Quantum Signature of a Squeezed Mechanical Oscillator

Chowdhury, A.; Vezio, P.; Bonaldi, M.; Borrielli, A.; Marino, F.; Morana, B.; Prodi, G. A.; Sarro, P. M.; Serra, E.; Marin, F.

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SUPPLEMENTAL MATERIAL

THEORETICAL MODEL

The linearized evolution equations for the intracavity field operator $\delta\hat{a}$ and the mechanical bosonic operator $\hat{b}$, in the frame rotating at frequency ω_L, are [1]

$$\delta\dot{\hat{a}} = \left(i\Delta - \frac{\kappa}{2}\right)\delta\hat{a} + ig_0\alpha(\hat{b} + \hat{b}^\dagger) + \sqrt{\kappa}\,\delta\hat{a}_\mathrm{in} \tag{1}$$

$$\dot{\hat{b}} = \left(-i\Omega_\mathrm{m}^0 - \frac{\Gamma_\mathrm{m}}{2}\right)\hat{b} + ig_0(\alpha^*\delta\hat{a} + \alpha\delta\hat{a}^\dagger) + \sqrt{\Gamma_\mathrm{m}}\,\hat{b}_\mathrm{th} \tag{2}$$

where $\Delta = \omega_\mathrm{L} - \omega_\mathrm{c}$ is the detuning with respect to the cavity resonance frequency ω_c, κ and Γ_m are the optical and mechanical decay rates, Ω_m^0 is the mechanical resonance frequency, g_0 is the single-photon opto-mechanical coupling rate, and α is the intracavity mean field. The input noise operators are characterized by the correlation functions

$$\langle\hat{a}_\mathrm{in}(t)\hat{a}_\mathrm{in}^\dagger(t')\rangle = \delta(t-t') \tag{3}$$

$$\langle\hat{a}_\mathrm{in}^\dagger(t)\hat{a}_\mathrm{in}(t')\rangle = 0 \tag{4}$$

$$\langle\hat{b}_\mathrm{th}(t)\hat{b}_\mathrm{th}^\dagger(t')\rangle = (\bar{n}_\mathrm{th}+1)\,\delta(t-t') \tag{5}$$

$$\langle\hat{b}_\mathrm{th}^\dagger(t)\hat{b}_\mathrm{th}(t')\rangle = \bar{n}_\mathrm{th}\,\delta(t-t') \tag{6}$$

where $\bar{n}_\mathrm{th}$ is the thermal occupation number.

We now consider an input field composed of two tones, shifted by $\pm\Omega_\mathrm{m}$ around ω_L. Here Ω_m is the effective resonance frequency, modified by the opto-mechanical interaction, that will be defined later in a self-consistent way. The mean value of the input field has the form

$$\alpha_\mathrm{in} = \alpha_-^\mathrm{in}\,e^{-i(\omega_\mathrm{L}-\Omega_\mathrm{m})t} + \alpha_+^\mathrm{in}\,e^{-i(\omega_\mathrm{L}+\Omega_\mathrm{m})t}\,. \tag{7}$$

The intracavity mean field, in the rotating frame, is $\alpha = \alpha_- e^{i\Omega_\mathrm{m}t} + \alpha_+ e^{-i\Omega_\mathrm{m}t}$, with amplitudes

$$\alpha_\pm = \alpha_\pm^\mathrm{in}\,\frac{\sqrt{\kappa_\mathrm{in}}}{-i(\Delta\pm\Omega_\mathrm{m}) + \kappa/2} \tag{8}$$

where κ_in is the input coupling rate. In the Fourier space, equation (1) can be written as

$$\delta\tilde{a}(\Omega) = \frac{1}{-i\Omega - i\Delta + \kappa/2}\left\{ig_0\left[\alpha_-\left(\tilde{b}(\Omega+\Omega_\mathrm{m}) + \tilde{b}^\dagger(\Omega+\Omega_\mathrm{m})\right) + \alpha_+\left(\tilde{b}(\Omega-\Omega_\mathrm{m}) + \tilde{b}^\dagger(\Omega-\Omega_\mathrm{m})\right)\right] + \sqrt{\kappa}\,\delta\tilde{a}_\mathrm{in}(\Omega)\right\} \tag{9}$$

