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de Kleijn, Niek; Wierstra, Felix

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# On the Maurer-Cartan simplicial set of a complete curved $A_\infty$ -algebra

Niek de Kleijn<sup>1</sup> · Felix Wierstra<sup>2</sup>

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## Abstract

In this paper, we develop the  $A_\infty$ -analog of the Maurer-Cartan simplicial set associated to an  $L_\infty$ -algebra and show how we can use this to study the deformation theory of  $\infty$ -morphisms of algebras over non-symmetric operads. More precisely, we first recall and prove some of the main properties of  $A_\infty$ -algebras like the Maurer-Cartan equation and twist. One of our main innovations here is the emphasis on the importance of the shuffle product. Then, we define a functor from the category of complete (curved)  $A_\infty$ -algebras to simplicial sets, which sends a complete curved  $A_\infty$ -algebra to the associated simplicial set of Maurer-Cartan elements. This functor has the property that it gives a Kan complex. In all of this, we do not require any assumptions on the field we are working over. We also show that this functor can be used to study deformation problems over a field of characteristic greater than or equal to 0. As a specific example of such a deformation problem, we study the deformation theory of  $\infty$ -morphisms of algebras over non-symmetric operads.

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✉ Felix Wierstra  
felix.wierstra@gmail.com  
Niek de Kleijn  
n.dekleijn@tudelft.nl

<sup>1</sup> Delft Institute of Applied Mathematics, Technical University Delft, Mekelweg 4, 2628 CD, Delft, The Netherlands

<sup>2</sup> Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Science Park, 105-107 Amsterdam, The Netherlands

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## 1 Introduction

The goal of this paper is to make the first steps towards an explicit description of deformation theory over a field of characteristic  $p \geq 0$ . In [23], Lurie showed, that over a field  $\mathbb{K}$  of characteristic  $p \geq 0$  there is an equivalence of  $\infty$ -categories between the category of formal  $E_n$ -moduli problems and the  $\infty$ -category of augmented  $E_n$ -algebras over  $\mathbb{K}$ . Although Lurie's results are great from a theoretical perspective, it can in practice be quite complicated to extract the relevant information about the moduli problem from the corresponding  $E_n$ -algebra.

In the characteristic 0 case, this problem is avoided by using  $L_\infty$ -algebras. In this case it is possible to get a very explicit equation, called the Maurer-Cartan equation, whose solutions correspond to the deformations. These solutions can be organized in a simplicial set, called the Deligne-Getzler-Hinich  $\infty$ -groupoid or Maurer-Cartan simplicial set, which encodes all information about the deformation problem. The goal of this paper, is to take a first step in making Lurie's work more explicit, we do this by defining and studying the Maurer-Cartan simplicial set for  $E_1$ -algebras, which are more commonly known as  $A_\infty$ -algebras. The main advantage here, is that we get an explicit description of the deformations associated to a deformation problem controlled by an  $A_\infty$ -algebra and that this theory works over every field regardless of the characteristic.

Associative algebras up to homotopy, also known as  $A_\infty$ -algebras, play an important role in many areas of mathematics and mathematical physics. They were originally defined in topology to study loop spaces, but later found applications in representation theory, algebraic geometry, string field theory, mathematical physics, etc. [11, 13, 28–30].

The goal of this paper, is to define and study the  $A_\infty$ -analog of the Deligne-Getzler-Hinich  $\infty$ -groupoid associated to a homotopy Lie algebra, also known as an  $L_\infty$ -

algebra. In [15], Getzler associates to every nilpotent  $L_\infty$ -algebra  $L$  a simplicial set  $MC_\bullet(L)$ . This simplicial set has many good properties and important applications. In [15], it is for example shown that  $MC_\bullet(L)$  is a Kan complex.

One important example of an application of the Maurer–Cartan simplicial set of an  $L_\infty$ -algebra, is that the Maurer–Cartan simplicial set can be used in deformation theory to encode the set of deformations of an object. In this case, the zero simplices of  $MC_\bullet(L)$ , which are the Maurer–Cartan elements of  $L$ , correspond to the deformations. Two deformations are equivalent if and only if the corresponding Maurer–Cartan elements are in the same path component of  $MC_\bullet(L)$ .

It is a well known philosophy, that over a field of characteristic 0 all deformation problems are controlled by the Maurer–Cartan elements in an  $L_\infty$ -algebra. This philosophy goes back to Deligne, Drinfeld, Feigin, Hinich, Kontsevich–Soibelman, Manetti, and many others, and was made precise by Lurie and Pridham (see [23] and [25]). Over a field of characteristic  $p > 0$  this is no longer true, especially since the theory of  $L_\infty$ -algebras has significant problems over a field of characteristic  $p \neq 0$ . The theory developed in this paper, tries to solve this gap in the case that the deformation problem is controlled by an  $A_\infty$ -algebra.

The goal of this paper, is to define an analog of the Maurer–Cartan simplicial set for complete  $A_\infty$ -algebras. Here it is important to note that to make sure that certain infinite sums converge, we need to impose some conditions on the  $A_\infty$ -algebras we work with. We will therefore only consider complete  $A_\infty$ -algebras. This is not a very serious restriction from the point of view of deformation theory. The class of complete  $A_\infty$ -algebras contains all  $A_\infty$ -algebras arising from  $E_1$ -formal moduli problems. If an  $A_\infty$ -algebra is complete for any filtration then it is also complete for the lower central series filtration. Thus the largest class of  $A_\infty$ -algebras we consider is the class of *pro-nilpotent  $A_\infty$ -algebras*. Similar to [15], we define a functor

$$MC_\bullet : A_\infty\text{-algebras} \longrightarrow \text{Simplicial Sets,}$$

from the category of complete  $A_\infty$ -algebras to the category of simplicial sets. As we explain in Remark 6.2, this Maurer–Cartan simplicial set can also be seen as a special case of Lurie’s dg-nerve construction. In [20], Lurie shows that this simplicial set is an  $\infty$ -category. In Sect. 6, we show that the Maurer–Cartan simplicial set is a Kan complex which implies that the nerve of a complete  $A_\infty$ -algebra is not just an  $\infty$ -category, but is in fact an  $\infty$ -groupoid.

Because the simplicial set  $MC_\bullet(A)$  is a Kan complex we can define an equivalence relation on the set of Maurer–Cartan elements of  $A$ , where we define two Maurer–Cartan elements to be homotopy equivalent if they are in the same path component of  $MC_\bullet(A)$ . This allows us to use the Maurer–Cartan simplicial set  $MC_\bullet(A)$  to study a deformation theory controlled by  $A_\infty$ -algebras.

In Sect. 7, we give an example of a deformation problem controlled by  $A_\infty$ -algebras. In this section, we explain how the deformation theory of  $\infty$ -morphisms over non-symmetric operads is controlled by  $A_\infty$ -algebras. Again, the main advantage here is that we no longer have any restrictions on the field we are working over.

## 1.1 Structure of this paper

This paper is structured as follows. In Sect. 2, we introduce the necessary preliminaries on coassociative coalgebras, like filtrations and the shuffle product, which we need to define  $A_\infty$ -algebras. In Sect. 3, we give the definition of  $A_\infty$ -algebras and define their morphisms. In Sect. 4, we describe the twist of an  $A_\infty$ -algebra using the shuffle product and define the Maurer-Cartan equation. In Sect. 5, we prove a few technical lemmas which are important for Sect. 6. In that section we define the Maurer-Cartan simplicial set associated to an  $A_\infty$ -algebra and show that it is a Kan complex. In Sect. 7, we apply the theory developed in this paper to the deformation theory of  $\infty$ -morphism of algebras over non-symmetric operads. We finish this paper with a comparison with other approaches, some possible directions for future work, and some open questions.

## 1.2 Conventions

In this paper, we will use the following conventions and notations. We will always work over a field  $\mathbb{K}$  of characteristic  $p \geq 0$  and always in the category of cochain complexes. We will use a cohomological grading on our cochain complexes, i.e. we use superscripts to indicate the degrees and the differential  $d$  will have degree  $+1$ . We will further assume that all our  $A_\infty$ -algebras are shifted  $A_\infty$ -algebras unless stated otherwise. This means that all the products  $Q_n$  will have degree  $+1$ , more details about this are given in Sect. 3.

The suspension of a cochain complex  $V$  is denoted by  $sV$  and is defined by  $sV^n = V^{n-1}$ . The linear dual of a cochain complex  $V$  is denoted by  $V^\vee$  and is defined as  $\text{Hom}_{\mathbb{K}}(V, \mathbb{K})$ .

All tensor products will be taken over the ground field  $\mathbb{K}$  unless stated otherwise. With the notation  $V^{\otimes n}$  we will denote  $V \otimes \cdots \otimes V$ , where  $V$  appears  $n$  times and by convention, we set  $V^{\otimes 0}$  equal to  $\mathbb{K}$ . We will also implicitly assume that we are using the Koszul sign rule, i.e. we assume that the isomorphism  $\tau : V \otimes W \rightarrow W \otimes V$  is given by  $\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v$ , for  $v \in V$  and  $w \in W$ .

## 2 Coassociative coalgebras

In this section, we recall the basic definitions for coassociative coalgebras. These coalgebras will play an important role in the definition of  $A_\infty$ -algebras. This section is mainly meant to fix notation and conventions. We will therefore assume that the reader is familiar with the basic notions of coassociative coalgebras. For more details we refer the reader to Section 1.2 of [22].

Recall that a coassociative coalgebra  $C$  is a cochain complex  $C$  together with a map  $\Delta : C \rightarrow C \otimes C$  which is coassociative and compatible with the differential. We say that a coassociative coalgebra  $C$  is counital if there is a map  $\epsilon : C \rightarrow \mathbb{K}$  such that this is a counit for the coproduct. A coassociative coalgebra  $C$  is called coaugmented if there is an additional map  $\eta : \mathbb{K} \rightarrow C$  given such that  $\eta$  is a morphism of coassociative coalgebras. Note that because  $\eta$  is a morphism of coassociative coalgebras, we get a

canonical splitting of  $C$  as  $C = \mathbb{K} \oplus \ker(\epsilon)$ . The ideal  $\ker(\epsilon)$  is often denoted by  $\bar{C}$  and is called the coaugmentation ideal. The coproduct on  $\bar{C}$  will be denoted by  $\bar{\Delta} : \bar{C} \rightarrow \bar{C} \otimes \bar{C}$ .

The splitting induced by the coaugmentation defines a pair of adjoint functors between the category of coaugmented coassociative coalgebras and non-counital coassociative coalgebras. The functor from coaugmented coassociative coalgebras is defined by sending  $C$  to its coaugmentation ideal  $\bar{C}$ . The adjoint of this functor is defined by sending a non-counital coassociative coalgebra  $\bar{C}$  to the coaugmented coassociative coalgebra  $C := \bar{C} \oplus \mathbb{K}$ , where the coaugmentation is defined as the inclusion of  $\mathbb{K}$  into  $C$  and the counit as the projection onto  $\mathbb{K}$ . It is a well known result, that these functors define an equivalence of categories.

