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Electromagnetism in the Electrical Engineering Classroom

Dominant trends in teaching classical electromagnetic field theory and innovation vectors.

This article explores some dominant trends in teaching classical electromagnetic (EM) field theory in electrical engineering (EE) undergraduate curricula. The acronym *EM* will be used interchangeably to designate either *electromagnetic* or *electromagnetism*. The intended significance will be evident from the context.

The focus of this article is on identifying innovation vectors, to equip students to understand and fittingly apply computational EM (CEM) tools as required when addressing present-day EM engineering challenges. Two aspects in need of critical reexamination are selected: the mathematic interpretation of EM field quantities and the conditions validating the use of bulk material properties. Both elements are inspected from the perspective of their incorporation into computational schemes. Additionally, the conceptual benefits of stressing the encompassing role played by special relativity in classical electrodynamics is highlighted. This survey produces guidelines for programmatically recalibrating expert training within EE curricula, primarily at the undergraduate level.

INTRODUCTION

EE has made true technological leaps in recent years, introducing unprecedented perspectives toward a green and (practically) fully interconnected society. Identifying the transforming solutions that society expects is unthinkable without an insightful understanding of the foundations of EE, and solid training in classical EM is one of the pillars of any EE curriculum. The undergraduate EE curriculum, with an emphasis on its electronic and telecommunication tracks, is what we scrutinize in this study.

While not claiming that they are exhaustive, our experiences with a wide range of European EE curricula show that the mainstream approach to teaching EM field theory starts by discussing the static fields, which are then used as a platform to discuss (quite basic) electrodynamics in the early bachelor's (B.Sc.) degree phase. Subsequently, the specialized courses in the master's (M.Sc.) degree curriculum focus on (advanced) EM propagation and scattering phenomena, as well as their applications.

These courses seldom, if ever, revisit basic EM concepts such as field quantities or influence of matter, so EE graduates' general perceptions owe a lot to the often-simplistic definitions given in early B.Sc. courses. Further, the increasing accessibility of custom-designed and commercial CEM tools brought them into the classroom, both as a supporting instrument [1]–[6] but also in an increasingly systematic manner [7]–[9], with [10] rightfully observing their decisive role in any present-day EM curriculum.

However, indiscriminate use of CEM and enclosed commercial platforms can become detrimental, as noted in the preface of [6]. Referring to M.Sc. students, our observation is that overuse of CEM leads to further obscuring the physical background of EM quantities and phenomena, resulting in not infrequent simulations that make perfect computational sense but have little, if any, physical justification. Interestingly enough, the insufficient preoccupation for the conceptual basis of the EM field extends into postgraduate (Ph.D.) education. For example, advanced courses in the European School of Antennas [80] program develop competencies in state-of-the-art design and modeling techniques, but seldom touch on the nature and physical significance of the computed quantities.

The results of this pedagogy are the following prevalent perceptions in the antenna engineering (AE) community, as summarized in the standard textbooks that most frequently provide the EM background for teaching AE [11]–[17].

1) EM quantities are vector quantities.
2) Maxwell's equations ([18]; see also the extremely informative historical perspectives in [19] and [20]) are introduced as such and live a life of their own.
3) Propagation in homogeneous and isotropic embedding (free space) and scattering from (im)penetrable objects are the only relevant EM aspects.
4) Inhomogeneity-related phenomena are confined to extremely small subdomains that are habitually accounted for via circuital models.
5) Time-harmonic operation is ubiquitous, with time-domain studies being, in fact, multifrequency analyses with little concern for the differences in the physical background and modeling instruments (with de Hoop [11] being a notable exception).

While it is principally sufficient to handle standard design and development tasks, this unilateral and occasionally simplistic view does not adequately equip the future EE (and, especially, AE) experts with the insight needed to tackle the great EM engineering challenges that lie ahead. These tests include either those posed by nanotechnologies, with nanospheres, nanodipoles, or dimers becoming critical enablers as optical antennas [21]–[27], or from emerging carbon-based nanoelectronics [28].

To address this situation, we critically reexamine the basic principles that underpin both undergraduate courses on classical EM and high(er)-level, system-engineering ones that build on classical EM. To this end, inspiration is drawn from reference physics textbooks on classical EM, with [29] being one of the most widely used and [30] and [31] offering an exceptional bridge between specialized physics curricula and EE profiles. To better serve the training of the experts, we focus on the following three aspects that we deem conditional for a proper understanding of EM phenomenology within the EE context:

1) the general framework, by also stressing the crucial interrelation between classical electrodynamics and classical mechanics via special relativity
2) the proper understanding of the EM field quantities, with an emphasis on duality and the geometrical interpretation of both the quantities themselves and the governing equations
3) the adequate understanding of bulk material parameters, with intrinsic limitations following from quantum mechanics and technological realizability.

Our analysis now proceeds by examining three aspects from a twofold perspective: the present state of undergraduate curriculum and the innovation vectors. The discussion will be confined to the conceptual foundations, with technical details intentionally left out as all these technicalities are elaborately covered in the cited references. Conclusions will be drawn at the end.

