Availability-Based Path Selection and Network Vulnerability Assessment

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Abstract—In data-communication networks, network reliability is of great concern to network operators and customers, since (1) the connection requested by the customer should obey the agreed-upon availability, otherwise the network operator or service provider may face liability costs as stipulated in the Service Level Agreement (SLA), and (2) it is important to determine the most vulnerable parts of a network to gain insight into where and how the network operator can increase the network reliability.

In this paper, we first study the problem of establishing a connection over at most \( k \) (partially) link-disjoint paths and for which the availability is no less than \( \delta \) \((0 < \delta \leq 1)\). We analyze the complexity of this problem in generic networks, Shared-Risk Link Group (SRLG) networks and multi-layer networks. We subsequently propose a polynomial-time heuristic algorithm and an exact Integer Non-Linear Program (INLP) for availability-based path selection. The proposed algorithms and two existing heuristic algorithms are compared in terms of acceptance ratio and running time. Subsequently, in three aforementioned types of networks, we study the problem of finding a (set of) network cut(s) for which the failure probability of its links is biggest.

Index Terms—Availability, Routing, Survivability, SRLG networks, Multi-layer networks, Min-cut.

I. INTRODUCTION

Due to the importance of data-communication networks, even short service disruptions may result in significant economic loss. Hence, survivability mechanisms to protect connections are called for. For instance, by allocating a pair of link-disjoint paths (instead of only one unprotected path), data are transported by the primary path, and upon link failure, can be switched to the backup path.

Ideally, a survivability mechanism should also take into account the reliability of links. For instance, if both primary and backup paths contain links that have a high probability to become unavailable, then proper protection cannot be provided. Connection availability, a value between 0 and 1, is therefore important and refers to the probability that a connection (including its survivability mechanism) is in the operating state during the requested life-time of the connection.

However, a survivability mechanism that does not allow for more than 2 link-disjoint paths for each connection may still fail to satisfy the customer’s availability requirement and \( k > 2 \) link-disjoint paths may be needed. Obviously, the bigger \( k \) is, the greater the availability of the connection could be, but also the greater the resource consumption (e.g., bandwidth) and hence price. This paper first deals with the Availability-Based Path Selection (ABPS) problem, which is to establish a connection over at most \( k > 0 \) (fully or partially) link-disjoint paths, for which the availability is at least \( \delta \) \((0 < \delta \leq 1)\).

Apart from considering how to provide a reliable connection to customers, it is also important for network operators to determine the most vulnerable part of the network, i.e., a subset of links with highest failure probability whose removal will disconnect the network. The network operator could then replace/strengthen those links in order to increase the network reliability. Hence, this paper also tackles this so-called Network Vulnerability Assessment (NVA) problem, which is to find a set of network cuts for which the failure probability of the links in a cut belonging to that set is highest.

Our key contributions are as follows:

- We consider the Availability-Based Path Selection (ABPS) problem in generic networks, Shared-Risk Link Group (SRLG) networks and multi-layer networks.
- We prove that, in general, the ABPS problem cannot be approximated in polynomial time.
- We propose a polynomial-time heuristic algorithm and an exact Integer Non-Linear Program (INLP) to solve the ABPS problem.
- We compare, via simulations, the proposed algorithms with two existing algorithms in terms of performance and running time.
- We consider the Network Vulnerability Assessment (NVA) problem in generic networks, SRLG networks and multi-layer networks.

The remainder of this paper is organized as follows. Related work is presented in Section II. Section III explains the calculation of availability for different path types: unprotected path, \( k \) fully link-disjoint and \( k \) partially link-disjoint. In Section IV, we formally define the Availability-Based Path Selection (ABPS) problem in generic networks and analyze its complexity. In Section V and VI, we consider the ABPS problem in SRLG networks and multi-layer networks, respectively. Section VII presents our heuristic routing algorithm and an exact INLP. Section VIII provides our simulation results. In Section IX, we study the Network Vulnerability Assessment problem in the three aforementioned types of networks. We conclude in Section X.

II. RELATED WORK

Availability-aware routing under both static and dynamic traffic demands has been extensively investigated [1], [2], [3], [4], [5], [6]. When the traffic matrix is given in advance (static
traffic), Zhang et al. [3] present a mathematical model to compute availability for different protection types (unprotected, dedicated protection and shared protection) for a given static traffic matrix. Furthermore, an Integer Linear Program (ILP) and a heuristic algorithm are proposed to find availability-aware paths. Tornatore et al. [4] address the availability design problem: to accommodate a given traffic matrix by using shared/dedicated protection paths. Song et al. [5] propose an availability-guaranteed routing algorithm, where different protection types are allowed. They define a new cost function for computing a backup path when the unprotected path fails to satisfy the availability requirement. She et al. [1] prove that for dedicated protection, finding two link-disjoint paths with maximal reliability (availability) is NP-hard. They also propose two heuristics for that problem. Luo et al. [6] analyze the problem of protection with different reliability, which is to find one unprotected path or dedicated protection path such that the cost of the whole path is minimized and the reliability requirement is satisfied. They subsequently propose an exact ILP as well as two approximation algorithms. However, the reliability (availability) calculation in [6] is different from the aforementioned papers, and assumes a single-link failure model. Assuming each link in the network has a failure probability (=1-availability), Lee and Modiano [2] minimize the total failure probability of unprotected, partially link-disjoint and fully link-disjoint paths by establishing INLPs. They further transform the proposed INLPs to ILPs by using linear approximations.

Different from the aforementioned articles, we target a more general problem, which is to find at most $k$ (fully or partially) link-disjoint paths for which the availability requirement is satisfied.

From the perspective of network reliability calculation, assuming each link in the network is associated with a failure probability value (=1-availability), Provan and Ball [7] prove the problem of computing the probability that the network stays connected is \#P-complete. Karger [9] proposes a Fully Polynomial Randomized Approximation Scheme (FPRAS) to solve this problem. There is also work focusing on how to mathematically model the availability of various network topologies or different protection segments/paths. Zou et al. [10] investigate how to mathematically calculate the availability of different types of network topologies, e.g., tree topology, double star, crown or triple star. Tornatore et al. [11] mathematically model the availability of segment protection (SP). In segment protection, a working path (WP) can be partitioned into several working segments (WSs) and each WS is protected by a backup segment (BS). Moreover, they consider two SP cases, namely (1) overlap SP, where different WSs can share the same link, and (2) no-overlap SP, where WSs are fully link-disjoint. By expressing the dual-link failure via a continuous time Markov chain, Mello et al. [12] approximately estimate the (un)availability of the shared protection path.

Regarding SRLG networks, Hu [13] proves that the problem of finding 2 SRLG-disjoint paths is NP-hard. To solve it, Hu [13] presents an exact ILP and Xu et al. [14] propose a trap-avoidance heuristic algorithm. However, the SRLG-disjoint routing problem is not the same as the one studied in this paper, due to Eq. (11) in Section V. Hence, the algorithms in [13] [14] cannot be used to effectively solve our problem.

In generic networks, the $(s,t)$ Min-Cut problem refers to partitioning the network into two disjoint subsets such that nodes $s$ and $t$ are in different subsets and the total weight of the cut links is minimized. According to [15], this problem can be solved by finding the maximum flow from $s$ to $t$. There is also a lot of work on the Min-Cut problem with no specified node pairs $(s,t)$. An enumeration of classical algorithms to solve the Min-Cut problem can be found in [16].