A coaugmented coassociative coalgebra is called conilpotent if the coradical filtration is exhaustive. The cofree coaugmented conilpotent coassociative coalgebra cogenerated by a cochain complex  $V$  is denoted by  $T^c(V)$  and is defined as follows. As a cochain complex, it is given by  $T^c(V) := \bigoplus_{n \geq 0} V^{\otimes n}$ , the coproduct is defined by deconcatenation. More explicitly,  $\Delta$  is given by

$$\Delta(a_1 \dots a_n) = 1 \otimes a_1 \dots a_n + a_1 \dots a_n \otimes 1 + \sum_{i=1}^{n-1} a_1 \dots a_i \otimes a_{i+1} \dots a_n.$$

Here and in the rest of this paper, we will, to avoid confusion, denote the tensor product  $a_1 \otimes \dots \otimes a_n$  in the tensor coalgebra by concatenation  $a_1 \dots a_n$ . The coaugmentation is given by the inclusion of  $\mathbb{K}$  as  $V^{\otimes 0}$  and the counit is defined as the projection onto  $V^{\otimes 0}$ . Since the element corresponding to  $V^{\otimes 0}$  plays a special role we will denote it by 1. As is explained in [22],  $T^c(V)$  is the cofree coaugmented conilpotent coassociative coalgebra cogenerated by  $V$ , thus any morphism  $C \rightarrow T^c(V)$  is determined by its image on its cogenerators, i.e. there is a bijection  $\text{Hom}_{\text{coalg}}(C, T^c(V)) \cong \text{Hom}_{\mathbb{K}}(C, V)$ . The cofree conilpotent coaugmented coassociative coalgebra is sometimes also called the tensor coalgebra.

Let  $(C, \Delta_C)$  and  $(D, \Delta_D)$  be two coassociative coalgebras. Then we can equip the tensor product  $C \otimes D$  with the structure of a coassociative coalgebra. The coproduct

$$\Delta_{C \otimes D} : C \otimes D \rightarrow C \otimes D \otimes C \otimes D$$

is given by

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\text{id} \otimes \tau \otimes \text{id}} C \otimes D \otimes C \otimes D,$$

where  $\tau : C \otimes D \rightarrow D \otimes C$  is the flip map.

As is described in Section 1.3 of [22], the cofree conilpotent coaugmented coassociative coalgebra can be equipped with a natural product called the shuffle product. This product plays an important role in the definition of the twist in Sect. 4 of this paper. The shuffle product is characterized by the following properties. It is a morphism of

coassociative coalgebras

$$\mu_{sh} : T^c(V) \otimes T^c(V) \rightarrow T^c(V),$$

and on cogenerators, it is given by

$$\mu_{sh} : T^c(V) \otimes T^c(V) \rightarrow V,$$

which is defined on  $V \otimes \mathbb{K} \oplus \mathbb{K} \otimes V \subset T^c(V) \otimes T^c(V)$  by

$$\mu_{sh}(1 \otimes v) = \mu_{sh}(v \otimes 1) = v$$

and is zero otherwise. Explicitly, we have

$$\mu_{sh}(v_1 \dots v_p \otimes v_{p+1} \dots v_{p+q}) = \sum_{\sigma \in \text{Sh}(p,q)} \epsilon(\sigma) v_{\sigma(1)} \dots v_{\sigma(p+q)},$$

where we denote by  $\epsilon(\sigma) = \epsilon(\sigma, v_1, \dots, v_{p+q})$  the Koszul sign and by  $\text{Sh}(p, q)$  the set of  $(p, q)$ -shuffles in the symmetric group on  $p + q$  letters. With the shuffle product the tensor coalgebra becomes a unital associative algebra, where the unit for the shuffle product is given by the element 1. It turns out that, the shuffle product and the coproduct satisfy the Hopf compatibility relation which is given by

$$\Delta \circ \mu_{sh} = (\mu_{sh} \otimes \mu_{sh}) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta \otimes \Delta),$$

where  $\tau$  is the flip map. So with the shuffle product and deconcatenation coproduct, the tensor coalgebra becomes a bialgebra.

A coderivation on a coalgebra  $(C, \Delta)$  is a linear map  $D : C \rightarrow C$  such that

$$(D \otimes \text{id} + \text{id} \otimes D) \Delta = \Delta D,$$

where we remind the reader that we always use the Koszul sign convention. A codifferential on the graded coalgebra  $C$  is a degree +1 coderivation  $Q$  such that  $Q^2 = 0$ . Note that we do *not* assume that  $Q(1) = 0$ . Since  $T^c(V)$  is cofree, it turns out that any coderivation  $D$  on it is uniquely determined by the composite

$$pr_V \circ D : T^c(V) \rightarrow T^c(V) \rightarrow V,$$

where  $pr_V$  simply denotes the projection onto its cogenerators.

Denote by

$$D_n : V^{\otimes n} \hookrightarrow T^c(V) \xrightarrow{D} T^c(V) \xrightarrow{pr_V} V$$

the weight  $n$  component of  $D$ . The coderivation  $D$  is determined by the  $D_n$ , by extending the formula

$$D(v_1 \dots v_n) := \mu_{sh}(D_0(1), v_1 \dots v_n) + \sum_{p=1}^n \sum_{i=0}^{n-p} v_1 \dots v_i D_p(v_{i+1} \dots v_{i+p}) v_{i+p+1} \dots v_n$$

to a linear map. This gives a bijection between  $\text{Coder}(T^c(V))$  and  $\text{Hom}_{\mathbb{K}}(T^c(V), V)$ .

### 3 $A_\infty$ -Algebras

In this section, we introduce shifted  $A_\infty$ -algebras which will be the main objects of study in this paper. The unshifted version of  $A_\infty$ -algebras was first introduced by Stasheff in the study of loop spaces (see [28,29]), and later found applications in many other areas of mathematics and mathematical physics. In this section and the next, we will recall some of the definitions and results of the theory  $A_\infty$ -algebras and offer a new perspective on the Maurer–Cartan equation and twist using the shuffle product. The main advantage of the use of the shuffle product is that the proofs become more conceptual, simpler, and immediately work over fields of arbitrary characteristic. We refer the reader to [21] for a nice introduction to  $A_\infty$ -algebras and Chapter 9 of [22] and the references therein for more details. The Maurer–Cartan equation and twist of an  $A_\infty$ -algebra are described in Chapter 3 of [14], we simplify their results by using the shuffle product.

#### 3.1 Definition of shifted $A_\infty$ -algebras

There are a few ways to define (shifted)  $A_\infty$ -algebras. In this section, we consider the definition given by a codifferential on the tensor coalgebra.

**Definition 3.1** Let  $A$  be a graded vector space, a (curved) shifted  $A_\infty$ -algebra structure on  $A$  is defined as a codifferential  $Q : T^c(A) \rightarrow T^c(A)$  on the cofree coaugmented conilpotent coassociative coalgebra cogenerated by  $A$ .

Since  $T^c(A)$  is cofree, every derivation is determined by its image on its cogenerators. A shifted  $A_\infty$ -algebra  $A$  is therefore equivalent to a sequence  $\{Q_n\}_{n \geq 0}$  of degree  $+1$  maps  $Q_n : A^{\otimes n} \rightarrow A$  satisfying a quadratic condition coming from  $Q^2 = 0$ . This gives rise to the “higher associativity” conditions

$$\sum_{\substack{a,b,c \geq 0 \\ a+b+c=n}} Q_{a+c+1} \circ (\text{id}^{\otimes a} \otimes Q_b \otimes \text{id}^{\otimes c}) = 0,$$

for all  $n \geq 0$ . A flat shifted  $A_\infty$ -algebra is one for which  $Q_0 = 0$ , i.e.  $Q(1) = 0$ . Because we will work in this paper, with curved algebras more often than with flat algebras, we will from now on assume that all  $A_\infty$ -algebras are curved unless



specifically stated otherwise, this is contrary to the usual convention. The associativity conditions above, show clearly that a flat shifted  $A_\infty$ -algebra such that  $Q_k = 0$  for  $k \geq 3$  is simply a shifted dg associative algebra. Note also, that a shifted  $A_\infty$ -algebra structure on  $A$  is equivalent to the usual notion of an unshifted  $A_\infty$ -algebra structure on  $sA$ . The multi-products in that setting are given by  $m_n = s \circ Q_n \circ (s^{-1})^{\otimes n}$ .

**Remark 3.2** An alternative description of flat shifted  $A_\infty$ -algebras is as algebras over the shifted  $A_\infty$ -operad. The shifted  $A_\infty$ -operad is defined as the operadic cobar construction on the coassociative cooperad  $As^c$  (concentrated in degree 0). This equivalence is a straightforward consequence of Theorem 10.1.13 of [22], since they assume that a coderivation maps 1 to 0 this is the same as requiring that the shifted  $A_\infty$ -algebra is flat.

**Definition 3.3** (Curvature of an  $A_\infty$ -Algebra) The curvature of an  $A_\infty$ -algebra  $(A, Q)$  is the element  $Q(1) = Q_0(1) \in A^1$ .

To facilitate certain infinite sums we will need a topology. To obtain this we consider a decreasing filtration of subspaces

$$A = F^1 A \supset F^2 A \supset \dots,$$

which satisfies  $\bigcap_k F^k A = \{0\}$ . This yields the metric topology given by the metric  $d(v, v) = 0$  and  $d(v, w) = 2^{-|v-w|}$  if  $v \neq w$ , where  $|x| = \max\{k \mid x \in F^k A\}$ . We also assume that the maps  $Q_n$  preserve this filtration in the sense that

$$Q_n(F^{i_1} A \otimes \dots \otimes F^{i_n} A) \subset F^{i_1 + \dots + i_n} A.$$

We will call a shifted  $A_\infty$ -algebra *complete* if this metric is complete. Note that a filtration  $F^i A$  on the graded vector space  $A$  induces a filtration  $F^i T^c(A)$  in the usual way

$$F^i T^c(A) = \bigoplus_{n \geq 0} \bigoplus_{\substack{i_1, \dots, i_n \in \mathbb{N} \\ i_1 + \dots + i_n = i}} F^{i_1} A \otimes \dots \otimes F^{i_n} A.$$

Note also that if  $v \in T^c(A)$  then

$$v = \sum_{n=0}^N \sum_{i=1}^{k_n} \lambda_{i,n} a_{i,1} \otimes \dots \otimes a_{i,n},$$

for some  $N, k_n \in \mathbb{N}$ ,  $\lambda_{i,n} \in \mathbb{K}$  and  $a_{i,j} \in A$ . Thus, there are numbers  $m_{i,j} \in \mathbb{Z}$  such that  $a_{i,j} \notin F^{m_{i,j}} A$  and thus, setting  $M = N \max\{m_{i,j}\}$ ,  $v \notin F^M T^c(A)$ . This implies that  $\bigcap_k F^k T^c(A) = \{0\}$ . Again, we consider  $T^c(A)$  as a metric space for the induced metric as above. This space is in general not complete and so we denote the completion of  $T^c(A)$  by  $\widehat{T^c(A)}$ . Note that since the structure maps all respect the filtration, the coalgebra structure and the codifferential extend uniquely to  $\widehat{T^c(A)}$  in

the appropriate completed sense, e.g. the coproduct maps into the completed tensor product, see Appendix A.