where we use $\tilde{O}$ to indicate the Fourier transformed of the operator $\hat{O}$, and $\tilde{O}^\dagger$ for the Fourier transformed of $\hat{O}^\dagger$. We now restrict our analysis to weak coupling, in which case the opto-mechanical damping rate and frequency shift of the mechanical oscillator (whose expressions will be given later) are much smaller than its resonance frequency. Therefore, in the equation (2) we just consider the quasi-resonant components in the opto-mechanical coupling term $g_0(\alpha^*\delta\hat{a} + \alpha\delta\hat{a}^\dagger)$, and the equation in the Fourier space can be written as

$$
\begin{aligned}
\big(-i\Omega + i\Omega_\mathrm{m}^0 + \Gamma_\mathrm{m}/2\big)\tilde{b}(\Omega) = \\
-g_0^2\Bigg[|\alpha_-|^2\,\tilde{b}(\Omega)\left(\frac{1}{-i\Omega - i\Delta + i\Omega_\mathrm{m} + \kappa/2} - \frac{1}{-i\Omega + i\Delta - i\Omega_\mathrm{m} + \kappa/2}\right) + \\
|\alpha_+|^2\,\tilde{b}(\Omega)\left(\frac{1}{-i\Omega - i\Delta - i\Omega_\mathrm{m} + \kappa/2} - \frac{1}{-i\Omega + i\Delta + i\Omega_\mathrm{m} + \kappa/2}\right) + \\
\alpha_-^*\alpha_+\,\tilde{b}^\dagger(\Omega - 2\Omega_\mathrm{m})\left(\frac{1}{-i\Omega - i\Delta + i\Omega_\mathrm{m} + \kappa/2} - \frac{1}{-i\Omega + i\Delta + i\Omega_\mathrm{m} + \kappa/2}\right)\Bigg] + \tilde{b}_\mathrm{in}(\Omega)
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
\tilde{b}_\mathrm{in}(\Omega) = \sqrt{\Gamma_\mathrm{m}}\,\tilde{b}_\mathrm{th}(\Omega) + ig_0\sqrt{\kappa}\Bigg[\alpha_-^*\frac{\delta\tilde{a}_\mathrm{in}(\Omega - \Omega_\mathrm{m})}{-i\Omega - i\Delta + i\Omega_\mathrm{m} + \kappa/2} + \alpha_+^*\frac{\delta\tilde{a}_\mathrm{in}(\Omega + \Omega_\mathrm{m})}{-i\Omega - i\Delta - i\Omega_\mathrm{m} + \kappa/2} + \\
+\alpha_-\frac{\delta\tilde{a}_\mathrm{in}^\dagger(\Omega + \Omega_\mathrm{m})}{-i\Omega + i\Delta - i\Omega_\mathrm{m} + \kappa/2} + \alpha_+\frac{\delta\tilde{a}_\mathrm{in}^\dagger(\Omega - \Omega_\mathrm{m})}{-i\Omega + i\Delta + i\Omega_\mathrm{m} + \kappa/2}\Bigg].
\end{aligned}
\tag{11}
$$

In the right hand side of Eq. (10), we notice the usual opto-mechanical effects of the two laser tones (first two terms inside square brackets), plus their coherent common interaction, proportional to the fields product $\alpha_-^*\alpha_+$, that originates the parametric squeezing. It can be directly calculated that this parametric effect is null for $\Delta = 0$, i.e., when the two tones are equally shifted with respect to the cavity resonance.

The total input noise source described by Eq. (11) includes thermal noise and back-action noise, the latter given by the terms into square brackets.