### III. Connection Availability

The availability of a system is the fraction of time the system is operational during the entire service time. Like [1], [2], [3], we first assume that, in generic networks, the links’ availabilities are uncorrelated/independent. If a connection is carried by a single (unprotected) path, its availability is equal to the path availability; if it is protected by $k \geq 2$ disjoint paths, the availability will be determined by these $k$ protection paths. The availability $A_j$ of a network component $j$ can be calculated as [10]:

$$A_j = \frac{MTTF}{MTTF + MTTR},$$

(1)

where $MTTF$ represents Mean Time To Failure and $MTTR$ denotes Mean Time To Repair. We assume that the link availability is equal to the product of availabilities of all its components (e.g., amplifiers).

#### A. Link Failure Scenarios

For simplicity, suppose there are two (fully) link-disjoint paths $p_1$ and $p_2$, and the availability of link $l$ is denoted as $A_l$ ($= 1 - failure \ probability$), where $0 < A_l \leq 1$. Then their total availability $A_{FD}^T$ can be computed based on the following scenarios:

- **Single-link failure**: Here it is assumed that all the links in the network have very low failure probability. In this context, a path $p$’s availability (denoted by $A_p$) is equal to its lowest traversed link availability (highest failure probability), i.e., $A_p = \min_{l \in p} A_l$. Using two disjoint paths (which is a conventional survivability mechanism) will therefore lead to a total connection availability of 1. However, this approach only works when all the links are highly reliable. In Appendix A, we will address the ABPS problem under the single-link failure scenario.

- **Multiple link failures**: This is a more general scenario where at one certain point in time, several links in the network may fail simultaneously. Hence, for a path $p$, its availability $A_p$ should take into account all its links’ availabilities, i.e., $A_p = \prod_{l \in p} A_l$. Consequently, $A_{FD}^T =
1 - (1 - A_{p_i})(1 - A_{p_j}), which indicates the probability that at least one path is available. In this paper, we assume multiple link failures may occur.

B. End-to-End Path Availability

If a path \( p \) contains the links \( l_1, l_2, l_3, \ldots, l_m \), and their corresponding availabilities are denoted by \( A_{l_1}, A_{l_2}, A_{l_3}, \ldots, A_{l_m} \), then the availability of this path (represented by \( A_p \)) is equal to:

\[
A_p = A_{l_1} \cdot A_{l_2} \cdot A_{l_3} \cdots A_{l_m}
\] (2)

If we take the \(-\log\) of the link availabilities, finding the path with the highest availability turns into a shortest path problem.

When, for a single connection, there are \( k \geq 2 \) link-disjoint paths \( p_1, p_2, \ldots, p_k \) with availabilities represented by \( A_{p_1}, A_{p_2}, \ldots, A_{p_k} \), the connection availability calculation can be divided into two cases, namely: (1) fully link-disjoint paths: these \( k \) paths have no links in common, and (2) partially link-disjoint paths: at least two of these \( k \) paths traverse at least one same link. In case (1), the availability (represented by \( A_{PD}^{k} \)) is:

\[
A_{PD}^{k} = 1 - \prod_{i=1}^{k}(1 - A_{p_i}) = \sum_{i=1}^{k} A_{p_i} - \sum_{0<i<j\leq k} A_{p_i} \cdot A_{p_j} + \sum_{0<i<j<l\leq k} A_{p_i} \cdot A_{p_j} \cdot A_{p_l} + \cdots + (-1)^{k-1} \prod_{i=1}^{k} A_{p_i}
\] (3)

Fig. 1: Network with different link availability values.

If we use Eq. (3) to calculate the availability for the partially link disjoint case, the availability of the overlapping links will be counted more than once. To amend this, we use a new operator \( \circ \), which is defined as follows:

\[
X_1 \cdot X_2 \cdots X_k \circ Y = \begin{cases} 
\prod_{i=1}^{k} X_i & \text{if } \exists X_i = Y \\
\sum_{i=1}^{k} X_i \cdot Y & \text{otherwise}
\end{cases}
\] (4)

where \( X_1, X_2, \ldots, X_k \) and \( Y \) represent different link availabilities. Therefore, the availability (represented by \( A_{PD}^{k} \)) of \( k \) partially link-disjoint paths can be calculated as follows:

\[
A_{PD}^{k} = 1 - \prod_{i=1}^{k}(1 - A_{p_i}) = 1 - (1 - A_{p_1}) \circ (1 - A_{p_2}) \circ \cdots \circ (1 - A_{p_k})
\] (5)

By merging Eq. (9) and Eq. (10), we ascertain that \( A_{PD}^{m+1} \geq A_{PD}^{m} \).
IV. ABPS PROBLEM AND COMPLEXITY

A. Problem Definition

Definition 1: Given a network represented by $G(N, L)$ where $N$ represents the set of $N$ nodes and $L$ denotes the set of $L$ links, and each link $l$ has its own availability value $A_l$. For a request represented by $r(s, t, \delta)$, where $s$ and $t$ denote the source and destination, respectively, and $\delta (0 < \delta \leq 1)$ represents the availability requirement, establish a connection over at most $k$ (partially) link-disjoint paths for which the availability is at least $\delta$.

A variant, called the Availability-Based Backup Path Selection (ABBPS) problem, is defined as:

Definition 2: Given an existing primary path $p$ from $s$ to $t$ and a requested availability $\delta$, find at most $k - 1$ paths that are fully or partially link-disjoint with $p$, such that the availability of these $k$ paths is no less than $\delta$.

B. Complexity Analysis

In this section, we study the complexity of the ABPS problem in generic networks. For the case $k = 1$, by taking the $-\log$ of the link availabilities, the ABPS problem turns into a shortest path problem, which is polynomially solvable.

Theorem 2: The ABPS problem is NP-hard for $k \geq 2$.

Proof: The case for partially link-disjoint paths can be reduced to the case of fully link-disjoint paths by a transformation such as in Fig. 2. More specifically, if we assume that all links in Fig. 2, except for $(s, s')$ and $(t', t)$, have availability less than $\delta$, then no link, except for $(s, s')$ and $(t', t)$, can be an unprotected link in the solution of the ABPS problem for the partially link disjoint case from $s$ to $t$. Hence, solving the fully link-disjoint ABPS problem from $s'$ to $t'$ is equivalent to solving the partially link-disjoint ABPS problem from $s$ to $t$. We therefore proceed to prove that the fully link disjoint variant for $k = 2$ is NP-hard. The proof for $k > 2$ follows analogously from the proof for $k = 2$.

Fig. 2: Reduction of ABPS problem from partially link disjoint to fully link disjoint.