**Remark 3.4** The *lower central series* filtration defined by

$$\mathcal{F}^i A = \sum_{n \geq 1} \sum_{\substack{i_1, \dots, i_n \\ i_1 + \dots + i_n = i}} Q_n(\mathcal{F}^{i_1} A \otimes \dots \otimes \mathcal{F}^{i_n} A),$$

for  $i > 1$ , is automatically preserved by the  $Q_n$ . If  $(A, Q)$  is moreover nilpotent, meaning that  $\mathcal{F}^i A = 0$  for sufficiently large  $i$ , then the filtration is also complete. In general, we may call a shifted  $A_\infty$ -algebra  $(A, Q)$  “pro-nilpotent” if the lower central series filtration is complete. Note that for a nilpotent shifted  $A_\infty$ -algebra we find that  $\widehat{T}^c(A) = \prod_{n \geq 0} (A)^{\otimes n}$ ; of course this is true for any filtration that terminates. Note also that, for any filtration  $F^\bullet A$  that is compatible with the  $A_\infty$ -structure, it is immediate that

$$F^n A \supset \mathcal{F}^n A$$

for all  $n \geq 1$ . This implies in particular that any sequence that is Cauchy for the metric induced by the lower central series filtration is also Cauchy for the metric induced by the filtration  $F^\bullet A$ . Thus any complete  $A_\infty$ -algebra is automatically pro-nilpotent. This means that the widest class of algebras for which the constructions of this paper work are the pro-nilpotent  $A_\infty$ -algebras.

**Definition 3.5** Let  $(A, Q_A)$  and  $(B, Q_B)$  be two curved  $A_\infty$ -algebras. An  $\infty$ -morphism of curved  $A_\infty$ -algebras, denoted by  $F : A \rightsquigarrow B$ , is a degree 0 morphism of counital coalgebras

$$F : T^c(A) \rightarrow T^c(B),$$

that commutes with the differentials, i.e. such that  $FQ_A = Q_B F$ . When  $(A, Q_A)$  and  $(B, Q_B)$  are filtered, we further require the morphism  $F$  to respect the induced filtrations on  $T^c(A)$  and  $T^c(B)$ .

Since maps to the cofree conilpotent coalgebra  $T^c(B)$  are determined by their projection on its cogenerators, the map  $F : T^c(A) \rightarrow T^c(B)$  is equivalent to a sequence of maps

$$F_n : A^{\otimes n} \rightarrow B,$$

for  $n \geq 0$ . The map  $F$  can be recovered from these maps via the following formulas

$$F(a_1 \dots a_n) = \sum_{p \geq 1} \sum_{\substack{k_1, \dots, k_p \geq 1 \\ k_1 + \dots + k_p = n}} F_{k_1}(a_1 \dots a_{k_1}) \dots F_{k_p}(a_{n-k_p+1} \dots a_n), \tag{3.1}$$

and the assertion that  $F(1) = 1$ .

When  $(A, Q_A)$  and  $(B, Q_B)$  are filtered by filtrations  $\{G^n A\}_{n \geq 1}$  and  $\{G^n B\}_{n \geq 1}$ , the condition that the morphism  $F : A \rightsquigarrow B$  respects the filtrations translates into the following requirement:

$$F_n \left( G^{i_1} A \otimes \cdots \otimes G^{i_n} A \right) \subseteq G^{i_1 + \cdots + i_n} B.$$

**Remark 3.6** Since we are using the cofree conilpotent coalgebra in Definition 3.5 and because the only grouplike element in the cofree conilpotent coalgebra is the element 1, it follows that  $F(1_A) = 1_B$ . So in other words, an  $\infty$ -morphism maps the curvature of  $A$  to the curvature of  $B$ . Note that this is no longer the case when we would replace the cofree conilpotent coalgebra by the completed cofree conilpotent coalgebra, i.e. when we would look at maps  $F' : \widehat{T^c(A)} \rightarrow \widehat{T^c(B)}$  it is no longer the case that  $F'(1_A) = 1_B$ , so  $\infty$ -morphisms will behave differently.

Finally, if  $(A, Q^A)$  is a flat shifted  $A_\infty$ -algebra, then  $(Q_1^A)^2 = 0$ . So it yields an underlying cochain complex  $(A, Q_1^A)$ . Any  $\infty$ -morphism  $F : (A, Q^A) \rightsquigarrow (B, Q^B)$  induces a map  $F_1 : (A, Q_1^A) \rightarrow (B, Q_1^B)$  of the underlying cochain complexes.

**Definition 3.7** Let  $(A, Q_A)$  and  $(B, Q_B)$  be flat shifted  $A_\infty$ -algebras. An  $\infty$ -morphism  $F : A \rightsquigarrow B$  is called an  $\infty$ -quasi-isomorphism if the arity one component

$$F_1 : A \rightarrow B,$$

is a quasi-isomorphism of chain complexes.

When  $A$  and  $B$  are filtered by filtrations  $\{G^n A\}_{n \geq 1}$  and  $\{G^n B\}_{n \geq 1}$ , we further require that the induced maps

$$F_1|_{G_n A} : G_n A \rightarrow G_n B,$$

are quasi-isomorphism for all  $n \geq 1$ . When  $A$  and  $B$  are filtered, we will always assume that  $\infty$ -morphisms respect the filtrations in this way and simply call this an  $\infty$ -isomorphism (so omitting the filtered in the terminology).

### 3.2 Extension of scalars

Given a unital differential graded associative algebra  $(C, m\mathbb{1}, d_C, \mu)$  and an  $A_\infty$ -algebra  $(A, Q)$ , it is possible to equip the tensor product  $A \otimes C$  with a new  $A_\infty$ -structure, which we call the extension of scalars by  $C$ . Using the identifications  $(A \otimes C)^{\otimes n} \cong A^{\otimes n} \otimes C^{\otimes n}$  (using the Koszul sign rule), the maps  $CQ_n : (A \otimes C)^{\otimes n} \rightarrow A \otimes C$  are given by

- $CQ_0(1) = Q_0(1) \otimes \mathbb{1}$ ;
- $CQ_1 = Q_1 \otimes \text{id} + \text{id} \otimes d_C$ ;
- $CQ_k = Q_k \otimes \mu^{(k)}$  for  $k \geq 2$ .

Here  $\mu^{(k)} : C^{\otimes k} \rightarrow C$ , for  $k \geq 2$ , is the  $(k - 1)$ -fold iterated product of  $C$  which is given by  $\mu^{(k)}(x_1, \dots, x_k) = \mu(x_1, \mu(x_2, \mu(\dots, x_k)))$ , for  $x_1, \dots, x_k \in C$ . Note that if  $A$  is flat then  $A \otimes C$  is automatically flat again.

**Proposition 3.8** *Let  $A$  be an  $A_\infty$ -algebra with filtration  $F^i A$  and  $C$  a finite dimensional associative algebra. The  $A_\infty$ -algebra  $A \otimes C$  is then equipped with the filtration*

$$F^i(A \otimes C) := (F^i A) \otimes C,$$

which is complete if  $A$  is complete.

**Proof** Suppose  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $A \otimes C$  and let  $B \subset C$  be a basis. Then there is a unique decomposition  $x_n = \sum_{b \in B} x_{n,b} \otimes b$  for each  $n$ . Since  $(x_n)$  is Cauchy we have that for each  $k \in \mathbb{N}$  there is  $N \in \mathbb{N}$  such that  $\forall n, m > N$ , we have  $x_n - x_m \in F_k A \otimes C$ , i.e.

$$\sum_{b \in B} (x_{n,b} - x_{m,b}) \otimes b \in F_k A \otimes C.$$

Thus we find that the sequences  $(x_{n,b})_{n \in \mathbb{N}}$  are Cauchy for each  $b \in B$ . By completeness of  $A$  these last sequences converge to elements  $x_b \in A$ , i.e. for each  $b \in B$  and  $k \in \mathbb{N}$  there is  $N \in \mathbb{N}$  such that for all  $n > N$  we have  $x_b - x_{n,b} \in F_k A$ . Thus by finiteness of the set  $B$ , we find that the sequence  $(x_n)$  converges to  $x = \sum_{b \in B} x_b \otimes b$ .  $\square$

### 4 MC elements and twisting

In this section, we define the twist of a complete curved shifted  $A_\infty$ -algebra  $A$ . Given an element  $x \in A^0$ , this is a procedure to construct a new  $A_\infty$ -algebra  $(A, Q^x)$  which has  $A$  as underlying graded vector space but the differential  $Q$  is twisted by the element  $x$ . The Maurer–Cartan equation then naturally appears as a flatness equation for this twisted structure.

As noted above, a shifted  $A_\infty$ -algebra is defined by a codifferential on the coaugmented cofree conilpotent coassociative coalgebra cogenerated by a  $\mathbb{Z}$ -graded vector space  $A$ . This last coalgebra is canonically a bialgebra for the shuffle product. Note that the shuffle product automatically respects a filtration induced by a filtration on  $A$ . So if  $(A, Q)$  is a complete shifted  $A_\infty$ -algebra, then we also have a bialgebra structure and codifferential on  $\widehat{T}^c(A)$ , again in the complete sense, see appendix A. From now on, we will assume completions whenever necessary but we will not differentiate in notation between maps and their unique extension to completions.

**Definition 4.1** (The exponential) The exponential map

$$e^- : A^0 \longrightarrow \widehat{T}^c(A)$$

is defined as  $e^x = \lim_{n \rightarrow \infty} \sum_{k=0}^n x^k$ , recall from Sect. 2 that  $x^k$  is a shorter notation for  $x^{\otimes k}$ .

Note that letting  $\mu_{sh}^{(k)}$  denote the  $(k-1)$ th iteration  $\mu_{sh}(\mu_{sh} \otimes \text{id}) \dots (\mu_{sh} \otimes \text{id}^{\otimes k-2})$ , for  $k \geq 2$ ,  $\mu_{sh}^{(0)} = 1$  and  $\mu_{sh}^{(1)} = \text{id}$ , we could write  $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} \mu_{sh}^{(k)}(x^k)$ , however the definition above makes perfect sense over a field of arbitrary characteristic.

**Lemma 4.2** *The operation  $\exp(x) : \widehat{T^c}(A) \rightarrow \widehat{T^c}(A)$  given by*

$$y \mapsto \mu_{sh}(e^x \otimes y)$$

for some element  $x \in A^0$  defines a coalgebra automorphism with inverse  $\exp(-x)$ .

**Proof** First of all, note that because of the Hopf compatibility relation the multiplication by  $e^x$  defines an endomorphism of  $\widehat{T^c}(V)$ . To show that it is an automorphism, first note that  $e^x$  satisfies

$$e^x \otimes e^x = \lim_{N \rightarrow \infty} \sum_{n=0}^N \sum_{k=0}^n x^k \otimes x^{n-k} = \lim_{N \rightarrow \infty} \sum_{n=0}^N \Delta(x^n) = \Delta(e^x),$$

i.e. it is a group-like element.

Similarly, we note that

$$\mu_{sh}(e^x \otimes e^{-x}) = \lim_{N \rightarrow \infty} \mu_{sh} \left( \sum_{k=0}^n \sum_{l=0}^k (-1)^l x^{k-l} \otimes x^l \right) = 1,$$

i.e.  $e^x$  is invertible with inverse  $e^{-x}$ .