Throughout our study, the position in the configuration is specified by the coordinates $\{x, y, z\}$ with respect to a background Cartesian reference frame with origin O and three mutually orthogonal unit vectors $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ that, in this order, form a right-handed system. The position vector is $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, with $|\boldsymbol{r}| = r$, and the time coordinate is t. The EM terminology used in this account is aligned with that used in EE [11]: $\boldsymbol{E}(\boldsymbol{r}, t) =$ electric field strength; $\boldsymbol{B}(\boldsymbol{r}, t) =$ magnetic flux density; $\boldsymbol{D}(\boldsymbol{r}, t) =$ electric flux density; $\boldsymbol{H}(\boldsymbol{r}, t) =$ magnetic field strength; $\boldsymbol{J}(\boldsymbol{r}, t) =$ the volume density of the electric (convection) current; $\boldsymbol{J}^{\text{ext}}(\boldsymbol{r}, t) =$ the volume density of the impressed (external) electric current; $\boldsymbol{\varepsilon}(\boldsymbol{r}) =$ permittivity [$\varepsilon(\boldsymbol{r})$ in the case of isotropic media]; $\boldsymbol{\mu}(\boldsymbol{r}) =$ permeability [$\mu(\boldsymbol{r})$ in the case of isotropic media]; in free space $\varepsilon = \varepsilon_0$; and $\mu = \mu_0$, with $c_0 = (\varepsilon_0 \mu_0)^{-1/2}$ representing the EM wave speed.

GENERAL FRAMEWORK

WHERE WE ARE

As clearly illustrated by textbooks [29]–[31], the physics approach to teaching EM is to first treat electrodynamics, with the central position of special relativity as the unifying bridge between Maxwell's electrodynamics and mechanics [30, p. 455] already highlighted at an early stage. The electrostatic and magnetostatic fields are then introduced as particular cases, by constantly stressing the intimate relationship between the two facets of the EM field [30, p. 477].

EE curricula follow a different path. The foundation of the EM training is laid down at the start of the B.Sc. program, by accounting for the limitations in the mathematic baggage of first-year students. (An observation in [32, p. vii] is quite in place.) Teaching EM starts with discussing static fields and then proceeds to a (more or less) brief examination of electrodynamics. EM wave propagation and scattering are primarily taught in basic M.Sc. courses by starting from Maxwell's equations, which are introduced axiomatically. Little attention is given to reexamining the nature of the EM field quantities, while deriving electrodynamics from special relativity is seldom, if at all, mentioned.

INNOVATION VECTORS

We deem that training of the kind discussed is insufficient to prepare the (M.Sc.) EE student for tackling advanced topics (such as integrated electronics and optics, critical elements in present-day AE), for which interdisciplinary dialogue, especially with fundamental physics branches, is indispensible. The following vectors can play an instrumental role in remedying this situation.

EM FIELD: THE BIG PICTURE

Any EE M.Sc.-level course should begin with properly defining the nature of the EM field, with [31, pp. 1–2] providing an excellent platform to this end. The EM field is ultimately a combination of $\boldsymbol{E}$ and $\boldsymbol{B}$ manifesting themselves in concert at every time instant t and point in space $\boldsymbol{r}$. The fields carry energy, momentum, and angular momentum. This perspective can (and should) already be stated during B.Sc.-level introductory courses.

An observation is due at this point. EE EM curriculum focuses on the field equations as a trampoline toward deriving transmission models that rely on power-transfer arguments, an approach primarily justified by standard physical measurement protocols. As a result, the energy considerations are elaborately covered in the curriculum. However, the EM momentum and angular momentum are largely overlooked due to an (apparent) lack of practical utility. Developing deep-space exploration instruments is likely to render this viewpoint obsolete. Insufficient familiarity with EM (angular) momentum entails a restrictive view of the EM field, with phenomena such as EM wave pressure difficult to comprehend and handle. We strongly advocate for insisting on the complete EM picture, in which field quantities and full electrodynamics form an interdependent, unitary whole.

EM FIELD AND COMPUTATIONAL TECHNIQUES

Solving the vast majority of EM problems requires applying some computational techniques [31, p. 302]. Paradoxically, the CEM fundamentals are seldom (if at all) a part of EM courses, so that M.Sc. (and even Ph.D.) students have a limited ability to design numerical analyses and interpret numerical results. While general and specific numerical methods are discussed in dedicated courses, it is the role of M.Sc. EM courses to discuss the following two aspects that are quintessential for any CEM technique.

1) Any meaningful discretization of the EM field quantities requires a thorough understanding of the nature of, and the interdependencies between, field quantities (see the "Field Quantities" section). By discretization, physical problems translate into mathematic models. Numeric results are representations of the physical states, which are constructed under specific assumptions that must be considered when assessing the adequacy of numeric results.
2) Numeric methods require some space–time discretization. For examining the propagation in homogeneous embedding + (im)penetrable scatterer configurations, a spatial discretization at the scatterers' boundaries offers an encompassing solution. However, the study of (highly) inhomogeneous configurations can only be carried out via local analysis techniques that call upon a full space–time discretization. The coarseness of the spatial discretization of such problems is selected for providing the level of detail required by the problem at hand. Unfortunately, it is less realized that this coarseness is bounded below by the limit of the validity of the macroscopic EM laws [31, pp. 286–289]. From an EE perspective, an even better lower bound is given by the scale at which macroscopic measurements are still feasible, which is termed the *mesoscopic scale* [33], [34]. A subdivision beyond these scales makes perfect sense from a computational point of view, but has no physical significance. Once a spatial discretization is selected, the temporal discretization is dictated by the mathematics of the employed numeric technique. In view of its outstanding relevance, the spatial discretization will be focused upon in the "Material Parameters" section.