We first introduce the NP-hard 3SAT problem [18] and then reduce the ABPS problem to it. The 3SAT problem is defined as: There is a boolean formula $C_1 \land C_2 \land ... \land C_m$, where $C_i$ denotes the $i$-th clause. Each clause contains 3 variables with an OR relation. The question is whether there is a truth assignment to the variables that simultaneously satisfies all $m$ clauses. Given a 3SAT instance, the graph construction follows similarly to [1]. Assume there are $n$ variables in the 3SAT instance. First, we create a lobe for each variable $x_i$, which is shown in Fig. 3, where $q_i$ represents the number of occurrences of variable $x_i$ in all the clauses. The availability value for each link is also shown in Fig. 3, where $0 < b < 1$. For each clause $C_i$ two nodes $y_i$ and $z_i$ are created and a link connects $z_i$ and $y_{i+1}$ with availability of 1, where $0 < i < m$. We assume that $s = x_1$ and $t = x_{n+1}$. Moreover, we draw a link $(s, y_1)$ with availability $a$ and a link $(z_m, t)$ with availability 1, where $0 < b < \frac{1}{2} < 1$. Fig. 4 depicts this process.

To relate the clause and variables in the constructed graph, we add the following links: (i) links $(y_j, u_k)$ and $(v_k, z_j)$ are added if the $k$-th occurrence of variable $x_i$ exists in clause $C_j$; or (ii) links $(y_j, v_k)$ and $(u_k, z_j)$ are added if the $k$-th occurrence of variable $x_i$ exists with a negation in the clause $C_j$.

For instance, a network corresponding to 3SAT instance $(x_1 \land x_2 \land \lnot x_4) \lor (\lnot x_1 \land x_2 \land x_3) \lor (x_2 \land x_3 \land x_4) \lor (\lnot x_1 \land \lnot x_3 \land x_4)$ is shown in Fig. 5. Based on the constructed graph, which

\[ C_1 \lor C_2 \lor ... \lor C_m \]

corresponds to a given 3SAT instance, we are asked to solve the ABPS problem for $k = 2$ and $\delta = a + b\theta - ab\eta$, where $q$ is the sum of occurrences for each variable in all the clauses, i.e., $q = \sum_{i=1}^{n} q_i$. Because one shortest path can at most have availability $a$, which is less than $\delta$, we have to find 2 link-disjoint paths. Next, we will prove that the fully link disjoint variant of the ABPS problem is NP-hard.

3SAT to ABPS: If there exists a truth assignment that satisfies all the clauses, then each clause $j$ has (at least) one variable with true or (negated) false assignment to make this clause true. Therefore, an upper subpath $y_j - u_k - v_k - z_j - y_{j+1}$ or a lower subpath $y_j - v_k - u_k - z_j - y_{j+1}$ will be selected.
By concatenating these \( m \) subpaths with \( s - y_1 \) and \( z_m - t \) we obtain one path (denoted by \( p_1 \)) with availability \( a \). Since each variable only has one truth assignment, \( p_1 \) cannot traverse both the upper subpath and lower subpath in the same lobe. Subsequently, we can get another fully link-disjoint path \( p_2 \): For each lobe \( i \) (corresponding to variable \( x_i \)), \( p_2 \) traverses the upper (lower) subpath with availability of \( b^i \) if \( p_1 \) goes through the link of lower (upper) subpath. The availability of \( p_2 \) is \( b^i = b^i \sum_{t=1}^{q}, \) therefore \( p_1 \) and \( p_2 \) together have availability of \( a + b^i - ab^i \), which satisfies the requirement \( \delta \).

**ABPS to 3SAT:** If there are two fully link-disjoint paths from \( s \) to \( t \) with availability no less than \( a + b^i - ab^i \), then one path must have availability \( a \). To understand this, assume that none of the two paths has availability \( a \); without loss of generality, we denote one path has availability of \( a^i b^i \), where \( c \) can be either 0 or 1 indicating whether link \((s, y_1)\) has been traversed, and \( e > 0 \) is the number of links that have availability \( b \). Since there exists only one link with availability \( a \), the other link-disjoint path has availability \( a^i e^i \), where \( c' \) is either 0 or 1 meaning whether link \((s, y_1)\) has been traversed and \( e' + c' \leq 1 \), and \( f > 0 \) is the number of links which have availability \( b \). Hence, the availability of these two paths is \( a^i b^i + a^i c^i b^i + a^i + c' b^i + f < b + b < a < \delta \), when \( b < \frac{2}{3} \).

Based on this analysis, there must exist one path \( p_1 \) from \( s \) to \( t \) with availability \( a \), which goes through \((s, y_1)\) and \((s, z_m)\) and the other links with availability of 1. To satisfy the availability requirement, there must also exist another fully link-disjoint path \( p_2 \) from \( s \) to \( t \) with availability of no less than \( b^i \). For each lobe, \( p_2 \) should traverse either the upper subpath or the lower subpath, otherwise \( p_1 \) and \( p_2 \) cannot be fully link disjoint. Therefore, \( p_2 \) will traverse the (entire) lower subpath if \( p_1 \) goes through link \((u_k^i, v_k^i)\) in the upper subpath, and traverse the (entire) upper subpath if \( p_1 \) goes through link \((\pi_k^i, \tau_k^i)\) in the lower subpath for each lobe \( x_i \). That is to say, \( p_1 \) cannot simultaneously traverse one link in the upper subpath and another link in the lower subpath for each same lobe. Consequently, \( p_1 \) either goes via an upper subpath \( y_j - u_{j+1}^i - v_{j+1}^i \) to set variable \( x_i \) to true or via a lower subpath \( y_j - \pi_{j+1}^i - \tau_{j+1}^i - z_j - y_{j+1} \) to set variable \( x_i \) to false for clause \( j \), where \( i = 1, 2, ..., n \) and \( j = 1, 2, ..., m \). Hence, all the \( m \) clauses can be simultaneously satisfied.

**Theorem 3:** The ABBPS problem is NP-hard for \( k \geq 2 \).

**Proof:** For \( k \geq 3 \), the ABBPS problem is equivalent to the ABPS problem for \( k - 1 \) fully or partially link-disjoint paths, and hence NP-hard. In Appendix B, we prove that the ABBPS problem is also NP-hard for \( k = 2 \).

We proceed to study the approximability of the ABPS problem.

**Theorem 4:** The ABPS problem for \( k \geq 2 \) cannot be approximated to arbitrary degree in polynomial time, unless \( P=NP \).

**Proof:** We can check in polynomial time whether a single path can accommodate the requested availability. Hence, the theorem is equivalent to: for a request \( r(s, t, \delta) \) and any constant number \( d > 1 \), there is no polynomial-time algorithm that can find at least 2, but at most \( k \), fully or partially link-disjoint paths from \( s \) to \( t \) with availability at least \( \frac{\delta}{d} \). We prove the theorem for the fully link disjoint variant\(^3\) of the ABPS problem for \( k = 2 \).

We will use a proof by contradiction and assume a polynomial-time approximation algorithm \( A \) exists for any \( d > 1 \). In the constructed graph based on the given 3SAT instance in Fig. 5 (also using the same notation and conditions), assume \( \delta = a + b^i - ab^i \), so algorithm \( A \) can find two fully link-disjoint paths with availability at least \( a + b^i - ab^i \).

Next, we prove that when \( 0 < b < \frac{2}{3} \), except for an exact solution, there exists no solution with availability no less than \( a + b^i - ab^i \). If the exact solution is not achieved by algorithm \( A \), according to our previous analysis, then one path must have availability of \( a^i b^i \) and the other path has availability of \( a^i c^i b^i \). Therefore, the availability of these two paths is equal to \( a^i b^i + a^i c^i b^i - a^i + c^i b^i + f \). For a given \( d \), we have \( a^i b^i + a^i c^i b^i - a^i + c^i b^i + f < b + b < 2b = 2h < \frac{2}{3} \) when \( 0 < b < \frac{2}{3} \).

