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TRANSVERSE AND LONGITUDINAL VIBRATIONS IN AXIALLY MOVING STRINGS

TRANSVERSE AND LONGITUDINAL VIBRATIONS IN AXIALLY MOVING STRINGS

Dissertation

for the purpose of obtaining the degree of doctor
at Delft University of Technology
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Chair of the Board for Doctorates
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by

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Keywords: Axially moving string, Resonance, Boundary excitation, Time-varying length, Singular perturbation, Averaging, Backstepping, Vibration control

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To my parents

CONTENTS

Summary

Varying-length cable systems are widely applied in a vast class of engineering problems which arise in industrial, civil, aerospatial, mechanical, and automotive applications. Due to external excitations, large oscillations can occur when cables are lifted up or down. This phenomenon is caused by resonance. In general, resonance is harmful, and can cause significient deformations and dynamic stresses in machinery and structures, and even can lead to accidents. Therefore, this doctoral dissertation is devoted to the study of transverse and longitudinal resonance phenomena and output feedback stabilization of varying-length cables.

Firstly, we are motivated by resonance phenomena occurring in a transversally vibrating cable, where one end of the cable is fixed, and the other one is attached to a spring for which the stiffness properties change in time (due to fatigue, temperature change, and so on). This problem may serve as a simplified model describing transverse or longitudinal vibrations as well as resonances in axially moving cables for which the length changes in time. By setting the frequency of the external force, and the time-dependent boundary coefficient in the Robin boundary condition, different kinds of resonances can be obtained. The first aim is to give the exact solution by using the method of d'Alembert and to determine wave reflections, in which we can divide the time domain into finite intervals. Then the resonance results can be analyzed by the constructed solution. The other goal is to give explicit approximations of the solution on long timescales by using the method of separation of variables, the method of d'Alembert, the averaging method, and multiple timescales perturbation methods. For problems with time-dependent coefficients in the Robin boundary condition, the analytical resonance results are all in agreement with those obtained by using a numerical method.

Next, we extend our analytical and numerical results to a real physical model of a flexible hoisting system, in which external disturbances exerted on the boundary can induce large vibrations. The dynamics is described by a wave equation on a slow time-varying spatial domain with a small harmonic boundary excitation at one end of the cable, and a moving mass at the other end. Due to the slow variation of the cable length, a singular perturbation problem arises. By using an averaging method, and an interior layer analysis, many resonance manifolds are detected. Further, a three time-scales perturbation method is used to construct formal asymptotic approximations of the solutions. It turns out that for a given boundary disturbance frequency, many oscillation modes jump up from order ε amplitudes to order $\sqrt{\varepsilon}$ amplitudes, where ε is a small parameter with $0 < \varepsilon << 1$. Moreover, numerical simulations are presented to verify the obtained analytical results.

Further, due to external excitation and loading conditions, the nonlinear interactions between transverse and longitudinal string motions may influence the vibration behavior in two directions when the hoisting conveyance is moving up or down. Therefore, we study both transverse and longitudinal oscillations and resonances in a hoisting system induced by boundary disturbances. The dynamics can be described by an initial-boundary value problem for a coupled system of nonlinear wave-equations on a slowly time-varying spatial domain. It will be shown how the boundary excitations and the nonlinear terms influence transverse and longitudinal vibrations of the system. Due to the slow variation of the cable length, a singular perturbation problem arises. By using an interior layer analysis many resonance manifolds are detected. It will be shown that resonances in the system are caused not only by boundary disturbances but also by nonlinear interactions. Based on these observations, a three time-scales perturbation method is used to approximate the solution of the initial-boundary value problem analytically. It turns out that for special frequencies in the boundary excitations and for certain parameter values of the longitudinal stiffness and the conveyance mass, many oscillation modes jump up from small to large amplitudes in the transverse and longitudinal directions. Moreover, numerical simulations are presented to verify the obtained analytical results.

Since these system vibrations may lead to structural failure by excessive strain in the moving process, we consider vibration stabilization of axially moving cable systems. We present an output feedback control design to stabilize an unstable moving cable subject to a spring-mass-dashpot boundary, where the control actuator is located at the other boundary of the cable. By constructing an invertible backstepping transformation, we design a state feedback controller to stabilize the system. Next, we present an observer to estimate the states of the system, and based on the estimated states, we design an output-feedback controller. The closed-loop system is proved to be exponentially stable by Lyapunov analysis. Numerical simulations are presented to verify the effectiveness of the proposed controller.

SAMENVATTING

Kabelsystemen met variabele lengte worden op grote schaal toegepast in een grote klasse van technische problemen die zich voordoen in industriële, civiele, luchtvaart en ruimtevaart gerelateerd, mechanische, en automobiel toepassingen. Als gevolg van externe krachten kunnen grote oscillaties optreden wanneer kabels worden op-of neerbewegen. Dit verschijnsel wordt veroorzaakt door resonantie. In het algemeen is resonantie schadelijk, en kan het aanzienlijke vervormingen en dynamische spanningen veroorzaken in machines en constructies, en zelfs leiden tot ongelukken. Daarom is deze doctoraalscriptie gewijd aan de studie van transversale en longitudinale resonantieverschijnselen en output feedback stabilisatie van kabels met variërende lengte.

In de eerste plaats zijn wij gemotiveerd door resonantieverschijnselen die zich voordoen in een transversaal trillende kabel, waarbij het ene uiteinde van de kabel is gefixeerd, en het andere is bevestigd aan een veer waarvan de stijfheidseigenschappen in de tijd veranderen (ten gevolge van vermoeiing, temperatuursverandering, enzovoort). Dit probleem kan dienen als een vereenvoudigd model voor het beschrijven van transversale of longitudinale trillingen en resonanties in axiaal bewegende kabels waarvan de lengte in de tijd verandert. Door de frequentie van de uitwendige kracht en de tijdsafhankelijke grenscoëfficiënt in de Robin-randvoorwaarde in te stellen, kunnen verschillende soorten resonanties worden verkregen. Het eerste doel is de exacte oplossing te geven met behulp van de methode van d'Alembert en golfreflecties te bepalen, waarbij we het tijdsdomein in eindige intervallen kunnen verdelen. Vervolgens kunnen de resonantieresultaten worden geanalyseerd aan de hand van de kaart van de geconstrueerde oplossing. Het andere doel is expliciete benaderingen te geven van de oplossing op lange tijdschalen door gebruik te maken van de methode van de scheiding van variabelen, de methode van d'Alembert, de middelingsmethode, en perturbatiemethoden gebaseerd op meerdere tijdschalen. Voor problemen met tijdsafhankelijke coëfficiënten in de Robin-randvoorwaarde zijn de analytische resonantieresultaten alle in overeenstemming met die verkregen met behulp van een numerieke methode.

Vervolgens breiden wij onze analytische en numerieke resultaten uit tot een reëel fysisch model van een buigingslijf kabel-lift-systeem, waarin externe verstoringen die aan de rand worden uitgeoefend grote trillingen kunnen induceren. De dynamica wordt beschreven door een golfvergelijking op een langzaam tijdvariërend ruimtelijk domein met een kleine harmonische grensexcitatie aan het ene eind van de kabel, en een bewegende massa aan het andere eind. Door de langzame variatie van de kabellengte ontstaat een singulier storingsprobleem. Door gebruik te maken van een middelingsmethode en een ïnterior layer"techniek, worden vele resonantiemanifolds gedetecteerd. Verder wordt een perturbatiemethode met drie tijdschalen gebruikt om formele asymptotische benaderingen van de oplossingen te construeren. Het blijkt dat voor een gegeven randstoringsfrequentie, vele oscillatiemodes in amplitudes toenemen van orde ε amplitudes naar orde $\sqrt{\varepsilon}$ amplitudes, waarbij ε een kleine parameter is met $0 < \varepsilon << 1$. Bovendien

worden numerieke simulaties gepresenteerd om de verkregen analytische resultaten te verifiëren.

Verder kunnen de niet-lineaire interacties tussen de transversale en longitudinale bewegingen van de snaar, als gevolg van externe krachten en belastingsomstandigheden, het trillingsgedrag in twee richtingen beïnvloeden wanneer de lift kabel op en neer beweegt. Daarom bestuderen we zowel transversale als longitudinale oscillaties en resonanties in een lift-kabel-systeem geïnduceerd door grensstoringen. De dynamica kan worden beschreven door een initieel-randwaardeprobleem voor een gekoppeld stelsel van niet-lineaire golfvergelijkingen op een langzaam tijdvariërend ruimtelijk domein. Er zal worden aangetoond hoe de randstoringen en de niet-lineaire termen de transversale en longitudinale trillingen van het systeem beïnvloeden. Door de langzame variatie van de kabellengte ontstaat een singulier perturbatieprobleem. Door gebruik te maken van een ïnterior layer"techniekworden vele resonantie-manifolds gedetecteerd. Er zal worden aangetoond dat resonanties in het systeem niet alleen worden veroorzaakt door randstoringen, maar ook door niet-lineaire interacties. Op basis van deze waarnemingen wordt een perturbatiemethode met drie tijdschalen gebruikt om de oplossing van het initieel-randwaardeprobleem an- alytisch te benaderen. Het blijkt dat voor speciale frequenties in de randexcitaties en voor bepaalde parameterwaarden van de longitudinale stijfheid en de transportmassa, vele oscillatiemodes in amplitudes toenemen van kleine naar grote amplitudes in de transversale en longitudinale richtingen. Bovendien worden numerieke simulaties gepresenteerd om de verkregen analytische resultaten te verifiëren.

Aangezien de trillingen van deze systemen kunnen leiden tot structureel falen door overmatige spanning in het bewegende proces, beschouwen wij de stabilisatie van trillingen van axiaal bewegende kabelsystemen. Wij stellen een ontwerp voor van een output feedback regelaar om een onstabiele bewegende kabel te stabiliseren met behulp van een veer-massa-dashpot systeem, waarbij de regelactuator zich aan de andere kant van de kabel bevindt. Door een inverteerbare backstepping transformatie te construeren, ontwerpen we een waarnemerssysteem om het systeem te stabiliseren. Vervolgens presenteren we een waarnemer om de toestanden van het systeem te schatten, en op basis van de geschatte toestanden ontwerpen we een output-feedback regelaar. Van het gesloten-lus systeem wordt exponentieel stabiliteit aangetoond met behulp van Lyapunov analyse. Numerieke simulaties worden gepresenteerd om de effectiviteit van de voorgestelde regelaar te verifiëren.

1

INTRODUCTION

1.1. BACKGROUND

With the last decades, elevator systems are widely used for transportation of objects to a large height or depth. Such systems consist of a drum, a head sheave, a driving motor, a flexible cable with time-varying length, and a cage moving along two guiding ropes. When the flexible cable's bending stiffness is not considered, the mathematical model for such systems can be described as an axially moving string with a time variable length [1]. Compared with rigid structures, the flexible cable has many advantages, such as low costs, high speeds and high load carrying capacities, which are applied in various engineering fields, for instance, elevators and hoisters [2] (see Figure 1.1), marine risers [3, 4], suspension bridges [5, 6], medical rescue systems [7], etc.

In lifting processes [8, 9] (see Figure 1.2), vibration-induced structural failure for elevator cables may occur due to external disturbances such as airflows or earthquakes, or due to other internal or external excitations. These failures are usually related to internal or external resonances [10, 11, 12, 13, 14]. Resonance refers to the phenomenon that a small periodic excitation can produce large vibrations when the frequency of the external or internal excitation is close to one of the natural frequencies of the system. In most cases resonance is harmful, it will not only lead to significant deformations and dynamic stress, but also lead to accidents. Therefore, it is important to develop advanced analytical models to figure out the nature of these large vibrations in moving media.

There are many characteristics in axially moving strings to classify vibrations. One of the classifications is based on the vibration directions. Vibrations can be divided into transverse and longitudinal. Most analytical solutions for displacements of moving strings focus on transverse vibrations, which are subject to classical boundary conditions. In this thesis, we consider the longitudinal vibrations in moving strings, which are subject to moving nonclassical boundary conditions. Compared to researches subject to classical boundary conditions, the analysis of axially moving systems with moving nonclassical boundary conditions is a challenging subject for study. Moreover, we not only consider the longitudinal vibrations, but also consider a non-linearly coupled transverse and longitudinal vibration problem for axially moving strings with time-varying length.

Figure 1.1: An example of a moving cable system

There is an abundance of analytical methods to determine exact solutions of string problems in mathematical physics, such as the method of separation of variables (SOV), or the (equivalent) Laplace transform method, which is used to solve initial-boundary value problem for a string equation on a bounded interval for various types of boundary conditions with constant coefficients. However, when a boundary condition with a time-dependent coefficient, or a time-varying interval, is considered in the problem, the afore-mentioned methods may not be applicable. Thus, it is necessary to develop analytical methods or to adapt existing methods to solve these types of problems from a mathematical view-point. Nowadays, with the development of computers, numerical simulations based on the discretized system models are always used to tackle the practical and complex mathematical problems. It is widely used in engineering and the physical sciences by using approximate solutions within specified error bounds rather than exact solutions. But, since the string problem mentioned above is described by infinite dimensional partial differential equations, the simple discretization with truncation may lead to inaccurate results on long timescales. Thus, in this thesis, perturbation methods give additional insight. These methods have a high amount of information and accuracy on long timescales compared with the discretized models. Usually by using perturbation methods, we can construct formal asymptotic approximations of the solutions for the problem. Based on the approximations, it can be seen clearly how each parameter affects the dynamic behavior of the solution for the system. Nevertheless, perturbation

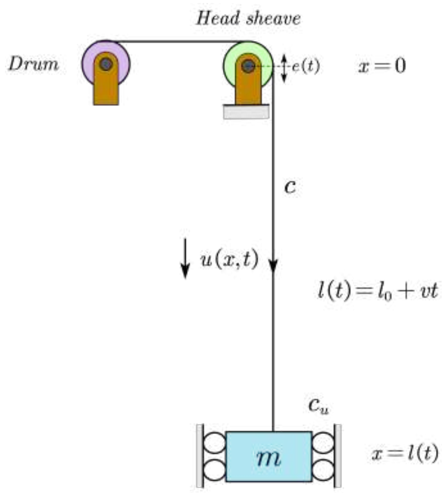

Figure 1.2: The longitudinal vibrating cable with time-varying cable length $l(t)$.

methods also have their disadvantages, for instance, the first order approximation of the solution does not always provide a required accuracy, or it is often not an easy task to construct approximations of high orders. Thus, the choice of the best methods to tackle the string problems depends on what we need, such as the mathematical model, the scope of the analysis, the accuracy for applications.

Moreover, we also consider the output feedback stabilization of the axially moving string system. There are many methods to achieve the vibration stabilization of axially moving strings or beams. One of the most useful methods for boundary controller is based on Lyapunov's method, by which control laws to reduce vibration energy to zero are derived using Lyapunov function candidates constructed by the total mechanical energy of the moving system. In this method, the controllers are required to follow the end causing vibration excitation, which is sometimes difficult to achieve in the practical implementation due to the inconvenient installation. Hence, the control system where control is applied at the end opposite to the instability is necessary to study. This is a more challenging task than the classical collocated "boundary damper" feedback control. Backstepping approach, which is proposed by Krstic, can deal with the proposed non-collocated stabilization problem efficiently. Its main principle is to offset the unstable terms of the system by variable transformations of partial differential equations, and by boundary feedback.

1.2. MATHEMATICAL MODELS

In this thesis we consider a set of one-dimensional initial-boundary value problem describing transverse, longitudinal vibrations as well as resonances in axially moving strings for which the length changes in time.

In chapter 2, we start with an simple initial-boundary value problem on a bounded, fixed interval for a one-dimensional and forced string equation subjected to a slowly time-varying Robin boundary condition. By using Hamilton's principle, the initial-boundary value problem is given by:

$$\begin{cases} \rho u_{tt}(x,t) - P u_{xx}(x,t) = \varepsilon A\cos(\omega t),\ 0 \le x \le L,\ t > 0, \\ u(0,t) = 0,\ t > 0, \\ P u_x(L,t) + k(t) u(L,t) = 0,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ 0 \le x \le L. \end{cases} \tag{1.1}$$

where $\varepsilon A\cos(\omega t)$ is a small external excitation with frequency ω, and $k(t)$ is a slowly time-varying coefficient. The boundary condition at $x = 0$ is a Dirichlet type of boundary condition, and the boundary condition at $x = L$ is a Robin type of boundary condition with a time-dependent coefficient $k(t)$. For given frequency of the external force ω and the time-dependent boundary coefficient $k(t)$, different kinds of solution behaviors and resonances can be obtained by using different methods. We mainly consider the following different cases:

- $k(t)$ is a constant;
- $k(t) = \frac{1}{1+\varepsilon t}$;
- $k(t) = k_0 + \varepsilon k_1 cos(\bar{\omega} t)$;
- $k(t) = 1 + \varepsilon t$.

where ε is a dimensionless small parameter.

In chapter 3, we study a real physical varying-length elevator system model, in which the longitudinal vibrations in an axially moving string system with time-varying length are considered subject to a small harmonic boundary excitation at one end and a moving nonclassical boundary condition at the other end. By using the Hamilton's principle, the initial-boundary value problem is given by:

$$\begin{cases} \rho(u_{tt} + 2v u_{xt} + v^2 u_{xx}) - EA u_{xx} + c(u_t + v u_x) = 0,\ 0 \le x \le l(t),\ t > 0, \\ [m(u_{tt} + 2v u_{xt} + v^2 u_{xx}) + EA u_x + c_u(u_t + v u_x)]|_{x=l(t)} = 0,\ t > 0, \\ u(0,t) = e(t),\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ 0 \le x \le l_0, \end{cases} \tag{1.2}$$

where the parameters v, c, c_u and the function $e(t)$, we make the following reasonable assumptions: the longitudinal velocity v is small compared to nominal wave velocity $\sqrt{\frac{EA}{\rho}}$; the viscous damping coefficients c and c_u are small; and the oscillation amplitudes $e(t)$ at $x = 0$ are small. Then, we can rewrite $v = \varepsilon v_0, c = \varepsilon c_0, c_u = \varepsilon c_{u0}, e(t) = \beta \sin(\alpha t)$ with $\beta = \varepsilon \beta_0$, where ε is a small parameter with $0 < \varepsilon << 1$, v_0, c_0, $c_{u,0}$, β_0 are positive constants and are of order 1. For convenience we only consider a non-accelerating cable, $l(t) = l_0 + \varepsilon v_0 t$, where l_0 is the initial cable length. It is also assumed that both initial conditions are $O(\varepsilon)$, that is, $u_0(x) = O(\varepsilon)$, and $u_1(x) = O(\varepsilon)$.

In chapter 4, we further study the real physical varying-length elevator system model, in which transverse and longitudinal oscillations and resonances in an axially moving string with time-varying length are both considered subject to small harmonic boundary excitations in transverse and longitudinal directions at one end and a moving nonclassical boundary condition at the other end. By the Hamilton's principle, the mathematical problem for the vibrating cable can be written as a non-linearly coupled initial boundary value problem for the transverse vibration:

$$\begin{cases} \rho(w_{tt}+2vw_{xt}+v^2w_{xx}+aw_x)-(Tw_x)_x+c_1(w_t+vw_x)-EA(zw_x)_x=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \beta_2 cos(\omega_2 t)<x<l(t),\ t>0, \\ w(l(t),t)=0, \quad t\geq 0, \\ w(\beta_2 cos(\omega_2 t),t)=\beta_1 cos(\omega_1 t+\alpha), \quad t\geq 0, \\ w(x,0)=w_0(x), \quad w_t(x,0)=w_1(x), \quad \beta_2<x<l_0, \end{cases} \tag{1.3}$$

and as an initial boundary value problem for the longitudinal vibration:

$$\begin{cases} \rho(u_{tt}+2vu_{xt}+v^2u_{xx}+au_x+a)+c_2(u_t+vu_x)-EAz_x=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \beta_2 cos(\omega_2 t)<x<l(t),\ t>0, \\ [m(u_{tt}+2vu_{xt}+v^2u_{xx}+au_x+a)+c_u(u_t+vu_x)+EAz]|_{x=l(t)}=0, \quad t\geq 0, \\ u(\beta_2 cos(\omega_2 t),t)=\beta_2 cos(\omega_2 t), \quad t\geq 0, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad \beta_2<x<l_0, \end{cases} \tag{1.4}$$

where $z=u_x+\frac{1}{2}w_x^2$ and

$$T(x,t)=[m+\rho(l(t)-x)]g, \quad \beta_2 cos(\omega_2 t)\leq x\leq l(t). \tag{1.5}$$

We use the following assumptions for the parameters and functions:

- The longitudinal velocity v is small compared to the wave velocities $\sqrt{\frac{EA}{\rho}}$ and $\sqrt{\frac{mg}{\rho}}$, that is, $v=\varepsilon v_0$;
- The nominal wave velocities $\sqrt{\frac{EA}{\rho}}$ and $\sqrt{\frac{mg}{\rho}}$ are of the same order of magnitude, that is, $\frac{EA}{mg}=O(1)$, $\sqrt{\frac{EA}{mg}}>1$, and $\frac{EA}{mg}$ is not near 1, i.e., $\sqrt{\frac{EA}{mg}}-1>O(\varepsilon)$;
- The cable mass ρL is small compared to the car mass m (L is the maximum length of the cable), that is, $\mu=\frac{\rho L}{m}=\varepsilon\mu_0$;
- The viscous damping parameters c_1, c_2, and c_u are small, that is, $c_1=\varepsilon c_{1,0}, c_2=\varepsilon c_{2,0}, c_u=\varepsilon c_{u,0}$;
- The fundamental excitations at the top of the elevator rope are small, and the longitudinal excitation is smaller than the transverse excitation, that is, $\beta_1=\varepsilon\beta_{1,0}, \beta_2=\varepsilon^2\beta_{2,0}$;
- The initial conditions $w_0(x)=\varepsilon h_0(x)$, $w_1(x)=\varepsilon h_1(x)$, $u_0(x)=\varepsilon^2 h_2(x)$ and $u_1(x)=\varepsilon^2 h_3(x)$;

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- For convenience we only consider a non-accelerating cable, that is, the cable length $l(t) = l_0 + vt$ and $a = 0$, where l_0 is the initial string length.

In the above assumptions, v_0, μ_0, $c_{1,0}$, $c_{2,0}$, $c_{u,0}$, $\beta_{1,0}$, $\beta_{2,0}$, α m, ρ, ω_1, ω_2, L and l_0 are positive constants and are of order 1, the functions $h_0(x), h_1(x), h_2(x), h_3(x)$ are of order 1, and ε is a small parameter with $0 < \varepsilon << 1$.

In chapter 5, we consider a moving string system with constant speed on a finite spatial domain subject to a spring-mass-dashpot attached at one end of the string. By the Hamilton's principle, the mathematical problem for the vibrating string can be written as:

$$\begin{cases} \rho(u_{tt} + 2vu_{xt} + v^2u_{xx}) - Tu_{xx} = 0,\ 0 \le x \le l,\ t > 0, \\ mu_{tt}(0,t) + Tu_x(0,t) + ku(0,t) + \rho vu_t(0,t) - \rho v^2u_x(0,t) = 0,\ t > 0, \\ Tu_x(l,t) + ku(l,t) + \rho vu_t(l,t) - \rho v^2u_x(l,t) = U(t),\ t > 0. \end{cases} \tag{1.6}$$

where $u(x,t)$ is the transverse displacement of the string at the coordinate x and the time t; l is the distance between two boundary ends; v is the traveling speed of the moving string; ρ is the mass density of the string; m is the mass of the spring-mass; T is the uniform tension of the string; k is the stiffness of the spring; $U(t)$ is the control force attached at $x = l$. Moreover, ρ, v, T, m, l and k are positive constants.

1.3. MATHEMATICAL METHODS

1.3.1. FOURIER SERIES

A Fourier series is a way of representing a periodic function as a (possibly infinite) sum of sine and cosine functions. It is analogous to a Taylor series, which represents functions as possibly infinite sums of monomial terms. The Fourier series, as well as its generalizations, is essential throughout the physical sciences since the trigonometric functions are eigenfunctions of the Laplacian, which appears in many physical equations.

Given a function $x(t)$ with period T, it can be expressed as infinite series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{ik(\frac{2\pi}{T})t}, \tag{1.7}$$

where

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-ik(\frac{2\pi}{T})t}dt. \tag{1.8}$$

For functions of two variables that are periodic in both variables, the trigonometric basis in the Fourier series is replaced by the spherical harmonics. For functions that are not periodic, the Fourier series is replaced by the Fourier transform. There are several common conventions for defining the Fourier transform of an integrable function f. One of them is:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}dx, \quad \forall \xi \in \mathbb{R}. \tag{1.9}$$

This is the customary form for generalizing to complex-valued functions.

1.3.2. METHOD OF D'ALEMBERT

Considering the initial value problem for the wave equation:

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t), \quad -\infty < x < +\infty, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 \le x \le l_0. \end{cases} \tag{1.10}$$

According to the formular of d'Alembert, the solution of (1.10) is given by

$$u(x,t) = \frac{u_0(x-at)+u_0(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} u_1(\bar{x})d\bar{x} + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\bar{x},\tau)d\bar{x}d\tau. \tag{1.11}$$

1.3.3. ADAPTED VERSION OF THE METHOD OF SEPARATION OF VARIABLES

We consider the homogeneous part of equation subject to the homogeneous boundary conditions:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, \; 0 < x < 1, \quad t \ge 0, \\ u(0,t) = 0, \quad u_x(1,t) + k(\varepsilon t)u(1,t) = 0,, \quad t \ge 0, \end{cases} \tag{1.12}$$

where ε is a small parameter with $0 < \varepsilon << 1$. Note that the coefficient $k(\varepsilon t)$ in the Robin boundary condition is slowly varying in time. So, in order to derive a solution of problem (1.12) , we define an extra slow time variable $\tau = \varepsilon t$, which will be treated independently from the variable t. Hence $u(x,t)$ becomes a new function $\bar{u}(x,t,\tau)$ and further problem (1.12) becomes

$$\begin{aligned} &\bar{u}_{tt}(x,t,\tau) + 2\varepsilon \bar{u}_{t\tau}(x,t,\tau) + \varepsilon^2 \bar{u}_{\tau\tau}(x,t,\tau) - \bar{u}_{xx}(x,t,\tau) = 0, \\ &\qquad\qquad\qquad\qquad 0 < x < 1, \quad t > 0, \quad \tau > 0, \\ &\bar{u}(0,t,\tau) = 0, \quad \bar{u}_x(1,t,\tau) + k(\tau)\bar{u}(1,t,\tau) = 0, \quad t \ge 0, \quad \tau \ge 0. \end{aligned} \tag{1.13}$$

By looking for a nontrivial solution $\bar{u}(x,t,\tau)$ in the form $T(t,\tau)X(x,\tau)$, the governing equations of (1.13) can be approximately written as

$$\begin{aligned} X(x,\tau)T_{tt}(t,\tau) + 2\varepsilon X(x,\tau)T_{t\tau}(t,\tau) + 2\varepsilon X_\tau(x,\tau)T_t(t,\tau) & \\ - X_{xx}(x,\tau)T(t,\tau) + O(\varepsilon^2) &= 0, \end{aligned}$$

or equivalently as

$$\frac{T_{tt}(t,\tau)}{T(t,\tau)} + O(\varepsilon) = \frac{X_{xx}(x,\tau)}{X(x,\tau)}, \; 0 < x < 1, \quad t \ge 0, \quad \tau \ge 0. \tag{1.14}$$

The $O(1)$ part of the left-hand side of equation (1.14) is a function of t and τ, and the right-hand side is a function of x and τ. To be equal, both sides need to be equal to a function of τ. Let this function be $-\lambda^2(\tau)$ (which will be defined later), so we get

$$\frac{T_{tt}(t,\tau)}{T(t,\tau)} = \frac{X_{xx}(x,\tau)}{X(x,\tau)} = -\lambda^2(\tau), \; 0 < x < 1, \quad t \ge 0, \quad \tau \ge 0,$$

implying:

$$X_{xx}(x,\tau) + \lambda^2(\tau)X(x,\tau) = 0, \; 0 < x < 1, \quad \tau \ge 0,$$

1

$$T_{tt}(t,\tau)+\lambda^2(\tau)T(t,\tau)=0,\ \ t\geq 0,\quad \tau\geq 0. \tag{1.15}$$

From the boundary condition (1.13), we obtain

$$\begin{aligned} &T(t,\tau)X(0,\tau)=0 \quad \Rightarrow \quad X(0,\tau)=0,\\ &T(t,\tau)X_x(1,\tau)+k(\tau)T(t,\tau)X(1,\tau)=0\\ &\qquad\qquad\qquad \Rightarrow \quad X_x(1,\tau)+k(\tau)X(1,\tau)=0. \end{aligned} \tag{1.16}$$

In accordance with the first equation for $X(x,\tau)$ in (1.15), a nontrivial solution $X_n(x,\tau)$ (satisfying (1.16)) is

$$X_n(x,\tau)=B_n(\tau)\sin(\lambda_n(\tau)x), \tag{1.17}$$

where $B_n(\tau)$ is a function of τ only, and $\lambda_n(\tau)$ is the n-th positive root of

$$\tan(\lambda_n(\tau))=-\frac{\lambda_n(\tau)}{k(\tau)}. \tag{1.18}$$

In should be observed that the eigenfunctions $X_n(x,\tau)$ are orthogonal on $0<x<1$. And so, the general solution of (1.12) can be expanded in the following form:

$$u(x,t)=\bar{u}(x,t,\tau)=\sum_{n=1}^{\infty}T_n(t,\tau)\sin(\lambda_n(\tau)x), \tag{1.19}$$

where the boundary conditions (1.12) are automatically satisfied.

1.3.4. AVERAGING METHOD

The idea of averaging as a computational technique, without proof of validity, originates from the 18 th century; it has been formulated very clearly by Lagrange in his study of the gravitational three-body problem as a perturbation of the two-body problem.

We assume that $f(t,x)$ is T-periodic in t, and consider the initial value problem:

$$\dot{x}=\varepsilon f(t,x)+\varepsilon^2 g(t,x,\varepsilon),\quad x(0)=x_0, \tag{1.20}$$

where ε is a small parameter with $0<\varepsilon<<1$. We introduce the average

$$f^0(y)=\frac{1}{T}\int_0^T f(t,y)dt. \tag{1.21}$$

In performing the integration y has been kept constant. Consider now the initial value problem for the averaged equation

$$\dot{y}=\varepsilon f^0(y),\quad y(0)=x_0. \tag{1.22}$$

1.3.5. MULTIPLE TIMESCALES METHOD

The method of multiple timescales is used to construct explicit and accurate approximate solutions for ordinary differential equations and partial differential equations on long timescales. Since a straight-forward expansion in ε may have secular terms (which are unbounded in time t), one or more new time variables $(t_0,\ t_1,\ t_2,...)$ are introduced and are treated independently. Then, it is assumed that the solution $u(x,t)$ of the perturbation problem can be expanded in a power series in ε, where ε is a small parameter, as follows:

$$u(x,t;\varepsilon) = u_0(x,t_0,t_1,t_2,...) + \varepsilon u_1(x,t_0,t_1,t_2,...) + O(\varepsilon^2). \tag{1.23}$$

In order to avoid unbounded (or secular) terms in the expansion, secularity conditions have to be found for u_i, where $i \in \mathbb{N}_+ \bigcup \{0\}$.

1.4. OUTLINE OF THE THESIS

THe thesis is organized as follows.

In chapter 1, a brief introduction to the subject is given.

In chapter 2, the initial–boundary value problem (1.1) is considered. It can serve as a simple model for the elevator system describing transverse or longitudinal vibrations as well as resonances in axially moving cables for which the length changes in time. The first aim is to give the exact solution for the problem by using the method of d'Alembert and wave reflections. Then the second aim is to construct approximate solutions for some cases with different time-dependent coefficients $k(t)$ by using the method of separation of variables, the method of d'Alembert, averaging method, and multiple timescales perturbation methods, respectively. Finally, numerical simulations are presented to verify the obtained analytical results.

In chapter 3, the initial–boundary value problem (1.2) is considered. Due to the slow variation of the cable length, a singular perturbation problem arises. By using an averaging method, and an interior layer analysis, many resonance manifolds are detected. Further, a three time-scales perturbation method is used to construct formal asymptotic approximations of the solutions. Finally, numerical simulations are presented to verify the obtained analytical results.

In chapter 4, the coupled nonlinear initial–boundary value problems (1.3) and (1.4) in transverse and longitudinal directions are considered. Due to the slow variation of the cable length, a singular perturbation problem arises. In order to deal with this problem, perturbation methods and an internal layer analysis are used in this chapter to approximate the vibrations and the resonances, including determining the resonance amplitudes and the size of the resonance zones. Based on this analysis, solutions of the coupled initial-boundary value problem for the transverse and the longitudinal motions can be predicted analytically. Finally, numerical simulations are presented to verify the obtained analytical results.

In chapter 5, the output feedback stabilization of the initial–boundary value problem (1.6) is considered. By constructing an invertible backstepping transformation, we design a state feedback controller to stabilize the system. Next, we present an observer to estimate the states of the system, and based on the estimated states, we design an output-feedback controller. The closed-loop system is proved to be exponentially stable by Lyapunov analysis. Finally, numerical simulations are presented to verify the effectiveness of the proposed controller.

2

Transverse resonances of a vibrating string with a time-dependent Robin boundary condition

2.1. Introduction

In this chapter we start with the resonance phenomena occurring in a transversally vibrating string (see Figure 2.1), where one end of the string is fixed, and the other one is attached to a spring for which the stiffness properties change in time (due to fatigue, temperature change, and so on). Mathematically, we will show how to (approximately) solve an initial-boundary value problem for a nonhomogeneous wave equation on a bounded, fixed interval with a Dirichlet type of boundary condition at one endpoint, and a Robin type of boundary condition with a time-dependent coefficient at the other end. Actually, the Robin boundary condition is an interesting one to study from the applicational and mathematical point of view. The wave equations involving a Robin type of boundary condition with a time-varying coefficient can be regarded as simple models for vibrations of elevator or mining cables in the study of axially moving strings with time-varying lengths. There is a lot of research on these types of problems. Chen et al. [15] considered an analytical vibration response in the time domain for an axially translating and laterally vibrating string with mixed boundary conditions. Further Chen et al. [16] investigated the exchange of vibrational energy of a finite length translating tensioned string model with mixed boundary conditions applying D'Alembert's principle and the reflection properties. Wang et al. [17] designed an output feedback controller to regulate the state of a wave equation on a time-varying spatial interval with an unknown boundary disturbance. Gaiko and van Horssen [18] considered lateral vibrations

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of a vertically moving string with in time harmonically varying length. Zhu and Wu [19] studied the transverse vibration of the translating string with sinusoidally varying velocities. Malookani and van Horssen [20] studied transverse vibrations of axially moving strings with time dependent velocity and further examined its instability. For more information on initial-boundary value problems for axially moving continua, the reader is referred to [21, 22, 23, 24, 25]. Also in other fields, Robin boundary conditions play an important role, and are sometimes called impedance boundary conditions in electromagnetic problems or convective boundary conditions in heat transfer problems [26, 27].

Usually the method of separation of variables (SOV), or the (equivalent) Laplace transform method is used to solve initial value problem for a wave equation on a bounded interval for various types of boundary conditions with constant coefficients [28, 29]. However, when a Robin boundary condition with a time-dependent coefficient is involved in the problem, the afore-mentioned methods are not applicable. For this reason, van Horssen and Wang in [30] employed the method of d'Alembert to solve a homogeneous wave equation involving Robin type of boundary conditions with time-dependent coefficients. This chapter is an extension of the study by van Horssen and Wang in [30]. The first aim of this chapter is to give the exact solution of the nonhomogenous problem by using the method of d'Alembert and wave reflections, in which we can divide the time domain into finite intervals of length 2. Then the resonance results can be analyzed by the map of the solution from t = 2n to t = 2(n + 1). The other goal is to give explicit approximations of the solution on long timescales by putting different values of $k(t)$.

- $k(t)$ is constant, resonances and exact solutions of this problem are solved by using the method of separation of variables (SOV).
- $k(t) = \frac{1}{1+\varepsilon t}$, resonance results are approximated by the map of solutions based on the method of d'Alembert.
- $k(t) = k_0 + \varepsilon k_1 \cos(\bar{\omega} t)$, resonances and explicit approximations of the solutions are obtained by using a two-timescales perturbation method.
- $k(t) = 1 + \varepsilon t$, an additional difficulty is introduced: for $t < O(\frac{1}{\varepsilon})$, εt is a small term, while for $t = O(\frac{1}{\varepsilon})$, εt is not a small term. So, by introducing an adapted version of the method of separation of variables, by using averaging and singular perturbation techniques, and by finally using a three time-scales perturbation method, resonances in the problem are detected and accurate, analytical approximations of the solutions of the problem are constructed.

