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An Ideal-Theoretic Criterion for Localization of an Unknown Number of Sources

Matthew W. Morency*, Sergiy A. Vorobyov† and Geert Leus*
*Faculty of Electrical Engineering, Delft University of Technology, Delft, The Netherlands
† Department of Signal Processing and Acoustics, Aalto University, Espoo, Finland

***Abstract*—Source localization is among the most fundamental problems in statistical signal processing. Methods which rely on the orthogonality of the signal and noise subspaces, such as Pisarenko's method, MUSIC, and root-MUSIC are some of the most widely used algorithms to solve this problem. As a common feature, these methods require both a-priori knowledge of the number of sources, and an estimate of the noise subspace. Both requirements are complicating factors to the practical implementation of the algorithms, and sources of potentially severe error. In this paper, we propose a new localization criterion based on the algebraic structure of the noise subspace. An algorithm is proposed which adaptively learns the number of sources and estimates their locations. Simulation results show significant improvement over root-MUSIC, even when the correct number of sources is provided to the root-MUSIC algorithm.**

I. Introduction

The problem of source localization in a noisy environment is one of the oldest and most well-studied problems in array processing [1]. Among the algorithms devised to solve this problem, subspace-based algorithms such as Pisarenko's method, MUSIC, and root-MUSIC have become ubiquitous [2]– [6]. Subspace based methods require two steps. First, the signal and noise subspaces must be estimated. Second, given the estimates of the signal and noise subspaces, the source locations are then derived with respect to some criterion, e.g. minimization of a cost function, or peak-finding. Most subspace based methods differ only in how to approach the second step. For example, MUSIC and root-MUSIC differ only in the criterion used to derive the target locations. However, it has been argued that the first step - estimating the signal and noise subspace - is far more crucial [4]. As such, substantial research efforts have been invested into providing robust estimates of signal and noise subspaces in a variety of challenging scenarios [4]– [6]. Estimation of the signal and noise subspaces is typically done through the eigen-decomposition of the autocorrelation matrix of a set of observations. Subspace estimation is thus reduced to a selection problem [4]– [6]. However, as vector spaces, the signal and noise subspaces have a dimension which is either assumed to be known a-priori, or, perhaps more practically, must first be estimated.

While the estimate of the signal and noise subspaces is taken from the eigenvectors of the observation autocorrelation matrix, the dimension of the signal subspace is typically inferred from the distribution of the eigenvalues [6]– [7]. For example, the dimension of the signal subspace can be taken to be the number of "dominant" eigenvalues. In recent work, the estimation of the number of sources has been considered as a multiple hypothesis test on the equality of eigenvalues [7]. In order to do this, multiple instances of the observation autocorrelation matrix must be generated, and multiple eigen-decompositions performed. The assumption which underlies both of these methods is that information about the number of sources is contained in the eigenvalues, while the eigenvectors themselves are ignored. In this paper we argue that, under certain assumptions, the algebraic structure of the eigenvectors themselves contains a great deal of information about the number of sources, as well as their locations. Specifically, the noise eigenvectors are argued to lie in a univariate polynomial ideal generated by a single element in the univariate polynomial ring. The degree of the generator is thus the number of targets, and the roots of the generator are their locations.

Throughout this paper bold upper case letters are matrices, bold lower-case letters are vectors, $\mathbb{K}$ is a base field, upper case letters are constants, lower case letters are variables, and $(\cdot)^H$ is the Hermitian transpose.

II. Preliminaries

A. Algebraic Geometry

Algebraic geometry is concerned with the relations between sets of polynomials called ideals (algebraic objects) and their associated zero loci called varieties (geometric objects) [8].

Definition II.1. *An ideal $\mathcal{I}$ in a commutative ring $R(+,\cdot)$ is a subgroup of $R(+,\cdot)$ with the property that $\forall a \in \mathcal{I},\ r \in R(+,\cdot), a \cdot r \in \mathcal{I}$.*

As an example, take the commutative ring of univariate polynomials over $\mathbb{C}$, written $\mathbb{C}[x]$, and as a subset, take the set of all polynomials with a common root at $\alpha \in \mathbb{C}$ [9].

Definition II.2. *An algebraic variety is a subset of $\mathbb{K}^N$ such that $V(\mathcal{I}) \triangleq \{\mathbf{p} \in \mathbb{K}^N | f(\mathbf{p}) = 0,\ \forall f \in \mathcal{I}\}$.*

Thus, an algebraic variety is described by the polynomials vanishing on it. One can similarly describe an ideal by the set on which every member vanishes [8].