The partially link disjoint variant follows analogously.

V. SHARED-RISK LINK GROUPS

In this section, we assume two types of failures/availabilities, namely Shared-Risk Link Group (SRLG) failures and single link failures/availabilities. A Shared-Risk Link Group (SRLG) [16] reflects that a certain set/group of links in a network will fail simultaneously. For instance, in optical networks, several fibers may reside in the same duct and a cut of the duct would cut all fibers in it. One duct in this context corresponds to one distinct SRLG. If each link is a single member of an SRLG, then no SRLGs exist. Hence the ABPS problem in SRLG networks includes as a special case the ABPS problem in generic networks as discussed in the previous section. Each link can belong to one or more SRLGs, and the links in the same SRLG will simultaneously fail when the corresponding SRLG fails. The probability of this happening (or not) is the SRLG failure (availability) probability. We assume there are \( g \) SRLGs in the network \( G(\mathcal{N}, \mathcal{L}) \), and that the failure probability of the \( i \)-th SRLG (represented by \( srlg_i \)) is denoted by \( \pi_i \), for \( 1 \leq i \leq g \). For a particular link \( l \in \mathcal{L} \), we denote by \( SR(l) \) the set of all SRLGs to which \( l \) belongs. Different from [2], where all SRLG events are assumed to be mutually exclusive, we assume that multiple SRLG events may occur simultaneously. The availability of a single path should incorporate the SRLG availabilities as well as the link availabilities. Consequently, the availability of path \( p \) can be calculated as:

\[
\prod_{i:srlg_i \cap p \neq \emptyset} (1 - \pi_i) \prod_{l \in p} A_l
\]  

\( ^3\)The partially link disjoint variant follows analogously.
where \( \prod_{i:srlg_i \notin p \neq \emptyset} (1 - \pi_i) \) in Eq. (11) is the contribution of all the traversed SRLGs, while \( \prod_{l \in p} A_l \) is the availability of path \( p \) under the condition that all its traversed SRLGs do not fail.

For example, in Fig. 6, suppose there are three SRLGs in the network with failure probabilities 0.1, 0.4 and 0.2, respectively, and all the links have availability 0.9. We calculate the availability of path \( s - a - b - t \), which traverses 2 SRLGs (\( srlg_1 \) and \( srlg_2 \)). The probability that both \( srlg_1 \) and \( srlg_2 \) do not fail is \((1 - 0.1) \times (1 - 0.2) \). Under this condition, all the links on path \( s - a - b - t \) have availability 0.9 and therefore path \( s - a - b - t \) has a total availability of \((1 - 0.1) \times (1 - 0.2) \times (0.9)^3 \) = 0.52488.

Fig. 6: Availability calculation in an SRLG network.

Next, we will prove that the single path variant of the ABPS problem in SRLG networks is NP-hard. To that end, we first introduce the Minimum Color Single-Path (MCSiP) problem, which is NP-hard [19]. Given a network \( G(N, \mathcal{C}) \), and given the set of colors \( \mathcal{C} = \{ c_1, c_2, ..., c_g \} \) where \( g \) is the total number of colors in \( G \), and given the color \( \{ c_l \} \) of every link \( l \in \mathcal{L} \), the Minimum Color Single-Path (MCSiP) problem is to find a path from source node \( s \) to destination node \( t \) that uses the least amount of colors.

**Theorem 5:** The ABPS problem is NP-hard in SRLG networks even for \( k = 1 \).

**Proof:** Assume we have a network where all the links have availability 1 when their SRLGs do not fail, and that there are \( g \) SRLGs with the same failure probability \( \frac{1}{g} \). Hence, a path’s availability is only determined by the number of SRLGs it traverses. If we denote one SRLG by one particular color, then the single-path ABPS problem in SRLG networks can be reduced to the MSCiP problem.

VI. MULTI-LAYER NETWORKS

In multi-layer (e.g., IP-over-WDM) networks or overlay networks, the abstract links in the logical layer are mapped to different physical links in the physical layer. In this context, two or more abstract links that contain the same physical links may have correlated availabilities or failure probabilities. Moreover, usually only the links in the logical layer are known in multi-layer networks.

Let us first consider the example of multi-layer networks shown in Fig. 7. In Fig. 7(a), the availability is labeled on each link in the physical layer, and the links in the logical layer are mapped to the links in the physical layer with the greatest availability. Suppose we want to find a most reliable unprotected path from \( s \) to \( t \) in Fig. 7(a). Since we are only aware of the links in the logical layer, we find that the most reliable path’s availability is 0.72 \( \times \) 0.4 = 0.288. However, the optimal solution is path \( s-b-t \) with availability 0.45 in the physical layer. The reason is that \( (s,a) \) and \( (a,t) \) in the logical layer share the same link \( (s,b) \), which leads to a lower availability value. Fig. 7(b) shows a similar example with Fig. 7(a), except that each link in the physical layer has one additional wavelength number. In the absence of wavelength conversion, it is required that the lightpath occupies the same wavelength on all links it traverses, which is referred to as the wavelength-continuity constraint in WDM-enabled networks. Now, suppose we want to find the most reliable lightpath from \( s \) to \( t \), that obeys the wavelength-continuity constraint. Clearly, if we are only aware of the links in the logical layer, the result is path \( s-a-t \) with availability 0.8 \( \times \) 0.45 = 0.36. However, since this path is mapped to the path \( s-a-b-t \) in the physical layer, it violates the wavelength-continuity constraint. The optimal solution is path \( s-a-t \) in the physical layer via wavelength \( \lambda_1 \). Its availability is 0.8 \( \times \) 0.3 = 0.24.

Fig. 7: A multi-layer network example.
can be used for more than two links. Next, we will prove the
NP-hardness of the ABPS problem for \( k = 1 \) in multi-layer
networks.

**Theorem 6:** The ABPS problem is NP-hard in multi-layer
networks even for \( k = 1 \).

**Proof:** When all the links are uncorrelated in multi-
layer networks, the ABPS with \( k = 1 \) problem is solvable
polynomial time. SRLG networks can be regarded as a special
case of increasing correlation in multi-layer networks, since
the links that share at least one common SRLG group (denote
this link set by \( L' \)) will have a greater availability than
\[
\prod_{l \in L'} \left( \prod_{i \in \text{srlg}_{l} \cap \pi} (1 - \pi_i) \cdot A_i \right).
\]
Since the single-path ABPS problem in SRLG networks is NP-hard, as we proved in
Theorem 5, the single-path ABPS problem in multi-layer
networks is also NP-hard.