The current chapter is organised as follows. In section 2.2 the problem is formulated by the Hamilton's principle. In section 2.3 the exact solution of the problem is tackled by the method of d'Alembert and wave reflections. In section 2.4, 2.5, 2.6 and in section 2.7 the cases of $k(t)$ is constant, $k(t) = \frac{1}{1+\varepsilon t}$, $k(t) = k_0 + \varepsilon k_1 cos(\bar{\omega} t)$ and $k(t) = 1 + \varepsilon t$ are analyzed to show the resonance results, respectively. And numerical simulations of above cases are presented to verify the obtained analytical results. As numerical method a standard finite difference method is used in this chapter for simplicity, but of course also more advanced methods such as the finite element method (as has been used in [31, 32, 33, 34, 35]) can be applied. Finally, in section 2.8 some concluding remarks are made.

2.2. FORMULATION OF THE PROBLEM

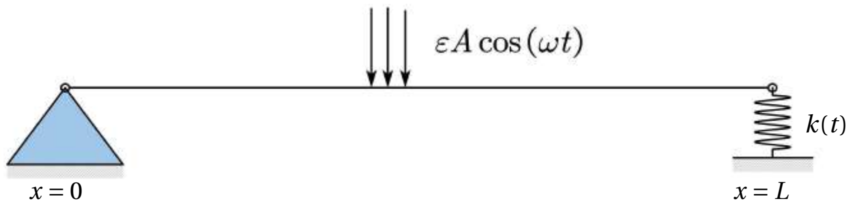

Figure 2.1: The transverse vibrating string with a time-varying spring-stiffness support at x=L, and an external force $\varepsilon A\cos(\omega t)$.

By using the Hamilton's principle [36], the governing equation of motion to describe the transversal vibration of a string as shown in Figure 2.1 can be derived, and is given by:

$$\rho u_{tt}(x,t) - P u_{xx}(x,t) = \varepsilon A\cos(\omega t),\ \ \omega > 0,\ \ 0 < x < L, \quad t > 0, \tag{2.1}$$

where ρ is the mass density, P is the axial tension (which is assumed to be constant), L is the distance between the supports, and u describes the lateral displacement of the string. The term $\varepsilon A\cos(\omega t)$ in (2.1) is a small external force acting on the whole string, where ε, ω and A are constants with $0 < \varepsilon << 1$, $\omega > 0$ and $A \in R$. The boundary conditions and initial conditions are given by:

$$u(0,t) = 0, \quad P u_x(L,t) + k(t)u(L,t) = 0, \quad t \geq 0, \tag{2.2}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x),\ \ 0 < x < L, \tag{2.3}$$

where $k(t)$ is the time-varying stiffness of the spring at $x = L$. The boundary condition at $x = 0$ is a Dirichlet type of boundary condition, and the boundary condition at $x = L$ is a Robin type of boundary condition with a time-dependent coefficient $k(t)$. Different choices of $k(t)$ lead to stiff changes of springs in time. This problem may also serve as a simplified model describing transverse or longitudinal vibrations as well as resonances in axially moving cables for which the length changes in time.

For simplicity, based on the Buckingham Pi theorem, the following dimensionless parameters are used:

$$\bar{x} = \frac{x}{L},\ \ \bar{u} = \frac{u}{L},\ \ \bar{t} = \frac{t}{L}\sqrt{\frac{P}{\rho}},\ \ \bar{k} = \frac{L}{P}k,\ \ \bar{\varepsilon} = \varepsilon L\sqrt{\frac{\rho}{P}},$$

$$\bar{A} = \frac{A}{\sqrt{\rho P}},\ \ \bar{\omega} = L\omega\sqrt{\frac{\rho}{P}},\ \ \bar{f} = \sqrt{\frac{P}{\rho}}\frac{f}{L^2},\ \ \bar{g} = \frac{g}{L},$$

by which, the governing equation (2.1), the boundary conditions (2.2), and the initial conditions (2.3) can be rewritten into the following non-dimensional form:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = \varepsilon A\cos(\omega t),\ \ 0 < x < 1, \quad t \geq 0, \\ u(0,t) = 0, \quad u_x(1,t) + k(t)u(1,t) = 0, \quad t \geq 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x),\ \ 0 < x < 1, \end{cases} \tag{2.4}$$

where the overbar notations are omitted for convenience.

2

2.3. THE METHOD OF D'ALEMBERT

IN this section, it will be shown how the well-known formula of d'Alembert can be used to obtain the solutions $u(x,t)$ of the initial-boundary value problem (2.4) with time-dependent coefficients $k(t)$ in Robin boundary conditions.

2.3.1. THE GENERAL SOLUTION

According to the method of d'Alembert (see [37]), the general solution to problem (2.4) without boundary conditions is given by

$$u(x,t) = \frac{1}{2}[f(x-t)+f(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds + \varepsilon A\int_0^t (t-\tau)cos(\omega\tau)d\tau. \tag{2.5}$$

It should be noted that the functions f and g are only defined on the interval [0,1]. To extend f and g on the whole domain $(-\infty,+\infty)$, the boundary conditions in problem (2.4) should be considered. By substituting Eq.(2.5) into boundary conditions, one obtains

$$f(t)+f(-t)+\int_{-t}^{t} g(s)ds + 2\varepsilon A\int_0^t (t-\tau)cos(\omega\tau)d\tau = 0, \tag{2.6}$$

$$\begin{aligned} f'(1+t)+f'(1-t)+g(1+t)-g(1-t)+k(t)[f(1+t)+f(1-t) \\ +\int_{1-t}^{1+t} g(s)ds + 2\varepsilon A\int_0^t (t-\tau)cos(\omega\tau)d\tau] \quad = \quad 0. \end{aligned} \tag{2.7}$$

Let

$$h(t) = f(t) + \int_0^t g(s)ds. \tag{2.8}$$

On the one hand, Eq.(2.6) can be transformed into

$$h(t) = -f(-t) - \int_{-t}^{0} g(s)ds - 2\varepsilon A\int_0^t (t-\tau)cos(\omega\tau)d\tau, \quad -1\le t\le 0, \tag{2.9}$$

which defines h on the interval [-1,0]. So $h(x)$ is now defined on the interval[-1,1]. On the other hand, it follows from Eq.(2.7) for $-1\le 1-t\le 1$ $(0\le t\le 2)$ that

$$\begin{aligned} & h'(1+t)+k(t)h(1+t) \\ = \quad & -f'(1-t)+g(1-t)-k(t)[f(1-t)+\int_{1-t}^{0} g(s)ds \\ & +2\varepsilon A\int_0^t (t-\tau)cos(\omega\tau)d\tau]. \end{aligned} \tag{2.10}$$

Multiplying both sides of (2.10) by the integrating factor $e^{\int_0^t k(s)ds}$, yields

$$\begin{aligned} & \frac{d(e^{\int_0^t k(s)ds}h(1+t))}{dt} \\ = \quad & e^{\int_0^t k(s)ds}[-f'(1-t)+g(1-t)-k(t)(f(1-t)+\int_{1-t}^{0} g(s)ds \\ & +2\varepsilon A\int_0^t (t-\tau)cos(\omega\tau)d\tau)]. \end{aligned} \tag{2.11}$$

Then we integrate (2.11) with respect to t from 0 to t to get

$$h(t+1) = e^{-\int_0^t k(s)ds}h(1) + e^{-\int_0^t k(s)ds}\int_0^t e^{\int_0^\tau k(s)ds}[-f'(1-\tau)+g(1-\tau) \\ -k(\tau)(f(1-\tau)+\int_{1-\tau}^0 g(s)ds + 2\varepsilon A\int_0^\tau (\tau-s)cos(\omega s)ds)]d\tau. \quad (2.12)$$

And so, the function h is defined on the interval [1,3]. By using Eq.(2.9) again, the expression for h on the interval [-3,-1] can be derived and h is now defined on [-3,3]. Further, referring to Eq.(2.8), the functions f and g can also be constructed on [-3,3]. By using Eq.(2.8), Eq.(2.9) and Eq.(2.12), we can again obtain the expression for f and g on the interval [-5,5]. Repeating this extension procedure over and over again, the expression for f(t) and g(t) can be found for all t with $-\infty \le t \le \infty$.

2.3.2. THE STATEMENT OF WAVE REFLECTION

The Nonhomogeneous wave equation we considered above in problem (2.4) has a propagation speed of 1, which implies that the vibration information at the point $x = x_i$, $t = 0$ will propagate into two different directions with speed 1, and the information will be back to the position x_i at $t = 2$ as shown in Figure 2.2 (a). Thus, by treating the information of the string at $t = 2$ as a new initial condition, we can then copy the extension steps as presented for the time-interval [0,2] for the next time interval of length 2, that is, $2 \le t \le 4$.

Furthermore, Figure 2.2 (b) shows the domain of dependence, from which we can see that it is sufficient to determine the response of the whole string at $0 \le t \le 2$ by the information at $x \in [-2,3]$, $t = 0$. By treating the state at $x \in [-2,3]$, t=2 as a new initial condition and using the same extension procedures, the solution $u(x,t)$, $2 \le t \le 4$, can also be obtained. Thus, we can calculate the solution of the nonhomogeneous problem up to every time by dividing the time domain into finite intervals of length 2.

2.3.3. NUMERICAL EXAMPLES

This section is devoted to presenting some numerical simulations on the behavior of the solution u(x,t) of problem (2.4) for t=2 and 4, respectively. Let us first choose the initial conditions as

$$\begin{cases} f(x) = sin(1.8x), \ 0 \le x \le 1, \\ g(x) = 0, \ 0 \le x \le 1, \end{cases} \quad (2.13)$$

and $k(t) = \frac{1}{t+1}$, $A = 1$, $\omega = 1.8$, $\varepsilon = 1$ as an example. The wave shape comparisons between the D'Alembert method and the finite difference method are shown in Figure 2.3. According to these figures, we can easily see that the results of the proposed method (the d'Alembert method) agree well with those of the finite difference method.

2.4. TIME-DEPENDENT COEFFICIENT $k(t)$ IS CONSTANT

IN this section, the method of separation of variables is used to obtain the exact solution of the initial boundary value problem (2.4), and the results are verified by numerical simulations.

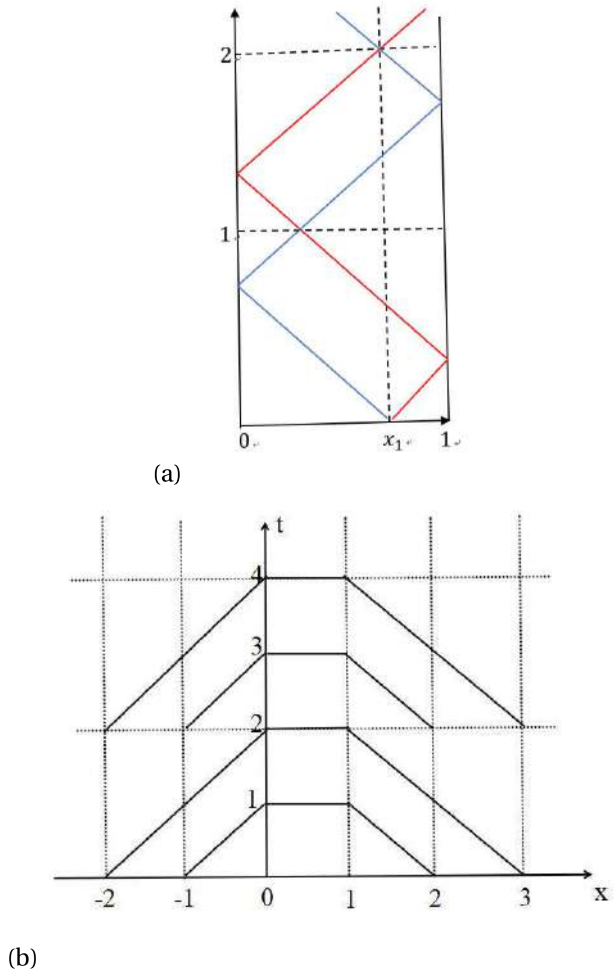

Figure 2.2: (a) Wave reflections. (b) Domain of dependence.

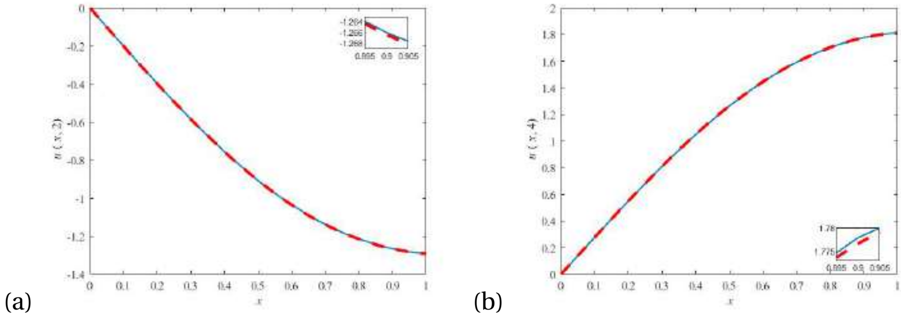

Figure 2.3: String shape comparisons when $k(t) = \frac{1}{t+1}$: the proposed method (solid line), the finite difference method (dashed line): u(x,2), u(x,4). (a) String shapes of u(x,2). (b) String shapes of u(x,4).

2.4.1. THE METHOD OF SEPARATION OF VARIABLES

To perform a resonance analysis for k(t) is constant, the method of separation of variables can be applied in this case, and the solution of problem (2.4) can be found as

$$u(x,t) = \sum_{i=1}^{\infty}[(A_i cos(\lambda_i t) + B_i sin(\lambda_i t) + \varepsilon A \int_0^t C_i(\tau) sin\lambda_i(t-\tau)d\tau) sin(\lambda_i x)], \qquad (2.14)$$

where

$$A_i = \frac{\int_0^1 f(x)sin(\lambda_i x)dx}{\int_0^1 sin^2(\lambda_i x)dx}, \quad B_i = \frac{\int_0^1 g(x)sin(\lambda_i x)dx}{\lambda_i \int_0^1 sin^2(\lambda_i x)dx}, \quad C_i(\tau) = \frac{cos(\omega\tau)\int_0^1 sin(\lambda_i x)dx}{\lambda_i \int_0^1 sin^2(\lambda_i x)dx}, \qquad (2.15)$$

and λ_i is the eigenvalue of the wave equation and satisfies

$$-\frac{1}{k}\lambda_i = tan(\lambda_i), \quad 0 < \lambda_1 < \lambda_2 < ... < \lambda_n .... \qquad (2.16)$$

Thus, the analytical result is that if $\omega = \lambda_i$, for a certain $i = 1,2,3,...$, then resonance behavior arises.

2.4.2. NUMERICAL EXAMPLES

In Figure 2.4, we give numerical results of the solution behaviors of the problem (2.4), which are found for the following parameters $k(t) = 1$, $A = 1$, $\omega = \lambda_1$ (λ_1 is given by (2.16)), and $\varepsilon = 0.01$. To better observe the resonance results, it is assumed that the initial conditions are given by

$$\begin{cases} f(x) = \varepsilon sin(1.7155x), \ 0 \le x \le 1, \\ g(x) = 0, \ 0 \le x \le 1. \end{cases} \qquad (2.17)$$

The displacement response of the problem (2.4) is given in Figure 2.4 (a), and the energy is given in Figure 2.4 (b), which are in complete agreement with the analytic results in (2.14).

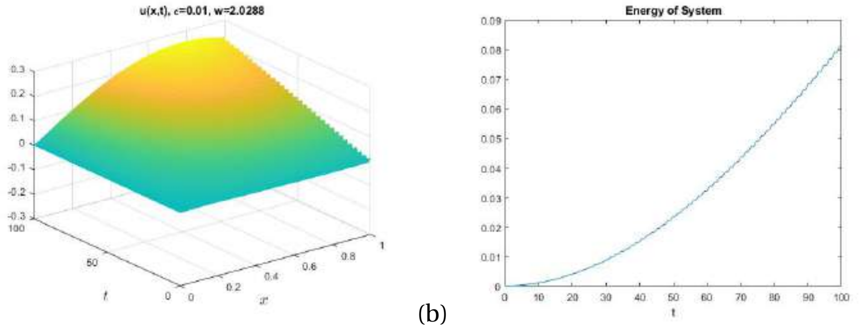

Figure 2.4: (a) The solution $u(x,t)$ of the problem (2.4). (b)The energy of the solution.

2.5. TIME-DEPENDENT COEFFICIENT $k(t) = \frac{1}{\varepsilon t+1}$

Considering the case of $k(t) = \frac{1}{\varepsilon t+1}$, the resonance results for initial-boundary value problem (2.4) can be analysed by the map of solutions based on the method of d'Alembert (in section 2.3) from t=2m to t=2(m+1). Further, the analytical results are verified by numerical simulations.

2.5.1. SOLUTION AND MAPPING

By treating the state at $t = 2m$, $u(x,2m)$, $u_t(x,2m)$, $m = 0,1,2...$ as new initial conditions, according to the method of d'Alembert and wave reflections, the solution $u(x,2m+\widetilde{t}), 0 \le \widetilde{t} \le 2$ can be written as

$$\begin{aligned} u(x,2m+\widetilde{t}) &= \frac{1}{2}[u(x-\widetilde{t},2m)+u(x+\widetilde{t},2m)] \\ &+\frac{1}{2}\int_{x-\widetilde{t}}^{x+\widetilde{t}} u_t(s,2m)ds + \varepsilon A\int_0^{\widetilde{t}}\int_{x-(\widetilde{t}-\tau)}^{x+(\widetilde{t}-\tau)} cos(\omega\tau)d\tau. \end{aligned} \tag{2.18}$$

Using the Fourier series, we observe that

$$\begin{aligned} & u(x,2m+\widetilde{t}) \\ = \; & \frac{1}{4\pi}\sum_{i=1}^{\infty} e^{i\lambda_i(x+\widetilde{t})}\int_0^1 e^{-i\lambda_i\xi}u(\xi,2m)d\xi + \frac{1}{4\pi}\sum_{i=1}^{\infty} e^{i\lambda_i(x+\widetilde{t})}\int_0^1 e^{-i\lambda_i\xi}\int_0^{\xi} u_t(s,2m)dsd\xi \\ & +\frac{1}{4\pi}\sum_{i=1}^{\infty} e^{i\lambda_i(x-\widetilde{t})}\int_0^1 e^{-i\lambda_i\xi}u(\xi,2m)d\xi - \frac{1}{4\pi}\sum_{i=1}^{\infty} e^{i\lambda_i(x-\widetilde{t})}\int_0^1 e^{-i\lambda_i\xi}\int_0^{\xi} u_t(s,2m)dsd\xi \\ & +\frac{\varepsilon A}{2\pi}\sum_{i=1}^{\infty}\int_0^{\widetilde{t}}[e^{i\lambda_i(x+(\widetilde{t}-\tau))} - e^{i\lambda_i(x-(\widetilde{t}-\tau))}]\int_0^1 e^{-i\lambda_i\xi}\int_0^{\xi} cos(\omega\tau)dsd\xi \\ = \; & \frac{1}{2\pi}\sum_{i=1}^{\infty} cos(\lambda_i\widetilde{t})e^{i\lambda x}\int_0^1 e^{-i\lambda_i\xi}u(\xi,2m)d\xi \\ & +\frac{1}{2\pi}\sum_{i=1}^{\infty} isin(\lambda_i\widetilde{t})e^{i\lambda_i x}\int_0^1 e^{-i\lambda_i\xi}\int_0^{\xi} u_t(s,2m)dsd\xi \\ & +\frac{\varepsilon A}{\pi}\sum_{i=1}^{\infty}\int_0^{\widetilde{t}}\int_0^1 \xi e^{i\lambda_i(x-\xi)}d\xi[e^{i(\omega-\lambda_i)\tau+i\lambda_i\widetilde{t}} \end{aligned}$$

$$-e^{i(\omega-\lambda_i)\tau - i\lambda_i\tilde{t}} - e^{i((\omega+\lambda_i)\tau-\lambda_i\tilde{t})} + e^{-i((\omega+\lambda_i)\tau-\lambda_i\tilde{t})}]d\tau, \tag{2.19}$$

where λ_i satisfies

$$-(1+2m\varepsilon)\lambda_i = tan\lambda_i, \tag{2.20}$$

and so satisfies equation (2.16) approximately at $k = k(t) = k(2m+\tilde{t})$. When we choose $\omega = \lambda_i$, for a certain i = 1,2,3,..., the last term of equation (2.19) can be written as

$$\begin{aligned}
&\frac{\varepsilon A}{\pi}\sum_{i=1}^{\infty}\int_0^{\tilde{t}} e^{i\lambda_i x}\int_0^1 \xi e^{-i\lambda_i\xi}d\xi[e^{i(\omega-\lambda_i)\tau+i\lambda_i\tilde{t}} - e^{i(\omega-\lambda_i)\tau-i\lambda_i\tilde{t}} - e^{i(\omega\tau-\lambda_i\tilde{t}+\lambda_i\tau)}\\
&+e^{-i(\omega\tau-\lambda_i\tilde{t}+\lambda_i\tau)}]d\tau\\
= &\frac{\varepsilon A}{\pi}\sum_{i=2}^{\infty}\int_0^{\tilde{t}} e^{i\lambda_i x}\int_0^1 e^{-i\lambda_i\xi}d\xi cos(\omega\tau)isin(\lambda_i(\tilde{t}-\tau))\\
&+\frac{\varepsilon A}{2\pi}\int_0^{\tilde{t}} e^{i\lambda_i x}\int_0^1 \xi e^{-i\lambda_i\xi}d\xi[isin(\omega\tau+\lambda_i(\tilde{t}-\tau))]d\tau\\
+ &\frac{i\varepsilon A}{4\pi}\tilde{t}sin(\omega\tilde{t})e^{i\lambda_i x}\int_0^1 \xi e^{-i\lambda_i\xi}d\xi.
\end{aligned} \tag{2.21}$$

And term "$\tilde{A}\tilde{t}sin(\omega\tilde{t})$" appear in (2.21), where $\tilde{A} = \frac{i\varepsilon A}{4\pi}e^{i\lambda_i x}\int_0^1 \xi e^{-i\lambda_i\xi}d\xi$. It implies that when the frequency of the external force is approximately equal to that of the homogeneous wave equation (2.4), i,e., $\omega = \lambda_i$ approximately, the resonance arises around $t = 2m$.

2.5.2. NUMERICAL EXAMPLES

Three different numerical examples are presented to verify the above analytical results.

- For $m = 0$ (and so, k(0)=1), note from Eq.(2.20) that $-\widetilde{\lambda_i} = tan\widetilde{\lambda_i}$, $\widetilde{\lambda_1} < \widetilde{\lambda_2} < \widetilde{\lambda_3} < ...$ By choosing $\omega = \widetilde{\lambda_i}$, the resonance arises around $t = 0$ in the above analysis. Numerically, let $k(t) = \frac{1}{\varepsilon t+1}$, $\varepsilon = 0.01$, $\omega = 2.0288 \approx \widetilde{\lambda_1}$, $A = 1$ and the initial conditions be (2.17), the solution behaviors and energy of the problem (2.4) are given in Figure 2.5, which turns out that the resonance arises from t=0 to t=40 approximately.

- For $m = \infty$ (and so, $k(\infty) = 0$), $\widetilde{\lambda_i} = \frac{\pi}{2} + (i-1)\pi$. By choosing $\omega = \widetilde{\lambda_i}$, a resonance arises as t is big enough in the above analysis. Numerically, let $k(t) = \frac{1}{\varepsilon t+1}$, $\varepsilon = 0.01$, $\omega = 1.5708 \approx \widetilde{\lambda_1}$, and the initial conditions be (2.17), the solution behaviors and energy of the problem (2.4) are given in Figure 2.6, which turns out that the resonance arises from t=200 to t=500 approximately.

- For $0 < m < \infty$, and so $-(1+2m\varepsilon)\widetilde{\lambda_i} = tan\widetilde{\lambda_i}$. By choosing $\omega = \widetilde{\lambda_i}$, a resonance arises around at $t = 2m$ in the above analysis. Numerically, let $k(t) = \frac{1}{\varepsilon t+1}$, $\varepsilon = 0.01$, $\omega = 1.82 \approx \widetilde{\lambda_1}$ at $m = 50$, and the initial conditions be (2.17), the solution behaviors and energy of problem (2.4) are given in Figure 2.7. It turns out that resonance arises from t=40 to t=180 approximately.

  These numerical examples are all in good agreement with those results in the method of d'Alembert.

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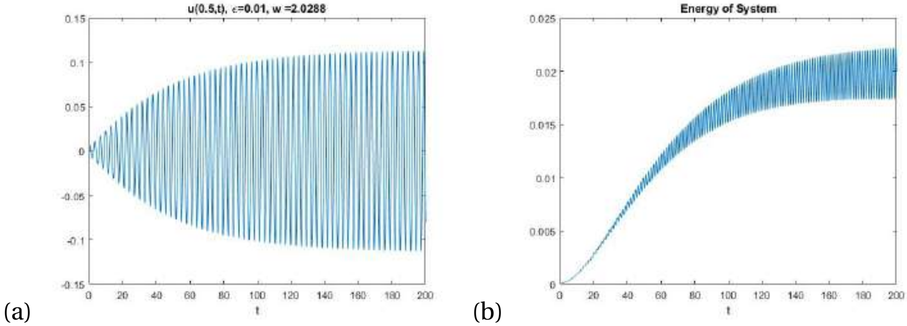

Figure 2.5: (a) The solution of the problem (2.4) for x=0.5, $\omega = 2.0288$. (b)The energy of solution as function of time t.

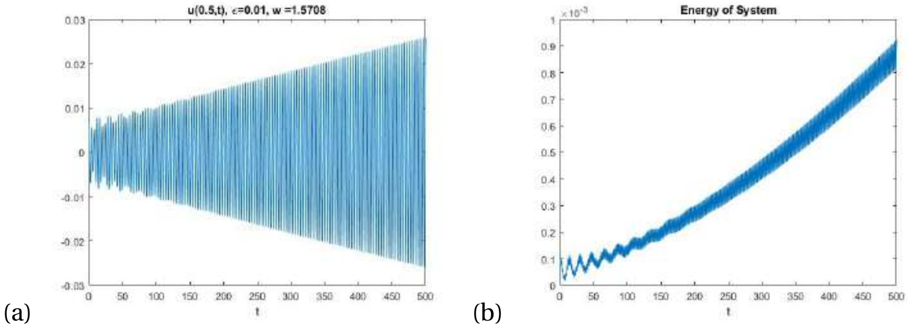

Figure 2.6: (a) The solution of the problem (2.4) for x=0.5, $\omega = 1.5708$. (b)The energy of solution as function of time t.

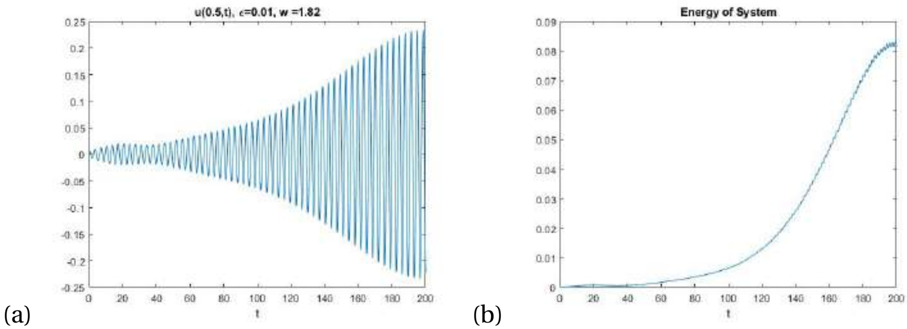

Figure 2.7: (a) The solution of the problem (2.4) for x=0.5, $\omega = 1.82$. (b)The energy of solution as function of time t.

2.6. TIME-DEPENDENT COEFFICIENT $k(t) = k_0 + \varepsilon k_1 cos(\bar{\omega} t)$

For $k(t) = k_0 + \varepsilon k_1 cos(\bar{\omega} t)$, the two-timescales perturbation method is used to construct approximations of solutions of the initial boundary value problem (2.4). Further, the analytical results are verified by numerical simulations.

2.6.1. TWO-TIMESCALES PERTURBATION METHOD

According to the two-time-scales perturbation method, we have to expand the solution in a Taylor series in ε as $u(x,t) \backsim u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + ...$, where $u_i(x,t) = O(1)$, $i = 1,2,3,....$ The approximation of the solution will contain secular terms, i.e., unbounded terms in time. It should be pointed out that the solution is bounded on a timescale of $O(\varepsilon^{-1})$. Consequently, we apply a two-timescales perturbation method to avoid the secular terms on long timescales by introducing an extra slow time variable $\tau = \varepsilon t$. Hence, $u(x,t)$ becomes a new function of x, t, and τ, i.e., $u(x,t) = w(x,t,\tau;\varepsilon)$. Using the time derivatives $\frac{d}{dt} \to \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}$ and $\frac{d^2}{dt^2} \to \frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2}{\partial \tau^2}$, we can rewrite the initial-boundary value problem (2.4) in terms of w as follows:

$$\begin{cases} w_{tt}(x,t,\tau;\varepsilon) - w_{xx}(x,t,\tau;\varepsilon) = \varepsilon[Acos(\omega t) - 2w_{t\tau}] - \varepsilon^2 w_{\tau\tau}, \ 0 < x < 1, \quad t > 0, \quad \tau > 0, \\ w(0,t,\tau;\varepsilon) = 0, \quad w_x(1,t,\tau;\varepsilon) + (k_0 + \varepsilon k_1 cos(\overline{\omega} t)) w(1,t,\tau;\varepsilon) = 0, \quad t > 0, \quad \tau > 0, \\ w(x,0,0;\varepsilon) = f(x), \ w_t(x,0,0;\varepsilon) + \varepsilon w_\tau(x,0,0;\varepsilon) = g(x), \ 0 < x < 1. \end{cases} \tag{2.22}$$

Then by substituting a power series expansion of the form $w(x,t,\tau) \backsim w_0(x,t,\tau) + \varepsilon w_1(x,t,\tau) + \varepsilon^2 w_2(x,t,\tau) + ...$ into the initial-boundary value problem (2.22) and equating the coefficients of like powers of ε, we obtain the $O(\varepsilon^n)-$ problems to solve for $n \in \mathbb{N}$. Meanwhile, the initial displacement and velocity can also be expanded in Taylor series in ε as $f(x) \backsim f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + ...$ and $g(x) \backsim g_0(x) + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + ...$, respectively.

THE $O(1)-$ PROBLEM.

Collecting the terms of $O(1)$ gives us the following unperturbed initial-boundary value problem:

$$\begin{cases} w_{0,tt}(x,t,\tau) - w_{0,xx}(x,t,\tau) = 0, \ 0 < x < 1, \quad t > 0, \quad \tau > 0, \\ w_0(0,t,\tau) = 0, \quad w_{0,x}(1,t,\tau) + k_0 w_0(1,t,\tau) = 0, \quad t > 0, \quad \tau > 0, \\ w_0(x,0,0) = f_0(x), \ w_{0,t}(x,0,0) = g_0(x), \ 0 < x < 1. \end{cases} \tag{2.23}$$

To find the solution of problem (2.23), the method of separation of variables can be used, yielding

$$w_0(x,t,\tau) = \sum_{n=1}^{\infty} [A_n(\tau) cos(\lambda_n t) + B_n(\tau) sin(\lambda_n t)] sin(\lambda_n x), \tag{2.24}$$

where the eigenvalues λ_n satisfy the transcendental equation

$$\frac{\lambda_n}{k_0} + tan(\lambda_n) = 0, \lambda_1 < \lambda_2 < ..., \tag{2.25}$$

and

$$A_n(0) = \frac{\int_0^1 f_0(x) sin(\lambda_n x) dx}{\int_0^1 sin^2(\lambda_n x) dx}, \qquad B_n(0) = \frac{\int_0^1 g_0(x) sin(\lambda_n x) dx}{\lambda_n \int_0^1 sin^2(\lambda_n x) dx}. \tag{2.26}$$

It should be observed that so far $w_0(x,t,\tau)$ contains undetermined functions $A_n(\tau)$ and $B_n(\tau)$. These functions will be used in the following $O(\varepsilon)-$ problem to avoid secular terms in $w_1(x,t,\tau)$.

The $O(\varepsilon)-$ Problem.

The $O(\varepsilon)-$ problem for w_1 is given by

$$\begin{cases} w_{1,tt}(x,t,\tau)-w_{1,xx}(x,t,\tau)=Acos(\omega t)-2w_{0,t\tau}(x,t,\tau),\ 0<x<1, \quad t>0, \quad \tau>0,\\ w_1(0,t,\tau)=0, \quad w_{1,x}(1,t,\tau)+k_0w_1(1,t,\tau)+k_1cos(\overline{\omega}t)w_0(1,t,\tau)=0, \quad t>0, \quad \tau>0,\\ w_1(x,0,0)=f_1(x),\ w_{1,t}(x,0,0)=g_1(x)-w_{0,\tau}(x,0,0),\ 0<x<1. \end{cases} \tag{2.27}$$

Let us introduce the following transformation:

$$w_1(x,t,\tau)=y(x,t,\tau)+xh(t,\tau), \tag{2.28}$$

where $h(t,\tau)=\frac{-k_1cos(\overline{\omega}t)w_0(1,t,\tau)}{1+k_0}$.