Definition II.3. *A polynomial ideal given a variety $V \in \mathbb{K}^N$ is a set of polynomials with the property that $\mathcal{I}(V) = \{f \in \mathbb{K}[x_1, \cdots, x_N] | f(\mathbf{p}) = 0,\ \forall \mathbf{p} \in V\}$.*

Ideals are generated by elements contained within them, much the same way that a vector space is spanned by linearly

independent vectors contained in the space. An ideal which is generated by a single element is a principal ideal [8].

Definition II.4. *A principal polynomial ideal generated by* f *is a set* $\langle f \rangle \triangleq \{h \in \mathbb{K}[x_1, \cdots, x_N] | h = f \cdot g,\ g \in \mathbb{K}[x_1, \cdots, x_N]\}$.

All univariate polynomial ideals are principal.

B. Data Model

Consider L independent Gaussian sources in the far-field impinging upon a uniform linear array of N antenna elements with inter-element spacing $\lambda_c/2$. The signal observed at the antenna array at time t can be written as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{A} \triangleq [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_l)]$, $[\mathbf{a}(\theta)]_n \triangleq \alpha^{n-1}$, $\alpha \triangleq e^{j\pi sin(\theta_l)}$, $\mathbf{s}(t) \sim \mathcal{N}(\mathbf{0}, \sigma_s^2 \mathbf{I}_L)$ are the source signals, and $\mathbf{n}(t) \sim \mathcal{N}(\mathbf{0}, \sigma_n^2 \mathbf{I}_N)$ is the sensor noise. Collecting T observations $\mathbf{x}(t)$, the sample covariance matrix (SCM) is written as

$$\mathbf{R}_{xx} \triangleq \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}^H(t) \tag{2}$$

$$\approx \mathbf{A}\mathbf{S}\mathbf{A}^H + \sigma_n^2 \mathbf{I}_N. \tag{3}$$

Under the aforementioned assumptions, $\mathbf{R}_{xx}$ has full rank almost surely if $T \geq N$. Since $\mathbf{R}_{xx}$ is Hermitian by definition, it has a full set of real eigenvalues, and an eigenbasis, allowing us to write

$$\mathbf{R}_{xx} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H \tag{4}$$

$$= \mathbf{Q}_s\mathbf{\Lambda}_s\mathbf{Q}_s^H + \mathbf{Q}_n\mathbf{\Lambda}_n\mathbf{Q}_n^H \tag{5}$$

where $\mathbf{Q}_s$,$\mathbf{\Lambda}_s$, and $\mathbf{Q}_n$,$\mathbf{\Lambda}_n$ are the matrices of eigenvectors and eigenvalues for the signal and noise subspaces respectively.

C. Subspace Based Methods

Based on the properties of (5), $\mathbf{Q}_s \perp \mathbf{Q}_n$, and as either $T \to \infty$ or $\sigma_n^2 \to 0$ $\mathcal{C}(\mathbf{Q}_s) = \mathcal{C}(\mathbf{A})$. This implies that $\mathbf{Q}_n^H \mathbf{a}(\theta_l) = \mathbf{0}, \forall l$ where θ_l are the source directions of arrival (DOA). It is this property which is exploited by subspace based methods such as Pisarenko's method, MUSIC, and root-Music. As was mentioned earlier, MUSIC and root-MUSIC differ only in how the DOA are retrieved from the subspace estimates. Given an estimate of the noise subspace $\mathbf{Q}_n$, both algorithms form the polynomial $J(\theta) = \mathbf{a}^H(\theta)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{a}(\theta)$. The MUSIC algorithm provides its estimates as

$$\hat{\theta}_{\text{MUSIC}} = \underset{\theta}{\text{argmax}} \frac{1}{J(\theta)} \tag{6}$$

The root-MUSIC algorithm first treats each column of $\mathbf{Q}_n$ as the coefficients of a polynomial in $\mathbb{C}[x]$, and factors each polynomial using a root-finding algorithm. Then, the roots which are closest to the unit circle are taken to be the roots which correspond to the targets.