VII. HEURISTIC AND EXACT ALGORITHMS

A. Heuristic Algorithm

**Algorithm 1 MMA** (\( G, s, t, \delta, k, I \))

1. Find one shortest path \( P_1 \), return it if the availability
   requirement is satisfied, otherwise go to Step 2.
2. \( ps \leftarrow 1 \), \( H \leftarrow P_1 \), \( P_b \leftarrow \emptyset \) and \( Q \leftarrow \emptyset \)
3. While \( ps \leq k \)
   4. \( P \leftarrow H \)
   5. For each path \( ap \in P \)
      6. \( P_b \leftarrow P \) and \( \text{counter} \leftarrow 0 \)
      7. While \( \text{counter} \leq I \)
         8. Randomly remove one link \( (u, v) \in ap \) and find
            one shortest path \( \psi_{u \rightarrow v} \) from \( u \) to \( v \).
         9. If it succeeds then
            10. Replace \( (u, v) \) with \( \psi_{u \rightarrow v} \) in \( ap \), denote as \( ap' \)
                \( P_b \), Remove\( (ap) \), \( P_b \), Add\( (ap') \), \( ap \leftarrow ap' \)
            12. Find another link-disjoint path \( p_2 \) with \( P_b \).
                Return \( \{p_2\} \cup P_b \) if \( \delta \) is met.
            14. For each link \( (u, v) \in ap' \)
                15. If its availability is at least \( \delta \) then
                    16. \( Q \), Add\( ((u, v)) \)
                17. While \( (Q \neq \emptyset) \)
                    18. \( (u, v) \leftarrow \text{EXTRACT-MIN}(Q) \)
                    19. Find a path \( p_3 \) which shares \( (u, v) \) with \( ap' \).
                        If \( (p_3 \in P_b) \) and \( \{p_3\} \cup P_b \) satisfy \( \delta \) then
                            22. \( \text{Return} \{p_3\} \cup P_b \)
                        else \( H \leftarrow \text{Max}_\text{Availability}(H, \{p_3\} \cup P_b) \)
                    23. \( \text{counter} \leftarrow \text{counter} + 1 \).
            24. \( ps \leftarrow ps + 1 \)

Our heuristic, called Min-Mins Algorithm (MMA) to solve
the ABPS problem in generic networks, SRLG networks and
multi-layer networks, is presented in Algorithm 1. Since
we want to use as least (and no more than \( k \)) link-disjoint paths
to satisfy the requested availability, we gradually increase the
number of paths.

The pseudo code to solve the ABPS problem in multi-layer
networks is similar to the one for generic networks, except
for the path availability calculation. Also, the pseudo code to
solve the ABPS problem in SRLG networks is similar to the
one in generic networks, and we will specify the differences
later. In what follows, we explain each step of the heuristic
algorithm. We assign link \( l \in L \) with the weight of \(-\log(A_l)\)
\(-\log(\prod_{i \in SRLG \setminus SRLG}(1 - \pi_i) \cdot A_i)\) for SRLG networks in MMA. If a
shortest path (represented by \( p_1 \)) in Step 1 fails to satisfy
the availability requirement, we keep it as the initial path
flow. In Step 2, we use \( ps \) to record the number of already
found link-disjoint paths. Initially \( ps \) is set to 1. \( H \) stores
the already found \( ps \) link-disjoint paths, and it is initially
assigned \( p_1 \). While \( ps \) is no greater than \( k \), Steps 3-24 continue
finding a solution. In Step 4, we assign to \( P \) the already found
paths \( H \). Based on \( P \), from Step 5 to Step 23, we each time
select one path \( ap \) from path set \( P \). We also use a variable,
denoted by \( \text{counter} \) in Algorithm 1, to record the number
of iterations. Initially, \( \text{counter} \) is set to 0. As long as the
number of iterations is less than an input value \( I \), Steps 7-23
proceed finding a solution based on \( ap \) and path set \( P_b \). The
(sub)path from \( u \) to \( v \) found by the algorithm is denoted by
\( \psi_{u \rightarrow v} \). In Step 8, we randomly remove one link \( (u, v) \) from
\( ap \), and we apply a shortest path algorithm from \( u \) to \( v \) to obtain a path \( \psi_{u \rightarrow v} \). By concatenating subpath \( \psi_{u \rightarrow v} \) and
the links of path \( ap \) except for \( (u, v) \), we obtain a new path \( ap' \).
Further, by substituting \( ap \) with \( ap' \) in \( P_b \), we have a new path set \( P_b \). After that, the algorithm tries to find \( P_b \)’s fully link-
disjoint path in Step 12. When solving the ABPS problem
in SRLG networks, since each SRLG only contributes once to
the path availability calculation, the link \( l \)’s weight is set to
\(-\log(\prod_{i \in \text{srlg}}(1 - \pi_i) \cdot A_i)\) before running a shortest
path algorithm in Step 12 (also the same for Step 19), where
\( \text{srlg} \) are the common traversed SRLGs between link \( l \) and
path set \( P_b \). If it fails to find \( p_2 \) or \( \{p_2\} \cup P_b \) cannot satisfy
the availability requirement, the algorithm tries to find a path
which is partially link disjoint with \( ap' \) (in Steps 13-22).

The general idea is that we first use a queue \( Q \) to store the links in
\( ap' \) whose availability is no less than \( \delta \) in Steps 14-16. After
that, as long as \( Q \) is not empty in Steps 17-22, each time
the link with the greatest availability in \( Q \) is extracted as the
unprotected link (represented by \( (u, v) \)), and then we remove
all the links traversed by \( ap' \) except for \( (u, v) \). Subsequently,
we find one shortest path \( \psi_{s \rightarrow t} \) from \( s \) to \( t \) (if it exists),
and find another shortest path \( \psi_{s \rightarrow t} \) from \( u \) to \( t \) (if it exists).
By concatenating \( \psi_{s \rightarrow t} \), \( (u, v) \), and \( \psi_{s \rightarrow t} \), we can get a new
path \( p_3 \), which is partially link disjoint with \( ap' \). If \( a \) and \( b \)
denote different sets of \( k > 1 \) link-disjoint paths, the function
Max_Availability\( (a, b) \) in Step 22 returns the one with greater
availability.

The time complexity of MMA can be computed as follows. Step 1 has a time complexity of \( O(N \log N + L) \). From Step
3 to Step 24, there are at most \( O(I) + O(2I) + \cdots + O(kI) =
O(k^2 I) \) iterations before the algorithm terminates. Steps 14-
16 have a time complexity of \( O(N) \) since a path contains
at most \( N - 1 \) links and therefore Steps 17-22 consume
\(O(N(N \log N + L))\) time. Finally, the whole time complexity of MMA is \(O(k^2 IN(N \log N + L))\).

### B. Exact INLP Formulation

In this subsection, we present an exact Integer Non-Linear Program (INLP) to solve the ABPS problem in generic, SRLG and multi-layer networks. We first solve the ABPS problem in generic networks and start by explaining the required notation and variables.

**INLP notation:**
- \(r(s, t, d)\): Traffic request, with source \(s\), destination \(t\) and requested availability \(d\).
- \(A_{i,j}\): Availability of link \((i, j)\).
- \(g\): The total number of SRLGs.
- \(\pi_{m}^{u}\): The failure probability of the \(m\)-th SRLG if link \((i, j)\) belongs to it, otherwise it is 0.

**INLP variable:**
- \(P_{r,i,j}^{u}\): Boolean variable equal to 1 if link \((i, j)\) is traversed by path \(u\) (1 \(\leq u \leq k\)) for request \(r\); 0 otherwise.