These two facts show that the map  $\exp(x)$  defines a coalgebra automorphism of  $\widehat{T^c}(A)$  with inverse  $\exp(-x)$ . □

**Lemma 4.3** *The operation  $Q^x = \exp(-x) \circ Q \circ \exp(x) : \widehat{T^c}(A) \rightarrow \widehat{T^c}(A)$  preserves the subspace  $T^c(A)$ .*

**Proof** Note that  $Q^x$  defines a codifferential by Lemma 4.2. Now, consider the coderivation  $\tilde{Q}^x$  given by the sequence of degree +1 maps

$$\tilde{Q}_n^x : A^{\otimes n} \rightarrow A,$$

defined by

$$\tilde{Q}_n^x(a_1 \dots a_n) = \sum_{p \geq 0} Q_{n+p}^x(\mu_{sh}(x^p \otimes a_1 \dots a_n)).$$

These are all well-defined by completeness of  $A$ . Clearly the restriction of  $Q^x$  coincides with  $\tilde{Q}^x$ , which proves the lemma. □

Lemmas 4.2 and 4.3 allow us to define the twist of a shifted  $A_\infty$ -algebra  $(A, Q)$ . This is done as follows.

**Definition 4.4** (Twisting) Let  $(A, Q)$  be a curved shifted  $A_\infty$ -algebra and let  $x \in A^0$ , then we define  $(A, Q^x)$ , the shifted  $A_\infty$ -algebra  $A$  twisted by the element  $x$ , as follows. The underlying graded vector space of  $(A, Q^x)$  is defined as  $A$ . The coderivation  $Q^x : T^c(A) \rightarrow T^c(A)$  is defined as the restriction of the map  $Q^x := \exp(-x) \circ Q \circ \exp(x)$  to  $T^c(A)$ .

Note that the formula for  $\tilde{Q}_n^x(a_1 \dots a_n)$  in the proof of Lemma 4.3 provides an explicit description of the twisted  $A_\infty$ -algebra structure.

**Definition 4.5** (Curvature of an Element) The curvature of an element  $x \in A^0$  in a shifted  $A_\infty$ -algebra  $(A, Q)$  is the element

$$\mathcal{R}(x) := \exp(-x)Q(e^x).$$

Note that since clearly  $\mathcal{R}(x) = Q^x(1)$ , it holds that  $\mathcal{R}(x) \in A^1 \subset T^c(A)$ . It will therefore not contain any terms of tensor weight higher than 1, so it can be seen that

$$\mathcal{R}(x) = \sum_{l \geq 0} Q_l(x^l)$$

and that  $\mathcal{R}(x)$  is simply the curvature of  $A^x$ .

**Lemma 4.6** (Bianchi Identity) For a shifted  $A_\infty$ -algebra  $(A, Q)$  and  $x \in A^0$ , we have

$$Q^x(\mathcal{R}(x)) = 0$$

or equivalently

$$\sum_{l \geq 1} Q_l \left( \mu_{sh} \left( x^{l-1} \otimes \mathcal{R}(x) \right) \right) = 0.$$

**Proof** Note that the first identity is obvious since

$$Q^x(\mathcal{R}(x)) = \exp(-x)Q(\exp(x)\exp(-x)Q(e^x)) = \exp(-x)Q^2(e^x) = 0.$$

Thus it is only left to show that it is equivalent to the second identity. This follows straightforwardly by considering the expression  $Q^x(\mathcal{R}(x)) = \mu_{sh}(e^{-x} \otimes Q(\mu_{sh}(e^x \otimes \mathcal{R}(x))))$  and realizing that this must be an element of  $A$ . □

**Definition 4.7** (Maurer–Cartan elements) Consider a shifted  $A_\infty$ -algebra  $(A, Q)$ . Then  $x \in A^0$  is called a Maurer–Cartan element (abbreviated to MC element) if

$$Q(e^x) = 0.$$

**Corollary 4.8** By invertibility of  $e^x$ , an element  $x \in A^0$  is an MC element if and only if  $\mathcal{R}(x) = 0$ . So, given a shifted  $A_\infty$ -algebra  $A$  and an element  $x \in A^0$ , the twisted algebra  $A^x$  is flat if and only if  $x$  is a Maurer–Cartan element.

**Definition 4.9** (Functoriality of Maurer–Cartan elements and twisting)

Given an  $\infty$ -morphism  $F$  of complete shifted  $A_\infty$ -algebras  $(A, Q)$  and  $(B, P)$  and an element  $x \in A^0$ , we define the element  $x_F \in B^0$  called  $F$ -associated to  $x$  as the solution to

$$e^{x_F} = F(e^x),$$

explicitly we have

$$x_F := \sum_{n \geq 1} F_n(x^n).$$

Note that  $x_F$  is well-defined by completeness of  $B$ . This way we may also define the  $\infty$ -morphism  $F^x : A^x \rightarrow B^{x_F}$  as

$$F^x := \exp(-x_F) \circ F \circ \exp(x).$$

Note finally that if  $x$  is a Maurer–Cartan element, then so is  $x_F$  and since it can be shown that  $x_{F \circ \tilde{F}} = (x_{\tilde{F}})_F$ , we obtain a functor associating to an  $A_\infty$ -algebra its set of Maurer–Cartan elements.

**Remark 4.10** Note that if  $F : A \rightarrow B$  is assumed to be strict in the definition above then  $x_F = F(x)$ . This implies that  $F^x$ ,  $F$  twisted by  $x$ , is the same as  $F$ , i.e. the maps  $F^x : A \rightarrow B$  and  $F : A \rightarrow B$  are the same.

## 5 Cochains on simplices

In this section, we prove some technical lemmas about the normalized cochains on the standard simplex. These lemmas will play an important role in the next section where they are used to define the Maurer–Cartan simplicial set associated to a complete (curved)  $A_\infty$ -algebra and are used to prove that it is a Kan complex. We assume that the reader is familiar with the basics of simplicial sets and otherwise we refer the reader to [10] or [16].

Denote by  $\Delta^n$  the standard  $n$ -simplex and recall that the collection of standard  $n$ -simplices forms a cosimplicial object in the category of simplicial sets. We denote this cosimplicial object by  $\Delta^\bullet$  and its coface and codegeneracy maps by  $d^i$  and  $s^j$ .

The normalized cochains on a simplicial set  $X$  are denoted by  $N^\bullet(X; \mathbb{K})$ , for notational simplicity we will drop the coefficients  $\mathbb{K}$  which we always assume to be a field. A basis for  $N^d(\Delta^n)$  is given by the cochains  $\phi_{i_0, \dots, i_d}$ , with  $0 \leq i_0 < \dots < i_d \leq n$ , where  $\phi_{i_0, \dots, i_d}$  denotes the cochain that evaluates to 1 on the subsimplex of  $\Delta^n$  with vertices  $i_0, \dots, i_d$  and is zero otherwise.

The normalized cochains form a unital associative algebra with the chain level cup product coming from the diagonal map  $X \rightarrow X \times X$ . Explicitly, this product is defined as follows. Let  $\phi \in N^i(X)$ ,  $\psi \in N^j(X)$  and  $x : \Delta^{i+j} \rightarrow \Delta^n$  be an  $(i + j)$ -simplex,

then  $\phi \cup \psi$  evaluated on  $x$  is given by

$$\phi \cup \psi(x) = \phi(x(0, \dots, i)) \cdot \psi(x(i, \dots, i + j)), \tag{5.1}$$

where  $x(0, \dots, i)$  is the image of the subsimplex of  $\Delta^{i+j}$  with vertices  $0, \dots, i$  and  $x(i, \dots, i + j)$  is the subsimplex with vertices  $i, \dots, i + j$ . The unit of this product is given by  $\mathbb{1} := \sum_{i=0}^n \varphi_i$ .

Since the normalized cochains are a contravariant functor, the collection of normalized cochains on the standard simplices form a simplicial object in the category of unital associative algebras. The face and degeneracy maps are denoted by  $\partial_i : N^\bullet(\Delta^n) \rightarrow N^\bullet(\Delta^{n-1})$  and  $\sigma_j : N^\bullet(\Delta^n) \rightarrow N^\bullet(\Delta^{n+1})$  and are defined by precomposing a cochain  $\phi$  with  $d^i$  (resp.  $s^j$ ). Let  $\varphi_I \in N^d(\Delta^n)$ , with  $I = (i_0, \dots, i_d)$ , be a basis element. Explicit formulas for the face and degeneracy maps are then given by  $\partial_i(\phi_I) = \sum_{J \in (d^i)^{-1}(I)} \phi_J$ , where the sum ranges over all elements in the inverse image of  $I$  under the map  $d^i$ , when the inverse image is empty we define this term to be zero. Because the maps  $d^i$  are injective, the inverse image will consist out of at most one term. For simplicity we will denote this term by  $\phi_{(d^i)^{-1}(I)}$ . The degeneracy maps are defined similarly by  $\sigma^j(\phi_I) = \sum_{J \in (s^j)^{-1}(I)} \phi_J$ .

The evaluation on  $e_i$ , the  $i$ th vertex of  $\Delta^n$ , will play a special role in what follows and we will therefore denote it by  $\tilde{\epsilon}_n^i : N^\bullet(\Delta^n) \rightarrow \mathbb{K}$ . We also define the map  $\epsilon_n^i : N^\bullet(\Delta^n) \rightarrow N^\bullet(\Delta^n)$  which is defined as the composition of  $\tilde{\epsilon}_n^i$  with the inclusion of the unit  $\mathbb{1} : \mathbb{K} \rightarrow N^\bullet(\Delta^n)$ . More explicitly this map is given by  $\epsilon_n^i(\varphi_i) = \mathbb{1} = \sum_{j=1}^n \varphi_j$  and zero otherwise. To prove that the Maurer–Cartan simplicial set, which we will define in the next section, is a Kan complex we need an explicit contraction on the level of the normalized cochains of  $\Delta^n$ . To do this, we define a contraction between the identity on  $N^\bullet(\Delta^n)$  and  $\epsilon_n^i$ .

The contraction

$$h_n^i : N^\bullet(\Delta^n) \longrightarrow N^{\bullet-1}(\Delta^n)$$

is defined by

$$h_n^i(\varphi_{j_0, \dots, j_d}) := \sum_{k=0}^d (-1)^i \delta_{i, j_k} \varphi_{j_0, \dots, \hat{j}_k, \dots, j_d},$$

where  $\hat{j}_k$  means that we omit the element  $j_k$  and  $\delta_{i, j_k}$  is the Kronecker delta.

**Lemma 5.1** *The map  $h_n^i$  is a contraction between  $id_{N^\bullet(\Delta^n)}$  and  $\epsilon_n^i$ , i.e.  $h_n^i$  satisfies the following formula*

$$dh_n^i + h_n^i d = id_{N^\bullet(\Delta^n)} - \epsilon_n^i.$$

The proof of the lemma is straightforward but tedious and is left to the reader.

If we have a complete  $A_\infty$ -algebra  $(A, Q_0, Q_1, Q_2, \dots)$ , then we can use the extension of scalars from Sect. 3.2 to form a simplicial object in the category of  $A_\infty$ -algebras.



This object is given by  $\{A \otimes N^\bullet(\Delta^n)\}_{n \geq 0}$  where the face and degeneracy maps are induced by the face and degeneracy maps of  $\{N^\bullet(\Delta^n)\}_{n \geq 0}$ . Similar to Sect. 3.2, we use the notation  $N_\bullet Q_n$  for the  $A_\infty$ -structure maps on  $A \otimes N^\bullet(\Delta^n)$ , we will often abuse notation by dropping the  $\bullet$ . By using the contraction maps  $h_n^i : N^\bullet(\Delta^n) \rightarrow N^{\bullet-1}(\Delta^n)$ , we can also define a ‘‘contraction’’ of  $A \otimes N^\bullet(\Delta^n)$ . To do this, we first introduce the maps  $E_n^i : A \otimes N^\bullet(\Delta^n) \rightarrow A \otimes N^\bullet(\Delta^n)$  which are the analogs of evaluation on the  $i$ th vertex of  $\Delta^n$ . This map is defined as  $E_n^i := \text{id}_A \otimes \epsilon_n^i$ .