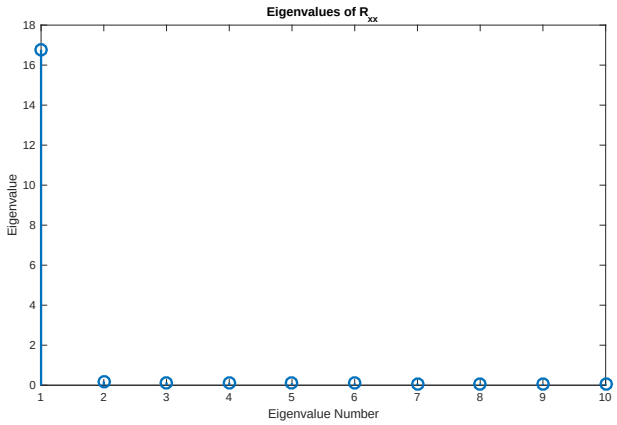

Fig. 1. Eigenvalues for one instance of R_{xx}. ULA of $N = 10$ elements, $T = 10$ snapshots, 10dB SNR, two sources located at 31^o and 32^o.

$$\hat{z}_{\text{root-MUSIC}} = \underset{z \in \mathbb{C}}{\text{argmin}}\ \mathbf{a}^H(z)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{a}(z) \tag{7}$$

Clearly, knowledge of the number of sources is required by both algorithms. As was discussed earlier, this knowledge is typically based on the distribution of the eigenvalues of $\mathbf{R}_{xx}$. However, this strategy quickly breaks down in non-ideal scenarios. Fig. 1 shows the eigenvalue distribution of a single instance of $\mathbf{R}_{xx}$ corresponding to a ULA of $N = 10$ elements, at 10 dB SNR, with two sources impinging from 31^o and 32^o. The difference between the second eigenvalue and the last eigenvalue is 0.1412, while the first eigenvalue is roughly 100 times larger than the second eigenvalue. It would be difficult to conclude that there is more than one target in this scenario on the basis of the eigenvalues alone.

However, under the data generation assumptions in subsection II-B, or more generally, the eigenvectors themselves have an algebraic structure which allows for the identification and separation of closely located sources.

III. Algebraic Structure of the Noise Subspace

Consider the root-MUSIC polynomial $\mathbf{a}^H(z)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{a}(z)$. The matrix $\mathbf{X} = \mathbf{Q}_n\mathbf{Q}_n^H$ is positive semidefinite (PSD) and thus $\mathbf{a}^H(z)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{a}(z)$ is a globally non-negative function over $\mathbb{C}$. Hilbert proved that a univariate polynomial is globally non-negative if and only if it is a sum of squares (SOS) polynomial [10]. Introducing a new variable $y_k(\theta) \triangleq \mathbf{a}(\theta)\mathbf{q}_k$, where $\mathbf{q}_k$ is the the k-th column of $\mathbf{Q}_n$, it becomes clear that

$$\mathbf{a}^H(\theta)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{a}(\theta) = \sum_{k=1}^{N-L} |y_k(\theta)|^2 \tag{8}$$

from which one can observe the SOS nature of the root-MUSIC polynomial. From (6) and (7) it's clear that root-MUSIC and MUSIC are searching for the points where (8) is 0. Since the matrix $\mathbf{X}$ has rank $N - L$ by definition, the quadratic form in (8) is a sum of $N - L$ globally non-negative functions, (8) can be 0 if and only if $y_k(\theta)$ are 0 for all k.

Clearly, $y_k(\theta) = \sum_{i=1}^{N-1} q_{k,i}\alpha^{*i}$, and so $y_k(\theta)$ can be zero if and only if the polynomial whose coefficients are those of $\mathbf{q}_k$ has a root at α. That is, $J(\theta) = 0$ if and only if the columns of $\mathbf{Q}_n$ lie in a univariate polynomial ideal. Under the assumptions mentioned in the previous section, due to the fact that $\mathbf{R}_{xx}$ has an eigenbasis, $\mathbf{Q}_n \perp \mathbf{Q}_s$ which implies that $\mathbf{a}(\theta_l)\mathbf{Q}_n = 0, \forall l$. Specifically, the polynomial ideal which describes the noise subspace is a function of the target locations, parametrized by α_l, and thus the generator of the polynomial ideal is

$$Q(x) = \prod_{l=1}^{L}(x - \alpha_l). \tag{9}$$

This yields an alternative subspace selection criterion: given N polynomials (the N eigenvectors of $\mathbf{R}_{xx}$), select a subset of polynomials which are closest to being in a univariate ideal.

