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An Iterative Sum-of-Squares Optimization for Static Output Feedback of Polynomial Systems

Simone Baldi[1]

***Abstract*— This work proposes an iterative procedure for static output feedback of polynomial systems based on Sum-of-Squares optimization. Necessary and sufficient conditions for static output feedback stabilization of polynomial systems are formulated, both for the global and for the local stabilization case. Since the proposed conditions are bilinear with respect to the decision variables, an iterative procedure is proposed for the solution of the stabilization problem. Every iteration is shown to improve the performance with respect to the previous one, even if convergence to a local minimum might occur. Since polynomial Lyapunov functions and control laws are considered, a Sum-of-Squares optimization approach is adopted. A numerical example illustrates the results.**

I. Introduction

The static output feedback (SOF) problem is still an active research topic in the control community: the reason why such control technique still receives so much attention is probably driven both by practical industrial needs, since static output feedback is the simplest control loop that can be realized in practice (e.g. for linear systems it simply amounts to finding a constant feedback gain via the available measurements), and by theoretical reasons, since several dynamic control design problems can be recast as static output feedback problems for a properly augmented system [1]. Despite the simplicity of this structure, the SOF problem has still challenging open issues for the systems and control community.

SOF stabilization of linear systems has been widely studied, e.g. in [2], [3], [4], [5], while research on SOF stabilization of (certain classes of) nonlinear systems has been increasing in the latest years, especially for nonlinear systems with polynomial vector fields. Recently the interest on polynomial systems has increased dramatically, possibly driven by two main reasons: one is that polynomial systems appear in a wide range of applications, spanning from biology to HVAC control to jet propulsion [6]; the second reason is the recent development of numerical tools based on sum-of-squares (SOS) decomposition for nonlinear analysis and controller synthesis [7]. Generally speaking, SOS is a generalization of the well-known linear matrix inequalities (LMI) methods to polynomial systems. There are toolboxes such as SOSTOOLS [8] that can recast the polynomial formulation into a Semidefinite programming which can be solved efficiently by solvers such as Sedumi [9] or SDPT3 [10].

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The work in nonlinear static output feedback can be distinguished into looking for (Lyapunov-based) sufficient conditions for stabilization and/or developing numerical algorithms for stabilization. In [11] a Hamilton-Jacobi framework was proposed for static output feedback of nonlinear systems, with a sufficient condition and a (partial) converse one. The work in [12] focuses on nonlinear systems with delayed disturbances, where a particular transformation and a Lyapunov-Razumikhin criterion are used to synthesize a sliding mode control. Several numerical algorithms for nonlinear static output feedback have been proposed in literature. Nonlinear static output feedback results have been derived for fuzzy systems: [13], [14] addresses static output feedback controllers for Takagi-Sugeno fuzzy models with linear and linear time-delay subsystems; for polynomial fuzzy systems a sum-of-squares approach is used in [15]. In [16] a static output feedback method with integral action is proposed for discrete-time polynomial systems. Other research directions on nonlinear static output feedback include sampled-data control systems consisting of a nonlinear plant in feedback with an output-feedback sampled-data polynomial controller [17].

A major difficulty, in linear or nonlinear static output feedback stabilization, is given by the non-convexity of the static output feedback solution set. In order to avoid non-convexity some simplifying assumptions must be made: for example in [18] the Lyapunov function is restricted to be only of function of states whose corresponding rows in the control matrix are zeroes. In doing so, it avoids the non-convexity of the static feedback design, but that makes the results more conservative.

In this approach we first develop a necessary and sufficient conditions for (global and local) stabilization via static output feedback of polynomial input-affine systems. Since the conditions are bilinear in the decision variables, an iterative algorithm based on sum-of-square decomposition will be developed to solve the stabilization problem: despite converge to the global minimum cannot be guaranteed, we can show that the solution of every iteration is feasible for the next one, so that at least convergence to a local minimum can be achieved. The work can be seen as an extension of [5] to polynomial nonlinear systems. It has to be underlined that an extension in this direction was done in [19], with the main differences that we adopt a more efficient sum-of-square decomposition, and furthermore we investigate local stabilization problem, which is relevant for nonlinear systems, where global stabilization cannot always be achieved.