EM FIELD AND SPECIAL RELATIVITY

Upon acquiring a proper understanding of the EM fundamentals, M.Sc. students should be introduced to the intimate relationship between EM field theory and special relativity. Without going into detail, EE M.Sc. students must conceptualize that the EM field is "relativity at low speeds" [31, p. 148], with special relativity bridging electrodynamics and mechanics [30, p. 455]. An adequate instrument to this end is discussing some didactic examples [35]–[39]. Further, we believe that a didactic benefit can be drawn from presenting landmark experiments that paved the way toward special relativity [40, Ch. 1]. The next level of conceptual complexity is deriving the EM field equations from special relativity. Admittedly, this would be too much of a detour in an EE undergraduate course although it can definitely be considered for Ph.D.-level courses. Interested students will find excellent self-study guidelines for understanding these topics in textbooks [29]–[31], [40]–[42].

FIELD QUANTITIES

One of the fundamental points that is insufficiently elucidated in the EE EM curriculum is the mathematic representation of the field quantities and its physical justification. This circumstance results in incomplete comprehension of the depth and complexity of the EM phenomena. By elaborating on the arguments in [43], this section will parallel the present prevalent framework in EE EM training and some alternatives, along two lines: the mathematic representation of the EM field quantities and the selection of the field quantities to be evaluated, based on the implicit complementarities manifesting between them. The next step will be to highlight the importance of judiciously selecting the spatial support to be employed when defining field quantities. These elements will eventually be combined into an easily comprehensible and versatile framework that is properly rooted in the physics of (highly) inhomogeneous configurations and can be directly mapped onto a computational scheme [44]. This framework also representss a highly apt didactic instrument for understanding the EM field in all its complexity and beauty.

FIELD QUANTITIES REPRESENTATION

EM theory and, subsequently, AE-related topics are practically always taught based on the field quantities represented as vectors that depend continuously on space and time coordinates and have a time-harmonic temporal dependence. (These field quantities with continuous space–time dependence are routinely used in local field equations that require the corresponding functions to be at least differentiable, if not twice-differentiable, as when wave equations are derived. From this perspective, stepping over to integrated field equations, as those employed in the "Space–Time Domain-Integrated EM Field Model" section, require the mathematically weaker integrability condition.) The continuous dependence on space and time is directly reflected in how the EM field quantities are constructed via Green's function representations, a strategy that seems natural in the typical AE scenario: localized sources; free-space propagation; (im)penetrable scatterers. Nonetheless, little, if any, attention is given to some (intrinsic) limitations of this ansatz.

The preference for the vector representation is justified mostly on a historic basis (see the excellent overview in [19] and the supplementary information in [45]) and by the facility of its (mathematic) handling. Moreover, in free space it enables a complete description of the EM field [31, pp. 1–2]. However, as soon as matter and material interfaces are present, the vector rendition encounters difficulties. From a strictly mathematic point of view, it leads to inconsistencies as concerns symmetry, as analyzed in [19]. Even more importantly, the continuous spatial dependence can result in (integrable) field singularities. While such behavior bears manifest conceptual benefits, as cogently demonstrated in [46], it is not amenable to physical measurements and cannot be reproduced numerically. Moreover, the singular behavior requires the EM field equations to preserve their form at indiscriminately small scales, a fact that collides with the matter's discrete structure at the (sub)atomic scale.

The Green's function type propagation also deserve discussion. To begin with, it assumes EM propagation through linear, homogeneous, isotropic embeddings or similar stratifications with convenient geometries. Strictly speaking, the conjunction of linearity, homogeneity, and isotropy applies solely to free space, although it can also be safely assumed for gases. However, these features only make sense in condensed matter by invoking bulk material properties, the validity of which is limited below by the scale at which atomic averaging is meaningful [31, Sec. 81]. A note is due at this point: Recently, there has been a pronounced trend in AE toward using artificially engineered materials that technologically consist of nonnegligibly small elements. Propagation through homogeneous, isotropic media (almost) never applies to such materials, although, sadly, bulk properties are incorrectly employed in many simulation-driven designs [49]. (Note that in a reduced number of cases, such as those presented in [47] and [48], some "averaged" materials parameters can be asymptotically inferred for artificially engineered. The validity of this inference is conditioned by the structure containing enough elements as to give rise to the relevant bulk behavior. This situation is largely equivalent to that described in [31, Sec. 81].)

In view of the vector representation with continuous space–time dependence encountering the difficulties mentioned, we now examine two possible alternatives.

DIFFERENTIAL FORMS REPRESENTATION

The starting point is the observation that the vector interpretation of the EM field quantities is, in fact, geometric in nature and its purpose is to indicate that these quantities not only have a magnitude but are also testing direction dependent. Once the geometric nature of this problem is recognized, the next step is to identify instruments that are suitable to manage it, implicitly accounting for the spatial discontinuities in the support of EM problems and, as much as possible, ensuring the metric invariance of the formulations.