**Flow conservation constraints:**

\[
\sum_{(i,j) \in \mathcal{L}} P_{r,i,j}^{u} - \sum_{(j,i) \in \mathcal{L}} P_{r,i,j}^{v} = \begin{cases} 
1, & i = s \\
-1, & i = t \\
0, & \text{otherwise}
\end{cases} \quad (12)
\]

\(\forall i \in \mathcal{N} \quad 1 \leq u \leq k\)

**Availability constraint:**

\[
\sum_{u=1}^{k} \prod_{(i,j) \in \mathcal{L}} \left( 1 - P_{r,i,j}^{u} + P_{r,i,j}^{u} A_{i,j} \right) - \\
\sum_{1 \leq u \leq k} \prod_{(i,j) \in \mathcal{L}} \min \left( 1 - P_{r,i,j}^{u} + P_{r,i,j}^{u} A_{i,j}, 1 - P_{r,i,j}^{u} + P_{r,i,j}^{u} A_{i,j} \right) \\
+ \cdots + (-1)^{k-1} \left( \prod_{(i,j) \in \mathcal{L}} \min \left( 1 - P_{r,i,j}^{u} + P_{r,i,j}^{u} A_{i,j} \right) \right) \geq \delta \quad (13)
\]

When both the flow conservation constraint (Eq. (12)) and the availability constraint (Eq. (13)) are satisfied, an optimal solution is found by the INLP; otherwise there is no solution. There is no objective (needed) in the proposed INLP, but one could include the objective of minimizing the number of paths (or links) used. Eq. (12) accounts for the flow conservation for each of the at most \(k\) paths. For a particular \(u^{th}\) path (1 \(\leq u \leq k\)), it ensures that (i) for the source node \(s\) of request \(r\), the outgoing traffic for each request is 1; (ii) for the destination node \(t\) of request \(r\), the incoming traffic is 1; and (iii) for an intermediate node which is neither source nor destination, its incoming traffic is equal to the outgoing traffic. Eq. (13) ensures that either the found single unprotected path or the (partially) link-disjoint paths should have availability no less than \(\delta\), according to the availability calculation of \(k\) link-disjoint paths in Eqs. (3) and (5). Since the overlapped link’s availability in the partially link-disjoint calculation according to Eq. (5) can only be counted once, we take the minimum value of the variables \(P_{r,i,j}^{u}\) for each link and then take the product over all the links for (partially) link-disjoint paths. We also note that Eq. (13) can simultaneously calculate the availability of the fully link disjoint variant, partially link disjoint variant and the unprotected variant. For instance when \(k = 2\), Eq. (13) becomes:

\[
\prod_{(i,j) \in \mathcal{L}} \left( 1 - P_{r,i,j}^{1} + P_{r,i,j}^{1} A_{i,j} \right) + \prod_{(i,j) \in \mathcal{L}} \left( 1 - P_{r,i,j}^{2} + P_{r,i,j}^{2} A_{i,j} \right) - \\
\prod_{(i,j) \in \mathcal{L}} \min \left( 1 - P_{r,i,j}^{1} + P_{r,i,j}^{1} A_{i,j}, 1 - P_{r,i,j}^{2} + P_{r,i,j}^{2} A_{i,j} \right) \geq \delta \quad (14)
\]

When \(P_{r,i,j}^{1} = P_{r,i,j}^{2}\) for all \((i, j) \in \mathcal{L}\), Eq. (14) is equal to

\[
\prod_{(i,j) \in \mathcal{L}} \left( 1 - P_{r,i,j}^{1} + P_{r,i,j}^{1} A_{i,j} \right) \geq \delta
\]

or

\[
\prod_{(i,j) \in \mathcal{L}} \left( 1 - P_{r,i,j}^{2} + P_{r,i,j}^{2} A_{i,j} \right) \geq \delta
\]

which is the availability constraint for a single unprotected path.

To solve the ABPS problem in SRLG networks, we need to slightly modify Eq. (13) (and keep flow conservation constraints Eq. (12) the same) by using

\[
\prod_{1 \leq m \leq g} \min_{u=1} \left( 1 - P_{r,i,j}^{u} + P_{r,i,j}^{u} \pi_{m}^{u} \right)
\]

to multiply the left side of Eq. (13), which is the non-failure probability of the SRLGs which at most \(k\) link-disjoint paths have traversed.

In multi-layer networks, the availability of a subset of links may not be equal to the product of their availabilities. Therefore, we need one more function \(f(L)\), which can return the joint availability of a subset of links \(L\) in multi-layer networks. The parameter of this function is a 0/1 link vector which contains \(L\) elements, where 1 denotes the link is present to be calculated and 0 means it is not. Consequently, to solve the ABPS problem in multilayer networks exactly, we could replace the operator \(\prod\) with \(f()\) in Eq. (13), and keep all the other notation and constraints the same as for the generic networks case.

### VIII. Simulation Results

#### A. Simulation Setup

![Fig. 8: USA carrier backbone network.](image_url)

We conduct simulations on two networks, one is USANet, displayed in Fig. 8, which is a realistic carrier backbone network consisting of 24 nodes and 43 links, and the other is GÉANT, shown in Fig. 9, which is the pan-European communications infrastructure serving Europe’s research and
networks, in multi-layer networks, on the same (individual) link availabilities as for generic networks, we assume that there are randomly been assigned to at most \( l \) links, if a one-degree node becomes the source or the destination of a certain request, the request can only be served by partial protection (or a single unprotected path). In this context, a feasible path may not exist in GÉANT, which will result in blocking. In terms of performance, the INLP achieves the highest AR. On the other hand, MMA shows a higher AR than the other two heuristics TRA and MRA (Fig. 10(a)).

For the high availability requirement scenario (shown in Figs. 10(b) and 10(c)), as expected, the AR of all these algorithms is lower than in the general availability requirement case. In this scenario, the INLP requires more time to find a solution, especially when a solution does not exist. In order to let the INLP return the result in a reasonable time, we set the time limit for it to serve one request to 50 minutes. Due to this reason, we can see that INLP has the lowest AR in USANet and often second highest AR in GÉANT. Meanwhile, MMA still has the highest AR in most of the cases.

The time limit for the INLP is even more constraining in the case of SRLG networks, leading to a very poor performance for SRLG networks. We have therefore omitted the results of the INLP in SRLG networks. Since the optimal solution rarely exists in the high availability requirement case, we only provide the simulation results for the heuristic algorithms in the general availability requirement case. Moreover, to have a fair comparison, we compare our algorithms with MRA and a modified TRA [2], which is a heuristic routing algorithm proposed for probabilistic SRLG networks. Its main idea is that after finding the first shortest path, the remaining link weights should be adjusted (We slightly change its link weight adjustment to be the same with the Step 12 of MMA for a fairer comparison), and then to find another link-disjoint shortest path. Fig. 11 shows that the proposed heuristic algorithm MMA still achieves higher AR than these two algorithms.

