m$ we have

$$\widetilde{\mu}(\bigvee_{n\geq 1}[A_n]) = \sum_{n\geq 1} \widetilde{\mu}([A_n]).$$

For convenience, we will often use $\Sigma(\mathrm{X})$ to denote both the underlying Boolean algebra ${}^{\mathcal{B}}/_{\mathcal{N}(\mathrm{X})}$ and the measure algebra itself. We will also write μ instead of $\widetilde{\mu}$.

Let $\Sigma(\mathrm{X})$ and $\Sigma(\mathrm{Y})$ be the measure algebras of the probability spaces $\mathrm{X} = (X, \mathcal{B}, \mu)$ and $\mathrm{Y} = (Y, \mathcal{C}, \nu)$ respectively. Given a measure-preserving map $\varphi : X \to Y$ of the probability spaces $\mathrm{X} = (X, \mathcal{B}, \mu)$ and $\mathrm{Y} = (Y, \mathcal{C}, \nu)$ respectively, it is easy to see that the map $\varphi^{-1} : \mathcal{C} \to \mathcal{B}$ induces a map $\varphi^* : {}^{\mathcal{C}}/_{\mathcal{N}(\mathrm{Y})} \to {}^{\mathcal{B}}/_{\mathcal{N}(\mathrm{X})}$ of the corresponding measure algebras, satisfying the following conditions for all $A, B \in {}^{\mathcal{C}}/_{\mathcal{N}(\mathrm{Y})}$:

a) $\varphi^*(A \vee B) = \varphi^*(A) \vee \varphi^*(B)$;
b) $\varphi^*(A^c) = (\varphi^*(A))^c$;
c) $\mu(\varphi^*(A)) = \nu(A)$.

For the reasons that will be clear later, we call φ^* a *homomorphism* of measure algebras.

A natural question is if the map φ is completely determined by φ^*. The following lemma gives us one implication. The proof is straightforward.

PROPOSITION 3.1.5. *Let $(X, \mathcal{B}, \mu), (Y, \mathcal{C}, \nu)$ be probability spaces and $\varphi, \psi : X \to Y$ be measure-preserving maps. Let $\varphi^*, \psi^* : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ be the corresponding homomorphisms of measure algebras. If φ and ψ are equivalent, then $\varphi^* = \psi^*$.*

In general, the converse of this lemma does not hold (see [EFHN15, Example 6.7] for a counterexample). However, the converse holds if we restrict ourselves to *standard* probability spaces.

3.2. Standard Probability Spaces

3.2.1. Abstract Measure Algebras. An **abstract measure algebra** is a pair $(\mathcal{M}, \mu)$, where $\mathcal{M}$ is a σ-complete Boolean algebra and $\mu : \mathcal{M} \to [0,1]$ satisfies the following assertions

a) $\mu(1) = 1$;
b) $\mu(a) = 0$ if and only if $a = 0$;
c) μ is σ-additive in the sense that for every sequence $(x_n)_{n\geq 1}$ of elements of $\mathcal{M}$ such that $x_n \wedge x_m = 0$ whenever $n \neq m$ we have

$$\mu(\bigvee_{n\geq 1} x_n) = \sum_{n\geq 1} \mu(x_n).$$

It will be essential later to have a metric space structure on abstract measure algebras.

Proposition 3.2.1. *Let $(\mathcal{M}, \mu)$ be an abstract measure algebra and ρ be the metric on $\mathcal{M}$ defined by*

$$\rho(a, b) := \mu(a \triangle b). \tag{3.2.1}$$

Then $(\mathcal{M}, \rho)$ is a complete metric space.

Proof. We sketch a proof from [GH09]. Verifying that ρ is a metric is trivial, we proceed to proving the completeness. Let $(p_n)_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{M}, \rho)$. Choose a subsequence $(p_{n_k})_{k\geq 1}$ such that for every $k \geq 1$

$$\rho(p_{n_k}, p_{n_{k+1}}) < \frac{1}{2^k}.$$

For every $i \geq 1$, let $r_i := \bigvee_{k=i}^{\infty} p_{n_k}$. Define

$$p := \bigwedge_{i\geq 1} r_i.$$

Then the subsequence $(p_{n_k})_{k\geq 1}$ converges to p, and hence so does $(p_n)_{n\geq 1}$. □

A measure algebra $(\mathcal{M}, \mu)$ is called **separable** if the corresponding metric space $(\mathcal{M}, \rho)$ is separable. Classical examples of separable abstract measure algebras are given by the measure algebras of 'nice' probability spaces.

Example 3.2.2. Let X be a Polish space, $\mathcal{B}$ be the Borel σ-algebra and μ be a probability measure on X. Then the measure algebra $\Sigma(\mathrm{X})$ of the probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$ is an abstract measure algebra, and we only need to prove that it is separable. Given a countable basis $(U_n)_{n\geq 1}$ of the topology on X, we take all finite intersections of sets in $(U_n)_{n\geq 1}$ and obtain a countable $\cap$-stable system $(V_n)_{n\geq 1}$. We have $\sigma((V_n)_{n\geq 1}) = \mathcal{B}$, and hence (see e.g. [EFHN15, Lemma B.15]) $([V_n])_{n\geq 1}$ is dense in the measure algebra $\Sigma(\mathrm{X})$.

We want to follow the 'categorical view', hence we need to define the arrows in our category of measure algebras. A **morphism** $\Phi : (\mathcal{M}, \mu) \to (\mathcal{N}, \nu)$

of abstract measure algebras is a map $\Phi : \mathcal{M} \to \mathcal{N}$ satisfying the following assertions for all $a, b \in \mathcal{M}$:

a) $\Phi(a \vee b) = \Phi(a) \vee \Phi(b)$;
b) $\Phi(a^c) = (\Phi(a))^c$;
c) $\mu(\Phi(a)) = \nu(a)$.

It is easy to show that every morphism of abstract measure algebras is necessarily injective (as the map of the Boolean algebras). Furthermore, it is an isometry and a σ-homomorphism (i.e. it respects countable meets of elements). We denote the set of all morphisms between abstract measure algebras $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ by $\mathrm{Hom}((\mathcal{M}, \mu), (\mathcal{N}, \nu))$. The category of separable abstract measure algebras, with the corresponding sets of morphisms defined above, will be denoted by SMAlg. Given a group Γ, SMAlg_Γ will, as usual, denote the category of representations of Γ on SMAlg with Γ-intertwining algebra homomorphisms as morphisms.

Example 3.2.3. Let $(\mathcal{M}, \mu)$ be an abstract measure algebra. Let $\mathcal{N} \subseteq \mathcal{M}$ be a σ-complete Boolean subalgebra. Then $(\mathcal{N}, \mu)$ is an abstract measure algebra and the identity map $\imath : \mathcal{N} \to \mathcal{M}$ is a morphism of abstract measure algebras.

Having the notion of a morphism of abstract measure algebras, we can talk about **isomorphism** of measure algebras. The following theorem (see [Fur81, Proposition 5.1] for a slightly different formulation of this result) simplifies checking if two abstract measure algebras are isomorphic.

Proposition 3.2.4. *Two measure algebras $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ are isomorphic if and only if $\mathcal{M}$ contains a dense Boolean subalgebra $\mathcal{M}_0$ and $\mathcal{N}$ contains a dense Boolean subalgebra $\mathcal{N}_0$ such that there is a bijection $\Phi : \mathcal{M}_0 \to \mathcal{N}_0$ satisfying the following assertions for all $a, b \in \mathcal{M}_0$:*

a) $\Phi(a \vee b) = \Phi(a) \vee \Phi(b)$*;*
b) $\Phi(a^c) = (\Phi(a))^c$*;*
c) $\mu(\Phi(a)) = \nu(a)$*.*

Furthermore, if such a bijection $\Phi : \mathcal{M}_0 \to \mathcal{N}_0$ exists, then it extends to an isomorphism of the measure algebras uniquely.

Proof. One implication is trivial. Conversely, suppose that $\mathcal{M}_0 \subseteq \mathcal{M}, \mathcal{N}_0 \subseteq \mathcal{N}$ and $\Phi : \mathcal{M}_0 \to \mathcal{N}_0$ satisfy the assertions of the theorem. Since $\mathcal{M}_0$ is dense in $\mathcal{M}$, for every $a \in \mathcal{M}$ there is a sequence $(a_n)_{n\geq 1}$ in $\mathcal{M}_0$ converging to a, i.e.

$$\lim_{n\to\infty} \mu(a_n \triangle a) = 0.$$

The sequence $(a_n)_{n\geq 1}$ is Cauchy and Φ is an isometry, hence the sequence $(\Phi(a_n))_{n\geq 1}$ is Cauchy as well. We let

$$\Phi(a) := \lim_{n\geq 1} \Phi(a_n),$$

and it is easy to see that this defines an extension of Φ to an isomorphism of measure algebras $\mathcal{M}$ and $\mathcal{N}$. □

We have seen that the measure algebras of 'nice' probability spaces are separable. The converse is true as well.

PROPOSITION 3.2.5 (Realization). *Let $(\mathcal{M},\mu)$ be a separable measure algebra. Then $(\mathcal{M},\mu)$ is isomorphic to the measure algebra $\Sigma(\mathrm{X}) = (\mathcal{B}/\mathcal{N}(\mathrm{X}),\nu)$ of some probability space $\mathrm{X} = (X,\mathcal{B},\nu)$, where X is a compact metric space and $\mathcal{B}$ is the Borel σ-algebra.*

PROOF. Let $(a_n)_{n\geq 1}$ be a dense subalgebra in $\mathcal{M}$ such that $a_n \neq a_m$ for $n \neq m$ (if the measure algebra is finite the statement of the theorem is trivial). Let $X := \{0,1\}^{\mathbb{N}}$, endowed with the product topology, and $\mathcal{B}$ be the Borel σ-algebra. For every $n \geq 1$, define the cylinder set

$$B_n := \{\omega \in X : \omega(n) = 1\}.$$

There exists a unique Borel probability measure ν on X such that, for every $N \geq 1$,

$$\nu(B_{n_1} \cap B_{n_2} \cap \cdots \cap B_{n_k}) := \mu(a_{n_1} \wedge a_{n_2} \wedge \cdots \wedge a_{n_k})$$

for all indices $n_1,\dots,n_k$ such that $1 \leq n_1 < n_2 < \cdots < n_k \leq N$. Let

$$\Sigma(\mathrm{X}) := (\mathcal{B}/\mathcal{N}(\mathrm{X}),\nu)$$

be the measure algebra of X. Let $\mathcal{M}_0 := (a_n)_{n\geq 1}$ and $\mathcal{B}_0 \subseteq \Sigma(\mathrm{X})$ be the countable algebra generated by $([B_n])_{n\geq 1}$. It is clear that the map

$$a_n \mapsto [B_n], \quad \forall n \geq 1$$

is a bijection $\Phi_0 : \mathcal{M}_0 \to \mathcal{B}_0$ of Boolean algebras, satisfying the requirements of Proposition 3.2.4. Then Φ_0 extends to an isomorphism Φ of measure algebras. □

3.2.2. Category of Standard Probability Spaces. In the previous section we have seen that the measure algebras of probability spaces on Polish spaces are separable, and, conversely, that each separable abstract measure algebra is in fact isomorphic to the measure algebra of a 'nice' probability space. However, not all probability spaces with separable measure algebras are 'nice' enough for our purposes, and we refer once again to [EFHN15, Example 6.7] for a counterexample. The correct definition of a 'nice' probability space is that of a *standard probability space.*

A measurable space $(X,\mathcal{B})$ is called a **standard Borel space** if there is a Polish topology $\mathcal{O}$ on X such that $\mathcal{B} = \sigma(\mathcal{O})$. A probability space $(X,\mathcal{B},\mu)$ is called a **Borel probability space** if $(X,\mathcal{B})$ is a standard Borel space. A probability space $\mathrm{X} = (X,\mathcal{B},\mu)$ is called a **standard probability space** if there is a Borel probability space $\mathrm{Y} = (Y,\mathcal{C},\nu)$ and an essentially invertible measure-preserving map $\varphi : X \to Y$. It follows immediately that the measure algebra of a standard probability space is separable.

First, we state the following proposition [EFHN15, Proposition 6.10], which gives the converse to Proposition 3.1.5.

PROPOSITION 3.2.6. *Let* $(X, \mathcal{B}, \mu), (Y, \mathcal{C}, \nu)$ *be standard probability spaces and* $\varphi, \psi : X \to Y$ *be measure-preserving maps. If* $\varphi^* = \psi^*$*, then* φ *and* ψ *are equivalent.*

Combining [EFHN15, Theorem 12.10] and [EFHN15, Theorem F.9], we deduce the following theorem. It tells us that, given standard probability spaces X and Y, each morphism $\Phi : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ of the corresponding abstract measure algebras is in fact of the form φ^* for some almost uniquely determined measure-preserving map φ.

THEOREM 3.2.7 (Von Neumann). *Let* $\mathrm{X} = (X, \mathcal{B}, \mu)$ *and* $\mathrm{Y} = (Y, \mathcal{C}, \nu)$ *be standard probability spaces. Let* $\Phi : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ *be a morphism of measure algebras. Then there is a* μ*-a.e. unique measure-preserving map* $\varphi : X \to Y$ *such that* $\varphi^* = \Phi$*.*

We can now define the category Prob of standard probability spaces. We let $\mathrm{Obj}(\mathsf{Prob})$ be the collection of standard probability spaces. Given standard probability spaces X and Y in $\mathrm{Obj}(\mathsf{Prob})$, we let

$$\mathrm{Hom}(\mathrm{X}, \mathrm{Y}) := \{[\varphi] : \varphi : X \to Y \text{ a measure-preserving map}\}.$$

It is easy to see that this indeed defines a set of morphisms. In fact, this is an instance of a *quotient category*, and we leave the details to the reader. Combining Proposition 3.1.5, Proposition 3.2.6 and Theorem 3.2.7 we see that, for all $\mathrm{X}, \mathrm{Y} \in \mathrm{Obj}(\mathsf{Prob})$,

$$\mathrm{Hom}(\mathrm{X}, \mathrm{Y}) = \{\Phi^{\mathrm{op}} : \Phi \in \mathrm{Hom}(\Sigma(\mathrm{Y}), \Sigma(\mathrm{X}))\}. \tag{3.2.2}$$

Here the superscript op in Φ^{op} means that, even though Φ is a morphism from $\Sigma(\mathrm{Y})$ to $\Sigma(\mathrm{X})$, the direction of Φ as a morphism between standard probability spaces is the opposite. Therefore, the equality in Equation (3.2.2) should be understood as follows: each equivalence class of measure-preserving maps from X to Y determines a unique morphism of measure algebras in $\mathrm{Hom}(\Sigma(\mathrm{Y}), \Sigma(\mathrm{X}))$, and, conversely, each morphism of measure algebras in $\mathrm{Hom}(\Sigma(\mathrm{Y}), \Sigma(\mathrm{X}))$ determines a unique equivalence class of measure-preserving maps from X to Y.

We will typically ignore the distinction between a measure-preserving map $\varphi : X \to Y$ and its equivalence class $[\varphi] \in \mathrm{Hom}(\mathrm{X}, \mathrm{Y})$. We will also sometimes write φ^{-1} (instead of φ^*) to denote the corresponding morphism of measure algebras to comply with the standard notation in ergodic theory.

3.3. Measure-preserving Dynamical Systems

3.3.1. Definition. Let Γ be a discrete group, $\mathrm{X} \in \mathsf{Prob}$ be a standard probability space and $\pi : \Gamma \to \mathrm{Aut}(\mathrm{X})$ be a group homomorphism. The pair

$$\mathbf{X} = (\mathrm{X}, \pi)$$

is called a **measure-preserving dynamical system**. When we want to stress that the acting group is Γ, we will sometimes say that this is a **measure-preserving Γ-system**. When we work with a measure-preserving Γ-system $\mathbf{X} = (\mathrm{X}, \pi)$ and the representation π is fixed, we will often write γ instead of

π_γ to denote the morphism $\pi_\gamma : \mathrm{X} \to \mathrm{X}$ for $\gamma \in \Gamma$. We will write γ^{-1} to denote the corresponding automorphism of the measure algebra $\Sigma(\mathrm{X})$. This coincides with the standard notation in dynamical systems, and typically does not cause any confusion. The representation of Γ^{op} in $\mathrm{Aut}(\Sigma(\mathrm{X}))$ induced by π will be denoted by π^*, thus

$$(\Sigma(\mathrm{X}), \pi^*) \in \mathsf{SMAlg}_{\Gamma^{\mathrm{op}}}.$$

The objects of the category Prob_Γ are, by definition, pairs (X, π), with X being a standard probability space and $\pi : \Gamma \to \mathrm{Aut}(\mathrm{X})$ being a group homomorphism. Let $\mathbf{X} = (\mathrm{X}, \pi), \mathbf{Y} = (\mathrm{Y}, \rho)$ be measure-preserving dynamical systems. We define $\mathrm{Hom}(\mathbf{X}, \mathbf{Y})$ as the set of all morphisms $\phi : \mathrm{X} \to \mathrm{Y}$ such that $\rho_\gamma \circ \phi = \phi \circ \pi_\gamma$ for all γ. That is, we require that the diagram

$$\begin{tikzcd} \mathrm{X} \arrow[r, "\phi"] & \mathrm{Y} \\ \mathrm{X} \arrow[u, "\pi_\gamma"] \arrow[r, "\phi"'] & \mathrm{Y} \arrow[u, "\rho_\gamma"'] \end{tikzcd}$$

commutes for all γ. Then $\mathbf{Y}$ is called a **factor** of $\mathbf{X}$, and ϕ is called a **factor map**.

Similar to topological dynamics, many interesting measure-preserving systems are $\mathbb{Z}$-systems. In this case we adapt the notation and write $(\mathrm{X}; \varphi)$ instead of (X, π), where $\varphi := \pi_1$ is an essentially invertible measure-preserving map. For instance, Examples 3.1.1 and 3.1.3 of measure-preserving transformations from Section 3.1.1 give us measure-preserving dynamical systems, which we call the **torus rotation** and the **skew-shift** respectively.

3.3.2. Continuous Models. The morphisms in the category of measure-preserving Γ-systems are, by definition, equivalence classes of measure-preserving maps (or, equivalently, 'opposites' of the morphisms of the corresponding measure algebras). It is sometimes more convenient to work with 'continuous models' of measure-preserving dynamical systems on topological spaces, where Γ acts by measure-preserving homeomorphisms and the morphisms that we are interested in are continuous, intertwining, measure-preserving maps. Fortunately, since all underlying probability spaces are standard, such models do always exist. Combining [Gla03, Theorem 2.15] and the results above, we deduce the following theorem.

THEOREM 3.3.1. *Let Γ be a discrete at most countable group. Let $\mathbf{X}, \mathbf{Y} \in \mathsf{Prob}_\Gamma$ be measure-preserving Γ-systems on standard probability spaces $\mathrm{X} = (X, \mathcal{B}, \mu)$ and $\mathrm{Y} = (Y, \mathcal{C}, \nu)$ respectively. Let $\varphi : \mathbf{X} \to \mathbf{Y}$ be a morphism. Then there exist*

1) *measure-preserving Γ-systems $\mathbf{X}'$ and $\mathbf{Y}'$ on compact metric spaces X' and Y', endowed with Borel probability measures μ' and ν' respectively, where Γ acts on X' and Y' by homeomorphisms;*
2) *a continuous and surjective factor map $\varphi' : \mathbf{X}' \to \mathbf{Y}'$;*

3) *isomorphisms* $\psi_1 : \mathbf{X} \to \mathbf{X}'$ *and* $\psi_2 : \mathbf{Y} \to \mathbf{Y}'$

which make the diagram

$$\begin{tikzcd} \mathbf{X}' \arrow[d, "\varphi'"'] & \mathbf{X} \arrow[l, "\psi_1"'] \arrow[d, "\varphi"] \\ \mathbf{Y}' & \mathbf{Y} \arrow[l, "\psi_2"] \end{tikzcd}$$

in the category Prob_Γ *commute.*

Proof. For the complete proof we refer to [Gla03, Theorem 2.15] and [Fur81, Theorem 5.15]. The main idea is to model $\mathbf{X}$ and $\mathbf{Y}$ on compact metric spaces via Proposition 3.2.5, but with a special choice of the dense sets in the corresponding measure algebras. With this special choice, the action of Γ^{op} on the measure algebras of X and Y induces an action of Γ on X' and Y' by measure-preserving homeomorphisms. The continuous factor map φ' is induced by the morphism $\varphi^* : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ of measure algebras, it is intertwining w.r.t. the action of Γ on X' and Y' and measure-preserving. Finally, Theorem 3.2.7 tells us that there are measure-preserving maps ψ_1, ψ_2, induced by the corresponding isomorphisms of measure algebras. □

A similar statement can be proved for a countable family of factors of $\mathbf{X}$. Furthermore, given $\mathbf{X} \in \mathsf{Prob}_\Gamma$ we will abuse the notation slightly and write $\pi_\gamma \cdot x$ or even $\gamma \cdot x$ for $x \in \mathrm{X}$ and $\gamma \in \Gamma$ to denote the action of Γ on X as if $\mathbf{X}$ was a continuous model already. Since all essential statements in ergodic theory are 'a.e.' statements and Theorem 3.3.1 gives an a.e. isomorphisms, this does not typically cause any confusion.

3.3.3. Ergodicity. Let $\mathbf{X} = (\mathrm{X}, \pi)$ be a measure-preserving Γ-system on a probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$. We say that $\mathbf{X}$ is **ergodic** (or that the measure μ on X is ergodic) if, for all $A \in \Sigma(\mathrm{X})$,

$$\gamma^{-1}A = A \quad \text{for all } \gamma \in \Gamma$$

implies that $A = 0$ or $A = 1$. That is, $\mathbf{X}$ is ergodic if only the trivial sets are essentially invariant under Γ.

We want to state a few equivalent definitions of ergodicity. A function $f \in \mathrm{L}^2(\mathrm{X})$ is called Γ-invariant if, for all $\gamma \in \Gamma$,

$$f \circ \pi_\gamma = f$$

as elements of $\mathrm{L}^2(\mathrm{X})$. We let

$$\operatorname{fix} \pi := \{f \in \mathrm{L}^2(\mathrm{X}) : f \text{ is } \Gamma\text{-invariant}\}.$$

It is clear $\operatorname{fix}\pi \subseteq \mathrm{L}^2(\mathrm{X})$ is a closed, Γ-invariant subspace containing the subspace of constant functions. We let $\mathbb{P} : \mathrm{L}^2(\mathrm{X}) \to \operatorname{fix}\pi$ be the orthogonal projection onto $\operatorname{fix}\pi$.

Proposition 3.3.2. *Let* $\mathbf{X}$ *be a measure-preserving* Γ*-system on a probability space* $\mathrm{X} = (X, \mathcal{B}, \mu)$*. The following assertions are equivalent:*

(i) $\mathbf{X}$ *is ergodic;*

(ii) $\dim \operatorname{fix} \pi = 1$*;*

(iii) *for every* $f \in \mathrm{L}^2(\mathrm{X})$,

$$\mathbb{P}f = \int f d\mu.$$

For the proof we refer the reader to [Gla03, Theorem 3.10] and to [EFHN15, Theorem 8.10].

3.3.4. Category of Factors. First of all, we show that every invariant, σ-complete subalgebra of the measure algebra of a system $\mathbf{X} \in \mathsf{Prob}_\Gamma$ determines a factor of $\mathbf{X}$. This statement is the 'measure algebra translation' of the standard definition of factors as invariant sub-σ-algebras.

PROPOSITION 3.3.3. *Let* Γ *be a discrete at most countable group. Let* $\mathbf{X} = (\mathrm{X}, \pi)$ *be a measure-preserving* Γ*-system on a standard probability space* $\mathrm{X} = (X, \mathcal{B}, \mu)$*. Let* $\mathcal{M} \subseteq \Sigma(\mathrm{X})$ *be a* σ*-complete,* Γ*-invariant Boolean subalgebra of* $\Sigma(\mathrm{X})$*. Then there is a factor* $\varphi : \mathbf{X} \to \mathbf{Y}$*, where* $\mathbf{Y} = (\mathrm{Y}, \rho)$ *is a measure-preserving system on* $\mathrm{Y} \in \mathsf{Prob}$*, and an isomorphism* $\Phi : (\mathcal{M}, \pi^*) \to (\Sigma(\mathrm{Y}), \rho^*)$*, which make the diagram*

$$\begin{tikzcd}
(\Sigma(\mathrm{X}), \pi^*) & \\
(\mathcal{M}, \pi^*) \arrow[u, "\imath"] \arrow[r, "\Phi"'] & (\Sigma(\mathrm{Y}), \rho^*) \arrow[ul, "\varphi^*"']
\end{tikzcd}$$

in the category $\mathsf{SMAlg}_{\Gamma^{\mathrm{op}}}$ *commute.*

PROOF. The proof follows from [Gla03, Theorem 2.15] and the fact that morphisms in the category of standard probability spaces are the opposites of morphisms of the underlying measure algebras. □

Now, let Prob_Γ be the category of measure-preserving Γ-systems. It is easy to see that all morphisms in this category are epimorphisms. Let $\mathbf{X} = (\mathrm{X}, \pi) \in \mathsf{Prob}_\Gamma$ be a fixed measure-preserving dynamical systems, and let $\mathsf{Fac}(\mathbf{X})$ be the associated category of factors. We want to prove that $\mathsf{Fac}(\mathbf{X})$ can be identified with the set of all Γ-invariant σ-complete Boolean subalgebras of $\Sigma(\mathrm{X})$. We refer to Section 2.3 for the abstract definition of the category $\mathsf{Fac}(\mathbf{X})$.

Let $\phi : \mathbf{X} \to \mathbf{Y}, \psi : \mathbf{X} \to \mathbf{Z}$ are equivalent factors, where 'equivalence' is understood in the sense of Section 2.3. Then the corresponding subalgebras $\varphi^*(\Sigma(\mathrm{Y})), \psi^*(\Sigma(\mathrm{Z})) \subseteq \Sigma(\mathrm{X})$ coincide. Conversely, if $\phi : \mathbf{X} \to \mathbf{Y}, \psi : \mathbf{X} \to \mathbf{Z}$ are factors such that the corresponding subalgebras of $\Sigma(\mathrm{X})$ coincide, then ϕ and ψ are equivalent. Finally, given a Γ-invariant σ-complete subalgebra $\mathcal{V}$ of $\Sigma(\mathrm{X})$, we use Proposition 3.3.3 to obtain a factor.

To understand the coproducts in $\mathsf{Fac}(\mathbf{X})$, we begin by describing a coproduct of two factors first. So let $\varphi : \mathbf{A} \to \mathbf{X}$ and $\psi : \mathbf{A} \to \mathbf{Y}$ be representatives of $[\varphi], [\psi] \in \mathsf{Fac}(\mathbf{A})$ with corresponding measure subalgebras $\varphi^*\Sigma(\mathrm{X})$ and $\psi^*\Sigma(\mathrm{Y})$ of the measure algebra $\Sigma(\mathrm{A})$ of $\mathbf{A}$. Consider the factor $\mathbf{Z}$ of $\mathbf{A}$ obtained by

1) intersecting the measure subalgebras $\varphi^*\Sigma(\mathrm{X})$ and $\psi^*\Sigma(\mathrm{Y})$, obtaining a measure subalgebra of $\Sigma(\mathrm{A})$;
2) using Proposition 3.3.3 to get a factor map $\chi : \mathbf{A} \to \mathbf{Z}$ with corresponding measure algebra $(\varphi^*\Sigma(\mathrm{X})) \cap (\psi^*\Sigma(\mathrm{Y}))$.

Then $\mathbf{Z} \in \mathsf{Prob}$ and $[\chi] \in \mathsf{Fac}(\mathbf{A})$ is the coproduct of $[\psi]$ and $[\varphi]$. A similar construction applies to arbitrary infinite coproducts (not necessarily just countable ones!).

3.4. Ergodic Theorems

One of the reasons why Følner sequences are of interest in this work is that they are 'good' for averaging group actions. We denote the averages by

$$\mathbb{E}_{g\in F} := \frac{1}{|F|}\sum_{g\in F}.$$

The simplest ergodic theorem for amenable group actions is the mean ergodic theorem.

THEOREM 3.4.1 (Mean ergodic theorem). *Let $\mathbf{X} = (\mathrm{X}, \pi)$ be a measure-preserving Γ-system, where the group Γ is amenable and $(F_n)_{n\geq 1}$ is a left Følner sequence. Then for every $f \in \mathrm{L}^2(\mathrm{X})$ we have*

$$\lim_{n\to\infty} \mathbb{E}_{g\in F_n} f \circ \pi_{\gamma^{-1}} = \mathbb{P}f,$$

where the convergence is understood in $\mathrm{L}^2(\mathrm{X})$-sense. If the system $\mathbf{X}$ is ergodic, then

$$\lim_{n\to\infty} \mathbb{E}_{g\in F_n} f \circ \pi_{\gamma} = \int f d\mu.$$

We refer for the proof to [Gla03, Theorem 3.33] However, it is not good enough for our purposes, because we will need *pointwise* convergence of the ergodic averages. Unlike the mean ergodic theorem, it is known that not every Følner sequence is good for the pointwise convergence of ergodic averages. The following important theorem was proved by E. Lindenstrauss in [Lin01].

THEOREM 3.4.2. *Let $\mathbf{X} = (\mathrm{X}, \pi)$ be a measure-preserving Γ-system, where the group Γ is amenable and $(F_n)_{n\geq 1}$ is a tempered left Følner sequence. Then for every $f \in \mathrm{L}^1(\mathrm{X})$ there is a Γ-invariant $\overline{f} \in \mathrm{L}^1(\mathrm{X})$ such that*

$$\lim_{n\to\infty} \mathbb{E}_{g\in F_n} f(g\cdot\omega) = \overline{f}(\omega)$$

for μ-a.e. $\omega \in \mathrm{X}$. If the system $\mathbf{X}$ is ergodic, then

$$\lim_{n\to\infty} \mathbb{E}_{g\in F_n} f(g\cdot\omega) = \int f d\mu$$

for μ-a.e. $\omega \in \mathrm{X}$.

We will need a weighted variant of this result. A function c on Γ is called a **good weight** for pointwise convergence of ergodic averages along a tempered left Følner sequence $(F_n)_{n\geq 1}$ in Γ if for every measure-preserving system $\mathbf{X} = (\mathrm{X}, \pi)$ and every $f \in \mathrm{L}^\infty(\mathrm{X})$ the averages

$$\mathbb{E}_{g\in F_n} c(g) f(g\cdot\omega)$$

converge as $n\to\infty$ for μ-a.e. $\omega\in\mathrm{X}$.

We will use a special case of the Theorem 1.3 from [ZK14].

THEOREM 3.4.3. *Let Γ be a group with a tempered Følner sequence $(F_n)_{n\geq 1}$. Then for every ergodic measure-preserving system $\mathbf{X} = (\mathrm{X}, \pi)$ and every $f \in \mathrm{L}^\infty(\mathrm{X})$ there exists a full measure subset $\widetilde{X} \subseteq X$ such that for every $x \in \widetilde{X}$ the map $g \mapsto f(g\cdot x)$ is a good weight for the pointwise ergodic theorem along $(F_n)_{n\geq 1}$.*

3.5. Remarks

3.5.1. Krylov-Bogolyubov Theorem. Let $\mathbf{X} = (\mathrm{X}, \pi)$ be a topological Γ-system on a compact metric space X, where the group Γ is discrete amenable. Let $\mathcal{B}$ be the Borel algebra. Then the set $\mathrm{M}^1(\mathrm{X})$ of Borel probability measures on X is a compact convex subset of the dual $\mathrm{C}(\mathrm{X})'$, endowed with weak-* topology. The action of Γ on X induces a left affine action of Γ on $\mathrm{M}^1(\mathrm{X})$ by

$$(\gamma\cdot\mu)(A) := \mu(\pi_\gamma^{-1}A) \quad \text{for all } A\in\mathcal{B}, \gamma\in\Gamma.$$

It follows from Proposition 1.5.1 that there is a fixed point $\mu\in\mathrm{M}^1(\mathrm{X})$, i.e. there exists an invariant measure. Furthermore, one can show [Gla03, Theorem 4.2] that the set of extreme points of $\mathrm{M}^1(\mathrm{X})$ is precisely the set of ergodic measures on X. Since the set of extreme points of $\mathrm{M}^1(\mathrm{X})$ is nonempty due to the Krein-Milman theorem, we deduce that each topological dynamical system can be endowed with an ergodic measure.