The sought for solution was offered by the differential forms algebra (also referred to as *exterior calculus*) [50]. Casting the classical EM field theory into a differential forms formalism originates in the pioneering works [51]–[54] and was elaborated upon in [55]. Differential forms were applied in EM over a very broad range, starting with basic formulations [56] up to high-level studies [57]. The main difference with respect to the prevalent, vector mathematic representation of EM quantities is dropping the continuous $\boldsymbol{r}$-dependence, which was unwarranted in this case, and replacing it with a small integrals interpretation.

The immediate consequence is that considering material parameters at a point becomes meaningless, matter being included in the formalism via parameters that are somehow averaged over small volumes. These ideas emerged in a large number of reference works mainly concerned with the computation of static and stationary fields [58]–[65]. Surprisingly, this approach had a reduced echo in AE, with [66] acting as one of the singular

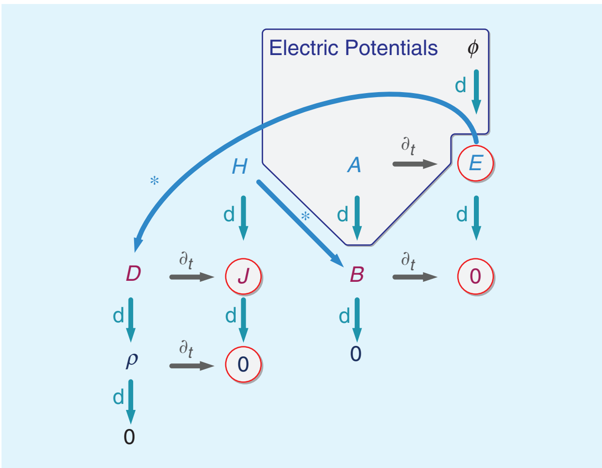

FIGURE 1. A diagram of Maxwell's equations using differential forms [54]. E and H are 1-forms, B, J, and D are 2-forms, and ρ is a 3-form; d stands for the exterior derivative [50, Sec. 8.1], ∂_t for the time derivative, and * for the Hodge operator [52], [53] (it is supplemented by a material parameter). The electric potentials are A (a 1-form) and ϕ (a 0-form). The $J \leftrightarrow E$ relation was intentionally left out of the scheme.

references that mentions the use of differential forms and [67] employing a Whitney forms discretization (a typical differential forms instrument), but within a vector wave equation. A similar situation also manifests itself in the realm of physics, with Hehl and Obukhov [68, p. 8] noting the scarcity of works on classical EM using exterior calculus.

For brevity, the differential forms expression of the EM field equations, suggestively summarized in Figure 1, is omitted here; readers are referred to the elaborate treatments in [51]–[55]. Furthermore, following arguments in [54], the analysis in [43] highlights the manner in which the symmetries in the scheme in Figure 1 reflect important EM field features and insist on the conceptual benefits entailed, as well as on the difficulties encountered in the computer-code implementation of this approach to interpreting EM field quantities.

Teaching EM via differential forms algebra can be rewarding (see [69] for the educational profits of this avenue). Such an effort can take advantage of a sufficient number of excellent textbooks using this instrument in EM theory [55], [59], [68], [70], [71]. Translating Maxwell's equations into this framework is formally very easy, as demonstrated by [54] and [55] (also see the summary in [43]). The most relevant conceptual benefits are 1) an articulated "big picture" (see Figure 1) in which the interdependencies between field sources and field quantities are clearly illustrated and 2) a direct translation of the formalism into numeric schemes.

The exterior calculus perspective on EM has similarities with the space–time domain-integrated field relations method [33] that we will elaborate upon in the "Space–Time Domain-Integrated EM Field Model" section. Note that this method was constructed, in essence, as a computational tool. Nonetheless, its physical justification is similar to using differential forms, with which it also shares a common point of view on material properties that are only accounted for in an averaged manner, and only over finite volumes.

TENSOR REPRESENTATION

An alternative interpretation of the EM field quantities is as tensors. This perspective is best illustrated by [72] (see also [41, Ch. 10]). This avenue also attempts to reconcile the geometric features of the mathematic representation with the physical fabric of the represented quantities. Recall that using a tensor formalism is at the core of special relativity, and that standard physics textbooks [29]–[32], [41] include substantial space on the algebra and, above all, on the physical substrate of the four-vectors and tensors describing relativistic quantities. It is also worth noting that de Hoop [72] explicitly advocated teaching EM via a tensor formalism, insisting on the insight benefits obtained at the price of using, in fact, quite basic mathematical tools. From this perspective, that formalism offers singular opportunities as a conceptualization vehicle and an educational tool, with classroom implementations testing the students' proficiency in comprehending and manipulating EM field notions.

SELECTION OF THE FIELD QUANTITIES

As we decidedly indicated at the beginning of the "Innovation Vectors" section, since duality and complementarity are at the core of the EM field concept, these primordial features should be stressed at the inception of any EM course. In EE curricula, preeminence is given to one field quantity or another for emphasizing the wave-like propagation of the EM field. However, the need to concurrently account for two complementary field quantities is the norm in CEM by now and should become mainstream in classrooms.

Selecting the correct combination of representative field quantities is crucial. In EE in general and especially in AE, there seems to be a preference for the $\boldsymbol{E} \leftrightarrow \boldsymbol{H}$ duality. This choice stems from the association of $\boldsymbol{E}$ with voltages and of $\boldsymbol{H}$ with electric currents (construed as the sources of the magnetic field). The combination of the two is put in correspondence with a circuital perspective on the operation of EM devices. In a CEM context, opting for the $\boldsymbol{E} \leftrightarrow \boldsymbol{H}$ duality is also justified by the (apparent) similarity in representation of the two quantities.