When the  $A_\infty$ -algebra  $A$  is flat, then as a map of cochain complexes, the map  $E_n^i$  is homotopic to the identity and an explicit homotopy  $H_n^i : A \otimes N^\bullet(\Delta^n) \rightarrow A \otimes N^{\bullet-1}(\Delta^n)$  is given by setting  $H_n^i(a \otimes \varphi) := a \otimes h_n^i(\varphi)$ , where  $a \otimes \varphi \in A \otimes N^\bullet(\Delta^n)$ . When  $A$  is not flat we can no longer speak about maps of cochain complexes but the maps  $H_n^i$  still have the following property.

**Proposition 5.2** *The maps  $H_n^i$ ,  $E_n^i$  and  $\text{id}_{A \otimes N^\bullet(\Delta^n)}$  satisfy the following equation*

$$H_n^i N Q_1 + N Q_1 H_n^i = \text{id}_{A \otimes N^\bullet(\Delta^n)} - E_n^i.$$

The proof of this proposition follows straightforwardly from Lemma 5.1. For the proof that the Maurer-Cartan simplicial set is a Kan complex, we need one more definition. For each  $0 \leq i \leq n$ , define the operator  $R_n^i : A \otimes N^\bullet(\Delta^n) \rightarrow A \otimes N^\bullet(\Delta^n)$  as  $R_n^i := N Q_1 \circ H_n^i$ . In the following lemmas, we give some of the properties of the operators  $R_n^i$  and  $E_n^i$ .

**Lemma 5.3** *The following identity holds*

$$E_n^i N Q_l(v) = N Q_l((E_n^i)^{\otimes l} v)$$

for all  $n, l \geq 0, 0 \leq i \leq n$  and  $v \in (A \otimes N^\bullet(\Delta^n))^{\otimes l}$ , where for  $l = 0$  we have used the convention that  $(E_n^i)^{\otimes 0} = \text{id}$ .

**Proof** To prove the lemma we distinguish three different cases, first we prove the lemma for  $l = 0, 1$  and then for  $l \geq 2$ . When  $l = 0$ , we find that  $E_n^i N Q_0 = N Q_0$  since  $N Q_0(1) = Q_0(1) \otimes \mathbb{1}$  and  $\epsilon_n^i(\mathbb{1}) = \mathbb{1}$ . When  $l = 1$ , the operation  $N Q_1$  is given by  $Q_1 \otimes \text{id} + \text{id} \otimes d_{N^\bullet(\Delta^n)}$ . Since  $E_n^i$  vanishes on all elements of degree greater than 0,  $E_n^i \circ (\text{id} \otimes d_{N^\bullet(\Delta^n)}) = 0$ . Because  $d_{N^\bullet(\Delta^n)}(\mathbb{1}) = 0$ , we also have that  $(\text{id} \otimes d_{N^\bullet(\Delta^n)}) E_n^i = 0$ . We are only left to show that  $E_n^i \circ (Q_1 \otimes \text{id}) = (Q_1 \otimes \text{id}) \circ E_n^i$  which is obvious.

The products  $N Q_l$  for  $l \geq 2$  are given by  $Q_l \otimes \cup^{(l)}$ . In this case we see that both sides of the equation vanish in case  $v$  contains any element of degree higher than 0 on the second tensor leg as a tensor factor. Furthermore, if we work in terms of generators on  $A \otimes N^0(\Delta^n)$  induced by the basis  $\varphi_j$ , for  $0 \leq j \leq n$ , of  $N^\bullet(\Delta^n)$ , we see that both sides vanish on elements  $v$  that are not of the form  $(\alpha_1 \otimes \varphi_i) \otimes \dots \otimes (\alpha_n \otimes \varphi_i)$ . For  $v = (\alpha_1 \otimes \varphi_i) \otimes \dots \otimes (\alpha_n \otimes \varphi_i)$  it is clear that  $E_n^i N Q_l(v) = N Q_l((E_n^i)^{\otimes l} v)$ , which proves the lemma. □

**Lemma 5.4** *The maps  $E_n^i$  and  $R_n^i$  satisfy the following identities:*

(i)

$$\partial_j E_n^i = \begin{cases} E_{n-1}^i \partial_j, & \text{if } i < j \\ E_{n-1}^{i-1} \partial_j, & \text{if } i > j. \end{cases}$$

(ii)

$$\partial_j R_n^i = \begin{cases} R_{n-1}^i \partial_j, & \text{if } i < j \\ R_{n-1}^{i-1} \partial_j, & \text{if } i > j. \end{cases}$$

**Proof** To prove part *i* of the lemma, first observe that  $E_{n-1}^i \partial_j$ ,  $E_{n-1}^{i-1} \partial_j$ , and  $\partial_j E_n^i$  vanish on all elements of the form  $\alpha \otimes \varphi$  when  $\varphi$  is of degree greater or equal than one. We therefore only need to prove part *i* of the lemma for elements of the form  $\alpha \otimes \varphi_k \in A \otimes N^0(\Delta^n)$ .

Explicit formulas for the face map  $\partial_j$  are given by

$$\partial_j(\varphi_{i_1, \dots, i_k}) = \varphi_{(dj)^{-1}(i_1, \dots, i_k)} = \varphi_{i_1, \dots, i_{m-1}, i_m-1, \dots, i_k-1},$$

if none of the  $i_l$  is equal to  $j$  and where  $i_{m-1}$  is the largest  $i_{m-1}$  smaller than  $j$ , if one of the  $i_l = j$ , then the face map is equal to zero.

After applying these formulas to  $\alpha \otimes \varphi_k$ , it is straightforward to see that part one of the lemma holds. More explicitly, on one side we get

$$\begin{aligned} \partial_j E_n^i(\alpha \otimes \varphi_k) &= \partial_j(\delta_{i,k} \alpha \otimes \mathbb{1}) \\ &= \delta_{i,k} \alpha \otimes \mathbb{1} \end{aligned}$$

and on the other side we get

$$E_n^i \partial_j(\alpha \otimes \varphi_k) = E_n^i(\alpha \otimes \varphi_{(dj)^{-1}(k)}),$$

which is non-zero if and only if  $k = i$ . So this is also equal to  $\delta_{i,k} \alpha \otimes \mathbb{1}$ . When  $j < i$ , the proof is the same except that the last term is non zero if and only if  $k$  is  $(i - 1)$  instead of  $i$ . This proves part *i* of the lemma.

To prove part *ii* of the lemma we need to show that  $R_{n-1}^i \partial_j = \partial_j R_n^i$  for  $i < j$  and  $R_{n-1}^{i-1} \partial_j = \partial_j R_n^i$  for  $i > j$ . We first show this for  $i < j$ . Let  $\varphi_I \in N^k(\Delta^n)$ , with  $I = (i_1, \dots, i_k)$  and let  $\alpha \in A$ . In this case we get the following sequence of equalities:

$$\begin{aligned} \partial_j R_n^i(\alpha \otimes \varphi_I) &= \partial_j N Q_1 H_n^i(\alpha \otimes \varphi_I) \\ &= \partial_j N Q_1(\alpha \otimes \varphi_{I \setminus \{i\}}), \end{aligned}$$

where we define  $\varphi_{I \setminus \{i\}}$  to be zero when the indexing set  $I$  does not contain the element  $i$ . When we continue we get the following terms

$$\begin{aligned} \partial_j N Q_1(\alpha \otimes \varphi_{I \setminus \{i\}}) &= \partial_j(Q_1(\alpha) \otimes \varphi_{I \setminus \{i\}}) \pm \sum_{l=0}^n \pm \varphi_{I \setminus \{i\} \cup l} \\ &= Q_1(\alpha) \otimes \varphi_{(d^j)^{-1}(I \setminus \{i\})} \pm \sum_{l=0}^n \pm \varphi_{(d^j)^{-1}(I \setminus \{i\}) \cup l}. \end{aligned}$$

Whenever we take the union of  $I$  and  $\{l\}$  such that  $l$  is already contained in  $I$ , then we set this term to zero. Because the term  $(d^j)^{-1}(I \setminus \{i\} \cup j)$  is zero and  $(d^j)^{-1}(l) = l - 1$  for  $l > j$ , this sum can be rewritten as

$$Q_1(\alpha) \otimes \varphi_{(d^j)^{-1}(I \setminus \{i\})} \pm \sum_{l=0}^{n-1} \pm \varphi_{(d^j)^{-1}(I \setminus \{i\}) \cup l}.$$

When we compute the  $R_{n-1}^i \partial_j(\alpha \otimes \varphi_I)$  side for  $i < j$ , we get the following.

$$\begin{aligned} R_{n-1}^i \partial_j(\alpha \otimes \varphi_I) &= N Q_1 H_{n-1}^i \alpha \otimes \varphi_{(d^j)^{-1}(I)} \\ &= N Q_1(\alpha \otimes \varphi_{(d^j)^{-1}(I) \setminus \{i\}}) \\ &= Q_1(\alpha) \otimes \varphi_{(d^j)^{-1}(I) \setminus \{i\}} \pm \sum_{l=0}^{n-1} \pm \alpha \otimes \varphi_{(d^j)^{-1}(I) \setminus \{i\} \cup l}. \end{aligned}$$

Because  $i < j$ , we have an equality between  $(d^j)^{-1}(I) \setminus \{i\}$  and  $(d^j)^{-1}(I \setminus \{i\})$ . This sum is therefore equal to

$$Q_1(\alpha) \otimes \varphi_{(d^j)^{-1}(I \setminus \{i\})} \pm \sum_{l=0}^{n-1} \pm \varphi_{(d^j)^{-1}(I \setminus \{i\}) \cup l},$$

which is  $\partial_j R_n^i(\alpha \otimes \varphi_I)$ . So for  $i < j$ , the lemma holds. When  $i > j$ , a similar arguments show that the lemma also holds in that case. We leave it to the reader to check that the signs agree as well. □

### 6 MC set of an $A_\infty$ -algebra

In this section, we will define and study the simplicial set of Maurer-Cartan elements associated to a complete shifted  $A_\infty$ -algebra  $(A, Q)$ . In the following, we will denote the set of Maurer-Cartan elements in a complete shifted  $A_\infty$ -algebra  $(A, Q)$  by  $MC(A, Q)$ .

**Definition 6.1** (Maurer-Cartan simplicial set) The Maurer-Cartan simplicial set  $MC_\bullet(A, Q)$  is given by the sets  $MC_n(A, Q) = MC(A \otimes N^\bullet(\Delta^n), NQ)$  with the face and degeneracy maps induced by those on  $A \otimes N^\bullet(\Delta^n)$ . Given an  $\infty$ -morphism  $F: A \rightarrow B$  we can consider the induced  $\infty$ -morphisms  $N^n F: A \otimes N^\bullet(\Delta^n) \rightarrow B \otimes N^\bullet(\Delta^n)$  given by  $N^n F_l = F_l \otimes \cup^{(l)}$  with the convention that  $\cup^{(1)} = \text{id}$ . Using

these, we find that  $MC_\bullet$  is a functor from the category of (curved)  $A_\infty$ -algebras with  $\infty$ -morphisms to simplicial sets.