Substituting Eq.(2.28) into Eq.(2.27), we obtain a new initial-boundary value problem for y.

$$\begin{cases} y_{tt}(x,t,\tau)-y_{xx}(x,t,\tau)=F(x,t,\tau),\ 0<x<1, \quad t>0, \quad \tau>0,\\ y(0,t,\tau)=0, \quad y_x(1,t,\tau)+k_0y(1,t,\tau)=0, \quad t>0, \quad \tau>0,\\ y(x,0,0)=f_1(x)+\frac{k_1}{1+k_0}f_0(1)x,\ 0<x<1,\\ y_t(x,0,0)=g_1(x)-w_{0,\tau}(x,0,0)+\frac{k_1}{1+k_0}g_0(1)x,\ 0<x<1, \end{cases} \tag{2.29}$$

where

$$\begin{aligned} F(x,t,\tau) &= -xh_{tt}(t,\tau)+Acos(\omega t)+2\sum_{i=1}^{\infty}\lambda_i[A_i'(\tau)sin(\lambda_i t)-B_i'(\tau)cos(\lambda_i t)]sin(\lambda_i x)\\ &= \sum_{i=1}^{\infty}x[\alpha_iA_i(\tau)cos((\overline{\omega}+\lambda_i)t)+\beta_iA_i(\tau)cos((\overline{\omega}-\lambda_i)t)\\ &+ \alpha_iB_i(\tau)sin((\overline{\omega}+\lambda_i)t)-\beta_iB_i(\tau)sin((\overline{\omega}-\lambda_i)t)]\\ &+ 2\sum_{i=1}^{\infty}\lambda_isin(\lambda_ix)[A_i'(\tau)sin(\lambda_it)-B_i'(\tau)cos(\lambda_it)]+Acos(\omega t), \end{aligned}$$

where λ_i satisfies Eq.(2.25), and

$$\alpha_i=-\frac{k_1(\overline{\omega}+\lambda_i)^2sin\lambda_i}{2(1+k_0)}, \quad \beta_i=-\frac{k_1(\overline{\omega}-\lambda_i)^2sin\lambda_i}{2(1+k_0)}. \tag{2.30}$$

Let us expand the unknown solution $y(x,t,\tau)$ in terms of the eigenfunctions of the problem as follows:

$$y(x,t,\tau)=\sum_{n=1}^{\infty}y_n(t,\tau)sin(\lambda_nx). \tag{2.31}$$

We use the eigenfunction expansion approach to solve problem (2.29). Then, by substituting the series (2.31) into Eq. (2.29), by multiplying the so-obtained equation with

$sin(\lambda_n x)$, by integrating then over the interval $0 < x < 1$, and by using the orthogonality of the eigenfunctions, we obtain:

$$\begin{aligned} y_{n,tt} + \lambda_n^2 y_n &= \sum_{i=1}^{\infty} \frac{\int_0^1 x sin(\lambda_n x) dx}{\int_0^1 sin^2(\lambda_n x) dx} [\alpha_i A_i(\tau) cos((\overline{\omega} + \lambda_i)t) + \beta_i A_i(\tau) cos((\overline{\omega} - \lambda_i)t) \\ &+ \alpha_i B_i(\tau) sin((\overline{\omega} + \lambda_i)t) - \beta_i B_i(\tau) sin((\overline{\omega} - \lambda_i)t)] \\ &+ 2\lambda_n [A_n'(\tau) sin(\lambda_n t) - B_n'(\tau) cos(\lambda_n t)] + \frac{A(1 - cos\lambda_n)}{\lambda_n \int_0^1 sin^2(\lambda_n x) dx} cos(\omega t) \\ &= G_n(t,\tau),\ t,\tau > 0. \end{aligned} \tag{2.32}$$

It is obvious that the right-hand side of (2.32) can contain resonant terms (which can lead to secular terms in $y_n(t,\tau)$, for n=1, 2, 3,...) depending on the positive roots λ_n of (2.25). For instance, the term in the right-hand side of (2.32) involving $A\cos(\omega t)$ is a resonant term for the M-th mode described by $y_M(t)$ when $\omega = \lambda_M$ for a fixed $M \in \mathbb{N}_+$. Similarly, terms involving $sin((\overline{\omega} + \lambda_i)t)$, $cos((\overline{\omega} + \lambda_i)t)$, $sin((\overline{\omega} - \lambda_i)t)$ or $cos((\overline{\omega} - \lambda_i)t)$ can be resonant when $\overline{\omega}$ is equal to a difference or a sum of fixed eigenvalues λ_I and λ_N, that is, when $\overline{\omega} = \lambda_I - \lambda_N$ or $\overline{\omega} = \pm(\lambda_I + \lambda_N)$ with I and N fixed integers. And so, we have to distinguish the following cases:

1. Assume that $\overline{\omega} \neq \lambda_I - \lambda_N, \overline{\omega} \neq \pm(\lambda_I + \lambda_N)$ for any $I, N = 1, 2, ...$

1.1 When $\omega \neq \lambda_n$ for all $n = 1, 2, ...$, there is no resonant term in the right-hand side of (2.32), and so it is sufficient to choose A_n and B_n such that

$$A_n'(\tau) = B_n'(\tau) \quad = \quad 0, \quad n = 1, 2, ..., \tag{2.33}$$

where $A_n(0)$ and $B_n(0)$ (n=1, 2, 3,...) are given by Eq.(2.26), respectively. In addition, it follows from Eq.(2.24) and from Eq.(2.33) that

$$w_0(x,t,\tau) = \sum_{n=1}^{\infty} [A_n(0) cos(\lambda_n t) + B_n(0) sin(\lambda_n t)] sin(\lambda_n x). \tag{2.34}$$

When the initial conditions in (2.4) are of $O(\varepsilon)$, the solution of $O(1)$ problem (2.23) :

$$w_0(x,t,\tau) = 0, \tag{2.35}$$

which implies that the solution of initial-boundary value problem (2.4) is of $O(\varepsilon)$.
1.2 When there exists M such that $\omega = \lambda_M$, then in order to avoid secular terms in the right-hand side of (2.32), we have to set

$$\begin{aligned} A_M'(\tau) &= 0, \qquad A_n'(\tau) = B_n'(\tau) = 0, \quad n \neq M, \\ B_M'(\tau) &= \frac{A(1 - cos\lambda_M)}{2\lambda_M^2 \int_0^1 sin^2(\lambda_M x) dx}, \end{aligned} \tag{2.36}$$

by which we observe that there exists M such that $B_M(\tau)$ is linearly increasing in τ:

$$B_M(\tau) = B_M(0) + \frac{A(1 - cos\lambda_M)}{2\lambda_M^2 \int_0^1 sin^2(\lambda_M x) dx} \tau. \tag{2.37}$$

It follows from Eq.(2.24) and from Eq.(2.36) that

$$\begin{aligned} w_0(x,t,\tau) &= \sum_{n\neq M}^{\infty}[A_n(0)cos(\lambda_n t)+B_n(0)sin(\lambda_n t)]sin(\lambda_n x) \\ &\quad +[A_M(0)cos(\lambda_M t)+B_M(\tau)sin(\lambda_M t)]sin(\lambda_M x). \end{aligned} \tag{2.38}$$

When the initial conditions in (2.4) are of $O(\varepsilon)$, the solution of $O(1)$ problem (2.23):

$$w_0(x,t,\tau) = B_M(\tau)sin(\lambda_M t)sin(\lambda_M x), \tag{2.39}$$

which implies that the solution of initial-boundary value problem (2.4) is of $O(1)$.

Before discussing the other cases, it is necessary to propose a lemma:

There exist no $I, N, i, n \in \mathbb{N}_+$, such that

$$\lambda_I + \lambda_N + \lambda_i = \lambda_n \tag{2.40}$$

holds, where λ_j (j = 1, 2, ...) satisfies the equation

$$\frac{\lambda_j}{k_0} + tan(\lambda_j) = 0, \quad 0 < k_0 < \infty. \tag{2.41}$$

Proof: In Figure 2.8, The lines with blue color, and green color are denoted by a_i and b_i,

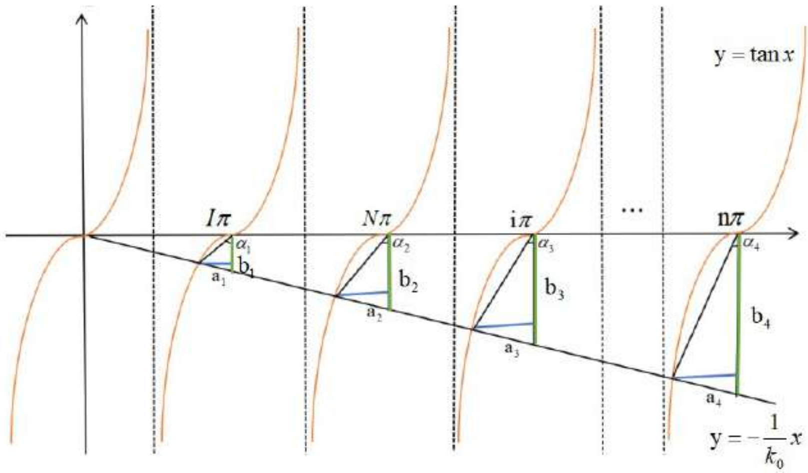

Figure 2.8: Graphical comparisons of eigenvalues for $k_0 = \infty$ and $0 < k_0 < \infty$ with I+N+i=n.

i=1,...,4, respectively.

From equation (2.41), when $k_0 = \infty$, $\lambda_n^0 = n\pi$ (λ_n is denoted by λ_n^0 to distinguish this λ_n from the other cases), it can be seen that (2.40) is satisfied. When $I + N + i = n$ and we have:

$$\lambda_I^0 + \lambda_N^0 + \lambda_i^0 = \lambda_{I+N+i}^0. \tag{2.42}$$

When $0 < k_0 < \infty$, λ_n is denoted by λ_n^1,

$$\begin{aligned}
&\lambda^1_{I+N+i} - (\lambda^1_I + \lambda^1_N + \lambda^1_i) \\
=\;& (\lambda^0_{I+N+i} - a_4) - [(\lambda^0_I - a_1) + (\lambda^0_N - a_2) + (\lambda^0_i - a_3)] \\
=\;& [\lambda^0_{I+N+i} - (\lambda^0_I + \lambda^0_N + \lambda^0_i)] + (a_1 + a_2 + a_3 - a_4) \\
=\;& a_1 + a_2 + a_3 - a_4. \qquad (2.43)
\end{aligned}$$

It follows from the relationship, I+N+i=n, that $b_1 + b_2 + b_3 = b_4$. Assume that the angles $\alpha_i (i = 1,2,3,4)$ satisfy $\widetilde{\alpha_1} = \widetilde{\alpha_2} = \widetilde{\alpha_3} = \alpha_4$, then we can obtain that the corresponding a_i satisfy $\widetilde{a_1} + \widetilde{a_2} + \widetilde{a_3} = a_4$, ($\alpha_i$, a_i are denoted by $\widetilde{\alpha_i}$ and $\widetilde{a_i}$ to distinguish them from that of the fact case). The fact is that $\alpha_i > \widetilde{\alpha_i}$, i=1,2,3, so then $a_1 + a_2 + a_3 > \widetilde{a_1} + \widetilde{a_2} + \widetilde{a_3} = a_4$. Hence, it follows from (2.43) that $\lambda^1_I + \lambda^1_N + \lambda^1_i < \lambda^1_{I+N+i}$ holds.
On the other hand, when $k_0 = 0$, $\lambda^*_n = \frac{\pi}{2} + (n-1)\pi$, λ_n is denoted by λ^*_n in this case, and so

$$\lambda^*_I + \lambda^*_N + \lambda^*_i = \lambda^*_{I+N+i-1}. \qquad (2.44)$$

From the same deduction, $\lambda^1_I + \lambda^1_N + \lambda^1_i > \lambda^1_{I+N+i-1}$ can be obtained.

Above all, $\lambda^1_{I+N+i-1} < \lambda^1_I + \lambda^1_N + \lambda^1_i < \lambda^1_{I+N+i}$, which implies that there is no $I, N, i, n \in \mathbb{N}_+$, such that $\lambda_I + \lambda_N + \lambda_i = \lambda_n$.

For $\overline{\omega} = \lambda_I - \lambda_N$ and $\overline{\omega} = \pm(\lambda_I + \lambda_N)$, these two cases can not occur at the same time.

2. Assume that there exist I and N, such that $\overline{\omega} = \lambda_I - \lambda_N$.

2.1 When $\omega \neq \lambda_n$ for any $n = 1,2,...$, it follows that A_n and B_n have to satisfy (in order to avoid secular terms in w_1)

$$\begin{aligned}
2\lambda_I \int_0^1 sin^2(\lambda_I x) dx A'_I(\tau) + \int_0^1 x sin(\lambda_I x) dx \alpha_N B_N(\tau) &= 0, \\
2\lambda_I \int_0^1 sin^2(\lambda_I x) dx B'_I(\tau) - \int_0^1 x sin(\lambda_I x) dx \alpha_N A_N(\tau) &= 0, \\
2\lambda_N \int_0^1 sin^2(\lambda_N x) dx A'_N(\tau) + \int_0^1 x sin(\lambda_N x) dx \beta_I B_I(\tau) &= 0, \\
2\lambda_N \int_0^1 sin^2(\lambda_N x) dx B'_N(\tau) - \int_0^1 x sin(\lambda_N x) dx \beta_I A_I(\tau) &= 0, \qquad (2.45)
\end{aligned}$$

which can be solved, yielding

$$\begin{aligned}
A_I(\tau) &= A_I(0)\cos(\sigma\tau) - \sqrt{\frac{\beta_I c_I d_N}{\alpha_N c_N d_I}} B_N(0)\sin(\sigma\tau), \\
B_N(\tau) &= A_I(0)\sqrt{\frac{\beta_I c_I d_N}{\alpha_N c_N d_I}} \sin(\sigma\tau) + B_N(0)\cos(\sigma\tau), \\
A_N(\tau) &= A_N(0)\cos(\sigma\tau) - \sqrt{\frac{\alpha_N c_N d_I}{\beta_I c_I d_N}} B_I(0)\sin(\sigma\tau), \\
B_I(\tau) &= A_N(0)\sqrt{\frac{\alpha_N c_N d_I}{\beta_I c_I d_N}} \sin(\sigma\tau) + B_I(0)\cos(\sigma\tau),
\end{aligned}$$

$$\sigma = \sqrt{\frac{\alpha_N \beta_I d_I d_N}{c_I c_N}}, \quad \frac{\alpha_N \beta_I d_I d_N}{c_I c_N} > 0, \tag{2.46}$$

where

2

$$c_i = 2\lambda_i \int_0^1 sin^2(\lambda_i x)dx, \quad d_i = \int_0^1 xsin(\lambda_i x)dx = \frac{1}{\lambda_i^2}(sin\lambda_i - \lambda_i cos\lambda_i), \quad i = I, N,$$

and,

$$A_n'(\tau) = B_n'(\tau) = 0, \quad n \neq I, N, \tag{2.47}$$

where $A_n(0)$ and $B_n(0)$ (n=1, 2, 3,...) are given by Eq.(2.26). From Eq.(2.24) and from Eq.(2.46)-(2.47), we can observe that when the initial conditions in (2.4) are of $O(\varepsilon)$, the solution of $O(1)$ problem (2.23):

$$w_0(x,t,\tau) = 0. \tag{2.48}$$

2.2 When there exists M such that $\omega = \lambda_M (M \neq N, I)$, it follows that A_n and B_n have to satisfy (in order to avoid secular terms in w_1)

$$\begin{aligned}
2\lambda_I \int_0^1 sin^2(\lambda_I x)dx A_I'(\tau) + \int_0^1 xsin(\lambda_I x)dx \alpha_N B_N(\tau) &= 0,\\
2\lambda_I \int_0^1 sin^2(\lambda_I x)dx B_I'(\tau) - \int_0^1 xsin(\lambda_I x)dx \alpha_N A_N(\tau) &= 0,\\
2\lambda_N \int_0^1 sin^2(\lambda_N x)dx A_N'(\tau) + \int_0^1 xsin(\lambda_N x)dx \beta_I B_I(\tau) &= 0,\\
2\lambda_N \int_0^1 sin^2(\lambda_N x)dx B_N'(\tau) - \int_0^1 xsin(\lambda_N x)dx \beta_I A_I(\tau) &= 0,
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
A_M'(\tau) &= 0, \qquad A_n'(\tau) = B_n'(\tau) = 0, \quad n \neq M, N, I,\\
B_M'(\tau) &= \frac{A(1 - cos\lambda_M)}{2\lambda_M^2 \int_0^1 sin^2(\lambda_M x)dx}.
\end{aligned} \tag{2.50}$$

The solution is the same as that in case 1.2 and 2.1, which can be obtained. And when the initial conditions in (2.4) are of $O(\varepsilon)$, the solution of $O(1)$ problem (2.23):

$$w_0(x,t,\tau) = B_M(\tau)sin(\lambda_M t)sin(\lambda_M x), \tag{2.51}$$

which implies that the solution of initial-boundary value problem (2.4) is of $O(\varepsilon)$.
2.3 When there exists an M such that $\omega = \lambda_M$ (M= N (or I)), it follows that A_n and B_n have to satisfy (in order to avoid secular terms in w_1)

$$\begin{aligned}
2\lambda_I \int_0^1 sin^2(\lambda_I x)dx A_I'(\tau) + \int_0^1 xsin(\lambda_I x)dx \alpha_N B_N(\tau) &= 0,\\
2\lambda_I \int_0^1 sin^2(\lambda_I x)dx B_I'(\tau) - \int_0^1 xsin(\lambda_I x)dx \alpha_N A_N(\tau) &= 0,
\end{aligned}$$

$$
\begin{aligned}
2\lambda_N \int_0^1 sin^2(\lambda_N x) dx A'_N(\tau) + \int_0^1 x sin(\lambda_N x) dx \beta_I B_I(\tau) &= 0,\\
2\lambda_N \int_0^1 sin^2(\lambda_N x) dx B'_N(\tau) - \int_0^1 x sin(\lambda_N x) dx \beta_I A_I(\tau) &= \frac{A c_N (1 - cos\lambda_N)}{2\lambda_N^2 \int_0^1 sin^2(\lambda_N x) dx}.
\end{aligned}
$$

which can be solved, yielding

$$
\begin{aligned}
A_I(\tau) &= A_I(0)\cos(\sigma\tau) - \sqrt{\frac{\beta_I c_I d_N}{\alpha_N c_N d_I}} B_N(0)\sin(\sigma\tau)\\
&\quad - \sqrt{\frac{\beta_I c_I d_N}{\alpha_N c_N d_I}} \frac{(1 - cos\lambda_N)}{c_N \lambda_N \sigma} A(1 - \cos(\sigma\tau)),\\
B_N(\tau) &= A_I(0)\sqrt{\frac{\beta_I c_I d_N}{\alpha_N c_N d_I}} \sin(\sigma\tau) + B_N(0)\cos(\sigma\tau) + \frac{A(1 - cos\lambda_N)}{c_N \lambda_N} \sin(\sigma\tau),\\
A_N(\tau) &= A_N(0)\cos(\sigma\tau) - \sqrt{\frac{\alpha_N c_N d_I}{\beta_I c_I d_N}} B_I(0)\sin(\sigma\tau),\\
B_I(\tau) &= A_N(0)\sqrt{\frac{\alpha_N c_N d_I}{\beta_I c_I d_N}} \sin(\sigma\tau) + B_I(0)\cos(\sigma\tau),
\end{aligned}
\tag{2.52}
$$

where σ, c_i, d_i can be obtained by (2.46).

$$
A'_n(\tau) = B'_n(\tau) = 0, \quad n \neq I, N(M), \tag{2.53}
$$

where $A_n(0)$ and $B_n(0)$ (n=1, 2, 3,...) are given by Eq.(2.26). Further when the initial conditions in (2.4) are of $O(\varepsilon)$, the solution of $O(1)$ problem (2.23):

$$
w_0(x, t, \tau) = A_I(\tau)\cos(\lambda_I t) sin(\lambda_I x) + B_N(\tau)\sin(\lambda_N t)\sin(\lambda_N x), \tag{2.54}
$$

which implies that the solution of initial-boundary value problem (2.4) is of $O(1)$.
3. Assume that there exist I and N, such that $\bar{\omega} = \lambda_I + \lambda_N$ (or $\bar{\omega} = -(\lambda_I + \lambda_N)$).

3.1 When $\omega \neq \lambda_n$ for any $n = 1, 2, ...$, then A_n and B_n have to satisfy

$$
\begin{aligned}
2\lambda_I \int_0^1 sin^2(\lambda_I x) dx A'_I(\tau) - \int_0^1 x sin(\lambda_I x) dx \beta_N B_N(\tau) &= 0,\\
2\lambda_I \int_0^1 sin^2(\lambda_I x) dx B'_I(\tau) - \int_0^1 x sin(\lambda_I x) dx \beta_N A_N(\tau) &= 0,\\
2\lambda_N \int_0^1 sin^2(\lambda_N x) dx A'_N(\tau) - \int_0^1 x sin(\lambda_N x) dx \beta_I B_I(\tau) &= 0,\\
2\lambda_N \int_0^1 sin^2(\lambda_N x) dx B'_N(\tau) - \int_0^1 x sin(\lambda_N x) dx \beta_I A_I(\tau) &= 0,
\end{aligned}
\tag{2.55}
$$

which can be solved, yielding

$$
\begin{aligned}
A_I(\tau) &= \frac{\sigma c_N}{2 d_N \beta_I} B_N(0)[e^{\sigma\tau} - e^{-\sigma\tau}] + \frac{1}{2} A_I(0)[e^{\sigma\tau} + e^{-\sigma\tau}],\\
B_N(\tau) &= \frac{\sigma c_I}{2 d_I \beta_N} A_I(0)[e^{\sigma\tau} - e^{-\sigma\tau}] + \frac{1}{2} B_N(0)[e^{\sigma\tau} + e^{-\sigma\tau}],
\end{aligned}
$$

$$\begin{aligned} A_N(\tau) &= \frac{\sigma c_I}{2d_I\beta_N}B_N(0)[e^{\sigma\tau}-e^{-\sigma\tau}]+\frac{1}{2}A_I(0)[e^{\sigma\tau}+e^{-\sigma\tau}], \\ B_I(\tau) &= \frac{\sigma c_N}{2d_N\beta_I}A_N(0)[e^{\sigma\tau}-e^{-\sigma\tau}]+\frac{1}{2}B_I(0)[e^{\sigma\tau}+e^{-\sigma\tau}], \end{aligned} \tag{2.56}$$

where σ, c_i, d_i can be obtained by (2.46).

$$A'_n(\tau)=B'_n(\tau) = 0, \quad n\neq I,N, \tag{2.57}$$

where $A_n(0)$ and $B_n(0)$ (n=1, 2, 3,...) are given by Eq.(2.26). Further when the initial conditions in (2.4) are of $O(\varepsilon)$, the solution of $O(1)$ problem (2.23):

$$w_0(x,t,\tau)=0, \tag{2.58}$$

which implies that the solution of initial-boundary value problem (2.4) is of $O(\varepsilon)$.
3.2 When $\omega=\lambda_M$ and $M\neq I,N$, then A_n and B_n have to satisfy

$$\begin{aligned} 2\lambda_I\int_0^1 sin^2(\lambda_I x)dxA'_I(\tau)-\int_0^1 xsin(\lambda_I x)dx\beta_N B_N(\tau) &= 0, \\ 2\lambda_I\int_0^1 sin^2(\lambda_I x)dxB'_I(\tau)-\int_0^1 xsin(\lambda_I x)dx\beta_N A_N(\tau) &= 0, \\ 2\lambda_N\int_0^1 sin^2(\lambda_N x)dxA'_N(\tau)-\int_0^1 xsin(\lambda_N x)dx\beta_I B_I(\tau) &= 0, \\ 2\lambda_N\int_0^1 sin^2(\lambda_N x)dxB'_N(\tau)-\int_0^1 xsin(\lambda_N x)dx\beta_I A_I(\tau) &= 0, \end{aligned} \tag{2.59}$$

$$\begin{aligned} A'_M(\tau) &= 0, \qquad A'_n(\tau)=B'_n(\tau)=0, \quad n\neq M,N,I, \\ B'_M(\tau) &= \frac{A(1-cos\lambda_M)}{2\lambda_M^2\int_0^1 sin^2(\lambda_M x)dx}. \end{aligned} \tag{2.60}$$

The solution is the same as that in case 1.2 and 3.1, which can be obtained. When the initial conditions in (2.4) are of $O(\varepsilon)$, the solution of $O(1)$ problem (2.23):

$$w_0(x,t,\tau)=B_M(\tau)\sin(\lambda_M t)\sin(\lambda_M x), \tag{2.61}$$

which implies that the solution of initial-boundary value problem (2.4) is of $O(1)$.
3.3 When $\omega=\lambda_M$ and $M=I(N)$, then A_n and B_n have to satisfy

$$\begin{aligned} 2\lambda_I\int_0^1 sin^2(\lambda_I x)dxA'_I(\tau)-\int_0^1 xsin(\lambda_I x)dx\beta_N B_N(\tau) &= 0, \\ 2\lambda_I\int_0^1 sin^2(\lambda_I x)dxB'_I(\tau)-\int_0^1 xsin(\lambda_I x)dx\beta_N A_N(\tau) &= \frac{Ac_I(1-cos\lambda_I)}{2\lambda_I^2\int_0^1 sin^2(\lambda_I x)dx}, \\ 2\lambda_N\int_0^1 sin^2(\lambda_N x)dxA'_N(\tau)-\int_0^1 xsin(\lambda_N x)dx\beta_I B_I(\tau) &= 0, \\ 2\lambda_N\int_0^1 sin^2(\lambda_N x)dxB'_N(\tau)-\int_0^1 xsin(\lambda_N x)dx\beta_I A_I(\tau) &= 0, \end{aligned} \tag{2.62}$$

which can be solved, yielding

$$A_I(\tau) = \frac{\sigma c_N}{2d_N\beta_I}B_N(0)[e^{\sigma\tau}-e^{-\sigma\tau}]+\frac{1}{2}A_I(0)[e^{\sigma\tau}+e^{-\sigma\tau}],$$

$$\begin{aligned} B_N(\tau) &= \frac{\sigma c_I}{2d_I\beta_N} A_I(0)[e^{\sigma\tau} - e^{-\sigma\tau}] + \frac{1}{2} B_N(0)[e^{\sigma\tau} + e^{-\sigma\tau}], \\ A_N(\tau) &= \frac{\sigma c_I}{2d_I\beta_N} B_N(0)[e^{\sigma\tau} - e^{-\sigma\tau}] + \frac{1}{2} A_I(0)[e^{\sigma\tau} + e^{-\sigma\tau}] \\ &+ \frac{Ac_I(1 - cos\lambda_I)}{2d_I\beta_N\lambda_I c_I}[e^{\sigma\tau} + e^{-\sigma\tau} - 2], \\ B_I(\tau) &= \frac{\sigma c_N}{2d_N\beta_I} A_N(0)[e^{\sigma\tau} - e^{-\sigma\tau}] + \frac{1}{2} B_I(0)[e^{\sigma\tau} + e^{-\sigma\tau}] \\ &+ \frac{A(1 - cos\lambda_I)}{2\sigma\lambda_I c_I}[e^{\sigma\tau} - e^{-\sigma\tau}], \end{aligned} \tag{2.63}$$

where σ, c_i, d_i can be obtained by (2.56).

$$A_n'(\tau) = B_n'(\tau) = 0, \quad n \neq I, N, \tag{2.64}$$

where $A_n(0)$ and $B_n(0)$ (n=1, 2, 3,...) are given by Eq.(2.26). Further when the initial conditions in (2.4) are of $O(\varepsilon)$, the solution of $O(1)$ problem (2.23):

$$w_0(x, t, \tau) = B_I(\tau)\sin(\lambda_I t) sin(\lambda_I x) + A_N(\tau)\cos(\lambda_N t)\sin(\lambda_N x), \tag{2.65}$$

which implies that the solution of initial-boundary value problem (2.4) is of $O(1)$.

The solution of Eq.(2.32) is given by a homogeneous solution and a particular one

$$y_n = C_n(\tau) cos(\lambda_n t) + D_n(\tau) sin(\lambda_n t) + E_n(t, \tau), \tag{2.66}$$

where

$$\begin{aligned} C_n(0) &= \frac{\int_0^1 [f_1(x) + \frac{k_1}{1+k_0} f_0(1)x] sin(\lambda_n x) dx}{\int_0^1 sin^2(\lambda_n x) dx}, \\ D_n(0) &= \frac{\int_0^1 [g_1(x) - w_{0,\tau}(x, 0, 0) + \frac{k_1}{1+k_0} g_0(1)x] sin(\lambda_n x) dx}{\lambda_n \int_0^1 sin^2(\lambda_n x) dx}, \\ E_n(t, \tau) &= cos(\lambda_n t) \int_0^t G_n(s, \tau) sin\lambda_n s ds + \frac{1}{\lambda_n} sin(\lambda_n t) \int_0^t G_n(s, \tau) cos\lambda_n s ds. \end{aligned}$$

The solution of the $O(\varepsilon)-$ problem now readily follows from Eq.(2.28), (2.31) and Eq.(2.66), yielding

$$\begin{aligned} w_1(x, t, \tau) &= \sum_{n=1}^{n=N} [C_n(\tau) cos(\lambda_n t) + D_n(\tau) sin(\lambda_n t) + E_n(t, \tau)] sin(\lambda_n x) \\ &- x \frac{k_1 cos(\overline{\omega} t) w_0(1, t, \tau)}{1 + k_0}. \end{aligned} \tag{2.67}$$

Threfore, we have constructed a formal approximation $u(x, t) = w(x, t, \tau) \backsim w_0(x, t, \tau) + \varepsilon w_1(x, t, \tau)$, where w_0 and w_1 are given by Eq.(2.24) and Eq.(2.67), respectively. Moreover, they are twice continuously differentiable with respect to x and t and infinitely many times with respect to τ. Only the first term in the expansion of the solution for the string problem is important from the physical point of view. We are not interested in high-order approximations; that is why we take $C_n(\tau) = C_n(0)$ and $D_n(\tau) = D_n(0)$ in Eq.(2.66). Otherwise, $C_n(\tau)$ and $D_n(\tau)$ can be found from the solvability conditions of the $O(\varepsilon^2)-$ problem.

Concluding, It follows from (2.35), (2.39), (2.48), (2.51), (2.54), (2.58), (2.61), (2.65) that when the initial conditions in (2.4) are of $O(\varepsilon)$, and the frequency of the boundary excitation $\omega = \lambda_i$, for a certain i = 1,2,3,..., solutions of initial boundary value problem (2.4) are increasing to $O(1)$ on timescales of $\frac{1}{\varepsilon}$, otherwise, the solution always keep $O(\varepsilon)$ on timescales of $\frac{1}{\varepsilon}$,

2.6.2. NUMERICAL EXAMPLES

As illustration of the obtained analytical results in section 2.6.1, we will now present some numerical results. The following parameter values are used: $A = 1$, $k_1 = 1$, $\varepsilon = 0.01$ and $k_0 = 1$. The initial conditions are assumed to be (2.17). There will be different behavior in the solution for different choices of the parameters ω and $\bar{\omega}$. In Figure 2.9 and Figure 2.10, we give an indication how the solutions can behave of the problem (2.4). Figure 2.9 (a) shows the solution behavior in case 1.1; Figure 2.9 (b) shows the solution behavior and resonance in case 1.2; Figure 2.10 (a) shows the solution behavior in case 2.3. Figure 2.10 (b) shows the solution behavior and resonance in case 3.3. These numerical examples are all in good agreement with those analytical results in section 2.6.1.

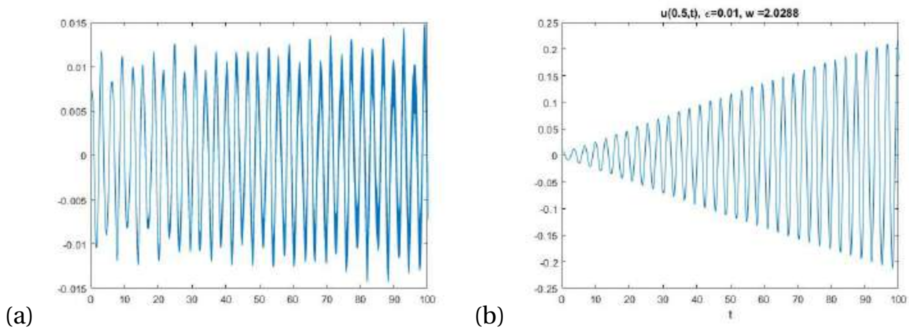

Figure 2.9: (a) The solution of the problem (2.4) for x=0.5, $\omega = \pi$, $\bar{\omega} = \frac{\pi}{3}$. (b)The solution of the problem (2.4) for x=0.5, $\omega = \lambda_1$,$\bar{\omega} = \frac{\pi}{3}$.

2.7. TIME-DEPENDENT COEFFICIENT $k(t) = 1 + \varepsilon t$

By putting $k(t) = 1 + \varepsilon t$ an additional difficulty is introduced: for $t < O(\frac{1}{\varepsilon})$, εt is a small term, while for $t = O(\frac{1}{\varepsilon})$, εt is not a small term. So, we need to analyse this problem from a new view-point. Firstly, since the coefficient changes slowly in time, we study the problem by an adapted version of the method of separation of variables, in which an extra independent slow time variable $\tau = \varepsilon t$ is defined, and $u(x, t)$ can be separated as $T(t, \tau)X(x, \tau)$. Then, by using the boundary conditions, the original partial differential

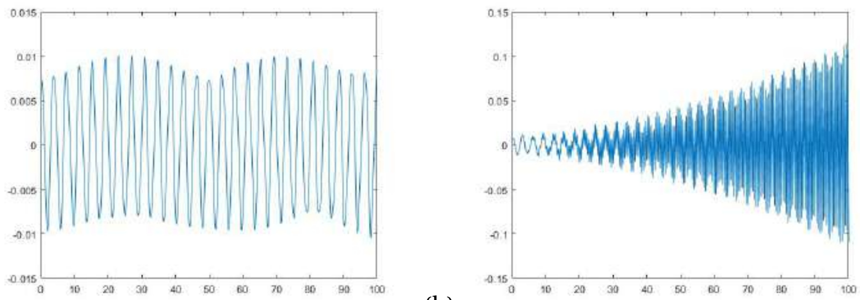

Figure 2.10: (a) The solution of the problem (2.4) for x=0.5, $\omega = \lambda_2, \bar{\omega} = \lambda_1 - \lambda_2$. (b)The solution of the problem (2.4) for x=0.5, $\omega = \lambda_1, \bar{\omega} = \lambda_1 + \lambda_2$.

equation can be transformed into linear ordinary differential equations with slowly varying (prescribed) frequencies. Unexpectedly (or not), the slow variation leads to a singular perturbation problem. By applying an interior layer analysis in the averaging procedure a resonance manifold is found. Three different scalings turn out to be present in the problem. For that reason, a three-timescales perturbation method is used to construct explicit approximations of the solutions of the initial-boundary value problem (2.4).