The rest of the paper is organized as follows: Section II presents the problem formulation for global stabilization via static output feedback: a necessary and sufficient condition is given, which is solved in Section III via an iterative SOS method. The extension to local stabilization is discussed in Section IV and Section V presents a numerical example. Section VI concludes the paper.

The notation of this paper is standard, with $X = X' > 0$ denoting a symmetric positive definite matrix, and $X = X' \geq 0$ denoting a symmetric positive semidefinite matrix. The prime symbol denotes transpose.

II. Problem Formulation

Let us take a polynomial input-affine system in the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the state, input and output, respectively, and $f(x)$ $g(x)$, $h(x)$ are polynomial functions of the state x. The purpose of the static output feedback stabilization is to find a static output feedback law $u = k(y) = k(h(x))$ that stabilizes the system (where k is also taken as a polynomial function of the state x). By choosing a vector of monomial $z(x)$ of sufficiently high order,

$$z(x) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n & x_1^2 & \cdots & x_1^a x_2^b \cdots x_n^c & \cdots \end{bmatrix}' \tag{2}$$

where the maximum order of $z(x)$ will depend on the degree of f, g and h, it is possible to write the functions in (1) as

$$\begin{aligned} f(x) &= \Phi z(x) \\ h(x) &= Hz(x) \\ k(h(x)) &= K\bar{H}z(x). \end{aligned} \tag{3}$$

Note that in (3) we used two different matrices H and $\bar{H}$ for the following reason. If the desire is to design the control input to be a linear function of y, then $u = KHz(x)$, i.e. $H = \bar{H}$. If the desire is to design the control input to be a higher order polynomial of y, then the appropriate structure of the matrix $\bar{H}$ must be found such that all the monomials of y can be represented as a polynomial function of x. The representation as in (3) can always be found, given a sufficiently high order of $z(x)$.

The problem of output feedback stabilization for the polynomial system (1) is recast into the problem of solving the following inequality (with some stability margin $\rho > 0$)

$$z'(x)\Big(\big[\Phi + g(x)K\bar{H}\big]' M(x)P + PM(x)\big[\Phi + g(x)K\bar{H}\big] + \rho P \Big) z(x) < 0, \tag{4}$$

where the jacobian $M(x) = dz(x)/dx$ is a polynomial matrix of appropriate dimension. Note that (4) has been derived by taking a Lyapunov function in the form $V(x) = z'(x)Pz(x)$. In the following, for brevity, we omit the argument x from z, M and g.

The following result holds

Lemma 1: Solving (4) is equivalent to solving the following inequality

$$z'\Big(\Phi' M' P + PM\Phi\Big) z - z' PMgg'M'Pz + \big(g'M'Pz + K\bar{H}z\big)' \big(g'M'Pz + K\bar{H}z\big) + z'\rho Pz < 0. \tag{5}$$

Proof:
(5) $\Rightarrow$ (4) This follows by observing that for (4) the following holds

$$(4) < (4) + z'\bar{H}'K'K\bar{H}z = (5). \tag{6}$$

(4) $\Rightarrow$ (5) This follows by observing that if (4) holds then there exist a $\gamma > 0$ such that

$$z'\Big(\big[\Phi + gK\bar{H}\big]' M'P + PM\big[\Phi + gK\bar{H}\big] + \rho P \Big) z + \frac{1}{\gamma^2} z'\bar{H}'K'K\bar{H}z < 0 \tag{7}$$