CHAPTER 4

Computability and Kolmogorov Complexity

The goal of this chapter is to provide the preliminaries on computability and complexity which will become essential in Part IV of the thesis.

We define computable functions and computable sets in Section 4.1. We take an 'informal' approach, calling a function computable if there is an algorithm that takes an argument as the input and produces the value of the function as the output. There are various ways to make this precise, for instance, via recursive functions or via Turing machines. We will not discuss these details here, since an intuitive understanding of an 'algorithm' would suffice. However, it is already apparent from this definition that not every function $f : \mathbb{N} \to \mathbb{N}$ is computable, because there are only countably many algorithms.

Very often we work with countable 'structures' that admit operations which are computable in certain sense. To formalize this idea, we introduce the notion of a computable space in Section 4.2. We define the category $\mathsf{CompSpc}$ of computable spaces by introducing appropriate morphisms between computable spaces. Once the notion of computability is available, we can proceed and define computable groups in Section 4.3, which are the groups where the multiplication operation is computable. Of course, classical groups such as the groups $\mathbb{Z}^d$ and the matrix groups (say, with rational entries) are computable when endowed with a certain natural indexing. Taking the notion of a Følner monotiling from Section 1.3, we introduce its computable version in Section 4.4 and provide some examples. We will see that the discrete Heisenberg group H_3 admits a computable Følner monotiling, and, in general, that every group $\mathsf{UT}_d(\mathbb{Z})$ of upper-triangular matrices of dimension $d \geq 2$ with integer entries does as well. As we will show later, these computable monotilings enjoy particularly nice regularity properties, hence the main theorems of the thesis from Chapter 7 hold for these groups.

At the end of the chapter we define plain Kolmogorov complexity and Kolmogorov complexity on word presheaves in Section 4.6 and 4.7 respectively. The first of these notions is classical and dates back to the work of Kolmogorov, while the second is suggested by the author in order to generalize the theorems of Brudno on entropy and complexity.

4.1. Computable Functions and Computable Sets

In this section we will discuss the standard notions of computability that are used in this work. We refer to Chapter 7 in [Hed04] for details, more definitions and proofs.

For a natural number k a k-ary **partial function** is any function of the form $f : D \to \mathbb{N} \cup \{0\}$, where D, the **domain of definition**, is a subset of $(\mathbb{N} \cup \{0\})^k$ for some natural k. A k-ary partial function is called **computable** if there exists an algorithm which takes a k-tuple of nonnegative integers $(a_1, a_2, \ldots, a_k)$, prints $f((a_1, a_2, \ldots, a_k))$ and terminates if $(a_1, a_2, \ldots, a_k)$ is in the domain of f, while yielding no output otherwise. A function is called **total**, if it is defined everywhere.

The term *algorithm* above stands, informally speaking, for a computer program. One way to formalize it is through introducing the class of *recursive functions*, and the resulting notion coincides with the class of functions computable on *Turing machines*. We do not focus on these questions in this work, and we will think about computability in an 'informal' way.

A set $A \subseteq \mathbb{N}$ is called **recursive** (or **computable**) if the indicator function $\mathbf{1}_A$ of A is computable. It is easy to see that finite and co-finite subsets of $\mathbb{N}$ are computable. Furthermore, for computable sets $A, B \subseteq \mathbb{N}$ their union and intersection are also computable. If a total function $f : \mathbb{N} \to \mathbb{N}$ is computable and $A \subseteq \mathbb{N}$ is a computable set, then $f^{-1}(A)$, the full preimage of A, is computable. The image of a computable set via a total computable bijection is computable, and the inverse of such a bijection is a computable function.

A sequence of subsets $(F_n)_{n\geq 1}$ of $\mathbb{N}$ is called **computable** if the total function $\mathbf{1}_{F_\cdot} : (n, x) \mapsto \mathbf{1}_{F_n}(x)$ is computable. It is easy to see that a total function $f : \mathbb{N} \to \mathbb{N}$ is computable if and only if the sequence of singletons $(\{f(n)\})_{n\geq 1}$ is computable in the sense above.

It is very often important to have a numeration of elements of a set by natural numbers. A set $A \subseteq \mathbb{N}$ is called **enumerable** if there exist a total computable surjective function $f : \mathbb{N} \to A$. If the set A is infinite, we can also require f to be injective. This leads to an equivalent definition because an algorithm computing the function f can be modified so that no repetitions occur in its output. Finite and cofinite sets are enumerable. It can be shown (Proposition 7.44 in [Hed04]) that a set A is computable if and only if both A and $\mathbb{N} \setminus A$ are enumerable. Furthermore, for a set $A \subsetneq \mathbb{N}$ the following are equivalent:

(i) A is enumerable;

(ii) A is the domain of definition of a partial recursive function.

4.2. Computable Spaces and Word Presheaves

The goal of this section is to introduce the notions of *computable space*, *computable function* between computable spaces and *word presheaf* over computable spaces. The complexity of sections of word presheaves and asymptotic complexity of sections of word presheaves are introduced in this section as well.

An **indexing** of a set X is an injective mapping $\imath : X \to \mathbb{N}$ such that $\imath(X)$ is a computable subset. Given an element $x \in X$, we call $\imath(x)$ the **index** of x. If $i \in \imath(X)$, we denote by x_i the element of X having index i. A **computable space** is a pair $(X, \imath)$ of a set X and an indexing $\imath$. Preimages of computable subsets of $\mathbb{N}$ under $\imath$ are called **computable subsets** of $(X, \imath)$. Each computable subset $Y \subseteq X$ can be seen as a computable space $(Y, \imath|_Y)$, where $\imath|_Y$ is the restriction of the indexing function. Of course, the set $\mathbb{N}$ with identity as an indexing function is a computable space, and the computable subsets of $(\mathbb{N}, \mathrm{id})$ are precisely the computable sets of $\mathbb{N}$ in the sense of Section 4.1.

Let $(X_1, \imath_1), (X_2, \imath_2), \ldots, (X_k, \imath_k), (Y, \imath)$ be computable spaces. A (total) function $f : X_1 \times X_2 \times \cdots \times X_k \to Y$ is called **computable** if the function $\widetilde{f} : \imath_1(X_1) \times \imath_2(X_2) \times \cdots \times \imath_k(X_k) \to \imath(Y)$ determined by the condition

$$\widetilde{f}(\imath_1(x_1), \imath_2(x_2), \ldots, \imath_k(x_k)) = \imath(f(x_1, x_2, \ldots, x_k))$$

for all $(x_1, x_2, \ldots, x_k) \in X_1 \times X_2 \times \cdots \times X_k$ is computable. This definition extends the standard definition of computability from Section 4.1 when the computable spaces under consideration are $(\mathbb{N}, \mathrm{id})$. A computable function $f : (X, \imath_1) \to (Y, \imath_2)$ is called a **morphism** between computable spaces. This yields the definition of the **category of computable spaces**. Let $(X, \imath_1)$, $(X, \imath_2)$ be computable spaces. The indexing functions $\imath_1$ and $\imath_2$ of X are called **equivalent** if $\mathrm{id} : (X, \imath_1) \to (X, \imath_2)$ is an isomorphism (i.e., a bijective morphism). It is clear that the classes of computable functions and computable sets do not change if we pass to equivalent indexing functions.

Given a computable space $(X, \imath)$, we call a sequence of subsets $(F_n)_{n\geq 1}$ of X **computable** if the function $\mathbf{1}_{F_\cdot} : \mathbb{N} \times X \to \{0, 1\}, (n, x) \mapsto \mathbf{1}_{F_n}(x)$ is computable. We will also need a special notion of computability for sequences of *finite* subsets of $(X, \imath)$. A sequence of finite subsets $(F_n)_{n\geq 1}$ of X is called **canonically computable** if there is an algorithm that, given n, prints the set $\imath(F_n)$ and halts. One way to make this more precise is by introducing the canonical index of a finite set. Given a finite set $A = \{x_1, x_2, \ldots, x_k\} \subset \mathbb{N}$, we call the number $\mathrm{I}(A) := \sum_{i=1}^{k} 2^{x_i}$ the **canonical index** of A. Hence a sequence of finite subsets $(F_n)_{n\geq 1}$ of X is canonically computable if and only if the total function $n \mapsto \mathrm{I}(\imath(F_n))$ is computable. It is easy to see that the following assertions are equivalent:

(i) The sequence of finite sets $(F_n)_{n\geq 1}$ is canonically computable;

(ii) The sequence of finite sets $(F_n)_{n\geq 1}$ is computable and the total function

$$n \mapsto \max\{m : m \in \imath(F_n)\}$$

is computable.

(iii) The sequence of finite sets $(F_n)_{n\geq 1}$ is computable and the total function

$$n \mapsto |F_n|$$

is computable.

Of course, a canonically computable sequence of finite sets is computable, but the converse is not true due to the fact that there is no effective way of determining how large a finite set with a given computable indicator function is. It is easy to see that the class of canonically computable sequences of finite sets does not change if we pass to an equivalent indexing. The proof of the following proposition is straightforward:

Proposition 4.2.1. *Let $(X, \imath)$ be a computable space. Then*

a) *If $(F_n)_{n\geq 1}, (G_n)_{n\geq 1}$ are computable (resp. canonically computable) sequences of sets, then the sequences of sets $(F_n \cup G_n)_{n\geq 1}$, $(F_n \cap G_n)_{n\geq 1}$ and $(F_n \setminus G_n)_{n\geq 1}$ are computable (resp. canonically computable).*

b) *If $(F_n)_{n\geq 1}$ is a canonically computable sequence of sets and $(G_n)_{n\geq 1}$ is a computable sequence of sets, then the sequence of sets $(F_n \cap G_n)_{n\geq 1}$ is canonically computable.*

Let $(X, \imath)$ be a computable space and Λ be a finite alphabet. A **word presheaf** $\mathcal{F}_\Lambda$ on X consists of

1) A set $\mathcal{F}_\Lambda(U)$ of Λ-valued functions defined on the set U for every computable subset $U \subseteq X$;
2) The restriction mapping $\rho_{U,V} : \mathcal{F}_\Lambda(U) \to \mathcal{F}_\Lambda(V)$ for each pair U, V of computable subsets such that $V \subseteq U$, that takes functions in $\mathcal{F}_\Lambda(U)$ and restricts them to the subset V.

It is easy to see that the standard 'presheaf axioms' are satisfied: $\rho_{U,U}$ is the identity on $\mathcal{F}_\Lambda(U)$ for every computable $U \subseteq X$, and for every triple $V \subseteq U \subseteq W$ we have that $\rho_{W,V} = \rho_{U,V} \circ \rho_{W,U}$. Elements of $\mathcal{F}_\Lambda(U)$ are called **sections** over U, or **words** over U. We will often write $s|_V$ for $\rho_{U,V}s$, where $s \in \mathcal{F}_\Lambda(U)$ is a section.

4.3. Computable Groups

In this section we provide the definitions of a computable group and a few related notions, connecting results from algebra with computability. This section is based on [Rab60].

Let Γ be a countable group with respect to the multiplication operation $*$. An indexing $\imath$ of Γ is called **admissible** if the function $* : (\Gamma, \imath) \times (\Gamma, \imath) \to (\Gamma, \imath)$ is a computable function in the sense of Section 4.2. A **computable group** is a pair $(\Gamma, \imath)$ of a group Γ and an admissible indexing $\imath$.

Of course, the groups $\mathbb{Z}^d$ and $\mathsf{UT}_d(\mathbb{Z})$ possess 'natural' admissible indexings. More precisely, for the group $\mathbb{Z}$ we fix the indexing

$$\imath : n \mapsto 2|n| + \mathbf{1}_{n\geq 0},$$

which is admissible. Here $\mathbf{1}_{n\geq 0}$ equals 1 if $n \geq 0$ and is zero otherwise. Next, it is clear that for every $d > 1$ the group $\mathbb{Z}^d$ possesses an admissible indexing function such that all coordinate projections onto $\mathbb{Z}$, endowed with the indexing function $\imath$ above, are computable. Similarly, for every $d \geq 2$ the group $\mathsf{UT}_d(\mathbb{Z})$ possesses an admissible indexing function such that for every pair of indices $1 \leq i, j \leq d$ the evaluation function sending a matrix $g \in \mathsf{UT}_d(\mathbb{Z})$ to its (i,j)-th entry is a computable function to $\mathbb{Z}$. We leave the details to the reader. It does not matter which admissible indexing function of $\mathbb{Z}^d$ or $\mathsf{UT}_d(\mathbb{Z})$ we use as long as it satisfies the conditions above, so from now on we assume that this choice is fixed.

The following lemma from [Rab60] shows that in a computable group taking the inverse is also a computable operation.

Lemma 4.3.1. *Let $(\Gamma, \imath)$ be a computable group. Then the function* inv : $(\Gamma, \imath) \to (\Gamma, \imath)$, $g \mapsto g^{-1}$ *is computable.*

$(\Gamma, \imath)$ is a computable space, and we can talk about computable subsets of $(\Gamma, \imath)$. A subgroup of Γ which is a computable subset will be called a **computable subgroup**. A homomorphism between computable groups that is computable as a map between computable spaces will be called a **computable homomorphism**. The proof of the proposition below is straightforward.

Proposition 4.3.2. *Let $(\Gamma, \imath)$ be a computable group. Then the following assertions holds*

1) *Given a computable set $A \subseteq \Gamma$ and a group element $g \in \Gamma$, the sets A^{-1}, gA and Ag are computable;*
2) *Given a computable (resp. canonically computable) sequence $(F_n)_{n\geq 1}$ of subsets of Γ and a group element $g \in \Gamma$, the sequences $(gF_n)_{n\geq 1}, (F_n g)_{n\geq 1}$ are computable (resp. canonically computable).*

It is interesting to see that a computable version of the 'First Isomorphism Theorem' also holds.

Theorem 4.3.3. *Let $(G, \imath)$ be a computable group and let $(H, \imath|_H)$ be a computable normal subgroup, where $\imath|_H$ is the restriction of the indexing function $\imath$ to H. Then there is a compatible indexing function $\imath'$ on the factor group ${}^{G}\!/_{H}$ such that the quotient map $\pi : (G, \imath) \to ({}^{G}\!/_{H}, \imath')$ is a computable homomorphism.*

For the proof we refer the reader to the Theorem 1 in [Rab60].

4.4. Computable Monotilings

Let $(\Gamma, \imath)$ be a computable group. A left Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ of Γ is called **computable** if the following assertions hold:

a) $(F_n)_{n\geq 1}$ is a canonically computable sequence of finite subsets of Γ;

b) $(\mathcal{Z}_n)_{n\geq 1}$ is a computable sequence of subsets of Γ.

First of all, let us show that the regular symmetric monotiling $([F_n,\mathcal{Z}_n])_{n\geq 1}$ of $\mathbb{Z}^d$ from Example 1.3.1 is computable.

EXAMPLE 4.4.1. Consider the group $\mathbb{Z}^d$ for some $d \geq 1$. We remind the reader that it is endowed with an admissible indexing such that all the coordinate projections $\mathbb{Z}^d \to \mathbb{Z}$ are computable. Then the Følner sequence $F_n = [0, 1, 2, \ldots, n-1]^d$ is canonically computable. Indeed, since the coordinate projections are computable, there is an algorithm that determines whether a given tuple $(x_1, \ldots, x_d)$ belongs to F_n. Since $|F_n| = n^d$, it suffices to find first n^d tuples $(x_1, \ldots, x_d)$ that belong to F_n, which can also be done 'computably'. Furthermore, the corresponding sets of centers equal $n\mathbb{Z}^d$ for every n, hence $([\mathcal{Z}_n, F_n])_{n\geq 1}$ is a computable symmetric Følner monotiling.

Next, we return to Example 1.3.2.

EXAMPLE 4.4.2. Consider the group $\mathsf{UT}_3(\mathbb{Z})$ and the monotiling $([F_n,\mathcal{Z}_n])_{n\geq 1}$ from Example 1.3.2 given by

$$\mathcal{Z}_n = \{(a,b,c) \in \mathsf{UT}_3(\mathbb{Z}) : a, b \in n\mathbb{Z}, c \in n^2\mathbb{Z}\}$$

and

$$F_n = \{(a,b,c) \in \mathsf{UT}_3(\mathbb{Z}) : 0 \leq a, b < n, 0 \leq c < n^2\}$$

for every $n \geq 1$. We define the projections $\pi_1, \pi_2, \pi_3 : \mathsf{UT}_3(\mathbb{Z}) \to \mathbb{Z}$ as follows. For every $g = (a,b,c) \in \mathsf{UT}_3(\mathbb{Z})$ we let

$$\begin{aligned}\pi_1(g) &:= a,\\ \pi_2(g) &:= b,\\ \pi_3(g) &:= c.\end{aligned}$$

The functions π_1, π_2, π_3 are computable. By definition, for every $(n,g) \in \mathbb{N} \times \mathsf{UT}_3(\mathbb{Z})$

$$\mathbf{1}_{\mathcal{Z}_\cdot}(n,g) = 1 \Leftrightarrow (\pi_1(g) \in n\mathbb{Z}) \wedge (\pi_2(g) \in n\mathbb{Z}) \wedge (\pi_3(g) \in n^2\mathbb{Z}),$$

hence the sequence of sets $(\mathcal{Z}_n)_{n\geq 1}$ is computable. It is also trivial to show that the sequence $(F_n)_{n\geq 1}$ is canonically computable. It follows that $([F_n,\mathcal{Z}_n])_{n\geq 1}$ is a computable symmetric Følner monotiling.

It is useful to have a computable way of determining which set in a Følner sequence is 'invariant enough'. The following propositions tells us that it is always possible for canonically computable Følner sequences. We will use this result later in Section 4.5.

PROPOSITION 4.4.3. *Let $(F_n)_{n\geq 1}$ be a canonically computable Følner sequence in a computable group $(\Gamma, \imath)$. Let K_i be the set of the first i elements of*

Γ *with respect to the indexing* $\imath$. *Then there is a computable function* $i \mapsto m_i$ *such that*

$$\max_{g \in K_i} \frac{|F_{m_i} \triangle g F_{m_i}|}{|F_{m_i}|} < \frac{1}{2i} \tag{4.4.1}$$

for all $i \geq 1$.

PROOF. For every i we let K_i be the finite set defined above. We let m_i be the first index such that ((4.4.1)) holds for all $g \in K_i$ (such m_i exists because $(F_n)_{n\geq 1}$ is a Følner sequence). □

In general, checking temperedness of a given canonically computable Følner sequence is not trivial. Lindenstrauss in [Lin01] proved that every Følner sequence has a tempered Følner subsequence. Furthermore, the construction of a tempered Følner subsequence from a given Følner sequence is 'algorithmic'. We provide his proof below, and we will use this result later in this section when discussing Følner monotilings of $\mathsf{UT}_d(\mathbb{Z})$ for $d > 3$.

PROPOSITION 4.4.4. *Let* $(F_n)_{n\geq 1}$ *be a canonically computable Følner sequence in a computable group* $(\Gamma, \imath)$. *Then there is a computable function* $i \mapsto n_i$ *such that the subsequence* $(F_{n_i})_{i\geq 1}$ *is a canonically computable tempered Følner subsequence.*

PROOF. We define n_i inductively as follows. Let $n_1 := 1$. If $n_1, \ldots, n_i$ have been determined, we set $\widetilde{F}_i := \bigcup_{j \leq i} F_{n_j}$. Take for n_{i+1} the first integer greater than $i+1$ such that

$$\left| F_{n_{i+1}} \triangle \widetilde{F}_i^{-1} F_{n_{i+1}} \right| \leq \frac{1}{\left| \widetilde{F}_i \right|}.$$

The function $i \mapsto n_i$ is total computable. It follows that

$$\left| \bigcup_{j \leq i} F_{n_j}^{-1} F_{n_{i+1}} \right| \leq 2 \left| F_{n_{i+1}} \right|,$$

hence the sequence $(F_{n_i})_{i\geq 1}$ is 2-tempered. Since the Følner sequence $(F_n)_{n\geq 1}$ is canonically computable and the function $i \mapsto n_i$ is computable, the Følner sequence $(F_{n_i})_{i\geq 1}$ is canonically computable and tempered. □

In case of the discrete Heisenberg group $\mathsf{UT}_3(\mathbb{Z})$ we were able to give simple formulas for the sequences $(F_n)_{n\geq 1}$ and $(\mathcal{Z}_n)_{n\geq 1}$, in particular, checking the computability was trivial. This is no longer the case when $d > 3$, and we will need the following lemma to check the computability of the sequence $(\mathcal{Z}_n)_{n\geq 1}$.

PROPOSITION 4.4.5. *Let* $(\Gamma, \imath)$ *be a computable group. Let* $([F_n, \mathcal{Z}_n])_{n\geq 1}$ *be a left Følner monotiling of* Γ *such that* $(F_n)_{n\geq 1}$ *is a canonically computable sequence of finite sets and* $\mathrm{e} \in F_n$ *for all* $n \geq 1$. *Then the following assertions are equivalent:*

(i) *There is a total computable function* $\phi : \mathbb{N}^2 \to \Gamma$ *such that*

$$\mathcal{Z}_n = \{\phi(n,1), \phi(n,2), \dots\}$$

for every $n \geq 1$.

(ii) *The sequence of sets* $(\mathcal{Z}_n)_{n\geq 1}$ *is computable.*

PROOF. The implication (ii)⇒(i) is clear. For the converse, note that to prove computability of the function $\mathbf{1}_{\mathcal{Z}_\cdot}$ we have to devise an algorithm that, given $n \in \mathbb{N}$ and $g \in \Gamma$, decides whether $g \in \mathcal{Z}_n$ or not. Let $\phi : \mathbb{N}^2 \to \Gamma$ be the function from assertion (i). Then the following algorithm answers the question. Start with $i := 1$ and compute $\mathrm{e}\phi(n,i), h_{1,n}\phi(n,i), \dots, h_{k,n}\phi(n,i)$, where $F_n = \{\mathrm{e}, h_{1,n}, \dots, h_{k,n}\}$ is the list of all (pairwise distinct) elements of F_n. This is possible since $(F_n)_{n\geq 1}$ is a canonically computable sequence of finite sets. If $g = \mathrm{e}\phi(n,i)$, then the answer is 'Yes' and we stop the program. If $g = h_{j,n}\phi(n,i)$ for some j, then the answer is 'No' and we stop the program. If neither is true, then we set $i := i+1$ and go to the beginning.

Since $\Gamma = F_n\mathcal{Z}_n$ for every n, the algorithm terminates for every input. □

4.5. Group Extension Lemma

In this section we introduce a 'technical' notion of a normal Følner monotiling and prove the group extension lemma due to Weiss.

We call a Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ **normal** if

a) $\frac{|F_n|}{\log n} \to \infty$ as $n \to \infty$;

b) $\mathrm{e} \in F_n$ for every n.

LEMMA 4.5.1 (Normalization). *Let* $(\Gamma, \imath)$ *be a computable group and* $([F_n, \mathcal{Z}_n])_{n\geq 1}$ *be a computable Følner monotiling. Then there is a computable function* $r_\cdot : \mathbb{N} \to \Gamma$ *and a computable function* $n_\cdot : \mathbb{N} \to \mathbb{N}$ *such that* $([F_{n_i} r_{n_i}^{-1}, r_{n_i}\mathcal{Z}_{n_i}])_{i\geq 1}$ *is a computable normal Følner monotiling.*

PROOF. Let r_i be the first element of the set F_i for every i when we view F_i as a subset of $\mathbb{N}$ via the indexing mapping $\imath$. Then $(F_n r_n^{-1})_{n\geq 1}$ is a Følner sequence such that $\mathrm{e} \in F_n r_n^{-1}$ for every n, and $([F_n r_n^{-1}, r_n\mathcal{Z}_n])_{n\geq 1}$ is a Følner monotiling. It is clear that we can pick the function $n_\cdot$ such that the growth condition is satisfied as well. □

The following theorem, whose proof is essentially due to B. Weiss [Wei01], shows that the class of computable groups admitting computable normal Følner monotilings is closed under group extensions.

THEOREM 4.5.2. *Let*

$$1 \to (E, \imath_E) \xrightarrow{\mathrm{id}} (F, \imath_F) \xrightarrow{\psi} (G, \imath_G) \to 1$$

be an exact sequence of computable groups such that id, ψ *are computable homomorphisms. Suppose that* $([E_k, \mathcal{Q}_k])_{k\geq 1}, ([G_m, \mathcal{S}_m])_{m\geq 1}$ *are computable normal Følner monotilings of the groups* $(E, \imath_E)$ *and* $(G, \imath_G)$ *respectively. Then there is a computable normal Følner monotiling* $([F_l, \mathcal{R}_l])_{l\geq 1}$ *in the group* $(F, \imath_F)$.

Proof. We begin by describing an auxiliary construction that provides us with 'computable' sections of computable sets in F over G. Let $T \subseteq F$ be a computable set. Let $\mathbf{1}_T$ be the characteristic function of T. Now we construct a characteristic function of a computable section T' of T as follows:

$$\mathbf{1}_{T'}(n) := \begin{cases} 1 & \text{if } \mathbf{1}_T(n) = 1 \text{ and } ((\forall\, l < n \quad \psi(n) \neq \psi(l)) \vee (n = \mathrm{e})); \\ 0 & \text{otherwise.} \end{cases}$$

That is, T' is the set of the first members of each E-coset in T except for the coset $\mathrm{e}E$, on which we pick e instead if $\mathrm{e} \in T$. Since the functions $\mathbf{1}_T, \psi$ are computable, it is easy to see that $\mathbf{1}_{T'}$ is computable as well. In particular, the function $x \mapsto \psi^{-1}(x)'$ from G to F is computable.

Let $l \in \mathbb{N}$ be fixed. Let $K_l = \{f_1, f_2, \ldots, f_l\} \subset F$ be the set of the first l elements of F with respect to the indexing $\imath_F$. We will describe an algorithm that yields a tile $F_l \subset F$ such that for all $g \in K_l$ we have

$$\frac{|F_l \triangle gF_l|}{|F_l|} \leq \frac{1}{l}.$$

It is easy to see that such a sequence $(F_l)_{l \geq 1}$ is a canonically computable Følner sequence. We will use Proposition 4.4.5 to show that the corresponding sequence of centers $(\mathcal{R}_l)_{l \geq 1}$ is computable, and it will be clear from the proof that the monotiling $([F_l, \mathcal{R}_l])_{l \geq 1}$ is normal as well.

Consider the finite set $\psi(K_l) \subset G$. Then the maximum $I_l \geq l$ of the indices of elements of $\psi(K_l)$ is a computable function of l. We let $Q_l := \{g_1, g_2, \ldots, g_{I_l}\}$ be the finite set of the first I_l elements of G. Let $m_\cdot : i \mapsto m_i$ be the computable function from the Proposition 4.4.3 applied to $(G, \imath_G)$ and $(G_m)_{m \geq 1}$, then the function $m^*_\cdot : l \mapsto \max(m_{I_l}, 2l)$ is a computable function.

Let $G_{m^*_l}$ be the corresponding Følner tile in G. Consider its preimage $\psi^{-1}(G_{m^*_l})$, which is a computable subset of F. We note that the sequence of sets $l \mapsto \psi^{-1}(G_{m^*_l})$ is computable because the sequence of sets $(G_m)_{m \geq 1}$ is computable and the functions $\psi, m^*_\cdot$ are computable. Let $T_l \subset \psi^{-1}(G_{m^*_l})$ be the computable section of $\psi^{-1}(G_{m^*_l})$ over G as defined above, then ψ is bijective as a map from T_l to $G_{m^*_l}$. Observe further that the sequence of *finite* sets $l \mapsto T_l$ is *canonically* computable. B. Weiss proved that if U is a sufficiently invariant monotile in E, then $T_l U$ is a sufficiently invariant monotile in F. Below we examine his construction closely.

Let $G^\circ_{m^*_l} := \{t \in G_{m^*_l} : Q_l t \subset G_{m^*_l}\} \subset G$ be the part of $G_{m^*_l}$ that stays within $G_{m^*_l}$ when shifted by elements of Q_l. It is clear that $T^\circ_l := \psi^{-1}(G^\circ_{m^*_l})' \subset T_l$, and that the sequence of finite sets $l \mapsto T^\circ_l$ is canonically computable. For all $x \in K_l$ and $t \in T^\circ_l$ we deduce that

$$xt = \lambda_l(x,t)\rho_l(x,t),$$

where $\lambda_l(x,t) \in T_l$ and $\rho_l(x,t) \in E$. The functions $\lambda_\cdot(\cdot,\cdot)$ and $\rho_\cdot(\cdot,\cdot)$ are uniquely determined by this condition and are partial computable. Consider the finite subset

$$P_l := \{\rho_l(x,t) : x \in K_l, t \in T^\circ_l\}.$$

The maximum index J_l of elements of this set is a computable function of l. Let $k_\cdot : i \mapsto k_i$ be the computable function from the Proposition 4.4.3 applied to $(E, \imath_E)$ and $(E_k)_{k\geq 1}$, then the function $k^*_\cdot : l \mapsto \max(k_{J_l}, m^*_l)$ is computable. Consider the Følner tile $E_{k^*_l}$, and let $E^\circ_{k^*_l} := \{s \in E_{k^*_l} : P_l s \subset E_{k^*_l}\}$ be the part of $E_{k^*_l}$ that stays in $E_{k^*_l}$ when shifted by elements of P_l. We claim that the tile

$$F_l := T_l E_{k^*_l}$$

is 'invariant enough'. Observe that

$$K_l T^\circ_l E^\circ_{k^*_l} \subset T_l E_{k^*_l}.$$

By definition, the set $G^\circ_{m^*_l}$ is large enough:

$$\left|G^\circ_{m^*_l}\right| \geq \left(1 - \frac{1}{2l}\right) \left|G_{m^*_l}\right|,$$

hence

$$|T^\circ_l| \geq \left(1 - \frac{1}{2l}\right) |T_l|.$$

Similarly,

$$\left|E^\circ_{k^*_l}\right| \geq \left(1 - \frac{1}{2l}\right) \left|E_{k^*_l}\right|,$$

and it follows that

$$\left|T^\circ_l E^\circ_{k^*_l}\right| \geq \left(1 - \frac{1}{2l}\right) \left|T_l E_{k^*_l}\right|.$$

We deduce that for every $g \in K_l$

$$\frac{|F_l \triangle g F_l|}{|F_l|} = \frac{\left|F_l \triangle g^{-1} F_l\right|}{|F_l|} \leq \frac{1}{l},$$

and this shows that F_l is 'invariant enough'. We have obtained a canonically computable Følner sequence $l \mapsto F_l$.

It remains to prove that for each l the set F_l is a tile and that the sequence of centers $(\mathcal{R}_l)_{l\geq 1}$ is computable. Let ϕ_E, ϕ_G be computable functions from the Proposition 4.4.5 applied to groups $(E, \imath_E), (G, \imath_G)$ respectively. Let $\theta : \mathbb{N}^2 \to F$ be the total computable function $(n, i) \mapsto \psi^{-1}(\phi_G(n, i))'$, i.e. we compute $\phi_G(n, i)$ first and then pick an element in its fiber in F. It is clear that

$$\{\phi_E(k^*_l, i)\theta(m^*_l, j)\}_{i,j\geq 1}$$

is a set of centers for the tile F_l. If $\nu : \mathbb{N} \to \mathbb{N}^2$ is any computable bijection, then the total computable function $\phi_F : (n, i) \mapsto \phi_E(k^*_n, \nu_1(i))\theta(m^*_n, \nu_2(i))$ satisfies the conditions of Proposition 4.4.5, and the proof is complete.