2.7.1. AN ADAPTED VERSION OF THE METHOD OF SEPARATION OF VARIABLES

FIRST of all, in the method of separation of variables we consider the homogeneous part of equation (2.4) subject to the homogeneous boundary conditions:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & 0 < x < 1, \quad t \geq 0, \\ u(0,t) = 0, \quad u_x(1,t) + k(t)u(1,t) = 0, & k(t) = 1 + \varepsilon t, \quad t \geq 0. \end{cases} \tag{2.68}$$

Note that the coefficient $k(t)$ in the Robin boundary condition is slowly varying in time. So, in order to derive a solution of problem (2.68) , we define an extra slow time variable $\tau = \varepsilon t$, which will be treated independently from the variable t. Hence $u(x,t)$ becomes a new function $\bar{u}(x,t,\tau)$ and further problem (2.68) becomes

$$\begin{aligned} &\bar{u}_{tt}(x,t,\tau) + 2\varepsilon \bar{u}_{t\tau}(x,t,\tau) + \varepsilon^2 \bar{u}_{\tau\tau}(x,t,\tau) - \bar{u}_{xx}(x,t,\tau) = 0, \\ &\qquad\qquad 0 < x < 1, \quad t > 0, \quad \tau > 0, \\ &\bar{u}(0,t,\tau) = 0, \quad \bar{u}_x(1,t,\tau) + (1+\tau)\bar{u}(1,t,\tau) = 0, \quad t \geq 0, \quad \tau \geq 0. \end{aligned} \tag{2.69}$$

By looking for a nontrivial solution $\bar{u}(x,t,\tau)$ in the form $T(t,\tau)X(x,\tau)$, the governing equations of (2.69) can be approximately written as

$$\begin{aligned} X(x,\tau)T_{tt}(t,\tau) + 2\varepsilon X(x,\tau)T_{t\tau}(t,\tau) + 2\varepsilon X_\tau(x,\tau)T_t(t,\tau) & \\ -X_{xx}(x,\tau)T(t,\tau) + O(\varepsilon^2) &= 0, \end{aligned}$$

or equivalently as

$$\frac{T_{tt}(t,\tau)}{T(t,\tau)} + O(\varepsilon) = \frac{X_{xx}(x,\tau)}{X(x,\tau)}, \quad 0 < x < 1, \quad t \geq 0, \quad \tau \geq 0. \tag{2.70}$$

The $O(1)$ part of the left-hand side of equation (2.70) is a function of t and τ, and the right-hand side is a function of x and τ. To be equal, both sides need to be equal to a function of τ. Let this function be $-\lambda^2(\tau)$ (which will be defined later), so we get

$$\frac{T_{tt}(t,\tau)}{T(t,\tau)} = \frac{X_{xx}(x,\tau)}{X(x,\tau)} = -\lambda^2(\tau), \; 0 < x < 1, \quad t \geq 0, \quad \tau \geq 0,$$

implying:

$$\begin{aligned} &X_{xx}(x,\tau) + \lambda^2(\tau)X(x,\tau) = 0, \; 0 < x < 1, \quad \tau \geq 0, \\ &T_{tt}(t,\tau) + \lambda^2(\tau)T(t,\tau) = 0, \; t \geq 0, \quad \tau \geq 0. \end{aligned} \tag{2.71}$$

From the boundary condition (2.69), we obtain

$$\begin{aligned} &T(t,\tau)X(0,\tau) = 0 \quad \Rightarrow \quad X(0,\tau) = 0, \\ &T(t,\tau)X_x(1,\tau) + (1+\tau)T(t,\tau)X(1,\tau) = 0 \\ &\qquad\qquad \Rightarrow \quad X_x(1,\tau) + (1+\tau)X(1,\tau) = 0. \end{aligned} \tag{2.72}$$

In accordance with the first equation for $X(x,\tau)$ in (2.71), a nontrivial solution $X_n(x,\tau)$ (satisfying (2.72)) is

$$X_n(x,\tau) = B_n(\tau)\sin(\lambda_n(\tau)x), \tag{2.73}$$

where $B_n(\tau)$ is a function of τ only, and $\lambda_n(\tau)$ is the n-th positive root of

$$\tan(\lambda_n(\tau)) = -\frac{\lambda_n(\tau)}{1+\tau}. \tag{2.74}$$

For $\tau = 0$ it is indicated in Figure 2.11 how $\lambda_n(0)$ can be obtained. In should be observed that the eigenfunctions $X_n(x,\tau)$ are orthogonal on $0 < x < 1$. And so, the general solution

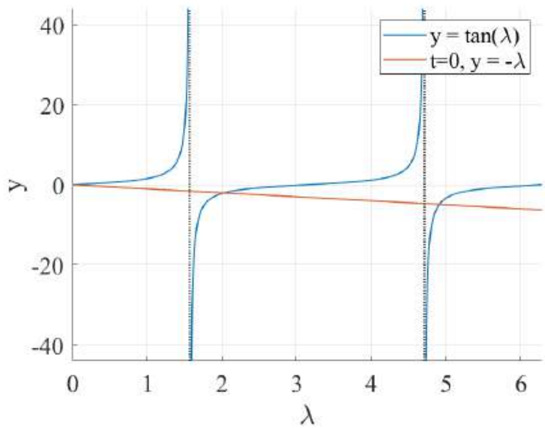

Figure 2.11: For t=0, intersection points of $y = \tan\lambda$ and $y = -\lambda$ are giving $\lambda_n(0)$.

of (2.4) can be expanded in the following form:

$$u(x,t) = \bar{u}(x,t,\tau) = \sum_{n=1}^{\infty} T_n(t,\tau)\sin(\lambda_n(\tau)x), \tag{2.75}$$

where the boundary conditions are automatically satisfied.
Substituting Eq.(2.75) into equation (2.4) yields

$$\begin{aligned}
\sum_{n=1}^{\infty}[(T_{n,tt}+2\varepsilon T_{n,t\tau}+\lambda_n^2(\tau)T_n)\sin(\lambda_n(\tau)x) \\
+2\varepsilon x\frac{d\lambda_n(\tau)}{d\tau}T_{n,t}\cos(\lambda_n(\tau)x)] &= \varepsilon A\cos(\omega t)+O(\varepsilon^2), \\
\sum_{n=1}^{\infty}[T_n(0,0)\sin(\lambda_n(0)x)] &= \varepsilon u_0(x), \\
\sum_{n=1}^{\infty}[(T_{n,t}(0,0)+\varepsilon T_{n,\tau}(0,0))\sin(\lambda_n(0)x) \\
+\varepsilon T_n(0,0)\frac{d\lambda_n(0)}{d\tau}x\cos(\lambda_n(0)x)] &= \varepsilon u_1(x).
\end{aligned} \tag{2.76}$$

Now, by multiplying the first equation in (2.76) with $\sin(\lambda_k(\tau)x)$, and the second and third equations in (2.76) with $\sin(\lambda_k(0)x)$, by integrating the so-obtained equation from $x=0$ to $x=1$, and by using the orthogonality properties of the sin-functions on $0<x<1$, we obtain the following differential equations for $k=1,2,3,...,$ and $t>0, \tau>0$:

$$\begin{cases}
T_{k,tt}+\lambda_k^2(\tau)T_k=\varepsilon[-2T_{k,t\tau}-2\sum_{n=1}^{\infty}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)T_{n,t}+Ad_k(\tau)cos(\omega t)] \\
\qquad +O(\varepsilon^2),\quad t\geq 0,\quad \tau\geq 0, \\
T_k(0,0)=\varepsilon\frac{\int_0^1 u_0(\xi)\sin(\lambda_k(0)\xi)d\xi}{\int_0^1\sin(\lambda_k(0)\xi)\sin(\lambda_k(0)\xi)d\xi}=\varepsilon F_k, \\
T_{k,t}(0,0)+\varepsilon T_{k,\tau}(0,0)=\varepsilon\frac{\int_0^1 u_1(\xi)\sin(\lambda_k(0)\xi)d\xi}{\int_0^1\sin(\lambda_k(0)\xi)\sin(\lambda_k(0)\xi)d\xi} \\
\qquad -\varepsilon\sum_{n=1}^{\infty}T_n(0,0)\frac{d\lambda_n(0)}{d\tau}\frac{\int_0^1\xi\cos(\lambda_n(0)\xi)\sin(\lambda_k(0)\xi)d\xi}{\int_0^1\sin(\lambda_k(0)\xi)\sin(\lambda_k(0)\xi)d\xi} \\
\qquad =\varepsilon G_k.
\end{cases} \tag{2.77}$$

where $c_{n,k}(\tau)$ and $d_k(\tau)$ are functions of τ, and are given by:

$$\begin{aligned}
c_{n,k}(\tau) &= \frac{\int_0^1 x\cos(\lambda_n(\tau)x)\sin(\lambda_k(\tau)x)dx}{\int_0^1\sin^2(\lambda_k(\tau)x)dx}, \\
c_{k,k}(\tau) &= \frac{\int_0^1 x\cos(\lambda_k(\tau)x)\sin(\lambda_k(\tau)x)dx}{\int_0^1\sin^2(\lambda_k(\tau)x)dx}, \\
d_k(\tau) &= \frac{\int_0^1\sin(\lambda_k(\tau)x)dx}{\int_0^1\sin^2(\lambda_k(\tau)x)dx}=\frac{4(1-\cos(\lambda_k(\tau)))}{2\lambda_k(\tau)-\sin(2\lambda_k(\tau))}.
\end{aligned} \tag{2.78}$$

To simplify the formula, we define a new dependent variable: $\tilde{T}_k(t)=T_k(t,\tau)$, yielding

$$\begin{cases}
\tilde{T}_{k,tt}+\lambda_k^2(\tau)\tilde{T}_k=\varepsilon[-2\Sigma_{n=1}^{\infty}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)\tilde{T}_{n,t}+Ad_k(\tau)\cos(\omega t)]+O(\varepsilon^2), \\
\tilde{T}_k(0)=\varepsilon F_k, \\
\tilde{T}_{k,t}(0)=\varepsilon G_k,
\end{cases} \tag{2.79}$$

where $\tau=\varepsilon t$, $t\geq 0$. In the next section we will use the averaging method to detect resonance zones in problem (2.79), and to determine time-scales which describe the solution of (2.79) accurately.

2.7.2. Averaging and resonance zones

The linear ordinary differential equation (2.79) with the slowly varying frequency $\lambda_k(\tau)$ as given by (2.74), can be analysed by making use of the averaging method. In this section it will be shown that an interior layer analysis (including a rescaling and balancing procedure) leads to a description of an (un-)expected resonance manifold and leads to time-scales which describe the solution of (2.79) sufficiently accurately. To apply the method of averaging to (2.79) the following standard transformations are introduced:

$$\phi_k(t) = \int_0^t \lambda_k(\varepsilon s)ds \quad \text{and} \quad \Phi = \omega t, \tag{2.80}$$

and $\tilde{T}_k(t), \tilde{T}_{k,t}(t)$ are described by $A_k(t), B_k(t)$ in the following way:

$$\begin{aligned} \tilde{T}_k(t) &= A_k(t)\sin(\phi_k(t)) + B_k(t)\cos(\phi_k(t)), \\ \tilde{T}_{k,t}(t) &= \lambda_k(\tau)A_k(t)\cos(\phi_k(t)) - \lambda_k(\tau)B_k(t)\sin(\phi_k(t)). \end{aligned} \tag{2.81}$$

Problem (2.79) can now be rewritten in the following problem:

$$\begin{cases} \dot{A}_k = \varepsilon[-\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{\lambda_k(\tau)} + 2c_{k,k}(\tau))A_k\cos^2(\phi_k(t)) \\ \qquad +\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))B_k\sin(2\phi_k(t)) \\ \qquad -2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)A_n\cos(\phi_n(t))\cos(\phi_k(t)) \\ \qquad +2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)B_n\sin(\phi_n(t))\cos(\phi_k(t)) \\ \qquad +\frac{Ad_k(\tau)}{2\lambda_k(\tau)}(\cos(\Phi+\phi_k(t)) + \cos(\Phi-\phi_k(t)))], \\ \dot{B}_k = \varepsilon[\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))A_k\sin(2\phi_k(t)) \\ \qquad -\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{\lambda_k(\tau)} + 2c_{k,k}(\tau))B_k\sin^2(\phi_k(t)) \\ \qquad +2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)A_n\cos(\phi_n(t))\sin(\phi_k(t)) \\ \qquad -2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)B_n\sin(\phi_n(t)t)\sin(\phi_k(t)) \\ \qquad -\frac{Ad_k(\tau)}{2\lambda_k(\tau)}(\sin(\Phi+\phi_k(t)) - \sin(\Phi-\phi_k(t)))], \\ \dot{\tau} = \varepsilon, \\ \dot{\Phi} = \omega, \\ \dot{\phi}_k = \lambda_k(\tau). \end{cases} \tag{2.82}$$

Resonance in (2.82), due to the external forcing with frequency ω, can be expected when $\dot{\Phi} - \dot{\phi}_k \approx 0$, or $\dot{\Phi} + \dot{\phi}_k \approx 0$. But since $\omega > 0$ and $\lambda_k(\tau) > 0$, resonance only will occur when

$$\omega \approx \lambda_k(\tau) \Leftrightarrow \tau \approx -\frac{\omega}{\tan\omega} - 1 \Leftrightarrow t \approx \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega} - 1). \tag{2.83}$$

Since $\lambda_k(\tau)$ satisfies (2.11), that is, $\tan(\lambda_k(\tau)) = -\frac{\lambda_k(\tau)}{1+\tau}$, it follows (see also Figure 2.11) that when t increases, then the value of $\lambda_k(\tau)$ increases. Besides, for t tending to infinity, $\lambda_k(\tau)$ tends to $k\pi$ (with $k = 1, 2, \ldots$). Therefore, $\lambda_k(\tau)$ is increasing in time and

$$0 < \lambda_k(0) \leq \lambda_k(\tau) < k\pi, \qquad \tan(\lambda_k(0)) = -\lambda_k(0). \tag{2.84}$$

From (2.84) we can then conclude that:

1. When the external force frequency ω is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \le \omega < \lambda_k(0)$, then no resonance will occur;
2. When the external force frequency ω satisfies $0 < \lambda_k(0) \le \omega < k\pi$, then resonance will occur around $t = \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega} - 1)$. Moreover, for $-\frac{\omega}{\tan\omega} - 1 > 0$, and $-\frac{\omega}{\tan\omega} - 1 = O(\varepsilon)$, the resonance time zone is around $\tilde{t}$ with $\tilde{t} = O(1)$; and when $-\frac{\omega}{\tan\omega} - 1 = O(1)$, the resonance time zone is around $\tilde{t}$ with $\tilde{t} = O(\frac{1}{\varepsilon})$.

2

As long as we stay out of the resonance time zone (or equivalently, the resonance manifold), the variables A_k and B_k are slowly varying in time. For that reason we can average the right-hand side of the equations in (2.82) over ϕ_k and Φ while keeping A_k and B_k constant. The averaged equation for A_k and B_k now become

$$\begin{cases} \dot{A}_k^a = -\varepsilon\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))A_k^a, \\ \dot{B}_k^a = -\varepsilon\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))B_k^a, \end{cases} \tag{2.85}$$

where the upper index a indicates that this is the averaged function. From the expression for $c_{k,k}$ in (2.78), we then obtain

$$\begin{aligned} \dot{A}_k^a &= -\varepsilon\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + \frac{\int_0^1 x\cos(\lambda_k(\tau)x)\sin(\lambda_k(\tau)x)dx}{\int_0^1 \sin^2(\lambda_k(\tau)x)dx})A_k^a \\ &= -\frac{\varepsilon}{2}\frac{d\lambda_k(\tau)}{d\tau}(\frac{d(\ln(\lambda_k(\tau)))}{d\lambda_k(\tau)} + \frac{d(\ln(\int_0^1 \sin^2(\lambda_k(\tau)x)dx))}{d\lambda_k(\tau)})A_k^a \\ &= -\frac{\varepsilon}{2}\frac{d\lambda_k(\tau)}{d\tau}(\frac{d(\ln(\lambda_k(\tau)))}{d\lambda_k(\tau)} + \frac{d(\ln(\frac{1}{2} - \frac{\sin(2\lambda_k(\tau))}{4\lambda_k(\tau)}))}{d\lambda_k(\tau)})A_k^a \\ &= -\frac{1}{2}(\frac{d(\ln(\frac{\lambda_k(\tau)}{2} - \frac{\sin(2\lambda_k(\tau))}{4}))}{dt})A_k^a, \end{aligned} \tag{2.86}$$

which implies that

$$A_k^a = \frac{C_1}{\sqrt{2\lambda_k(\tau) - \sin(2\lambda_k(\tau))}}, \quad B_k^a = \frac{C_2}{\sqrt{2\lambda_k(\tau) - \sin(2\lambda_k(\tau))}}, \tag{2.87}$$

with

$$C_1 = \frac{\varepsilon G_k\sqrt{2\lambda_k(0) - \sin(2\lambda_k(0))}}{\lambda_k(0)}, \quad C_2 = \varepsilon F_k\sqrt{2\lambda_k(0) - \sin(2\lambda_k(0))}, \tag{2.88}$$

where G_k and F_k are given in (2.77).
Hence, outside the resonance manifold the solution of system (2.79) is given by

$$\begin{aligned} \tilde{T}_k(t) &= \frac{\varepsilon G_k\sqrt{2\lambda_k(0) - \sin(2\lambda_k(0))}}{\lambda_k(0)\sqrt{2\lambda_k(\tau) - \sin(2\lambda_k(\tau))}}\sin(\phi_k(t)) \\ &+ \frac{\varepsilon F_k\sqrt{2\lambda_k(0) - \sin(2\lambda_k(0))}}{\sqrt{2\lambda_k(\tau) - \sin(2\lambda_k(\tau))}}\cos(\phi_k(t)), \end{aligned} \tag{2.89}$$

where $\phi_k(t)$ is defined in (2.80).

2

When ω satisfies $\lambda_k(0) \leq \omega < k\pi$ for a certain k (with $k = 1, 2, ...$) then a resonance zone will occur. We introduce

$$\psi = \Phi(t) - \phi_k(t), \tag{2.90}$$

and rescale $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$ with $\bar{\tau} = O(1)$ and $\tau_k = -\frac{\omega}{\tan\omega} - 1$. System (2.82) then becomes:

$$\begin{cases} \dot{A}_k = \varepsilon[-\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{\lambda_k(\tau)} + 2c_{k,k}(\tau))A_k\cos^2(\phi_k(t)) \\ \quad +\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))B_k\sin(2\phi_k(t)) \\ \quad -2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)A_n\cos(\phi_n(t))\cos(\phi_k(t)) \\ \quad +2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)B_n\sin(\phi_n(t))\cos(\phi_k(t)) \\ \quad +\frac{Ad_k(\tau)}{2\lambda_k(\tau)}(\cos(\Phi+\phi_k(t)) + \cos(\psi)], \\ \dot{B}_k = \varepsilon[\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))A_k\sin(2\phi_k(t)) \\ \quad -\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{\lambda_k(\tau)} + 2c_{k,k}(\tau))B_k\sin^2(\phi_k(t)) \\ \quad +2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)A_n\cos(\phi_n(t)t)\sin(\phi_k(t)) \\ \quad -2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)B_n\sin(\phi_n(t))\sin(\phi_k(t)) \\ \quad -\frac{Ad_k(\tau)}{2\lambda_k(\tau)}(\sin(\Phi+\phi_k(t)) - \sin(\psi)], \\ \dot{\tau} = \varepsilon, \\ \dot{\Phi} = \omega, \\ \dot{\bar{\tau}} = \frac{\varepsilon}{\delta(\varepsilon)}, \\ \dot{\psi} = \omega - \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}). \end{cases} \tag{2.91}$$

To simplify (2.91) it should be observed that for $\tau = \tau_k + \delta(\varepsilon)\bar{\tau}$ we have

$$\begin{aligned} \dot{\psi} &= \omega - \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}) = \lambda_k(\tau_k) - \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}) \\ &= \lambda_k(\tau_k) - (\lambda_k(\tau_k) + \delta(\varepsilon)\bar{\tau}\frac{d\lambda_k}{d\tau}|_{\tau=\tau_k} + O(\delta^2(\varepsilon))) \\ &= -\delta(\varepsilon)\bar{\tau}\frac{d\lambda_k}{d\tau}|_{\tau=\tau_k} + O(\delta^2(\varepsilon)). \end{aligned} \tag{2.92}$$

By differentiating (2.74), that is, $\tan(\lambda_k(\tau)) = -\frac{\lambda_k(\tau)}{1+\tau}$ with respect to τ, we obtain

$$\frac{1}{\cos^2\lambda_k}\frac{d\lambda_k}{d\tau} = \frac{-1}{1+\tau}\frac{d\lambda_k}{d\tau} + \frac{\lambda_k}{(1+\tau)^2}. \tag{2.93}$$

And so, it follows from (2.93) that

$$\begin{aligned} \frac{d\lambda_k}{d\tau}|_{\tau=\tau_k} &= \frac{\lambda_k}{(1+\tau)}\cdot\frac{\cos^2\lambda_k}{1+\tau+\cos^2\lambda_k}|_{\tau=\tau_k} \\ &= -\frac{\sin 2\omega}{2(1+\tau_k+\cos^2\omega)} = \frac{\sin^2\omega}{\omega - \sin\omega\cos\omega}, \end{aligned} \tag{2.94}$$

which implies for $\dot{\psi}$ (see (2.92)):

$$\dot{\psi} = -\frac{\sin^2\omega}{\omega - \sin\omega\cos\omega}\delta(\varepsilon)\bar{\tau} + O(\delta^2(\varepsilon)). \tag{2.95}$$

From (2.94) and from $\lambda_k(0) \le \omega < k\pi$, we obtain $\frac{\sin^2\omega}{\omega-\sin\omega\cos\omega} \neq 0$.

Based on (2.95) it now follows from (2.95) that a balance in system (2.91) occurs when $\frac{\varepsilon}{\delta(\varepsilon)} = \delta(\varepsilon)$, and this implies for the averaging procedure in the resonance zone that $\delta(\varepsilon) = \sqrt{\varepsilon}$. So, together with $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$, it follows from (2.91) that

$$\bar{\tau} = \sqrt{\varepsilon}(t-t_k), \qquad t_k = \frac{\tau_k}{\varepsilon}. \tag{2.96}$$

2

Further, from (2.95), we obtain

$$\psi(t) = \psi(t_k) - \frac{1}{2}\alpha\varepsilon(t-t_k)^2, \quad \alpha = \frac{\sin^2\omega}{\omega - \sin\omega\cos\omega}. \tag{2.97}$$

Hence, in the resonance zone, we can write

$$\cos(\psi(t)) = \cos(-\frac{1}{2}\alpha\varepsilon(t-t_k)^2 + \omega t_k - \phi_k(t_k)), \quad t_k = -\frac{1}{\varepsilon}(\frac{\omega}{\tan\omega}+1). \tag{2.98}$$

So taking into account (2.96), let us average system (2.91) over the fast variables. Then, the averaged equations for A_k and B_k become

$$\begin{cases} \dot{A}_k^a = -\varepsilon\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))A_k^a + \varepsilon\frac{Ad_k(\tau)}{2\lambda_k(\tau)}\cos(\psi), \\ \dot{B}_k^a = -\varepsilon\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))B_k^a + \varepsilon\frac{Ad_k(\tau)}{2\lambda_k(\tau)}\sin(\psi), \end{cases} \tag{2.99}$$

where the upper index a indicates that this is the averaged function.
It follows from (2.98) and (2.99) that A_k^a can be written as

$$A_k^a = \frac{C_1}{l_k(\varepsilon t)} + \frac{A\varepsilon}{l_k(\varepsilon t)}\int_0^t h_k(\varepsilon\bar{t})\cos[-\frac{1}{2}\alpha\varepsilon(\bar{t}-t_k)^2 + \omega t_k - \phi_k(t_k)]d\bar{t}, \tag{2.100}$$

where C_1 is given by (2.88) and

$$l_k(\varepsilon t) = \sqrt{2\lambda_k(\varepsilon t) - \sin(2\lambda_k(\varepsilon t))}, \tag{2.101}$$

$$h_k(\varepsilon\bar{t}) = \frac{l_k(\varepsilon\bar{t})}{2\lambda_k(\varepsilon\bar{t})}d_k(\varepsilon\bar{t}) = \frac{2(1-\cos(\lambda_k(\varepsilon\bar{t})))}{\lambda_k(\varepsilon\bar{t})\sqrt{2\lambda_k(\varepsilon\bar{t}) - \sin(2\lambda_k(\varepsilon\bar{t}))}}. \tag{2.102}$$

For $\bar{t} = t_k + O(\frac{1}{\sqrt{\varepsilon}})$, $\tau_k = \varepsilon t_k$,

$$\begin{aligned} h_k(\varepsilon\bar{t}) &= h_k(\varepsilon t_k + O(\sqrt{\varepsilon})) = h_k(-\frac{\omega}{\tan\omega} - 1 + O(\sqrt{\varepsilon})) \\ &= h_k(-\frac{\omega}{\tan\omega} - 1) + O(\sqrt{\varepsilon})\cdot\frac{dh_k(a)}{da}|_{a=\tau_k} + \text{h.o.t}, \end{aligned} \tag{2.103}$$

where $\frac{dh_k(a)}{da}|_{a=\tau_k}$ is bounded due to (2.94). Then,

$$A_k^a = \frac{C_1}{l_k(\varepsilon t)} + \frac{\varepsilon A h_k(-\frac{\omega}{\tan\omega}-1)}{l_k(\varepsilon t)}\int_0^t \cos[-\frac{1}{2}\alpha\varepsilon(\bar{t}-t_k)^2 + \omega t_k - \phi_k(t_k)]d\bar{t} + \text{h.o.t}, \tag{2.104}$$

2

where C_1 is given by (2.88). We know that $\frac{C_1}{l_k(\varepsilon t)} = O(\varepsilon)$ and $\frac{Ah_k(-\frac{\omega}{\tan\omega}-1)}{l_k(\varepsilon t)} = O(1)$, and so it is important to consider the order of

$$\varepsilon \int_0^t \cos[-\frac{1}{2}\alpha\varepsilon(t-t_k)^2 + \omega t_k - \phi_k(t_k)]d\bar{t}. \tag{2.105}$$

By setting $u = \sqrt{\frac{1}{2}\alpha\varepsilon(t-t_k)}$, we obtain

$$\begin{aligned} &\varepsilon \int_0^t \cos[-\frac{1}{2}\alpha\varepsilon(t-t_k)^2 + \omega t_k - \phi_k(t_k)]d\bar{t} \\ = &\sqrt{\frac{2\varepsilon}{\alpha}} \int_{\sqrt{\frac{\alpha\varepsilon}{2}}(-t_k)}^{\sqrt{\frac{\alpha\varepsilon}{2}}(t-t_k)} \cos(-u^2 + \omega t_k - \phi_k(t_k))du \\ = &\sqrt{\frac{2\varepsilon}{\alpha}} \cos(\omega t_k - \phi_k(t_k))C_{Fr}(t) + \sqrt{\frac{2\varepsilon}{\alpha}} \sin(\omega t_k - \phi_k(t_k))S_{Fr}(t), \end{aligned} \tag{2.106}$$

where

$$C_{Fr}(t) = \int_{\sqrt{\frac{\alpha\varepsilon}{2}}(-t_k)}^{\sqrt{\frac{\alpha\varepsilon}{2}}(t-t_k)} \cos(u^2)du,\ S_{Fr}(t) = \int_{\sqrt{\frac{\alpha\varepsilon}{2}}(-t_k)}^{\sqrt{\frac{\alpha\varepsilon}{2}}(t-t_k)} \sin(u^2)du,\ t_k = \frac{\tau_k}{\varepsilon}. \tag{2.107}$$

When $0 \le t_k \le O(\frac{1}{\sqrt{\varepsilon}})$,

$$\begin{cases} 0 \le C_{Fr}(t) \le O(1), & 0 \le t \le t_k + O(\frac{1}{\sqrt{\varepsilon}}); \\ C_{Fr}(t) = O(1), & t > t_k + O(\frac{1}{\sqrt{\varepsilon}}). \end{cases} \tag{2.108}$$

When $t_k > O(\frac{1}{\sqrt{\varepsilon}})$,

$$\begin{cases} C_{Fr}(t) = O(\sqrt{\varepsilon}), & 0 \le t < t_k - O(\frac{1}{\sqrt{\varepsilon}}); \\ O(\sqrt{\varepsilon}) \le C_{Fr}(t) \le O(1), & t_k - O(\frac{1}{\sqrt{\varepsilon}}) \le t \le t_k + O(\frac{1}{\sqrt{\varepsilon}}); \\ C_{Fr}(t) = O(1), & t > t_k + O(\frac{1}{\sqrt{\varepsilon}}). \end{cases} \tag{2.109}$$

In the same way, $S_{Fr}(t,\hat{\alpha})$ also satisfies (2.108) and (2.109). So, the presence of the functions $C_{Fr}(t)$ and $C_{Fr}(t)$ causes resonance jumps in the system. $C_{Fr}(t)$ and $S_{Fr}(t)$ are plotted for $\varepsilon = 0.01$, $\omega = 2.2889$, and so $t_k = 100$ in Figure 2.12.

Above all, it follows from (2.104) that

$$\begin{aligned} A_k \quad = \quad & \frac{2\sqrt{2\varepsilon}A(1-\cos(\lambda_k(\varepsilon t)))}{\sqrt{\alpha}\lambda_k(\varepsilon t)(2\lambda_k(\varepsilon t) - \sin(2\lambda_k(\varepsilon t)))}\Big|_{t=-\frac{\omega}{\varepsilon\tan\omega}-\frac{1}{\varepsilon}} \\ & \cdot[\cos(\omega t_k - \phi_k(t_k))C_{Fr}(t) + \sin(\omega t_k - \phi_k(t_k))S_{Fr}(t)] + O(\varepsilon), \end{aligned} \tag{2.110}$$

and

$$\begin{cases} A_k = O(\varepsilon), & 0 \le t < t_k - O(\frac{1}{\sqrt{\varepsilon}}); \\ O(\varepsilon) \le A_k \le O(\sqrt{\varepsilon}), & t_k - O(\frac{1}{\sqrt{\varepsilon}}) \le t \le t_k + O(\frac{1}{\sqrt{\varepsilon}}); \\ A_k = O(\sqrt{\varepsilon}), & t > t_k + O(\frac{1}{\sqrt{\varepsilon}}). \end{cases} \tag{2.111}$$

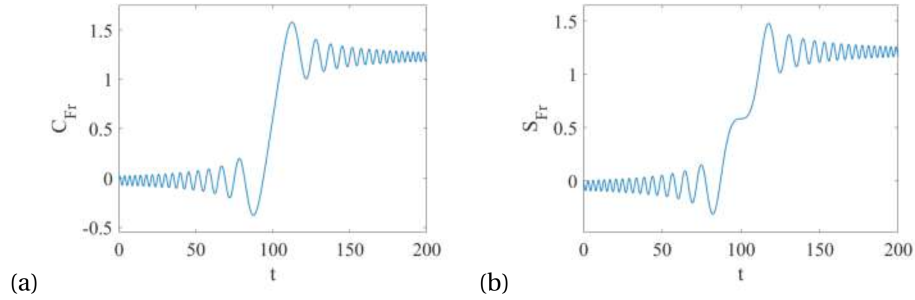

Figure 2.12: (a) $C_{Fr}(t)$ (b)$S_{Fr}(t)$ have a resonance jump from $O(\sqrt{\varepsilon})$ to $O(1)$ around t=100.

Similarly, B_k also satisfies (2.111). So, in the resonance zone,

$$\begin{aligned}
& \tilde{T}_k(t) \\
= \; & \sqrt{\varepsilon}M_1[(\cos(\omega t_k - \phi_k(t_k))C_{Fr}(t) + \sin(\omega t_k - \phi_k(t_k))S_{Fr}(t))\sin(\phi_k(t)) \\
+ \; & (\sin(\omega t_k - \phi_k(t_k))C_{Fr}(t) - \cos(\omega t_k - \phi_k(t_k))S_{Fr}(t))\cos(\phi_k(t))] + O(\varepsilon) \\
= \; & \sqrt{\varepsilon}M_1[(\cos(\omega t_k)C_{Fr}(t) + \sin(\omega t_k)S_{Fr}(t))\sin(\int_{t_k}^{t}\lambda_k(\varepsilon\bar{t})d\bar{t}) \\
+ \; & (\sin(\omega t_k)C_{Fr}(t) - \cos(\omega t_k)S_{Fr}(t))\cos(\int_{t_k}^{t}\lambda_k(\varepsilon\bar{t})d\bar{t})] + O(\varepsilon),
\end{aligned} \tag{2.112}$$

where $C_{Fr}(t)$ and $S_{Fr}(t)$ are given in (2.107), and

$$M_1 = \frac{2\sqrt{2}A(1-\cos(\lambda_k(\varepsilon t)))}{\sqrt{\alpha}\lambda_k(\varepsilon t)(2\lambda_k(\varepsilon t) - \sin(2\lambda_k(\varepsilon t)))}\Big|_{t=-\frac{\omega}{\varepsilon\tan\omega}-\frac{1}{\varepsilon}}. \tag{2.113}$$

Hence, when the external force frequency ω is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \le \omega < \lambda_k(0)$, for all k=1,2,..., (where $\lambda_k(0)$ satisfies (2.84)), then no resonance will occur, and for an $O(\varepsilon)$ external excitation there is only an $O(\varepsilon)$ response, which is described in detail in (2.89). When the external force frequency ω satisfies $\lambda_k(0) \le \omega < k\pi$, for a certain k (with k=1,2,...) and $\lambda_k(0)$ satisfies (2.84), then a resonance will occur for t near $-\frac{\omega}{\varepsilon\tan(\omega)} - \frac{1}{\varepsilon}$, and for the $O(\varepsilon)$ external excitation an $O(\sqrt{\varepsilon})$ amplitude response will occur, which is described in detail in (3.33).

In the next section, the occurrence of the (un)expected timescales will be used to construct accurate approximation of the solution for problem (2.76) and for the original problem (2.68) when a resonance zone exists. When a resonance zone does not exist then the solution of problem (2.68) will remain $O(\varepsilon)$ for $t = O(\varepsilon^{-1})$.

2.7.3. THREE-TIMESCALES PERTURBATION METHOD

IN the section 2.7.2, it was shown that (under certain condition on the external frequency ω) resonance can occur around time $t = \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega} - 1)$. For this reason, we rescale t by defining $t = \tilde{t} + \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega} - 1)$, and $\tau = \varepsilon\tilde{t} - \frac{\omega}{\tan\omega} - 1$. Thus, problem (2.79) can be rewritten in $\tilde{t}$ as follows

$$\begin{cases} \tilde{T}_{k,\tilde{t}\tilde{t}} + \lambda_k^2(\tau)\tilde{T}_k = \varepsilon[-2\Sigma_{n=1}^{n=\infty}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)\tilde{T}_{n,\tilde{t}} \\ \qquad\qquad +Ad_k(\tau)\cos(\omega\tilde{t}+\frac{\omega}{\varepsilon}(-\frac{\omega}{\tan\omega}-1))] + O(\varepsilon^2), \\ \tilde{T}_k(\frac{1}{\varepsilon}(\frac{\omega}{\tan\omega}+1)) = \varepsilon F_k, \\ \tilde{T}_{k,\tilde{t}}(\frac{1}{\varepsilon}(\frac{\omega}{\tan\omega}+1)) = \varepsilon G_k. \end{cases} \tag{2.114}$$

We study problem (2.114) in detail under the assumption that ω is such that a resonance zone exits for the k^{th} oscillation mode. The application of the straightforward expansion method to solve (2.114) will result in the occurrence of so-called secular terms which cause the approximations of the solutions to become unbounded on long timescales. For this reason, to remove secular terms, we introduce three timescales $t_0 = \tilde{t}$, $t_1 = \sqrt{\varepsilon}\tilde{t}$, $t_2 = \varepsilon\tilde{t}$. The time-scale $t_1 = \sqrt{\varepsilon}\tilde{t}$ is introduced because of the size of the resonance zone which has been found in the previous section, and the other two time-scales are the natural scalings for weakly nonlinear equations such as (2.114). By using the three timescales perturbation method, the function $\tilde{T}_k(\tilde{t};\sqrt{\varepsilon})$ is supposed to be a function of t_0, t_1 and t_2,

$$\tilde{T}_k(\tilde{t};\sqrt{\varepsilon}) = w_k(t_0,t_1,t_2;\sqrt{\varepsilon}). \tag{2.115}$$

By substituting (2.115) into (2.114), we obtain the following equations up to $O(\varepsilon\sqrt{\varepsilon})$:

$$\begin{cases} \frac{\partial^2 w_k}{\partial t_0^2} + \lambda_k^2(t_2)w_k + 2\sqrt{\varepsilon}\frac{\partial^2 w_k}{\partial t_0\partial t_1} + \varepsilon(2\frac{\partial^2 w_k}{\partial t_0\partial t_2} + \frac{\partial^2 w_k}{\partial t_1^2}) + 2\varepsilon\sqrt{\varepsilon}\frac{\partial^2 w_k}{\partial t_1\partial t_2} \\ = \varepsilon[-2\Sigma_{n=1}^{n=\infty}\frac{d\lambda_n(t_2)}{dt_2}c_{n,k}(t_2)\frac{\partial w_n}{\partial t_0} + Ad_k(t_2)\cos(\omega(t_0-a))] \\ \quad -2\varepsilon\sqrt{\varepsilon}[\Sigma_{n=1}^{n=\infty}\frac{d\lambda_n(t_2)}{dt_2}c_{n,k}(t_2)\frac{\partial w_n}{\partial t_1}], \\ w_k(a,b,c;\sqrt{\varepsilon}) = \varepsilon F_k, \\ \frac{\partial w_k}{\partial t_0}(a,b,c;\sqrt{\varepsilon}) + \sqrt{\varepsilon}\frac{\partial w_k}{\partial t_1}(a,b,c;\sqrt{\varepsilon}) + \varepsilon\frac{\partial w_k}{\partial t_2}(a,b,c;\sqrt{\varepsilon}) = \varepsilon G_k, \end{cases} \tag{2.116}$$

where

$$a = \frac{1}{\varepsilon}(\frac{\omega}{\tan(\omega)}+1), \quad b = \frac{1}{\sqrt{\varepsilon}}(\frac{\omega}{\tan(\omega)}+1), \quad c = \frac{\omega}{\tan(\omega)}+1. \tag{2.117}$$

By using a three-timescales perturbation method, $w_k(t_0,t_1,t_2;\sqrt{\varepsilon})$ will be approximated by the formal asymptotic expansion

$$\begin{aligned} w_k(t_0,t_1,t_2;\sqrt{\varepsilon}) &= \sqrt{\varepsilon}w_{k,0}(t_0,t_1,t_2;\sqrt{\varepsilon}) + \varepsilon w_{k,1}(t_0,t_1,t_2;\sqrt{\varepsilon}) \\ &\quad +\varepsilon\sqrt{\varepsilon}w_{k,2}(t_0,t_1,t_2;\sqrt{\varepsilon}) + O(\varepsilon^2). \end{aligned} \tag{2.118}$$

It is reasonable to assume this solution form since the function $w_k(t_0,t_1,t_2;\sqrt{\varepsilon})$ analytically depends on$\sqrt{\varepsilon}$, and we are interested in approximations of the solution of Eq.(2.4), when the initial conditions and the external excitation are of $O(\varepsilon)$. By substituting (2.118) into problem (2.116), and after equating the coefficients of like powers in $\sqrt{\varepsilon}$, we obtain as:

the $O(\sqrt{\varepsilon})$-problem:

$$\frac{\partial^2 w_{k,0}}{\partial t_0^2} + \lambda_k^2(t_2)w_{k,0} = 0, \quad w_{k,0}(a,b,c) = 0, \quad \frac{\partial w_{k,0}}{\partial t_0}(a,b,c) = 0, \tag{2.119}$$

the $O(\varepsilon)$-problem:

$$\begin{aligned}\frac{\partial^2 w_{k,1}}{\partial t_0^2} + \lambda_k^2(t_2) w_{k,1} &= -2\frac{\partial^2 w_{k,0}}{\partial t_0 \partial t_1} + Ad_k(t_2)\cos(\omega(t_0-a)),\\ w_{k,1}(a,b,c) = F_k,\ \frac{\partial w_{k,1}}{\partial t_0}(a,b,c) &= -\frac{\partial w_{k,0}}{\partial t_1}(a,b,c) + G_k,\end{aligned} \tag{2.120}$$

and the $O(\varepsilon\sqrt{\varepsilon})$-problem:

$$\begin{aligned}\frac{\partial^2 w_{k,2}}{\partial t_0^2} + \lambda_k^2(t_2) w_{k,2} &= -2\frac{\partial^2 w_{k,1}}{\partial t_0 \partial t_1} - 2\frac{\partial^2 w_{k,0}}{\partial t_0 \partial t_2} - \frac{\partial^2 w_{k,0}}{\partial t_1^2}\\ &\quad -2\Sigma_{n=1}^{\infty}\frac{d\lambda_n(t_2)}{d\tau} c_{n,k}(t_2)\frac{\partial w_{n,0}}{\partial t_0},\\ w_{k,2}(a,b,c) = 0,\ \frac{\partial w_{k,2}}{\partial t_0}(a,b,c) &= -\frac{\partial w_{k,0}}{\partial t_2}(a,b,c) - \frac{\partial w_{k,1}}{\partial t_1}(a,b,c).\end{aligned} \tag{2.121}$$

The $O(\sqrt{\varepsilon})-$ problem has as solution

$$w_{k,0}(t_0,t_1,t_2;\sqrt{\varepsilon}) = C_{k,1}(t_1,t_2)\sin(\lambda_k(t_2)t_0) + C_{k,2}(t_1,t_2)\cos(\lambda_k(t_2)t_0), \tag{2.122}$$

where $C_{k,1}$ and $C_{k,2}$ are still unknown functions of the slow variables t_1 and t_2, and they can be determined by avoiding secular terms in the solutions of the $O(\varepsilon)-$ and $O(\varepsilon\sqrt{\varepsilon})-$ problems. By using the initial conditions (2.119), it follows that $C_{k,1}(b,c) = C_{k,2}(b,c) = 0$. Now, we shall solve the $O(\varepsilon)-$ problem (2.120).