$$z'\Big(\Phi'M'P + PM\Phi + \rho P\Big)z - \gamma^2 zPMgg'M'Pz + (\gamma g'M'Pz + \frac{1}{\gamma}K\bar{H}z)'(\gamma g'M'Pz + \frac{1}{\gamma}K\bar{H}z) < 0 \tag{8}$$

$$\gamma^2 z'\Big(\Phi'M'P + PM\Phi + \rho P\Big)z - \gamma^4 zPMgg'PM'z + (\gamma^2 g'M'Pz + K\bar{H}z)'(\gamma^2 g'M'Pz + K\bar{H}z) < 0, \tag{9}$$

and (5) follows by defining $\hat{P} = \gamma^2 P$. *End proof.*

The expression in (5) is quadratic in P and cannot be solved by standard Semidefinite Programming. In order to solve this problem we introduce an auxiliary positive definite symmetric matrix $X = X'$ such that

$$z'XMgg'M'Pz + z'PMgg'M'Xz - z'XMgg'M'Xz < z'PMgg'M'Pz, \tag{10}$$

where (10) is valid since

$$z'(X - P)Mgg'M'(X - P)z > 0 \tag{11}$$

The observation is that if we find two matrices X and P such that

$$z'\Big(\Phi'M'P + PM\Phi\Big)z + z'\Psi z + z'\big(g'M'P + K\bar{H}\big)'\big(g'M'P + K\bar{H}\big)z < 0 \tag{12}$$

where Ψ is defined as

$$\Psi = XMgg'M'P + PMgg'M'X - XMgg'M'X + \rho P. \tag{13}$$

Then (5) holds automatically, and thus (4) holds and the static output feedback problem has been solved. So, the idea is to develop an iterative method to find X, P, K and ρ so that (12) holds.

III. ITERATIVE METHOD

The following generalized Schur complement is recalled [20]

Lemma 2: The following inequality is valid

$$-p(x)+g'(x)g(x)<0,\ \forall x \tag{14}$$

if and only if

$$-p(x)+2g'(x)s-s's<0,\ \forall x,s \tag{15}$$

where s is a variable of appropriate dimension.

Using the generalized Schur complement find that (12) is equivalent to

$$z'\Big(\Phi'M'P+PM\Phi+\Psi\Big)z+ 2z'\left(g'M'P+K\bar{H}\right)'s-s's<0 \tag{16}$$

$\forall s,x$, which makes (16) linear with respect to the decision variables P and K. Note that Ψ is the same as in (13). We exploit this transformation in order to develop an iterative algorithm over P, X and K, as shown in Algorithm 1 (where all inequalities have to be intended as relaxed to Sum-of-Squares conditions).

Algorithm 1 Global Static Output Feeback Stabilization

Initialize:
Given an initial X
Optimization 1:
Solve for P, K and ρ
  $\bar{\rho}=\max\rho$
  s.t.
  $z'\Big(\Phi'M'P+PM\Phi+\Psi\Big)z+2z'\left(g'M'P+K\bar{H}\right)'s-s's<0$
  $\Psi=\bar{\rho}P-XMgg'M'P-PMgg'M'X+XMgg'M'X$
  $P>0$
If $\bar{\rho}>0$ **problem solved** and return the desired controller gains K.
Otherwise, **goto** *Optimization 2*.
Optimization 2:
Given $\bar{\rho}$ and X from *Optimization 1*
Solve for P
  $\bar{P}=\arg\min tr(P)$
  s.t.
  $z'\Big(\Phi'M'P+PM\Phi+\Psi\Big)z+2z'\left(g'M'P+K\bar{H}\right)'s-s's<0$
  $\Psi=\bar{\rho}P-XMgg'M'P-PMgg'M'X+XMgg'M'X$
  $P>0$
If $\|X-\bar{P}\|<\kappa$, with κ a prescribed tolerance, the synthesis problem may not be solvable, **Stop**.
Else, **goto** *Optimization 1*, using as a new X the $\bar{P}$ just found.