Similar to SRLG networks, the exact INLP in multi-layer networks is very time consuming. We therefore omit the results of the INLP in multi-layer networks. Fig. 12 provides the

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**Results**

We first evaluate the performance of the algorithms in terms of Acceptance Ratio (AR) in generic networks. Acceptance ratio (AR) is defined as the percentage of the number of accepted requests over all the requests. We first analyze the general availability requirement case: In USANet, all the algorithms achieved an AR of 1. We therefore omit the figure of the general availability performance for USANet. However, this is not the case for the GÉANT topology. From Fig. 10(a), we can see that the performance of all algorithms is under 0.95. Since GÉANT is not as well connected as USANet is, some nodes in GÉANT only have degree one (e.g., nodes 3, 8, etc.), if a one-degree node becomes the source or the destination of a certain request, the request can only be served by partial protection (or a single unprotected path). In this context, a feasible path may not exist in GÉANT, which will result in blocking. In terms of performance, the INLP achieves the highest AR. On the other hand, MMA shows a higher AR than the other two heuristics TRA and MRA (Fig. 10(a)).

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**Fig. 9: GÉANT pan-European research network.**

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in the general availability requirement case.

IX. NETWORK VULNERABILITY ASSESSMENT

A. Problem Definition and Complexity Analysis

Finding the most vulnerable part of a network as well as the previously considered problem of availability-based path selection are both important elements for network robustness. In this section, we study the \((s,t)\) Network Vulnerability Assessment (NVA) problem. That is, find one or a set of (equal-weight) network cuts whose failure probability of the links in the cut is highest. A network cut refers to a set of links, whose removal will result in the disconnection of the network. Formally, the NVA problem can be defined as follows:

**Definition 3:** The \((s,t)\) Network Vulnerability Assessment (NVA) problem: Given is a network \(G(N, L)\), and each link \(l \in L\) is associated with a failure probability \(f_l = 1 - A_l\). Given a source \(s\) and a target \(t\), find an \(s - t\) cut \(C\) for which \(\prod_{l \in C} f_l\) is maximized. In case there are multiple cuts of highest weight all of them should be returned.

When the node pair \((s,t)\) is not specified, we denote this problem as the NVA problem, which can be solved by solving the \((s,t)\) NVA problem at most \(N - 1\) times. Therefore, these two problems share the same hardness.

We use \(-\log(f_l)\) for the weight of link \(l\) in the network. In generic networks, the NVA problem can be solved by finding all the min-cuts in \(O(L^2N + N^2L)\) time according to [21]. On the other hand, we will prove that the NVA problem in SRLG networks is NP-hard. Recall that in SRLG networks, introduced in Section V, two types of failures/availabilities should be incorporated, namely Shared-Risk Link Group (SRLG) failures and single link failures/availabilities, therefore the probability that all the links belonging to path \(p\) fail simultaneously (denoted by \(F(p)\)) can be calculated as:

\[
1 - \prod_{i:r \in p, r \neq \emptyset} (1 - \pi_i) + \prod_{i \in p} f_i - \prod_{i:r \in p, r \neq \emptyset} (1 - \pi_i) \prod_{i \in p} f_i
\]  

Finally, in Table I, we present the (average) running times per request for these four algorithms in generic, SRLG and multi-layer networks. It shows that the INLP is significantly more time consuming than the three polynomial-time heuristics. On the other hand, MMA has only a slightly higher running time than MRA and TRA, but it pays off by having a higher AR as shown in Figs. 10-12. Another observation is that, for the same algorithm in the same network, the running time is higher for the high availability requirement case than in the general availability requirement case.
Fig. 12: AR of heuristics in multi-layer networks: (a)-(b) general availability requirement; (c)-(d) high availability requirement.

Please note that Eq. (15) is different from the failure probability of a path in SRLG networks: 1-Eq. (11).

Before we prove the NP-hardness of the NVA problem, let us first study the Maximum Color Path Selection (MCPS) problem. Contrary to the MCSiP problem, the MCPS problem is to find a single path such that it uses the largest amount of colors.

**Theorem 7:** The Maximum Color Path Selection (MCPS) problem is NP-hard.

**Proof:** We distinguish two cases, namely, (1) all the links have exclusive colors, i.e., there does not exist any color that is shared/overlapped by two or more links, and (2) two or more links may contain the same color(s).

Case (1): The MCPS problem is equivalent to the NP-hard Longest Path problem [18], which is to find a path from s to t such that its weight is maximized.

![Fig. 13: Reduction of the MCPS problem to the Disjoint Connecting Paths problem.](image)

Links (a,b) and (c,d) together contain totally g distinct colors.

Case (2): We first introduce the Disjoint Connecting Paths problem [18]. Given a network $G(\mathcal{N}, \mathcal{L})$, and a collection of disjoint node pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_z, t_z)$, does $G$ contain $z$ mutually link-disjoint paths, one connecting $s_i$ and $t_i$ for each $i, 1 \leq i \leq z$? This problem is proved NP-hard when $z \geq 3$, and we reduce the MCPS problem to it. In Fig. 13, assume that there are in total $g$ colors and links $(a, b)$ and $(c, d)$ share $0 < x < g$ common colors, but they together contain $g$ distinct colors. Except for these two links, the other links are assigned $0 < y < g$ colors, but there does not exist two or more links containing $g$ distinct colors. In this context, finding a path from $s$ to $t$ with the largest amount of colors is equivalent to finding three mutually link-disjoint paths between three node pairs $(s, o), (b, c) \text{ and } (d, t)$.

**Theorem 8:** The Network Vulnerability Assessment (NVA) problem in SRLG Networks is NP-hard.

![Fig. 14: Example network.](image)

![Fig. 15: Reduction of the (s,t) NVA problem in SRLG networks to the MCPS problem.](image)

**Proof:** It is equivalent to prove that the (s,t) NVA problem is NP-hard.

In Fig. 14, assume all link availabilities are 1 and all links have non-zero SRLG failure probabilities, except for links $(x_i, x_{i+1})$ and $(z_i, z_{i+1})$ which have 0 SRLGs, where $1 \leq i \leq n-1$. Assume there are $g$ SRLG events in total, and each SRLG event occurs with a probability of $\frac{1}{g}$. In this context, for a path $p$ without links $(x_i, x_{i+1})$ and $(z_i, z_{i+1})$, according to Eq. (15), $F(p)$ can be calculated as:

$$1 - (1 - \frac{1}{g})^m$$  (16)

where $m$ is the total number of distinct SRLGs traversed by $p$. Therefore, to maximize Eq. (16) one needs to maximize $m$, i.e., to find a path with the greatest probability that all its traversed links fail simultaneously. This is equivalent to finding a path having the largest number of distinct SRLGs. We want to solve the $(x_1, z_1)$ NVA problem. Based on Fig. 14, we first...
derive Fig. 15 with the same nodes except that we add one more node \( s \). We regard \( s = y_0 \) and \( t = y_n \). The link weight in Fig. 15 is set as follows: \( (y_{i-1}, x_i) \) and \( (y_{i-1}, z_i) \) have 0 SRLGs, while \( (x_i, z_i) \) and \( (z_i, y_i) \) have the same SRLGs as in Fig. 14, where \( 1 \leq i \leq n \). In Fig. 15, we are asked to solve the MCPS problem from the source \( s \) to the destination \( t \).