□

4.6. Plain Kolmogorov Complexity

Finally, we can introduce the Kolmogorov complexity for finite words. Let A be a computable partial function defined on a domain D of finite binary words with values in the set of all finite words over a finite alphabet Λ. The set of all finite binary words, denoted by $\{0,1\}^*$, is defined as

$$\{0,1\}^* := \bigcup_{n\geq 0} \{0,1\}^n,$$

and, similarly, the set of all finite Λ-words, denoted by Λ^*, is defined as

$$\Lambda^* := \bigcup_{n\geq 0} \Lambda^n.$$

Of course, we have defined computable functions on subsets of $(\mathbb{N} \cup \{0\})^k$ with values in $\mathbb{N} \cup \{0\}$ above, but this can be easily extended to (co)domains of finite words over finite alphabets. We can think of A as a 'decompressor' that takes compressed binary descriptions (or 'programs') in the domain, and decompresses them to finite words over the alphabet Λ. The **Kolmogorov complexity** of a finite word ω with respect to A is defined as follows:

$$\mathrm{K}^0_A(\omega) := \inf\{l(p) : A(p) = w\},$$

where $l(p)$ denotes the length of the description. If some word ω_0 does not admit a compressed version, then we let $\mathrm{K}^0_A(\omega_0) = \infty$. The **average Kolmogorov complexity** with respect to A is defined by

$$\overline{\mathrm{K}}^0_A(\omega) := \frac{\mathrm{K}^0_A(\omega)}{l(\omega)},$$

where $l(\omega)$ is the length of the word ω. Intuitively speaking, this quantity tells how effective the compressor A is when describing the word ω.

Of course, some decompressors are intuitively better than some others. This is formalized by saying that A_1 is **not worse** than A_2 if there is a constant c such that for all words ω

$$\mathrm{K}^0_{A_1}(\omega) \leq \mathrm{K}^0_{A_2}(\omega) + c. \tag{4.6.1}$$

A theorem of Kolmogorov [Kol65] says that there exists a decompressor A^* that is optimal, i.e. for every decompressor A there is a constant c such that for all words ω we have

$$\mathrm{K}^0_{A^*}(\omega) \leq \mathrm{K}^0_{A}(\omega) + c.$$

An optimal decompressor is not unique, so from now on we let A^* be a fixed optimal decompressor.

The notion of Kolmogorov complexity can be extended to words defined on finite subsets of $\mathbb{N}$, and this will be essential in the following sections. More precisely, let $X \subseteq \mathbb{N}$ be a finite subset, $\imath_X : X \to \{1, 2, \ldots, \operatorname{card} X\}$ an increasing bijection, Λ a finite alphabet, A a decompressor and $\omega \in \Lambda^Y$ a word defined on some set $Y \supseteq X$, $Y \subseteq \mathbb{N}$. Then we let

$$\mathrm{K}_A(\omega, X) := \mathrm{K}^0_A(\omega \circ \imath_X^{-1}). \tag{4.6.2}$$

and

$$\overline{\mathrm{K}}_A(\omega, X) := \frac{\mathrm{K}^0_A(\omega \circ \imath_X^{-1})}{\operatorname{card} X}. \tag{4.6.3}$$

We call $\mathrm{K}_A(\omega, X)$ the **Kolmogorov complexity** of ω over X with respect to A, and $\overline{\mathrm{K}}_A(\omega, X)$ is called the **mean Kolmogorov complexity** of ω over X with respect to A. If a decompressor A_1 is not worse than a decompressor A_2 with some constant c, then for all X, ω above

$$\mathrm{K}_{A_1}(\omega, X) \leq \mathrm{K}_{A_2}(\omega, X) + c.$$

If $X \subseteq \mathbb{N}$ is an infinite subset and $(F_n)_{n\geq 1}$ is a sequence of finite subsets of X such that $\operatorname{card} F_n \to \infty$, then the **asymptotic Kolmogorov complexity** of $\omega \in \Lambda^X$ with respect to $(F_n)_{n\geq 1}$ and a decompressor A is defined by

$$\widehat{\mathrm{K}}_A(\omega) := \limsup_{n\to\infty} \overline{\mathrm{K}}_A(\omega|_{F_n}, F_n).$$

The dependence on the sequence $(F_n)_{n\geq 1}$ is omitted in the notation. It is easy to see that for every decompressor A and $\omega \in \Lambda^X$

$$\widehat{\mathrm{K}}_{A^*}(\omega) \leq \widehat{\mathrm{K}}_A(\omega). \tag{4.6.4}$$

From now on, we will (mostly) use the optimal decompressor A^* and write $\mathrm{K}(\omega, X)$, $\overline{\mathrm{K}}(\omega, X)$ and $\widehat{\mathrm{K}}(\omega)$ omitting an explicit reference to A^*.

When estimating the Kolmogorov complexity of words we will often have to encode nonnegative integers using binary words. We will now fix some notation that will be used later. When n is a nonnegative integer, we write $\underline{\mathrm{n}}$ for the **binary encoding** of n and $\overline{\mathrm{n}}$ for the **doubling encoding** of n, i.e., if

$$b_l b_{l-1} \ldots b_0$$

is the binary expansion of n, then $\underline{\mathrm{n}}$ is the binary word

$$\mathrm{b}_l \mathrm{b}_{l-1} \ldots \mathrm{b}_0$$

of length $l+1$ and $\overline{\mathrm{n}}$ is the binary word

$$\mathrm{b}_l \mathrm{b}_l \mathrm{b}_{l-1} \mathrm{b}_{l-1} \ldots \mathrm{b}_0 \mathrm{b}_0$$

of length $2l+2$. We denote the length of the binary word w by $l(\mathrm{w})$, and it is clear that $l(\underline{\mathrm{n}}) \leq \lfloor \log n \rfloor + 1$ and $l(\overline{\mathrm{n}}) \leq 2\lfloor \log n \rfloor + 2$. We write $\widehat{\mathrm{n}}$ for the encoding $\overline{l(\underline{\mathrm{n}})}01\underline{\mathrm{n}}$ of n, i.e., the encoding begins with the length of the binary word $\underline{\mathrm{n}}$ encoded using doubling encoding, then the delimiter 01 follows, then the word $\underline{n}$. It is clear that $l(\widehat{\mathrm{n}}) \leq 2\lfloor \log(\lfloor \log n \rfloor + 1) \rfloor + \lfloor \log n \rfloor + 5$. This encoding enjoys the following property: given a binary string

$$\widehat{x}_1 \widehat{x}_2 \ldots \widehat{x}_l,$$

the integers $x_1, \ldots, x_l$ are unambiguously restored. We will call such an encoding a **simple prefix-free encoding** (see [LV08, Section 1.11.1] for more information on prefix codes).

4.7. Kolmogorov Complexity on Word Presheaves

We have introduced Kolmogorov complexity of words supported on subsets of $\mathbb{N}$ in the previous section, now we want to extend this by introducing complexity of sections. Let $(X, \imath)$ be a computable space and let $\mathcal{F}_\Lambda$ be a word presheaf over $(X, \imath)$. Let $U \subseteq X$ be a finite set and $\omega \in \mathcal{F}_\Lambda(U)$. Then we define the **Kolmogorov complexity** of $\omega \in \mathcal{F}_\Lambda(U)$ by

$$\mathrm{K}(\omega, U) := \mathrm{K}(\omega \circ \imath^{-1}, \imath(U)) \tag{4.7.1}$$

and the **mean Kolmogorov complexity** of $\omega \in \mathcal{F}_\Lambda(U)$ by

$$\overline{\mathrm{K}}(\omega, U) := \overline{\mathrm{K}}(\omega \circ \imath^{-1}, \imath(U)). \tag{4.7.2}$$

The quantities on the right hand side here are defined in the Equations (4.6.2) and (4.6.3) respectively (which are special cases of the more general definition when the computable space X is $(\mathbb{N}, \mathrm{id})$).

Let $(F_n)_{n\geq 1}$ be a sequence of finite subsets of X such that $\operatorname{card} F_n \to \infty$. Then we define **asymptotic Kolmogorov complexity** of a section $\omega \in \mathcal{F}_\Lambda(X)$ along the sequence $(F_n)_{n\geq 1}$ by

$$\widehat{\mathrm{K}}(\omega) := \limsup_{n\to\infty} \overline{\mathrm{K}}(\omega|_{F_n}, F_n).$$

The dependence on the sequence $(F_n)_{n\geq 1}$ is omitted in the notation for $\widehat{\mathrm{K}}$, but it will be always clear from the context which sequence we take.

Let $(F_n)_{n\geq 1}$ be a sequence of finite subsets of X such that $\operatorname{card} F_n \to \infty$ as $n \to \infty$. Then we define **asymptotic Kolmogorov complexity** of the word presheaf $\mathcal{F}_\Lambda$ along the sequence $(F_n)_{n\geq 1}$ by

$$\widetilde{\mathrm{K}}(\mathcal{F}_\Lambda) := \limsup_{n\to\infty} \max_{\omega\in\mathcal{F}_\Lambda(F_n)} \overline{\mathrm{K}}(\omega, F_n).$$

The dependence on the sequence is omitted in the notation, but it will always be clear from the context which sequence we take. If A' is some decompressor, then it follows from the optimality of A^* that

$$\widetilde{\mathrm{K}}(\mathcal{F}_\Lambda) \leq \widetilde{\mathrm{K}}_{A'}(\mathcal{F}_\Lambda).$$

We close this section with an interesting result on the invariance of the asymptotic Kolmogorov complexity of sections. It says that the asymptotic Kolmogorov complexity of a section $\omega \in \mathcal{F}_\Lambda(X)$ does not change if one passes to an equivalent indexing.

THEOREM 4.7.1 (Invariance of asymptotic complexity). *Let $\imath_1, \imath_2$ be equivalent indexing functions of a set X. Let $(F_n)_{n\geq 1}$ be a sequence of finite subsets of X such that*

a) *$(F_n)_{n\geq 1}$ is a canonically computable sequence of sets in $(X, \imath_1)$;*

b) *$\frac{\operatorname{card} F_n}{\log n} \to \infty$ as $n \to \infty$.*

Let $\omega \in \mathcal{F}_\Lambda(X)$. Then

$$\limsup_{n\to\infty} \overline{\mathrm{K}}(\omega|_{F_n} \circ \imath_1^{-1}, \imath_1(F_n)) = \limsup_{n\to\infty} \overline{\mathrm{K}}(\omega|_{F_n} \circ \imath_2^{-1}, \imath_2(F_n)),$$

i.e. asymptotic Kolmogorov complexity of ω *does not change when we pass to an equivalent indexing.*

PROOF. Since the indexing functions $\imath_1, \imath_2$ are equivalent, there is a computable bijection $\phi : \imath_2(X) \to \imath_1(X)$ such that $\phi(\imath_2(x)) = \imath_1(x)$ for all $x \in X$. Furthermore, the sequence $(F_n)_{n\geq 1}$ is canonically computable in $(X, \imath_2)$.

Let n be fixed. By the definition,

$$\overline{\mathrm{K}}(\omega|_{F_n} \circ \imath_1^{-1}, \imath_1(F_n)) = \frac{\mathrm{K}^0_{A^*}((\omega|_{F_n} \circ \imath_1^{-1}) \circ \imath^{-1}_{\imath_1(F_n)})}{\operatorname{card} F_n},$$

where $\omega|_{F_n} \circ \imath_1^{-1}$ is seen as a word on $\imath_1(F_n) \subseteq \mathbb{N}$ and $\widetilde{\omega}_1 := (\omega|_{F_n} \circ \imath_1^{-1}) \circ \imath^{-1}_{\imath_1(F_n)}$ is a word on $\{1, 2, \ldots, \operatorname{card} F_n\} \subseteq \mathbb{N}$. Let p_1 be an optimal description of $(\omega|_{F_n} \circ \imath_1^{-1}) \circ \imath^{-1}_{\imath_1(F_n)}$. Similarly, $\widetilde{\omega}_2 := (\omega|_{F_n} \circ \imath_2^{-1}) \circ \imath^{-1}_{\imath_2(F_n)}$ is a word on $\{1, 2, \ldots, \operatorname{card} F_n\}$. It is clear that $\widetilde{\omega}_1$ is a permutation of $\widetilde{\omega}_2$, hence we can describe $\widetilde{\omega}_2$ by giving the description of $\widetilde{\omega}_1$ and saying how to permute it to obtain $\widetilde{\omega}_2$. We make this intuition formal below.

We define a new decompressor A'. The domain of definition of A' consists of the programs of the form

$$\bar{\mathrm{l}}01\mathrm{p}, \tag{4.7.3}$$

where $\bar{\mathrm{l}}$ is the doubling encoding of an integer l and p is an input for A^*. The decompressor works as follows. Compute the subsets $\imath_1(F_l)$ and $\imath_2(F_l)$ of $\mathbb{N}$. Let $\overline{\phi}$ be the element of $\mathrm{Sym}_{\operatorname{card} F_n}$ such that the diagram

$$\begin{tikzcd}
\imath_1(F_n) \arrow[d, "\imath_{\imath_1(F_n)}"'] & & \imath_2(F_n) \arrow[ll, "\phi"'] \arrow[d, "\imath_{\imath_2(F_n)}"] \\
\{1, 2, \ldots, \operatorname{card} F_n\} & & \{1, 2, \ldots, \operatorname{card} F_n\} \arrow[ll, "\overline{\phi}"']
\end{tikzcd}$$

commutes. Compute the word $\omega' := A^*(\mathrm{p})$, and if $\operatorname{card} F_l \neq l(\omega')$ the algorithm terminates without producing output. Otherwise, the word $\omega' \circ \overline{\phi}$ is printed. It follows that there is a constant c such that the following holds: for all $l \in \mathbb{N}$ and for all words ω' of length $\operatorname{card} F_l$ we have

$$\mathrm{K}^0_{A^*}(\omega' \circ \overline{\phi}) \leq \mathrm{K}^0_{A^*}(\omega') + 2 \log l + c,$$

where $\overline{\phi}$ is the permutation of $\{1, 2, \ldots, \operatorname{card} F_l\}$ defined above.

Finally, consider the program $\mathrm{p}' := \bar{\mathrm{n}}01\mathrm{p}_1$, then $A'(\mathrm{p}') = \widetilde{\omega}_2$. We deduce that $\mathrm{K}^0_{A^*}(\widetilde{\omega}_2) \leq \mathrm{K}^0_{A^*}(\widetilde{\omega}_1) + 2 \log n + c$. The statement of the theorem follows trivially. □

To simplify the notation below, we adopt the following convention. We say explicitly what indexing function we use when introducing a computable space, but later, when the indexing is fixed, we often omit the indexing function from the notation and think about computable spaces as computable subsets of $\mathbb{N}$. Words defined on subsets of a computable space become words defined on subsets of $\mathbb{N}$.

Part III

Entropy Theory for Actions of Amenable Groups

CHAPTER 5

Kolmogorov-Sinai and Topological Entropies

The term entropy originated in physics in the works of R. Clausius, N. L. S. Carnot and L. Boltzmann. Later, C. E. Shannon in [Sha48] introduced entropy in the information theory as a means of measuring the amount of information coming from a data source. The Kolmogorov-Sinai entropy, which is an adaptation of the Shannon entropy for studying dynamical systems, was originally introduced for $\mathbb{Z}$-systems. It is an important invariant - i.e. it remains constant on isomorphism classes of dynamical systems - and enjoys other useful properties. The topological entropy was defined later for $\mathbb{Z}$-systems by R. L. Adler, mimicking the Kolmogorov-Sinai entropy. Later, both notions of entropy were extended to amenable group actions, relying on the lemma of Ornstein and Weiss from Section 1.4. For actions of amenable groups, both types of entropy enjoy very nice properties: for instance, entropy decreases when passing to a factor of a system, entropy is additive in a sense that entropy of the product of two systems is the sum of entropies, and so on.

This chapter is structured as follows. First, we discuss the Kolmogorov-Sinai entropy. In Section 5.1.2 we define the Kolmogorov-Sinai entropy, prove the most basic properties that will be used later and provide some basic examples. In Section 5.2 we state the Shannon-McMillan-Breiman theorem and some of its consequences. Next, we proceed to topological entropy. We provide the definitions in Section 5.3 and prove some basic properties of the topological entropy. In Section 5.4 we give some basic examples. We close the chapter with Section 5.5, where we provide some additional remarks and sketch a proof of the Kolmogorov-Sinai generator theorem (Proposition 5.1.4).

5.1. Kolmogorov-Sinai Entropy

5.1.1. Shannon Entropy of a Partition. To introduce the Kolmogorov-Sinai entropy of measure-preserving systems for amenable group actions we need to remind the reader of the notion of the Shannon entropy first.

Let $\eta_0 : [0, 1] \to \mathbb{R}_{\geq 0}$ be the continuous extension of the function

$$t \mapsto -t \log(t)$$

defined on $(0, 1]$. Here, as usual, log is the binary logarithm. It is easy to see that η_0 is strictly concave. Let $\mathfrak{P} \subsetneq \ell^1(\mathbb{N}; \mathbb{R})$ be the closed subset of all probability vectors. We define the function $\eta_\infty : \mathfrak{P} \to [0, \infty]$ by

$$\eta_\infty((p_n)_{n\geq 1}) := \sum_{n\geq 1} \eta_0(p_n).$$

LEMMA 5.1.1. *η_∞ is concave, i.e. for all $\overline{p} = (p_n)_{n\geq 1}, \overline{q} = (q_n)_{n\geq 1} \in \mathfrak{P}$ and for all $\alpha \in (0,1)$*

$$\eta_\infty(\alpha\overline{p} + (1-\alpha)\overline{q}) \geq \alpha\eta_\infty(\overline{p}) + (1-\alpha)\eta_\infty(\overline{q}).$$

PROOF. This follows from the concavity of η_0. □

Let $\alpha = \{A_1, \ldots, A_n\}$ be a finite measurable partition of a probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$. The function $\omega \mapsto \alpha(\omega)$, mapping a point $\omega \in X$ to the atom of the partition α containing ω, is defined almost everywhere. The **information function** of α is defined as

$$I_\alpha(\omega) := -\sum_{i=1}^{n} \mathbf{1}_{A_i}(\omega) \log \mu(A_i) = -\log(\mu(\alpha(\omega))).$$

Then $I_\alpha \in \mathrm{L}^\infty(\mathrm{X})$. The **Shannon entropy of a partition** α is defined by

$$\begin{aligned} H_{\mathrm{Sh}}(\alpha) : &= -\sum_{i=1}^{n} \mu(A_i) \log(\mu(A_i)) = \eta_\infty((\mu(A_i)_{i=1}^n)) = \\ &= \int I_\alpha d\mu \end{aligned}$$

The Shannon entropy of a finite measurable partition is always a nonnegative real number.

Let α, β be two finite measurable partitions of X. Then

$$\alpha \vee \beta := \{A \cap B : A \in \alpha, B \in \beta\}$$

is a finite measurable partition of X as well, which we call the **join** of α and β. If for every atom $A \in \alpha$ there exists an atom $B \in \beta$ such that $\mu(A \setminus B) = 0$, i.e. $A \subseteq B$ modulo null sets, then we say that α is **finer** than β, or that α **refines** β, and write $\alpha \geq \beta$.

The following lemma summarizes a few crucial properties of the Shannon entropy.

LEMMA 5.1.2. *Let X be a probability space and α, β be finite measurable partitions. The following assertions hold:*

a) *If α is finer than β, then*

$$H_{\mathrm{Sh}}(\beta) \leq H_{\mathrm{Sh}}(\alpha);$$

b) $H_{\mathrm{Sh}}(\alpha \vee \beta) \leq H_{\mathrm{Sh}}(\alpha) + H_{\mathrm{Sh}}(\beta)$.

PROOF. The first statement is easily proved using subadditivity of the function η_0. The second statement is less trivial, we refer to [Gla03, Proposition 14.16] or [Dow11, Corollary 1.6.8] for the proof. Later in Chapter 6, Proposition 6.3.1 we will prove a slightly more general version of this 'subadditivity' result. □

5.1.2. Entropy of Measure-preserving Systems. Let $\mathbf{X} = (\mathrm{X}, \pi)$ be a measure-preserving Γ-system on a probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$, where the discrete amenable group Γ acts on X. The goal is to define the 'dynamical' entropy of a partition with respect to the action of Γ. First, for every element $g \in \Gamma$ and every finite measurable partition α we define a finite measurable partition $g^{-1}\alpha$ by

$$g^{-1}\alpha := \{g^{-1}A : A \in \alpha\}.$$

Next, for every finite subset $F \subseteq \Gamma$ and every finite measurable partition α we define the partition

$$\alpha^F := \bigvee_{g \in F} g^{-1}\alpha.$$

Let $(F_n)_{n\geq 1}$ be a Følner sequence in Γ and α be a finite measurable partition of X. Then the limit

$$h_{\mathsf{Prob}}(\alpha, \pi) := \lim_{n\to\infty} \frac{H_{\mathrm{Sh}}(\alpha^{F_n})}{|F_n|}$$

exists. The limite is a nonnegative real number independent of the choice of a Følner sequence due to Ornstein-Weiss lemma (see Proposition 1.4.2, the assertions of the lemma are easily checked using Lemma 5.1.2). The limit $h_{\mathsf{Prob}}(\alpha, \pi)$ is called the **Kolmogorov-Sinai entropy of α with respect to π**. We define the **Kolmogorov-Sinai entropy** of a measure-preserving system $\mathbf{X} = (\mathrm{X}, \pi)$ by

$$h_{\mathsf{Prob}}(\mathbf{X}) := \sup\{h_{\mathsf{Prob}}(\alpha, \pi) : \alpha \text{ a finite measurable partition of X}\}.$$

The first basic property of the Kolmogorov-Sinai entropy is that it decreases when one passes to factors.

PROPOSITION 5.1.3. *Let $\mathbf{X} = (\mathrm{X}, \pi)$ and $\mathbf{Y} = (\mathrm{Y}, \rho)$ be measure-preserving Γ-systems and $\varphi : \mathbf{X} \to \mathbf{Y}$ be a factor map. Then*

$$h_{\mathsf{Prob}}(\mathbf{Y}) \leq h_{\mathsf{Prob}}(\mathbf{X}).$$

PROOF. Let α be an arbitrary finite measurable partition of Y. Then $\varphi^{-1}\alpha$ is a finite measurable partition of X, and, additionally,

$$h_{\mathsf{Prob}}(\alpha, \rho) = h_{\mathsf{Prob}}(\varphi^{-1}\alpha, \pi).$$

Since α is arbitrary, taking the supremum over all such α on both sides completes the proof of the statement. □

Computing the supremum in the definition of the Kolmogorov-Sinai entropy might be very nontrivial. Fortunatelly, there is a result which simplifies this computation. Given a measure-preserving Γ-system $\mathbf{X} = (\mathrm{X}, \pi)$, a finite measurable partition α of $\mathrm{X} = (X, \mathcal{B}, \mu)$ is called a **generator** for the system $\mathbf{X}$ if, up to null sets, we have

$$\sigma\Big(\bigvee_{g\in\Gamma} g^{-1}\alpha\Big) = \mathcal{B}.$$

Proposition 5.1.4. *Let $\mathbf{X} = (\mathrm{X}, \pi)$ be a measure-preserving Γ-system. If α is a generator for the system $\mathbf{X}$, then*

$$h_{\mathsf{Prob}}(\mathbf{X}) = h_{\mathsf{Prob}}(\alpha, \pi).$$

For the sketch of the proof we refer the reader to Section 5.5.2.

Example 5.1.5. We return now to Example 3.1.4. Let $\Lambda = \{1, 2, \dots, k\}$ be a finite alphabet and let $p = (p_1, p_2, \dots, p_k)$ be a probability vector. Let

$$\mathrm{X} := \Lambda^{\mathbb{Z}}$$

be the measurable space carrying the Borel structure coming from the product topology, and the transformation $\varphi : \mathrm{X} \to \mathrm{X}$ be the left shift. Consider the partition $\alpha = (A_1, A_2, \dots, A_k)$, where for every $i = 1, \dots, k$

$$A_i = \{\omega \in \mathrm{X} : \omega(0) = p_i\}.$$

It is easy to see that α is a generating partition. Using Proposition 5.1.4, we conclude that

$$h_{\mathsf{Prob}}(\mathbf{X}) = -\sum_{i=1}^{k} p_i \log p_i.$$

5.2. Shannon-McMillan-Breiman Theorem

We will need the Shannon-McMillan-Breiman theorem for amenable group actions. For the proof see [Lin01].

Theorem 5.2.1. *Let $\mathbf{X} = (\mathrm{X}, \pi)$ be an ergodic measure-preserving system and α be a finite partition of X. Assume that $(F_n)_{n\geq 1}$ is a tempered Følner sequence in Γ such that $\frac{|F_n|}{\log n} \to \infty$ as $n \to \infty$. Then there is a constant $h'_{\mathsf{Prob}}(\alpha, \pi)$ such that*

$$\frac{I_{\alpha^{F_n}}(\omega)}{|F_n|} \to h'_{\mathsf{Prob}}(\alpha, \pi) \tag{5.2.1}$$

as $n \to \infty$ for μ-a.e. $\omega \in \mathrm{X}$ and in $\mathrm{L}^1(\mathrm{X})$.

Integrating both sides of the Equation (5.2.1) with respect to μ, we deduce that

$$\frac{h_{\mathrm{Sh}}(\alpha^{F_n})}{|F_n|} \to h_{\mathsf{Prob}}(\alpha, \pi) = h'_{\mathsf{Prob}}(\alpha, \pi).$$

as $n \to \infty$. The Shannon-McMillan-Breiman theorem has the following important corollary that will be used in the proof of Theorem 7.3.5 ([Gla03, Corollary 14.36]).

Corollary 5.2.2. *Let $\mathbf{X} = (\mathrm{X}, \pi)$ be an ergodic measure-preserving system on a space $\mathrm{X} = (X, \mathcal{B}, \mu)$, and let $(F_n)_{n\geq 1}$ be a tempered Følner sequence in Γ such that $\frac{|F_n|}{\log n} \to \infty$ as $n \to \infty$. If α is a finite partition, then, given $\varepsilon > 0$ and $\delta > 0$, there exists n_0 such that the following assertions hold:*

a) *For all* $n \geq n_0$

$$2^{-|F_n|(h_{\mathsf{Prob}}(\alpha,\pi)+\varepsilon)} \leq \mu(A) \leq 2^{-|F_n|(h_{\mathsf{Prob}}(\alpha,\pi)-\varepsilon)}$$

for all atoms $A \in \alpha^{F_n}$ *with the exception of a set of atoms whose total measure is less than* δ.

b) *For all* $n \geq n_0$

$$2^{-|F_n|(h_{\mathsf{Prob}}(\alpha,\pi)+\varepsilon)} \leq \mu(\alpha^{F_n}(\omega)) \leq 2^{-|F_n|(h_{\mathsf{Prob}}(\alpha,\pi)-\varepsilon)}$$

for all but at most δ *fraction of elements* $\omega \in \mathrm{X}$.

PROOF. By Theorem 5.2.1, $\frac{I_{\alpha^{F_n}}(\omega)}{|F_n|} \to h_{\mathsf{Prob}}(\alpha,\pi)$ for a.e. ω and hence also in measure. Thus, given $\varepsilon, \delta > 0$ as above, there is n_0 such that for all $n \geq n_0$ we have

$$\mu\{\omega \in \mathrm{X} : \left|\frac{I_{\alpha^{F_n}}(\omega)}{|F_n|} - h_{\mathsf{Prob}}(\alpha,\pi)\right| \geq \varepsilon\} < \delta.$$

It is now clear that both assertions follow. □

5.3. Topological Entropy

5.3.1. Topological Entropy of Covers. Let $\alpha = \{A_1, \ldots, A_n\}$ be a finite open cover of a topological space X. The **topological entropy of a cover** α is defined by

$$H_{\mathsf{Top}}(\alpha) := \log \min\{\operatorname{card} \beta : \beta \subseteq \alpha \text{ a subcover}\}.$$

Here log denotes, as usual, the binary logarithm. The entropy of a cover is always a nonnegative real number. We say that a finite open cover α is **finer** than a finite open cover β if for every $B \in \beta$ there exists $A \in \alpha$ such that $A \subseteq B$. If α, β are two finite open covers, then

$$\alpha \vee \beta := \{A \cap B : A \in \alpha, B \in \beta\}$$

is a finite open cover as well. It is clear that $\alpha \vee \beta$ is finer than α and β.

LEMMA 5.3.1. *Let* $\mathrm{X} \in \mathsf{Top}$ *be a topological space and* α, β *be finite open covers. The following assertions hold:*

a) *If* α *is finer than* β, *then*

$$H_{\mathsf{Top}}(\beta) \leq H_{\mathsf{Top}}(\alpha);$$

b) $H_{\mathsf{Top}}(\alpha \vee \beta) \leq H_{\mathsf{Top}}(\alpha) + H_{\mathsf{Top}}(\beta)$.

PROOF. The proof follows trivially from the definitions. □

5.3.2. Topological Entropy of Dynamical Systems. Given a topological dynamical system $\mathbf{X} = (\mathrm{X}, \pi)$, where the discrete amenable group Γ acts on the topological space X on the left by homeomorphisms, we can also define the (dynamical) entropy of a cover. For every element $g \in \Gamma$ and every finite open cover α we define a finite open cover $g^{-1}\alpha$ by

$$g^{-1}\alpha := \{g^{-1}A : A \in \alpha\}.$$

Next, for every finite subset $F \subseteq \Gamma$ and every finite open cover α we define a finite open cover

$$\alpha^F := \bigvee_{g \in F} g^{-1}\alpha.$$

Let $(F_n)_{n \geq 1}$ be a Følner sequence in Γ and α be a finite open cover. The limit

$$h_{\mathsf{Top}}(\alpha, \pi) := \lim_{n \to \infty} \frac{H_{\mathsf{Top}}(\alpha^{F_n})}{|F_n|}$$

exists. The limit is a nonnegative real number independent of the choice of a Følner sequence due to the lemma of D. S. Ornstein and B. Weiss (see Proposition 1.4.2, the assertions of the lemma are easily checked using Lemma 5.3.1). The limit $h_{\mathsf{Top}}(\alpha, \pi)$ is called the **topological entropy of α with respect to π**. Finally, the **topological entropy** of a topological system $\mathbf{X} = (\mathrm{X}, \pi)$ is defined by

$$h_{\mathsf{Top}}(\mathbf{X}) := \sup\{h_{\mathsf{Top}}(\alpha, \pi) : \alpha \text{ a finite open cover of } \mathrm{X}\}.$$

Similar to the Kolmogorov-Sinai entropy, the topological entropy decreases along arrows.

PROPOSITION 5.3.2. *Let $\mathbf{X} = (\mathrm{X}, \pi), \mathbf{Y} = (\mathrm{Y}, \rho)$ be topological Γ-systems. Let $\varphi : \mathbf{X} \to \mathbf{Y}$ be a factor map. Then*

$$h_{\mathsf{Top}}(\mathbf{Y}) \leq h_{\mathsf{Top}}(\mathbf{X}).$$

PROOF. Let α be an arbitrary finite open cover of Y. Then $\varphi^{-1}\alpha$ is a finite open cover of X, and, additionally,

$$h_{\mathsf{Top}}(\alpha, \rho) = h_{\mathsf{Top}}(\varphi^{-1}\alpha, \pi).$$

Taking the supremum over all finite covers, we complete the proof. □

Let $\mathbf{X} \in \mathsf{Top}_\Gamma$. We say that a finite open cover α of X is a **topological generator** of $\mathbf{X}$ if, for every open cover β of X, there exists a finite subset $F \subseteq \Gamma$ such that

$$\bigvee_{f \in F} f^{-1}\alpha \geq \beta.$$

In place of the Kolmogorov-Sinai generator theorem, we will use the following result to compute the topological entropy.