Note that Optimization 1 implies the solution of a bisection algorithm due to the product of P and ρ. The solution of Optimization 2 gives a new X for the next iteration.

Remark 1: Let the system be stabilizable under static output feedback, so that inequality (12), (13) will have a solution (thanks to the condition in (10)). Assume that a solution to Optimization 1 exists at the first iteration. Then for any following iteration, the existence of the solution is guaranteed by inequality (12), (13) and the sequence of solutions $\bar{\rho}_i$ to Optimization 1 will be non decreasing. In fact, if the following inequality holds

$$z'\Big(\Phi'M'P_i+P_iM\Phi\Big)z+z'\Psi_iz+ z'\left(g'M'P_i+K\bar{H}\right)'\left(g'M'P_i+K\bar{H}\right)z<0, \tag{17}$$

for some $P_i>0$ at iteration i (Ψ_i is defined accordingly), then the following is also true

$$z'\Big(\Phi'M'P_i+P_iM\Phi\Big)z-z'P_iMgg'M'P_iz+z'\rho P_iz\quad z'\left(g'M'P_i+K\bar{H}\right)'\left(g'M'P_i+K\bar{H}\right)z<0, \tag{18}$$

which means that at iteration $i+1$ Optimization 1 is feasible with $P_{i+1}=\bar{P}_i$, $\rho_{i+1}=\bar{\rho}_i$. This proves that the sequence of solutions $\bar{\rho}_i$ to Optimization 1 will be nondecreasing.

Remark 2: A nondecreasing sequence of solutions $\bar{\rho}_i$ to Optimization 1 does not guaranteed that $\bar{\rho}_i>0$ eventually. Neither it is guaranteed convergence to the a global minimum. In general convergence will be affected by the initial choice for X.

Remark 3: In this work the inequality (12) is relaxed to

$$-z'\Big(\Phi'M'P+PM\Phi+\Psi\Big)z- 2z'\left(g'M'P+K\bar{H}\right)'s+s's\ \text{ is SOS} \tag{19}$$

The condition in [19] is less efficient than (19), in view of the fact that the following result is used in [19]: (12) is relaxed to

$$-v'\begin{bmatrix} z'\Big(\Phi'M'P+PM\Phi+\Psi\Big)z & * \\ (g'M'P+K\bar{H})z & -I \end{bmatrix}v'\ \text{ is SOS} \tag{20}$$

which clearly involves more decision variables, thus increasing computational complexity. Since the gap between semi-definiteness and sum-of-squares has been shown to increase at higher dimensions [21], the relaxation (20) might lead to increased conservativeness.

IV. LOCAL STABILITY

Global stabilization is not always possible with nonlinear systems, e.g. if multiple equilibria are present: for this reason, it is interesting to derive regional conditions for stabilization. At first the definition for (local) regional stabilization is given [22, Sect. 8.2]:

Definition 1: The origin of $\dot{x}=f(x)+g(x)k(x)$ is regionally stable if it is (locally) asymptotically stable in a given region G which is a subset of the region of attraction, i.e.

$$\lim_{t\to\infty}x(t)=0,\quad \forall x(0)\in G, \tag{21}$$

An example of a given region G is e.g., $G \subset \Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ where Ω_c is an estimate of the region of attraction.

The next proposition provides a sufficient condition for the regional stabilization of the origin:

Proposition 1: If it exist a Lyapunov function $V(x)$ and a multiplier $m(x)$ such that the following is satisfied

$$\begin{aligned} &\dot{V}(x) - m(x)(V(x) - c) < 0 \\ &m(x) > 0, \end{aligned} \tag{22}$$

with $c > 0$, then the origin is regionally stable, where $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ is an estimate of the region of attraction.
Proof:
The proof is straightforward via Lyapunov arguments as in [22, Sect. 8.2] *End proof.*