The standard opto-mechanical interaction is parametrized by the optical damping rate Γ_opt, defined as [1]

$$
\begin{aligned}
\Gamma_\mathrm{opt} = 2g_0^2\,\mathrm{Re}\Bigg[|\alpha_-|^2\left(\frac{1}{-i\Omega - i\Delta + i\Omega_\mathrm{m} + \kappa/2} - \frac{1}{-i\Omega + i\Delta - i\Omega_\mathrm{m} + \kappa/2}\right) + \\
|\alpha_+|^2\left(\frac{1}{-i\Omega - i\Delta - i\Omega_\mathrm{m} + \kappa/2} - \frac{1}{-i\Omega + i\Delta + i\Omega_\mathrm{m} + \kappa/2}\right)\Bigg],
\end{aligned}
\tag{12}
$$

and by a frequency shift that determines the effective resonance frequency Ω_m according to the equation

$$
\begin{aligned}
\Omega_\mathrm{m} = \Omega_\mathrm{m}^0 + g_0^2\,\mathrm{Im}\Bigg[|\alpha_-|^2\left(\frac{1}{-i\Omega - i\Delta + i\Omega_\mathrm{m} + \kappa/2} - \frac{1}{-i\Omega + i\Delta - i\Omega_\mathrm{m} + \kappa/2}\right) + \\
|\alpha_+|^2\left(\frac{1}{-i\Omega - i\Delta - i\Omega_\mathrm{m} + \kappa/2} - \frac{1}{-i\Omega + i\Delta + i\Omega_\mathrm{m} + \kappa/2}\right)\Bigg].
\end{aligned}
\tag{13}
$$

The total damping rate is $\Gamma_{\mathrm{eff}} = \Gamma_{\mathrm{m}} + \Gamma_{\mathrm{opt}}$. Its expression coincides with that given in the main text if we define the total opto-mechanical coupling strength $g^2 = g_0^2\left(|\alpha_-|^2 + |\alpha_+|^2\right)$, the ratio between intracavity powers $\epsilon_{\mathrm{c}} = |\alpha_-|^2/(|\alpha_-|^2 + |\alpha_+|^2)$, and using the quasi-resonant frequency condition $\Omega \simeq \Omega_{\mathrm{m}}$. With the same condition, Eq. (10) simplifies to

$$\big(-i\Omega + i\Omega_{\mathrm{m}} + \Gamma_{\mathrm{eff}}/2\big)\tilde{b}(\Omega) = -\frac{\Gamma_{\mathrm{par}}}{2}e^{i\phi}\,\tilde{b}^{\dagger}(\Omega - 2\Omega_{\mathrm{m}}) + \tilde{b}_{\mathrm{in}}(\Omega) \tag{14}$$

where

$$\Gamma_{\mathrm{par}} = \frac{4g_0^2|\alpha_+||\alpha_-|\Delta}{\Delta^2 + \kappa^2/4} \tag{15}$$

and $\phi = \pi/2 + \arg[\alpha_-^*\alpha_+]$. Using the notation defined above, the expression for Γ_{par} coincides with Eq. (4) of the main text. Moving to the frame rotating at Ω_{m} by means of the transformation

$$\hat{b}_R = \hat{b}\,e^{i\Omega_{\mathrm{m}}t} \qquad \hat{b}_R^{\dagger} = \hat{b}^{\dagger}e^{-i\Omega_{\mathrm{m}}t} \tag{16}$$

and, for Fourier transformed operators,

$$\tilde{b}_R(\Omega) = \tilde{b}\,(\Omega + \Omega_{\mathrm{m}}) \qquad \tilde{b}_R^{\dagger}(\Omega) = \tilde{b}^{\dagger}(\Omega - \Omega_{\mathrm{m}}) \tag{17}$$

and defining the frequency with respect to the mechanical resonance $\omega = \Omega - \Omega_{\mathrm{m}}$, Eq. (14) and its Hermitian conjugate can be written in the form of the system of coupled linear equations

$$\begin{pmatrix} -i\omega + \frac{\Gamma_{\mathrm{eff}}}{2} & \frac{\Gamma_{\mathrm{par}}}{2}e^{i\phi} \\ \frac{\Gamma_{\mathrm{par}}}{2}e^{-i\phi} & -i\omega + \frac{\Gamma_{\mathrm{eff}}}{2} \end{pmatrix} \begin{pmatrix} \tilde{b}_R \\ \tilde{b}_R^{\dagger} \end{pmatrix} = \begin{pmatrix} \tilde{b}_{\mathrm{in}} \\ \tilde{b}_{\mathrm{in}}^{\dagger} \end{pmatrix}. \tag{18}$$