Nonetheless, in physics there is no doubt that $\boldsymbol{E} \leftrightarrow \boldsymbol{B}$ is the proper duality [30, p. 477]. The fundamental justification of this choice is dictated by special relativity: $\boldsymbol{E}$ and $\boldsymbol{B}$ are the quantities intervening in the four-vectors and tensors providing the relativistic description of the EM field [29]–[32], [41]. But opting for $\boldsymbol{B}$ (as opposed to $\boldsymbol{H}$) has important benefits from an engineering perspective as well. As indicated in [44], operating with $\boldsymbol{E}$ and $\boldsymbol{B}$ in a CEM context entails exclusively evaluating field quantities that are continuous across any (locally) smooth interface. It must be noted that any imposed discontinuity of the applicable field components of $\boldsymbol{E}$ and $\boldsymbol{B}$ requires invoking active magnetic charge distributions or currents (with "active" being interpreted as in [11, Sec. 18.3]). While induced magnetic charges or currents do serve a purpose in CEM, imposing them requires acknowledging their physical existence and all available observations compellingly contradict this (see also [68, p. 3]).

HIERARCHY AND DUALITY IN $\mathbb{R}^3$ OBJECTS

Recognizing that defining the EM field quantities in a somewhat integrated manner, as opposed to assigning them a value at any given point, presents clear conceptual advantages, it becomes important to carefully examine the nature of the spatial support of those aimed-at integrals. This problem received an elegant and conclusive solution in [73] and [74], in which a general topological hierarchy of 3D manifolds was constructed and the physical properties of the EM quantities were mapped on those manifolds. The core ideas of this theory are summarized in [43] and those arguments are now briefly reiterated for the convenience of the reader.

The framework in [73] distinguishes between configuration variables associated with an inner orientation and source variables associated with an outer orientation (see Figure 2). Field quantities only make sense in integrated form; consequently, this framework is termed the *finite formulation of the EM field*. Duality plays a fundamental role in that construction: the field quantities have dual topological supports that directly find their counterpart in the duality of field quantities. As a result, any

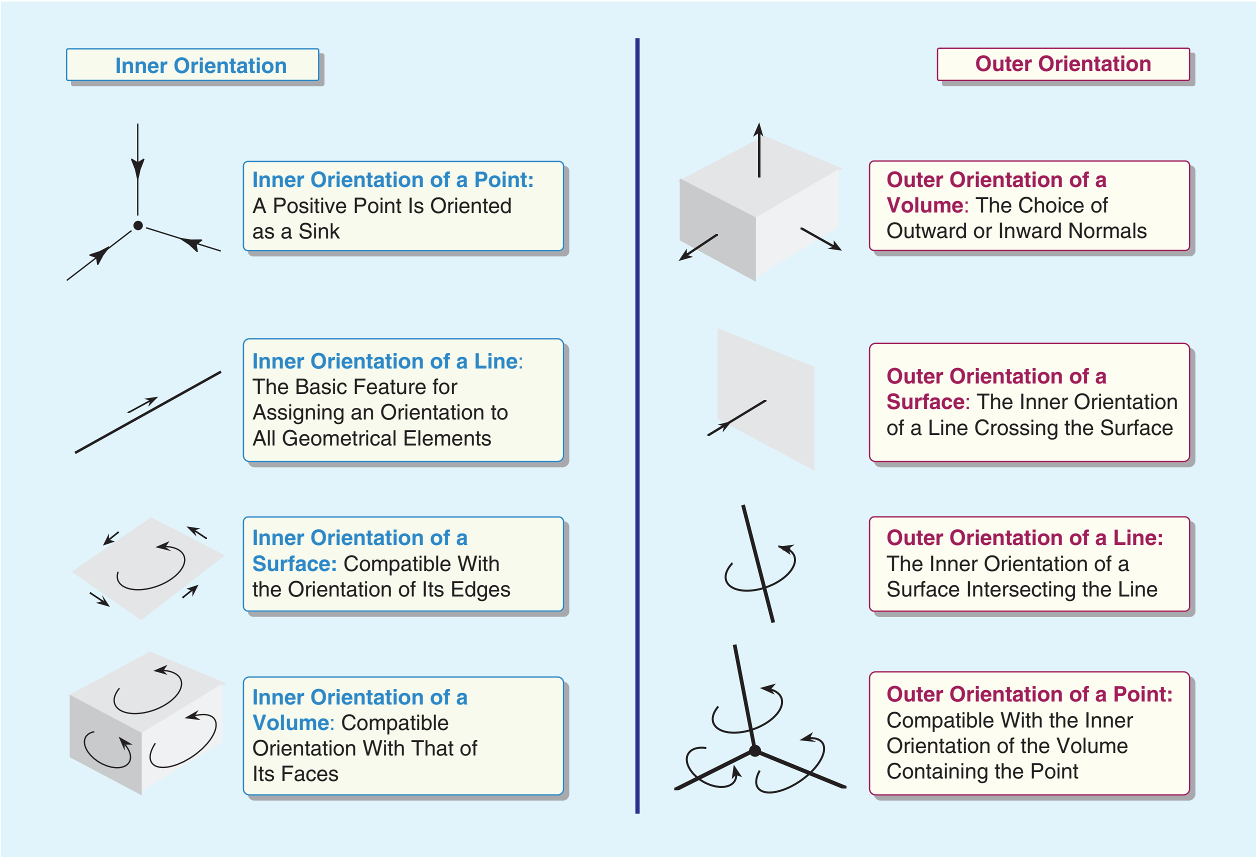

FIGURE 2. The inner and outer orientations, according to the scheme in [73, Fig. 2]. The inner orientation is fully defined along a line, surface, etc., while the outer orientation requires crossing a line, surface, etc. Except for points, all orientations are consistently derived from the inner orientation on a line. The duality between geometric elements is illustrated. (Used with permission from [73].)