PROPOSITION 5.3.3. *Let $\mathbf{X} \in \mathsf{Top}_\Gamma$ and α be a topological generator of $\mathbf{X}$. Then*

$$h_{\mathsf{Top}}(\mathbf{X}) = h_{\mathsf{Top}}(\alpha, \pi).$$

Proof. We begin with an argument similar to the proof of the Kolmogorov-Sinai generator theorem in Section 5.5.2. The group Γ admits a Følner sequence $(F_n)_{n\geq 1}$ such that

a) $\Gamma = \bigcup_{n\geq 1} F_n$;

b) the sequence $(F_n)_{n\geq 1}$ is monotone increasing.

We fix this Følner sequence and use the independence of the entropies on the choice of a Følner sequence. Furthermore, it is easy to see that if $(F_n)_{n\geq 1}$ is a Følner sequence and $E \subseteq \Gamma$ is a finite set, then $(EF_n)_{n\geq 1}$ is a Følner sequence as well.

For every $k \geq 1$

$$h_{\mathsf{Top}}(\alpha^{F_k}, \pi) = \lim_{n\to\infty} \frac{H_{\mathsf{Top}}\left(\bigvee_{g\in F_n} g^{-1} \bigvee_{h\in F_k} h^{-1}\alpha\right)}{|F_n|} =$$

$$= \lim_{n\to\infty} \frac{H_{\mathsf{Top}}\left(\bigvee_{g\in F_n, h\in F_k} (hg)^{-1}\alpha\right)}{|F_n|} = \lim_{n\to\infty} \frac{H_{\mathsf{Top}}\left(\bigvee_{f\in F_k F_n} f^{-1}\alpha\right)}{|F_n|} =$$

$$= \lim_{n\to\infty} \frac{H_{\mathsf{Top}}\left(\bigvee_{f\in F_k F_n} f^{-1}\alpha\right)}{|F_k F_n|} = h_{\mathsf{Top}}(\alpha, \pi),$$

where in the last equality we use the independence of $h_{\mathsf{Top}}(\alpha, \pi)$ on the choice of the Følner sequence once again.

Let β be an arbitrary finite open cover of X. There exists a finite set $F \subseteq \Gamma$ such that $\alpha^F \geq \beta$, hence there exists $k \geq 1$ such that $\alpha^{F_k} \geq \beta$. Thus for every $n \geq 1$ we have $(\alpha^{F_k})^{F_n} \geq \beta^{F_n}$. This implies that

$$h_{\mathsf{Top}}(\beta, \pi) = \lim_{n\to\infty} \frac{H_{\mathsf{Top}}\left(\bigvee_{g\in F_n} g^{-1}\beta\right)}{|F_n|} \leq \lim_{n\to\infty} \frac{H_{\mathsf{Top}}\left(\bigvee_{g\in F_n} g^{-1}\alpha^{F_k}\right)}{|F_n|} =$$
$$= h_{\mathsf{Top}}(\alpha^{F_k}, \pi) = h_{\mathsf{Top}}(\alpha, \pi).$$

Since β is arbitrary, the proof is complete. □

We use the topological generator theorem to compute the topological entropy of subshifts.

5.4. Examples

Our main objects of interest in this work are subshifts, so we will now study the topological entropy of a subshift.

Example 5.4.1. Let Γ be an amenable group, $\Lambda = \{1, 2, \ldots, k\}$ be a finite alphabet and $\mathrm{X} \subseteq \Lambda^\Gamma$ be a subshift. For every set $F \subseteq \Gamma$, let $\mathcal{F}_\Lambda(F) := \{\omega|_F :$

$\omega \in \mathrm{X}\}$ be the set of all restrictions of the words in X to the set F. We want to prove that

$$h_{\mathsf{Top}}(\mathbf{X}) = \lim_{n\to\infty} \frac{\log \operatorname{card} \mathcal{F}_\Lambda(F_n)}{|F_n|}.$$

Consider the finite open cover

$$\alpha_\Lambda := \{A_1, \dots, A_k\},\ A_i := \{\omega \in \mathrm{X} : \omega(\mathrm{e}) = i\} \text{ for } i = 1, \dots, k.$$

of X. Then α_Λ is a generating cover. Indeed, the cylinder sets in X form the basis of the topology, hence for every finite open cover β there is a (possibly infinite) open cover γ consisting of cylinder sets such that $\gamma \geq \beta$. We pick a finite subcover $\gamma' \subseteq \gamma$, then $\gamma' \geq \beta$ as well. Since all elements of the finite open cover γ' are cylinder sets, we can find a finite set $F \subseteq \Gamma$ such that $\alpha_\Lambda^F \geq \gamma'$, and hence $\alpha_\Lambda^F \geq \beta$.

It remains to observe that for every finite set $F \subseteq \Gamma$ the equality

$$H_{\mathsf{Top}}(\alpha_\Lambda^F) = \log \operatorname{card} \mathcal{F}_\Lambda(F)$$

holds, and the proof is complete.

In some simple cases, this proposition allows to compute the entropy of a subshift immediately.

Example 5.4.2. Let Λ be a finite alphabet and Γ be a group. We define a compact Hausdorff space

$$\mathrm{X} := \Lambda^\Gamma,$$

carrying the product topology. Let $\mathbf{X} = (\mathrm{X}, \pi)$ be the right shift transformation from Example 2.2.4. Then it is easy to see that

$$h_{\mathsf{Top}}(\mathbf{X}) = \log k.$$

5.5. Remarks

5.5.1. Logarithm in the Shannon Entropy. In the definition of the Shannon entropy in Section 5.1 we have requested that the logarithm in the definition of the function η_0 is the binary logarithm. Very often another base is chosen instead. For our purposes using binary logarithm is the most appropriate definition because of the connection with the Kolmogorov complexity.

5.5.2. Kolmogorov-Sinai Generator Theorem. A proof of the classical generator theorem can be found in [Gla03, Theorem 14.33]. We will now sketch a slight adaption of this proof for general amenable groups for the reader's convenience. Note that any amenable group Γ admits a Følner sequence $(F_n)_{n\geq 1}$ such that

a) $\Gamma = \bigcup_{n\geq 1} F_n$;

b) the sequence $(F_n)_{n\geq 1}$ is monotone increasing.

We fix this Følner sequence and use the independence of the entropies on the choice of a Følner sequence. Furthermore, it is easy to see that if $(F_n)_{n\geq 1}$ is a Følner sequence and $E \subseteq \Gamma$ is a finite set, then $(EF_n)_{n\geq 1}$ is a Følner sequence as well.

First of all, we state the following proposition. The proof for arbitrary amenable groups is a straightforward modification of [Gla03, Proposition 14.22].

PROPOSITION 5.5.1. *Let $\mathbf{X} = (\mathrm{X}, \pi) \in \mathsf{Prob}_\Gamma$, where Γ is an arbitrary amenable group. For two finite partitions α, β of X we have*

$$h_{\mathsf{Prob}}(\alpha, \pi) \leq h_{\mathsf{Prob}}(\beta, \pi) + H_{\mathrm{Sh}}(\alpha|\beta).$$

Here $H_{\mathrm{Sh}}(\alpha|\beta) \in \mathbb{R}_{\geq 0}$ is the entropy of the partition α relative to the partition β. Intuitively speaking, it is a measure of the amount of information we obtain from observing α, once we know the outcome of β. Furthermore, one can show [Gla03, Proposition 14.16] that

$$H_{\mathrm{Sh}}(\alpha|\beta) = H_{\mathrm{Sh}}(\alpha \vee \beta) - H_{\mathrm{Sh}}(\beta).$$

If β is finer than α, then $H_{\mathrm{Sh}}(\alpha|\beta) = 0$. In general, one can define the conditional entropy $H_{\mathrm{Sh}}(\alpha|\mathcal{C})$ of a partition α relative to a subalgebra $\mathcal{C} \subseteq \mathcal{B}$. Then, if α is $\mathcal{C}$-measurable, $H_{\mathrm{Sh}}(\alpha|\mathcal{C}) = 0$ (see [Gla03, Proposition 14.18]). We refer to [Gla03, Chapter 14, Section 2] and [Dow11] for a rigorous treatment of the relative Kolmogorov-Sinai entropy.

We are now able to sketch the proof of the generator theorem.

PROPOSITION 5.5.2. *If α is a generator for the dynamical system $\mathbf{X}$ then*

$$h_{\mathsf{Prob}}(\mathbf{X}) = h_{\mathsf{Prob}}(\alpha, \pi).$$

PROOF. For every $k \geq 1$

$$\begin{aligned} h_{\mathsf{Prob}}(\alpha^{F_k}, \pi) &= \lim_{n\to\infty} \frac{H_{\mathrm{Sh}}\left(\bigvee_{g\in F_n} g^{-1} \bigvee_{h\in F_k} h^{-1}\alpha\right)}{|F_n|} = \lim_{n\to\infty} \frac{H_{\mathrm{Sh}}\left(\bigvee_{g\in F_n, h\in F_k} (hg)^{-1}\alpha\right)}{|F_n|} = \\ &= \lim_{n\to\infty} \frac{H_{\mathrm{Sh}}\left(\bigvee_{f\in F_k F_n} f^{-1}\alpha\right)}{|F_n|} = \lim_{n\to\infty} \frac{H_{\mathrm{Sh}}\left(\bigvee_{f\in F_k F_n} f^{-1}\alpha\right)}{|F_k F_n|} = h_{\mathsf{Prob}}(\alpha, \pi), \end{aligned}$$

where in the last equality we use the independence of $h_{\mathsf{Prob}}(\alpha, \pi)$ on the choice of the Følner sequence once again. Let β be an arbitrary finite partition. Then, for every $n \geq 1$, we have by Proposition 5.5.1

$$h_{\mathsf{Prob}}(\beta, \pi) \leq h_{\mathsf{Prob}}(\alpha^{F_n}, \pi) + H_{\mathrm{Sh}}(\beta|\alpha^{F_n}) = h_{\mathsf{Prob}}(\alpha, \pi) + H_{\mathrm{Sh}}(\beta|\alpha^{F_n})$$

As n tends to infinity, $H_{\mathrm{Sh}}(\beta|\alpha^{F_n})$ tends to zero because α is a generator (see [Gla03, Theorem 14.28]). $\square$

5.5.3. Entropy of Subshifts and Dimension. A classical result of H. Furstenberg [Fur67] asserts that the topological entropy of a subshift $\mathrm{X} \subseteq \Lambda^{\mathbb{Z}}$ equals the Hausdorff dimension of the subshift X as a metric space. A generalization of this result to $\mathbb{Z}^d$-subshifts was considered in [Sim15]. We will now briefly explain this result.

Let X be a metric space. The s-dimensional Hausdorff measure of X is defined by

$$\mu_s(\mathrm{X}) := \lim_{\varepsilon \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^s,$$

where $\operatorname{diam}(U)$ is the diameter of U. $\mathcal{U}$ here runs over covers of X of diameter less or equal to ε. The Hausdorff dimension of X is

$$\dim(\mathrm{X}) := \inf\{s \geq 0 : \mu_s(\mathrm{X}) = 0\}.$$

Given a subshift $\mathrm{X} \subseteq \Lambda^{\mathbb{Z}^d}$ where the group $\mathbb{Z}^d$ is endowed with the standard Følner sequence $(F_n)_{n \geq 1}$, we define a metric ρ on X by setting, for all $x, y \in \mathrm{X}$, $\rho(x, y) := 2^{-|F_n|}$ where $n \geq 1$ is the largest integer such that $x|_{F_n} = y|_{F_n}$. After these preparations, we can now state

Proposition 5.5.3. *Let* $\mathrm{X} \subseteq \Lambda^{\mathbb{Z}^d}$ *be a subshift, endowed with the metric defined above. Then*

$$h_{\mathsf{Top}}(\mathbf{X}) = \dim(\mathrm{X}).$$

We refer to [Sim15] for more details.

5.5.4. Sofic Entropy. We have mentioned in Section 1.5.4 that not all countable groups are amenable, and that finitely generated free groups F_n give a counterexample. The entropy theory that we have developed so far does not carry over to the actions of such groups. A substitute is the sofic entropy theory, which gives entropy for sofic group actions. The measure-theoretic sofic entropy is developed in [Bow10], and the topological sofic entropy is developed in [KL11]. When the acting group is amenable, the sofic measure-preserving entropy coincides with the classical 'amenable' Kolmogorov-Sinai entropy. However, in general sofic entropy does not enjoy some of the useful properties such the monotonicity when passing to factors.

CHAPTER 6

Measurement Functors and Palm Entropy

In the previous chapters we have discussed the topological and the Kolmogorov-Sinai entropies for amenable group actions. A natural question is whether there is a common generalization of these theories. Such a generalization for $\mathbb{Z}$-systems was indeed discovered by Günther Palm in his PhD dissertation. The goal of his research was finding a generalization of these theories in the language of functional analysis.

We review his approach briefly. To every topological dynamical system on a topological space K one can associate the corresponding Koopman representation on the Banach lattice $\mathrm{C}(\mathrm{K})$ of continuous functions. Similarly, to every measure-preserving dynamical system on a probability space X one can associate the corresponding Koopman representation on the Banach lattice $\mathrm{L}^1(\mathrm{X})$ of integrable functions. These classical lattices are nowadays called *Banach lattices of observables.* Taking the set of closed Banach lattice ideals of these lattices, one retrieves the collection of closed and measurable sets respectively, and also the dynamics on the underlying spaces by using some appropriate duality theorems. Both types of lattices are examples of Banach lattices with quasi-interior points (see Section 6.4.1 for the definition). It is well-known that the set of closed ideals of a Banach lattice is in fact a distributive lattice under some natural lattice operations. It is easy to see that an action of $\mathbb{Z}$ on a Banach lattice with quasi-interior point induces an action on its distributive lattice of closed ideals. For this action, Palm introduced a concept of entropy that coincides with the classical notions of entropy when the underlying Banach lattices with quasi-interior points are the classical lattices of observables. For the details we refer to [Pal76]. In this work, however, we significantly deviate from the original approach for the following reasons.

Firstly, we have not seen much usage of Banach lattices with quasi-interior points in the theory of dynamical systems so far. It is not clear why these lattices should be the proper setup for a 'very general entropy theory', since they appear to be a pure functional analysis phenomenon. Unfortunately, representation results such as the Kakutani representation theorem [Sch74] do not answer all the relevant questions. For instance, the entropy is defined by looking at the induced action on the distributive lattice of closed lattice ideals. However, the structure of this lattice of ideals is in general not very well described through representation results. Thus the structure of the lattices of open (measurable) sets becomes substantially more obscure when we pass to the lattice of observables.

Secondly, in both topological and Kolmogorov-Sinai entropy theories, the entropy of a system is greater or equal than the entropy of its factors, i.e., entropy decreases along the arrows of the category. However, we were not able to prove this statement in the setting of Palm's theory - there is no notion of a factor to start with - and there is a 'structural' counterexample at the level of distributive lattices that we shall provide at the end of this chapter. Hence, the structure of an abstract distributive lattice is not sufficient, and the additional structure that we see when considering lattices of closed ideals of Banach lattices with quasi-interior points is obscure. Therefore, we chose to shift the focus away from Banach lattices and search for a way to impose additional structure on lattices via an explicit 'functorial' correspondence instead.

Finally, the theories of entropy for general discrete amenable group actions based on the Ornstein-Weiss lemma were not even developed at the time of Palm's work. These considerations have lead to the abstract approach described in this chapter. The chapter is based on [Mor15a].

6.1. Category of Measured Lattices with Localization

6.1.1. Definitions. For convenience we use the term **distributive lattice** for distributive lattices with 0 ('bottom') and 1 ('top') such that $0 \neq 1$. The set of all finite subsets of a set X is denoted by $\mathcal{P}_0(\mathrm{X})$. When talking about lattice embeddings of distributive lattices, we assume that these embeddings respect the top and bottom elements. Let V be a distributive lattice. A finite subset $\alpha \in \mathcal{P}_0(V)$ is called a **cover** if $\sup \alpha = 1$. Clearly, lattice embeddings map covers to covers. The set of all covers of V is denoted by Cov_V. The set of all distributive sublattices of V is denoted by $\mathrm{Lat}(V)$. The set of all distributive sublattices of V containing a given cover α is denoted by $\mathrm{Lat}_\alpha(V)$.

We will now prepare the key ingredients for the definition of the category of measured lattices with localization. A function $m : V \to \mathbb{R}_{\geq 0}$ is called a **measurement function** if it satisfies the following conditions:

a) $m(0) = 0, m(1) \neq 0$;
b) $m(a) = 0 \Rightarrow m(a \vee b) = m(b)$ for all $b \in V$.

Measurement functions tell us how big or 'likely' the elements of V are. A function $\Omega : \mathrm{Cov}_V \to \mathrm{Lat}(V)$ is called a **localization function** if for every cover $\alpha \in \mathrm{Cov}_V$ the sublattice $\Omega(\alpha) \subseteq V$ contains α. Later we will think of $\Omega(\alpha)$ as 'the smallest subsystem which realizes α'.

The category ML of **measured lattices with localization** is defined as follows. Objects of the category ML are all triples (V, m, Ω), where

a) V is a distributive lattice;
b) $m : V \to \mathbb{R}_{\geq 0}$ is a measurement function;
c) $\Omega : \mathrm{Cov}_V \to \mathrm{Lat}(V)$ is a localization function.

To complete the definition of the category of measured lattices with localization we also need to describe the arrows. Let $\mathrm{V} = (V, m_1, \Omega_1), \mathrm{W} = (W, m_2, \Omega_2)$ be a pair of distributive lattices with localization. Let $\Phi : W \to V$ be a lattice embedding such that

a) Φ preserves the measurement function, i.e. $m_1(\Phi(a)) = m_2(a)$ for all elements $a \in W$;
b) Φ preserves the localization function, i.e. $\Omega_1(\Phi(\alpha)) = \Phi(\Omega_2(\alpha))$ for all covers $\alpha \in \mathrm{Cov}_W$.

Then we call Φ^{op} a morphism between V and W and write $\mathrm{Hom}(\mathrm{V}, \mathrm{W})$ for the collection of all morphisms obtained this way. The superscript 'op' in Φ^{op} above indicates that even though Φ is a mapping from W to V, the morphism of lattices determined by Φ points in the opposite direction. We will typically define morphisms by defining corresponding lattice embeddings, so we write 'op' to avoid confusion about direction of morphism. We have chosen to 'switch the arrows' so that the arrows in ML have the same direction as the arrows in some category C that is being 'represented' on ML (see Section 6.2). Note that all morphisms in ML are epimorphisms.

Let $W \subseteq V$ be a sublattice of a measured distributive lattice with localization (V, m, Ω). We call W a **local sublattice** of the lattice V if $\Omega(\mathrm{Cov}_W) \subseteq \mathrm{Lat}(W)$, that is, if for every cover $\alpha \in \mathrm{Cov}_W$ the corresponding localization $\Omega(\alpha)$ is actually a sublattice of $W \subseteq V$. Hence if $W \subseteq V$ is a local sublattice, we obtain a measured lattice with localization (W, m, Ω) from (V, m, Ω) by restricting the functions m and Ω. We denote by LocLat(V) the set of local sublattices of $\mathrm{V} = (V, m, \Omega)$, and by $\mathrm{LocLat}_\alpha(\mathrm{V})$ the set of local sublattices containing a cover $\alpha \in \mathrm{Cov}_V$. Both of these sets are nonempty, since they contain at least V. Intuitively speaking, we are only interested in local sublattices of a lattice V because they correspond to 'subsystems' in the category ML. If $\mathrm{V} = (V, m, \Omega)$ is a measured lattice with localization, we will often write $\mathrm{Cov}_{\mathrm{V}}$ to denote the set Cov_V of covers of V.

We now provide the classical examples of lattices with localization. These are not the simplest examples possible, but are the most important ones for the purposes of this work. We will also return to these examples when discussing measurement functors at the end of Section 6.2.

6.1.2. Topological lattices. Consider the category Top of nonempty compact Hausdorff spaces with surjective continuous maps as morphisms. Objects of this category are the pairs $\mathrm{X} = (X, \mathcal{U})$, where $\mathcal{U}$ is a compact Hausdorff topology on a set X. The nonempty collection of open sets $\mathcal{U}$ is a distributive lattice under the operations of set union and intersection. The measured lattice with localization $(\mathcal{U}, m, \Omega)$ associated with X can now be introduced. The measurement function m is defined by

$$m(A) := \begin{cases} 1 & \text{if } A \text{ is a nonempty;} \\ 0 & \text{otherwise.} \end{cases}$$

The localization function Ω maps an open cover $\alpha \in \mathrm{Cov}_{\mathcal{U}}$ to the smallest topology generated by α. We denote by $\overline{\mathrm{X}}$ the lattice $(\mathcal{U}, m, \Omega)$ obtained this way. It is easy to see that every morphism $\phi : \mathrm{X} \to \mathrm{Y}$ in Top induces a morphism $\overline{\phi} : \overline{\mathrm{X}} \to \overline{\mathrm{Y}}$ by taking preimages of open sets. Hence the correspondence $\mathrm{X} \mapsto \overline{\mathrm{X}}$ is in fact a covariant functor $\mathrm{M}^0_{\mathsf{Top}}$ between Top and ML.

6.1.3. Measure lattices. Now we consider the category Prob of standard probability spaces with equivalence classes of measure-preserving maps as morphisms. Objects of this category are triples $\mathrm{X} = (X, \mathcal{B}, \mu)$, and $\mathcal{M} := \Sigma(\mathrm{X})$ is the measure algebra of a standard probability space X. The nonempty collection of equivalence classes of measurable sets $\mathcal{M}$ is a distributive lattice under operations of union and intersection. We introduce the measured lattice with localization $(\mathcal{M}, m, \Omega)$, associated with X. The measurement function is defined by $m(a) := \mu(a)$ for all $a \in \mathcal{M}$. The localization function Ω takes a cover $\alpha \in \mathrm{Cov}_{\mathcal{M}}$ and maps it to the smallest σ-complete Boolean algebra containing α. We denote by $\overline{\mathrm{X}}$ the lattice $(\mathcal{M}, m, \Omega)$ obtained this way. Every morphism $\phi : \mathrm{X} \to \mathrm{Y}$ in Prob induces a morphism $\overline{\phi} : \overline{\mathrm{X}} \to \overline{\mathrm{Y}}$ by taking preimages of measurable sets. Hence the correspondence $\mathrm{X} \mapsto \overline{\mathrm{X}}$ is in fact a covariant functor $\mathrm{M}^0_{\mathsf{Prob}}$ between Prob and ML.

6.1.4. Representations on the Category of Lattices with Localization. Now consider the associated category of representations ML_Γ, and call it the category of **abstract dynamical lattices**. The objects of ML_Γ are pairs (V, π) of a measured distributive lattice with localization $\mathrm{V} = (V, m, \Omega)$ and a representation π of Γ in $\mathrm{Aut}(\mathrm{V})$. To simplify the notation we will often write $(V, m, \Omega; \pi)$ in place of $((V, m, \Omega), \pi)$ to denote abstract dynamical lattices. Given $(\mathrm{V}, \pi) \in \mathrm{Obj}(\mathsf{ML}_\Gamma)$ we write $\mathrm{LocLat}(\mathrm{V}, \pi)$ to denote the set of all Γ-invariant local sublattices W of V, and similarly we write $\mathrm{LocLat}_\alpha(\mathrm{V}, \pi)$ to denote the set of all local Γ-invariant sublattices of V containing a cover $\alpha \in \mathrm{Cov}_V$. Every Γ-invariant local sublattice $W \subseteq V$ yields an abstract dynamical lattice $\mathrm{W} = (W, m, \Omega; \rho)$, where we take ρ to be the restriction of the representation π to W. If $\imath : W \to V$ is the inclusion mapping for lattices W and V as above, then $\imath^{\mathrm{op}} \in \mathrm{Hom}_\Gamma(\mathrm{V}, \mathrm{W})$ is a factor map. Conversely, every factor W of V corresponds to a Γ-invariant local sublattice of V. We will now make this relation between sublattices and factors more precise by introducing some language from category theory.

The sets of Γ-invariant local sublattices $\mathrm{LocLat}(\mathrm{V}, \pi)$ and $\mathrm{LocLat}_\alpha(\mathrm{V}, \pi)$ have canonical structures of small categories. We will now explain this structure for $\mathrm{LocLat}_\alpha(\mathrm{V}, \pi)$ and define the category $\mathsf{Fac}_\alpha(\mathrm{V}, \pi)$ (with obvious modifications in the definition of $\mathsf{Fac}(\mathrm{V}, \pi)$). The set of objects $\mathrm{Obj}(\mathsf{Fac}_\alpha(\mathrm{V}, \pi))$ is by definition $\mathrm{LocLat}_\alpha(\mathrm{V}, \pi)$. For two objects $\mathbf{X}, \mathbf{Y} \in \mathrm{Obj}(\mathsf{Fac}_\alpha(\mathrm{V}, \pi))$ with corresponding Γ-invariant local sublattices $X, Y \subseteq V$ we set

$$\mathrm{Hom}(\mathbf{X}, \mathbf{Y}) := \begin{cases} \imath^{\mathrm{op}} & \text{if } Y \overset{\imath}{\subseteq} X \text{ is a sublattice} \\ \varnothing & \text{otherwise.} \end{cases}$$

So in particular for any two objects $\mathbf{X}, \mathbf{Y} \in \mathrm{Obj}(\mathsf{Fac}_\alpha(\mathrm{V}, \pi))$ the set of morphisms consists of at most one arrow, and it might be empty. Thus a final object in a subcategory D of $\mathsf{Fac}_\alpha(\mathrm{V}, \pi)$ is a Γ-invariant local sublattice of V containing α that is contained in any other lattice in D. Furthermore, there can be only one final object. Category $\mathsf{Fac}_\alpha(\mathrm{V}, \pi)$ is a full subcategory of $\mathsf{Fac}(\mathrm{V}, \pi)$. Finally, it is easy to see that $\mathsf{Fac}(\mathrm{V}, \pi)$ that we have just defined is in fact the category of factors of (V, π) in the sense of Section 2.3.

6.1.5. Products and Coproducts of Local Sublattices. Let (V, π) be an abstract dynamical lattice and $\mathsf{Fac}(\mathrm{V}, \pi)$ be the category of its Γ-invariant local sublattices. Firstly, we have a look at the products in $\mathsf{Fac}(\mathrm{V}, \pi)$. Let X, Y be Γ-invariant local sublattices of V. Let

$$\mathrm{Z} := \bigcap\{\mathrm{W} : \mathrm{W} \text{ is a local invariant sublattice of V containing X and Y}\}.$$

An intersection of any family of invariant local sublattices is local and invariant as well, and checking the universal property is straightforward. Hence $\mathrm{Z} = \mathrm{X} \prod \mathrm{Y}$. Secondly, we want to understand the coproducts of sublattices. Let $\{\mathrm{X}_\lambda\}_{\lambda \in \Lambda}$ be a family of local invariant sublattices of V. Let

$$\mathrm{W} := \bigcap_{\lambda \in \Lambda} \mathrm{X}_\lambda.$$

Once again W is a local invariant sublattice and the universal property is also satisfied. Hence $\mathrm{W} = \coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda$.

6.2. Measurement Functors

Now we can introduce the notion of a measurement functor. Let C be a category and let C_Γ be the associated category of representations. A covariant functor $\mathrm{M} : \mathsf{C}_\Gamma \to \mathsf{ML}_\Gamma$ is called a **measurement functor** if it satisfies the following condition for every object $(\mathrm{A}, \pi) \in \mathrm{Obj}(\mathsf{C}_\Gamma)$:

> If $\mathrm{M}(\mathrm{A}, \pi) = (\overline{\mathrm{A}}, \overline{\pi}) \in \mathrm{Obj}(\mathsf{ML}_\Gamma)$, then for every cover $\alpha \in \mathrm{Cov}_{\overline{\mathrm{A}}}$ the corresponding localization $\Omega(\alpha)$ within the lattice $\overline{\mathrm{A}}$ is precisely the minimal sublattice containing α that can be realized by the functor.

We explain what these requirements mean. The objects of the category C_Γ are pairs (A, π), where $\mathrm{A} \in \mathrm{Obj}(\mathsf{C})$ and $\pi : \Gamma \to \mathrm{Aut}(\mathrm{A})$ is a group homomorphism. So, for every object $(\mathrm{A}, \pi) \in \mathrm{Obj}(\mathsf{C}_\Gamma)$ applying the functor M yields a dynamical lattice

$$\mathrm{M}(\mathrm{A}, \pi) = (\overline{\mathrm{A}}, \overline{\pi}) \in \mathrm{Obj}(\mathsf{ML}_\Gamma).$$

Here $\overline{\mathrm{A}} \in \mathrm{Obj}(\mathsf{ML})$ is a measured lattice with localization, and $\overline{\pi}$ is a representation of Γ in the group $\mathrm{Aut}(\overline{\mathrm{A}})$.

Morphisms in $\mathrm{Hom}((\mathrm{A}, \pi), (\mathrm{B}, \rho))$ are those morphisms ψ in $\mathrm{Hom}(\mathrm{A}, \mathrm{B})$ that satisfy $\psi \circ \pi_\gamma = \rho_\gamma \circ \psi$ for all $\gamma \in \Gamma$. For every such $\psi \in \mathrm{Hom}_\Gamma(\mathrm{A}, \mathrm{B})$ the functor M defines a Γ-equivariant morphism $\mathrm{M}(\psi) : \overline{\mathrm{A}} \to \overline{\mathrm{B}}$. This means that $\mathrm{M}(\psi) \circ \overline{\pi}_\gamma = \overline{\rho}_\gamma \circ M(\psi)$ for every $\gamma \in \Gamma$, i.e. the diagram

$$\begin{tikzcd} \overline{\mathrm{A}} \arrow[r, "\mathrm{M}(\psi)"] & \overline{\mathrm{B}} \\ \overline{\mathrm{A}} \arrow[u, "\overline{\pi}_\gamma"] \arrow[r, "\mathrm{M}(\psi)"'] & \overline{\mathrm{B}} \arrow[u, "\overline{\rho}_\gamma"'] \end{tikzcd}$$

commutes.

Now we explain the main condition. For that we need to introduce some language first. Let M be a covariant functor as above, $(\mathrm{A}, \pi) \in \mathrm{Obj}(\mathsf{C}_\Gamma)$, $\mathrm{M}(\mathrm{A}, \pi) = (\overline{\mathrm{A}}, \overline{\pi})$ and let $\alpha \in \mathrm{Cov}_{\overline{\mathrm{A}}}$ be a cover. Then $\overline{\mathrm{A}}$ is a measured lattice with localization, and $\mathsf{Fac}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$ is the associated category of Γ-invariant local sublattices of $\overline{\mathrm{A}}$ containing α. Some of these sublattices are coming from factors of the system (A, π) via applying M - for example $\overline{\mathrm{A}}$ itself - while others may not. To make this precise, consider the categories $\mathsf{Fac}(\overline{\mathrm{A}}, \overline{\pi})$, $\mathsf{Fac}(\mathrm{A}, \pi)$ of factors of $\overline{\mathrm{A}}$ and A respectively. Observe that the functor M induces a functor $\widetilde{\mathrm{M}} : \mathsf{Fac}(\mathrm{A}, \pi) \to \mathsf{Fac}(\overline{\mathrm{A}}, \overline{\pi})$ in the following manner. Given an epimorphism $\phi : \mathrm{A} \to \mathrm{B}$ in C_Γ, it is mapped to the equivalence class of epimorphism $[\mathrm{M}(\phi)]$ in $\mathsf{Fac}(\overline{\mathrm{A}}, \overline{\pi})$, and this definition is independent of the representative of ϕ in $[\phi] \in \mathsf{Fac}(\mathrm{A}, \pi)$. In this definition we are only using that the functor M preserves commutative diagrams, in particular, that it preserves isomorphisms. We define the subcategory

$$\mathsf{SpFac}^{\mathrm{M}}(\overline{\mathrm{A}}, \overline{\pi}) \subseteq \mathsf{Fac}(\overline{\mathrm{A}}, \overline{\pi})$$

of **spacial sublattices** w.r.t. the functor M as the range of $\widetilde{\mathrm{M}}$. In general this subcategory is not full, i.e. there can be Γ-equivariant lattice embeddings that are not spacial. Since the category $\mathsf{Fac}(\overline{\mathrm{A}}, \overline{\pi})$ is in fact the category of local Γ-invariant sublattices of $\overline{\mathrm{A}}$ (via the identification that was explained at the end of Section 6.1.4), we can talk about the subcategory of spacial sublattices

$$\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi}) \subseteq \mathsf{SpFac}^{\mathrm{M}}(\overline{\mathrm{A}}, \overline{\pi})$$

containing the cover α, which is a full subcategory of $\mathsf{SpFac}^{\mathrm{M}}(\overline{\mathrm{A}}, \overline{\pi})$.