This problem (outside as well as inside the resonance manifold) can be written as

$$\begin{aligned}&\frac{\partial^2 w_{k,1}}{\partial t_0^2} + \lambda_k^2(t_2) w_{k,1}\\ = \ &-2\lambda_k(t_2)[\frac{\partial C_{k,1}}{\partial t_1}\cos(\lambda_k(t_2)t_0) - \frac{\partial C_{k,2}}{\partial t_1}\sin(\lambda_k(t_2)t_0)] + Ad_k(t_2)\cos(\omega(t_0-a)),\end{aligned}$$

$$\begin{aligned}w_{k,1}(a,b,c) &= F_k,\\ \frac{\partial w_{k,1}}{\partial t_0}(a,b,c) &= -\frac{\partial w_{k,0}}{\partial t_1}(a,b,c) + G_k.\end{aligned} \tag{2.123}$$

By introducing the transformation $(w_{k,1}, w_{k,1,t_0}) \to (D_{k,1}, D_{k,2})$ with

$$\begin{aligned}&w_{k,1}(t_0,t_1,t_2;\sqrt{\varepsilon})\\ = \ &D_{k,1}(t_0,t_1,t_2;\sqrt{\varepsilon})\sin(\lambda_k(t_2)t_0) + D_{k,2}(t_0,t_1,t_2;\sqrt{\varepsilon})\cos(\lambda_k(t_2)t_0),\\ &w_{k,1,t_0}(t_0,t_1,t_2;\sqrt{\varepsilon})\\ = \ &\lambda_k(t_2)[D_{k,1}(t_0,t_1,t_2;\sqrt{\varepsilon})\cos(\lambda_k(t_2)t_0) - D_{k,2}(t_0,t_1,t_2;\sqrt{\varepsilon})\sin(\lambda_k(t_2)t_0)],\end{aligned}$$

it follows that the partial differential equation in (2.123) is equivalent with the system

$$\begin{cases}\dot{D}_{k,1} = -\frac{\partial C_{k,1}}{\partial t_1}[\cos(2\lambda_k(t_2)t_0)+1] + \frac{\partial C_{k,2}}{\partial t_1}\sin(2\lambda_k(t_2)t_0)\\ \quad + \frac{Ad_k(t_2)}{2\lambda_k(t_2)}[\cos(\omega t_0 - \omega a + \lambda_k(t_2)t_0) + \cos(\omega t_0 - \omega a - \lambda_k(t_2)t_0)],\\ \dot{D}_{k,2} = \frac{\partial C_{k,1}}{\partial t_1}\sin(2\lambda_k(t_2)t_0) - \frac{\partial C_{k,2}}{\partial t_1}[1-\cos(2\lambda_k(t_2)t_0)]\\ \quad - \frac{Ad_k(t_2)}{2\lambda_k(t_2)}[\sin(\omega t_0 - \omega a + \lambda_k(t_2)t_0) - \sin(\omega t_0 - \omega a - \lambda_k(t_2)t_0)],\end{cases} \tag{2.124}$$

where the overdot represents differentiation with respect to t_0, that is, $\dot{} = \frac{\partial}{\partial t_0}$.

Outside of the resonance zone, whether it exists or not it should be observed that the last terms in both equations in (2.124) do not give rise to secular terms in $D_{k,1}$ and $D_{k,2}$. To avoid secular terms, $C_{k,1}$ and $C_{k,2}$ have to satisfy the following conditions

$$\frac{\partial C_{k,1}}{\partial t_1} = 0, \quad \frac{\partial C_{k,2}}{\partial t_1} = 0. \tag{2.125}$$

Then, $C_{k,1}(t_1, t_2) = \bar{C}_{k,1}(t_2)$, and $C_{k,2}(t_1, t_2) = \bar{C}_{k,2}(t_2)$.

Inside the resonance zone, we observe that $\cos(\omega t_0 - \omega a - \lambda_k(t_2)t_0)$ and $\sin(\omega t_0 - \omega a - \lambda_k(t_2)t_0)$ might cause secular terms. In accordance with the resonance timescale of $O(\frac{1}{\sqrt{\varepsilon}})$, see (2.95), it is convenient to rewrite the arguments of the cos – and sin – function as

$$\omega t_0 - \omega a - \lambda_k(t_2)t_0 = -\frac{\alpha}{2}t_1^2 - \omega a,$$

where α is given by (2.97). Accordingly, the solution of $w_{k,1}$ in Eq.(2.123) has unbounded terms in t_0 unless

$$\begin{aligned} -2\lambda_k(t_2)\frac{\partial C_{k,1}}{\partial t_1} + Ad_k(t_2)\cos[-\frac{\alpha}{2}t_1^2 - \omega a] = 0, \\ 2\lambda_k(t_2)\frac{\partial C_{k,2}}{\partial t_1} - Ad_k(t_2)\sin[-\frac{\alpha}{2}t_1^2 - \omega a] = 0, \end{aligned} \tag{2.126}$$

which implies that

$$\begin{aligned} C_{k,1}(t_1,t_2) &= \bar{C}_{k,1}(t_2) + \frac{A\cos(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{C}_{Fr}(t_1) - \frac{A\sin(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{S}_{Fr}(t_1), \\ C_{k,2}(t_1,t_2) &= \bar{C}_{k,2}(t_2) - \frac{A\sin(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{C}_{Fr}(t_1) - \frac{A\cos(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{S}_{Fr}(t_1), \end{aligned} \tag{2.127}$$

where

$$\bar{C}_{Fr}(t) = \int_{\sqrt{\frac{\alpha}{2}}b}^{\sqrt{\frac{\alpha}{2}}t}\cos(x^2)dx, \quad \text{and} \quad \bar{S}_{Fr}(t) = \int_{\sqrt{\frac{\alpha}{2}}b}^{\sqrt{\frac{\alpha}{2}}t}\sin(x^2)dx, \tag{2.128}$$

and which are the well-known Fresnel integrals. Thus, it follows from (2.124) that

$$\begin{aligned} D_{k,1}(t_0,t_1,t_2) &= \bar{D}_{k,1}(t_1,t_2) + \frac{Ad_k(t_2)}{2\lambda_k(t_2)(\omega+\lambda_k(t_2))}\sin(\omega t_0 - \omega a + \lambda_k(t_2)t_0) \\ &\quad + \frac{Ad_k(t_2)}{2\lambda_k(t_2)(\omega-\lambda_k(t_2))}\sin(\omega t_0 - \omega a - \lambda_k(t_2)t_0), \\ D_{k,2}(t_0,t_1,t_2) &= \bar{D}_{k,2}(t_1,t_2) + \frac{Ad_k(t_2)}{2\lambda_k(t_2)(\omega+\lambda_k(t_2))}\cos(\omega t_0 - \omega a + \lambda_k(t_2)t_0) \\ &\quad - \frac{Ad_k(t_2)}{2\lambda_k(t_2)(\omega-\lambda_k(t_2))}\cos(\omega t_0 - \omega a - \lambda_k(t_2)t_0). \end{aligned} \tag{2.129}$$

where $\bar{D}_{k,1}$ and $\bar{D}_{k,2}$ are still unknown functions of the slow variables t_1 and t_2. The undetermined behaviour with respect to t_1 and t_2 can be used to avoid secular terms in the $O(\varepsilon\sqrt{\varepsilon})$– problem (2.121), and in the high order problems.

Taking into account the secularity conditions, the general solution of the $O(\varepsilon)$ equation is given by

$$w_{k,1}(t_0, t_1, t_2; \sqrt{\varepsilon}) = D_{k,1}(t_0, t_1, t_2)\sin(\lambda_k(t_2)t_0) + D_{k,2}(t_0, t_1, t_2)\cos(\lambda_k(t_2)t_0), \quad (2.130)$$

where $D_{k,1}(t_0, t_1, t_2)$ and $D_{k,1}(t_0, t_1, t_2)$ are given by (2.129) and

$$D_{k,2}(a,b,c) = F_k, \quad D_{k,1}(a,b,c) = -\frac{\partial w_{k,0}}{\partial t_1}(a,b,c) + G_k. \quad (2.131)$$

The $O(\varepsilon\sqrt{\varepsilon})-$ problem (2.121) outside and inside the resonance manifold can be written as

$$\begin{aligned}
\frac{\partial^2 w_{k,2}}{\partial t_0^2} + \lambda_k^2(t_2)w_{k,2} \;=\; & -2\lambda_k(t_2)[\frac{\partial D_{k,1}}{\partial t_1}\cos(\lambda_k(t_2)t_0) - \frac{\partial D_{k,2}}{\partial t_1}\sin(\lambda_k(t_2)t_0)] \\
& -2\lambda_k(t_2)[\frac{\partial C_{k,1}}{\partial t_2}\cos(\lambda_k(t_2)t_0) - \frac{\partial C_{k,2}}{\partial t_2}\sin(\lambda_k(t_2)t_0)] \\
& +[\frac{\partial^2 C_{k,1}}{\partial t_1^2}\sin(\lambda_k(t_2)t_0) + \frac{\partial^2 C_{k,2}}{\partial t_1^2}\cos(\lambda_k(t_2)t_0)] \\
& -2\Sigma_{n=1}^{n=\infty}\frac{d\lambda_n(t_2)}{dt_2}c_{n,k}(t_2)\lambda_k(t_2)[C_{n,1}\cos(\lambda_k(t_2)t_0) \\
& -C_{n,2}\sin(\lambda_k(t_2)t_0)],
\end{aligned}$$

$$\begin{aligned}
w_{k,2}(a,b,c) &= 0, \\
\frac{\partial w_{k,2}}{\partial t_0}(a,b,c) &= -\frac{\partial w_{k,0}}{\partial t_2}(a,b,c) - \frac{\partial w_{k,1}}{\partial t_1}(a,b,c). \quad (2.132)
\end{aligned}$$

To avoid secular terms in the solution $w_{k,2}$ in Eq.(2.132), the following conditions have to be imposed

$$\begin{aligned}
-2\lambda_k(t_2)\frac{\partial D_{k,1}}{\partial t_1} - 2\lambda_k(t_2)\frac{\partial C_{k,1}}{\partial t_2} + \frac{\partial^2 C_{k,2}}{\partial t_1^2} - 2c_{k,k}(t_2)\lambda_k(t_2)\frac{d\lambda_k(t_2)}{dt_2}C_{k,1} = 0, \\
2\lambda_k(t_2)\frac{\partial D_{k,2}}{\partial t_1} + 2\lambda_k(t_2)\frac{\partial C_{k,2}}{\partial t_2} + \frac{\partial^2 C_{k,1}}{\partial t_1^2} + 2c_{k,k}(t_2)\lambda_k(t_2)\frac{d\lambda_k(t_2)}{dt_2}C_{k,2} = 0. \quad (2.133)
\end{aligned}$$

Next, we analyse this equation (2.133) inside and outside the resonance manifold.

Inside the resonance zone, substituting (2.127) and (2.129) into (2.133), we obtain the following secularity conditions:

$$\begin{aligned}
& -2\frac{\partial \bar{D}_{k,1}}{\partial t_1} - 2\frac{d\bar{C}_{k,1}}{dt_2} - 2c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,1} \\
& -\frac{d(\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)})}{dt_2}(\bar{C}_{Fr}(t_1) - \bar{S}_{Fr}(t_1)) + \frac{Ad_k(t_2)}{\lambda_k(t_2)}\alpha t_1\sin[-\alpha t_1^2 - \omega a] \\
& +c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)}(\bar{S}_{Fr}(t_1) - \bar{C}_{Fr}(t_1)) \;=\; 0, \quad (2.134)
\end{aligned}$$

and

$$
\begin{aligned}
2\frac{\partial \bar{D}_{k,2}}{\partial t_1}+2\frac{d\bar{C}_{k,2}}{dt_2}+2c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,2} & \\
-\frac{d(\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)})}{dt_2}(\bar{C}_{Fr}(t_1)+\bar{S}_{Fr}(t_1))-\frac{Ad_k(t_2)}{\lambda_k(t_2)}\alpha t_1\cos[-\alpha t_1^2-\omega a] & \\
-c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)}(\bar{S}_{Fr}(t_1)+\bar{C}_{Fr}(t_1)) &= 0. \quad (2.135)
\end{aligned}
$$

Solving (2.134) and (2.135) for $\bar{D}_{k,1}$ and $\bar{D}_{k,2}$, we observe that the solution will be unbounded in t_1, due to terms which are only depending on t_2. Therefore, to have secular-free solutions for $\bar{D}_{k,1}$ and $\bar{D}_{k,2}$, the following conditions have to be imposed independently

$$
\frac{\partial\bar{C}_{k,1}}{\partial t_2}+c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,1}=0, \quad \frac{\partial\bar{C}_{k,2}}{\partial t_2}+c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,2}=0, \qquad (2.136)
$$

together with

$$
\begin{aligned}
-2\frac{\partial \bar{D}_{k,1}}{\partial t_1}-\frac{d(\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)})}{dt_2}(\bar{C}_{Fr}(t_1)-\bar{S}_{Fr}(t_1)) & \\
+\frac{Ad_k(t_2)}{\lambda_k(t_2)}\alpha t_1\sin[-\alpha t_1^2-\omega a] & \\
+c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)}(\bar{S}_{Fr}(t_1)-\bar{C}_{Fr}(t_1)) &= 0, \qquad (2.137)
\end{aligned}
$$

and

$$
\begin{aligned}
2\frac{\partial \bar{D}_{k,2}}{\partial t_1}+\frac{d(\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)})}{dt_2}(\bar{C}_{Fr}(t_1)-\bar{S}_{Fr}(t_1)) & \\
-\frac{Ad_k(t_2)}{\lambda_k(t_2)}\alpha t_1\cos[-\alpha t_1^2-\omega a] & \\
-c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)}(\bar{S}_{Fr}(t_1)+\bar{C}_{Fr}(t_1)) &= 0. \qquad (2.138)
\end{aligned}
$$

where $\bar{D}_{k,1}$ and $\bar{D}_{k,2}$ can be found by integration of (2.137) and (2.138), but we omit the details because of cumbersome expressions.

Next, from (2.136) we obtain the functions $\bar{C}_{k,1}(t_2)$ and $\bar{C}_{k,2}(t_2)$:

$$
\bar{C}_{k,1}(t_2)=m_1e^{-\int_c^{t_2}c_{kk}(s)\frac{d\lambda_k(s)}{ds}ds}, \quad \bar{C}_{k,2}(t_2)=m_2e^{-\int_c^{t_2}c_{kk}(s)\frac{d\lambda_k(s)}{ds}ds}, \qquad (2.139)
$$

where m_1 and m_2 are constants. Since $C_{k,1}(b,c)=0$ and $C_{k,2}(b,c)=0$, together with (2.127), this implies that $\bar{C}_{k,1}(t_2), \bar{C}_{k,2}(t_2)$ are both identically equal to zero.

So, in the resonance zone,

$$
\begin{aligned}
& w_{k,0}(t_0,t_1,t_2;\sqrt{\varepsilon}) \\
= & [\frac{A\cos(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{C}_{Fr}(t_1)-\frac{A\sin(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{S}_{Fr}(t_1)]\sin(\lambda_k(t_2)t_0)
\end{aligned}
$$

$$
\begin{aligned}
& -[\frac{A\cos(\omega a) d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)} S_{Fr}(t_1) + \frac{A\sin(\omega a) d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)} \bar{C}_{Fr}(t_1)] \cos(\lambda_k(t_2) t_0) \\
= \ & M_1 [(\cos(\omega a)\bar{C}_{Fr}(t_1) - \sin(\omega a)\bar{S}_{Fr}(t_1)) \sin(\lambda_k(t_2) t_0) \\
& -(\cos(\omega a)\bar{S}_{Fr}(t_1) + \sin(\omega a)\bar{C}_{Fr}(t_1)) \cos(\lambda_k(t_2) t_0)] + h.o.t, \qquad (2.140)
\end{aligned}
$$

where M_1 is given by (2.113). Outside the resonance zone, it follows from (2.125) and (2.133) that

2

$$
\frac{\partial \bar{C}_{k,1}}{\partial t_2} + c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,1} = 0, \quad \frac{\partial \bar{C}_{k,2}}{\partial t_2} + c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,2} = 0,
$$

which implies that

$$
\bar{C}_{k,1}(t_2) = m_1 e^{-\int_c^{t_2} c_{kk}(s)\frac{d\lambda_k(s)}{ds}ds} \quad \bar{C}_{k,2}(t_2) = m_2 e^{-\int_c^{t_2} c_{kk}(s)\frac{d\lambda_k(s)}{ds}ds}, \qquad (2.141)
$$

where m_1 and m_2 are constants. Since $C_{k,1}(b,c) = 0$, $C_{k,2}(b,c) = 0$, this implies that $\bar{C}_{k,1}(t_2)$, and $\bar{C}_{k,2}(t_2)$ are identically equal to zero outside the resonance zone.

Now we can solve the $O(\varepsilon\sqrt{\varepsilon})$– problem, where

$$
w_{k,2}(t_0, t_1, t_2; \sqrt{\varepsilon}) = E_{k,1}(t_0, t_1, t_2)\sin(\lambda_k(t_2)t_0) + E_{k,2}(t_0, t_1, t_2)\cos(\lambda_k(t_2)t_0), \qquad (2.142)
$$

where $E_{k,1}$ and $E_{k,2}$ are still unknown functions of the variable t_0, and the slow variables t_1, t_2, and they can be obtained by avoiding secular terms from higher order problems. Moreover,

$$
\begin{aligned}
E_{k,2}(a,b,c) &= 0, \\
E_{k,1}(a,b,c) &= -\frac{\partial w_{k,0}}{\partial t_2}(a,b,c) - \frac{\partial w_{k,1}}{\partial t_1}(a,b,c). \qquad (2.143)
\end{aligned}
$$

Note that in Eq.(2.130) and Eq.(2.142), $D_{k,1}, D_{k,2}, E_{k,1}, E_{k,2}$ are yet also undetermined functions. All these unknown functions can be determined from the $O(\varepsilon^2)$– problem and $O(\varepsilon^2\sqrt{\varepsilon})$– problem. At this moment, only the first term in the expansion of the solution for the string problem is important from the physical point of view. We are not interested in high-order approximations; that is why we take

$$
\begin{aligned}
\bar{D}_{k,1}(t_1,t_2) = \bar{D}_{k,1}(b,c), \quad \bar{D}_{k,2}(t_1,t_2) = \bar{D}_{k,2}(b,c), \\
E_{k,1}(t_0,t_1,t_2) = E_{k,1}(a,b,c), \quad E_{k,2}(t_0,t_1,t_2) = E_{k,2}(a,b,c). \qquad (2.144)
\end{aligned}
$$

Thus, an approximation of the solution of Eq.(2.4) is given by

$$
\begin{aligned}
& u(x,t) \\
= \ & \sum_{n=1}^{\infty} [\sqrt{\varepsilon} w_{n,0}(t_0,t_1,t_2;\sqrt{\varepsilon}) + \varepsilon w_{n,1}(t_0,t_1,t_2;\sqrt{\varepsilon}) \\
& + \varepsilon\sqrt{\varepsilon} w_{n,2}(t_0,t_1,t_2;\sqrt{\varepsilon})] \sin(\lambda_n(\tau)x) + O(\varepsilon^2), \qquad (2.145)
\end{aligned}
$$

where $w_{k,0}$, $w_{k,1}$ and $w_{k,2}$ are given by (2.122), (2.130) and (2.142).

- When the external force frequency ω is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \le \omega < \lambda_k(0)$, with $\lambda_k(0)$ given by (2.84), for all $k = 1, 2, ...$, it follows from (2.125) and (2.141) that $w_{k,0}(t_0, t_1, t_2; \sqrt{\varepsilon}) = 0$. Further from (2.115) and (2.118), we obtain $\tilde{T}_k(t) = O(\varepsilon)$, i.e., for the $O(\varepsilon)$ external excitation, there is an $O(\varepsilon)$ response. This case can be referred to as the non-resonant case.

2

- When the external force frequency ω satisfies $\lambda_k(0) \le \omega < k\pi$, with $\lambda_k(0)$ given by (2.84), for a fixed $k = 1, 2, \ldots$, it follows from (2.140) that $w_{n,0}(t_0, t_1, t_2; \sqrt{\varepsilon})_{n \neq k} = 0$ and

$$\begin{aligned} & w_{k,0}(t_0, t_1, t_2; \sqrt{\varepsilon}) \\ = \; & M_1[(\cos(\omega a)\bar{C}_{Fr}(t_1) - \sin(\omega a)\bar{S}_{Fr}(t_1))\sin(\lambda_k(t_2)t_0) \\ & -(\cos(\omega a)\bar{S}_{Fr}(t_1) + \sin(\omega a)\bar{C}_{Fr}(t_1))\cos(\lambda_k(t_2)t_0)], \end{aligned} \tag{2.146}$$

where M_1 satisfies (2.113). For the solution $u(x,t)$ of the original problem (2.130) this implies a resonance jump from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$ around $t = \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega} - 1)$ in the k-th oscillation mode.

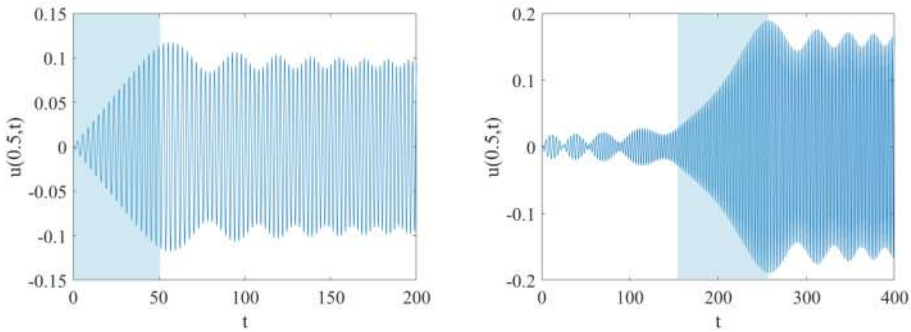

Figure 2.13: (a) The solution $u(0.5,t)$ of the system with $\omega = 2.0917$, and the resonance time $t \approx 20$, $\lambda_1(\varepsilon \cdot 20) = \omega$. (b) The solution $u(0.5,t)$ of the system with $\omega = 2.4556$, and the resonance time $t \approx 200$, $\lambda_1(\varepsilon \cdot 200) = \omega$.

It should be observed that $\tilde{T}_k(t)$ as calculated by using a three-timescales perturbation method agrees well with the approximation as calculated by using the averaging method. Further, by using the first term $w_{k,0}(t_0, t_1, t_2; \sqrt{\varepsilon})$ as $u(x,t) = [\sqrt{\varepsilon}w_{k,0} + O(\varepsilon)]\sin(\lambda_k(\tau)x)$ with $\lambda_k(0) \le \omega < k\pi$ and $\lambda_k(0)$ given by (2.84), $u(0.5,t)$ is plotted for $\varepsilon = 0.01$, $A = 1$, $\omega = 2.0917$, and so $t_k = 20$ in Figure 2.13 (a) and $u(0.5,t)$ is plotted for $\varepsilon = 0.01$, $A = 1$, $\omega = 2.4556$, and so $t_k = 200$ in Figure 2.13 (b).

2.7.4. Numerical examples

In this section the finite difference method is used to present numerical approximations of the vibration response and energy of the string. The computations are performed by using the parameters $\varepsilon = 0.01$, $A = 1$. The initial conditions are assumed to be given by (2.17). There will be different behavior in the amplitude response of the solution for different choices of the parameter ω. Note that the following numerical results are computed based on $O(\varepsilon)$ approximations of the equations. Higher order terms in the equations are neglected due to their unimportant contribution in the solution. By using (2.83), and according to our analytical results, resonance occurs around times

$$t = -\frac{\omega}{\varepsilon \cdot \tan\omega} - \frac{1}{\varepsilon}. \tag{2.147}$$

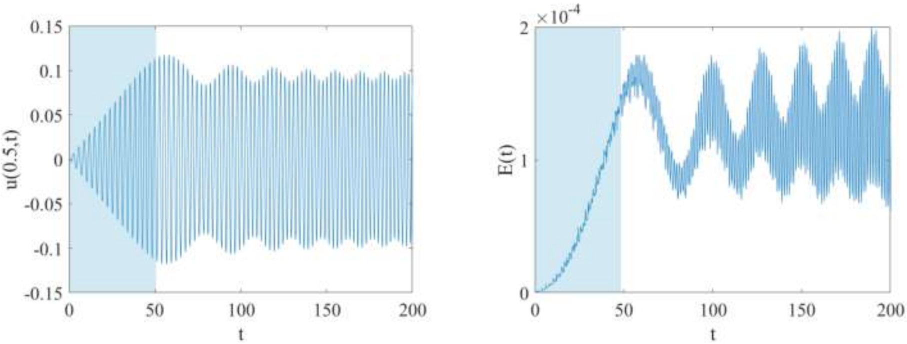

Figure 2.14: The solution $u(0.5, t)$ (a) and the energy $E(t)$. (b) of the system with $\omega = 2.0917$, and the resonance time $t \approx 20$, $\lambda_1(\varepsilon \cdot 20) = \omega$.

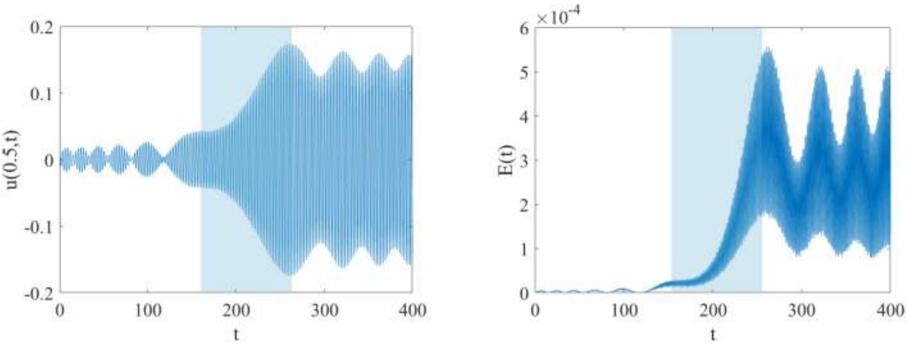

Figure 2.15: The solution $u(0.5, t)$ (a) and the energy $E(t)$. (b) of the system with $\omega = 2.4556$, and the resonance time $t \approx 200$, $\lambda_1(\varepsilon \cdot 200) = \omega$.

When the external force frequency ω is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \leq \omega < \lambda_k(0)$, with $\lambda_k(0)$ given by (2.84), for all $k = 1, 2, \ldots$ Figure 2.14 shows the displacement at $x = 0.5$ and the vibratory energy of the system, respectively, for times up to t=200 for $\omega = 2.0917$. We observe that for $t \approx 20$ the response amplitudes of the vibration become of order $\sqrt{\varepsilon}$ from order ε at $t = 0$. Similarly, Figure 2.15 shows the displacement and vibratory energy of the system for times up to t=400 for $\omega = 2.4556$. Again we observe that the response amplitudes of the vibration become of order $\sqrt{\varepsilon}$ but now for $t \approx 200$. When the external force frequency ω is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \leq \omega < \lambda_k(0)$, with $\lambda_k(0)$ given by (2.84), for all $k = 1, 2, \ldots$, Figure 2.16 shows the displacement and vibratory energy of the system for times up to t=400 for $\omega = 1.58$, but now there is no resonance and the response amplitudes of the vibration are still of order ε. These numerical results are in agreement with our results as presented in section 2.7.2 and in section 2.7.3. Moreover, in these figures, the shadowed bands represent the resonance zones, which have the size of $(O(\frac{1}{\sqrt{\varepsilon}}))$ as was also

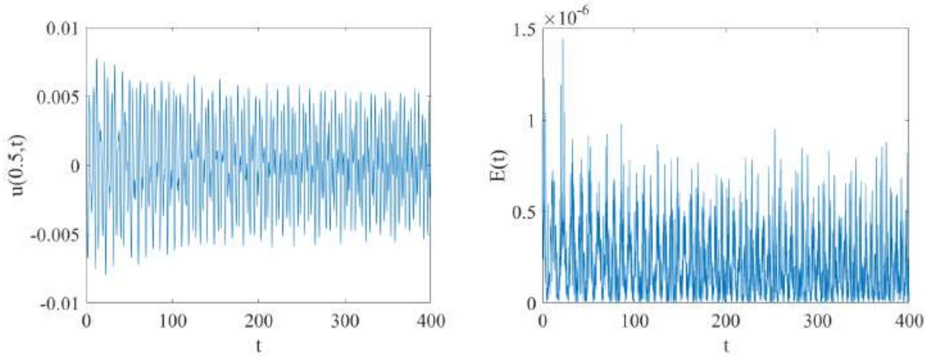

Figure 2.16: The solution $u(0.5,t)$ (a) and the energy $E(t)$. (b) of the system with $\omega = 1.5$, there is no resonance.

obtained analytically. Therefore, from Figure 2.14, Figure 2.15 and Figure 2.16, we can conclude that the general dynamic behavior of the solution as approximated numerically is in complete agreement with the analytic approximations as obtained in section 2.7.2 and in section 2.7.3.

2.8. CONCLUSIONS

IN this chapter resonances in a transversally vibrating string are studied. A small, externally applied and harmonic force with frequency ω is acting on the whole string. The string is fixed at one end, and at the other end a spring is attached for which the stiffness slowly varies in time. Firstly, based on the d'Alembert formula and on the boundary conditions, the initial conditions are extended on the whole x-domain. Taking into account the wave travelling speed and the total reflection time, the time domain is divided into smaller intervals of fixed length, so that the initial conditions extension procedure for each interval coincides with the previous ones. In this way one can obtain in a rather straightforward way an analytical expression for the solution on the time-interval [0, 2n] with n= 1, 2, 3, ..., N and N not too large. Moreover, four choices are made for the slowly varying stiffness, but also other choices can be made and a similar analysis as presented in this chapter can be given. By assuming that the small external force is of order ε and by assuming that the initial values are also small and of order ε, it is shown in this chapter that resonances can occur for certain values of ω. To obtain these results, the method of separation of variables (SOV), the method of d'Alembert, and the adapted version of the method of separation of variables is introduced, and perturbation methods, (such as averaging methods, singular perturbation techniques, and multiple timescales perturbation methods) are used. Explicit, and accurate approximations of solutions of the initial-boundary value problem are constructed. All approximations are valid on time-scales of order ε^{-1}. Also a finite difference method is applied to construct numerical approximations of the solution of the initial-boundary value problem. These numerical approximations are in full agreement with the analytically obtained approximations.

APPENDIX A NUMERICAL APPROXIMATION

Firstly, we introduce a uniform mesh Δx, a constant discretization time Δt, and a rectangular mesh consisting of points (x_i, t_j) with $x_i = i\Delta x$ and $t_j = j\Delta t$, where $i = 1,2,3,...,N$, $j = 1,2,...,$ with $N\Delta x = 1$. Following the finite difference method and by using the Taylor series expansion, the second order space and time derivatives can be discretized by

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{(\Delta x)^2} + O((\Delta x)^2), \tag{2.148}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{(\Delta x)^2} + O((\Delta t)^2). \tag{2.149}$$

Substituting the finite difference formulae into Eq. (2.4), and rearranging the terms, we end up with the linear iterative system

$$u_{i,j+1} = \sigma^2 u_{i+1,j} + 2(1-\sigma^2)u_{i,j} + \sigma^2 u_{i-1,j} - u_{i,j-1} + \varepsilon A\cos(\omega t_j), \tag{2.150}$$

where $\sigma = \frac{\Delta t}{\Delta x}$. From the boundary condition in (2.4) it follows that

$$u_{0,j} = 0, \quad u_{n,j} = \frac{u_{n-1,j}}{1 + k(t_j)\Delta x}. \tag{2.151}$$

Let us introduce the following vector: $U^{(j)} = [u_{1,j}, u_{2,j}, ..., u_{n-2,j}, u_{n-1,j}]^T$, $S^{(j)} = [\underbrace{\bar{s}_j, \bar{s}_j, ..., \bar{s}_j}_{\text{n-1 times}}]^T$, $\bar{s}_j = \varepsilon Acos(\omega t_j)$,

$$B = \begin{pmatrix} 2(1-\sigma^2) & \sigma^2 & 0 & \cdots & \cdots & 0 \\ \sigma^2 & 2(1-\sigma^2) & \sigma^2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma^2 & 2(1-\sigma^2) & \sigma^2 \\ 0 & \cdots & \cdots & 0 & \sigma^2 & 2(1-\sigma^2) + \frac{\sigma^2}{1+k(t_j)\Delta x} \end{pmatrix},$$

then the iteration process can be rewritten in the following matrix form

$$U^{(j+1)} = BU^{(j)} - U^{(j-1)} + S^{(j)}, \tag{2.152}$$

where the initial conditions imply:

$$u_{i,0} = f_i = f(x_i), \quad u_{i,1} = \frac{1}{2}\sigma^2 f_{i+1} + (1-\sigma^2)f_i + \frac{1}{2}\sigma^2 f_{i-1} + \Delta t g_i. \tag{2.153}$$

3

LONGITUDINAL RESONANCES FOR A VERTICALLY MOVING CABLE

3.1. INTRODUCTION

IN this chapter we study a real physical varying-length elevator system model, in which the longitudinal vibrations in an axially moving cable with time-varying length are considered subject to a small harmonic boundary excitation at one end of the cable and a moving nonclassical boundary condition at the other end. This elevator system consists of a drum, a head sheave, a driving motor, a moving conveyance, and a flexible cable with time-varying length $l(t)$. The upper end of the vertical cable is located at $x = e(t)$, where the small displacement $e(t)$ of this upper end is supposed to be generated by the catenary system (consisting of drum, head sheave) in vertical direction. A flexible cable lets the conveyance run up and down (see Figure 3.1).

Most analytical solutions for moving cables focus on transversal displacements or classical boundary conditions. Tan and Ying in [38] analyzed the axially moving cable based on wave propagetion subject to a classical boundary condition. Zhu in [39] considered a class of translating media with moving Dirichlet boundary conditions. Sandilo and van Horssen in [40] studied auto-resonance phenomena in a space time-varying mechanical system with a moving Dirichlet boundary condition. Gaiko and van Horssen in [41] considered transverse vibrations of a traveling cable subject to a moving Dirichlet boundary condition with boundary damping, and in [42] the authors further discussed resonances and vibrations in an elevator cable system due to boundary sway. Recently, researchers started to study longitudinal vibrations of moving cables with moving nonclassical boundary conditions. Wang et al. in [43] investigated the longitudinal responses of the two hoisting ropes, and the model is calculated numerically with different coefficients and excitations. Bao et al. in [44] evaluated longitudinal vibration of flexible hoisting systems with time-varying length by using The Galerkin's method. Ghayesh

A slightly varied version of this chapter have been published in Applied Mathematical Modelling **111**, 44-62 (2022).

in [45] numerically investigated the coupled longitudinal–transverse nonlinear dynamics of an axially accelerating beam. Wang et al. in [46] investigated a coupled dynamic model for a flexible guiding hoisting system and presented the response of the system by numerical simulations. Wang et al. in [47] studied the axial vibration suppression in a partial differential equation model for an ascending mining elevator cable system. These studies mainly focus on numerical simulations, and not on an analytical, mathematical analysis. For more information on numerical results for axially moving continua, the reader is referred to [48, 49, 50, 51, 22, 52, 53]. Compared to the analysis of systems subject to classical boundary conditions, the analytical study of axially moving systems with moving nonclassical boundary conditions is a challenging subject for research. And compared to chapter 2, the model in this chapter is physically relevant, including a fundamental excitation in a boundary condition, a time-varying interval $(0, l(t))$, second order derivatives in a boundary condition, viscous damping, spatiotemporally varying tension, longitudinal stiffness and so on. An adapted version of the method of separation of variables, an averaging method, singular perturbation techniques, and a three time-scales perturbation method are applied to construct accurate, analytical approximations of the solutions of the problem. For the aforementioned reasons, averaging, determining the resonance zones, and constructing accurate approximations of solutions are much harder than for the problem as studied in chapter 2. And in chapter 2 we concluded that when the external force frequency satisfies a certain condition, then the resonance will occur in one oscillation mode only and no resonance will occur in the other modes, i.e., resonance emerges for only one time internal. However, in this chapter, based on a perturbation analysis of the formulated, mathematical problem for the cable equation, we come to the conclusion that for a given arbitrary excitation frequency, many oscillation modes jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$ amplitudes, i.e., resonance emerges for many times and the size of the resonance zone is of $O(\frac{1}{\sqrt{\varepsilon}})$. This analytical result is accurate and valuable for real applications.