Remark 4: In case the control designer requires the estimate of the region of attraction to be of a desired shape (e.g. described by a given polynomial function $r(x)$), then condition (22) can be modified into

$$\begin{aligned} &\dot{V}(x) - m_1(x)(V(x) - c) < 0 \\ &V(x) - c - m_2(x)(r(x) - b) > 0 \\ &m_1(x) > 0 \\ &m_2(x) > 0, \end{aligned} \tag{23}$$

with $c, b > 0$, where the second inequality guarantees that $\{x \in \mathbb{R}^n : r(x) \leq b\} \subseteq \{x \in \mathbb{R}^n : V(x) \leq c\}$

The following lemma is an extension of Lemma 1 for local regional stability

Lemma 3: The following condition for local regional stabilization via output feedback (with some stability margin $\rho > 0$)

$$\begin{aligned} z'\Big(&\big[\Phi + gK\bar{H}\big]' MP + PM\big[\Phi + gK\bar{H}\big] \\ &+ \rho P \Big) z - m\big(z'Pz - c\big) < 0 \end{aligned} \tag{24}$$

is equivalent to

$$\begin{aligned} &z'\Big(\Phi' M' P + PM\Phi\Big) z + z'\rho P z - z' PMgg' M' P z \\ &+ \big(g' M' P z + K\bar{H} z\big)' \big(g' M' P z + K\bar{H} z\big) \\ &- m\big(z'Pz - c\big) < 0, \end{aligned} \tag{25}$$

where the estimate of the region of attraction is given in both cases by $\Omega_c = \{x \in \mathbb{R}^n : z'(x) P z(x) \leq c\}$
Proof
By following the same reasoning as Lemma 1
(25) $\Rightarrow$ (24) This follows by observing that for (24) the following holds

$$(24) < (24) + z'\bar{H}' K' K \bar{H} z = (25). \tag{26}$$

(24) $\Rightarrow$ (25) This follows by observing that if (24) holds then there exist a $\gamma > 0$ such that

$$\begin{aligned} &z'\Big(\big[\Phi + gK\bar{H}\big]' M' P + PM\big[\Phi + gK\bar{H}\big] + \rho P \Big) z \\ &\quad + \frac{1}{\gamma^2} z' \bar{H}' K' K \bar{H} z - m\big(z'Pz - c\big) < 0 \end{aligned} \tag{27}$$

$$\begin{aligned} &z'\Big(\Phi' M' P + PM\Phi + \rho P\Big) z - \gamma^2 z PMgg' M' P z + \\ &\quad (\gamma g' M' P z + \frac{1}{\gamma} K\bar{H} z)'(\gamma g' M' P z + \frac{1}{\gamma} K\bar{H} z) \\ &\quad - m\big(z'Pz - c\big) < 0 \end{aligned} \tag{28}$$

$$\begin{aligned} &\gamma^2 z'\Big(\Phi' M' P + PM\Phi + \rho P\Big) z - \gamma^4 z PMgg' PM' z + \\ &\quad (\gamma^2 g' M' P z + K\bar{H} z)'(\gamma^2 g' M' P z + K\bar{H} z) \\ &\quad - \gamma^2 m\big(z'Pz - c\big) < 0, \end{aligned} \tag{29}$$

and (25) follows by defining $\hat{P} = \gamma^2 P$. *End proof.*

Note that condition (22) is bilinear with respect to the decision variables, i.e. the Lyapunov function $V(x)$ and the multiplier $m(x)$. If Algorithm 1 wants to be extended to local stabilization, an extra iterative step in necessary in order to deal with this bilinearity. The resulting algorithm is presented in Algorithm 2 (where all inequalities have to be intended as relaxed to Sum-of-Squares conditions). The starting point for the algorithm is an initial Lyapunov function that can certify local stability. This initial Lyapunov function can be achieved by running the method in [5] for the linearized version of the system. From there, first, for a fixed Lyapunov function, the largest estimate of the region of attraction is sought (Optimization 1); then, for a fixed multiplier, the maximum stability margin inside the region is sought (Optimization 2, which requires a bisection algorithm); then a new matrix X is sought for the next iteration (Optimization 3).