This language allows us to explain what the main requirement says. Namely, for every object $(\mathrm{A}, \pi) \in \mathrm{Obj}(\mathsf{C}_\Gamma)$ we consider its image $\mathrm{M}(\mathrm{A}, \pi) = (\overline{\mathrm{A}}, \overline{\pi})$, where $\overline{\mathrm{A}} = (A, m, \Omega)$ is a measured lattice with localization, and require that for every cover $\alpha \in \mathrm{Cov}_{\overline{\mathrm{A}}}$ the lattice $\Omega(\alpha)$ is the final object in the category $\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$.

Hence $\Omega(\alpha)$ is the spacial sublattice canonically embedded in any other spacial sublattice containing α via a spacial (i.e. corresponding to a factor in C_Γ) morphism. In particular, this requirement shows that $\Omega(\alpha)$ does depend on the representation π, since it is Γ-invariant w.r.t. $\overline{\pi}$.

Examples 6.2.1. We will now provide some trivial examples of spacial and non-spacial sublattices. For simplicity we will take Γ to be trivial and work in the category Prob. Let $\mathrm{X} = (\mathrm{X}, \mathcal{M}, \mu)$ be the probability space on the set

$\mathrm{X} := \{a, b, c\}$ with $\mathcal{M} := \mathcal{P}(\mathrm{X})$ being the Boolean algebra of all subsets of X and μ be the uniform probability distribution. We consider the measured lattice with localization $\overline{\mathrm{X}}$ associated to X that was given in Section 6.1.3. Consider the sublattice $V := \{0, \{a, b\}, \{b\}, \{b, c\}, 1\}$ of $\mathcal{M}$. Then V is not a spacial sublattice because it is not a Boolean algebra. Conversely, any subalgebra of $\mathcal{M}$ defines a factor of X in Prob and thus is an example of a spacial sublattice.

Given a covariant functor $\mathrm{M} : \mathsf{C}_\Gamma \to \mathsf{ML}_\Gamma$, the main condition in the definition of a measurement functor is rather nontrivial, and it is not clear whether the final object exists in the category $\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$. When it exists, we know that it is unique. We now provide an abstract condition that simplifies the verification of the second requirement.

Proposition 6.2.2 (Coproduct stability condition). *Let* $\mathrm{M} : \mathsf{C}_\Gamma \to \mathsf{ML}_\Gamma$ *be a covariant functor. Let* $(\mathrm{A}, \pi) \in \mathsf{C}_\Gamma$, $(\overline{\mathrm{A}}, \overline{\pi}) = \mathrm{M}(\mathrm{A}, \pi)$, *and let* $\alpha \in \mathrm{Cov}_{\overline{\mathrm{A}}}$. *Suppose the following holds:*

1) *the category* $\mathsf{Fac}(\mathrm{A}, \pi)$ *is small and admits infinite coproducts;*
2) *if* $\{\mathrm{X}_\lambda\}_{\lambda \in \Lambda}$ *is a collection of objects in* $\mathsf{Fac}(\mathrm{A}, \pi)$ *such that for every index* $\lambda \in \Lambda$ *the object* $\widetilde{\mathrm{M}}(\mathrm{X}_\lambda)$ *is in* $\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$, *then the image of the coproduct* $\widetilde{\mathrm{M}}(\coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda)$ *is in* $\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$ *as well.*

Then $\widetilde{\mathrm{M}}(\coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda)$ *is the final object in category* $\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$, *where* $\{\mathrm{X}_\lambda\}_{\lambda \in \Lambda}$ *is the set of all objects in* $\mathsf{Fac}(\mathrm{A}, \pi)$ *such that* $\widetilde{\mathrm{M}}(\mathrm{X}_\lambda)$ *is in* $\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$ *for all indices* λ.

Proof. The second condition implies that $\widetilde{\mathrm{M}}(\coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda)$ is indeed in $\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$, i.e. it is a Γ-invariant spacial local sublattice of $\overline{\mathrm{A}}$, containing α.

To show that $\widetilde{\mathrm{M}}(\coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda)$ is the final object in $\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$, we need to show that for every $\mathrm{Z} \in \mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}}, \overline{\pi})$ there is a spacial morphism $\psi : \mathrm{Z} \to \widetilde{\mathrm{M}}(\coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda)$. By the definition of the category of spacial sublattices, we have $\mathrm{Z} = \widetilde{\mathrm{M}}(\mathrm{X}_0)$ for some factor X_0 of A. Then X_0 belongs to the collection of factors $\{\mathrm{X}_\lambda\}_{\lambda \in \Lambda}$ from the statement of the theorem, and so there is a morphism $\psi_0 : \mathrm{X}_0 \to \coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda$ in category $\mathsf{Fac}(\mathrm{A}, \pi)$. Then $\widetilde{\mathrm{M}}(\psi_0) : \mathrm{Z} \to \widetilde{\mathrm{M}}(\coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda)$ is the spacial morphism as required.

In the language of lattices this implies that $\widetilde{\mathrm{M}}(\coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda)$ is a spacial sublattice containing α, and it is embedded in any other spacial sublattice with this property. □

This proposition has the following corollary, whose proof is straightforward.

Corollary 6.2.3. *Let* $\mathrm{M} : \mathsf{C}_\Gamma \to \mathsf{ML}_\Gamma$ *be a covariant functor satisfying the conditions of Proposition 6.2.2. Then* M *is a measurement functor if and* *only if for every* $(\mathrm{A},\pi) \in \mathrm{Obj}(\mathsf{C}_\Gamma)$ *with the corresponding abstract dynamical lattice* $(\overline{\mathrm{A}},\overline{\pi}) = \mathrm{M}(\mathrm{A},\pi)$ *and the localization function* Ω *we have for every cover* $\alpha \in \mathrm{Cov}_{\overline{\mathrm{A}}}$ *that* $\Omega(\alpha) = \widetilde{\mathrm{M}}(\coprod_{\lambda\in\Lambda} \mathrm{X}_\lambda)$*, where* $\{\mathrm{X}_\lambda\}_{\lambda\in\Lambda}$ *is the set of all objects in* $\mathsf{Fac}(\mathrm{A},\pi)$ *such that* $\widetilde{\mathrm{M}}(\mathrm{X}_\lambda)$ *is in* $\mathsf{SpFac}^{\mathrm{M}}_\alpha(\overline{\mathrm{A}},\overline{\pi})$ *for all indices* λ*.*

There is always a trivial measurement functor $\mathrm{M}_0 : \mathsf{C}_\Gamma \to \mathsf{ML}_\Gamma$ that maps a representation (A,π) of Γ to a trivial representation $(\mathrm{V}_0,\mathbf{id})$ of Γ. Here $\mathrm{V}_0 = (V,m,\Omega)$ is a measured lattice with localization with $V = \{0,1\}$, $m(1) = 1, m(0) = 0$ and $\Omega(\{0,1\}) = V$. We now want to understand the measurement functors arising from topological and measure-preserving dynamical systems.

6.2.1. Topological Dynamics. Consider the category Top of compact Hausdorff spaces with surjective continuous maps as morphisms and the associated category Top_Γ, which is just the category of topological dynamical systems. Objects of Top_Γ are pairs (A,π), where A is a compact topological space and $\pi : \Gamma \to \mathrm{Aut}(\mathrm{A})$ is a homomorphism from Γ to the group of homeomorphisms of A. We will describe the construction of a measurement functor $\mathrm{M}_{\mathsf{Top}} : \mathsf{Top}_\Gamma \to \mathsf{ML}_\Gamma$ that eventually leads to the definition of topological entropy.

Given (A,π) as above, A is a nonempty topological space $\mathrm{A} = (K,\mathcal{U})$ with a compact topology $\mathcal{U}$ on a set K, define $\mathrm{M}_{\mathsf{Top}}(\mathrm{A},\pi) := (\mathcal{U},m,\Omega;\overline{\pi})$. Here $\mathcal{U}$ is the distributive lattice of open sets with 0 being the empty set and 1 being the whole space K, m takes the value 1 on all $a \in \mathcal{U}, a \neq 0$ and $m(0) = 0$. A finite cover $\alpha \in \mathrm{Cov}_{\mathcal{U}}$ is then simply a finite cover of K by open sets. We define the group homomorphism $\overline{\pi}$ by

$$\overline{\pi}_\gamma : U \mapsto \pi_\gamma^{-1} U \quad \text{for all } \ U \in \mathcal{U}, \gamma \in \Gamma.$$

For every $\alpha \in \mathrm{Cov}_{\mathcal{U}}$, we let $\Omega(\alpha)$ be the minimal Γ-invariant topology $\mathcal{V} \subseteq \mathcal{U}$ on K that contains the family of open sets α. Note that this topology is compact and has a basis of open sets

$$\{\overline{\pi}_{\gamma_1} a_1 \cap \overline{\pi}_{\gamma_2} a_2 \cap \cdots \cap \overline{\pi}_{\gamma_n} a_n\},$$

where n runs through $\mathbb{N}$, $\gamma_1,\ldots,\gamma_n$ run over Γ, $a_1,\ldots,a_n$ run over α. This topology is in general not Hausdorff. We describe the action of $\mathrm{M}_{\mathsf{Top}}$ on morphisms. Let (A,π) and (B,ρ) in Top_Γ be topological dynamical systems with $\mathrm{A} = (K,\mathcal{U}), \mathrm{B} = (L,\mathcal{V})$ such that

$$\mathrm{M}_{\mathsf{Top}}(\mathrm{A},\pi) = (\mathcal{U},m_A,\Omega_A;\overline{\pi}), \quad \mathrm{M}_{\mathsf{Top}}(\mathrm{B},\rho) = (\mathcal{V},m_B,\Omega_B;\overline{\rho}).$$

Let $\phi : \mathrm{A} \to \mathrm{B}$ be a Γ-equivariant morphism, that is a surjective continuous map $\phi : K \to L$ commuting with the action of Γ, then we define morphism $\mathrm{M}_{\mathsf{Top}}(\phi)$ as follows. Consider the distributive lattice embedding $\Phi : \mathcal{V} \to \mathcal{U}$, $a \mapsto \phi^{-1}a$ for $a \in \mathcal{V}$. Then by the surjectivity of ϕ the set $\Phi(a)$ is nonempty if and only if a is nonempty, thus $m_A(\Phi(a)) = m_B(a)$. Let $\alpha \subseteq \mathcal{V}$ be an open cover. Then $\Omega_B(\alpha)$ is the minimal Γ-invariant topology $\mathcal{V}' \subseteq \mathcal{V}$ on L that contains the family of open sets α, and we have $\Phi(\Omega_B(\alpha)) = \phi^{-1}\mathcal{V}' \subseteq \mathcal{U}$. The topology $\phi^{-1}\mathcal{V}'$ has the basis

$$\{\Phi(\overline{\rho}_{\gamma_1}a_1 \cap \overline{\rho}_{\gamma_2}a_2 \cap \cdots \cap \overline{\rho}_{\gamma_n}a_n) : n \in \mathbb{N}, \gamma_1, \ldots, \gamma_n \in \Gamma,$$
$$a_1, \ldots, a_n \in \alpha\}.$$

Similarly, $\Omega_A(\Phi(\alpha))$ is the topology with basis sets of the form

$$\overline{\pi}_{\gamma_1}(\Phi a_1) \cap \overline{\pi}_{\gamma_2}(\Phi a_2) \cap \cdots \cap \overline{\pi}_{\gamma_n}(\Phi a_n).$$

The map ϕ is Γ-equivariant, so

$$\overline{\pi}_{\gamma_1}(\Phi a_1) \cap \overline{\pi}_{\gamma_2}(\Phi a_2) \cap \cdots \cap \overline{\pi}_{\gamma_n}(\Phi a_n) =$$
$$= \Phi(\overline{\rho}_{\gamma_1}a_1 \cap \overline{\rho}_{\gamma_2}a_2 \cap \cdots \cap \overline{\rho}_{\gamma_n}a_n)$$

and hence the topologies coincide. This shows that $\Phi^{\mathrm{op}} \in \mathrm{Hom}_\Gamma((\mathrm{A}, \pi), (\mathrm{B}, \rho))$.

The fact that $\mathrm{M}_{\mathsf{Top}}(\phi \circ \psi) = \mathrm{M}_{\mathsf{Top}}(\phi) \circ \mathrm{M}_{\mathsf{Top}}(\psi)$ is also easily verified. Hence, $\mathrm{M}_{\mathsf{Top}}$ is a well-defined covariant functor.

We explain briefly how one can use Proposition 6.2.2 to verify the main condition in the definition of a measurement functor. First of all, $\mathsf{Fac}(\mathrm{A}, \pi)$ is a small category. Secondly, if one follows all the steps in the construction of the coproduct of two factors, one notes that the resulting subtopology contains a given cover α if both factors do. Thus we can apply Proposition 6.2.2 and Corollary 6.2.3, since $\Omega(\alpha)$ defined above coincides with $\widetilde{\mathrm{M}}(\coprod_{\lambda \in \Lambda} \mathrm{X}_\lambda)$ from the proposition.

6.2.2. Measure-preserving Dynamics. Let us return to the category Prob of standard probability spaces with equivalence classes of measure-preserving maps as morphisms and the associated category Prob_Γ. In what follows, a measurement functor $\mathrm{M}_{\mathsf{Prob}} : \mathsf{Prob}_\Gamma \to \mathsf{ML}_\Gamma$ is constructed that can be used to define the Kolmogorov-Sinai entropy. Given a probability-preserving dynamical system (X, π), where the standard probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$ has the measure algebra $\mathcal{M} := \Sigma(\mathrm{X})$ and the probability measure μ, define $\mathrm{M}_{\mathsf{Prob}}(\mathrm{X}, \pi) := (\mathcal{M}, m, \Omega; \overline{\pi})$. Here $\mathcal{M}$ carries the structure of a distributive lattice with 0 being the empty set and 1 being the whole space X, and $m(a) := \mu(a)$ for all $a \in \mathcal{M}$. We define group the homomorphism $\overline{\pi}$ by $\overline{\pi}_\gamma : U \mapsto \pi_\gamma^{-1}U$ for $U \in \mathcal{M}, \gamma \in \Gamma$. A finite cover $\alpha \in \mathrm{Cov}_{\mathcal{M}}$ is a finite cover of X by measurable sets modulo null sets, so for $\Omega_{\mathcal{M}}(\alpha)$ we take the minimal Γ-invariant σ-complete Boolean subalgebra of $\mathcal{M}$ on X that contains α. Note that this subalgebra equals the closure $\mathrm{cl}(\mathcal{A}_0)$, for $\mathcal{A}_0$ being the algebra of unions of sets of the form

$$\bigcap_{i=1}^{n} \overline{\pi}_{\gamma_i}a_i \cap \bigcap_{i=n+1}^{n+m} \overline{\pi}_{\gamma_i}a_i^c,$$

where $n, m \in \mathbb{N}, \gamma_1, \ldots, \gamma_{n+m} \in \Gamma, a_1, \ldots, a_{n+m} \in \alpha$.

Let $(\mathrm{X}, \pi), (\mathrm{Y}, \rho) \in \mathsf{Prob}_\Gamma$ be probabilistic dynamical systems such that

$$\mathrm{M}_{\mathsf{Prob}}(\mathrm{X}, \pi) = (\mathcal{A}, m_A, \Omega_A; \overline{\pi}), \quad \mathrm{M}_{\mathsf{Prob}}(\mathrm{Y}, \rho) = (\mathcal{B}, m_B, \Omega_B; \overline{\rho}).$$

Let $\phi : \mathrm{X} \to \mathrm{Y}$ be a Γ-equivariant morphism, then we define the morphism $\mathrm{M}_{\mathsf{Prob}}(\phi)$ as follows. Consider the measure algebra embedding $\Phi : \mathcal{B} \to \mathcal{A}$, $a \mapsto \phi^{-1}a$ for $a \in \mathcal{B}$. Then $m_A(\Phi(a)) = m_B(a)$, since ϕ is measure-preserving. For a finite cover $\alpha \in \mathrm{Cov}_{\mathcal{B}}$ the Boolean algebra $\Phi\Omega_B(\alpha)$ equals $\Phi\mathrm{cl}(\mathcal{B}_0)$. Here $\mathcal{B}_0$ is the algebra of unions of sets of the form $\bigcap_{i=1}^{n} \overline{\pi}_{\gamma_i} a_i \cap \bigcap_{i=n+1}^{n+m} \overline{\pi}_{\gamma_i} a_i^c$, where $n, m \in \mathbb{N}, \gamma_1, \ldots, \gamma_{n+m} \in \Gamma, a_1, \ldots, a_{n+m} \in \alpha$. But $\mathrm{cl}(\Phi\mathcal{B}_0) = \Phi\mathrm{cl}(\mathcal{B}_0)$ because Φ is an isometry; so $\Phi\mathrm{cl}(\mathcal{B}_0)$ coincides with the closure of the algebra $\Phi\mathcal{B}_0$, which is just $\Omega_A(\Phi\alpha)$.

It is also easy to see that the functor $\mathrm{M}_{\mathsf{Prob}}$ respects composition of morphisms, thus it is a covariant functor. The main condition in the definition of a measurement functor follows from a similar application of Proposition 6.2.2 and Corollary 6.2.3.

6.3. Palm Entropy

6.3.1. Entropy of Abstract Dynamical Lattices. In this subsection we let

$$\mathrm{V} = (V, m, \Omega; \pi)$$

be an arbitrary abstract dynamical lattice in ML_Γ with a representation π of a discrete amenable group Γ. Since morphisms in ML are defined as opposites of the corresponding lattice embeddings (Section 6.1.1), the representation π of Γ determines (canonically) a left action of Γ^{op} on V by lattice embeddings. Also, the representation π is fixed throughout the largest part of this subsection, hence we suppress π in the notation when possible and write $gx, g \in \Gamma, x \in V$ for the action of Γ^{op}. In particular, we have the identity $(fg)x = g(fx)$ for all $f, g \in \Gamma$ and $x \in V$. For an arbitrary Γ-invariant sublattice W of V we have $\mathrm{Cov}_W \subseteq \mathrm{Cov}_V$.

Since V is a distributive lattice, it is equipped with a partial ordering relation $\leq$. To describe how covers are related to each other we need an ordering of covers as well. We cannot get a useful partial ordering relation on the set of covers in general, but there is a 'natural' relation that is *not* antisymmetric. Namely, the set Cov_V is equipped with a quasiorder relation $\succeq$ defined by

$$\beta \succeq \alpha \iff \forall\, b \in \beta \;\; \exists\, a \in \alpha \;\text{ such that } a \geq b.$$

This coincides with the definition from the theory of topological entropy, where an open cover $\mathcal{U}$ is said to be finer than an open cover $\mathcal{V}$ if for every open set $A \in \mathcal{U}$ there is an open set $B \in \mathcal{V}$ such that $A \subseteq B$. If $\alpha \succeq \beta$, we say that α is **finer** than β, or that α **refines** β.

Given covers $\alpha, \beta \in \mathrm{Cov}_W$ we define the **join** of these covers by

$$\alpha \vee \beta := \{a \wedge b : a \in \alpha, b \in \beta\} \in \mathrm{Cov}_W.$$

The binary operation $\vee$ is associative and commutative, hence we can also talk about joins of finite sets of covers. It is easy to see that

$$\alpha \vee \beta \succeq \alpha \;\; \forall \alpha, \beta \in \mathrm{Cov}_V. \tag{6.3.1}$$

Similar to the remark made above, we cannot call a join of covers a supremum, but it is easy to see that $\gamma \succeq (\alpha \vee \beta)$ if and only if $\gamma \succeq \alpha$ and $\gamma \succeq \beta$[1]. Furthermore,

$$\alpha \vee \beta \succeq \alpha \quad \forall \alpha, \beta \in \mathrm{Cov}_V. \tag{6.3.2}$$

For an element $f \in \Gamma$ and a cover $\alpha \in \mathrm{Cov}_W$ define $f\alpha := \{fx : x \in \alpha\} \in \mathrm{Cov}_W$. This yields an action of Γ^{op} on the set of covers Cov_W, hence for all $f, g \in \Gamma$ we have $(fg)\alpha = g(f\alpha)$. Given a finite subset $F \subset \Gamma$ define $\alpha^F := \bigvee_{f \in F} f\alpha \in \mathrm{Cov}_W$. It is clear that the action of Γ^{op} on Cov_W preserves the preorder relation $\succeq$. Furthermore, $\beta \succeq g^{-1}\alpha$ if and only if $g\beta \succeq \alpha$. Also, for all covers $\alpha, \beta \in \mathrm{Cov}_W$ and $g \in \Gamma$

$$\begin{aligned} g\alpha \vee g\beta &= \{ga \wedge gb : a \in \alpha, b \in \beta\} = \\ &= g\{a \wedge b : a \in \alpha, b \in \beta\} = g(\alpha \vee \beta). \end{aligned}$$

Let $\alpha \in \mathrm{Cov}_V$, and let $W \in \mathrm{Lat}_\alpha(V)$ be a Γ-invariant sublattice containing α. Our goal now is to define the entropy of a cover α relative to the sublattice W. Define the **total mass** of a cover α by

$$S(\alpha) := \sum_{a \in \alpha} m(a).$$

Since α is a cover, there exists an element $a \in \alpha$ with $m(a) > 0$, hence $S(\alpha)$ is always strictly positive. Define

$$h^*(\alpha) := -\sum_{a \in \alpha} \frac{m(a)}{S(\alpha)} \log \frac{m(a)}{S(\alpha)}. \tag{6.3.3}$$

By a standard convention, we assume that $0 \cdot \log 0 = 0$. Then $h^*(\alpha)$ is always a nonnegative real number.

For a cover $\alpha \in \mathrm{Cov}_V$ we denote the number of nonzero elements of α by

$$N(\alpha) := |\{a \in \alpha : m(a) \neq 0\}|.$$

Then $N(\alpha) \geq 1$ and $N(g\alpha) = N(\alpha)$ for every $g \in \Gamma$. Furthermore, $h^*(\alpha) \leq \log N(\alpha)$. Given a cover $\alpha \in \mathrm{Cov}_V$ and an invariant sublattice $W \in \mathrm{Lat}_\alpha(V)$ as above, we define

$$\hat{h}_W(\alpha) := \sup\{h^*(\beta) : \beta \in \mathrm{Cov}_W \text{ such that } \beta \succeq \alpha, N(\beta) \leq N(\alpha)\}. \tag{6.3.4}$$

Of course,

$$h^*(\alpha) \leq \hat{h}_W(\alpha) \leq \log N(\alpha).$$

Finally, the function $h.(\cdot)$ with values in $\mathbb{R}_{\geq 0}$ can be introduced. It will be used together with the Ornstein-Weiss Lemma to define the entropy of a

[1] It follows from this remark that $\alpha \vee \beta$ is in fact a product of α and β in a suitably defined 'quasi-order category' $(\mathrm{Cov}_V, \succeq)$.

dynamical lattice $((V, m, \Omega), \pi)$ in Proposition 6.3.2. Given a cover $\alpha \in \mathrm{Cov}_V$ and an invariant sublattice $W \in \mathrm{Lat}_\alpha(V)$ as above, let
(6.3.5)

$$h_W(\alpha) := \inf\{\sum_{j=1}^{n} \hat{h}_W(\beta_j) : n \in \mathbb{N},\ \beta_1, \ldots, \beta_n \in \mathrm{Cov}_W \text{ such that } \bigvee_{j=1}^{n} \beta_j \succeq \alpha\}.$$

Since $\alpha \in \mathrm{Cov}_W$,

$$0 \leq h_W(\alpha) \leq \hat{h}_W(\alpha) \leq \log N(\alpha),$$

thus $h_W(\alpha)$ is always a nonnegative real number. Hence we can define a function $f_{\alpha,W} : \mathcal{P}_0(\Gamma) \to \mathbb{R}_{\geq 0}$ by $f_{\alpha,W}(F) := h_W(\alpha^F)$. We are now able to prove the main proposition.

Proposition 6.3.1. *For each cover $\alpha \in \mathrm{Cov}_V$ and each invariant sublattice $W \in \mathrm{Lat}_\alpha(V)$ the function $f_{\alpha,W}$ satisfies the conditions of the Ornstein-Weiss lemma (Proposition 1.4.2).*

Proof. (i) Let us show that $f_{\alpha,W}$ is monotone, i.e. for two arbitrary finite subsets $F_1 \subseteq F_2 \subset \Gamma$ implies that $f_{\alpha,W}(F_1) \leq f_{\alpha,W}(F_2)$. Indeed, let $F_3 = F_2 \setminus F_1$, then $\alpha^{F_2} = \alpha^{F_1} \vee \alpha^{F_3} \succeq \alpha^{F_1}$ by equation ((6.3.2)). Thus

$$h_W(\alpha^{F_1}) = \inf\{\sum_{j=1}^{n} \hat{h}_W(\beta_j) : n \in \mathbb{N},\ \beta_1, \ldots, \beta_n \in \mathrm{Cov}_W \text{ such that } \bigvee_{j=1}^{n} \beta_j \succeq \alpha^{F_1}\}$$

$$\leq \inf\{\sum_{j=1}^{n} \hat{h}_W(\beta_j) : n \in \mathbb{N},\ \beta_1, \ldots, \beta_n \in \mathrm{Cov}_W \text{ such that } \bigvee_{j=1}^{n} \beta_j \succeq \alpha^{F_1} \vee \alpha^{F_3}\}$$

$$= h_W(\alpha^{F_2}).$$

(ii) Now we show that $f_{\alpha,W}$ is subadditive, i.e. $f_{\alpha,W}(F_1 \cup F_2) \leq f_{\alpha,W}(F_1) + f_{\alpha,W}(F_2)$ holds for two arbitrary finite subsets $F_1, F_2 \subset \Gamma$. For that observe that $\alpha^{F_1} \vee \alpha^{F_2} \succeq \alpha^{F_1 \cup F_2}$, hence $h_W(\alpha^{F_1 \cup F_2}) \leq h_W(\alpha^{F_1} \vee \alpha^{F_2})$ by a monotonicity argument. Thus it suffices to show that for any two $\alpha, \beta \in \mathrm{Cov}_W$ the inequality $h_W(\alpha \vee \beta) \leq h_W(\alpha) + h_W(\beta)$ holds.

Indeed, for any $k, l \in \mathbb{N}$ and sequences of covers $(\beta'_{i_2})_{i_2=1}^{k}, (\beta''_{i_3})_{i_3=1}^{l}$ in Cov_W such that $\bigvee_{i_2=1}^{k} \beta'_{i_2} \succeq \alpha$ and $\bigvee_{i_3=1}^{l} \beta''_{i_3} \succeq \beta$ the sequence of covers $\beta'_1, \beta'_2, \ldots, \beta'_k, \beta''_1, \beta''_2, \ldots, \beta''_l$ is also in Cov_W and its join refines $\alpha \vee \beta$. Hence

$$h_W(\alpha \vee \beta) \leq h_W(\alpha) + h_W(\beta).$$

(iii) Finally, we prove that $f_{\alpha,W}$ is right-invariant, i.e. $f_{\alpha,W}(Fg) = f_{\alpha,W}(F)$ holds for every finite $F \subsetneq \Gamma$ and every element $g \in \Gamma$. Observe that

$$\bigvee_{f \in Fg} f\alpha = \bigvee_{f \in F} (fg)\alpha = g\left(\bigvee_{f \in F} f\alpha\right).$$

Hence it suffices to show that $h_W(g\alpha) = h_W(\alpha)$ holds for arbitrary $\alpha \in \mathrm{Cov}_V, g \in \Gamma$ and W being arbitrary Γ-invariant sublattice containing α.

Indeed,

$$h_W(g\alpha) = \inf\{\sum_{j=1}^{n} \hat{h}_W(\beta_j) : n \in \mathbb{N},\ \beta_1, \ldots, \beta_n \in \mathrm{Cov}_W \text{ such that } \bigvee_{j=1}^{n} \beta_j \succeq g\alpha\}$$

$$= \inf\{\sum_{j=1}^{n} \hat{h}_W(g(g^{-1}\beta_j)) : n \in \mathbb{N},\ \beta_1, \ldots, \beta_n \in \mathrm{Cov}_W \text{ such that } \bigvee_{j=1}^{n} (g^{-1}\beta_j) \succeq \alpha\}$$

hence it suffices to show that $\hat{h}_W(g\alpha) = \hat{h}_W(\alpha)$ for every $\alpha \in \mathrm{Cov}_V, g \in \Gamma$ and W arbitrary Γ-invariant sublattice containing α. Indeed,

$$\begin{aligned}\hat{h}_W(g\alpha) &= \sup\{h^*(\beta) : \beta \in \mathrm{Cov}_W \text{ such that } \beta \succeq g\alpha, N(\beta) \leq N(g\alpha)\} = \\ &= \sup\{h^*(g(g^{-1}\beta)) : g^{-1}\beta \in \mathrm{Cov}_W \text{ such that } g^{-1}\beta \succeq \alpha, N(\beta) \leq N(\alpha)\},\end{aligned}$$

thus we only need to show that $h^*(g\alpha) = h^*(\alpha)$ for any $\alpha \in \mathrm{Cov}_V, g \in \Gamma$. This in turn follows from the definition of a morphism in the category ML, more precisely we use that morphisms preserves measurement function.

□

Now we are ready to define the dynamical entropy of a cover $\alpha \in \mathrm{Cov}_V$ relative to a sublattice W containing α.

Proposition 6.3.2 (Entropy of a cover relative to sublattice). *Consider an abstract dynamical lattice $(V, m, \Omega; \pi)$ with a representation π of a discrete amenable group Γ, and let $(F_n)_{n\in\mathbb{N}}$ be a Følner sequence in Γ. Then for all $\alpha \in \mathrm{Cov}_V$ and all invariant sublattices $W \in \mathrm{Lat}_\alpha(V)$ the limit*

$$\lim_{n\to\infty} \frac{h_W(\alpha^{F_n})}{|F_n|} =: h_W(\alpha, \pi) \tag{6.3.6}$$

exists, is nonnegative and is independent of the Følner sequence $(F_n)_n$.

Proof. Follows from Proposition 6.3.1 and Proposition 1.4.2. □

Now we mimic the definitions of both Kolmogorov-Sinai and topological entropies and define the entropy h_{ML} of a dynamical lattice $(V, m, \Omega; \pi)$ by

$$h_{\mathsf{ML}}(V, m, \Omega; \pi) := \sup\{h_{\Omega(\alpha)}(\alpha, \pi) : \alpha \in \mathrm{Cov}_V\}.$$

This notion of entropy enjoys a very useful monotonicity property that we will use in the next section to prove some of the key results.