IV. Proposed Algorithm

In the infinite sample case, the model of $\mathbf{R}_{xx}$ would be exact. That is, $\mathbf{R}_{xx} = \mathbf{A}\mathbf{S}\mathbf{A}^H + \sigma_n^2\mathbf{I}$ exactly, and thus the noise subspace would lie in a polynomial ideal. Thus one could blindly estimate the noise subspace by computing the greatest common divisor (GCD) between two randomly selected columns of $\mathbf{Q}$. If two noise eigenvectors were selected, then the GCD would be a polynomial, and its factors, α_l, would be the complex generators corresponding to the source locations. Any other selection (signal-signal, signal-noise) would result in a constant GCD, which would require another iteration of the algorithm, with new choices of columns of $\mathbf{Q}$.

However, one only has access to finite samples and thus the noise eigenvectors are perturbed from being in a univariate ideal, and thus lie in an ϵ-ideal [12].

The ϵ-ideal structure of $\mathbf{Q}_n$ implies that each noise eigenvector has, as a factor, the perturbed generator

$$Q(x) = \prod_{l=1}^{L}(x - \alpha_l + \epsilon_l) \tag{10}$$

where ϵ_l are small, random perturbations [2]. Then, the problem of finding an approximate GCD can be seen as finding $\hat{\alpha}_l$ such that

$$\hat{Q}(x) = \prod_{l=1}^{L}(x - \hat{\alpha}_l) \tag{11}$$

is as "close" to (10) as possible, in some appropriate sense. What we observe are the coefficients of (10) as some of the columns of $\mathbf{Q}$. However, it does not necessarily follow that small perturbations in coefficients correspond to small perturbations in roots. It is indeed possible for small perturbations in roots to result in large perturbations in the coefficients of a polynomial, e.g. Wilkinson's polynomial. Furthermore, what we're interested in most is $\hat{\alpha}_l$ which best explain the target locations. As such, we opt to deal with the roots of the eigenvectors as polynomials to decide "closeness" to (10), as opposed to the coefficients themselves.

Towards this end, we adopt the approach of root-clustering. There is a wide variety of clustering algorithms within the literature of Machine Learning [13]. Because we are looking for tight clusters, we adopt the hierarchical approach of agglomerative clustering. Agglomerative clustering starts by considering each data point as its own cluster. At each step, it calculates for each cluster the closest neighbor data point. The cluster which has the closest neighbor is then merged with this neighbor to form a new cluster, while the distance between the original cluster and the new member is stored. This distance is called a "dissimilarity," and can, in general, be any pseudonorm. This process continues until all the data points are in one large cluster, with all clusters are ranked by their dissimilarity in a tree-structure known as a "Dendrogram." Fig. 2 shows a dendrogram for a system consisting of a ULA with $N = 10$ elements, and $L = 2$ sources impinging from 30^o and -40^o. Two clusters of 8 roots each with a dissimilarity close to 0 are observed.

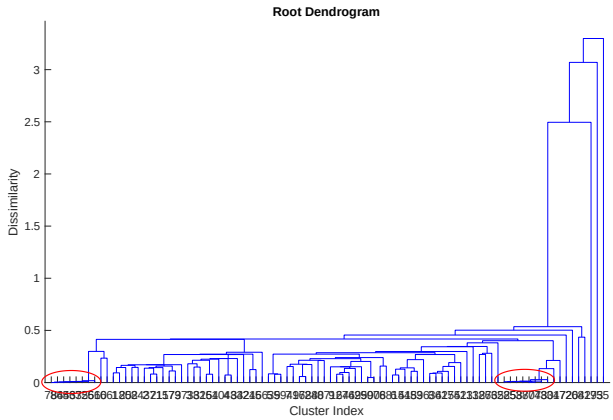

Fig. 2. Root dendrogram corresponding to ULA with $N = 10$ elements, $L = 2$ targets at 30^o and -40^o. Target clusters are highlighted with red ellipses.

Assuming there are L distinct targets, then there must be $N - L$ eigenvectors of the form (10). Thus, the roots corresponding to the source locations form L tight clusters of $N - L$ roots each. Thus, not only do the roots contain information about the source location, but the number of roots in each cluster contains information about the number of sources. It is this information which is used to separate closely located sources. Specifically, if two sources are closely located, they will be grouped into the same cluster. According to (10), a valid cluster cannot have more than $N-1$ roots, since the signal must occupy a subspace with at least dimension 1 to which the resulting $N - 1$ dimensional subspace must be orthogonal. Thus, tight clusters with more than $N - 1$ roots are deemed to correspond to multiple closely located targets. An algorithm which exploits this criterion to simultaneously learn the number of sources, and estimate their locations is given in Algorithm 1 given the roots of every eigenvector of $\mathbf{Q}$ stored in a vector $\mathbf{r}$. The dissimilarity within a cluster, and the cardinality of a cluster are given by $d(\cdot)$ and $|\cdot|$ respectively.