V. Numerical Example

The following example has been selected to illustrate the results of the method

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 + x_2 u \\ \dot{x}_2 &= x_1 + 2x_2 + x_1^2 + x_1^2 x_2 + u \end{aligned} \tag{30}$$

The maximum order of the monomial is chosen as 3, which also accounts for the maximum order of the vector field f. This means that

$$z(x) = [x_1 \;\; x_2 \;\; x_1^2 \;\; x_1 x_2 \;\; x_2^2 \;\; x_1^3 \; x_1^2 x_2 \;\; x_1 x_2^2 \;\; x_2^3]' \tag{31}$$

At first it is assumed that the state x is completely measurable, so that the static output feedback problem is nothing but a static state feedback problem. No global solution could be found for this system, so that Algorithm 2 was run. Starting with $X = I$, the algorithm terminates after 4 iterations with $\bar{\rho} = 1.9897$ and

$$\begin{aligned} u \;=\; & 23.1680 x_1 - 102.9800 x_2 - 0.0612 x_1^2 + 78.0630 x_1 x_2 \\ & + 0.8709 x_2^2 + 4.7724 x_1^3 - 170.1000 x_1^2 x_2 \\ & - 2.4245 x_1 x_2^2 - 229.5300 x_2^3 \end{aligned} \tag{32}$$

Algorithm 2 Local Static Output Feedback Stabilization

Initialize:
Given an initial X and an initial Lyapunov function P (e.g. a quadratic Lyapunov function coming from the linearized version of the plant)
$\bar{\rho}$ sufficiently small and $\bar{c}^{-} = 0$
Optimization 1:
With fixed P, $\bar{\rho}$ and X, solve for c, K and m
  $\bar{c} = \max c$
  s.t.
  $z'\left(\Phi' M' P + P M \Phi + \Psi\right) z + 2z'\left(g' M' P + K\bar{H}\right)' s - s's - m(z'Pz - c) < 0$
  $\Psi = \bar{\rho} P - XMgg'M'P - PMgg'M'X + XMgg'M'X$
  $P > 0, \qquad m > 0$
If $\bar{c} - \bar{c}^{-} > \kappa_c$, with κ_c a prescribed tolerance, **goto** *Optimization 2* and update $\bar{c}^{-}$ with the new $\bar{c}$.
Else, **goto** *Optimization 3*
Optimization 2:
With fixed m and $\bar{c}$, solve for P, K and ρ
  $\bar{\rho} = \max \rho$
  s.t.
  $z'\left(\Phi' M' P + P M \Phi + \Psi\right) z + 2z'\left(g' M' P + K\bar{H}\right)' s - s's - m(z'Pz - c) < 0$
  $\Psi = \bar{\rho} P - XMgg'M'P - PMgg'M'X + XMgg'M'X$
  $P > 0$
If $\bar{\rho} > 0$ **problem solved** and return the desired controller gains K.
Otherwise, **goto** *Optimization 1*.
Optimization 3:
Given $\bar{\rho}$, m and X from *Optimization 1*
Solve for P
  $\bar{P} = \arg\min tr(P)$
  s.t.
  $z'\left(\Phi' M' P + P M \Phi + \Psi\right) z + 2z'\left(g' M' P + K\bar{H}\right)' s - s's - m(z'Pz - c) < 0$
  $\Psi = \bar{\rho} P - XMgg'M'P - PMgg'M'X + XMgg'M'X$
  $P > 0$
If $\|X - \bar{P}\| < \kappa$, with κ a prescribed tolerance, the synthesis problem may not be solvable, **Stop**.
Else, **goto** *Optimization 1*, using as a new X the $\bar{P}$ just found.