Proposition 6.3.3. *Let $\mathrm{A} = (W, m_W, \Omega_W)$ and $\mathrm{B} = (V, m_V, \Omega_V)$ be measured lattices with localization with representations π, ρ of Γ. Let $(\mathrm{A}, \pi) \xrightarrow{\psi} (\mathrm{B}, \rho)$ be a morphism in $\mathrm{Hom}_\Gamma(\mathrm{A}, \mathrm{B})$. Then $h_{\mathsf{ML}}(\mathrm{A}, \pi) \geq h_{\mathsf{ML}}(\mathrm{B}, \rho)$.*

Proof. Let $\psi = \Phi^{\mathrm{op}}$, where Φ is the corresponding embedding. Then

$$\begin{aligned}
&h_{\mathsf{ML}}(W, m_W, \Omega_W; \pi) = \\
&= \sup\{h_{\Omega_W(\alpha)}(\alpha, \pi) : \alpha \in \mathrm{Cov}_W\} \geq \sup\{h_{\Omega_W(\Phi(\alpha))}(\Phi(\alpha), \pi) : \alpha \in \mathrm{Cov}_V\} = \\
&\overset{(*)}{=} \sup\{h_{\Phi(\Omega_V(\alpha))}(\Phi(\alpha), \pi) : \alpha \in \mathrm{Cov}_V\} \overset{(**)}{=} \sup\{h_{\Omega_V(\alpha)}(\alpha, \rho) : \alpha \in \mathrm{Cov}_V\} = \\
&= h_{\mathsf{ML}}(V, m_V, \Omega_V; \rho).
\end{aligned}$$

Apart from the equality $(**)$, the proof is rather straightforward. In the equality $(*)$ we use that $\Omega_W(\Phi(\alpha)) = \Phi(\Omega_V(\alpha))$ by the definition of morphisms in ML. In the proof of $(**)$ we have used that $m_W(\Phi(a)) = m_V(a)$ for all $a \in \alpha \in \mathrm{Cov}_V$ and that the morphism Φ is a lattice homomorphism intertwining the action of Γ. Hence the entropy of α computed with respect to $\Omega(\alpha)$ in the lattice V equals the entropy of $\Phi(\alpha)$ computed with respect to $\Phi(\Omega(\alpha))$ in the lattice W.

□

This result admits a categorical interpretation.

Corollary 6.3.4. *The correspondence $h_{\mathsf{ML}} : (\mathrm{A}, \pi) \mapsto h_{\mathsf{ML}}(\mathrm{A}, \pi) \in [0, \infty]$ is a covariant functor from ML_Γ to the poset category $[0, \infty]$ of extended positive reals.*

Proof. Objects of $[0, \infty]$ are extented positive reals. For $x, y \in [0, \infty]$ the set $\mathrm{Hom}(x, y)$ consists of exactly one arrow $\geq$ if and only if $x \geq y$, and is empty otherwise. Then the statement follows from the previous proposition and the transitivity of the $\geq$ relation on the poset $[0, \infty]$. □

6.3.2. Entropy of Representations on Abstract Categories. Let C be a category, C_Γ be the associated category of representations of a discrete amenable group Γ, and $\mathrm{M} : \mathsf{C}_\Gamma \to \mathsf{ML}_\Gamma$ be a measurement functor. We define the entropy of the representation $(\mathrm{A}, \pi) \in \mathsf{C}_\Gamma$ associated with the measurement functor M by

$$h((\mathrm{A}, \pi), \mathrm{M}) := h_{\mathsf{ML}}(\mathrm{M}(\mathrm{A}, \pi)). \tag{6.3.7}$$

An important property of topological and Kolmogorov-Sinai entropy is that it decreases when passing to factors. One of the main results of this chapter is that our abstractly defined entropy also does decrease when moving down the arrows.

Proposition 6.3.5 (Left Entropic Inequality). *Let $(\mathrm{A}, \pi) \xrightarrow{\psi} (\mathrm{B}, \rho)$ be a morphism in $\mathrm{Hom}_\Gamma(\mathrm{A}, \mathrm{B})$. Then $h((\mathrm{A}, \pi), \mathrm{M}) \geq h((\mathrm{B}, \rho), \mathrm{M})$.*

Proof. Suppose $\mathrm{M}(\mathrm{A}, \pi) = (\overline{\mathrm{A}}, \overline{\pi})$, $\mathrm{M}(\mathrm{B}, \rho) = (\overline{\mathrm{B}}, \overline{\rho})$. Here $\overline{\mathrm{A}} = (W, m_W, \Omega_W)$, $\overline{\mathrm{B}} = (V, m_V, \Omega_V)$ are measured lattices with localization and $\mathrm{M}(\psi) = \Phi^{\mathrm{op}} : \overline{\mathrm{A}} \to \overline{\mathrm{B}}$ is the image of morphism ψ. Then the statement follows immediately from Proposition 6.3.3. □

In the notation for the entropy of $(\mathrm{A}, \pi) \in \mathsf{C}_\Gamma$ we always say explicitly what measurement functor we use. It is natural to ask how different measurement functors can be compared, i.e., what could be the 'arrows' between measurement functors. And if there is a morphism between two different measurement functors, do we also get monotonicity of the corresponding entropies?

To answer these questions we recall first the standard notion of a **natural transformation** between functors. So let for the moment C, D be two arbitrary categories and $\mathcal{F}, \mathcal{G} : \mathsf{C} \to \mathsf{D}$ be covariant functors. A family of morphisms $\alpha = \{\alpha_{\mathrm{X}}\}_{\mathrm{X} \in \mathrm{Obj}(\mathsf{C})}$ in D, where $\alpha_{\mathrm{X}} : \mathcal{F}(\mathrm{X}) \to \mathcal{G}(\mathrm{X})$ for every $\mathrm{X} \in \mathrm{Obj}(\mathsf{C})$, is called a natural transformation between functors $\mathcal{F}$ and $\mathcal{G}$ if for every morphism $\phi : \mathrm{X} \to \mathrm{Y}$ in C the diagram

$$\begin{tikzcd} \mathcal{F}(\mathrm{X}) \arrow[r, "\mathcal{F}(\phi)"] \arrow[d, "\alpha_{\mathrm{X}}"'] & \mathcal{F}(\mathrm{Y}) \arrow[d, "\alpha_{\mathrm{Y}}"] \\ \mathcal{G}(\mathrm{X}) \arrow[r, "\mathcal{G}(\phi)"'] & \mathcal{G}(\mathrm{Y}) \end{tikzcd}$$

commutes. Then we write $\alpha : \mathcal{F} \to \mathcal{G}$, i.e. we call α an arrow between functors. This can be justified by defining the **functor category** $[\mathsf{C}, \mathsf{D}]$ whose objects are covariant functors from C to D and whose arrows are natural transformations[2]. Given functors $\mathcal{F}, \mathcal{G}, \mathcal{H}$ with natural transformations $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{H}$, we see that $\gamma = \{\beta_{\mathrm{X}}\alpha_{\mathrm{X}}\}_{\mathrm{X} \in \mathrm{Obj}(\mathsf{C})}$ is a natural transformation between $\mathcal{F}$ and $\mathcal{H}$. We call a natural transformation $\alpha : \mathcal{F} \to \mathcal{G}$ a **natural equivalence** if α_{X} is an isomorphism for every $\mathrm{X} \in \mathrm{Obj}(\mathsf{C})$.

We return back to the main topic, so now C is some category and C_Γ is the associated category of representations of a discrete amenable group Γ. Then the collection of all measurement functors from C_Γ to ML_Γ is a full subcategory of $[\mathsf{C}_\Gamma, \mathsf{ML}_\Gamma]$. We denote the category of measurement functors from C_Γ to ML_Γ by $[\mathsf{C}_\Gamma, \mathsf{ML}_\Gamma]_{\mathfrak{M}}$. Observe that for a nontrivial[3] category C the categories C_Γ and $[\mathsf{C}_\Gamma, \mathsf{ML}_\Gamma]_{\mathfrak{M}}$ are also nontrivial. Indeed, there is always a trivial measurement functor M_0. It is easy to see that the entropy measured with respect to the functor M_0 is identically zero on C_Γ.

We now prove the second key 'monotonicity' result.

PROPOSITION 6.3.6 (Right Entropic Inequality). *Let* $\alpha : \mathrm{M} \to \mathrm{N}$ *be a natural transformation between measurement functors* $\mathrm{M}, \mathrm{N} : \mathsf{C}_\Gamma \to \mathsf{ML}_\Gamma$. *Then for every object* $(\mathrm{A}, \pi) \in \mathsf{C}_\Gamma$ *we have* $h((\mathrm{A}, \pi), \mathrm{M}) \geq h((\mathrm{A}, \pi), \mathrm{N})$. *If* α *is a natural equivalence, then* $h((\mathrm{A}, \pi), \mathrm{M}) = h((\mathrm{A}, \pi), \mathrm{N})$.

PROOF. Let $\mathrm{M}(\mathrm{A}, \pi) = (\overline{\mathrm{A}}, \overline{\pi})$, $\mathrm{N}(\mathrm{A}, \pi) = (\overline{\mathrm{B}}, \overline{\rho})$ where $\overline{\mathrm{A}} = (W, m_W, \Omega_W)$, $\overline{\mathrm{B}} = (V, m_V, \Omega_V)$ are measured lattices with localization endowed with representations $\overline{\pi}, \overline{\rho}$ of Γ respectively.

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[2]There is a subtlety here: the collection of all natural transformations between functors is not necessarily a small set, so the 'category of functors' is not necessarily locally small.

[3]I.e. nonempty.

Then by definition $h((\mathrm{A},\pi),\mathrm{M}) = h_{\mathsf{ML}}(\overline{\mathrm{A}},\overline{\pi})$ and $h((\mathrm{A},\pi),\mathrm{N}) = h_{\mathsf{ML}}(\overline{\mathrm{B}},\overline{\rho})$. Since α is a natural transformation, we obtain a morphism $\alpha_{(\mathrm{A},\pi)} : (\overline{\mathrm{A}},\overline{\pi}) \to (\overline{\mathrm{B}},\overline{\rho})$ in the category ML_Γ. Then the statement follows from Proposition 6.3.3. □

Of course, we have not used the commutative diagram from the definition of a natural transformation in this proof[4], but we will have to use it in order to derive much stronger results, giving a better functorial interpretation of entropy, below.

Given categories C_Γ and $[\mathsf{C}_\Gamma,\mathsf{ML}_\Gamma]_{\mathfrak{M}}$, we can define a **product category** $\mathsf{C}_\Gamma \times [\mathsf{C}_\Gamma,\mathsf{ML}_\Gamma]_{\mathfrak{M}}$ as follows. Objects of $\mathsf{C}_\Gamma \times [\mathsf{C}_\Gamma,\mathsf{ML}_\Gamma]_{\mathfrak{M}}$ are all pairs (A,M), where A is in C_Γ and M is a measurement functor in $[\mathsf{C}_\Gamma,\mathsf{ML}_\Gamma]_{\mathfrak{M}}$. The collection of morphisms between objects (A,M) and (B,N) is given by the collection of all pairs (ϕ,α), where $\phi : \mathrm{A} \to \mathrm{B}$ and $\alpha : \mathrm{M} \to \mathrm{N}$. Given morphisms $(\phi,\alpha) : (\mathrm{A},\mathrm{M}) \to (\mathrm{B},\mathrm{N})$ and $(\psi,\beta) : (\mathrm{B},\mathrm{N}) \to (\mathbf{C},\mathrm{L})$, we define their composition componentwise by $(\psi\phi,\beta\alpha) : (\mathrm{A},\mathrm{M}) \to (\mathbf{C},\mathrm{L})$.

Given the category $\mathsf{C}_\Gamma \times [\mathsf{C}_\Gamma,\mathsf{ML}_\Gamma]_{\mathfrak{M}}$, one naturally obtains an **evaluation bifunctor** ev (the construction, however, is not specific to the categories C_Γ and ML_Γ). On objects, it is given as

$$(\mathrm{A},\mathrm{M}) \overset{\mathrm{ev}}{\longmapsto} \mathrm{M}(\mathrm{A}),$$

where (A,M) is in $\mathsf{C}_\Gamma \times [\mathsf{C}_\Gamma,\mathsf{ML}_\Gamma]_{\mathfrak{M}}$. For objects $(\mathrm{A},\mathrm{M}),(\mathrm{B},\mathrm{N})$ the functor works on the corresponding morphisms by

$$(\phi,\alpha) \overset{\mathrm{ev}}{\longmapsto} \alpha_{\mathrm{B}}\mathrm{M}(\phi),$$

where (ϕ,α) is a morphism in $\mathrm{Hom}((\mathrm{A},\mathrm{M}),(\mathrm{B},\mathrm{N}))$.

We claim that this is indeed a covariant functor, i.e., it respects composition of morphisms. That is, if $(\phi,\alpha) : (\mathrm{A},\mathrm{M}) \to (\mathrm{B},\mathrm{N})$ and $(\psi,\beta) : (\mathrm{C},\mathrm{N}) \to (\mathrm{C},\mathrm{L})$, we want that

$$\mathrm{ev}((\psi\phi),(\beta\alpha)) \overset{\mathrm{def}}{=\!=} \alpha_{\mathrm{C}}\beta_{\mathrm{C}}\mathrm{M}(\psi\phi) = \beta_{\mathrm{C}}\mathrm{N}(\psi)\alpha_{\mathrm{B}}\mathrm{M}(\phi). \tag{6.3.8}$$

Applying the definition of the natural transformation α to the morphism ϕ, we conclude that the diagram

$$\begin{tikzcd}
\mathrm{M}(\mathrm{A}) \arrow[r, "\mathrm{M}(\phi)"] \arrow[d, "\alpha_{\mathrm{A}}"'] & \mathrm{M}(\mathrm{B}) \arrow[d, "\alpha_{\mathrm{B}}"] \\
\mathrm{N}(\mathrm{A}) \arrow[r, "\mathrm{N}(\phi)"'] & \mathrm{N}(\mathrm{B})
\end{tikzcd}$$

commutes, hence we obtain identity $\alpha_{\mathrm{B}}\mathrm{M}(\phi) = \mathrm{N}(\phi)\alpha_{\mathrm{A}}$. Applying the definition of the natural transformation α to the morphism $\psi\phi$, we deduce that the

[4] I.e. it would suffice to have an infranatural transformation of measurement functors.

diagram

$$\begin{tikzcd} \mathrm{M}(\mathrm{A}) \arrow[r, "\mathrm{M}(\psi\phi)"] \arrow[d, "\alpha_{\mathrm{A}}"'] & \mathrm{M}(\mathrm{C}) \arrow[d, "\alpha_{\mathrm{C}}"] \\ \mathrm{N}(\mathrm{A}) \arrow[r, "\mathrm{N}(\psi\phi)"'] & \mathrm{N}(\mathrm{C}) \end{tikzcd}$$

commutes, hence we obtain the identity $\mathrm{N}(\psi\phi)\alpha_{\mathrm{A}} = \alpha_{\mathrm{C}}\mathrm{M}(\psi\phi)$. Substituting these identities in the equation ((6.3.8)), we get

$$\beta_{\mathrm{C}}\mathrm{N}(\psi)\alpha_{\mathrm{B}}\mathrm{M}(\phi) = \beta_{\mathrm{C}}\mathrm{N}(\psi)\mathrm{N}(\phi)\alpha_{\mathrm{A}} = \beta_{\mathrm{C}}\mathrm{N}(\psi\phi)\alpha_{\mathrm{A}} = \beta_{\mathrm{C}}\alpha_{\mathrm{C}}\mathrm{M}(\psi\phi),$$

which shows that ev is indeed a covariant functor.

Combining these results, we arrive at

COROLLARY 6.3.7. *Entropy is a (bi)functor:*

$$\mathsf{C}_\Gamma \times [\mathsf{C}_\Gamma, \mathsf{ML}_\Gamma]_{\mathfrak{M}} \overset{h(\cdot,\cdot)}{\longrightarrow} [0, \infty].$$

PROOF. This follows immediately, since $((\mathrm{A}, \pi), \mathrm{M}) \mapsto h((\mathrm{A}, \pi), \mathrm{M})$ is a composition of the evaluation bifunctor ev : $\mathsf{C}_\Gamma \times [\mathsf{C}_\Gamma, \mathsf{ML}_\Gamma]_{\mathfrak{M}} \to \mathsf{ML}_\Gamma$ and $h_{\mathsf{ML}} : \mathsf{ML}_\Gamma \to [0, \infty]$, which was show to be a functor in Corollary 6.3.4. □

It is interesting to observe that one can derive the corollary above in a more general - though also completely informal - setting. Consider some category of physical systems $\mathsf{PhysSys}$, some category of observables Obs and the associated category of measurement functors $[\mathsf{PhysSys}, \mathsf{Obs}]_{\mathfrak{M}}$, which is just some full subcategory of $[\mathsf{PhysSys}, \mathsf{Obs}]$. Then evaluation ev : $(\mathrm{A}, \mathrm{M}) \mapsto \mathrm{M}(\mathrm{A})$ is *still* a bifunctor from $\mathsf{PhysSys} \times [\mathsf{PhysSys}, \mathsf{Obs}]_{\mathfrak{M}}$ to Obs. Suppose furthermore that we are given a poset of complexity values $\mathsf{CompVal}$ and a functor $\tilde{h} : \mathsf{Obs} \to \mathsf{CompVal}$. Then the 'complexity' defined via $(\mathrm{A}, \mathrm{M}) \mapsto \tilde{h}(\mathrm{M}(\mathrm{A}))$ is a (bi)functor from $\mathsf{PhysSys} \times [\mathsf{PhysSys}, \mathsf{Obs}]_{\mathfrak{M}}$ to $\mathsf{CompVal}$.

6.3.3. Comparison with the Classical Notions of Entropy. In this subsection we intend to compare the entropies defined via the measurement functors $\mathrm{M}_{\mathsf{Prob}}$ and $\mathrm{M}_{\mathsf{Top}}$ (introduced in Section 6.2) with the classical notions of Kolmogorov-Sinai and topological entropy for amenable group actions.

PROPOSITION 6.3.8. *Let* $\mathrm{X} = (X, \mathcal{B}, \mu)$ *be a standard probability space with measure algebra* $\mathcal{M} := \Sigma(\mathrm{X})$*,* $(\mathrm{X}, \pi) \in \mathrm{Obj}(\mathsf{Prob}_\Gamma)$ *be a measure-preserving* Γ*-system, with* Γ *being discrete amenable. Denote by* $h_{\mathsf{Prob}}(\mathrm{X}, \pi)$ *the Kolmogorov-Sinai entropy of the system* (X, π)*. Then*

$$h_{\mathsf{Prob}}(\mathrm{X}, \pi) = h((\mathrm{X}, \pi), \mathrm{M}_{\mathsf{Prob}}).$$

PROOF. Recall that the classical entropy of a finite partition α of X with respect to a probability measure μ is defined as

$$h_{\mathsf{Prob}}(\alpha, \pi) = \lim_{n\to\infty} \frac{1}{|F_n|} H_{\mathrm{Sh}}(\alpha^{F_n}), \tag{6.3.9}$$

where $H_{\mathrm{Sh}}(\beta) = -\sum\limits_{b\in\beta} \mu(b)\log\mu(b)$ and $(F_n)_n$ is a Følner sequence. Then the classical Kolmogorov-Sinai entropy of the system (X, π) is given by

$$h_{\mathsf{Prob}}(\mathrm{X}, \pi) = \sup\{h_{\mathsf{Prob}}(\alpha, \pi) : \alpha \text{ finite partition of } \mathrm{X}\}. \tag{6.3.10}$$

Let $(\overline{\mathrm{X}}, \overline{\pi}) = \mathrm{M}_{\mathsf{Prob}}(\mathrm{X}, \pi)$ be the abstract dynamical lattice associated to the system (X, π), where $\overline{\mathrm{X}} = (\mathcal{M}, \mu, \Omega)$ is the measured lattice with localization. We view the measure algebra $\mathcal{M}$ as a distributive lattice. The measure μ plays the role of a measurement function, and for a cover $\alpha \in \mathrm{Cov}_{\mathcal{M}}$ the localization $\Omega(\alpha)$ is the smallest Γ-invariant measure subalgebra of $\mathcal{M}$ containing the family of sets α.

Observe that if $\alpha \in \mathrm{Cov}_{\mathcal{M}}$ is a cover, then there exists a partition $\alpha' \in \mathrm{Cov}_{\Omega(\alpha)}$ with $\alpha' \succeq \alpha$ and $\Omega(\alpha') = \Omega(\alpha)$. We call any such α' a *generating disjoint refinement* of α. It is essential in this observation that $\Omega(\alpha)$ is a subalgebra containing α, it allows to find the required refinement without leaving the sublattice $\Omega(\alpha)$. So, since $\alpha' \succeq \alpha$ and $\Omega(\alpha') = \Omega(\alpha)$, the inequality $h_{\Omega(\alpha')}(\alpha') \geq h_{\Omega(\alpha)}(\alpha)$ holds. Hence

$$\begin{aligned} h(V, m, \Omega) &= \sup\{h_{\Omega(\alpha)}(\alpha, \Gamma) : \alpha \in \mathrm{Cov}_{\mathcal{M}}\} = \\ &= \sup\{h_{\Omega(\alpha)}(\alpha, \Gamma) : \alpha \text{ is a partition of } \mathrm{X}\}. \end{aligned}$$

It follows that it is enough to show that for all partitions α of X and all invariant measure subalgebras W of $\mathcal{M}$ containing α we have $h^*(\alpha) = h_W(\alpha)$, since, clearly, $h^*(\alpha) = h_\mu(\alpha)$ for partitions. For any partition α and any invariant measure subalgebra W containing α it follows by the pigeonhole principle that $\hat{h}_W(\alpha) = h^*(\alpha)$. For an arbitrary cover $\beta \in \mathrm{Cov}_W$ there is a *non-growing disjoint refinement* $\beta' \in \mathrm{Cov}_W$, i.e. a partition β' such that $\beta' \succeq \beta$ and $N(\beta') \leq N(\beta)$. Then, clearly, $\hat{h}_W(\beta') \leq \hat{h}_W(\beta)$.

It is obvious that for any partition α and any invariant measure subalgebra W containing α we have

$$h_W(\alpha) = \inf\{\sum_{j=1}^{n} \hat{h}_W(\beta_j) : \bigvee_{j=1}^{n} \beta_j \succeq \alpha, n \in \mathbb{N}, \forall j\ \beta_j \in \mathrm{Cov}_W\} \leq h^*(\alpha).$$

Now, pick any sequence $(\beta_j)_{j=1}^n$ of covers in Cov_W such that $\bigvee\limits_{j=1}^{n} \beta_j \succeq \alpha$, and consider the respective non-growing disjoint refinements $(\beta_j')_{j=1}^n$. Then $\sum\limits_{j=1}^{n} \hat{h}_W(\beta_j') \leq \sum\limits_{j=1}^{n} \hat{h}_W(\beta_j)$ and it follows that it suffices to take the infinum over the sequences $(\beta_j)_{j=1}^n$ of partitions of X. It only remains to observe that for such $\alpha, (\beta_j)_{j=1}^n$

$$h^*(\alpha) \leq h^*(\bigvee_{j=1}^{n} \beta_j) \leq \sum_{j=1}^{n} h^*(\beta_j)$$

by the standard monotonicity and subadditivity properties of h_μ on partitions.

$\square$

Now we prove a similar statement for topological dynamical systems and the measurement functor $\mathrm{M}_{\mathsf{Top}}$.

PROPOSITION 6.3.9. *Let* $\mathrm{X} = (X,\mathcal{U})$ *be in* Top*, and let* $(\mathrm{X},\pi) \in \mathrm{Obj}(\mathsf{Top}_\Gamma)$ *be topological* Γ*-system, where* Γ *is discrete amenable. Denote by* $h_{\mathsf{Top}}(\mathrm{X},\pi)$ *the topological entropy of the system* (X,π)*. Then*

$$h_{\mathsf{Top}}(\mathrm{X},\pi) = h((\mathrm{X},\pi),\mathrm{M}_{\mathsf{Top}}).$$

PROOF. Recall that the topological entropy of a finite open cover α of X is defined as

$$h_{\mathsf{Top}}(\alpha,\pi) = \lim_{n\to\infty} \frac{1}{|F_n|} H_{\mathsf{Top}}(\alpha^{F_n}), \tag{6.3.11}$$

where $H_{\mathsf{Top}}(\alpha) = \log\min\{|\beta| : \beta \subseteq \alpha \text{ is a subcover}\}$ and $(F_n)_n$ is a Følner sequence. Then the topological entropy of the system (X,π) is

$$h_{\mathsf{Top}}(\mathrm{X},\pi) = \sup\{h_{\mathsf{Top}}(\alpha,\pi) : \alpha \text{ finite open cover of K}\}. \tag{6.3.12}$$

Let $(\overline{\mathrm{X}},\overline{\pi}) = \mathrm{M}_{\mathsf{Top}}(\mathrm{X},\pi)$ be the abstract dynamical lattice associated to system (X,π), where (V,m,Ω) is the measured lattice with localization. Distributive lattice V is the lattice of open subsets of X, m is equal to 1 everywhere except for the empty set, $m(\varnothing) = 0$, and for an open cover α we have defined $\Omega(\alpha)$ as the smallest Γ-invariant topology containing family of open sets α.

Let α be a finite open cover, and let W be any invariant topology, containing α. It is clear that $h^*(\alpha) = \log N(\alpha)$ and $\hat{h}_W(\alpha) = \log N(\alpha)$ as well. For a minimal subcover α' of α one concludes $\alpha' \succeq \alpha$ and $\alpha' \subseteq W$, so

$$\begin{aligned} h_W(\alpha) &= \inf\{\sum_{i=1}^n \hat{h}_W(\beta_i) : \bigvee_{i=1}^n \beta_i \succeq \alpha, n\in\mathbb{N}, \forall i\ \beta_i \in \mathrm{Cov}_W\} \leq \\ &\leq \hat{h}_W(\alpha') = H_{\mathsf{Top}}(\alpha) \end{aligned}$$

Now, pick an arbitrary sequence $(\beta_i)_{i=1}^n$ of covers in Cov_W such that $\bigvee_{i=1}^n \beta_i \succeq \alpha$. Then

$$H_{\mathsf{Top}}(\alpha) \leq H_{\mathsf{Top}}(\bigvee_{i=1}^n \beta_i) \leq \sum_{i=1}^n H_{\mathsf{Top}}(\beta_i) \leq \sum_{i=1}^n \hat{h}_W(\beta_i)$$

by the standard monotonicity and subadditivity properties of H. It follows that $H_{\mathsf{Top}}(\alpha) = h_W(\alpha)$ and the proof of the statement is complete. □

6.4. Remarks

6.4.1. The Construction of Palm. In this section we explain the original construction of Palm for Banach lattices with quasi-interior point. For more details on Banach lattice theory we refer the reader to [Sch74]. For simplicity, we will now sketch the original work of Palm [Pal76] and not discuss its generalizations to amenable group actions. In this section we use the original notation of Palm.

Let E be a Banach lattice. An element $u \geq 0$ of E is called a *quasi-interior* point if the principal ideal E_u is dense in E. The principal ideal E_u is defined as

$$E_u := \{x \in E : |x| \leq \lambda u \text{ for some } \lambda > 0\}.$$

An *abstract dynamical system* is a triple (E, u, T), where E is a Banach lattice with quasi-interior point u and $T : E \to E$ is a Banach lattice automorphism satisfying $Tu = u$.

A *dynamical lattice* is a triple (V, m, f), where

1) V is a distributive lattice with 0 and 1;
2) $m : V \to \mathbb{R}_{\geq 0}$ satisfies $m(0) = 0$ and $m(a) = 0 \Rightarrow m(a \vee b) = m(b)$ for all $a, b \in V$;
3) $f : V \to V$ is a lattice homomorphism such that $m(a) = 0 \Rightarrow m(f(a)) = 0$ for every $a \in V$.

Now, let (E, u, T) be an abstract dynamical system. We let

1) V be the lattice of all closed lattice ideals in E;
2) $m : V \to \mathbb{R}_{\geq 0}$ be the function $I \mapsto \sup\{\|x\| : x \in I \cap [0, u]\}$;
3) $f : V \to V$ be the lattice automorphism $I \mapsto \{T(I)\}$, i.e., the closed lattice ideal generated by $T(I)$.

The triple (V, m, f) is called the *dynamical lattice of closed ideals* associated to (E, u, T).

In topological dynamics, we have $E = \mathrm{C}(\mathrm{X})$, where X is a compact Hausdorff space, $u := \mathbf{1}$ and $T := T_\varphi$ a Koopman operator with $\varphi \in \mathrm{Aut}(\mathrm{X})$. Then V is the distributive lattice of open sets. For every $a \in V$, $m(a) = 1$ iff $a \neq \varnothing$ and $m(\varnothing) = 0$. The lattice automorphism f is given by φ^{-1}.

In measurable dynamics, we have $E = \mathrm{L}^1(\mathrm{X})$, where $\mathrm{X} = (X, \mathcal{B}, \mu)$ is a probability space, $u := \mathbf{1}$ and $T := T_\varphi$ a Koopman operator with $\varphi \in \mathrm{Aut}(\mathrm{X})$. Then V is isomorphic the measure algebra $\Sigma(\mathrm{X})$. For every $a \in V$, $m(a) = \mu(a)$. The lattice automorphism f is given by φ^{-1}.

The main distinction between the Palm's notion of entropy and ours is the absence of localization in the original work. Given a cover $\alpha \in \mathrm{Cov}_V$ we define

$$\hat{h}(\alpha) := \sup\{h^*(\beta) : \beta \in \mathrm{Cov}_V \text{ such that } \beta \succeq \alpha, N(\beta) \leq N(\alpha)\}.$$

Given a cover $\alpha \in \mathrm{Cov}_V$, let

$$h(\alpha) := \inf\{\sum_{j=1}^{n} \hat{h}(\beta_j) : n \in \mathbb{N},\ \beta_1, \ldots, \beta_n \in \mathrm{Cov}_V \text{ such that } \bigvee_{j=1}^{n} \beta_j \succeq \alpha\}.$$

The entropy of a cover $\alpha \in \mathrm{Cov}_V$ with respect to the lattice homomorphism f is defined as

$$h(\alpha, f) := \liminf_{n \to \infty} \frac{h(\alpha^n)}{n},$$

and the entropy of an abstract dynamical lattice (V, m, f) is given by

$$h(V, m, f) := \sup\{h(\alpha, f) : \alpha \in \mathrm{Cov}_V\}.$$

One can show that if (E, u, T) is a topological or a measure-preserving abstract dynamical system, then $h(V, m, f)$ equals the topological or the Kolmogorov-Sinai entropy respectively.

6.4.2. Failure of Monotonicity. We will now show that if one takes Palm's original notion of the entropy of a measured distributive lattice *without localization*, then, in general, entropy of a sublattice can be bigger than the entropy of a whole lattice. We take Γ to be trivial, this is not essential for our purposes. So consider measured distributive lattice $\mathrm{V} = (V, m)$, where V is the distributive lattice of all subsets of $\{a, b, c\}$, and μ is a probability measure taking value *very* close to 1 on atom b. Then by the properties of Palm's entropy we have

$$\tilde{h}(\mathrm{V}) := \sup\{h(\alpha) : \alpha \in \mathrm{Cov}_V\} \approx 0,$$

since V is *very* close to a singleton in the category Prob.

Now, let $\mathrm{W} = (W, m)$ be the sublattice of V consisting of the subsets

$$\varnothing, \{a, b\}, \{b\}, \{b, c\}, \{a, b, c\}$$

and carrying the induced measurement function. Then W is no longer an object of Prob, it is closer to a topological system on $\{a, b, c\}$ with W being the lattice of opens. It is straightforward to see that such system has Palm's entropy

$$\tilde{h}(\mathrm{W}) := \sup\{h(\alpha) : \alpha \in \mathrm{Cov}_W\} \approx 1,$$

which is attained at the cover $\alpha := \{\{a, b\}, \{b, c\}\}$.