The determinant of the system matrix is

$$\mathcal{D} = \left(-i\omega + \frac{\Gamma_+}{2}\right)\left(-i\omega + \frac{\Gamma_-}{2}\right) \tag{19}$$

where

$$\Gamma_{\pm} = \Gamma_{\mathrm{eff}} \pm \Gamma_{\mathrm{par}} \tag{20}$$

and the solutions of the system can be written as

$$\tilde{b}_R = \frac{1}{\mathcal{D}}\left[\left(-i\omega + \frac{\Gamma_{\mathrm{eff}}}{2}\right)\tilde{b}_{\mathrm{in}} - \frac{\Gamma_{\mathrm{par}}}{2}e^{i\phi}\,\tilde{b}_{\mathrm{in}}^{\dagger}\right] \tag{21}$$

$$\tilde{b}_R^{\dagger} = \frac{1}{\mathcal{D}}\left[\left(-i\omega + \frac{\Gamma_{\mathrm{eff}}}{2}\right)\tilde{b}_{\mathrm{in}}^{\dagger} - \frac{\Gamma_{\mathrm{par}}}{2}e^{-i\phi}\,\tilde{b}_{\mathrm{in}}\right]. \tag{22}$$

The correlation functions for the input noise source of Eq. (11) are obtained from Eqs. (3)-(6) by considering that $\langle \hat{O}(t)\hat{O}^{\dagger}(t')\rangle = c\,\delta(t-t')$ implies $\langle \tilde{O}(\Omega)\tilde{O}^{\dagger}(\Omega')\rangle = 2\pi c\,\delta(\Omega+\Omega')$:

$$\frac{1}{2\pi}\langle \tilde{b}_{\rm in}(-\Omega)\tilde{b}^{\dagger}_{\rm in}(\Omega)\rangle = \Gamma_{\rm m}(\bar{n}_{\rm th}+1) + A^{+} \tag{23}$$

$$\frac{1}{2\pi}\langle \tilde{b}^{\dagger}_{\rm in}(-\Omega)\tilde{b}_{\rm in}(\Omega)\rangle = \Gamma_{\rm m}\,\bar{n}_{\rm th} + A^{-} \tag{24}$$

$$\frac{1}{2\pi}\left\langle \tilde{b}_{\rm in}(-\Omega)\tilde{b}_{\rm in}(\Omega)\right\rangle = \frac{1}{2\pi}\left\langle \tilde{b}^{\dagger}_{\rm in}(-\Omega)\tilde{b}^{\dagger}_{\rm in}(\Omega)\right\rangle^{*} = -g_0^2\kappa\frac{\alpha_-^{*}\alpha_+}{\Delta^2+\kappa^2/4} \tag{25}$$

where the Stokes and anti-Stokes rates due to the two tones are [1]

$$A^{+} = g_0^2\kappa\left[\frac{|\alpha_-|^2}{\Delta^2+\kappa^2/4} + \frac{|\alpha_+|^2}{(\Delta+2\Omega_{\rm m})^2+\kappa^2/4}\right] \tag{26}$$

$$A^{-} = g_0^2\kappa\left[\frac{|\alpha_-|^2}{(\Delta-2\Omega_{\rm m})^2+\kappa^2/4} + \frac{|\alpha_+|^2}{\Delta^2+\kappa^2/4}\right] \tag{27}$$

and it can be verified that $\Gamma_{\rm opt} = A^{-} - A^{+}$.