**Remark 6.2** Note that this Maurer–Cartan simplicial set can be seen as a special case of Lurie’s dg-nerve construction (see Construction 1.3.1.6 and Remark 1.3.1.7 of [20]). This can be seen as follows, first note that an  $A_\infty$ -algebra  $A$  can be seen as an  $A_\infty$ -category with one object and the underlying chain complex of  $A$  as the chain complex set of morphisms. The compositions are then given by multiplications. Using this we get a dg-category to which we can apply Lurie’s dg-nerve construction. As Lurie shows in Proposition 1.3.1.10, the simplicial set we obtain this way is an  $\infty$ -category, i.e. we can fill all the inner horns. In this paper, we improve Lurie’s result by showing that when the  $A_\infty$ -algebra  $A$  is complete (pro-nilpotent), then we do not get just an  $\infty$ -category, but in fact an  $\infty$ -groupoid.

**Remark 6.3** In characteristic 0, we may associate to any  $A_\infty$ -algebra an  $L_\infty$ -algebra by symmetrization. Thus, we arrive naturally at the question of comparing the Maurer–Cartan simplicial set constructed in [15, 19] and the one presented in this paper. As mentioned in the introduction, the question of whether these are homotopy equivalent remains open. The main issue is caused by the fact that we use the normalized cochains, instead of the polynomial de Rham forms, on the standard  $n$ -simplex. If we would have used the polynomial de Rham forms in the definition above (and stayed in characteristic 0), then all results in this paper would go through and we would have an isomorphism of the Maurer–Cartan simplicial sets of an  $A_\infty$ -algebra and the corresponding  $L_\infty$ -algebra.

The problem is that one cannot use the polynomial de Rham forms in the characteristic non-zero case (since they do not satisfy the Poincaré lemma) and one cannot use the normalized cochains in the  $L_\infty$  case (because they are not commutative). The comparison of the  $L_\infty$  and  $A_\infty$  cases in characteristic 0 comes down to comparing the  $A_\infty$ -algebra arising by extending scalars by polynomial de Rham forms with the one arising by extending scalars by normalized cochains.

In [15], Getzler shows that the analogous simplicial set for an  $L_\infty$ -algebra is a Kan complex. We will proceed to show, that the Maurer–Cartan simplicial set associated to a complete shifted  $A_\infty$ -algebra is a Kan complex as well. In fact, Getzler’s methods also work in the  $A_\infty$ -case and we will therefore be brief in the proofs. To do this recall the maps

$$R_n^i = NQ_1 \circ H_n^i: A \otimes N^\bullet(\Delta^n) \longrightarrow A \otimes N^\bullet(\Delta^n)$$

and note the following corollary of Proposition 5.2.

**Corollary 6.4** *Suppose that  $A$  is a complete shifted  $A_\infty$ -algebra, then for all  $x \in MC_n(A, Q)$  we have the decomposition*

$$x = E_n^i x + R_n^i x - \sum_{k \geq 2} H_n^i NQ_k(x^k)$$

for all  $0 \leq i \leq n$ .

**Proof** This follows from Proposition 5.2, the MC equation for  $x$ , and the fact that  $H_n^i N Q_0(1) = 0$ . □

In the following, we will define  $mc_n^i(A, Q)$  as  $mc_n^i(A, Q) := \text{Im } R_n^i$ .

**Lemma 6.5** *Given any (curved) complete shifted  $A_\infty$ -algebra, the map*

$$MC_n(A, Q) \longrightarrow MC(A, Q) \times mc_n^i(A, Q),$$

*given by  $x \mapsto (E_n^i x, R_n^i x)$ , is a bijection for all  $0 \leq i \leq n$  and all  $n \geq 0$ . Here we implicitly equate  $MC(A, Q)$  and  $MC_0(A, Q)$ .*

**Proof** To show surjectivity, fix  $(e, r) \in MC(A, Q) \times mc_n^i(A, Q)$  and consider the sequence defined recursively by

$$\alpha_{k+1} = \alpha_0 - \sum_{l \geq 2} H_n^i N Q_l(\alpha_k^l),$$

where  $\alpha_0 = e + r$ . This sequence is a Cauchy sequence in  $A \otimes N^\bullet(\Delta^n)$  and so we may consider its limit  $\alpha = \lim_{k \rightarrow \infty} \alpha_k \in A \otimes N^\bullet(\Delta^n)$  by completeness of  $A$  and Proposition 3.8. By definition of the  $\alpha_k$ , we have

$$\alpha = \alpha_0 - \sum_{l \geq 2} H_n^i N Q_l(\alpha^l), \quad E_n^i \alpha = e \quad \text{and} \quad R_n^i \alpha = r.$$

This implies that

$$\begin{aligned} N Q_1 \alpha &= N Q_1 \alpha_0 - \sum_{l \geq 2} N Q_1 H_n^i N Q_l(\alpha^l) \\ &= \sum_{l \geq 2} H_n^i N Q_1 N Q_l(\alpha^l) - \sum_{l \geq 2} N Q_l(\alpha^l) - N Q_0(1). \end{aligned}$$

Here we used Lemma 5.3, Proposition 5.2, and the fact that  $e$  is a Maurer-Cartan element. It means, we find that

$$\mathcal{R}(\alpha) = H_n^i N Q_1 \mathcal{R}(\alpha) = - \sum_{l \geq 1} H_n^i N Q_{l+1}(\alpha^l \otimes \mathcal{R}(\alpha)) = 0,$$

where the second equality follows from the Bianchi identity (Lemma 4.6) and the third identity follows from the fact that  $\bigcap_k F^k A = \{0\}$ . Note that we have now proved surjectivity of  $x \mapsto (E_n^i x, R_n^i x)$ .

It is left to show injectivity. So suppose  $\alpha, \beta \in MC_n(A, Q)$  and  $(E_n^i \alpha, R_n^i \alpha) = (E_n^i \beta, R_n^i \beta)$ . Then by Corollary 6.4, we find that

$$\alpha - \beta = \sum_{l \geq 2} H_n^i N Q_l(\beta^l - \alpha^l) = \sum_{l \geq 2} \sum_{k=0}^{l-1} H_n^i N Q_l(\beta^k (\beta - \alpha) \alpha^{k-l-1}) = 0,$$

where the final equality follows again from the fact that  $\bigcap_k F^k A = \{0\}$ . Thus, we find that  $\alpha = \beta$  which shows injectivity.  $\square$

**Proposition 6.6** *Suppose that  $f: (A, Q) \rightarrow (B, P)$  is a surjective strict morphism between complete shifted  $A_\infty$ -algebras, then the induced map  $f: MC_\bullet(A, Q) \rightarrow MC_\bullet(B, P)$  is a Kan fibration.*

**Proof** For  $0 \leq i \leq n$ , let  $\beta \in sSet(\Lambda_i^n, MC_\bullet(A, Q))$  and let  $\gamma$  be an  $n$ -simplex in  $MC_\bullet(B, P)$  such that  $\partial_j \gamma = f(\partial_j \beta)$  for  $j \neq i$ . The map  $f \otimes \text{id}: A \otimes N^\bullet(\Delta^n) \rightarrow B \otimes N^\bullet(\Delta^n)$  is a surjective map of simplicial Abelian groups and therefore it is a Kan fibration. Thus there exists an element  $\rho \in (A \otimes N^\bullet(\Delta^n))^0$  such that  $\partial_j \rho = \partial_j \beta$  for all  $j \neq i$  and  $f \otimes \text{id}(\rho) = \gamma$ . Let  $\alpha \in MC_\bullet(A, Q)$  be the unique element with  $E_n^i \alpha = E_n^i \rho$  and  $R_n^i \alpha = R_n^i \rho$  given by Lemma 6.5. By Lemma 5.4, we find that  $E_n^i \partial_j \alpha = E_n^i \partial_j \beta$  and  $R_n^i \partial_j \alpha = R_n^i \partial_j \beta$  for  $i \neq j$ . Thus, by Lemma 6.5, we find that  $\partial_j \alpha = \partial_j \beta$  for  $j \neq i$  and  $\alpha$  fills the horn  $\beta$  in  $MC_\bullet(A, Q)$ . The facts that  $f$  is a strict morphism,  $f \otimes \text{id}(\rho) = \gamma$ ,  $E_n^i \alpha = E_n^i \rho$  and  $R_n^i \alpha = R_n^i \rho$  show that  $E_n^i f \otimes \text{id}(\alpha) = E_n^i \gamma$  and  $R_n^i f \otimes \text{id}(\alpha) = R_n^i \gamma$ . Thus by Lemma 6.5, we find that  $f(\alpha) = \gamma$  and the proposition follows, since  $\gamma$  and  $\beta$  were arbitrary.  $\square$

**Corollary 6.7** *Since any complete shifted  $A_\infty$ -algebra admits a strict map to the trivial shifted  $A_\infty$ -algebra 0, we find that  $MC_\bullet(A, Q)$  is always a Kan complex.*

**Proposition 6.8** *Suppose  $A$  and  $B$  are flat shifted  $A_\infty$ -algebras concentrated in degrees  $-1$  and below, suppose further that  $f: A \rightarrow B$  is a strict quasi-isomorphism between them, then the induced map on Maurer–Cartan simplicial sets is a homotopy equivalence.*

The proof of this proposition can be taken *mutatis mutandis* from [15].

**Remark 6.9** Finally, we should note that Dolgushev–Rogers expanded the theory developed by Getzler in [6]. In particular they proved a version of the proposition above that does not need the restriction imposed on degree and, moreover, it allows for  $\infty$ -morphisms instead of only strict morphism. The authors are of the opinion, that a similar result can be proved also in the case of  $A_\infty$ -algebras, however it would be a considerable addition to the present paper to prove it here. Thus we chose to postpone this to future work.

## 7 Application: the deformation theory of $\infty$ -morphisms of algebras over non-symmetric operads

In the last section of this paper, we apply the theory developed in this paper to the deformation theory of  $\infty$ -morphisms of algebras over non-symmetric operads. The main advantage of our theory is, that everything now also works over a field of arbitrary characteristic and not just over a field of characteristic 0. Most of this section is the non-symmetric version of the results of [26] and [27]. Since all the proofs are completely analogous to the proofs in those papers, we will omit most of them.

In the remainder of this paper, we assume that all operads and cooperads are non-symmetric. We further assume that all operads and cooperads are reduced, i.e.  $\mathcal{P}(0) = 0$  and  $\mathcal{P}(1) = \mathbb{K}$  (resp.  $\mathcal{C}(0) = 0$  and  $\mathcal{C}(1) = \mathbb{K}$ ). We further assume that all cooperads and coalgebras are conilpotent.

## 7.1 $\infty_\alpha$ -morphisms

In this section, we recall the definition of homotopy morphisms relative to an operadic twisting morphism  $\alpha : \mathcal{C} \rightarrow \mathcal{P}$ , between a cooperad  $\mathcal{C}$  and an operad  $\mathcal{P}$ . We call these relative homotopy morphisms  $\infty_\alpha$ -morphisms. The main motivation for  $\infty_\alpha$ -morphisms is, that when  $\alpha$  is Koszul they can be used to describe the homotopy category of  $\mathcal{P}$ -algebras (resp. nilpotent  $\mathcal{C}$ -coalgebras). For example, not every quasi-isomorphism of  $\mathcal{P}$ -algebras has a strict homotopy inverse, but it always has an  $\infty_\alpha$ -homotopy inverse.