We call this counterexample 'structural' because of its nature: within a representation of a probabilistic system V as a measured lattice there exists a sublattice W, which is not of a probabilistic origin. This has lead us to the idea of introducing the localized entropy $h_{\cdot}(\cdot)$. If one was to compute the localized entropy $h_W(\alpha)$ with W, α as above, one would also find that $h_W(\alpha) \approx 1$; but not if one takes $h_V(\alpha)$ instead. The reason is once again 'structural': V is the minimal spacial sublattice containing α, while W is not spacial at all.

The notion of a factor of an abstract dynamical system is missing in the original work of Palm. Thus the question remains if there is such a notion of factor so that the Palm's entropy for abstract dynamical systems would decrease 'along the arrows'.

6.4.3. Extending Sofic Entropy. This work was partially motivated by a recent paper [KL11], where an 'operator algebra' approach was used to define sofic measure-theoretic and sofic topological entropies. The arguments share many similarities, thus it appears natural to ask whether Palm's approach can give a common generalization for sofic entropies. However, at the moment it is not clear if such a common generalization exists.

Part IV

Entropy and Complexity

CHAPTER 7

The Theorems of Brudno

Originally, entropy in mathematics was introduced by C. E. Shannon as a certain measure of the amount of information coming from a data source. In particular, the well-known 'lossless coding theorem' implies that, under certain assumptions on a compression algorithm and a data source, entropy of the data source equals the optimal compression ratio attainable with the compression algorithm. Later, the notion of Shannon entropy was modified by A. N. Kolmogorov and Ya. G. Sinai in order to study measure-preserving dynamical systems (see Section 5.1).

In Chapter 4 we discussed Kolmogorov complexity, which is an alternative notion of the amount of information. A natural question is whether one can adapt Kolmogorov complexity for studying dynamical systems as well. For $\mathbb{N}$-systems, some answers to these questions were given by A. A. Brudno in his papers [Bru74] and [Bru82]. In the first article, the relation of topological entropy and Kolmogorov complexity for subshifts was studied, while the second paper focused on Kolmogorov-Sinai entropy and Kolmogorov complexity for subshifts and more general measure-preserving $\mathbb{N}$-systems. The goal of this chapter is to provide the generalizations of these theorems of Brudno on entropy and Kolmogorov complexity beyond the known case of subshifts on $\mathbb{Z}$. The presentation of these results is based on [Mor15b] and [Mor15c].

This chapter is structured as follows. The first theorem of Brudno for subshifts over groups admitting computable Følner monotilings is proved in Section 7.1. Next, we proceed to the second theorem. Our goal is to prove a generalization of the second theorem of Brudno for subshifts over groups which posses particularly nice computable symmetric Følner monotilings, which we call *regular*. This requirement is stronger than normality, namely we require that the Følner sequence is tempered and two-sided and that the corresponding sets of centers are 'good' for taking pointwise ergodic averages. The notion of a regular Følner monotiling is introduced in Section 7.2, where we also provide some examples. The main theorem is proved in Section 7.3.

7.1. The First Theorem

The goal of this section is to prove the following:

THEOREM 7.1.1. *Let* $(\Gamma, \imath)$ *be a computable group with a fixed computable normal Følner monotiling* $([F_n, \mathcal{Z}_n])_{n\geq 1}$. *Let* $\mathbf{X} = (\mathrm{X}, \Gamma)$ *be a subshift on* Γ

and $\mathcal{F}_\Lambda$ *be the associated word presheaf on* Γ. *Then*

$$\widetilde{\mathrm{K}}(\mathcal{F}_\Lambda) = h_{\mathsf{Top}}(\mathbf{X}),$$

where the asymptotic complexity of the word presheaf $\mathcal{F}_\Lambda$ *is computed along the sequence* $(F_n)_{n\geq 1}$.

The proof is split into two parts, establishing respective inequalities in Theorems 7.3.5 and 7.3.7. We now use Example 5.4.1, which we now recall.

PROPOSITION 7.1.2. *Let* $\mathbf{X}$ *be a subshift of* Λ^Γ. *Then*

$$h_{\mathsf{Top}}(\mathbf{X}) = \lim_{n\to\infty} \frac{\log \operatorname{card} \mathcal{F}_\Lambda(F_n)}{|F_n|}.$$

We will now establish the first inequality. The proof is essentially the same as the original one from [Bru74].

THEOREM 7.1.3. *In the setting of Theorem 7.3.1 we have*

$$h_{\mathsf{Top}}(\mathbf{X}) \leq \widetilde{\mathrm{K}}(\mathcal{F}_\Lambda) \tag{7.1.1}$$

PROOF. By the definition, we have to show that

$$\lim_{n\to\infty} \frac{\log \operatorname{card} \mathcal{F}_\Lambda(F_n)}{|F_n|} \leq \limsup_{n\to\infty} \max_{\omega\in\mathcal{F}_\Lambda(F_n)} \overline{\mathrm{K}}(\omega, F_n).$$

Suppose that $\widetilde{\mathrm{K}}(\mathcal{F}_\Lambda) < t$ for some $t \geq 0$. Then there exists n_0 such that for all $n \geq n_0$ and all $\omega \in \mathcal{F}_\Lambda(F_n)$ the inequality $\mathrm{K}(\omega, F_n) \leq t\,|F_n|$ holds. There are at most $2^{t|F_n|+1}$ valid binary programs for the decompressor A^* of length at most $t\,|F_n|$, hence $\operatorname{card} \mathcal{F}_\Lambda(F_n) \leq 2^{t|F_n|+1}$. Taking the logarithm shows that for all $n \geq n_0$ we have

$$\frac{\log \operatorname{card} \mathcal{F}_\Lambda(F_n)}{|F_n|} \leq t + \frac{1}{|F_n|},$$

and this completes the proof. □

The proof of the second inequality requires more work. The proof we provide is based on the idea of the proof of Lemma 5.1 from [Sim15].

THEOREM 7.1.4. *In the setting of Theorem 7.3.1 we have*

$$h_{\mathsf{Top}}(\mathbf{X}) \geq \widetilde{\mathrm{K}}(\mathcal{F}_\Lambda) \tag{7.1.2}$$

PROOF. By the definition, we have to show that

$$\limsup_{n\to\infty} \max_{\omega\in\mathcal{F}_\Lambda(F_n)} \overline{\mathrm{K}}(\omega, F_n) \leq \lim_{n\to\infty} \frac{\log \operatorname{card} \mathcal{F}_\Lambda(F_n)}{|F_n|}.$$

The alphabet Λ is finite, so we encode each letter of the alphabet Λ using precisely $\lfloor \log \operatorname{card} \Lambda \rfloor + 1$ bits. We fix this encoding. Then binary words of length $N\,(\lfloor \log \operatorname{card} \Lambda \rfloor + 1)$ are unambiguously interpreted as Λ-words of length N. We will now describe a decompressor $A^!$ that will be used to prove the theorem. The decompressor is defined on the domain of binary programs of the form

$$\mathrm{p} = \widehat{\mathrm{k}}\widehat{\mathrm{n}}\widehat{\mathrm{N}}\widehat{\mathrm{L}}\widehat{\mathrm{l}}\mathrm{w}_1\mathrm{w}_2\ldots\mathrm{w}_N\mathrm{v}\widehat{\mathrm{i}_1}\widehat{\mathrm{i}_2}\ldots\widehat{\mathrm{i}_s}. \tag{7.1.3}$$

Here $\widehat{\mathrm{k}}, \widehat{\mathrm{n}}, \widehat{\mathrm{N}}, \widehat{\mathrm{L}}, \widehat{\mathrm{l}}$ are the simple prefix-free encodings of the natural numbers k, n, N, L, l. The binary words $\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_N$ have all length L. The words $\widehat{\mathrm{i}_1}, \widehat{\mathrm{i}_2}, \ldots, \widehat{\mathrm{i}_s}$ encode some natural numbers $i_1, i_2, \ldots, i_s$ that are required to be less or equal to N. Finally, v is a binary word of length l. Observe that programs of the form (7.1.3) are indeed unambiguously interpreted.

The decompressor $A^!$ works as follows. First, given k, n above, the finite sets

$$F_k, F_n, \mathrm{int}^1_{F_k}(F_n), \mathrm{int}^1_{F_k}(F_n) \cap \mathcal{Z}_k \subseteq \mathbb{N}$$

are computed. We let $I_{k,n} := \mathrm{int}^1_{F_k}(F_n) \cap \mathcal{Z}_k$ and compute the set

$$\Delta_{k,n} := F_k \left(\mathrm{int}^1_{F_k}(F_n) \cap \mathcal{Z}_k\right) \subseteq F_n.$$

We treat N binary words $\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_N$ of length L as encodings of Λ-words $w_1, w_2, \ldots, w_N$ of length F_k. If $L \neq |F_k| \left(\lfloor \log \mathrm{card}\, \Lambda \rfloor + 1\right)$ the algorithm terminates without producing output. The words $w_1, w_2, \ldots, w_N$ form the *dictionary* that we will use to encode parts of the words. We require that $s = \left|\mathrm{int}^1_{F_k}(F_n) \cap \mathcal{Z}_k\right|$ and

$$l = |F_n \setminus \Delta_{k,n}| \left(\lfloor \log \mathrm{card}\, \Lambda \rfloor + 1\right),$$

and the algorithm terminates without producing output if this does not hold. Otherwise, the binary word v of length l is seen as a binary encoding of the Λ-word v of length $|F_n \setminus \Delta_{k,n}|$.

We will now compute a Λ-word ω defined on F_n. The set $\mathrm{int}^1_{F_k}(F_n) \cap \mathcal{Z}_k$ is ordered as a subset of $\mathbb{N}$. For j-th element $g_j \in \mathrm{int}^1_{F_k}(F_n) \cap \mathcal{Z}_k$ we require that $\omega|_{F_k g_j} \circ \imath^{-1}_{F_k g_j} = w_{i_j}$, where $j = 1, 2, \ldots, s$. That is, we require that the restriction of ω to the subset $F_k g_j$ coincides with i_j-th element of the dictionary for every j. It is clear that this determines the restriction $\omega|_{\Delta_{k,n}}$, and it remains to describe $\omega|_{F_n \setminus \Delta_{k,n}}$. We require that $\omega|_{F_n \setminus \Delta_{k,n}} \circ \imath^{-1}_{F_n \setminus \Delta_{k,n}} = v$. The decompressor $A^!$ prints the Λ-word $\omega \circ \imath^{-1}_{F_n}$.

Fix $k \geq 1$ and $\varepsilon > 0$. Let n_0 be such that for all $n \geq n_0$ we have

$$\frac{|F_n \setminus \Delta_{k,n}|}{|F_n|} \leq \varepsilon.$$

Let $\omega \in \mathcal{F}_\Lambda(F_n)$. We use the following program to encode ω. We let $N := \mathrm{card}\, \mathcal{F}_\Lambda(F_k)$, $L := |F_k| \left(\lfloor \log \mathrm{card}\, \Lambda \rfloor + 1\right)$ and $w_1, w_2, \ldots, w_N$ be the list of words $v \circ \imath^{-1}_{F_k}$ for $v \in \mathcal{F}_\Lambda(F_k)$ (say, in lexicographic order). For every $g_j \in \mathrm{int}^1_{F_k}(F_n) \cap \mathcal{Z}_k$ and every $x \in F_k$ note that

$$\omega|_{F_k g_j}(x g_j) = (g_j \cdot \omega)|_{F_k}(x),$$

where $g_j \cdot \omega \in \mathrm{X}$ by invariance. Hence we let i_j be the index of the word $(g_j \cdot \omega)|_{F_k} \circ \imath^{-1}_{F_k}$ in the dictionary $w_1, w_2, \ldots, w_N$ for every $j = 1, 2, \ldots, \left|\mathrm{int}^1_{F_k}(F_n) \cap \mathcal{Z}_k\right|$.

Finally, we let v be the remainder $\omega|_{F_n \setminus \Delta_{k,n}} \circ \imath^{-1}_{F_n \setminus \Delta_{k,n}}$ and l be the length of the binary encoding of the word v. It is clear that the program (7.1.3) with the parameters determined above does describe $\omega|_{F_n}$.

It remains to estimate the length of p. It is easy to see that

$$\begin{aligned} l(\mathrm{p}) \leq l(\widehat{\mathrm{k}}) + l(\widehat{\mathrm{n}}) + l(\widehat{\mathrm{N}}) + l(\widehat{\mathrm{L}}) + l(\widehat{\mathrm{l}}) + \operatorname{card} \mathcal{F}_\Lambda(F_k)\,|F_k|\,(\lfloor \log \operatorname{card} \Lambda \rfloor + 1) + \\ + |F_n \setminus \Delta_{k,n}|\,(\lfloor \log \operatorname{card} \Lambda \rfloor + 1) + |I_{k,n}|\, l(\widehat{\mathrm{N}}), \end{aligned}$$

and taking the limit as $n \to \infty$ we see (using Proposition 1.3.3) that

$$\begin{aligned} \limsup_{n\to\infty} \max_{\omega \in \mathcal{F}_\Lambda(F_n)} \overline{\mathrm{K}}(\omega, F_n) \leq \varepsilon\,(\lfloor \log \operatorname{card} \Lambda \rfloor + 1) + \\ + \frac{2\lfloor \log(\lfloor \log \operatorname{card} \mathcal{F}_\Lambda(F_k) \rfloor + 1) \rfloor + \lfloor \log \operatorname{card} \mathcal{F}_\Lambda(F_k) \rfloor + 5}{|F_k|}. \end{aligned}$$

Since k, ε are arbitrary, the conclusion follows. □

It is clear that the Theorem 7.1.1 now follows from Theorem 7.3.5 and Theorem 7.3.7.

7.2. Regular Følner monotilings

For the purposes of this work we need to introduce special Følner monotilings where one can 'average' along the intersections $F_n \cap \mathcal{Z}_k$ for every fixed k and $n \to \infty$. This, together with some other requirements, leads to the following definition. We call a left Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ **regular** if the following assumptions hold:

a) the sequence $(F_n)_{n\geq 1}$ is a tempered two-sided Følner sequence;
b) for every $k \in \mathbb{N}$ the function $\mathbf{1}_{\mathcal{Z}_k} \in \ell_\infty(\Gamma)$ is a good weight for pointwise convergence of ergodic averages along the sequence $(F_n)_{n\geq 1}$;
c) $\frac{|F_n|}{\log n} \to \infty$ as $n \to \infty$;
d) $\mathrm{e} \in F_n$ for every n.

From the definition of a regular Følner monotiling and the preceeding results we can immediately derive the following proposition.

THEOREM 7.2.1. *Let $([F_n, \mathcal{Z}_n])_{n\geq 1}$ be a regular Følner monotiling. Then for every measure-preserving Γ-system $\mathbf{X} = (\mathrm{X}, \pi)$ on a space $\mathrm{X} = (X, \mathcal{B}, \mu)$, every $f \in \mathrm{L}^\infty(\mathrm{X})$ and every $k \geq 1$ the limits*

$$\begin{aligned} |F_k| \lim_{n\to\infty} \mathbb{E}_{g \in F_n} \mathbf{1}_{\mathcal{Z}_k} f(g \cdot \omega) = \lim_{n\to\infty} \mathbb{E}_{g \in F_n \cap \mathcal{Z}_k} f(g \cdot \omega) = \\ = \lim_{n\to\infty} \mathbb{E}_{g \in \mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{r}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k} f(g \cdot \omega) \end{aligned}$$

exist and coincide for μ-a.e. $\omega \in \mathrm{X}$.

PROOF. Existence of the limit on the left hand side follows from the definition of a good weight and the definition of a regular Følner monotiling, equality of the limits follows from Proposition 1.3.3. □

Of course, our motivating example for the notion of a regular Følner monotiling is the Example 1.3.1. Below we explain why the corresponding indicator functions $\mathbf{1}_{\mathcal{Z}_k}$ are good weights for every k. Checking the remaining conditions for the regularity of the Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is straightforward.

EXAMPLE 7.2.2. Let Γ be an amenable group with a fixed tempered Følner sequence $(F_n)_{n\geq 1}$, $H \leq \Gamma$ be a finite index subgroup. Let $F \subseteq \Gamma$ be a fundamental domain for left cosets of H. Then $[F, H]$ is a left monotiling of Γ. Furthermore, the indicator function $\mathbf{1}_H$ is a good weight. To see this, consider the ergodic system $\mathbf{X} := (\Gamma/H, \pi)$, where Γ acts on the left on the finite set Γ/H with the normalized counting measure $|\cdot|$. by

$$g(fH) := gfH, \;\; f \in F, \;\; g \in \Gamma.$$

Let $f := \mathbf{1}_{\mathrm{e}H} \in \mathrm{L}^\infty(\Gamma/H)$ and $x := \mathrm{e}H \in \Gamma/H$. Then $\mathbf{1}_H(g) = f(g \cdot x)$ for all $g \in \Gamma$ and the statement follows from Theorem 3.4.3 [1].

We are now able return to some of the examples of the computable monotilings from the previous sections and prove the regularity.

EXAMPLE 7.2.3. Consider the group $\mathbb{Z}^d$ for some $d \geq 1$. Since $\mathcal{Z}_n = n\mathbb{Z}^d$ is a finite index subgroup, we conclude that $([\mathcal{Z}_n, F_n])_{n\geq 1}$ is a computable regular symmetric Følner monotiling.

Next, we return to the monotiling of the discrete Heisenberg group.

EXAMPLE 7.2.4. Consider the group $\mathsf{UT}_3(\mathbb{Z})$ and the monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ from Example 1.3.2 given by

$$\mathcal{Z}_n = \{(a, b, c) \in \mathsf{UT}_3(\mathbb{Z}) : a, b \in n\mathbb{Z}, c \in n^2\mathbb{Z}\}$$

and

$$F_n = \{(a, b, c) \in \mathsf{UT}_3(\mathbb{Z}) : 0 \leq a, b < n, 0 \leq c < n^2\}$$

for $n \geq 1$. $\mathcal{Z}_n$ is a finite index subgroup with a fundamental domain F_n, so it follows that $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is a computable regular symmetric Følner monotiling.

In this last example we will demonstrate, referring to the work [GS02] for details, that the groups $\mathsf{UT}_d(\mathbb{Z})$ for $d > 3$ have computable regular symmetric Følner monotilings as well.

EXAMPLE 7.2.5. Let an integer $d > 3$ be fixed. Let u_{ij} be the matrix whose entry with the indices (i, j) is 1, and where all the other entries are zero. Let $T_{ij} := I + u_{ij}$ for all $1 \leq i, j \leq d$. Let p be a prime number. For every $m \in \mathbb{N}$ consider the subgroup $\mathcal{Z}_m$ of $\mathsf{UT}_d(\mathbb{Z})$ generated by $T_{ij}^{p^{m(j-i)}}$ for all indices (i, j), $i < j$. Then $\mathcal{Z}_m$ is an enumerable subset. There exists a total computable function $\phi : \mathbb{N}^2 \to \mathsf{UT}_d(\mathbb{Z})$ such that

$$\mathcal{Z}_m = \{\phi(m, 1), \phi(m, 2), \phi(m, 3), \dots\}$$

[1] One can also prove this directly without referring to Theorem 3.4.3

for all $m \geq 1$.

$\mathcal{Z}_m$ is finite a index subgroup of $\mathsf{UT}_d(\mathbb{Z})$ for every m. The fundamental domain ρ_m for $\mathcal{Z}_m$ can be written as

$$\begin{aligned}\rho_m :=& \{T_{d-1,d}^{k_{d-1,d}} \cdots T_{1,d}^{k_{1,d}} : \\ & l_{d-1,d}(m) \leq k_{d-1,d} \leq L_{d-1,d}(m), \ldots, l_{1,d}(m) \leq k_{1,d} \leq L_{1,d}(m)\},\end{aligned}$$

where

$$l_{i,j}(m) = -\lfloor \frac{p^{m(j-i)}}{2} \rfloor, \quad L_{i,j}(m) = \lfloor \frac{p^{m(j-i)}+1}{2} \rfloor.$$

It is clear that the sequence of sets $m \mapsto \rho_m$ is canonically computable. Furthermore, it is shown in [GS02] that $(\rho_m)_{m\geq 1}$ is a two-sided Følner sequence. Computablity of the Følner monotiling $([\rho_m, \mathcal{Z}_m])_{m\geq 1}$ follows from Proposition 4.4.5.

The fact that the Følner monotiling $([\rho_m, \mathcal{Z}_m])_{m\geq 1}$ is symmetric is clear since $\mathcal{Z}_m$ is a subgroup for every m. The fact that for each m the function $\mathbf{1}_{\mathcal{Z}_m}$ is a good weight along a tempered subsequence of $(\rho_m)_{m\geq 1}$ follows from Example 7.2.2. It is clear that we can ensure the growth conditions by picking a subsequence $(n_i)_{i\geq 1}$ computably such that $([\rho_{n_i}, \mathcal{Z}_{n_i}])_{i\geq 1}$ is a computable regular symmetric Følner monotiling.

7.3. The Second Theorem

We recall that by a **subshift** (X, Γ) we mean a closed Γ-invariant subset X of Λ^Γ, where Λ is the finite **alphabet** of X. The left action of the group Γ on X is given by

$$(g \cdot \omega)(x) := \omega(xg) \quad \forall x, g \in \Gamma, \omega \in \mathrm{X}.$$

The words consisting of letters from the alphabet Λ will often be called **Λ-words**. Of course, we can assume without loss of generality that $\Lambda = \{1, 2, \ldots, k\}$ for some k. When an invariant probability measure μ is fixed on X, we will often denote by $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ the associated measure-preserving system. We can associate a word presheaf $\mathcal{F}_\Lambda$ to the subshift (X, Γ) by setting

$$\mathcal{F}_\Lambda(F) := \{\omega|_F : \omega \in \mathrm{X}\}. \tag{7.3.1}$$

That is, $\mathcal{F}_\Lambda(F)$ is the set of all restrictions of words in X to the set F for every computable F.

The main result of this section is

THEOREM 7.3.1. *Let $(\Gamma, \imath)$ be a computable group with a fixed computable regular symmetric Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$. Let (X, Γ) be a subshift on Γ, $\mu \in \mathrm{M}^1_\Gamma(\mathrm{X})$ be an ergodic measure and $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ be the associated measure-preserving system. Then*

$$\widehat{\mathrm{K}}(\omega) = h_{\mathsf{Prob}}(\mathbf{X})$$

for μ-a.e. $\omega \in \mathrm{X}$, where the asymptotic complexity is computed with respect to the sequence $(F_n)_{n\geq 1}$.

The proof is split into two parts, establishing respective inequalities in Theorems 7.3.5 and 7.3.7. From now on, we follow more or less the strategy of the Brudno's original paper [Bru82].

Given a subshift $\mathrm{X} \subseteq \Lambda^{\Gamma}$ with an invariant measure μ on the alphabet $\Lambda = \{1, \ldots, k\}$, we define the partition

$$\alpha_{\Lambda} := \{A_1, \ldots, A_k\},\ A_i := \{\omega \in \mathrm{X} : \omega(\mathrm{e}) = i\} \text{ for } i = 1, \ldots, k.$$

Then α_{Λ} is, clearly, a generating partition. We will use the Kolmogorov-Sinai generator theorem (see Section 5.5.2), which we recall below.

PROPOSITION 7.3.2. *Let* $\mathrm{X} \subseteq \Lambda^{\Gamma}$ *be a subshift,* μ *be an invariant measure on* X*,* $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ *be the associated measure-preserving system and* α_{Λ} *be the partition defined above. Then*

$$h_{\mu}(\alpha_{\Lambda}, \Gamma) = h_{\mathsf{Prob}}(\mathbf{X}).$$

Given a word $\omega \in \mathrm{X}$ and a finite subset $F \subseteq \Gamma$, we will set

$$X_F(\omega) := \{\sigma \in \mathrm{X} : \sigma|_F = \omega|_F\},$$

i.e., $X_F(\omega)$ is the cylinder of all words in X that coincide with ω when restricted to F. Note that

$$X_F(\omega) = \left(\bigvee_{g \in F} g^{-1}\alpha_{\Lambda}\right)(\omega) = \alpha_{\Lambda}^{F}(\omega), \tag{7.3.2}$$

i.e. the cylinder set $X_F(\omega)$ is precisely the atom of the partition α_{Λ}^{F} that contains ω.

The alphabet Λ is finite, so we encode each letter of the alphabet Λ using precisely $\lfloor \log \operatorname{card} \Lambda \rfloor + 1$ bits. Then binary words of length $N\,(\lfloor \log \operatorname{card} \Lambda \rfloor + 1)$ are unambiguously interpreted as Λ-words of length N.

7.3.1. Part A. The first step is proving that the Kolmogorov complexity of a word over Γ is shift-invariant. In the proof below it will become apparent why we need a computable structure on the group and why we require the Følner sequence to be computable. In the proof below we view Γ as a computable subset of $\mathbb{N}$ such that the multiplication is computable.

THEOREM 7.3.3 (Shift invariance). *Let* $(\Gamma, \imath)$ *be a computable amenable group with a fixed canonically computable right Følner sequence* $(F_n)_{n \geq 1}$ *such that* $\frac{|F_n|}{\log n} \to \infty$ *as* $n \to \infty$*. Let* (X, Γ) *be a subshift and* $\omega \in \mathrm{X}$ *be a word on* Γ*. Then for every* $g \in \Gamma$

$$\widehat{\mathrm{K}}(\omega) = \widehat{\mathrm{K}}(g \cdot \omega),$$

where the asymptotic complexity is computed with respect to the sequence $(F_n)_{n \geq 1}$.

PROOF. We will prove the following claim: for arbitrary $g \in \Gamma$

$$\widehat{\mathrm{K}}(g \cdot \omega) = \limsup_{n \to \infty} \frac{\mathrm{K}^0_{A^*}((g \cdot \omega)|_{F_n} \circ \imath_{F_n}^{-1})}{|F_n|} \leq \widehat{\mathrm{K}}(\omega).$$

It is trivial to see that the statement of the theorem follows from this claim. Speaking informally, our idea behind the proof of the claim is that the sets F_n and $F_n g^{-1}$ are almost identical for large enough n. The word $(g \cdot \omega)|_{F_n \cap F_n g^{-1}}$ can be encoded using the knowledge of the word $\omega|_{F_n}$ and the *computable* action by g that 'permutes' a part of the word $\omega|_{F_n}$. To encode the word $(g \cdot \omega)|_{F_n}$ we also need to treat the part outside the intersection. We use the fact that our Følner sequence is computable, i.e. there is an algorithm that, given n, will print the set F_n. But then we also know the remainder $F_n \setminus F_n g^{-1}$, which is endowed with the ambient numbering of $\Gamma \subseteq \mathbb{N}$. Hence we can simply list additionally the $\left|F_n \setminus F_n g^{-1}\right|$ corrections we need to make, which takes little space compared to $|F_n|$. Together this implies that the complexity of $(g \cdot \omega)|_{F_n}$ can be asymptotically bounded by the complexity of $\omega|_{F_n}$. Below we make this intuition formal.

Recall that A^* is a fixed asymptotically optimal decompressor in the definition of Kolmogorov complexity K. We now introduce a new decompressor $A^\dagger$ on the domain of programs of the form

$$\bar{\mathrm{s}}01\mathrm{w}01\bar{\mathrm{n}}01\bar{\mathrm{m}}01\mathrm{p}, \tag{7.3.3}$$

where $\bar{\mathrm{s}}$ is a doubling encoding of a nonnegative integer s, and w is a binary encoding of a Λ-word υ of length s, hence $l(\mathrm{w}) = s(\lfloor \log \operatorname{card} \Lambda \rfloor + 1)$. Next, $\bar{\mathrm{n}}$ and $\bar{\mathrm{m}}$ are doubling encodings of some natural numbers n, m. The remainder p is required to be a valid input for A^*. Observe that programs of this form (Equation (7.3.3)) are unambiguously interpreted.

The decompressor $A^\dagger$ is defined as follows. Let $g := g_m$ be the element of the computable group $(\Gamma, \imath)$ with index m, and let $F := F_n$ be the n-th element of the canonically computable Følner sequence $(F_n)_{n \geq 1}$. We compute the set $D := F \setminus F g^{-1}$, which is seen as a subset of $\mathbb{N}$ with induced ordering. Further, we compute the word $\widetilde{\omega}_1 := A^*(\mathrm{p})$. The increasing bijection $\imath_F : F \to \{1, 2, \ldots, |F|\}$ maps the subsets $F \cap F g^{-1}$ and $F g \cap F$ of F to subsets Y_1, Y_2 of $\{1, 2, \ldots, |F|\}$. The right multiplication R_g on Γ is computable and restricts to a bijection from $F \cap F g^{-1}$ to $F g \cap F$, so let $\widetilde{R_g}$ be the bijection making the diagram

$$\begin{tikzcd}
F \cap F g^{-1} \arrow[r, "\imath_F"] \arrow[d, "R_g"'] & Y_1 \arrow[d, "\widetilde{R_g}"] \\
F g \cap F \arrow[r, "\imath_F"] & Y_2
\end{tikzcd}$$

commute. The output of $A^\dagger$ is produced as follows. For $x \in Y_1 \subseteq \{1, 2, \ldots, |F|\}$ we set $\widetilde{\omega}_2(x) := \widetilde{\omega}_1(\widetilde{R_g}(x))$, and the algorithm terminates without producing output if $\widetilde{R_g}(x) > l(\widetilde{\omega}_1)$ for some x. It is left to describe $\widetilde{\omega}_2$ on the remainder $Y_0 := \{1, 2, \ldots, |F|\} \setminus Y_1$. We let $\widetilde{\omega}_2|_{Y_0} := \upsilon \circ \imath_{Y_0}$, where $\imath_{Y_0} : Y_0 \to \{1, 2, \ldots, \operatorname{card} Y_0\}$ is an increasing bijection. The algorithm prints nothing and terminates if $\operatorname{card} Y_0 \neq s$, otherwise the word $\widetilde{\omega}_2$ is printed.

Let $(g \cdot \omega)|_{F_n} \circ \imath_{F_n}^{-1}$ be the word on $\{1, 2, \dots, |F_n|\}$ that we want to encode, where $g \in \Gamma$ has index m. Let p_n be an optimal description for $\omega|_{F_n} \circ \imath_{F_n}^{-1}$ with respect to A^*. Let υ be the word $(g \cdot \omega)|_{F_n \setminus F_n g^{-1}} \circ \imath_{F_n \setminus F_n g^{-1}}^{-1}$. To encode the word $(g \cdot \omega)|_{F_n} \circ \imath_{F_n}^{-1}$ using $A^\dagger$, consider the program

$$\widetilde{\mathrm{p}}_n := \bar{\mathrm{s}}01\mathrm{w}01\bar{\mathrm{n}}01\bar{\mathrm{m}}01\mathrm{p}_n,$$

where w is the binary encoding of the Λ-word υ and $s = \left|F_n \setminus F_n g^{-1}\right|$. It is trivial to see that $A^\dagger(\widetilde{\mathrm{p}}_n) = (g \cdot \omega)|_{F_n} \circ \imath_{F_n}^{-1}$.