Fig. 1 depicts the evolution of the state, of the state norm and of the input with the controller (32). The exponential bound given by $\bar{\rho} = 1.9897$ is indicated by a dashed line.

As a further example, we assume now that only x_2 is measurable, which leads to the following $\bar{H}$ matrix

$$\bar{H} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (33)$$

Starting with $X = I$, the algorithm terminates after 5 iterations

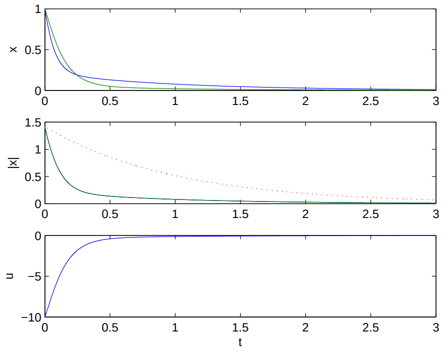

Fig. 1: Evolution of the state, of the state norm and of the input with the controller (32).

with

$$u = -22.8540x_2 - 15.4830x_2^2 - 40.1450x_2^3 \qquad (34)$$

Fig. 2 depicts the evolution of the state, of the state norm and of the input with the controller (34).

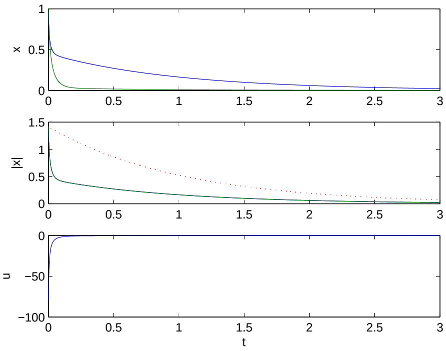

Fig. 2: Evolution of the state, of the state norm and of the input with the controller (34).

As a final example, we assume now that only $x_1 - 5x_2$ is measurable, which leads to the following $\bar{H}$ matrix

$$\bar{H} = \begin{bmatrix} 1 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 25 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -15 & 75 & -125 \end{bmatrix} \qquad (35)$$

Starting with $X = I$, the algorithm terminates after 4 iterations with

$$u = 7.2304(x_1 - 5x_2) + 0.2665(x_1 - 5x_2)^2 + 3.9124(x_1 - 5x_2)^3 \qquad (36)$$

Fig. 3 depicts the evolution of the state, of the state norm and of the input with the controller (36).

VI. Conclusions

This work proposed an iterative procedure for static output feedback of polynomial systems based on Sum-of-Squares optimization. Necessary and sufficient conditions for static output feedback stabilization of polynomial systems were

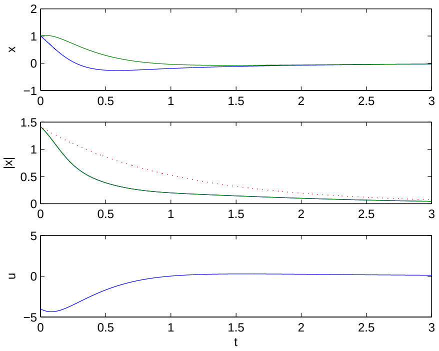

Fig. 3: Evolution of the state, of the state norm and of the input with the controller (36).

formulated, both for the global and for the local stabilization case. Since the proposed conditions are bilinear with respect to the decision variables, an iterative procedure has been proposed for the solution of the stabilization problem. Every iteration is shown to improve the performance with respect to the previous one, even if convergence to a local minimum might occur. Since polynomial Lyapunov functions and control laws are considered, a Sum-of-Squares optimization approach has been adopted. A numerical example illustrated the results.

Future work might include at least two directions: the first one is adding input saturation constraints to the formulation, so that regional stabilization can be achieved also taking into account the unsaturated and saturated regions; the second one is investigating rational Lyapunov functions for local stabilization, by resorting to the Zubov's method.

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