To define these  $\infty_\alpha$ -morphisms, we need the bar and cobar construction relative to an operadic twisting morphism. We will not recall those here and refer the reader to Chapter 11 of [22]. The bar construction relative to a twisting morphism  $\alpha : \mathcal{C} \rightarrow \mathcal{P}$  is denoted by  $B_\alpha$  and the cobar construction relative to  $\alpha$  is denoted by  $\Omega_\alpha$ .

**Definition 7.1** Let  $\alpha : \mathcal{C} \rightarrow \mathcal{P}$  be an operadic twisting morphism from a cooperad  $\mathcal{C}$  to an operad  $\mathcal{P}$ . Let  $C$  and  $C'$  be  $\mathcal{C}$ -coalgebras and let  $A$  and  $A'$  be  $\mathcal{P}$ -algebras.

- (i) An  $\infty_\alpha$ -morphism,  $\Psi : C' \rightsquigarrow C$ , from  $C'$  to  $C$ , is defined as a  $\mathcal{P}$ -algebra map  $\Psi : \Omega_\alpha C' \rightarrow \Omega_\alpha C$ .
- (ii) An  $\infty_\alpha$ -morphism,  $\Phi : A \rightsquigarrow A'$ , from  $A$  to  $A'$ , is defined as a  $\mathcal{C}$ -coalgebra map  $\Phi : B_\alpha A \rightarrow B_\alpha A'$ .

On the set of  $\infty_\alpha$ -morphisms from a  $\mathcal{P}$ -algebra  $A$  (resp.  $\mathcal{C}$ -coalgebra  $C'$ ) to a  $\mathcal{P}$ -algebra  $A'$  (resp.  $\mathcal{C}$ -coalgebra  $C$ ), we can define a notion of homotopy equivalence. This is done by defining a model structure on the categories of  $\mathcal{P}$ -algebras and  $\mathcal{C}$ -coalgebras.

On the category of  $\mathcal{P}$ -algebras, we define a model structure in which the weak equivalences are given by quasi-isomorphisms, the fibrations by degree-wise surjective maps, and the cofibrations are the maps with the left lifting property with respect to acyclic fibrations. A proof that this is a model structure can be found as Theorem 1.7 in [18].

On the category of  $\mathcal{C}$ -coalgebras, we define a model structure in which the weak equivalences are created by the cobar construction, i.e. a map  $f : C' \rightarrow C$  is a weak equivalence if the induced map  $\Omega_\alpha f : \Omega_\alpha C' \rightarrow \Omega_\alpha C$  is a quasi-isomorphism of  $\mathcal{P}$ -algebras. The cofibrations are the degree-wise injective maps and fibrations are the maps with the right lifting property with respect to the acyclic cofibrations. This model structure was originally defined by Vallette in [31] for the case that the twisting morphism  $\alpha$  is Koszul. This was generalized to general twisting morphisms by Drummond-Cole and Hirsh in [9].

Since the categories of  $\mathcal{P}$ -algebras and  $\mathcal{C}$ -coalgebras are model categories, we have a notion of homotopy between the maps. To make this explicit we need a path object for  $\mathcal{P}$ -algebras and a cylinder object for  $\mathcal{C}$ -coalgebras. To define these objects, recall that  $N_\bullet(\Delta^1)$ , the normalized chains on the 1-simplex as a coassociative coalgebra,

is given by the following coalgebra (see Definition 3.1 of [31]),  $N_\bullet(\Delta^1)$  is given by  $\mathbb{K}a \oplus \mathbb{K}b \oplus \mathbb{K}c$ , with  $|a| = |b| = 0$  and  $|c| = 1$ . The coproduct is given by  $\Delta(a) = a \otimes a$ ,  $\Delta(b) = b \otimes b$  and  $\Delta(c) = a \otimes c + c \otimes b$ , the differential is given by  $d(a) = d(b) = 0$  and  $d(c) = b - a$ .

Similar to Sect. 3.2, we can equip the tensor product of a  $\mathcal{C}$ -coalgebra  $C$  and a coassociative coalgebra  $A$  with the structure of a  $\mathcal{C}$ -coalgebra. Denote by  $As^c$  the non-symmetric cooperad encoding coassociative coalgebras. The coproduct on  $C \otimes A$  is then given by

$$\begin{aligned} \Delta_{C \otimes A} &= C \otimes A \xrightarrow{\Delta_C \otimes \Delta_A} (C \circ C) \otimes (As^c \circ A) \\ &\xrightarrow{\cong} (C \otimes As^c) \circ (C \otimes A) \xrightarrow{\cong} C \otimes C \otimes A, \end{aligned}$$

where  $\circ$  denotes the non-symmetric composition product. In the last line, we use that we have a canonical isomorphism between  $C \otimes As^c$  and  $C$ . For more details about this isomorphism see the dual version of Theorem 7.6.

**Lemma 7.2** (i) *Let  $A$  be a  $\mathcal{P}$ -algebra, then  $A \otimes N^\bullet(\Delta^1)$  is a good path object for  $A$ .*  
(ii) *Let  $C$  be a  $\mathcal{C}$ -coalgebra, then  $C \otimes N_\bullet(\Delta^1)$  is a good cylinder object for  $C$ .*

Using the cylinder and path objects from Lemma 7.2, we can define the notion of homotopy between  $\infty_\alpha$ -morphisms.

**Definition 7.3** (i) Let  $\Psi, \Psi' : C' \rightsquigarrow C$  be two  $\infty_\alpha$ -morphisms of  $\mathcal{C}$ -coalgebras, then we call them homotopic if they are homotopic in the model category of  $\mathcal{P}$ -algebras. In other words, if there exists a morphism  $H : C' \otimes N_\bullet(\Delta^1) \rightarrow C$  such that the restriction to  $C' \otimes a$  is  $\Psi$  and the restriction of  $H$  to  $C' \otimes b$  is  $\Psi'$ .  
(ii) Let  $\Phi, \Phi' : A \rightarrow A'$  be two  $\infty_\alpha$ -morphisms of  $\mathcal{P}$ -algebras, then we call them homotopic if they are homotopic in the model category of  $\mathcal{C}$ -coalgebras, i.e. if there exists a map  $H : A \rightarrow A' \otimes N^\bullet(\Delta^1)$ , such the projection on the first vertex is  $\Phi$  and the projection on the second vertex is  $\Phi'$ .

### 7.2 $A_\infty$ -convolution algebras and the deformation theory of $\infty_\alpha$ -morphisms

Let  $\mathcal{C}$  be a cooperad,  $\mathcal{P}$  be an operad and  $\alpha : \mathcal{C} \rightarrow \mathcal{P}$  be an operadic twisting morphism. Recall from [2], that the convolution operad  $\text{Hom}(\mathcal{C}, \mathcal{P})$  is defined as follows. The arity  $n$  component,  $\text{Hom}(\mathcal{C}, \mathcal{P})(n)$ , of this operad is defined as  $\text{Hom}(\mathcal{C}(n), \mathcal{P}(n))$ , the space of linear maps from the arity  $n$  component of  $\mathcal{C}$  to the arity  $n$  component of  $\mathcal{P}$ . Let  $f \in \text{Hom}(\mathcal{C}, \mathcal{P})(n)$  and  $g_1, \dots, g_n \in \text{Hom}(\mathcal{C}, \mathcal{P})$ , with  $g_i \in \text{Hom}(\mathcal{C}, \mathcal{P})(m_i)$ , then we define the convolution map by the following sequence of maps

$$\begin{aligned} \mathcal{C} &\xrightarrow{\Delta_{\mathcal{C}}} (C \circ C) \twoheadrightarrow \mathcal{C}(n) \otimes \mathcal{C}(m_1) \otimes \dots \otimes \mathcal{C}(m_n) \xrightarrow{f \otimes g_1 \otimes \dots \otimes g_n} \\ &\mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \dots \otimes \mathcal{P}(m_n) \xrightarrow{\gamma_{\mathcal{P}}} \mathcal{P}(m_1 + \dots + m + n), \end{aligned}$$

where  $\Delta_{\mathcal{C}}$  is the decomposition map of  $\mathcal{C}$  and  $\gamma_{\mathcal{P}}$  is the composition map of  $\mathcal{P}$ . Recall from [22] Section 6.4, that a twisting morphism  $\alpha : \mathcal{C} \rightarrow \mathcal{P}$  is a Maurer–Cartan



element in the pre-Lie algebra associated to the convolution operad. The following theorem is the non-symmetric analog of Lemma 4.1 and Theorem 7.1 of [32], see also Section 4 of [26].

**Theorem 7.4** *Let  $\mathcal{C}$  be a cooperad and let  $\mathcal{P}$  be an operad. Then there exists a bijection*

$$\text{Hom}_{\mathcal{O}\mathcal{P}}(A_\infty, \text{Hom}(\mathcal{C}, \mathcal{P})) \cong \text{Tw}(\mathcal{C}, \mathcal{P}),$$

*between the set of operad morphisms from the  $A_\infty$ -operad to  $\text{Hom}(\mathcal{C}, \mathcal{P})$  and the set of operadic twisting morphisms from  $\mathcal{C}$  to  $\mathcal{P}$ .*

Using the fact that when we have a  $\mathcal{C}$ -coalgebra  $C$  and a  $\mathcal{P}$ -algebra  $A$ , then  $\text{Hom}(C, A)$  is a  $\text{Hom}(\mathcal{C}, \mathcal{P})$ -algebra (see Proposition 7.1 of [32]), we have the following corollary.

**Corollary 7.5** *Let  $C$  be a  $\mathcal{C}$ -coalgebra and let  $A$  be a  $\mathcal{P}$ -algebra, then  $\text{Hom}(C, \mathcal{P})$  with the products coming from the twisting morphism  $\alpha$  is a flat  $A_\infty$ -algebra. The differential  $Q_1$  applied to a map  $f$  is given by  $Q_1(f) = d_A \circ f + (-1)^{|f|} f \circ d_C$ . The products  $Q_n$ , for  $n \geq 2$ , are defined by  $Q_n : \text{Hom}(C, A)^{\otimes n} \rightarrow \text{Hom}(C, A)$  is*

$$Q_n(f_1, \dots, f_n)(x) := \gamma_A(\alpha \otimes f_1 \otimes \dots \otimes f_n) \Delta_C^n(x),$$

*where  $\Delta_C^n : C \rightarrow \mathcal{C}(n) \otimes C^{\otimes n}$  is the arity  $n$  part of the coproduct of  $C$  and  $\gamma_A : \mathcal{P} \circ A \rightarrow A$  is the product of  $A$ . We denote  $\text{Hom}(C, A)$  with this  $A_\infty$ -structure by  $\text{Hom}^\alpha(C, A)$ .*

This  $A_\infty$ -structure has the additional property that, the Maurer-Cartan elements in the  $A_\infty$ -algebra  $\text{Hom}^\alpha(C, A)$  correspond to the twisting morphisms relative to  $\alpha$ . This is the non-symmetric analog of Theorem 2.4 of [27].