The length of the program $\widetilde{\mathrm{p}}_n$ can be estimated by

$$l(\widetilde{\mathrm{p}}_n) \le \left|F_n \setminus F_n g^{-1}\right| (\log \operatorname{card} \Lambda + 1) + 2 \log \left|F_n \setminus F_n g^{-1}\right| + c + 2 \log n + 2 \log m + l(\mathrm{p}_n),$$

where c is some constant. By the definition of complexity of sections

$$\widehat{\mathrm{K}}(g \cdot \omega) = \limsup_{n \to \infty} \frac{\mathrm{K}^0_{A^*}((g \cdot \omega)|_{F_n} \circ \imath_{F_n}^{-1})}{|F_n|}.$$

Using that the optimal decompressor A^* is not worse than $A^\dagger$ (Equation (4.6.1)), we conclude that

$$\begin{aligned} &\mathrm{K}^0_{A^*}((g \cdot \omega)|_{F_n} \circ \imath_{F_n}^{-1}) \le \mathrm{K}^0_{A^\dagger}((g \cdot \omega)|_{F_n} \circ \imath_{F_n}^{-1}) + C \le \\ &\le \left|F_n \setminus g^{-1} F_n\right| \cdot (\log \operatorname{card} \Lambda + 1) + 2 \log \left|F_n \setminus F_n g^{-1}\right| + 2 \log n + l(\mathrm{p}_n) + C' \end{aligned}$$

for some constants C, C' independent of n and ω. Taking the limits yields

$$\limsup_{n \to \infty} \frac{\mathrm{K}^0_{A^*}((g \cdot \omega)|_{F_n} \circ \imath_{F_n}^{-1})}{|F_n|} \le \limsup_{n \to \infty} \frac{\mathrm{K}^0_{A^*}(\omega|_{F_n} \circ \imath_{F_n}^{-1})}{|F_n|}.$$

This completes the proof of the claim, and therefore the proof of the theorem. □

Of course, in the proof above we have not used that X is closed. From now on we will omit explicit reference to the sequence $(F_n)_{n \ge 1}$ when talking about $\widehat{\mathrm{K}}$. The proof of the following proposition is essentially the same to the original one in [Bru82].

PROPOSITION 7.3.4. *Let $(\Gamma, \imath)$ be a computable amenable group with a fixed canonically computable right Følner sequence $(F_n)_{n \ge 1}$ such that $\frac{|F_n|}{\log n} \to \infty$ as $n \to \infty$. Let (X, Γ) be a subshift. For every $t \in \mathbb{R}_{\ge 0}$ the sets*

$$\begin{aligned} E_t &:= \{\omega \in \mathrm{X} : \widehat{\mathrm{K}}(\omega) = t\}, \\ L_t &:= \{\omega \in \mathrm{X} : \widehat{\mathrm{K}}(\omega) < t\}, \\ G_t &:= \{\omega \in \mathrm{X} : \widehat{\mathrm{K}}(\omega) > t\} \end{aligned}$$

are measurable and shift-invariant.

PROOF. Invariance of the sets above follows from the previous proposition. We will now prove that the set L_t is measurable, the measurability of other sets is proved in a similar manner. Observe that

$$L_t := \{\omega : \widehat{\mathrm{K}}(\omega) < t\} = \bigcup_{k\geq 1} \bigcup_{N\geq 1} \bigcap_{n>N} \{\omega : \mathrm{K}^0_{A^*}(\omega|_{F_n} \circ \imath^{-1}_{F_n}) < (t - \frac{1}{k})\,|F_n|\},$$

and the sets $\{\omega : \mathrm{K}^0_{A^*}(\omega|_{F_n} \circ \imath^{-1}_{F_n}) < (t - \frac{1}{k})\,|F_n|\}$ are measurable as finite unions of cylinder sets. □

We are now ready to prove the Kolmgorov complexity of almost every word dominates the Kolmogorov-Sinai entropy of an ergodic subshift. The proof below is a slight adaption of the original one from [Bru82].

THEOREM 7.3.5. *Let $(\Gamma, \imath)$ be a computable group with a canonically computable tempered two-sided Følner sequence $(F_n)_{n\geq 1}$ such that $\frac{|F_n|}{\log n} \to \infty$. Let (X, Γ) be a subshift on Γ, $\mu \in \mathrm{M}^1_\Gamma(\mathrm{X})$ be an ergodic Γ-invariant probability measure, and $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ be the associated measure-preserving system. Then $\widehat{\mathrm{K}}(\omega) \geq h_{\mathsf{Prob}}(\mathbf{X})$ for μ-a.e. ω.*

PROOF. Suppose this is false, and let

$$R := \{\omega : \widehat{\mathrm{K}}(\omega) < h_{\mathsf{Prob}}(\mathbf{X})\}$$

be the measurable set of words whose complexity is strictly smaller than the entropy $h_{\mathsf{Prob}}(\mathbf{X})$. By the assumption, $\mu(R) > 0$. The measure μ is ergodic and the set R is invariant, hence $\mu(R) = 1$. For every $i \geq 1$ let

$$R_i := \{\omega : \widehat{\mathrm{K}}(\omega) < h_{\mathsf{Prob}}(\mathbf{X}) - \frac{1}{i}\},$$

then $R = \bigcup_{i\geq 1} R_i$ and the sets R_i are measurable and invariant for all i. It follows that there exists an index i_0 such that $\mu(R_{i_0}) = 1$. For every $l \geq 1$ define the set

$$Q_l := \{\omega : \mathrm{K}^0_{A^*}(\omega|_{F_i} \circ \imath^{-1}_{F_i}) < \left(h_{\mathsf{Prob}}(\mathbf{X}) - \frac{1}{i_0}\right)|F_i| \text{ for all } i \geq l\},$$

then Q_l is a measurable set for every $l \geq 1$ and $R_{i_0} = \bigcup_{l\geq 1} Q_l$. Let $1 > \delta > 0$ be fixed. The sequence of sets $(Q_l)_{l\geq 1}$ is monotone increasing, hence there is l_0 such that for all $l \geq l_0$ we have $\mu(Q_l) > 1 - \delta$.

Let $\varepsilon < \min(\frac{1}{i_0}, 1 - \delta)$ be positive. Let $n_0 := n_0(\varepsilon) \geq l_0$ such that for all $n \geq n_0$ we have the decomposition $\mathrm{X} = A_n \sqcup B_n$, where $\mu(B_n) < \varepsilon$ and for all $\omega \in A_n$ the inequality

$$2^{-|F_n|(h_{\mathsf{Prob}}(\mathbf{X})+\varepsilon)} \leq \mu(\alpha^{F_n}_\Lambda(\omega)) \leq 2^{-|F_n|(h_{\mathsf{Prob}}(\mathbf{X})-\varepsilon)} \tag{7.3.4}$$

holds. Such n_0 exists due to Corollary 5.2.2. For every $l \geq n_0$, we partition the sets Q_l in the following way:

$$Q^A_l := Q_l \cap A_l;$$
$$Q^B_l := Q_l \cap B_l.$$

It is clear that for every $l \geq n_0$

$$\mu(Q_l^B) < \varepsilon;$$
$$\mu(Q_l^A) \geq 1-\delta - \varepsilon > 0.$$

By the definition of the set Q_l^A, for all $l \geq n_0$ and all $\omega \in Q_l^A$ we have

$$\mathrm{K}^0_{A^*}(\omega|_{F_l} \circ \imath_{F_l}^{-1}) \leq |F_l|\,(h_{\mathsf{Prob}}(\mathbf{X}) - \frac{1}{i_0}).$$

This allows, for every $l \geq n_0$, to estimate the cardinality of the set of all restrictions of words in Q_l^A to F_l as

$$\left|\{\omega|_{F_l} : \omega \in Q_l^A\}\right| \leq 2^{|F_l|(h_{\mathsf{Prob}}(\mathbf{X}) - \frac{1}{i_0})+1},$$

which can be seen by counting all binary programs of length at most $|F_l|(h_{\mathsf{Prob}}(\mathbf{X}) - \frac{1}{i_0})$. Combining this with Equation (7.3.4), we deduce that

$$\mu(Q_l^A) \leq 2^{|F_l|(h_{\mathsf{Prob}}(\mathbf{X}) - \frac{1}{i_0})+1} \cdot 2^{-|F_l|(h_{\mathsf{Prob}}(\mathbf{X}) - \varepsilon)} \leq 2^{|F_l|(\varepsilon - \frac{1}{i_0})+1}.$$

This implies that $\mu(Q_l^A) \to 0$ as $l \to \infty$, since $|F_l| \to \infty$ and $\varepsilon - \frac{1}{i_0} < 0$. This contradicts to the estimate

$$\mu(Q_l^A) \geq 1 - \delta - \varepsilon$$

for all $l \geq n_0$ above. □

7.3.2. Part B. In this part of the proof we shall derive the converse inequality (that the Kolmogorov complexity of almost every word is less or equal than the Kolmogorov-Sinai entropy), which is technically more difficult to prove. We begin with a preliminary lemma.

LEMMA 7.3.6. *Let $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ be an ergodic measure-preserving system, where the group Γ admits a regular symmetric Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$. Let $(\beta_k)_{k\geq 1}$ be a sequence of finite partitions of X, where $\beta_k = \{B_1^k, B_2^k, \ldots, B_{M_k}^k\}$ for all $k \geq 1$. For all $k \geq 1$, $h \in \Gamma$, $m \in \{1, 2, \ldots, M_k\}$ let*

$$\pi_{n,m}^{k,h}(\omega) := \mathbb{E}_{g \in F_n \cap \mathcal{Z}_k} \mathbf{1}_{B_m^k}((gh)\cdot\omega) \tag{7.3.5}$$

and

$$\tilde{\pi}_{n,m}^{k,h}(\omega) := \mathbb{E}_{g \in \mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{r}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k} \mathbf{1}_{B_m^k}((gh)\cdot\omega). \tag{7.3.6}$$

Then the following assertions hold:

a) *For μ-a.e. $\omega \in \mathrm{X}$ the limit*

$$\pi_m^{k,h}(\omega) := \lim_{n\to\infty} \pi_{n,m}^{k,h}(\omega) = \lim_{n\to\infty} \tilde{\pi}_{n,m}^{k,h}(\omega)$$

exists for all $k \geq 1$, $m \in \{1, 2, \ldots, M_k\}$ and $h \in \Gamma$.

b) *For μ-a.e. $\omega \in \mathrm{X}$ and all $k \geq 1$ there exists $h := h_k(\omega) \in F_k^{-1}$ such that*

$$-\sum_{m=1}^{M_k} \pi_m^{k,h}(\omega) \log \pi_m^{k,h}(\omega) \leq H_{\mathrm{Sh}}(\beta_k).$$

Proof. The first assertion follows from the definition of a regular Følner monotiling, Theorem 7.2.1 and countability of Γ.

For the second assertion, observe that for μ-a.e. ω, all $k \geq 1$ and all $m \in \{1, 2, \ldots, M_k\}$

$$\frac{1}{|F_k|} \sum_{h \in F_k^{-1}} \pi_m^{k,h}(\omega) = \lim_{n \to \infty} \mathbb{E}_{g \in F_n} \mathbf{1}_{B_m^k}(g \cdot \omega),$$

since, for every $k \geq 1$, $[\mathcal{Z}_k, F_k^{-1}]$ is a right monotiling,

$$(\mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{r}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k) F_k^{-1} \subseteq F_n$$

for all $n \geq 1$ and

$$\frac{\left|(\mathrm{int}^{\mathrm{l}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{r}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k) F_k^{-1}\right|}{|F_n|} \to 1$$

as $n \to \infty$.

Using ergodicity of $\mathbf{X}$, we deduce that for μ-a.e. ω, all $k \geq 1$ and all $m \in \{1, 2, \ldots, M_k\}$

$$\frac{1}{|F_k|} \sum_{h \in F_k^{-1}} \pi_m^{k,h}(\omega) = \lim_{n \to \infty} \mathbb{E}_{g \in F_n} \mathbf{1}_{B_m^k}(g \cdot \omega) = \mu(B_m^k),$$

and the second assertion follows by the concavity of the entropy. □

We are now ready to prove the converse inequality. The proof is based on essentially the same idea of 'frequency encoding', but the technical details differ quite a bit.

Theorem 7.3.7. *Let $(\Gamma, \imath)$ be a computable group with a fixed computable regular symmetric Følner monotiling $([F_n, \mathcal{Z}_n])_{n \geq 1}$. Let (X, Γ) be a subshift on Γ, $\mu \in \mathrm{M}^1_\Gamma(\mathrm{X})$ be an ergodic measure and $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ be the associated measure-preserving system. Then $\widehat{\mathrm{K}}(\omega) \leq h_{\mathsf{Prob}}(\mathbf{X})$ for μ-a.e. ω.*

Proof. We will now describe a decompressor $A^!$ that will be used to encode restrictions of the words in X. The decompressor $A^!$ is defined on the domain of the programs of the form

$$\mathrm{p} := \overline{\mathrm{s}}01\overline{\mathrm{t}}01\overline{\mathrm{f}}_1 01 \ldots \overline{\mathrm{f}}_L 0110 \overline{\mathrm{r}} 01 \mathrm{w} 01 \underline{\mathrm{N}}. \tag{7.3.7}$$

Here $\overline{\mathrm{s}}, \overline{\mathrm{t}}, \overline{\mathrm{r}}$ are doubling encodings of some natural numbers s, t, r. Words $\overline{\mathrm{f}}_1, \ldots, \overline{\mathrm{f}}_L$, where we require that $L = (\mathrm{card}\,\Lambda)^{|F_s|}$, are doubling encodings of nonnegative integers $f_1, \ldots, f_L$. The word w encodes a Λ-word v of length r. The word $\underline{\mathrm{N}}$ encodes[2] a natural number N. Observe that this interpretation is not ambiguous. Let

$$\{\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_L\}$$

be the list of all Λ-words of length $|F_s|$ ordered lexicographically.

[2]We stress that we use a binary encoding here and not a doubling encoding.

The decompressor $A^!$ works as follows. From s and t compute the finite subsets

$$F_s, F_t, \mathrm{int}^{\mathrm{l}}_{F_s}(F_t) \cap \mathrm{int}^{\mathrm{r}}_{F_s^{-1}}(F_t)$$

of $\mathbb{N}$. Compute the finite set

$$I_{s,t} := \mathrm{int}^{\mathrm{l}}_{F_s}(F_t) \cap \mathrm{int}^{\mathrm{r}}_{F_s^{-1}}(F_t) \cap \mathcal{Z}_s$$

of centers of monotiling $[F_s, \mathcal{Z}_s]$. Next, for every $h \in I_{s,t}$ compute the tile $T_h := F_s h \subseteq F_t$ centered at h. We compute the union

$$\Delta_{s,t} := \bigcup_{h \in I_{s,t}} T_h \subseteq F_t$$

of all such tiles.

We will construct a Λ-word σ on the set F_t, then $\widetilde{\sigma} := \sigma \circ \imath_{F_t}^{-1}$ yields a word on $\{1, 2, \ldots, |F_t|\}$. The word σ is computed as follows. First, we describe how to compute the restriction $\sigma|_{\Delta_{s,t}}$. For every $h \in I_{s,t}$ the word $\sigma \circ \imath_{T_h}^{-1}$ is a word on $\{1, 2, \ldots, |F_s|\}$, hence it coincides with one of the words

$$\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_L$$

introduced above. We require that the word $\widetilde{\omega}_i$ occurs exactly f_i times for every $i \in \{1, \ldots, L\}$. This amounts to saying that the word $\sigma|_{\Delta_{s,t}}$ has the *collection of frequencies* $f_1, f_2, \ldots, f_L$. Of course, this does not determine $\sigma|_{\Delta_{s,t}}$ uniquely, but only up to a certain permutation. Let $\mathcal{F}_{\Lambda,\mathrm{p}}$ be the set of all Λ-words on $\Delta_{s,t}$ having collection of frequencies $f_1, f_2, \ldots, f_L$. If $\sum_{j=1}^{L} f_j \neq |I_{s,t}|$ the algorithm terminates and yields no output, otherwise $\mathcal{F}_{\Lambda,\mathrm{p}}$ is nonempty. The set $\mathcal{F}_{\Lambda,\mathrm{p}}$ is ordered lexicographically (recall that $\Delta_{s,t}$ is a subset of $\mathbb{N}$). It is clear that

$$\operatorname{card} \mathcal{F}_{\Lambda,\mathrm{p}} = \frac{|I_{s,t}|!}{f_1! f_2! \ldots f_L!} \tag{7.3.8}$$

Thus to encode $\sigma|_{\Delta_{s,t}}$ it suffices to give the index $N_{\mathcal{F}_{\Lambda,\mathrm{p}}}(\sigma|_{\Delta_{s,t}})$ of $\sigma|_{\Delta_{s,t}}$ in the set $\mathcal{F}_{\Lambda,\mathrm{p}}$. We require that $N_{\mathcal{F}_{\Lambda,\mathrm{p}}}(\sigma|_{\Delta_{s,t}}) = N$, and this together with the collection of frequencies $f_1, f_2, \ldots, f_L$ determines the word $\sigma|_{\Delta_{s,t}}$ uniquely. If $N > \operatorname{card} \mathcal{F}_{\Lambda,\mathrm{p}}$, the algorithm terminates without producing output.

Now we compute the restriction $\sigma_{F_t \setminus \Delta_{s,t}}$. Since $F_t \setminus \Delta_{s,t}$ is a finite subset of $\mathbb{N}$, we can simply list the values of σ in the order they appear on $F_t \setminus \Delta_{s,t}$. That is, we require that

$$\sigma|_{F_t \setminus \Delta_{s,t}} \circ \imath^{-1}_{F_t \setminus \Delta_{s,t}} = \upsilon,$$

and the algorithm terminates without producing output if $r \neq \operatorname{card}(F_t \setminus \Delta_{s,t})$.

For all $k \geq 1$, let

$$\{\widetilde{\omega}^k_1, \widetilde{\omega}^k_2, \ldots, \widetilde{\omega}^k_{M_k}\}$$

be the list of all Λ-words of length $|F_k|$ ordered lexicographically. Here $M_k = (\operatorname{card} \Lambda)^{|F_k|}$ for all k. For all $k \geq 1$ and $i \in \{1, \ldots, M_k\}$ define the cylinder sets

$$B^k_i := \{\omega \in \mathrm{X} : \omega|_{F_k} \circ \imath^{-1}_{F_k} = \widetilde{\omega}^k_i\},$$

and let $\beta_k := \{B_1^k, B_2^k, \ldots, B_{M_k}^k\}$ be the corresponding partition of X into cylinder sets for every k. We apply Lemma 7.3.6 to the system $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ and the sequence of partitions $(\beta_k)_{k\geq 1}$. This yields a full measure subset $\mathrm{X}_0 \subseteq \mathrm{X}$ such that for all $\omega \in \mathrm{X}_0$ and all $k \geq 1$ there is an element $h' := h_k(\omega) \in F_k^{-1}$ such that

$$-\sum_{m=1}^{M} \pi_m^{k,h'}(\omega) \log \pi_m^{k,h'}(\omega) \leq H_{\mathrm{Sh}}(\beta_k). \tag{7.3.9}$$

Let $\omega \in \mathrm{X}_0$, $k \geq 1$ be arbitrary fixed and $h' := h_k(\omega) \in F_k^{-1}$ be the group element given by Lemma 7.3.6. Because of the shift-invariance of Kolmogorov complexity we have $\widehat{\mathrm{K}}(h' \cdot \omega) = \widehat{\mathrm{K}}(\omega)$. We will show that

$$\widehat{\mathrm{K}}(\omega) = \widehat{\mathrm{K}}(h' \cdot \omega) \leq \frac{H_{\mathrm{Sh}}(\beta_k)}{|F_k|},$$

then, since $H_{\mathrm{Sh}}(\beta_k) = H_{\mathrm{Sh}}(\alpha_\Lambda^{F_k})$ for all $k \geq 1$, taking the limit as $k \to \infty$ completes the proof of the theorem.

For the moment let n be arbitrary fixed. Observe that for all $i \in \{1, \ldots, M_k\}$

$$\widetilde{\pi}_{n,i}^{k,h'}(\omega) = \frac{1}{|I_{k,n}|} \sum_{h \in I_{k,n}} \mathbf{1}_{B_i^k}(h \cdot (h' \cdot \omega)),$$

i.e. $|I_{k,n}|\, \widetilde{\pi}_{n,i}^{k,h'}(\omega)$ equals the number of times the translates of the word $\widetilde{\omega}_i^k$ along the set $I_{k,n}$ appear in the word $(h' \cdot \omega)|_{\Delta_{k,n}}$. It follows by the definition of the algorithm $A^!$ that the following program describes the word $(h' \cdot \omega)|_{F_n} \circ \imath_{F_n}^{-1}$:

$$\mathrm{p} := \overline{\mathrm{k}}01\overline{\mathrm{n}}01\overline{\mathrm{f}}_1 01 \ldots \overline{\mathrm{f}}_{M_k} 0110 \overline{\mathrm{r}} 01 \mathrm{w} 01 \underline{\mathrm{N}}$$

Here $\overline{\mathrm{f}}_i$ is the doubling encoding of $|I_{k,n}|\, \widetilde{\pi}_{n,i}^{k,h'}(\omega)$ for all $i \in \{1, \ldots, M_k\}$. The binary word w encodes the word $\upsilon = (h' \cdot \omega)|_{F_n \setminus \Delta_{k,n}} \circ \imath_{F_n \setminus \Delta_{k,n}}^{-1}$ of length r and $\underline{\mathrm{N}}$ encodes the index of $(h' \cdot \omega)|_{\Delta_{k,n}}$ in the set $\mathcal{F}_{\Lambda,\mathrm{p}}$.

We will now estimate the length $l(\mathrm{p})$ of the program p above. We begin by estimating the length of the word $\overline{\mathrm{f}}_1 01 \ldots \overline{\mathrm{f}}_{M_k}$. Observe that

$$f_j \leq |I_{k,n}| \leq \frac{|F_n|}{|F_k|} \quad \text{for every } j = 1, \ldots, M_k,$$

hence $l(\overline{\mathrm{f}}_1 01 \ldots \overline{\mathrm{f}}_{M_k}) = o(|F_n|)$. Next, we estimate the length of the word w. Since $(F_n)_{n\geq 1}$ is a Følner sequence, we conclude that $l(\mathrm{w}) = o(|F_n|)$. It is clear that $l(\overline{\mathrm{n}}) \leq 2\lfloor \log n \rfloor + 2 = o(|F_n|)$, since $\frac{|F_n|}{\log n} \to \infty$. Finally, we estimate $l(\underline{\mathrm{N}})$. Of course, $l(\underline{\mathrm{N}}) \leq \log \frac{|I_{k,n}|!}{f_1! f_2! \ldots f_{M_k}!} + 1$. We use Stirling's approximation to deduce that

$$\log \frac{|I_{k,n}|!}{f_1! f_2! \ldots f_{M_k}!} \leq -\sum_{j=1}^{M_k} f_j \log \frac{f_j}{|I_{k,n}|} + o(|F_n|).$$

Hence we can estimate the length of p by

$$l(\mathrm{p}) \leq o(|F_n|) - \sum_{j=1}^{M_k} f_j \log \frac{f_j}{|I_{k,n}|}.$$

Since $f_i = |I_{k,n}|\, \widetilde{\pi}^{k,h'}_{n,i}(\omega)$ for every $i = 1, \ldots, M_k$, we deduce that

$$l(\mathrm{p}) \leq o(|F_n|) - |I_{k,n}| \sum_{j=1}^{M_k} \widetilde{\pi}^{k,h'}_{n,j}(\omega) \log \widetilde{\pi}^{k,h'}_{n,j}(\omega).$$

Dividing both sides by $|F_n|$ and taking the limit as $n \to \infty$, we use Lemma 7.3.6 and Proposition 1.3.3 to conclude that

$$\limsup_{n\to\infty} \frac{\mathrm{K}^0_{A^!}((h'\cdot\omega)|_{F_n} \circ \imath^{-1}_{F_n})}{|F_n|} \leq \frac{H_{\mathrm{Sh}}(\beta_k)}{|F_k|}.$$

By the optimality of A^* we deduce that

$$\widehat{\mathrm{K}}(h'\cdot\omega) = \limsup_{n\to\infty} \frac{\mathrm{K}^0_{A^*}((h'\cdot\omega)|_{F_n} \circ \imath^{-1}_{F_n})}{|F_n|} \leq \frac{H_{\mathrm{Sh}}(\beta_k)}{|F_k|}$$

and the proof is complete. □

7.4. Remarks

7.4.1. The First Theorem of Brudno for Arbitrary Amenable Groups. As far as we know, no work generalizing the theorems of Brudno beyond the groups $\mathbb{Z}$ and $\mathbb{Z}^d$ has been published yet. However, it was pointed out to us by A. Shen after publishing the preprints [Mor15b], [Mor15c] that Andrei Alpeev considered the generalization of the results of S. G. Simpson from [Sim15] in his master thesis 'Entropy and Kolmogorov complexity for subshifts over amenable groups'. One of the main theorems of that thesis asserts that the first theorem of Brudno holds for subshifts over arbitrary computable amenable groups. However, that work has not been published yet and is only available in Russian. The advantage of working with computable Følner monotilings as we do in this thesis is that the proof of the first theorem of Brudno becomes essentially simple, but it is not clear whether computable Følner monotilings exist for all computable amenable groups.

7.4.2. Regularity and Subsequential Ergodic Theorems. Suppose that $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is a left Følner monotiling. Proposition 1.3.3 asserts that for every fixed k

$$\frac{\left|\mathrm{int}^1_{F_k}(F_n) \cap \mathcal{Z}_k\right|}{|F_n|} \to \frac{1}{|F_k|}$$

as $n \to \infty$, which implies 'positive density' of the set $\mathcal{Z}_k$. However, even for $\mathbb{Z}$-actions the pointwise convergence along positive density subsequences might fail, we refer to [Kre85, §8.2, Section 2] for the details. This is why in the definition of a regular Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ we require that for every $k \geq 1$ the indicator function $\mathbf{1}_{\mathcal{Z}_k}$ is a good weight for the pointwise convergence of ergodic averages.

Summary

This thesis is dedicated to studying the theory of entropy and its relation to the Kolmogorov complexity. Originating in physics, the notion of entropy was introduced to mathematics by C. E. Shannon as a way of measuring the rate at which information is coming from a data source. There are, however, a few different ways of telling how much information there is: an alternative approach to quantifying the amount of information is the Kolmogorov complexity, which was proposed by A. N. Kolmogorov.

The Shannon entropy is the key ingredient in the definition of the Kolmogorov-Sinai entropy of a measure-preserving systems. Roughly speaking, the Kolmogorov-Sinai entropy is the expected amount of information in 'Shannon sense' that one obtains per unit of time by observing the evolution of a measure-preserving system. In topological dynamics, the topological entropy takes place of the Kolmogorov-Sinai entropy. For metrizable systems, the topological entropy measures the exponential growth rate of the number of distinguishable partial orbits of length n as n tends to infinity. Originally defined for $\mathbb{Z}$-actions, the 'classical' theories of entropy were later extended to actions of amenable groups. We provide a necessary background on amenable groups, topological/measure-preserving dynamics and the entropy theory in Chapters 1, 2, 3 and 5.

The main focus of this thesis is extending the following results. First of all, a common generalization of the topological and the Kolmogorov-Sinai entropy theories for $\mathbb{Z}$-systems was suggested by G. Palm. We provide an abstract generalization of the work of Palm for actions of discrete amenable groups in the language of measurement functors in Chapter 6.

Secondly, we investigate the connection of entropy and Kolmogorov complexity. Originally, the equality between the topological entropy and a certain quantity measuring maximal asymptotic Kolmogorov complexity of the trajectories was established by A. A. Brudno for subshifts over $\mathbb{Z}$. Later, he proved the equality of the Kolmogorov-Sinai entropy and the asymptotic Kolmogorov complexity of almost every trajectory for ergodic subshifts over $\mathbb{Z}$. We provide a generalization of these results as follows. Firstly, in Chapter 4 we give a background on computability and Kolmogorov complexity and, further, introduce computable Følner monotilings, which are central in our extensions of Brudno's results. We treat the 'first' and the 'second' theorems of Brudno in Chapter 7. The first theorem is generalized for subshifts over computable groups admitting computable Følner monotilings, while the second theorem is proved under the

assertion of regularity of the monotiling, which we introduce in Chapter 7 as well.

Samenvatting

Het doel van dit proefschrift is het bestuderen van entropie en de relatie tussen entropie en Kolmogorov complexiteit. Het begrip van entropie, dat oorspronkelijk uit de natuurkunde komt, is geintroduceerd in de wiskunde door C. E. Shannon als een maat van de informatiedichtheid gegenereerd door een informatiebron. Er zijn nog een paar andere manieren om te bepalen hoeveel informatie er is. Kolmogorov complexiteit, geintroduceerd door A. N. Kolmogorov, is een alternatieve benadering tot dit probleem.

De Shannon entropie wordt gebruikt om de Kolmogorov-Sinai entropie van een dynamisch systeem met een invariante maat te definieren. In grote lijnen, de Kolmogorov-Sinai entropie is de verwachte hoeveelheid informatie (volgens Shannon) die we per tijdseenheid krijgen door waarneming van het gedrag van het systeem. De topologische entropie speelt dezelfde rol in de studie van topologische dynamische systemen. Als het systeem metriseerbaar is, meet de topologische entropie de exponentiële groei van het aantal onderscheidbare trajecten van lengte n in het systeem als $n \to \infty$. De 'klassieke' theorieën van entropie zijn oorspronkelijk ontwikkeld voor $\mathbb{Z}$-groepsacties, en zijn later uitgebreid voor de groepsacties van amenable groepen. In Hoofdstukken 1, 2, 3 en 5 wordt de benodigde achtegrond informatie met betrekking tot amenable groepen, dynamische systemen en entropie gegeven.

Dit proefschrift is vooral gericht op de uitbreiding van de volgende resultaten. Ten eerste, een gemeenschappelijke generalisatie van de topologische en de Kolmogorov-Sinai entropie theorieën voor $\mathbb{Z}$-systemen is ontwikkeld door G. Palm. We geven een abstracte generalisatie van de resultaten van Palm voor groepsacties van discrete amenable groepen, waarvoor we het begrip van maat functoren ontwikkelen.

Ten tweede, onderzoeken we de relatie tussen entropie en Kolmogorov complexiteit. De gelijkheid tussen de topologische entropie en een bepaalde maat van 'maximale asymptotische Kolmogorov complexiteit' van de trajecten voor $\mathbb{Z}$-subshiften is bewezen door A. A. Brudno. Later toonde hij ook de gelijkheid tussen de Kolmogorov-Sinai entropie en de asymptotische Kolmogorov complexiteit aan voor bijna elk traject van een ergodische $\mathbb{Z}$-subshift. We ontwikkelen een generalisatie van deze resultaten als volgt. Ten eerste geven we de achtegrond in berekenbaarheid en Kolmogorov complexiteit in Hoofdstuk 4. We introduceren ook de berekenbare Følner monobetegelingen, die we later in onze uitbreidingen van de resultaten van Brudno gebruiken. We behandelen de 'eerste' en de 'tweede' stelling van Brudno in Hoofdstuk 7. De eerste stelling wordt veralgemenizeerd voor subshifts op berekenbare groepen die een berekenbare Følner monobetegeling hebben. De tweede stelling wordt bewezen voor ergodische subshifts op de berekenbare groepen die een *reguliere* berekenbare Følner monobetegeling hebben, die we in Hoofdstuk 7 intoduceren.

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Curriculum Vitæ

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List of Publications

N. Moriakov, Computable Følner monotilings and the second theorem of Brudno, *Submitted to Ergodic theory Dynam. Systems*, Online at http://arxiv.org/abs/1510.03833, 2016.

N. Moriakov, Fluctuations of ergodic averages for actions of groups of polynomial growth, *Submitted to Studia Mathematica*, Online at http://arxiv.org/abs/1608.05033, 2016.

M. Haase, N. Moriakov, On systems with quasi-discrete spectrum, *Preparing for publication*, Online at http://arxiv.org/abs/1509.08961, 2015.

N. Moriakov, Computable Følner monotilings and a theorem of Brudno I, Online at http://arxiv.org/abs/1509.07858, 2015.

N. Moriakov, Categories of measurement functors. Entropy of discrete amenable group representations on abstract categories. Entropy as a bifunctor into $[0, \infty]$, Online at http://arxiv.org/abs/1509.07836, 2015.

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