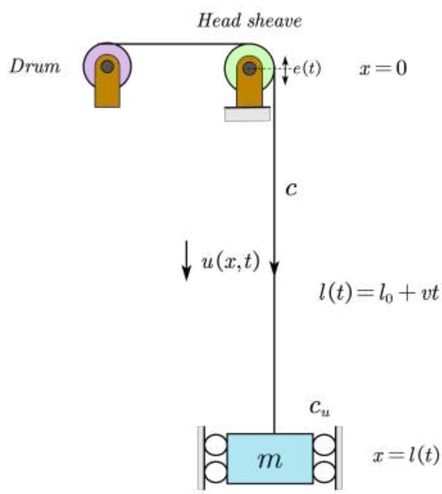

Figure 3.1: The longitudinal vibrating cable with time-varying cable length $l(t)$.

The chapter is organised as follows. In section 3.2 the problem is formulated and some transformations are introduced to simplify the originally formulated problem. In section 3.3 an interior layer analysis is presented. By introducing an adapted version of the method of separation of variables, by using averaging and singular perturbation techniques, the resonance zones are detected and the scalings are determined in the problem. By using these scalings, in section 3.4 a three time-scales perturbation method is used to construct accurate, analytical approximations of the solutions of the problem. In section 3.5 numerical approximations are presented by using a central finite difference scheme, which are in full agreement with the obtained, analytical approximations. In section 3.6 we draw some conclusions based on the analytical and numerical results and also we discuss future research.

3

3.2. FORMULATION OF THE PHYSICAL PROBLEM

Nomenclature:	
$u(x,t)$	the longitudinal displacement of the cable
$l(t)$	the length of the cable
$v = \dot{l}(t)$	the longitudinal velocity of the cable, v is assumed to be a constant.
ρ	the mass density of the cable
m	the mass of the conveyance
EA	the longitudinal stiffness, E Young's elasticity modulus, A the cross-sectional area of the cable
$T(x,t)$	the spatiotemporally varying tension in the cable
c	viscous damping coefficient in the cable
g	gravity
E_{gs}	initial gravitational potential energy
c_u	viscous damping coefficient
$e(t)$	the generated longitudinal displacement at the top of the vertical cable

By using the Hamilton's variational principle, the longitudinal vibrations of the axially moving rope in Figure 3.1 are described by the following initial boundary value problem (see Appendix B.1):

$$\begin{cases} \rho(u_{tt} + 2vu_{xt} + v^2 u_{xx}) - EAu_{xx} + c(u_t + vu_x) = 0, \ 0 \le x \le l(t), \ t > 0, \\ [m(u_{tt} + 2vu_{xt} + v^2 u_{xx}) + EAu_x + c_u(u_t + vu_x)]|_{x=l(t)} = 0, \ t > 0, \\ u(0,t) = e(t), \ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \ 0 \le x \le l_0. \end{cases} \tag{3.1}$$

For the parameters v, c, c_u and the function $e(t)$, we make the following reasonable assumptions: the longitudinal velocity v is small compared to nominal wave velocity $\sqrt{\frac{EA}{\rho}}$; the viscous damping coefficients c and c_u are small; and the oscillation amplitudes $e(t)$ at $x = 0$ are small. Then, we can rewrite $v = \varepsilon v_0, c = \varepsilon c_0, c_u = \varepsilon c_{u0}, e(t) = \beta \sin(\alpha t)$ with $\beta = \varepsilon \beta_0$, where ε is a small parameter with $0 < \varepsilon << 1$. And $l(t) = l_0 + \varepsilon v_0 t$, where l_0 is the initial cable length. It is also assumed that both initial conditions are $O(\varepsilon)$, that is, $u_0(x) = O(\varepsilon)$, and $u_1(x) = O(\varepsilon)$.

To put problem (3.1) in a non-dimensional form, the following dimensionless parameters will be used:

$$u^* = \frac{u}{L},\ x^* = \frac{x}{L},\ t^* = \frac{t}{L}\sqrt{\frac{EA}{\rho}},\ l^* = \frac{l}{L},\ v^* = v\sqrt{\frac{\rho}{EA}},$$
$$c^* = \frac{cL}{\sqrt{EA\rho}}, c_u^* = \frac{Lc_u}{m}\sqrt{\frac{EA}{\rho}},\ \beta^* = \frac{\beta}{L},\ \alpha^* = L\alpha\sqrt{\frac{\rho}{EA}},\ u_0^* = \frac{u_0}{L},\ u_1^* = \sqrt{\frac{\rho}{EA}}u_1,$$

where L is the maximum length of the cable. The equations of motion in non-dimensional form then become:

$$\begin{cases} u_{tt} - u_{xx} = -2vu_{xt} - v^2u_{xx} - c(u_t + vu_x), \quad 0 \le x \le l(t),\ t > 0, \\ u_{tt}(l(t),t) + \frac{\rho L}{m}u_x(l(t),t) = [-2vu_{xt} - v^2u_{xx} - c_u(u_t + vu_x)]|_{x=l(t)}, \quad t > 0, \\ u(0,t) = e(t) = \beta\sin(\alpha t), \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 \le x \le l_0, \end{cases} \tag{3.2}$$

where m, ρ, α, β, L and l_0 are positive constants, and where the asterisks (indicating the dimensionless variables and parameters) are omitted in problem (3.2) for convenience.

In order to simplify the integration of (3.2), it is convenient to transform the time-varying spatial domain $[0, l(t)]$ to a fixed domain $[0,1]$ by introducing a new independent spatial coordinate $\xi = \frac{x}{l(t)}$. Since the function $u(x,t)$ becomes a new function $\tilde{u}(\xi,t)$, all the partial derivatives have to be transformed in accordance with this new variable ξ as follows:

$$u_x = \frac{1}{l}\tilde{u}_\xi,\ u_{xx} = \frac{1}{l^2}\tilde{u}_{\xi\xi},\ u_t = -\xi\frac{v}{l}\tilde{u}_\xi + \tilde{u}_t,$$
$$u_{xt} = -\frac{v}{l^2}\tilde{u}_\xi + \frac{1}{l}\tilde{u}_{\xi t} - \xi\frac{v}{l^2}\tilde{u}_{\xi\xi},\ u_{tt} = \tilde{u}_{tt} - 2\frac{v}{l}\xi\tilde{u}_{\xi t} + \frac{v^2}{l^2}\xi^2\tilde{u}_{\xi\xi} + \frac{2v^2}{l^2}\xi\tilde{u}_\xi.$$

Substituting these derivatives into (3.2), we obtain the following problem for $\tilde{u}(\xi,t)$:

$$\begin{cases} \tilde{u}_{tt} - \frac{1}{l^2}\tilde{u}_{\xi\xi} = \frac{2v}{l}\xi\tilde{u}_{\xi t} - \frac{2v}{l}\tilde{u}_{\xi t} - c\tilde{u}_t + O(\varepsilon^2), \quad 0 \le \xi \le 1,\ t > 0, \\ \tilde{u}_{tt}(1,t) + \frac{\rho L}{ml}\tilde{u}_\xi(1,t) = [\frac{2v}{l}\xi\tilde{u}_{\xi t} - \frac{2v}{l}\tilde{u}_{\xi t} - c_u\tilde{u}_t]|_{\xi=1} + O(\varepsilon^2), \quad t > 0, \\ \tilde{u}(0,t) = \tilde{e}(t) = \beta\sin(\alpha t), \quad t > 0, \\ \tilde{u}(\xi,0) = \tilde{u}_0(\xi), \quad \tilde{u}_t(\xi,0) = \tilde{u}_1(\xi), \quad 0 \le \xi \le 1, \end{cases} \tag{3.3}$$

where $l = l(t)$, $\tilde{u}_0(\xi) = u_0(\xi l_0)$, and $\tilde{u}_1(\xi) = u_1(\xi l_0) - \varepsilon\frac{\xi v_0}{l_0}u_\xi(\xi,0)$.

In the following sections, we will construct analytical approximations of the solution of problem (3.3) on a time-scale of order $\frac{1}{\varepsilon}$ by an internal layer analysis and a three time-scales perturbation method. Moreover, to verify the analytical results, in subsection 3.5 we will compare these analytical approximations with numerically obtained approximations.

3.3. INTERNAL LAYER ANALYSIS

IN this section, we determine resonance manifolds and their corresponding timescales by an adapted version of the method of separation of variables, by using averaging and singular perturbation techniques.

3.3.1. TRANSFORMATIONS

The partial differential equation in (3.3) has in the left-hand side of the equation a variable coefficient $\frac{1}{l^2}$. To remove this variable coefficient the Liouville-Green transformation (or equivalently the WKBJ method [54, 55]) is used by introducing a new time-like variable $s(t)$ with

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{l(t)}. \tag{3.4}$$

Substituting the derivative into (3.3), we obtain the problem for $\bar{u}(\xi, s) = \tilde{u}(\xi, t)$ (see Appendix B.2). Further, in order to eliminate the non-homogeneous terms up to order ε^2 in the boundary condition at $\xi = 0$ and $\xi = 1$ in (3.3), the following transformation is used:

$$\bar{u}(\xi, s) = W(\xi, s) + \varepsilon \frac{m\xi}{\rho L}(c_0 - c_{u0})W_s(1, s) + \bar{e}(s) + O(\varepsilon^2). \tag{3.5}$$

Thus, in order to obtain an order ε accurate approximation of the solution of $\bar{u}(\xi, s)$, it is necessary and sufficient to construct an order ε accurate approximation of the solution of $W(\xi, s)$. From (3.5) and Appendix B, it follows that $W(\xi, s)$ has to satisfy:

$$\begin{cases} W_{ss} - W_{\xi\xi} = \varepsilon[(v_0 - c_0\hat{l})W_s + 2v_0(\xi - 1)W_{\xi s} + \beta_0\alpha^2\hat{l}^2 \sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s} - 1)) \\ \qquad - \frac{m\xi(c_0 - c_{u0})}{\rho L} W_{\xi\xi s}(1, s)] + O(\varepsilon^2), \quad 0 \le \xi \le 1,\ s > 0, \\ W_{\xi\xi}(1, s) + \frac{\rho L\hat{l}}{m} W_\xi(1, s) = O(\varepsilon^2), \quad W(0, s) = O(\varepsilon^2), \quad s > 0, \\ W(\xi, 0) = W_0(\xi), \quad W_s(\xi, 0) = W_1(\xi), \quad 0 \le \xi \le 1, \end{cases} \tag{3.6}$$

where $W_0(\xi) = f(\xi) - \bar{e}(0) + O(\varepsilon^2)$, $W_1(\xi) = g(\xi) - \bar{e}_s(0) + O(\varepsilon^2)$. So the problem (3.3) is transformed into a simplified problem (3.6). In the following sections, accurate, analytical approximations of the solution $W(\xi, s)$ of problem (3.6) are constructed, and by using (3.4) and (3.5), accurate approximations of $\tilde{u}$ of problem (3.3) can be obtained.

3.3.2. AN ADAPTED VERSION OF THE METHOD OF SEPARATION OF VARIABLES

First of all, in order to make the method of separation of variables applicable to problem (3.6), we consider problem (3.6) by neglecting the $O(\varepsilon)$ terms, that is,

$$\begin{cases} W_{ss} - W_{\xi\xi} = 0, \quad 0 \le \xi \le 1,\ s > 0, \\ W(0, s) = 0, \quad W_{\xi\xi}(1, s) + \frac{\rho L\hat{l}(s)}{m} W_\xi(1, s) = 0, \quad s > 0, \\ W(\xi, 0) = W_0(\xi), \quad W_s(\xi, 0) = W_1(\xi), \quad 0 \le \xi \le 1, \end{cases} \tag{3.7}$$

where it should be noted that $\hat{l}(s) = l_0\mathrm{e}^{\varepsilon v_0 s}$. By defining a slow time variable $\tau = \varepsilon s$, which will be treated independently from the variable s, and so by defining $\bar{l}(\tau) = l_0\mathrm{e}^{v_0\tau}$, function $W(\xi, s)$ becomes a new function $W^*(\xi, s, \tau)$, and problem (3.7) becomes:

$$\begin{cases} W^*_{ss}(\xi, s, \tau) + 2\varepsilon W^*_{s\tau}(\xi, s, \tau) + \varepsilon^2 W^*_{\tau\tau}(\xi, s, \tau) - W^*_{\xi\xi}(\xi, s, \tau) = 0, \\ W^*(0, s, \tau) = 0, \quad W^*_{\xi\xi}(1, s, \tau) + \frac{\rho L\bar{l}(\tau)}{m} u^*_\xi(1, s, \tau) = 0, \quad \bar{l}(\tau) = l_0 e^{v_0\tau}, \\ W^*(\xi, 0, 0) = W_0(\xi), \quad W^*_s(\xi, 0, 0) + \varepsilon W^*_\tau(\xi, 0, 0) = W_1(\xi), \end{cases} \tag{3.8}$$

where $0 \leq \xi \leq 1$ and $s, \tau > 0$. Now let $T(s,\tau)X(\xi,\tau)$ be a nontrivial solution of (3.8). The general solution of (3.6) can be expanded in the following form (see Appendix B.3):

$$W(\xi, s) = \sum_{n=1}^{\infty} \bar{T}_n(s,\tau)\sin(\lambda_n(\tau)\xi), \tag{3.9}$$

where $\lambda_n(\tau)$ is the n-th positive root of

$$tan(\lambda_n(\tau)) = \frac{\rho L\bar{l}(\tau)}{m}\frac{1}{\lambda_n(\tau)}, \qquad \bar{l}(\tau) = l_0 e^{v_0\tau}, \tag{3.10}$$

and $\bar{T}_k(s,\tau)$ for $k = 1,2,3,\ldots$, $s > 0, \tau > 0$ have to satisfy:

$$\begin{cases} \bar{T}_{k,ss} + \lambda_k^2(\tau)\bar{T}_k = -2\varepsilon\bar{T}_{k,s\tau} + \varepsilon(v_0 - c_0\bar{l}(\tau))\bar{T}_{k,s} - 2\sum_{n=1}^{\infty}\varepsilon c_{n,k}^1(\tau)\frac{d\lambda_n(\tau)}{d\tau}\bar{T}_{n,s} \\ \qquad +2\sum_{n=1}^{\infty}\varepsilon v_0 c_{n,k}^2(\tau)\bar{T}_{n,s} + \sum_{n=1}^{\infty}\varepsilon\frac{m(c_0-c_{uo})}{\rho L}c_{n,k}^3(\tau)\bar{T}_{n,s} \\ \qquad +\varepsilon\beta_0\alpha^2\bar{l}^2(\tau)d_k(\tau)\sin(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s}-1)), \ \ t,\tau \geq 0, \\ \bar{T}_k(0,0) = \frac{\int_0^1 \sigma(0,\xi)W_0(\xi)\sin(\lambda_k(0)\xi)d\xi}{\int_0^1\sigma(0,\xi)\sin^2(\lambda_k(0)\xi)d\xi} = F_k, \\ \bar{T}_{k,s}(0,0) + \varepsilon\bar{T}_{k,\tau}(0,0) = -\varepsilon\sum_{n=1}^{\infty} T_n(0,0)\frac{d\lambda_n(\tau)}{d\tau}|_{\tau=0}\frac{\int_0^1\sigma(0,\xi)\xi\sin(\lambda_n(0)\xi)\cos(\lambda_k(0)\xi)d\xi}{\int_0^1\sigma(0,\xi)\sin^2(\lambda_k(0)\xi)d\xi} \\ \qquad + \frac{\int_0^1\sigma(0,\xi)W_1(\xi)\sin(\lambda_k(0)\xi)d\xi}{\int_0^1\sigma(0,\xi)\sin^2(\lambda_k(0)\xi)d\xi} \\ \qquad = G_k, \end{cases} \tag{3.11}$$

where $c_{n,k}^1(\tau)$, $c_{n,k}^2(\tau)$, $c_{n,k}^3(\tau)$ and $d_k(\tau)$ are functions of τ, and are given by:

$$\begin{aligned} c_{n,k}^1(\tau) &= \frac{\int_0^1\sigma(\tau,\xi)\xi\cos(\lambda_n(\tau)\xi)\sin(\lambda_k(\tau)\xi)d\xi}{\int_0^1\sigma(\tau,\xi)\sin^2(\lambda_k(\tau)\xi)d\xi}, \\ c_{n,k}^2(\tau) &= \frac{\lambda_n(\tau)\int_0^1\sigma(\tau,\xi)(\xi-1)\cos(\lambda_n(\tau)\xi)\sin(\lambda_k(\tau)\xi)d\xi}{\int_0^1\sigma(\tau,\xi)\sin^2(\lambda_k(\tau)\xi)d\xi}, \\ c_{n,k}^3(\tau) &= \frac{\lambda_n^2(\tau)\sin(\lambda_n(\tau))\int_0^1\sigma(\tau,\xi)\xi\sin(\lambda_k(\tau)\xi)d\xi}{\int_0^1\sigma(\tau,\xi)\sin^2(\lambda_k(\tau)\xi)d\xi}, \\ d_k(\tau) &= \frac{\int_0^1\sigma(\tau,\xi)\sin(\lambda_k(\tau)\xi)d\xi}{\int_0^1\sigma(\tau,\xi)\sin^2(\lambda_k(\tau)\xi)d\xi}. \end{aligned} \tag{3.12}$$

To simplify the formulas, we define a new dependent variable $\tilde{T}_k(s) = \bar{T}_k(s,\tau)$, for $k = 1,2,3,\ldots$, yielding:

$$\begin{cases} \tilde{T}_{k,ss} + \lambda_k^2(\tau)\tilde{T}_k = \varepsilon(v_0 - c_0\bar{l}(\tau))\tilde{T}_{k,s} - 2\sum_{n=1}^{\infty}\varepsilon(c_{n,k}^1(\tau)\frac{d\lambda_n(\tau)}{d\tau} - v_0 c_{n,k}^2(\tau))\tilde{T}_{n,s} \\ \qquad +\sum_{n=1}^{\infty}\varepsilon\frac{m(c_0-c_{uo})}{\rho L}c_{n,k}^3(\tau)\tilde{T}_{n,s} + \varepsilon\alpha^2\beta_0\bar{l}^2(\tau)d_k(\tau)\sin(\frac{\alpha l_0}{\varepsilon v_0}(e^{v_0\tau}-1)) \\ \qquad +O(\varepsilon^2), \\ \tilde{T}_k(0) = F_k, \\ \tilde{T}_{k,s}(0) = G_k, \end{cases} \tag{3.13}$$

where $F_k = O(\varepsilon)$, $G_k = O(\varepsilon)$, $s \geq 0$ and $\tau = \varepsilon s$. In the next subsection we will use the averaging method to detect resonance zones in problem (3.13), and to determine time-scales which describe the solutions of (3.13) accurately.

3.3.3. AVERAGING AND RESONANCE ZONES

The solution of the linear ordinary differential equation (3.13) with the slowly varying frequencies $\lambda_k(\tau)$ as given by (3.10), can be approximated by using the averaging method. In this section, by an interior layer analysis (including a rescaling and balancing procedure), the slowly varying frequencies $\lambda_k(\tau)$ lead to a description of many resonance manifolds and lead to time-scales which describe the solution of (3.13) sufficiently accurate. For the sake of convenience let us introduce the following standard transformations:

$$\phi_k(s) = \int_0^s \lambda_k(\varepsilon\bar{s})\mathrm{d}\bar{s} \quad \text{and} \quad \Phi = \frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau} - 1), \tag{3.14}$$

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and according to an adapted version of the Lagrange variation of constants method, we assume that $\tilde{T}_k(s), \tilde{T}_{k,s}(s)$ are described by $A_k(s), B_k(s)$ in the following way:

$$\begin{aligned} \tilde{T}_k(s) &= A_k(s)\sin(\phi_k(s)) + B_k(s)\cos(\phi_k(s)), \\ \tilde{T}_{k,s}(s) &= \lambda_k(\tau)A_k(s)\cos(\phi_k(s)) - \lambda_k(\tau)B_k(s)\sin(\phi_k(s)). \end{aligned} \tag{3.15}$$

Then, by substituting (4.19) into problem (3.13), we obtain the following problem (where the dot $\cdot$ represents differentiation with respect to s):

$$\begin{cases} \dot{A}_k = \tilde{A}_k(s) + \varepsilon\frac{\alpha^2\beta_0\bar{l}^2 d_k(\tau)}{2\lambda_k(\tau)}(\sin(\Phi+\phi_k) + \sin(\Phi-\phi_k)), \\ \dot{B}_k = \tilde{B}_k(s) + \varepsilon\frac{\alpha^2\beta_0\bar{l}^2 d_k(\tau)}{2\lambda_k(\tau)}(\cos(\Phi+\phi_k) - \cos(\Phi-\phi_k)), \\ \dot{\tau} = \varepsilon, \\ \dot{\Phi} = \alpha l_0\mathrm{e}^{v_0\tau}, \\ \dot{\phi_k} = \lambda_k(\tau), \end{cases} \tag{3.16}$$

where

$$\begin{aligned} \tilde{A}_k(s) &= \frac{1}{2}\varepsilon(v_0 - c_0\bar{l}(\tau))[A_k(s)(\cos(2\phi_k(s)) + 1) - B_k(s)\sin(2\phi_k(s))] \\ &\quad +\varepsilon\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{2\lambda_k}[B_k(s)\sin(2\phi_k(s)) - A_k(s)(\cos(2\phi_k(s)) + 1)] \\ &\quad -\varepsilon\eta_{k,k}(\tau)[A_k(s)(\cos(2\phi_n(s)) + 1) - B_k(s)\sin(2\phi_k(s))] \\ &\quad -2\varepsilon\sum_{n\neq k}\frac{\lambda_n(\tau)}{\lambda_k(\tau)}\eta_{n,k}(\tau)[A_n(s)\cos(\phi_n(s))\cos(\phi_k(s)) - B_n(s)\sin(\phi_n(s))\cos(\phi_k(s))], \\ \tilde{B}_k(s) &= -\frac{1}{2}\varepsilon(v_0 - c_0\bar{l}(\tau))[A_k(s)\sin(2\phi_k(s)) - B_k(s)(1 - \cos(2\phi_k(s)))] \\ &\quad +\varepsilon\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{2\lambda_k}[A_k(s)\sin(2\phi_k(s)) - B_k(s)(1 - \cos(2\phi_k(s)))] \\ &\quad +\varepsilon\eta_{k,k}(\tau)[A_k(s)\sin(2\phi_k(s)) - B_k(s)(1 - \cos(2\phi_n(s)))] \\ &\quad +2\varepsilon\sum_{n\neq k}\frac{\lambda_n(\tau)}{\lambda_k(\tau)}\eta_{n,k}(\tau)[A_n(s)\cos(\phi_n(s))\sin(\phi_k(s)) \\ &\quad -B_n(s)\sin(\phi_n(s))\sin(\phi_k(s))], \end{aligned} \tag{3.17}$$

and $\eta_{n,k}(\tau) = c^1_{n,k}(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} - v_0 c^2_{n,k}(\tau) - \frac{m(c_0 - c_{uo})}{2\rho L}c^3_{n,k}(\tau)$. Resonance in (3.16), can be expected when $\dot{\Phi} - \dot{\phi_k} \approx 0$, or when $\dot{\Phi} + \dot{\phi_k} \approx 0$. But since $\alpha l_0 e^{v_0\tau}$ and $\lambda_k(\tau) > 0$, resonance only will occur when

$$\alpha l_0\mathrm{e}^{v_0\tau} \approx \lambda_k(\tau). \tag{3.18}$$

Since $\lambda_k(\tau)$ satisfies (3.10), that is, $tan(\lambda_k(\tau)) = \frac{\rho L l_0 e^{\nu_0 \tau}}{m}\frac{1}{\lambda_k(\tau)}$, it follows that resonance occurs when

$$\lambda_k \approx \arctan(\frac{\rho L}{\alpha m}) + (k-1)\pi, \quad k = 1, 2, ..., \tag{3.19}$$

corresponding to the manifold τ around τ_k with

$$\tau_k = \frac{1}{\nu_0}\ln(\frac{1}{\alpha l_0}\lambda_k) = \frac{1}{\nu_0}\ln(\frac{\arctan(\frac{\rho L}{\alpha m}) + (k-1)\pi}{\alpha l_0}), \quad \lambda_k \geq \alpha l_0, \quad k = 1, 2, ... \tag{3.20}$$

From (3.20), we can conclude that no matter what the frequency is, there will be many resonance manifolds.

Outside the resonance manifold, we can average the right-hand side of the equations in (3.16) over ϕ_k and Φ while keeping A_k and B_k fixed [56, 57, 58]. Note that $\tilde{A}_k(s)$ and $\tilde{B}_k(s)$ are slowly varying, therefore they will not average out. The last terms of the first and second equations in (3.16) is the fast varying terms outside the resonance manifolds, therefore they will average out. Thus, the averaged equation for A_k and B_k now become

$$\begin{cases} \dot{A}_k^a = [\frac{1}{2}\varepsilon(\nu_0 - c_0\bar{l}(\tau)) - \varepsilon c_{k,k}^1(\tau)\frac{d\lambda_k(\tau)}{d\tau} + \varepsilon\nu_0 c_{k,k}^2(\tau) + \varepsilon\frac{m(c_0 - c_{uo})}{2\rho L}c_{k,k}^3(\tau) - \varepsilon\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{2\lambda_k}]A_k^a, \\ \dot{B}_k^a = [\frac{1}{2}\varepsilon(\nu_0 - c_0\bar{l}(\tau)) - \varepsilon c_{k,k}^1(\tau)\frac{d\lambda_k(\tau)}{d\tau} + \varepsilon\nu_0 c_{k,k}^2(\tau) + \varepsilon\frac{m(c_0 - c_{uo})}{2\rho L}c_{k,k}^3(\tau) - \varepsilon\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{2\lambda_k}]B_k^a, \end{cases} \tag{3.21}$$

where the upper index a indicates that this is the averaged function. From the expression for $c_{k,k}^1 > 0$, $c_{k,k}^2 = -\frac{1}{2}$ and $c_{k,k}^3 = 0$ in (3.12), we then obtain

$$A_k^a(s) = \frac{G_k}{\lambda_k(0)}e^{-\int_0^\tau \zeta(\varrho)\mathrm{d}\varrho}, \quad B_k^a(s) = F_k e^{-\int_0^\tau \zeta(\varrho)\mathrm{d}\varrho}, \tag{3.22}$$

with

$$\zeta(\tau) \quad = \quad \frac{1}{2}c_0\bar{l}(\tau) + c_{k,k}^1(\tau)\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau} + \frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{2\lambda_k}, \tag{3.23}$$

and $G_k = O(\varepsilon)$, $F_k = O(\varepsilon)$ are given in (3.13). Hence, outside the resonance manifold the solution of system (3.13) is given by

$$\tilde{T}_k(s) \quad = \quad \frac{G_k}{\lambda_k(0)}\mathrm{e}^{-\int_0^{\varepsilon s}\zeta(\varrho)\mathrm{d}\varrho}\sin(\phi_k(s)) + F_k\mathrm{e}^{-\int_0^{\varepsilon s}\zeta(\varrho)\mathrm{d}\varrho}\cos(\phi_k(s)), \tag{3.24}$$

where $s = O(\frac{1}{\varepsilon})$. Observe that outside the resonance zone $\tilde{T}_k(s)$ remains order ε.

To study the behavior of the solution in the resonance zone we introduce $\psi = \Phi(t) - \phi_k(t)$ and rescale $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$ with $\bar{\tau} = O(1)$ and τ_k is given by (3.20). System (3.13) then becomes:

$$\begin{cases} \dot{A}_k = \tilde{A}_k(s) + \varepsilon\frac{\alpha^2\beta_0\bar{l}^2(\tau)d_k(\tau)}{2\lambda_k(\tau)}(\sin(\Phi + \phi_k) + \sin(\psi)), \\ \dot{B}_k = \tilde{B}_k(s) + \varepsilon\frac{\alpha^2\beta_0\bar{l}^2(\tau)d_k(\tau)}{2\lambda_k(\tau)}(\cos(\Phi + \phi_k) - \cos(\psi)), \end{cases} \tag{3.25}$$

combined with the slow/fast variables

$$\begin{cases} \dot{\tau} = \varepsilon, \\ \dot{\bar{\tau}} = \frac{\varepsilon}{\delta(\varepsilon)}, \\ \dot{\Phi} = \lambda_k(\tau_k)\mathrm{e}^{\nu_0\delta(\varepsilon)\bar{\tau}}, \\ \dot{\phi}_k = \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}), \\ \dot{\psi} = \lambda_k(\tau_k)\mathrm{e}^{\nu_0\delta(\varepsilon)\bar{\tau}} - \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}) = (\nu_0\lambda_k(\tau_k) - \frac{\mathrm{d}\lambda_k}{\mathrm{d}\tau}|_{\tau=\tau_k})\delta(\varepsilon)\bar{\tau} + O(\delta^2(\varepsilon)), \end{cases} \tag{3.26}$$

where $\tilde{A}_k(s)$ and $\tilde{B}_k(s)$ are given by (3.17). By differentiating (3.10) with respect to τ, we obtain

$$\begin{aligned}\frac{1}{\cos^2(\lambda_k(\tau))}\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau} &= \frac{\rho L v_0 \bar{l}(\tau)}{m\lambda_k(\tau)} - \frac{\rho L \bar{l}(\tau)}{m\lambda_k^2(\tau)}\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\\ \Rightarrow \frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau} &= \frac{\rho L v_0 \bar{l}(\tau)\lambda_k(\tau)\cos^2(\lambda_k(\tau))}{m\lambda_k^2(\tau)+\rho L\bar{l}(\tau)\cos^2(\lambda_k(\tau))}.\end{aligned} \tag{3.27}$$

This implies for $\dot{\psi}$ (see (3.26)) that

$$\dot{\psi} = \gamma\delta(\varepsilon)\bar{\tau} + O(\delta^2(\varepsilon)), \quad \gamma = \frac{m v_0 \lambda_k^3(\tau_k)}{m\lambda_k^2(\tau_k)+\rho L\bar{l}(\tau_k)\cos^2(\lambda_k(\tau_k))} \neq 0. \tag{3.28}$$

It now follows from (3.26) and (3.28) that a balance in system (3.26) occurs by choosing $\frac{\varepsilon}{\delta(\varepsilon)} = \delta(\varepsilon)$, that is, $\delta(\varepsilon) = \sqrt{\varepsilon}$. This is the size of the resonance zone. So, together with $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$, it follows from (3.26) that

$$\bar{\tau} = \sqrt{\varepsilon}(s - s_k), \qquad s_k = \frac{\tau_k}{\varepsilon}. \tag{3.29}$$

Further, from (3.28), we obtain $\psi(s) = \psi(s_k) + \frac{1}{2}\gamma\varepsilon(s-s_k)^2$. Hence, in the resonance zone, we can write

$$\sin(\psi(s)) = \sin\!\Big(\frac{1}{2}\gamma\varepsilon(s-s_k)^2 + \frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1) - \phi_k(s_k)\Big), \qquad s_k = \frac{\tau_k}{\varepsilon}, \tag{3.30}$$

where τ_k is given by (3.20). So, let us average system (3.25) over the fast variables. Then, the averaged equations for A_k and B_k become

$$\dot{A}_k^a = -\varepsilon\zeta(\tau)A_k^a + \frac{\varepsilon\alpha^2\beta_0\bar{l}^2(\tau)d_k(\tau)}{2\lambda_k(\tau)}\sin(\psi(s)), \dot{B}_k^a = -\varepsilon\zeta(\tau)B_k^a - \frac{\varepsilon\alpha^2\beta_0\hat{l}^2(\tau)d_k(\tau)}{2\lambda_k(\tau)}\cos(\psi(s)), \tag{3.31}$$

where the upper index a indicates that this is the averaged function. It follows from (3.30) and (3.31) that A_k^a can be written as

$$\begin{aligned}A_k^a(s) \;=\; &\frac{G_k}{\lambda_k(0)}\mathrm{e}^{-\int_0^{\varepsilon s}\zeta(\varrho)\mathrm{d}\varrho}\\ &+\varepsilon\alpha^2\beta_0\mathrm{e}^{-\int_0^{\varepsilon s}\zeta(\varrho)\mathrm{d}\varrho}\int_0^s \frac{\bar{l}^2(\varepsilon\bar{s})d_k(\varepsilon\bar{s})}{2\lambda_k(\varepsilon\bar{s})}\mathrm{e}^{-\int_0^{-\varepsilon\bar{s}}\zeta(\varrho)\mathrm{d}\varrho}\sin[\frac{1}{2}\gamma\varepsilon(\bar{s}-s_k)^2\\ &+\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1)-\phi_k(s_k)]\mathrm{d}\bar{s},\end{aligned}$$

where $\zeta(\tau)$ is given by (3.23). For $\bar{s} = s_k + O(\frac{1}{\sqrt{\varepsilon}})$, $\tau_k = \varepsilon s_k$, we can observe that

$$\frac{\bar{l}^2(\varepsilon\bar{s})d_k(\varepsilon\bar{s})}{2\lambda_k(\varepsilon\bar{s})}\mathrm{e}^{-\int_0^{-\varepsilon\bar{s}}\zeta(\varrho)\mathrm{d}\varrho} = \frac{\bar{l}^2(\varepsilon s_k)d_k(\varepsilon s_k)}{2\lambda_k(\varepsilon s_k)}\mathrm{e}^{-\int_0^{-\varepsilon s_k}\zeta(\varrho)\mathrm{d}\varrho} + O(\sqrt{\varepsilon}).$$

Then, it follows from (3.32) that

$$A_k^a(s) \;=\; \frac{G_k}{\lambda_k(0)}\mathrm{e}^{-\int_0^{\varepsilon s}\zeta(\varrho)\mathrm{d}\varrho}$$

$$+\frac{\varepsilon\alpha^2\beta_0\bar{l}^2(\tau_k)d_k(\tau_k)}{2\lambda_k(\tau_k)}\int_0^s sin[\frac{1}{2}\gamma\varepsilon(\bar{s}-s_k)^2+\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k)]d\bar{s}$$
$$+\text{high order terms in } \varepsilon.$$

By setting $u=\sqrt{\frac{1}{2}\gamma\varepsilon}(\bar{s}-s_k)$, we obtain

$$\begin{aligned}
&\varepsilon\int_0^s \sin[\frac{1}{2}\gamma\varepsilon(\bar{s}-s_k)^2+\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon l_1 s_k}-1)-\phi_k(s_k)]d\bar{s}\\
=&\sqrt{\varepsilon}\bar{\alpha}\int_{-\sqrt{\varepsilon}\bar{\beta}s_k}^{\sqrt{\varepsilon}\bar{\beta}(s-s_k)}\sin(u^2+\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k))du\\
=&\sqrt{\varepsilon}\bar{\alpha}\sin(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k))C_{Fr}(s,s_k)\\
&+\sqrt{\varepsilon}\bar{\alpha}\cos(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k))S_{Fr}(s,s_k),
\end{aligned}$$

where γ is given by (3.28), and where $\bar{\alpha}=\sqrt{\frac{2}{\gamma}}$, $\bar{\beta}=\sqrt{\frac{\gamma}{2}}$ and

$$C_{Fr}(s,s_k)=\int_{-\sqrt{\varepsilon}\bar{\beta}s_k}^{\sqrt{\varepsilon}\bar{\beta}(s-s_k)}\cos(u^2)du,\quad S_{Fr}(s,s_k)=\int_{-\sqrt{\varepsilon}\bar{\beta}s_k}^{\sqrt{\varepsilon}\bar{\beta}(s-s_k)}\sin(u^2)du,\quad s_k=\frac{\tau_k}{\varepsilon}. \tag{3.32}$$

Actually the presence of the Fresnel functions $C_{Fr}(s)$ and $S_{Fr}(s)$ cause resonance jumps in the system. The integrals $C_{Fr}(s)$ and $S_{Fr}(s)$ are plotted in Figure 3.2 with $\varepsilon=0.01$, $\bar{\beta}=1$, and $s_k=100$, respectively. B_k^a can also be approximated in a similar expression as

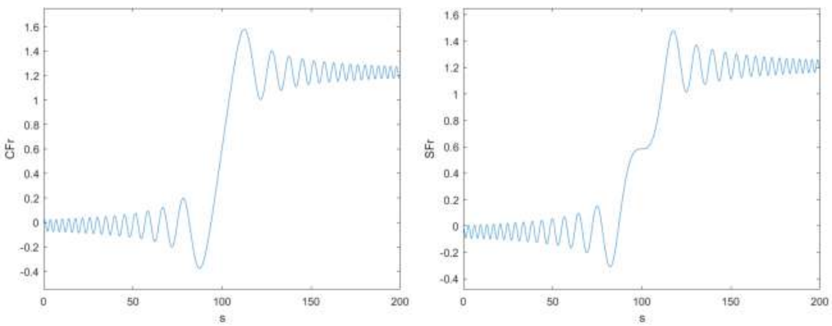

Figure 3.2: (a) $C_{Fr}(s,s_k)$ has a resonance jump from $O(\sqrt{\varepsilon})$ to $O(1)$ around s=100. (b) $S_{Fr}(s,s_k)$ has a resonance jump from $O(\sqrt{\varepsilon})$ to $O(1)$ around s=100.

for A_k^a. So, in the resonance zone, the solution of $\tilde{T}_k(s)$ for problem (3.13) is given by

$$\begin{aligned}
\tilde{T}_k(s) =& \sqrt{\varepsilon}M_k[(\sin(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k))C_{Fr}(s,s_k)\\
&+\cos(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k))S_{Fr}(s,s_k))\sin(\phi_k(s))-(\cos(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)\\
&-\phi_k(s_k))C_{Fr}(s,s_k)-\sin(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k))S_{Fr}(s,s_k))\cos(\phi_k(s))]
\end{aligned}$$

$+O(\varepsilon),$

where $C_{Fr}(s,s_k)$ and $S_{Fr}(s,s_k)$ are given in (3.32), and

$$M_k = \frac{\bar{\alpha}\alpha^2\beta_0\bar{l}^2(\tau_k)d_k(\tau_k)}{2\lambda_k(\tau_k)}, \tag{3.33}$$

3

where $\bar{\alpha}$ is given in (3.32). Thus, the resonance always occurs for s near s_k and the size of the resonance zone in s is of $O(\frac{1}{\sqrt{\varepsilon}})$. For $O(\varepsilon)$ initial conditions and for an $O(\varepsilon)$ external, harmonic excitation, an $O(\sqrt{\varepsilon})$ amplitude modal response will occur. And for a fixed fundamental excitation frequency α, many resonance manifolds arise. The solution $\bar{u}(\xi,s)$ in (3.5) (see also Appendix B.3 (3.65)) is given by

$$\begin{aligned}\bar{u}(\xi,s) \quad = \quad & \sum_{k=1}^{\infty}\sqrt{\varepsilon}M_k[(\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1))C_{Fr}(s,s_k)+\cos(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}\\ & -1))S_{Fr}(s,s_k))\sin(\int_{s_k}^{s}\lambda_k(\varepsilon\bar{s})\mathrm{d}\bar{s})+(-\cos(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1))C_{Fr}(s,s_k)\\ & +\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1))S_{Fr}(s,s_k))\cos(\int_{s_k}^{s}\lambda_k(\varepsilon\bar{s})\mathrm{d}\bar{s})]\sin(\lambda_k(\varepsilon s)\xi)\\ & +O(\varepsilon).\end{aligned} \tag{3.34}$$

In the next section, the timescales as found by using the averaging method in this section will be used again to construct accurate approximations of the solutions for problem (3.13) by using a three-timescales perturbation method.