computational scheme must make use of dual meshes and the supports of the integrals that involve configuration and source variables cannot be collocated. It is noteworthy to observe that this strategy meant a return to the principles pioneered in [75] for hexahedral meshes. Nonetheless, the finite formulation of EM field was also applied to unstructured grids via Voronoy mesh duality.

SPACE–TIME DOMAIN-INTEGRATED EM FIELD MODEL

The arguments presented were combined in the space–time domain-integrated field relations in EM, as proposed in [34]. It admittedly is a primarily computational construction but with a solid physical background. That method was revisited in [44] and its physical basis was refined further. In view of their value as an insightful didactic instrument, we summarize the applicable main choices and equations, as given in [34] and [44].

The following prerequisites apply.

1) The method is designed to analyze highly inhomogeneous domains inside which inhomogeneity is (possibly) preserved down to the mesoscopic scale (see the definition in the "Innovation Vectors" section).
2) A simplicial (tetrahedral) mesh is constructed at mesoscopic scale. The mesh fits tightly on the boundaries of the subdomains where material continuity can be assumed. Strictly for computational purposes, the simplicial mesh should be a Delaunay one and mesh refinements are exclusively performed to ensure this condition. Note that the Delaunay meshing is not required from a conceptual point of view.
3) Define the constructed simplicial mesh as the primal mesh and its barycentric dual [73] as the dual mesh.
4) Discretize EM field quantities via consistently linear edge and face expansion functions [76] on the primal mesh: $\boldsymbol{E}$ and $\boldsymbol{H}$ via edge expansion functions and $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{J}$, and $\boldsymbol{J}^{\text{ext}}$ via face expansion functions. In this manner, field quantities are defined on the boundary of the simplicial cells only and then are extrapolated into the cells' interior. (Algebraic topology ensures the possibility to employ a consistently linear spatial expansion based on the limiting values of the expanded quantities upon approaching nodes, edges, and faces.) This expansion is needed for performing line, surface, and volume integrations of the local EM field quantities.
5) Complement the spatial discretization by a linear time discretization.

Two elements are quintessential to this scheme: the simplicial mesh and using exclusively linear interpolations. Since only (somehow) integrated quantities can be subject to physical measurements, it is logical to select the simplest integration supports, namely lines, triangles, and tetrahedra. Further, local quantities can only be inferred from those integral quantities via some interpolation/extrapolation procedure and the linear interpolation offers the natural instrument to this end. Computationally, the conjunction of simplicial decomposition and linear interpolation allows expressing all entailed integrals analytically, with obvious computational effectiveness benefits.

By following the reasoning in [34], but making use of the customary volume densities of (impressed) electric currents, the space–time domain-integrated field relations are

$$\int_{\partial\mathcal{S}\times\mathcal{T}} \hat{\boldsymbol{\tau}}\cdot\boldsymbol{E}(\boldsymbol{r},t)\,\mathrm{d}L\mathrm{d}t + \int_{\mathcal{S}} \hat{\boldsymbol{n}}\cdot\boldsymbol{B}(\boldsymbol{r},t)\,\mathrm{d}A\Big|_{\partial\mathcal{T}} = 0 \tag{1}$$

$$\int_{\partial\mathcal{S}\times\mathcal{T}} \hat{\boldsymbol{\tau}}\cdot\boldsymbol{H}(\boldsymbol{r},t)\,\mathrm{d}L\mathrm{d}t - \int_{\mathcal{S}} \hat{\boldsymbol{n}}\cdot\boldsymbol{D}(\boldsymbol{r},t)\,\mathrm{d}A\Big|_{\partial\mathcal{T}} - \int_{\mathcal{S}} \hat{\boldsymbol{n}}\cdot\boldsymbol{J}(\boldsymbol{r},t)\,\mathrm{d}A|_{\partial\mathcal{T}} = \int_{\mathcal{S}} \hat{\boldsymbol{n}}\cdot\boldsymbol{J}^{\mathrm{ext}}(\boldsymbol{r},t)\,\mathrm{d}A\Big|_{\partial\mathcal{T}}. \tag{2}$$

The following notations are used in the space–time domain-integrated field relations: $\mathcal{D}$ = a bounded domain with piecewise smooth boundary $\partial\mathcal{D}$; $\mathcal{S}$ = a simply connected subsurface of $\partial\mathcal{D}$ with piecewise smooth boundary $\partial\mathcal{S}$; $\hat{\boldsymbol{n}}$ = the unit vector along the outward normal to $\partial\mathcal{D}$ (the orientation on $\partial\mathcal{S}$ and that of $\hat{\boldsymbol{n}}$ are related by means of the right screw rule); $\hat{\boldsymbol{\tau}}$ = the unit vector along the tangent to $\partial\mathcal{S}$; and $\mathcal{T}$ = a bounded time interval with boundary $\partial\mathcal{T} = \{t_1, t_2\}$, the notation $|_{\partial\mathcal{T}}$ being used for $f(t)|_{\partial\mathcal{T}} = f(t_2) - f(t_1)$, with $f(\cdot)$ an arbitrary function.