The algorithm searches for tight clusters, that are below a (small) dissimilarity level δ. Then, among these, it searches for clusters with 3 or more roots to filter roots which are

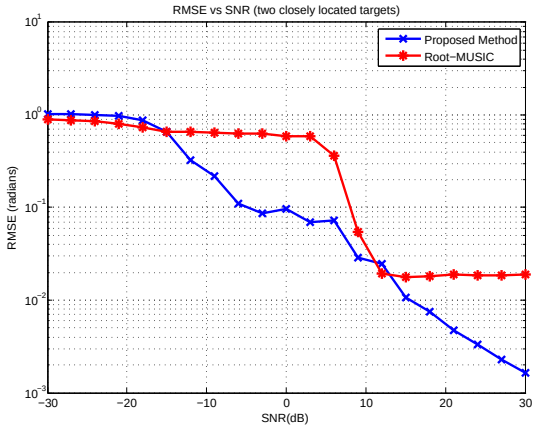

Fig. 3. Performance comparison between Algorithm 1 and root-MUSIC.

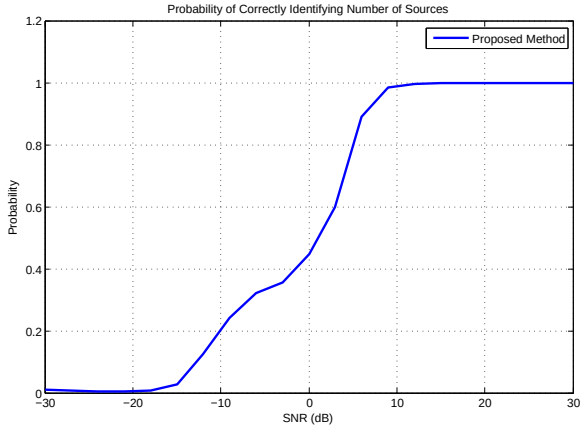

Fig. 4. Probability of correctly learning the number of sources from the number of roots in the detected clusters.

procedure ROOT CLUSTERING($\mathbf{r}$)
  Agglomerative clustering on $\mathbf{r}$ into clusters $d(c_i) < \delta$
  while $\exists\ |c_i| > 2$ **do**
    if $|c_i| > N - 1$ **then**
      Separate c_i using linear classifier
      Return median coordinates of resulting clusters
    else
      Return median coordinates of c_i
    end if
  end while
end procedure

randomly clustered together. The if condition checks whether the number of roots in the detected cluster is consistent with (10), and splits them using a linear classifier if the number of roots is inconsistent. The estimate of the target location α_l is then the Karcher mean of the roots in each cluster, as the solution set is the unit circle which is a Riemannian manifold.

V. Simulation Results

To test the performance of Algorithm 1, we consider the following scenario: two Gaussian uncorrelated, equal power sources are impinging upon a ULA of 10 elements from $\theta_1 = 31^o$ and $\theta_1 = 32.5^o$ respectively. The SCM is formed from $T = 100$ snapshots. Fig. 1 presents the eigenvalue distribution of a single instance of the SCM in this scenario. We compare the performance of Algorithm 1 with the root-MUSIC algorithm provided with the correct number of sources. Algorithm 1 learns the number of sources by detecting a cluster of roots larger than 9. It then splits the detected cluster using a linear classifier defined by the line between the origin and the mean coordinates of the cluster. Figure 3 demonstrates a significant performance gain compared to the root-MUSIC algorithm in both the low and high SNR regions. In the low SNR region, the sources have merged, and thus Algorithm 1 only detects the single composite target specified by the model. In the high SNR region, Algorithm 1 correctly separates the sources and produces an estimate based on all of the available information, as opposed to only a single root in the case of root-MUSIC. At 10dB SNR, the proposed algorithm correctly separates the targets 99% of the time.

VI. Conclusions

A new criterion for the simultaneous detection and localization of an unknown number of sources has been introduced, and a new algorithm which leverages this criterion was proposed. The proposed algorithm leverages underlying algebraic structure in the array processing model. The proposed algorithm has been compared to root-MUSIC, where the correct number of sources was provided to root-MUSIC. Significant performance benefits were observed in the case of closely located targets, in both high and low SNR conditions.

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