3.4. FORMAL APPROXIMATION

3.4.1. ANALYSIS RESULTS BY USING A THREE-TIMESCALES PERTURBATION METHOD

IN this section the solution of problem (3.13) will be approximated by using a three-timescales perturbation method. This method can be applied to construct more accurate approximations of the solutions for problem (3.13) and can be applied to test the accuracy of the analytical results as obtained in the previous sections. It will turn out that the approximation as constructed in this section coincides up to order $\sqrt{\varepsilon}$ with the approximation as constructed in the previous section by using the averaging method. The Liouville-Green transformation and the following standard transformations are introduced (for fixed k) to study problem (3.13):

$$\tilde{T}_{k,s} = \lambda_k(\tau)\hat{T}_{k,\phi_k}, \qquad \tilde{T}_{k,ss} = \lambda_k^2(\tau)\hat{T}_{k,\phi_k\phi_k} + \varepsilon\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\hat{T}_{k,\phi_k}, \tag{3.35}$$

where $\tilde{T}_k(s) = \hat{T}_k(\phi_k(s))$ and $\phi_k(s)$ is given by (3.14). Substituting the transformations (3.35) into (3.13), we obtain the following problem for $\hat{T}_k(\phi_k)$:

$$\begin{cases} \hat{T}_{k,\phi_k\phi_k} + \hat{T}_k = -\varepsilon \frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau} \frac{1}{\lambda_k^2(\tau)} \hat{T}_{k,\phi_k} + \varepsilon(v_0 - c_0\bar{l}(\tau)) \frac{1}{\lambda_k(\tau)} \hat{T}_{k,\phi_k} + 2\sum_{n=1}^{\infty} \varepsilon \frac{v_0 c_{n,k}^2(\tau)}{\lambda_n(\tau)} \hat{T}_{n,\phi_n} \\ \qquad -2\sum_{n=1}^{\infty} \varepsilon c_{n,k}^1(\tau) \frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} \frac{1}{\lambda_n(\tau)} \hat{T}_{n,\phi_n} + \sum_{n=1}^{\infty} \varepsilon \frac{m(c_0 - c_{uo})}{\rho L} c_{n,k}^3(\tau) \frac{1}{\lambda_n(\tau)} \hat{T}_{n,\phi_n} \\ \qquad +\varepsilon\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k^2(\tau)} \sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau} - 1)) + O(\varepsilon^2), \ \ t \geq 0, \\ \hat{T}_k(0) = F_k, \\ \hat{T}_{k,\phi_k}(0) = \frac{G_k}{\lambda_k(0)}, \end{cases} \tag{3.36}$$

where $\tau = \varepsilon s$ is a function of ϕ_k. In the previous section, it was shown that (under certain conditions on the fundamental excitation frequency α) resonance can occur around times s_k, for $k = 1, 2, \ldots$. In order to construct accurate approximations in the neighborhood of s_k, we rescale s with $s = \tilde{s} + s_k$, $\tau = \varepsilon\tilde{s} + \tau_k$, and $\phi_k(s) = \phi_k(\tilde{s} + s_k) = \tilde{\phi}_k(\tilde{s}) = \int_{-s_k}^{\tilde{s}} \lambda_k(\tau_k + \varepsilon\bar{s})\mathrm{d}\bar{s}$. So, problem (3.36) can be rewritten for the function $\hat{T}_k(\tilde{\phi}_k)$ in:

$$\begin{cases} \hat{T}_{k,\tilde{\phi}_k\tilde{\phi}_k} + \hat{T}_k = -\varepsilon \frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau} \frac{1}{\lambda_k^2(\tau)} \hat{T}_{k,\tilde{\phi}_k} + \varepsilon(v_0 - c_0\bar{l}(\tau)) \frac{1}{\lambda_k(\tau)} \hat{T}_{k,\tilde{\phi}_k} + 2\sum_{n=1}^{\infty} \varepsilon \frac{v_0 c_{n,k}^2(\tau)}{\lambda_n(\tau)} \hat{T}_{n,\tilde{\phi}_n} \\ \qquad -2\sum_{n=1}^{\infty} \varepsilon c_{n,k}^1(\tau) \frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} \frac{1}{\lambda_n(\tau)} \hat{T}_{n,\tilde{\phi}_n} + \sum_{n=1}^{\infty} \varepsilon \frac{m(c_0 - c_{uo})}{\rho L} c_{n,k}^3(\tau) \frac{1}{\lambda_n(\tau)} \hat{T}_{n,\tilde{\phi}_n} \\ \qquad +\varepsilon\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k^2(\tau)} \sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau} - 1)) + O(\varepsilon^2), \ \ t \geq 0, \\ \hat{T}_k(0) = F_k, \\ \hat{T}_{k,\tilde{\phi}_k}(0) = \frac{G_k}{\lambda_k(0)}, \end{cases} \tag{3.37}$$

where τ is a function of $\tilde{\phi}_k$. Next we study problem (3.37) in detail. The application of the straightforward expansion method to solve (3.37) will result in the occurrence of so-called secular terms which cause the approximations of the solutions to become unbounded on long timescales. And it has been shown in the previous section that the $O(\varepsilon)$ excitation can produce timescale of $O(\sqrt{\varepsilon})$. Therefore, to avoid these secular terms, we introduce three timescales $\tilde{s}_0 = \tilde{s}$, $\tilde{s}_1 = \sqrt{\varepsilon}\tilde{s}$, $\tilde{s}_2 = \varepsilon\tilde{s}$, $\tau = \tilde{s}_2 + \tau_k$, and so $\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}$ are introduced as follows:

$$\tilde{\phi}_{k,0} = \int_a^{\tilde{s}_0} \lambda_k(\tau_k + \varepsilon\bar{s})\mathrm{d}\bar{s}, \quad \tilde{\phi}_{k,1} = \int_b^{\tilde{s}_1} \lambda_k(\tau_k + \sqrt{\varepsilon}\bar{s})\mathrm{d}\bar{s}, \quad \tilde{\phi}_{k,2} = \int_c^{\tilde{s}_2} \lambda_k(\tau_k + \bar{s})\mathrm{d}\bar{s},$$

where $a = -s_k, \quad b = -\sqrt{\varepsilon}s_k, \quad c = -\varepsilon s_k$. These scalings are based on the size of the resonance zone (which has been found in the previous section), and on the natural scalings for weakly nonlinear equations such as (3.37). By using the three timescales perturbation method, the function $\hat{T}_k(\tilde{\phi}_k;\sqrt{\varepsilon})$ is supposed to be a function of $\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}$, that is,

$$\hat{T}_k(\tilde{\phi}_k;\sqrt{\varepsilon}) = w_k(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2};\sqrt{\varepsilon}). \tag{3.38}$$

By substituting (3.38) into (3.37), we obtain the following equations up to $O(\varepsilon\sqrt{\varepsilon})$:

$$\begin{cases} \frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,0}^2} + w_k + 2\sqrt{\varepsilon}\frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,1}} + \varepsilon(2\frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,2}} + \frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,1}^2}) + 2\varepsilon\sqrt{\varepsilon}\frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,1}\partial\tilde{\phi}_{k,2}} \\ = \varepsilon[-\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{\lambda_k^2(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,0}} + (v_0 - c_0\bar{l}(\tau))\frac{1}{\lambda_k(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,0}} + \alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k^2(\tau)}\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau}-1)) \\ -2\sum_{n=1}^{\infty}(c_{n,k}^1(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} - v_0c_{n,k}^2(\tau) - \frac{m(c_0-c_{uo})}{2\rho L}c_{n,k}^3(\tau))\frac{1}{\lambda_n(\tau)}\frac{\partial w_n}{\partial\tilde{\phi}_{n,0}}] \\ +\varepsilon\sqrt{\varepsilon}[-\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{\lambda_k^2(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,1}} + (v_0 - c_0\bar{l}(\tau))\frac{1}{\lambda_k(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,1}} \\ -2\sum_{n=1}^{\infty}(c_{n,k}^1(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} - v_0c_{n,k}^2(\tau) - \frac{m(c_0-c_{uo})}{2\rho L}c_{n,k}^3(\tau))\frac{1}{\lambda_n(\tau)}\frac{\partial w_n}{\partial\tilde{\phi}_{n,1}}], \\ w_k(0,0,0;\sqrt{\varepsilon}) = F_k, \\ \frac{\partial w_k}{\partial\tilde{\phi}_{k,0}}(0,0,0;\sqrt{\varepsilon}) + \sqrt{\varepsilon}\frac{\partial w_k}{\partial\tilde{\phi}_{k,1}}(0,0,0;\sqrt{\varepsilon}) + \varepsilon\frac{\partial w_k}{\partial\tilde{\phi}_{k,2}}(0,0,0;\sqrt{\varepsilon}) = \frac{G_k}{\lambda_k(0)}, \end{cases} \tag{3.39}$$

where $F_k = \varepsilon\bar{F}_k$ and $G_k = \varepsilon\bar{G}_k$ are $O(\varepsilon)$, and τ is a function of $\tilde{\phi}_{k,2}$. By using a three-timescales perturbation method, the function $w_k(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon})$ is approximated by the formal asymptotic expansion

$$\begin{aligned} w_k(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon}) &= \sqrt{\varepsilon}w_{k,0}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon}) + \varepsilon w_{k,1}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon}) \\ &\quad +\varepsilon\sqrt{\varepsilon}w_{k,2}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon}) + O(\varepsilon^2). \end{aligned} \tag{3.40}$$

By substituting (3.40) into problem (3.39), and after equating the coefficients of like powers in $\sqrt{\varepsilon}$, we obtain as:

the $O(\sqrt{\varepsilon})$-problem:

$$\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,0}^2} + w_{k,0} = 0,\ w_{k,0}(0,0,0) = 0, \quad \frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,0}}(0,0,0) = 0, \tag{3.41}$$

the $O(\varepsilon)$-problem:

$$\begin{aligned} \frac{\partial^2 w_{k,1}}{\partial\tilde{\phi}_{k,0}^2} + w_{k,1} &= -2\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,1}} + \alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k^2(\tau)}\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau}-1)), \\ w_{k,1}(0,0,0) &= \bar{F}_k, \quad \frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,0}}(0,0,0) = -\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,1}}(0,0,0) + \frac{\bar{G}_k}{\lambda_k(0)}, \end{aligned} \tag{3.42}$$

and the $O(\varepsilon\sqrt{\varepsilon})$-problem:

$$\begin{aligned} \frac{\partial^2 w_{k,2}}{\partial\tilde{\phi}_{k,0}^2} + w_{k,2} &= -2\frac{\partial^2 w_{k,1}}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,1}} - 2\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,2}} - \frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,1}^2} + [(v_0 - c_0\bar{l}(\tau))\lambda_k(\tau) \\ &\quad -\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}]\frac{1}{\lambda_k^2(\tau)}\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,0}} - 2\sum_{n=1}^{\infty}[c_{n,k}^1(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} - v_0c_{n,k}^2(\tau) \\ &\quad -\frac{m(c_0-c_{uo})}{2\rho L}c_{n,k}^3(\tau)]\frac{1}{\lambda_n(\tau)}\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,0}}, \\ w_{k,2}(0,0,0) &= 0, \quad \frac{\partial w_{k,2}}{\partial\tilde{\phi}_{k,0}}(0,0,0) = -\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,2}}(0,0,0) - \frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,1}}(0,0,0). \end{aligned} \tag{3.43}$$

The $O(\sqrt{\varepsilon})$– problem has as solution

$$w_{k,0}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon}) = C_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})\sin(\tilde{\phi}_{k,0}) + C_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})\cos(\tilde{\phi}_{k,0}), \tag{3.44}$$

where $C_{k,1}$ and $C_{k,2}$ are still unknown functions of the slow variables $\tilde{\phi}_{k,1}$ and $\tilde{\phi}_{k,2}$, and they can be determined by avoiding secular terms in the solutions of the $O(\varepsilon)-$ and the $O(\varepsilon\sqrt{\varepsilon})-$ problems (see Appendix B.4). Before entering the resonance zone, the following result is found:

$$C_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) = 0, \quad C_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) = 0, \tag{3.45}$$

and inside of the resonance zone, to avoid secular terms in the solution $w_{k,1}$ and $w_{k,2}$, it turns out that

$$\begin{aligned} C_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) &= \bar{\alpha}\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{2\lambda_k(\tau)}[\sin(\vartheta(s_k))\bar{C}_{Fr}(\tilde{s}_1)+\cos(\vartheta(s_k))\bar{S}_{Fr}(\tilde{s}_1)], \\ C_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) &= \bar{\alpha}\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{2\lambda_k(\tau)}[\cos(\vartheta(s_k))\bar{C}_{Fr}(\tilde{s}_1)-\sin(\vartheta(s_k))\bar{S}_{Fr}(\tilde{s}_1)], \end{aligned} \tag{3.46}$$

where γ is given by (3.28), $\bar{\alpha}$ and $\bar{\beta}$ are given by (3.32),

$$\begin{aligned} \vartheta(s_k) &= \frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k), \\ \bar{C}_{Fr}(\tilde{s}_1) &= \int_{\bar{\beta}b}^{\bar{\beta}\tilde{s}_1}\cos(u^2)\mathrm{d}u, \quad \bar{S}_{Fr}(\tilde{s}_1) = \int_{\bar{\beta}b}^{\bar{\beta}\tilde{s}_1}\sin(u^2)\mathrm{d}u. \end{aligned} \tag{3.47}$$

Further, to obtain more accurate approximations of problem (3.39), the $O(\varepsilon)-$ problem and the $O(\varepsilon\sqrt{\varepsilon})-$ problem can also be solved by using a similar analysis as for the $O(\sqrt{\varepsilon})-$ problem in Appendix B.4. At this moment, only the first term in the expansion of the solution for the cable problem is important from the physical point of view. So, to shorten the paper, we are not interested in high-order approximations.

Thus, from (3.40), an approximation of the solution of Eq.(3.39) is given by $w(\xi,s) = \sum_{n=1}^{\infty}\sqrt{\varepsilon}w_{n,0}+O(\varepsilon)$, where $w_{n,0}$, is given by (3.44). It follows from (3.20), for a given value of α, that around $\tau=\tau_n=\frac{1}{v_0}\ln(\frac{arctan(\frac{\rho L}{\alpha m})+(n-1)\pi}{\alpha l_0})$ the n-th oscillation mode jumps up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. For such a jump the inequality $arctan(\frac{\rho L}{\alpha m})+(n-1)\pi \geq \alpha l_0$ needs to be satisfied. This implies that it might occur that the first few modes do not show this jump, but all that the higher order modes do. Before entering the resonance zone for the n-th oscillation mode $w_{n,0}\equiv 0$ and in the resonance zone the n-th oscillation mode $w_{n,0}$ is given by

$$\begin{aligned} & w_{n,0}(\tilde{\phi}_{n,0},\tilde{\phi}_{n,1},\tilde{\phi}_{n,2};\sqrt{\varepsilon}) \\ = & \frac{1}{2}\alpha^2\beta_0\bar{l}^2(\tau_n)\frac{d_n(\tau_n)}{\lambda_n(\tau_n)}\bar{\alpha}[\sin(\vartheta(s_n))\bar{C}_{Fr}(\tilde{s}_1)+\cos(\vartheta(s_n))\bar{S}_{Fr}(\tilde{s}_1)]\sin(\tilde{\phi}_{n,0}) \\ & -\frac{1}{2}\alpha^2\beta_0\bar{l}^2(\tau_n)\frac{d_n(\tau_n)}{\lambda_n(\tau_n)}\bar{\alpha}[\cos(\vartheta(s_n))\bar{C}_{Fr}(\tilde{s}_1)-\sin(\vartheta(s_n))\bar{S}_{Fr}(\tilde{s}_1)]\cos(\tilde{\phi}_{n,0}) \\ = & M_n[(\sin(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_n}-1))\bar{C}_{Fr}(\tilde{s}_1)+\cos(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_n}-1))\bar{S}_{Fr}(\tilde{s}_1))\sin(\tilde{\phi}_{n,0}-\phi_n(s_n)) \\ & +(-\cos(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_n}-1))\bar{C}_{Fr}(\tilde{s}_1) \\ & +\sin(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_n}-1))\bar{S}_{Fr}(\tilde{s}_1))\cos(\tilde{\phi}_{n,0}-\phi_n(s_n))], \end{aligned} \tag{3.48}$$

where $\tilde{\phi}_{k,0} - \phi_k(s_k) = \int_{s_k}^{s} \lambda_k(\varepsilon\bar{s})\mathrm{d}\bar{s}$, γ, $\bar{\alpha}$, $\vartheta(s_n)$, $\bar{C}_{Fr}(\tilde{s}_1)$ and $\bar{S}_{Fr}(\tilde{s}_1)$ are given by (3.28), (3.32), (3.47) and M_n is given by (3.33). The solution $w_{n,0}$ in (3.48) implies a resonance jump from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$ around τ_n in the n-th oscillation mode. Thus, the solution $\bar{u}(\xi, s)$ in (3.5) (see also Appendix B (3.65)) is given by

$$\bar{u}(\xi,s) \quad = \quad \sum_{k=1}^{\infty} \sqrt{\varepsilon} w_{k,0}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon}) \sin(\lambda_k(\varepsilon s)\xi) + O(\varepsilon). \tag{3.49}$$

The solution in (3.49) is in agreement with (3.34), that is, the one obtained by the averaging method. All in all, introducing the following notation $\chi(t) = \frac{1}{\varepsilon v_0}\ln(\frac{l(t)}{l_0})$, according to the solution of $\bar{u}(\xi, s)$ in (3.49), we obtain as approximation for the solution $\tilde{u}(\xi, t)$ of problem (3.3)

3

$$\begin{aligned}
\tilde{u}(\xi,t) \quad = \quad & \sum_{k=1}^{\infty} \sqrt{\varepsilon} M_k [(\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1))\bar{C}_{Fr}(\sqrt{\varepsilon}(\chi(t)-s_k)) \\
& +\cos(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1))\bar{S}_{Fr}(\sqrt{\varepsilon}(\chi(t)-s_k)))\sin(\int_{s_k}^{\chi(t)} \lambda_k(\varepsilon\bar{s})\mathrm{d}\bar{s}) \\
& +(-\cos(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1))\bar{C}_{Fr}(\sqrt{\varepsilon}(\chi(t)-s_k)) \\
& +\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1))\bar{S}_{Fr}(\sqrt{\varepsilon}(\chi(t)-s_k)))\cos(\int_{s_k}^{\chi(t)} \lambda_k(\varepsilon\bar{s})\mathrm{d}\bar{s})]\sin(\lambda_k(\varepsilon\chi(t))\xi) \\
& +O(\varepsilon),
\end{aligned} \tag{3.50}$$

where $s_k = \frac{\tau_k}{\varepsilon}$ and τ_k is given by (3.20), $\bar{C}_{Fr}, \bar{S}_{Fr}(\tilde{s}_1)$ are given by (3.47), M_k is given by (3.33) and $\xi = \frac{x}{l(t)}$.

3.4.2. NUMERICAL RESULTS

In this section we will present numerical simulations of the vibration response as computed and based on the analytical expressions (3.50). The computations are performed by using the following parameters:

$$\varepsilon = 0.01,\ l_0 = 3,\ v_0 = 1,\ c_0 = 2,\ c_{uo} = 1,\ \rho = 1,\ m = 10,\ L = 10,\ \beta_0 = 1,\ \alpha = 1. \tag{3.51}$$

For simplicity, let us assume that only the initial displacement is prescribed, so that

$$\tilde{u}_0(\xi) = \varepsilon\sin(1.5\xi), \quad \tilde{u}_1(\xi) = 0, \quad 0 \le \xi \le 1. \tag{3.52}$$

It is worth mentioning that the following numerical results are computed based on $O(\varepsilon)$ approximations. Higher-order approximations are neglected due to their insignificant and small contribution to the solution. By using (3.18), we see that the resonance occurs around time instants s_k satisfying

$$\alpha l_0 \mathrm{e}^{v_0 \tau} = \lambda_k, \quad \tau = \varepsilon s. \tag{3.53}$$

By using the Liouville-Green transformation with $\frac{ds}{dt} = \frac{1}{l(t)}$, we obtain $\alpha l(t) = \lambda_k$, $\quad l(t) = l_0 + \varepsilon v_0 t$, which implies that

$$t_k = \frac{\lambda_k - \alpha l_0}{\varepsilon \alpha v_0}, \quad k \in \mathbb{N}, \tag{3.54}$$

where λ_k is given by (3.19). From the analysis in section 4.1, we observe that the resonance times depend on the mode numbers k. Resonance for the first oscillation mode does not occur in this numerical example. For the second, third and forth oscillation modes, resonance emerges for times $t_2 \approx 92.7, t_3 \approx 406.8, t_4 \approx 721.0$, respectively. The solution $\tilde{u}(\xi,t)$ in (3.50), and its corresponding energy are illustrated in Figure 3.3, respectively.

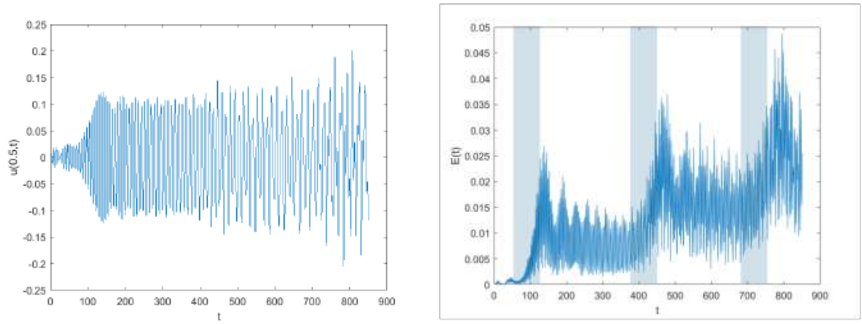

Figure 3.3: Perturbation method: (a) Displacements of the mid-point of the cable. (b) The energy of the cable. The shadowed bands represent the resonance zones.

3.5. NUMERICAL APPROXIMATIONS

IN this section we will directly integrate problem (3.3) with a numerical method. To solve (3.3) numerically, we first rewrite (3.3) as

$$\begin{cases} \tilde{u}_{tt} - \frac{1}{l^2}\tilde{u}_{\xi\xi} = \frac{2v}{l}\xi\tilde{u}_{\xi t} - \frac{2v}{l}\tilde{u}_{\xi t} - c\tilde{u}_t + O(\varepsilon^2), \quad 0 \le \xi \le 1,\ t > 0, \\ \tilde{u}_{\xi\xi}(1,t) + \frac{\rho L l}{m}\tilde{u}_\xi(1,t) = [(c-c_u)l^2\tilde{u}_t]|_{\xi=1} + O(\varepsilon^2), \quad \tilde{u}(0,t) = \tilde{e}(t) = \beta\sin(\alpha t), \quad t > 0, \\ \tilde{u}(\xi,0) = \tilde{u}_0(\xi), \quad \tilde{u}_t(\xi,0) = \tilde{u}_1(\xi), \quad 0 \le \xi \le 1. \end{cases} \tag{3.55}$$

By using the transformation $\tilde{u}(\xi,t) = \check{u}(\xi,t) + \beta\sin(\alpha t) + \xi\frac{ml}{\rho L}(c-c_u)\check{u}_t(1,t)$, problem (3.55) can be written as

$$\begin{cases} \check{u}_{tt} - \frac{1}{l^2}\check{u}_{\xi\xi} = \frac{2v}{l}(\xi-1)\check{u}_{\xi t} - c\check{u}_t + \alpha^2\beta\sin(\alpha t) - \frac{m\xi(c-c_u)}{\rho L l}\check{u}_{\xi\xi t}(1,t) + O(\varepsilon^2), \\ \qquad 0 \le \xi \le 1,\ t > 0, \\ \check{u}_{\xi\xi}(1,t) + \frac{\rho L l}{m}\check{u}_\xi(1,t) = O(\varepsilon^2), \quad \check{u}(0,t) = 0, \quad t > 0, \\ \check{u}(\xi,0) = \check{u}_0(\xi), \quad \check{u}_t(\xi,0) = \check{u}_1(\xi), \quad 0 \le \xi \le 1, \end{cases} \tag{3.56}$$

where $0 \le \xi \le 1$ and $t > 0$. For problem (3.56), we first discretize the partial differential equation in (3.56) in the $\xi-$ coordinate by using a central finite difference scheme. Then, we rewrite the so-obtained discretized equation in a matrix form and use the numerical time integration method of Crank-Nicolson(see Appendix B.5). We will use the same parameter values (3.51) and initial conditions (3.52) as for the analytic approximation, which is presented in the previous section (see also Figure 3.3). Figure 4.3 show the displacements at $\xi = 0.5$ and the vibratory energy of the cable, respectively, for times up to t = 850.

Comparison with Figure 3.3: both Figure 3.3 and Figure 4.3 illustrate that resonances emerge at times $t_1 \approx 92.7, t_2 \approx 406.8, t_3 \approx 721.0$. In the resonance zones the displacements and the energy increase, and between these zones, stay constant (approximately). Around the first resonance time t_1, the displacement amplitudes jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. Around the second resonance time t_2 and the third resonance time t_3, the amplitudes change again at the $O(\sqrt{\varepsilon})$ level, where ε is a small parameter with $\varepsilon = 0.01$. Moreover, we can observe that, between the resonance times, the frequency ranges are similar in Figure 3.3 and Figure 4.3, and the sizes of the resonance zones are of $O(\frac{1}{\sqrt{\varepsilon}})$. Thus, the numerical simulations in Figure 4.3 agree very well with the analytical results as presented in Figure 3.3, respectively.

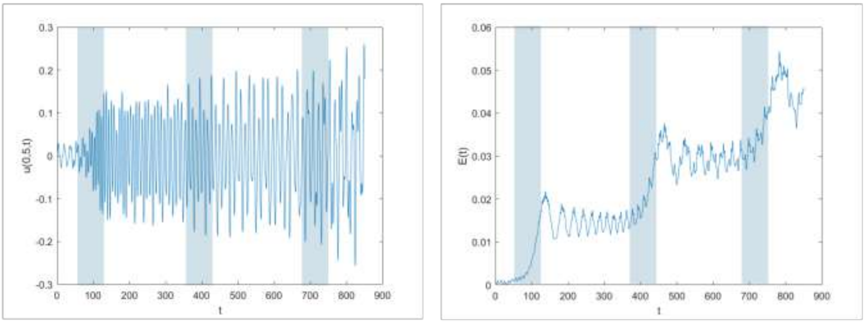

Figure 3.4: Numerical method: (a) Displacements of the mid-point of the cable. (b) The energy of the cable. The shadowed bands represent the resonance zones.

3.6. CONCLUSIONS

In this chapter, the longitudinal vibrations and associated resonances in an elevator system due to a harmonic excitation at one of its boundaries have been studied. The problem is described by a partial differential equation (PDE) on a time-varying spatial interval with a small harmonic disturbance at one end and a moving nonclassical boundary condition at the other end. By assuming that the small harmonic boundary disturbance is of order ε and by assuming that the initial values are also small and of order ε, it is shown in this paper that for a given arbitrary boundary disturbance frequency, many oscillation modes jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. To obtain these results an adapted version of the method of separation of variables is introduced and presented, and perturbation methods, (such as averaging methods, singular perturbation techniques, and multiple timescales perturbation methods) are used. Furthermore, explicit, and accurate approximations of the solution of the initial-boundary value problem are constructed. These approximations are valid on time-scales of order ε^{-1}. Also approximations of the solution of the initial-boundary value problem are computed by using a numerical method.

These numerical approximations are in full agreement with the analytically obtained approximations. The presented methods clearly indicate how more complicated problems can now be treated analytically. Also more complicated boundary conditions and changes of cable length over time can be included in the analysis of these problems. Finally, it should be remarked that we intend to apply the presented analytical approach to nonlinearly coupled transverse and longitudinal vibrations of axially moving cables. For these problems the partial differential equations, the boundary conditions and the nonlinear terms are expected to give challenges, which might be solved by applying the approach as has been presented in this chapter.

APPENDIX B

APPENDIX B.1 THE DERIVATION OF MOTION (3.1)

According to Figure 3.1, the partial differential equation (PDE) can be derived by Hamilton's variational principle:

$$\int_{t_1}^{t_2} (\delta E_k(t) - \delta E_p(t) + \delta W_c(t))\mathrm{d}t = 0. \tag{3.57}$$

The Kinetic energy $E_k(t)$ can be represented as $E_k(t) = \frac{1}{2}\rho\int_0^{l(t)}(\frac{\mathrm{D}u}{\mathrm{D}t}+v)^2\mathrm{d}x + \frac{1}{2}m(\frac{\mathrm{D}u}{\mathrm{D}t}+v)^2|_{x=l(t)}$, the Potential energy $E_p(t)$ can be expressed as $E_p(t) = \frac{1}{2}EA\int_0^{l(t)} u_x^2\mathrm{d}x + \int_0^{l(t)} Tu_x\mathrm{d}x + E_{gs} - \int_0^{l(t)}\rho g u\mathrm{d}x - mgu|_{x=l(t)}$, and

$$\begin{aligned}\delta E_k(t) - \delta E_p(t) \quad = \quad & \rho\int_0^{l(t)}(\frac{\mathrm{D}u}{\mathrm{D}t}+v)\delta\frac{\mathrm{D}u}{\mathrm{D}t}\mathrm{d}x + m(\frac{\mathrm{D}u}{\mathrm{D}t}+v)\delta\frac{\mathrm{D}u}{\mathrm{D}t}|_{x=l(t)} \\ & -[EA\int_0^{l(t)} u_x\delta u_x\mathrm{d}x + \int_0^{l(t)} T\delta u_x\mathrm{d}x \\ & -\int_0^{l(t)}\rho g\delta u\mathrm{d}x - mg\delta u|_{x=l(t)}],\end{aligned} \tag{3.58}$$

where the operator $\frac{\mathrm{D}u}{\mathrm{D}t}$ is defined as $\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = u_t + vu_x$. The virtual work δW_c done by the distributed and the lumped damping force is given by

$$\delta W_c(t) = -\int_0^{l(t)} c\frac{\mathrm{D}u}{\mathrm{D}t}\delta u\mathrm{d}x - c_u\frac{\mathrm{D}u}{\mathrm{D}t}\delta u|_{x=l(t)}. \tag{3.59}$$

By substituting the equations (3.58)-(3.59) into (3.57), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^{l(t)}\rho(\frac{\mathrm{D}u}{\mathrm{D}t}+v)\delta\frac{\mathrm{D}u}{\mathrm{D}t}\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} m(\frac{\mathrm{D}u}{\mathrm{D}t}+v)\delta\frac{\mathrm{D}u}{\mathrm{D}t}|_{x=l(t)}\mathrm{d}t \\ -EA&\int_{t_1}^{t_2}\int_0^{l(t)} u_x\delta u_x\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2}\int_0^{l(t)} T\delta u_x\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2}\int_0^{l(t)}\rho g\delta u\mathrm{d}x\mathrm{d}t \\ +&\int_{t_1}^{t_2} mg\delta u|_{x=l(t)}\mathrm{d}t - \int_{t_1}^{t_2}\int_0^{l(t)} c\frac{\mathrm{D}u}{\mathrm{D}t}\delta u\mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2} c_u\frac{\mathrm{D}u}{\mathrm{D}t}\delta u|_{x=l(t)}\mathrm{d}t \quad = \quad 0.\end{aligned} \tag{3.60}$$

By integrating by parts the integrals in (3.60) it then follows that (3.60) can be rewritten in:

$$\int_{t_1}^{t_2}\int_0^{l(t)}[-\rho(u_{tt}+2vu_{xt}+v^2u_{xx}) + EAu_{xx} + T_x + \rho g - c(u_t+vu_x)]\delta u\mathrm{d}x\mathrm{d}t$$

$$+\int_{t_1}^{t_2}[-m(u_{tt}+2vu_{xt}+v^2u_{xx}+au_x)-EAu_x-T+mg-c_u(u_t+vu_x)]\delta u|_{x=l(t)}\mathrm{d}t$$
$$+\int_{t_1}^{t_2}[\rho v(u_t+vu_x+v)+EAu_x+T]\delta u|_{x=0}\mathrm{d}t \quad = \quad 0.$$

So, the initial boundary value problem of the system can be obtained as

$$\rho(u_{tt}+2vu_{xt}+v^2u_{xx})-EAu_{xx}-T_x-\rho g+c(u_t+vu_x)=0, \quad 0\le x\le l(t),\ t>0, \tag{3.61}$$
$$[m(u_{tt}+2vu_{xt}+v^2u_{xx})+EAu_x+T-mg+c_u(u_t+vu_x)]|_{x=l(t)}=0, \quad t>0, \tag{3.62}$$
$$EAu_x+T+\rho v(u_t+vu_x+v)|_{x=0}=0, \quad t>0. \tag{3.63}$$

Note that (3.62) and (3.63) are the natural boundary conditions. However, the natural boundary condition (3.63) is not appropriate for our problem, since the cable at the top has an assumed and prescribed displacement $e(t)$, which is supposed to be generated by the catenary system (consisting of drum, head sheave) in vertical direction. Thus, the boundary condition is given by $u(e(t),t)=e(t)$, $t\ge 0$. By using the Taylor expansion for $u(x,t)$ in x for $x=0$, and by assuming that $e(t)$ and $u(x,t)$ are small, the boundary condition

$$e(t)=u(e(t),t)=u(0,t)+e(t)\frac{\partial u}{\partial x}(0,t)+O(e^2(t))$$

can be approximated by $u(0,t)=e(t)$. Since the tension $T(x,t)$ is given by $T(x,t)=[m+\rho(l(t)-x)]g$, $0\le x\le l(t)$, it then follows that the initial boundary value problem for the axially moving hoisting rope is given by (3.1).