These relations are supplemented with the following volume (source) integral relations:

$$\int_{\partial\mathcal{D}} \hat{\boldsymbol{n}}\cdot\boldsymbol{B}(\boldsymbol{r},t)\,\mathrm{d}A\Big|_{\partial\mathcal{T}} = 0 \tag{3}$$

$$\int_{\partial\mathcal{D}} \hat{\boldsymbol{n}}\cdot\boldsymbol{D}(\boldsymbol{r},t)\,\mathrm{d}A\Big|_{\partial\mathcal{T}} = \int_{\mathcal{D}} \hat{\boldsymbol{n}}\cdot\boldsymbol{J}(\boldsymbol{r},t)\,\mathrm{d}A\Big|_{\partial\mathcal{T}} + \int_{\mathcal{D}} \hat{\boldsymbol{n}}\cdot\boldsymbol{J}^{\mathrm{ext}}(\boldsymbol{r},t)\,\mathrm{d}A\Big|_{\partial\mathcal{T}}. \tag{4}$$

As usually, $\int_{\partial\mathcal{D}} \hat{\boldsymbol{n}}\cdot\boldsymbol{D}(\boldsymbol{r},t)\,\mathrm{d}A$ is interpreted as the total (time-dependent) electric charge enclosed by $\partial\mathcal{D}$. Inside domains where no volume charge density is stored, as is the case in the vast majority of (C)EM relevant configurations, (4) has a zero right-hand side term as in [34], with the usual current continuity condition being readily entailed. The space–time domain-integrated field relations must be complemented by suitable $\boldsymbol{E}\leftrightarrow\boldsymbol{D}$, $\boldsymbol{E}\leftrightarrow\boldsymbol{J}$ and $\boldsymbol{B}\leftrightarrow\boldsymbol{H}$ mappings, with [33], [34], and [44] providing the guidelines.

Equations (3) and (4) generalize the standard Gauss's laws. However, as stressed in [34], they are in fact space–time integrated compatibility relations since a summation of (1) and (2) applied to any subsurface composing $\partial\mathcal{D}$ automatically yields (3) and (4). Two important observations apply to these space–time domain-integrated relations.

1) In line with [73] and [74], (1) and (3) are written for curves and surfaces with an inner orientation, while (2) and (4) for surfaces with an outer orientation. In view of the assumed prerequisites, (1) and (3) are associated with elements of the primal mesh, whereas (2) and (4) are associated with elements of the dual mesh.
2) Relations (1) and (3) have the dimension of action rated by charge and (2) and (4) have that charge, with action and charge being the fundamental mechanical and electrical physical observables (as discussed in Rumsey [77]), respectively. At this point, we note that [31, p. 273] speculated on the benefits (and beauty) of a CEM method using action as a basic quantity and observed the unavailability of such a formulation; the space–time domain-integrated EM field model may offer that missing tool.

In the classroom, the space–time domain-integrated relations framework offers a unique instrument for demonstrating the geometric fabric of the EM field equations while requiring extremely basic mathematic tools. Furthermore, its close relation to the manner in which physical measurements are effectuated renders it very tangible and easily comprehensible. Moreover, judiciously interrelating integral quantities with some local values (via linear interpolation) provides a path toward defining field quantities, while precluding any singularity. At a higher conceptual level, constructing the field quantities via the consistently linear expansions associated with simplexes of the primal mesh enables expressing straightforwardly surface integrals on any configurational surface. In particular, it allows expressing the integrals needed for applying the EM reciprocity theorems [11, Ch. 28], a fundamental result in classical EM field theory.

MATERIAL PARAMETERS

The last aspect to be elucidated is accounting for material parameters. Here it is noted that most AE works concentrate on very simple, homogeneous, and isotropic media. Moreover, the examined configurations often consist of contrasting scatterers that are immersed in an embedding (free space or a simple dielectric with permittivity $\varepsilon = \varepsilon_r\varepsilon_0$, with the relative permittivity ε_r as a scalar constant). However, the technological evolutions mentioned in the "Introduction" make use of extremely fragmented media, possibly down to a mesoscopic level. The materials in these subdomains are themselves nontrivial, such as the strongly anisotropic artificial dielectrics proposed in [47] and [48]. As a result, examining the adequate manners to include intricate material properties in EM formalisms is fully justified.

WHERE WE ARE

The mainstream approach in EE is to account for bulk material parameters inside subdomains of arbitrary shapes and bounded by sharp transition boundaries (interfaces) where the standard, local interface boundary conditions are applied; see, for example, [11, Sec. 20.1], [12, Sec. 1.5], [14, Sec. 11.4], and [15, Sec. 1.5). The relevant material properties are inferred via (often implicit) averaging strategies, an approach that concurs with the manner in which macroscopic measurements are performed. This avenue is not free of caveats, with the most evident being the legitimacy of the averaging procedure that yields the employed bulk property. We highlight two situations when this legitimacy is debatable (other doubtful situations may also be identified).