Since we want to find a cut that separates \( x_1 \) and \( z_1 \), any cuts in the form of \( (x_i, y_i) \) and \( (y_i, z_i) \), where \( 1 \leq i \leq n \) are not part of the optimal solution. Moreover, considering the link in the form of \( (x_i, x_{i+1}) \) or \( (y_i, y_{i+1}) \) has 0 SRLGs and single link availability 1, which means its availability is 1 or failure probability is 0, it cannot lead to the optimal solution as well. Based on the above observations, any feasible cut \( C \) should contain one link either \( (x_i, y_i) \) or \( (y_i, z_i) \), for all \( 1 \leq i \leq n \). We prove in the following that the \((s,t)\) NVA problem in Fig. 14 can be reduced to the MCPS problem in Fig. 15 in polynomial time, where, for simplicity, we assume only one optimal cut exists in the \((s,t)\) NVA problem.

\((s,t)\) NVA to MCPS: An optimal solution of the \((s,t)\) NVA problem should be composed of either \( (x_i, y_i) \) or \( (y_i, z_i) \), where \( 1 \leq i \leq n \). Let \( C_{NVA} \) reflect the set of links in the optimal solution. Because \( C_{NVA} \) has the largest amount of distinct SRLGs, \( C_{NVA} \) together with \( (s, x_1) \) or \( (s, z_1) \) forms a path from \( s \) to \( t \) with the maximum number of SRLGs.

MCPS to \((s,t)\) NVA: Let \( R_{MCPS} \) denote the set of links in the optimal solution of the MCPS problem. Because \( R_{MCPS} \) has the largest amount of SRLGs, let \( C_{NVA} = R_{MCPS} \setminus \{(y_i, x_{i+1}), (y_i, z_{i+1})\} \). Considering that the links \( (y_i, x_{i+1}) \) and \( (y_i, z_{i+1}) \) have 0 SRLGs, \( C_{NVA} \) also has the largest amount of SRLGs. Therefore solving the MCPS problem yields a solution to the \((s,t)\) NVA problem.

**Theorem 9:** The NVA problem in multi-layer networks is NP-hard.

**Proof:** For any two links \( l \) and \( m \) in a multi-layer network, we have that \( A_l \odot A_m = (1 - f_l) \odot (1 - f_m) = 1 - f_m - f_l + f_l \odot f_m \). Similar to Theorem 6, we can also prove that it is NP-hard to find one path from the source to the destination in multi-layer networks such that the probability that all its links fail is maximized. Subsequently, we can prove that the NVA problem in multi-layer networks is NP-hard, which follows analogously with Theorem 8. We therefore omit the details.

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**X. Conclusion**

The availability of a connection represents how reliably a connection can carry data from a source to a destination. In this paper, we have first studied the Availability-Based Path Selection (ABPS) problem, which is to establish a connection over at most \( k \) (fully or partially) link-disjoint paths, such that the total availability is no less than \( \delta \) (\( 0 < \delta \leq 1 \)). We have proved that, in general, the ABPS problem is NP-hard and cannot be approximated in polynomial time for \( k \geq 2 \), unless \( P = NP \). We have further proved that in SRLG networks and multi-layer networks, even the single-path (\( k = 1 \)) variant of the ABPS problem is NP-hard.

We have proposed a polynomial-time heuristic algorithm and an exact INLP to solve the ABPS problem in generic networks, SRLG networks and multi-layer networks. Via simulations, we have found that our heuristic algorithm outperforms two existing algorithms in terms of acceptance ratio with only slightly higher running time. On the other hand, the running time of the exact INLP is significantly larger (by several orders of magnitude) than the running time of the heuristic algorithms.

Finally, we have proved that the Network Vulnerability Assessment (NVA) problem, which is to find a cut of the network for which the failure probability of all its links is highest, is solvable in polynomial time in generic networks, but NP-hard to solve in SRLG networks and multi-layer networks.

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**References**


Algorithm 2 ABPSSingleLinkFailure\((G, s, t, \delta, k)\)

1. Eliminate the links with availability less than \(\delta\) on \(G\), thereby obtaining a new graph \(G'\).
2. Run Dijkstra’s algorithm on \(G'\) from \(s\) to \(t\). If the solution is found then return the result; Else if \(k > 1\), continue; Otherwise output there is no solution.
3. Run Suurballe’s algorithm [20] on \(G\) from \(s\) to \(t\). Return the result if the solution is found, otherwise continue.
4. Create another (parallel) link between \(u\) and \(v\) with the same availability \(A(u,v)\), for each \((u,v) \in E\) if \(A(u,v) \geq \delta\). The graph is denoted as \(G''\).
5. Run Suurballe’s algorithm [20] on \(G''\) from \(s\) to \(t\). Return the result if the solution is found, otherwise output there is no solution.

![Fig. 16: Reduction of the ABBPS problem with partially link-disjoint paths to the Disjoint Connecting Paths problem.](image)

The links have an availability of 1. We now want to find a partially link-disjoint path with GP such that their combined availability is no less than 1. Since the requested availability is 1, only link \((a,b)\) and link \((c,d)\) can be unprotected in an optimal solution. Suppose that when link \((a,b)\) is eliminated, there do not exist paths from node \(s\) to nodes \(b, c, d\) and \(t\), and when link \((c,d)\) is eliminated, there are no paths from node \(b\) to nodes \(d\) and \(t\). In this context, to solve the partially link-disjoint ABBPS problem for \(k = 2\), both \((a,b)\) and \((c,d)\) should be unprotected in the optimal solution. This is equivalent to finding three pairs of link-disjoint paths between node pairs \((s,a)\), \((b,c)\) and \((d,t)\) (i.e., the Disjoint Connecting Paths problem). Hence, solving the partially link-disjoint ABBPS problem for \(k = 2\) yields a solution to the NP-hard Disjoint Connecting Paths problem.

\[\text{Algorithm 2 ABPSSingleLinkFailure}\]

**Appendix B**

**Hardness of the ABBPS Problem for** \(k=2\)

For some variants, like the fully link disjoint case, the ABBPS problem is polynomially solvable\(^4\), however it is NP-hard in its general setting.

**Theorem 10:** The partially link-disjoint ABBPS problem for \(k = 2\) is NP-hard.

**Proof:** We provide a proof for when \(k = 2\). As it is shown in Fig. 16, assume we are given a path (denoted by GP) \(s-a-b-c-d-t\) with the availability labeled on each link and all the other links have availability of 1. We now want to find a partially link-disjoint path with GP such that their combined availability is no less than 1. Since the requested availability is 1, only link \((a, b)\) and link \((c, d)\) can be unprotected in an optimal solution. Suppose that when link \((a, b)\) is eliminated, there do not exist paths from node \(s\) to nodes \(b, c, d\) and \(t\), and when link \((c, d)\) is eliminated, there are no paths from node \(b\) to nodes \(d\) and \(t\). In this context, to solve the partially link-disjoint ABBPS problem for \(k = 2\), both \((a, b)\) and \((c, d)\) should be unprotected in the optimal solution. This is equivalent to finding three pairs of link-disjoint paths between node pairs \((s, a)\), \((b, c)\) and \((d, t)\) (i.e., the Disjoint Connecting Paths problem). Hence, solving the partially link-disjoint ABBPS problem for \(k = 2\) yields a solution to the NP-hard Disjoint Connecting Paths problem.

\[\text{Algorithm 2 ABPSSingleLinkFailure}\]

\(^4\)Trivially, by looking for an unprotected path with maximum availability in the network where all the links from the primary path are excluded.