The spectra of the Stokes and anti-Stokes motional sidebands are finally calculated from Eqs. (21)-(22) using the correlation functions given above, and are respectively

$$S_{\hat{b}^{\dagger}_R\hat{b}^{\dagger}_R} = \frac{1}{2\pi}\langle \tilde{b}_R(-\omega)\tilde{b}^{\dagger}_R(\omega)\rangle = \frac{\Gamma_{\rm eff}}{2}\left[\frac{1+\bar{n}-s/2}{\omega^2+\Gamma_-^2/4} + \frac{1+\bar{n}+s/2}{\omega^2+\Gamma_+^2/4}\right] \tag{28}$$

$$S_{\hat{b}_R\hat{b}_R} = \frac{1}{2\pi}\langle \tilde{b}^{\dagger}_R(-\omega)\tilde{b}_R(\omega)\rangle = \frac{\Gamma_{\rm eff}}{2}\left[\frac{\bar{n}+s/2}{\omega^2+\Gamma_-^2/4} + \frac{\bar{n}-s/2}{\omega^2+\Gamma_+^2/4}\right] \tag{29}$$

as in Eqs. (5)-(6) of the main text, where we have dropped the subscript R to simplify the notation, and form the integrals over ω of the different Lorentzian components we can derive the Eqs. (1)-(2) of the main text. The squeezing parameter is $s = \Gamma_{\rm par}/\Gamma_{\rm eff}$ and the oscillator effective phonon number in the absence of parametric effect is

$$\bar{n} = \frac{\Gamma_{\rm m}\,\bar{n}_{\rm th} + \Gamma_{\rm opt}\,\bar{n}_{BA}}{\Gamma_{\rm eff}} \tag{30}$$

with $\bar{n}_{BA} = A^{+}/\Gamma_{\rm opt}$.

A quadrature X_θ of the oscillator is defined as $X_\theta = (e^{i\theta}\,\hat{b}_R + e^{-i\theta}\,\hat{b}^{\dagger}_R)/2$. The quadrature operator can be calculated in the Fourier space from Eqs. (21)-(22), obtaining

$$\tilde{X}_\theta = \frac{1}{2\mathcal{D}}\left[e^{i\theta}\,\tilde{b}_{\rm in}\left(-i\omega+\frac{\Gamma_{\rm eff}}{2}-\frac{\Gamma_{\rm par}}{2}e^{-i(2\theta+\phi)}\right) + e^{-i\theta}\,\tilde{b}^{\dagger}_{\rm in}\left(-i\omega+\frac{\Gamma_{\rm eff}}{2}-\frac{\Gamma_{\rm par}}{2}e^{i(2\theta+\phi)}\right)\right]. \tag{31}$$

Minimal and maximal fluctuations characterize the quadratures defined respectively by $2\theta+\phi=0$ and $2\theta+\phi=\pi$. These quadratures are defined in the main text as $Y \equiv X_{-\phi/2}$ and $X \equiv X_{-\phi/2+\pi/2}$. Their operators are

$$Y = \frac{e^{-i\phi/2}\,\tilde{b}_{\rm in} + e^{i\phi/2}\,\tilde{b}^{\dagger}_{\rm in}}{2\left(-i\omega+\frac{\Gamma_+}{2}\right)} \qquad X = \frac{i\left(e^{-i\phi/2}\,\tilde{b}_{\rm in} - e^{i\phi/2}\,\tilde{b}^{\dagger}_{\rm in}\right)}{2\left(-i\omega+\frac{\Gamma_-}{2}\right)} \tag{32}$$

and the associated spectra are

$$S_{YY} = \frac{\Gamma_{\text{eff}}(2\bar{n}+1)}{4\left(\omega^2 + \frac{\Gamma_+^2}{4}\right)} \qquad S_{XX} = \frac{\Gamma_{\text{eff}}(2\bar{n}+1)}{4\left(\omega^2 + \frac{\Gamma_-^2}{4}\right)} \,. \tag{33}$$

The integrals of the spectra give the variances $\sigma_Y^2 = \sigma_0^2/(1+s)$ and $\sigma_X^2 = \sigma_0^2/(1-s)$ with $\sigma_0^2 = (2\bar{n}+1)/4$.

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[1] M. Aspelmeyer, T. J. Kippenberg, F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).