NATURAL MEDIA

Bulk (electrical) properties can only be inferred above a scale that greatly exceeds that at which the large atomic variations manifest themselves, with [31, Sec. 81] giving a lower limit of about 10 nm, above which quantum fluctuations can be deemed "washed out." For most present-day technologies, there is a safe margin down to that limit. However, pushing toward higher operational frequencies and integrated circuits miniaturization may compress that margin. An immediate example is provided by [23], [27], and especially [24] that operate with nanoparticles with shape details coming very close to the 10-nm limit. Another example is inferring circuital parameters from surface roughness in the case of devices manufactured in sub-32-nm technologies. The validity of this strategy is challenged not only by the stochastic character of the surface's shape, an aspect that is already accounted for, but also by the fact that many of the surface irregularities may contain insufficient atoms to justify the use of an averaged, bulk material property.

ARTIFICIALLY ENGINEERED MEDIA

An even more pressing need to adequately understand bulk material properties arises in the realm of artificially engineered media, which are generically referred to as *metamaterials*. These purposefully created materials hold promise for exceptional EM properties, some of them not encountered in nature, that allow developing devices with unprecedented features. As a result, a vast research effort is being invested in metamaterials, with exotic material properties being readily used in CEM simulation-driven studies and designs. Interestingly enough, metamaterials are routinely accounted for in CEM via bulk material properties. However, most of such materials consist of lattices of (resonant) structures of macroscopic dimensions, which leads to the issue of the legitimacy of averaging for deriving bulk properties for such media. With strict respect to the use of metamaterials in CEM, the following situations can occur.

1) The domain occupied by metamaterials has dimensions commensurable with those of the constructive elements. In such situations, inferring bulk properties via averaging is questionable, the more so when the simulation technique makes use of submeshing. These difficulties can be circumvented by resorting to alternative, standard, methodologies that explicitly avoid invoking any (unphysical) bulk material parameters [49].
2) The domain for which the metamaterial behavior is presumed is too small to accommodate any technologically realizable lattice. In these cases, although applying a numerical analysis with a presupposed bulk property is computationally legitimate, the technological impossibility renders that study completely irrelevant.
3) The domain for which the metamaterial behavior is presumed is sufficiently large to allowing averaging and the (CEM-employed) bulk properties are physically justifiable. Nevertheless, such structures are necessarily electromagnetically (very) large and, as such, are amenable to analyses via instruments such as the ones described in [78] and [79] that do not invoke bulk material parameters.

In view of the continuously increasing interest in artificially engineered media and the expected increased involvement of students in design efforts making use of them, it is important to insist on these aspects during (basic) EM courses to preclude misconceptions and false expectations.

INNOVATION VECTORS

This review has clearly demonstrated the need to include an in-depth discussion on bulk behavior in any EE (AE) training. The basic elements of this discussion should be as follows.

- At the technological macroscopic level, any bulk property must rely on averaging over a sufficient number of constructive elements that are expected to collectively offer a certain EM capability. Should this not be the case, using bulk properties is not warranted; this also holds when the applied numerical technique requires submeshing, the size of the mesh elements not endorsing the averaging.
- At the microscopic level, the lower limit for which quantum fluctuations can be averaged (about 10 nm) inherently sets a lower limit for the applicability of bulk properties. Here, too, the need to submesh may raise supplementary concerns about the validity of using bulk material properties.

CONCLUSIONS

We have scrutinized some trends in teaching classical EM field theory within the EE undergraduate curricula. Our analysis yielded a number of innovation vectors. The first concerns the general framework, with the crucial interrelation between classical electrodynamics and classical mechanics via special relativity being highlighted as a necessary addition to the current programs. Subsequently, upon noting the pervasive interpretation of the EM field quantities as vector functions that depend continuously on space and time coordinates, and have a time-harmonic temporal dependence, we put forward a differential forms representation, a tensor representation, and a space–time domain-integrated field relations formalism as alternative interpretations.

In all these cases, a time-domain dependence was assumed and two of the approaches circumvented the continuous dependence on the space coordinates. Our survey insisted on the dualities and complementarities that manifest themselves in the classical EM theory. The treatment of the mathematic modeling was complemented by discussing a general topological hierarchy of 3D manifolds that provides support for expressing the (somehow) integrated field equations. The the space–time domain-integrated field relations in EM, in essence, a computational framework, was singled out as a didactic avenue to gain insight and proficiency for handling classical EM field concepts. The third vector concerned the appropriate understanding and the physically justifiable use of bulk material properties. Upon noting misconceptions, some rules were formulated to adequately account for bulk properties. In natural media, averaging must always be performed at a scale at which quantum fluctuations are "washed out," with a diameter of 10 nm as a practical lower bound. In artificially engineered media, the lower limit is dictated by the electrical size of the elements that collectively offer a specific property. In most practical cases, the scale at which averaging must be performed is so large that accounting for bulk material properties is actually impractical.

Including these elements in undergraduate curricula will allow their programmatic recalibration, especially in the context of pervasive use of computational instruments in education. Moreover, we deem their discussion conditional for equipping future EM experts with an adequate platform for understanding and correctly applying CEM tools, as required to address the emerging EM engineering challenges.

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