**Theorem 7.6** *Let  $\alpha : \mathcal{C} \rightarrow \mathcal{P}$  be an operadic twisting morphism and let  $C$  be a  $\mathcal{C}$ -coalgebra and  $A$  a  $\mathcal{P}$ -algebra. Then the following statements hold.*

(i) *We have bijections between the following sets*

$$\text{Hom}_{\mathcal{C}\text{-coalgebras}}(C, B_\alpha A) \cong MC(\text{Hom}^\alpha(C, A)) \cong \text{Hom}_{\mathcal{P}\text{-algebras}}(\Omega_\alpha C, A).$$

- (ii) *Two  $\mathcal{C}$ -coalgebra morphisms  $f, g : C \rightarrow B_\alpha A$  are homotopic in the category of  $\mathcal{C}$ -coalgebras, if and only if the corresponding Maurer-Cartan elements are gauge equivalent.*
- (iii) *Two  $\mathcal{P}$ -algebra morphisms  $f, g : \Omega_\alpha C \rightarrow A$  are homotopic in the category of  $\mathcal{P}$ -algebras if and only if the corresponding Maurer-Cartan elements are gauge equivalent.*

Note that, compared to the proof of Theorem 2.4 of [27], the proof in the non-symmetric case is significantly easier because  $N^\bullet(\Delta^1)$  and  $N_\bullet(\Delta^1)$  are both finite dimensional.

Using Theorem 7.6, we can now define the deformation complex of  $\infty_\alpha$ -morphisms for non-symmetric operads. This problem was initially stated by M. Kontsevich in his 2017 Séminaire Bourbaki, and was answered for symmetric operads over a field of characteristic 0 by D. Robert-Nicoud and the second author in Definition 2.7 of [27]. Using the theory developed in this paper, we can also answer this question for (co)algebras over a field of arbitrary characteristic.

**Definition 7.7** Let  $\alpha : \mathcal{C} \rightarrow \mathcal{P}$  be an operadic twisting morphism.

- (i) Let  $A$  and  $A'$  be two  $\mathcal{P}$ -algebras, the deformation complex of  $\infty_\alpha$ -morphisms from  $A$  to  $A'$  is defined as the  $A_\infty$ -algebra  $\text{Hom}^\alpha(B_\alpha A, A')$ .
- (ii) Let  $C'$  and  $C$  be two  $\mathcal{C}$ -coalgebras, the deformation complex of  $\infty_\alpha$ -morphisms from  $C'$  to  $C$  is defined as the  $A_\infty$ -algebra  $\text{Hom}^\alpha(C', \Omega_\alpha C)$ .

Because of Theorem 7.6, the Maurer–Cartan elements in this deformation  $A_\infty$ -algebra correspond indeed to the  $\infty_\alpha$ -morphisms between  $A$  and  $A'$  (resp  $C'$  and  $C$ ), and the notion of gauge equivalence corresponds to the relation of homotopy equivalence. This is therefore the correct deformation complex. For more details see the discussion after Definition 2.7 of [27].

As stated earlier, the main advantage of working with  $A_\infty$ -algebras is that we no longer need any restrictions on the ground field we are working over.

## 8 Comparison with other approaches, possible applications and some open questions

We finish the paper by comparing our constructions with some other approaches, sketching some possible applications, and stating some open questions.

### 8.1 Comparison with other approaches

Most of the constructions in this paper are analogous to the  $L_\infty$ -case. For example the way that we treat our description of the  $A_\infty$ -twisting procedure is based on Dolgushev's thesis (see [4]) and the construction of the Maurer–Cartan simplicial set is based on [15]. It seems plausible that many other results from the theory of  $L_\infty$ -algebras can also be generalized to the  $A_\infty$ -case (for a few possible generalizations see the next section about future work). One of the most important advantages of the work in this paper, is that we do not need the characteristic 0 assumption on the field we are working over.

There are however, also a few papers which explicitly deal with versions of some of the constructions we use for  $A_\infty$ -algebras. The Maurer–Cartan equation for curved  $A_\infty$ -algebras and the twisting for curved  $A_\infty$ -algebras have appeared before in for example [14]. There are several other papers that also explicitly deal with curved  $A_\infty$ -algebras, see for example the paper by Hamilton and Lazarev [17] and the paper by Nicolás [24]. Another area in which twisting of  $A_\infty$ -algebras is defined is in string field theory (see for example [11]). Another method to define the twist was defined by Dotsenko, Shadrin and Vallette in [8] using the pre-Lie deformation theory and

the gauge group from [7]. This work has the disadvantage that their pre-Lie approach uses formulas that involve factorials and therefore not immediately work over more general fields. Although these problems can be solved for  $A_\infty$ -algebras, the method using the shuffle product, which we have defined in this paper, has the advantage that everything works immediately without any modifications.

The Maurer-Cartan simplicial set and equivalence between Maurer-Cartan elements for non-curved  $A_\infty$ -algebras has also appeared before. In [3], Chuang, Holstein, and Lazarev define a notion of a Maurer-Cartan element in an associative algebra and the relation of strong homotopy between these Maurer-Cartan elements. It seems that for associative algebras our definitions coincide. In Section 8 of [1], Behrend and Getzler also define a Maurer-Cartan simplicial set of a finite dimensional associative algebra, we improve their results for (pro)nilpotent algebras in several ways. First of all we extend it to  $A_\infty$ -algebras and allow them to be curved. The second improvement is, that by working with filtered  $A_\infty$ -algebras we do not need their restrictions on the dimensions. Note however, that it also implies we only consider pronilpotent  $A_\infty$ -algebras, while they also consider highly non-nilpotent algebras such as square matrices. The last improvement, is that we give an explicit proof that the simplicial set obtained is a Kan complex, which was not done in [1].

## 8.2 Possible applications, future work and open questions

In this section, we sketch some further applications and open questions associated to this paper.

Given an  $A_\infty$ -algebra  $A$ , we can form two different simplicial sets. One by the construction described in this paper and one by taking the Maurer-Cartan simplicial set of the  $L_\infty$ -algebra corresponding to  $A$  with all the higher commutators as  $L_\infty$ -operations. Although these simplicial sets are clearly not isomorphic, we do expect them to be homotopy equivalent. This question will be the topic of future work.

A possible further application of the theory developed in this paper is the non-symmetric version of the paper [5]. In this paper Dolgushev, Hoffnung and Rogers, show that over a field of characteristic 0 the category of homotopy algebras over a symmetric operad with  $\infty$ -morphisms, is enriched over (filtered)  $L_\infty$ -algebras. By using the Maurer-Cartan simplicial set of an  $L_\infty$ -algebra, this implies that the category of homotopy algebras is also enriched over simplicial sets. It seems highly likely, that the theory developed in this paper can be used to answer the question: What do homotopy algebras over a non-symmetric operad form? The main advantage of the theory developed in this paper, is that it does not require us to work over a field of characteristic 0 and would answer this question in a much larger generality than in [5]. For the sake of brevity, we do not work out the details of this construction and leave it as a topic for future work.

The methods developed in this paper do unfortunately not allow us to answer the question: What do homotopy algebras over a symmetric operad over a field of characteristic  $p > 0$  form? This question still remains open and will be a topic of future research.

Another question that is not treated in full detail in this paper, is the behavior of the Maurer–Cartan simplicial set with respect to quasi-isomorphisms. It seems reasonable to expect that an analog of the Dolgushev–Rogers Theorem (see Theorem 1.1 of [6]) also holds for the Maurer–Cartan simplicial set of an  $A_\infty$ -algebra. This theorem would state that filtered  $\infty$ -quasi-isomorphisms between filtered  $A_\infty$ -algebras would induce homotopy equivalences between the corresponding Maurer–Cartan simplicial sets. Since there are several technical differences between our setting and their setting, this generalization is not completely straightforward and we will make this a topic of future work.

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## Appendix A: Completed Coalgebras

In this appendix, we will fix what we mean by a coalgebra structure on the completion  $\widehat{T^c(A)}$  of the coaugmented conilpotent coassociative cofree coalgebra cogenerated by the complete filtered  $\mathbb{Z}$ -graded vector space  $A$ . In fact, it comes from the following general notion. We may consider the category of filtered  $\mathbb{Z}$ -graded vector spaces  $V$  that are complete for the filtration topology. Note that we subsume here the property of the filtration that  $\bigcap_{i \geq 0} F^i V = \{0\}$ . As is described in Section 7.3 of [12], we may equip this category with a monoidal structure by considering the completed tensor product  $\widehat{\otimes}$ , namely we equip the tensor product of two complete filtered vector spaces with the induced filtration and then complete for the corresponding filtration topology. A coalgebra in the completed sense thus means a coalgebra object in this category.

Let us be explicit in the case of the tensor coalgebra since this is the only case that occurs in the present paper. First of all, we observe that there exists a canonical isomorphism of (filtered)  $\mathbb{Z}$ -graded vector spaces

$$\widehat{T^c(A)} \widehat{\otimes} \widehat{T^c(A)} \cong \overline{T^c(A) \otimes T^c(A)},$$

where on the right hand side the overline indicates completion in the induced filtration topology on  $T^c(A) \otimes T^c(A)$ . This follows from the fact that the tensor product of two Cauchy sequences is a Cauchy sequence and the inclusion  $T^c(A) \otimes T^c(A) \hookrightarrow \widehat{T^c(A)} \widehat{\otimes} \widehat{T^c(A)}$  of filtered vector spaces. Now the map

$$\Delta : T^c(A) \longrightarrow \overline{T^c(A) \otimes T^c(A)},$$

yields the unique extension

$$\hat{\Delta}: \widehat{T^c(A)} \longrightarrow \overline{T^c(A) \otimes T^c(A)}.$$

Similarly to the above, we find that

$$\widehat{T^c(A)} \hat{\otimes} \widehat{T^c(A)} \hat{\otimes} \widehat{T^c(A)} \cong \overline{T^c(A) \otimes T^c(A) \otimes T^c(A)}.$$

Thus, it is easily seen that  $(\hat{\Delta} \otimes \text{id})\hat{\Delta}$  and  $(\text{id} \otimes \hat{\Delta}) \circ \hat{\Delta}$  are both extensions of

$$\Delta^{(3)}: T^c(A) \rightarrow \overline{T^c(A) \otimes T^c(A) \otimes T^c(A)},$$

where  $(\Delta \otimes \text{id}) \circ \Delta = \Delta^{(3)} = (\text{id} \otimes \Delta) \circ \Delta$  denotes the iterated coproduct. So, since such an extension is unique, we see indeed that  $\widehat{T^c(A)}$  forms a coassociative coalgebra in the completed sense.

The rest of the structures needed in this article follow similarly. Given an  $A_\infty$ -algebra structure  $Q$ , we can again extend it uniquely to a map

$$\hat{Q}: \widehat{T^c(A)} \longrightarrow \widehat{T^c(A)}$$

and, similar to the coassociativity condition on  $\hat{\Delta}$ , it can be shown that  $\hat{Q}$  defines a coderivation in the completed sense, i.e. where we replace all tensor products in the commuting diagram corresponding to the coderivation property by completed tensor products. Furthermore, we find that  $\hat{Q}^2 = 0$ . Finally, we should consider the shuffle product, which as the comultiplication and coderivations gives rise to the unique extended map

$$\hat{\mu}_{sh}: \overline{T^c(A) \otimes T^c(A)} \longrightarrow \widehat{T^c(A)}.$$

Again, this map satisfies associativity and together with  $\hat{\Delta}$  satisfies the Hopf compatibility condition in the completed sense.

For notational simplicity, we have chosen to drop the hats on  $\hat{\Delta}$ ,  $\hat{Q}$ , etc in the main body of this article.

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