APPENDIX B.2 TRANSFORMATION TO A FIXED DOMAIN

By introducing a new time-like variable $s(t)$ with $\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{1}{l(t)}$, $l(t)=\hat{l}(s)=l_0\mathrm{e}^{\varepsilon v_0 s}$, All partial derivatives then become $s=\frac{1}{\varepsilon v_0}\ln(\frac{l(t)}{l_0})$, $\tilde{u}_t=\frac{1}{\hat{l}}\bar{u}_s$, $\tilde{u}_{\xi t}=\frac{1}{\hat{l}}\bar{u}_{\xi s}$, $\tilde{u}_{tt}=\frac{1}{\hat{l}^2}\bar{u}_{ss}-\frac{v}{\hat{l}}\bar{u}_s$, $\tilde{e}(t)=\beta\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s}-1))$, where $\tilde{u}(\xi,t)=\bar{u}(\xi,s)$. Substituting these derivatives into (3.3), we obtain the following problem for $\bar{u}(\xi,s)$:

$$\begin{cases}\bar{u}_{ss}-\bar{u}_{\xi\xi}=v\bar{u}_s+2v\xi\bar{u}_{\xi s}-2v\bar{u}_{\xi s}-c\hat{l}\bar{u}_s+O(\varepsilon^2), & 0\le\xi\le 1,\ s>0,\\ \bar{u}_{ss}(1,s)+\frac{\rho L\hat{l}}{m}\bar{u}_\xi(1,s)=[v\hat{l}\bar{u}_s+2v\xi\bar{u}_{\xi s}-2v\bar{u}_{\xi s}-c_u\hat{l}\bar{u}_s]|_{\xi=1}+O(\varepsilon^2), & s>0,\\ \bar{u}(0,s)=\beta\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s}-1)), & s>0,\\ \bar{u}(\xi,0)=f(\xi), \quad \bar{u}_s(\xi,0)=g(\xi), & 0\le\xi\le 1,\end{cases} \tag{3.64}$$

where $\hat{l}=\hat{l}(s)$, $f(\xi)=\tilde{u}_0(\xi)$ and $g(\xi)=l_0\tilde{u}_1(\xi)$. By using the PDE, the boundary condition at $\xi=1$ can be rewritten and we obtain from (3.64) the following problem:

$$\begin{cases}\bar{u}_{ss}-\bar{u}_{\xi\xi}=v\bar{u}_s+2v\xi\bar{u}_{\xi s}-2v\bar{u}_{\xi s}-c\hat{l}\bar{u}_s+O(\varepsilon^2), & 0\le\xi\le 1,\ s>0,\\ \bar{u}_{\xi\xi}(1,s)+\frac{\rho L\hat{l}}{m}\bar{u}_\xi(1,s)=(c-c_u)\hat{l}\bar{u}_s(1,s)+O(\varepsilon^2), & s>0,\\ \bar{u}(0,s)=\bar{e}(s)=\beta\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s}-1)), & s>0,\\ \bar{u}(\xi,0)=f(\xi), \quad \bar{u}_s(\xi,0)=g(\xi), & 0\le\xi\le 1.\end{cases} \tag{3.65}$$

APPENDIX B.3 AN ADAPTED VERSION OF THE METHOD OF SEPARATION OF VARIABLES

By substituting $T(s,\tau)X(\xi,\tau)$ into the partial differential equation in (3.8),we obtain

$$\frac{T_{ss}(s,\tau)}{T(s,\tau)} + O(\varepsilon) = \frac{X_{\xi\xi}(\xi,\tau)}{X(\xi,\tau)}, \quad 0 \le \xi \le 1, \quad s > 0, \quad \tau > 0. \tag{3.66}$$

The $O(1)$ part of the left-hand side of equation (3.66) is a function of s and τ, and the right-hand side is a function of ξ and τ. To be equal, both sides need to be equal to a function of τ. Let this function be $-\lambda^2(\tau)$ (which will be defined later), so we obtain from (3.66) by neglecting terms of order ε: $\frac{T_{ss}(s,\tau)}{T(s,\tau)} = \frac{X_{\xi\xi}(\xi,\tau)}{X(\xi,\tau)} = -\lambda^2(\tau)$, implying:

$$X_{\xi\xi}(\xi,\tau) + \lambda^2(\tau)X(\xi,\tau) = 0, \quad T_{ss}(s,\tau) + \lambda^2(\tau)T(s,\tau) = 0, \; 0 \le \xi \le 1, \; s > 0, \; \tau > 0. \tag{3.67}$$

In accordance with the first equation for $X(\xi,\tau)$ in (3.67) and boundary conditions in (3.8), a nontrivial solution $X_n(\xi,\tau)$ is

$$X_n(\xi,\tau) = B_n(\tau)\sin(\lambda_n(\tau)\xi), \tag{3.68}$$

where $B_n(\tau)$ is an arbitrary function of τ only, and $\lambda_n(\tau)$ is given by (3.10). Assuming that $\frac{\rho L l_0}{m} = 1$, the values of $\lambda_n(0)$ can be obtained in Figure 3.5. It should be observed that the eigenfunctions $X_n(\xi,\tau)$ are orthogonal on $0 < \xi < 1$. And so, The general solution

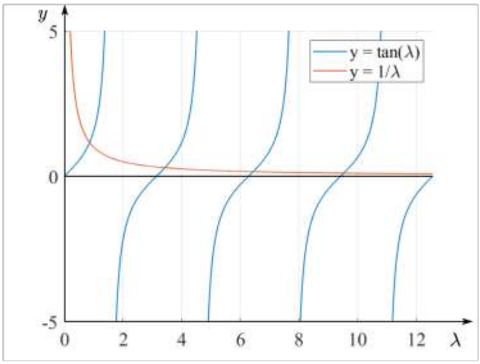

Figure 3.5: For s=0, intersection points of $y = tan\lambda$ and $y = \frac{1}{\lambda}$ are giving $\lambda_n(0)$.

of (3.7) - (3.8) can be expanded in the form in (3.9). By substituting Eq.(3.9) into the nonhomogeneous governing equation and initial conditions in (3.6), we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} [(\bar{T}_{n,ss} + 2\varepsilon\bar{T}_{n,s\tau} + \lambda_n^2(\tau)\bar{T}_n)\sin(\lambda_n(\tau)\xi) + 2\varepsilon\xi\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau}\bar{T}_{n,s}\cos(\lambda_n(\tau)\xi)] \\
= &\sum_{n=1}^{\infty} \varepsilon[(v_0 - c_0\bar{l}(\tau))\bar{T}_{n,s}\sin(\lambda_n(\tau)\xi) + 2(\xi-1)v_0\lambda_n(\tau)\bar{T}_{n,s}cos(\lambda_n(\tau)\xi) \\
&+ \frac{m(c_0 - c_{uo})\lambda_n^2(\tau)\xi}{\rho L}\bar{T}_{n,s}\sin(\lambda_n(\tau))] + \varepsilon\beta_0\alpha^2\bar{l}^2(\tau)\sin(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s} - 1)) + O(\varepsilon^2),
\end{aligned}$$

$$\sum_{n=1}^{\infty}[\bar{T}_n(0,0)\sin(\lambda_n(0)\xi)] = W_0(\xi),$$
$$\sum_{n=1}^{\infty}[(\bar{T}_{n,s}(0,0)+\varepsilon\bar{T}_{n,\tau}(0,0))\sin(\lambda_n(0)\xi)$$
$$+\varepsilon\bar{T}_n(0,0)\frac{d\lambda_n(0)}{d\tau}\xi\cos(\lambda_n(0)\xi)] = W_1(\xi). \quad (3.69)$$

Now, let $\sigma(\tau,\xi) = 1+\frac{2m}{\rho L\bar{I}(\tau)}\delta(\xi-1)$ be a weight function, where $\delta(\xi-1)$ is the Dirac delta function (with $\delta(\xi-1)=0$ for $\xi\neq 1$, and $\int_0^1\delta(\xi-1)\mathrm{d}\xi=\frac{1}{2}$). By multiplying the first equation in (3.69) by $\sigma(\tau,\xi)\sin(\lambda_k(\tau)\xi)$, and the second and third equations in (3.69) with $\sigma(0,\xi)\sin(\lambda_k(0)\xi)$, by integrating the so-obtained equation from $\xi=0$ to $\xi=1$, and by using the fact that the $\sin(\lambda_k(\tau)\xi)$ functions subject to the inner product with weight function $\sigma(\tau,\xi)$ are orthogonal on $0\leq\xi\leq 1$, it follows that $\bar{T}_k(s,\tau)$ for $k=1,2,3,\ldots$, and $s>0,\tau>0$ have to satisfy (3.11).

APPENDIX B.4 THE CONSTRUCTION OF THE FUNCTIONS $C_{k,1}$ AND $C_{k,2}$

First of all, by using the initial conditions in Eq.(3.41), it follows that $C_{k,1}(0,0)=C_{k,2}(0,0)=0$. Then, we shall solve the $O(\varepsilon)-$ problem (3.42). This problem (outside as well as inside the resonance manifold) can be written as

$$\frac{\partial^2 w_{k,1}}{\partial\tilde{\phi}_{k,0}^2}+w_{k,1} = -2[\frac{\partial C_{k,1}}{\partial\tilde{\phi}_{k,1}}\cos(\tilde{\phi}_{k,0})-\frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,1}}\sin(\tilde{\phi}_{k,0})]$$
$$+\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k^2(\tau)}\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau}-1)), \quad (3.70)$$
$$w_{k,1}(0,0,0) = \bar{F}_k,\quad \frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,0}}(0,0,0) = -\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,1}}(0,0,0)+\bar{G}_k. \quad (3.71)$$

Outside of the resonance zone, it should be observed that the last term in Eq. (3.70) does not give rise to secular terms in $w_{k,1}$. To avoid secular terms outside the resonance zone, it follows from (3.70) that $C_{k,1}$ and $C_{k,2}$ have to satisfy the following conditions $\frac{\partial C_{k,1}}{\partial\tilde{\phi}_{k,1}}=0$, $\frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,1}}=0$, which has as solutions

$$C_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})=\bar{C}_{k,1}(\tilde{\phi}_{k,2}),\quad C_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})=\bar{C}_{k,2}(\tilde{\phi}_{k,2}), \quad (3.72)$$

where $\bar{C}_{k,1}$ and $\bar{C}_{k,2}$ are still unknown functions of the slow variable $\tilde{\phi}_{k,2}$, and can be used to avoid secular terms in the $O(\varepsilon\sqrt{\varepsilon})-$ problem (3.43). Since $C_{k,1}(0,0)=C_{k,2}(0,0)=0$, this implies that $\bar{C}_{k,1}(0)=\bar{C}_{k,2}(0)=0$. Now we consider the $O(\varepsilon)$ equation inside the resonance zone and observe that inside the resonance zone, the last term in Eq. (3.70) gives rise to secular terms in $w_{k,1}$. According to (3.30), we can write $\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau}-1)) = \sin(\frac{1}{2}\gamma\tilde{s}_1^2+\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k}-1)-\phi_k(s_k)+\tilde{\phi}_{k,0})$, where γ is given by (3.28). So we can rewrite Eq. (3.70) inside the resonance zone as

$$\frac{\partial^2 w_{k,1}}{\partial\tilde{\phi}_{k,0}^2}+w_{k,1} = [-2\frac{\partial C_{k,1}}{\partial\tilde{\phi}_{k,1}}$$

$$
\begin{aligned}
&+\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k^2(\tau)}\sin(\frac{1}{2}\gamma\bar{s}_1^2+\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k))]\cos(\tilde{\phi}_{k,0})\\
&+[2\frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,1}}\\
&+\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k^2(\tau)}\cos(\frac{1}{2}\gamma\bar{s}_1^2+\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k))]\sin(\tilde{\phi}_{k,0}).
\end{aligned}
$$

In order to remove secular terms, it follows that $C_{k,1}$ and $C_{k,2}$ have to satisfy

3

$$
\begin{aligned}
-2\frac{\partial C_{k,1}}{\partial\tilde{\phi}_{k,1}}+\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k^2(\tau)}\sin(\frac{1}{2}\gamma\bar{s}_1^2+\vartheta(s_k)) &= 0,\\
2\frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,1}}+\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k^2(\tau)}\cos(\frac{1}{2}\gamma\bar{s}_1^2+\vartheta(s_k)) &= 0,
\end{aligned}
$$

where $\vartheta(s_k)=\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s_k}-1)-\phi_k(s_k)$ and $\frac{\partial C_{k,i}}{\partial\tilde{\phi}_{k,1}}=\frac{1}{\lambda_k(\tau)}\frac{\partial C_{k,i}}{\partial\bar{s}_1}$, $i=1,2$. Thus,

$$
\begin{aligned}
C_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) &= \bar{C}_{k,1}(\tilde{\phi}_{k,2})+\frac{1}{2}\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{\lambda_k(\tau)}\int_b^{\bar{s}_1}\sin(\frac{1}{2}\gamma\bar{s}_1^2+\vartheta(s_k))d\bar{s}_1\\
&= \bar{C}_{k,1}(\tilde{\phi}_{k,2})+\tilde{F}(\bar{s}_1,\tau),\\
C_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) &= \bar{C}_{k,2}(\tilde{\phi}_{k,2})-\tilde{G}(\bar{s}_1,\tau),
\end{aligned}
\tag{3.73}
$$

where

$$
\begin{aligned}
\tilde{F}(\bar{s}_1,\tau) &= \bar{\alpha}\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{2\lambda_k(\tau)}[\sin(\vartheta(s_k))\bar{C}_{Fr}(\bar{s}_1)+\cos(\vartheta(s_k))\bar{S}_{Fr}(\bar{s}_1)],\\
\tilde{G}(\bar{s}_1,\tau) &= \bar{\alpha}\alpha^2\beta_0\bar{l}^2(\tau)\frac{d_k(\tau)}{2\lambda_k(\tau)}[\cos(\vartheta(s_k))\bar{C}_{Fr}(\bar{s}_1)-\sin(\vartheta(s_k))\bar{S}_{Fr}(\bar{s}_1)],
\end{aligned}
\tag{3.74}
$$

and where $\bar{C}_{k,1}$ and $\bar{C}_{k,2}$ are still unknown functions of the slow variables $\tilde{\phi}_{k,2}$. The undetermined behaviour with respect to $\tilde{\phi}_{k,2}$ can be used to avoid secular terms in the $O(\varepsilon\sqrt{\varepsilon})-$ problem (3.43). Taking into account the secularity conditions, the general solution of $w_{k,1}$ is given by

$$
w_{k,1}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon})=D_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})\sin(\tilde{\phi}_{k,0})+D_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})\cos(\tilde{\phi}_{k,0}),
\tag{3.75}
$$

where $D_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})$ and $D_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})$ are unknown functions of $\tilde{\phi}_{k,1}$ and $\tilde{\phi}_{k,2}$. By using the initial conditions in Eq.(3.42), the values of $D_{k,1}(0,0)$ and $D_{k,2}(0,0)$ are given by the following equations $D_{k,1}(0,0)=\frac{\bar{G}_k}{\lambda_k(0)}-\frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,1}}(0,0)$, $D_{k,2}(0,0)=\bar{F}_k$.

The $O(\varepsilon\sqrt{\varepsilon})-$ problem (3.43) outside and inside the resonance manifold can be written as

$$
\begin{aligned}
\frac{\partial^2 w_{k,2}}{\partial\tilde{\phi}_{k,0}^2}+w_{k,2} &= -2[\frac{\partial D_{k,1}}{\partial\tilde{\phi}_{k,1}}\cos(\tilde{\phi}_{k,0})-\frac{\partial D_{k,2}}{\partial\tilde{\phi}_{k,1}}\sin(\tilde{\phi}_{k,0})]\\
&\quad -2[\frac{\partial C_{k,1}}{\partial\tilde{\phi}_{k,2}}\cos(\tilde{\phi}_{k,0})-\frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,2}}\sin(\tilde{\phi}_{k,0})]\\
&\quad -[\frac{\partial^2 C_{k,1}}{\partial\tilde{\phi}_{k,1}^2}\sin(\tilde{\phi}_{k,0})+\frac{\partial^2 C_{k,2}}{\partial\tilde{\phi}_{k,1}^2}\cos(\tilde{\phi}_{k,0})]
\end{aligned}
$$

$$
\begin{aligned}
&+[(v_0 - c_0\bar{l}(\tau))\lambda_k(\tau) - \frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}]\frac{1}{\lambda_k^2(\tau)}[C_{k,1}\cos(\tilde{\phi}_{k,0}) - C_{k,2}\sin(\tilde{\phi}_{k,0})]\\
&-2[c^1_{k,k}(\tau)\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau} - v_0 c^2_{k,k}(\tau)\\
&-\frac{m(c_0-c_{uo})}{2\rho L}c^3_{k,k}(\tau)]\frac{1}{\lambda_k(\tau)}[C_{k,1}\cos(\tilde{\phi}_{k,0}) - C_{k,2}\sin(\tilde{\phi}_{k,0})]\\
&-2\sum_{n\neq k}^{\infty}[c^1_{n,k}(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} - v_0 c^2_{n,k}(\tau)\\
&-\frac{m(c_0-c_{uo})}{2\rho L}c^3_{n,k}(\tau)]\frac{1}{\lambda_n(\tau)}[C_{n,1}\cos(\tilde{\phi}_{n,0}) - C_{n,2}\sin(\tilde{\phi}_{n,0})]
\end{aligned}
$$

$$
w_{k,2}(0,0,0) = 0, \quad \frac{\partial w_{k,2}}{\partial\tilde{\phi}_{k,0}}(0,0,0) = -\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,2}}(0,0,0) - \frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,1}}(0,0,0). \tag{3.76}
$$

To avoid secular terms in the solution $w_{k,2}$ of Eq.(3.76), outside the resonance zone, it follows from (3.76) that $D_{k,1}$, $D_{k,2}$, $\bar{C}_{k,1}$, and $\bar{C}_{k,2}$ have to satisfy:

$$
-2\frac{\partial D_{k,1}}{\partial\tilde{\phi}_{k,1}} - 2\frac{\partial\bar{C}_{k,1}}{\partial\tilde{\phi}_{k,2}} + \bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}\bar{C}_{k,1} = 0, \quad 2\frac{\partial D_{k,2}}{\partial\tilde{\phi}_{k,1}} + 2\frac{\partial\bar{C}_{k,2}}{\partial\tilde{\phi}_{k,2}} - \bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}\bar{C}_{k,2} = 0, \tag{3.77}
$$

where $\bar{\zeta}(\tau) = (v_0 - c_0\bar{l}(\tau)) - \frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{\lambda_k(\tau)} - 2c^1_{k,k}(\tau)\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau} + 2v_0c^2_{k,k}(\tau) + 2\frac{m(c_0-c_{uo})}{2\rho L}c^3_{k,k}(\tau)$. If we solve Eqs. (3.77) for $D_{k,1}$ and $D_{k,2}$ and integrate Eqs.(3.77) with respect to $\tilde{\phi}_{k,1}$, we observe that the solutions will be unbounded in $\tilde{\phi}_{k,1}$ due to terms which are only depending on $\tilde{\phi}_{k,2}$. Therefore, to have secular-free solutions, the following conditions have to be imposed independently

$$
\frac{\partial\bar{C}_{k,1}}{\partial\tilde{\phi}_{k,2}} - \frac{1}{2}\bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}\bar{C}_{k,1} = 0, \quad \frac{\partial\bar{C}_{k,2}}{\partial\tilde{\phi}_{k,2}} - \frac{1}{2}\bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}\bar{C}_{k,2} = 0. \tag{3.78}
$$

For $\frac{\partial\bar{C}_{k,i}}{\partial\tilde{\phi}_{k,2}} = \frac{1}{\lambda_k(\tau)}\frac{\partial\bar{C}_{k,i}}{\partial\tilde{s}_2}$, $i = 1,2$, we obtain $\bar{C}_{k,1} = \bar{C}_{k,1}(0)\mathrm{e}^{\int_0^\tau \frac{1}{2}\bar{\zeta}(\varrho)\mathrm{d}\varrho}$, $\bar{C}_{k,2} = \bar{C}_{k,2}(0)\mathrm{e}^{\int_0^\tau \frac{1}{2}\bar{\zeta}(\varrho)\mathrm{d}\varrho}$. Since $\bar{C}_{k,1}(0) = \bar{C}_{k,2}(0) = 0$ and $\lambda_k(\tau)$ is bounded, it follows from Eq. (3.78) that outside the resonance zone

$$
C_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) = 0, \quad C_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) = 0. \tag{3.79}
$$

Inside the resonance zone, to avoid secular terms in the solution $w_{k,2}$ of Eq.(3.76), the following conditions have to be imposed

$$
-2\frac{\partial D_{k,1}}{\partial\tilde{\phi}_{k,1}} - 2\frac{\partial\bar{C}_{k,1}}{\partial\tilde{\phi}_{k,2}} - 2\frac{\partial\tilde{F}(\tilde{s}_1,\tau)}{\partial\tilde{\phi}_{k,2}} + \frac{1}{\lambda_k^2(\tau)}\frac{\partial^2\tilde{G}(\tilde{s}_1)}{\partial\tilde{s}_1^2} + \bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}(\bar{C}_{k,1} + \tilde{F}(\tilde{s}_1,\tau)) = 0, \tag{3.80}
$$

$$
2\frac{\partial D_{k,2}}{\partial\tilde{\phi}_{k,1}} + 2\frac{\partial\bar{C}_{k,2}}{\partial\tilde{\phi}_{k,2}} - 2\frac{\partial\tilde{G}(\tilde{s}_1,\tau)}{\partial\tilde{\phi}_{k,2}} - \frac{1}{\lambda_k^2(\tau)}\frac{\partial^2\tilde{F}(\tilde{s}_1)}{\partial\tilde{s}_1^2} - \bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}(\bar{C}_{k,2} - \tilde{G}(\tilde{s}_1,\tau)) = 0. \tag{3.81}
$$

If we solve Eqs. (3.80) and (3.81) for $D_{k,1}$ and $D_{k,2}$ and integrate respect to $\tilde{\phi}_{k,1}$, we observe that the solutions will be unbounded in $\tilde{\phi}_{k,1}$ due to terms which are only depending on $\tilde{\phi}_{k,2}$. Therefore, to have secular-free solutions, the following conditions have to

3

3

be imposed independently

$$\frac{\partial \bar{C}_{k,1}}{\partial \tilde{\phi}_{k,2}} = \frac{1}{2}\bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}\bar{C}_{k,1}, \quad \frac{\partial \bar{C}_{k,2}}{\partial \tilde{\phi}_{k,2}} = \frac{1}{2}\bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}\bar{C}_{k,2}. \tag{3.82}$$

Since $\bar{C}_{k,1}(0) = \bar{C}_{k,2}(0) = 0$, it follows from Eq. (3.82) that inside the resonance zone

$$\begin{aligned} &\bar{C}_{k,1}(\tilde{\phi}_{k,2}) = 0, \quad \bar{C}_{k,2}(\tilde{\phi}_{k,2}) = 0, \\ &C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = \tilde{F}(\tilde{s}_1, \tau), \quad C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = \tilde{G}(\tilde{s}_1, \tau), \end{aligned} \tag{3.83}$$

where $\tilde{F}(\tilde{s}_1, \tau)$ and $\tilde{G}(\tilde{s}_1, \tau)$ are given by (3.74). Thus, we obtain the functions of $C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2})$ and $C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2})$ in (3.79) and (3.83). Similarly, we also can obtain the solution $w_{k,1}$ of $O(\varepsilon)$ problem and the solution $w_{k,2}$ of $O(\varepsilon\sqrt{\varepsilon})$ problem by using the above analysis. In order to shorten the paper, this derivation is omitted.

APPENDIX B.5 DISCRETIZATION AND ENERGY

To solve (3.56) numerically, it is convenient to rewrite the second order partial differential equation as a system of two coupled first-order partial differential equations:

$$\check{u}_t = \check{v}, \quad \check{v}_t = \frac{1}{l^2}\check{u}_{\xi\xi} + \varepsilon[\frac{2v_0}{l}(\xi-1)\check{v}_\xi - c_0\check{v} + \alpha^2\beta\sin(\alpha t) - \frac{m\xi(c-c_u)}{\rho L l}\check{v}_{\xi\xi}(1,t)]. \tag{3.84}$$

Next, let us use mesh grids $\xi_j = (j-1)\Delta\xi$ for $j = 1, 2, \ldots, n, n+1$ with $n\Delta\xi = 1$. By introducing the differences, $\check{u}_{\xi\xi}(\xi_j, t) = \frac{\check{u}_{j+1}-2\check{u}_j+\check{u}_{j-1}}{(\Delta\xi)^2} + O((\Delta\xi)^2)$, $\check{v}_\xi(\xi_j, t) = \frac{\check{v}_{j+1}-\check{v}_{j-1}}{2\Delta\xi} + O((\Delta\xi)^2)$, $\check{v}_{\xi\xi}(\xi_j, t) = \frac{\check{v}_{j+1}-2\check{v}_j+\check{v}_{j-1}}{(\Delta\xi)^2} + O((\Delta\xi)^2)$, it follows how system (3.84) can be discretized, yielding:

$$\begin{cases} \frac{d\check{u}}{dt}(\xi_j, t) = \check{v}_j, \\ \frac{d\check{v}}{dt}(\xi_j, t) = r(\check{u}_{j+1} - 2\check{u}_j + \check{u}_{j-1}) + q_j(\check{v}_{j+1} - \check{v}_{j-1}) - \varepsilon c_0 v_j + p_j\check{v}_n - p_j\check{v}_{n-1} \\ \qquad +\varepsilon\alpha^2\beta_0\sin(\alpha t), \end{cases}$$

where $r = \frac{1}{l^2(\Delta\xi)^2}$, $q_j = \frac{\varepsilon v_0(\xi_j-1)}{l\Delta\xi}$, $p_j = \frac{\varepsilon m(c_0-c_{u0})\xi_j}{\rho L(2+\Delta\xi)\Delta\xi}$ for j=1,2,..,n. Further,

$$R = \begin{pmatrix} -2r & r & 0 & \cdots & \cdots & 0 \\ r & -2r & r & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & r & -2r & r \\ 0 & \cdots & \cdots & 0 & c & -c \end{pmatrix} \in \mathbb{R}^{n\times n}, \text{ where } c = \frac{2\rho L}{l(2m+\rho L l\Delta\xi)\Delta\xi}, \text{ and}$$

$$P = \begin{pmatrix} -\varepsilon c_0 & q_1 & 0 & \cdots & 0 & -p_1 & p_1 \\ -q_2 & -\varepsilon c_0 & q_2 & \cdots & 0 & -p_2 & p_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -q_{n-2} & -\varepsilon c_0 & q_{n-2}-p_{n-2} & p_{n-2} \\ 0 & \cdots & 0 & 0 & -q_{n-1} & -\varepsilon c_0 - p_{n-1} & q_{n-1}+p_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & d_n - q_n - p_n & e_n - \varepsilon c_0 \end{pmatrix} \in \mathbb{R}^{n\times n},$$

where $d_n = \frac{\rho L l \Delta\xi - 2m}{2m + \rho L l \Delta\xi}$, $e_n = \frac{4m}{2m+\rho L l \Delta\xi} + p_n$. The four matrices $\emptyset$, I, R and P compose the system matrix M:

$$M = \begin{pmatrix} \emptyset & I \\ R & P \end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

where $\emptyset$ is the zero matrix, and I is the identity matrix. In addition, let us introduce the following vector: $w = (u_1(\xi_1,t), u_2(\xi_2,t), ..., u_n(\xi_n,t), v_1(v_1,t), v_2(\xi_2,t), ..., v_n(\xi_n,t))^T$, $s = (\underbrace{0,0,...,0}_{\text{n times}}, \underbrace{\bar{s},\bar{s},...,\bar{s}}_{\text{n times}})^T$, where $\bar{s} = \varepsilon\alpha^2\beta_0\sin(\alpha t)$. So, system (3.84) can be written in the following matrix form: $\frac{dw}{dt} = Mw + s$. In order to perform a time integration, we apply the Crank-Nicolson method. Introducing the mesh grid in time, $t_k = k\Delta t$ for k=1,2,...,n, we obtain

$$w^{k+1} = Dw^k + \frac{\Delta t}{2}(I - \frac{\Delta t}{2}M^{k+1})^{-1}(s^{k+1} + s^k), \tag{3.85}$$

where I is the identity matrix and $I \in \mathbb{R}^{2n\times 2n}$ and $D = (I - \frac{\Delta t}{2}M^{k+1})^{-1}(I + \frac{\Delta t}{2}M^k)$.

The total mechanical energy of the problem (8) is given by

$$E(t) = \frac{1}{2}\int_0^{l(t)} [\rho(u_t + vu_x)^2 + EAu_x^2]dx + \frac{m}{2}[u_t(l(t),t) + vu_x(l(t),t)]^2.$$

Using the dimensionless quantities, we rewrite the energy in a dimensionless form:

$$E(t) = \frac{1}{2}EAL\int_0^{l(t)} [(u_t + vu_x)^2 + u_x^2]dx + \frac{EAm}{2\rho}[u_t(l(t),t) + vu_x(l(t),t)]^2.$$

In order to define the energy on the interval (0,1), we obtain problem (10) by using the following transformation $\xi = \frac{x}{l(t)}$:

$$\begin{aligned} E(t) &= \frac{EAL}{2l(t)}\int_0^1 [(l(t)\tilde{u}_t + (1-\xi)v\tilde{u}_\xi)^2 + \tilde{u}_\xi^2]d\xi \\ &+ \frac{EAm}{2\rho l^2(t)}[l(t)\tilde{u}_t(1,t) + (1-\xi)v\tilde{u}_\xi(1,t)]^2. \end{aligned} \tag{3.86}$$

4

Transverse and longitudinal resonances for an elevator system

4.1. Introduction

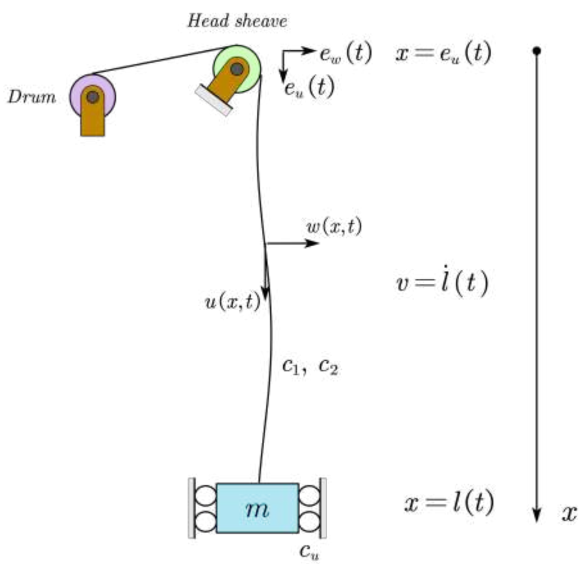

Figure 4.1: Coupled transverse-longitudinal vibrating cable with time-varying cable length.

In chapter 3, we considered the longitudinal vibrations in an axially moving cable with time-varying length subject to a small harmonic boundary excitation at one end of the cable and a moving loading mass at the other end. Due to boundary excitation and loading conditions, the nonlinear interactions between transverse and longitudinal string motions may influence the vibration behavior in two directions when the elevator conveyance is moving up or down. Some research has been conducted on similar types of problems by using numerical simulations. Crespo et al. in [59] introduced a stationary high-rise elevator cable system model and presented its numerical simulations. Wang et al. in [60] studied the dynamic behavior of the multi-cable double drum winding hoister with unbalance factors, the conveyance eccentricity and the drum radius inconformity by simulation. Cao et al. in [61] analysed coupled vibrations of rope-guided hoisting system with tension difference between two guiding ropes by simulations. Wang et al. in [46] investigated a coupled dynamic model for a flexible guiding hoisting system and presented the response of the system by numerical simulations. For more information on numerical results for coupled transverse and longitudinal dynamics of axially moving continua, the reader is referred to [62, 63, 64, 65, 66].

In this chapter, we will construct analytical approximate solutions for the nonlinear coupled transverse and longitudinal vibration string problem with time-varying length. The elevator system considered in this chapter is described by a vertically translating string with a time-varying length and a mass attached at one of the ends of the string. The time-varying length of the string is given by $l(t) = l_0 + \varepsilon v_0 t$, where l_0 and v_0 are constants, and where ε is a dimensionless small parameter. It is assumed that the axial velocity of the string is small compared to nominal wave velocity, and that the string mass is small compared to cage mass. The system is excited at the upper end by small displacements in the horizontal and vertical directions from its equilibrium position caused by, for instance, wind forces (see Figure 4.1). By Hamilton's principle, the model can be written as a coupled system of nonlinear wave equations (in transverse and longitudinal directions) on a slowly time-varying spatial domain. The string is excited at a boundary by two harmonic functions in the horizontal and in the vertical directions. The main objective of this chapter is to study how the boundary excitations and nonlinear interactions between the two motion-directions influence the vibration behavior in the transverse and in the longitudinal directions for the moving string. In contrast to previous research, where only the transverse or the longitudinal vibration behavior was studied the (reader is referred to [67, 68, 69, 70]), the coupled model is more accurate. However, the appearance of nonlinear and coupled terms increases the complexity of the system analysis. In order to deal with this difficulty, perturbation methods and an internal layer analysis are used in this chapter to approximate the vibrations and the resonances, including determining the resonance-amplitudes and the size of the resonance zones. Based on this analysis, solutions of the coupled initial-boundary value problem for the transverse and the longitudinal motions can be predicted analytically. To the best of our knowledge, the results about analytical approximations of the solutions have not been proposed for the coupled transverse and longitudinal vibrations of the moving cable system until now.

The remaining part of this chapter is organized as follows. In section 4.2 the problem is formulated. In section 4.3 the problem is reformulated from a partial differential equations formulation to an ordinary differential equations formulation by using the method of separation of variables. Many resonance manifolds for the transverse and longitudinal motions are detected by an inner layer analysis. In section 4.4 approximate solutions are constructed analytically for the transverse and longitudinal motions by using a three time-scales perturbation method. In section 4.5 some numerical approximations are presented by using a central finite difference scheme to validate the theoretical results from section 4.4. Finally, in the last section we draw some conclusions.

4.2. FORMULATION OF THE PHYSICAL SYSTEM

In this section the mathematical model of the elevator system is described and the equations for the transverse and the longitudinal motions of the system are derived and explained.

Nomenclature:	
$w(x,t)$	the transverse displacement
$u(x,t)$	the longitudinal displacement
$l(t)$	the length of the elevator cable
$v = \dot{l}(t)$	the longitudinal velocity of the elevator cable
$a = \ddot{l}(t)$	the longitudinal acceleration of the hoisting cable
ρ	the linear density of the elevator cable
m	the mass of the elevator conveyance
EA	the longitudinal stiffness, E Young's elasticity modulus, A the cross-sectional area
$T(x,t)$	the spatiotemporally varying tension in elevator cable
c_1, c_2	transverse and longitudinal viscous damping coefficients in elevator cable
g	the standard gravity
E_{gs}	initial gravitational potential energy
c_u	longitudinal viscous damping coefficient in elevator conveyance
$e_w(t)$, $e_u(t)$	the transverse and longitudinal fundamental excitations at the top of the elevator cable $e_w(t) = \beta_1 cos(\omega_1 t + \alpha)$, $e_u(t) = \beta_2 cos(\omega_2 t)$
β_1, β_2	the amplitudes of the transverse and longitudinal fundamental excitations
α	primary phase of the transverse fundamental excitation

By using Hamilton's principle, the mathematical problem for the vibrating cable (Figure 4.1) can be written as an initial boundary value problem for the transverse vibration (see also Appendix C.1):

$$\begin{cases} \rho(w_{tt} + 2vw_{xt} + v^2 w_{xx} + aw_x) - (Tw_x)_x + c_1(w_t + vw_x) - EA(zw_x)_x = 0, \\ \qquad\qquad\qquad\qquad \beta_2 \cos(\omega_2 t) < x < l(t),\ t > 0, \\ w(l(t),t) = 0, \quad t \ge 0, \\ w(\beta_2 \cos(\omega_2 t), t) = \beta_1 \cos(\omega_1 t + \alpha), \quad t \ge 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad \beta_2 < x < l_0, \end{cases} \tag{4.1}$$

and as an initial boundary value problem for the longitudinal vibration:

$$\begin{cases} \rho(u_{tt}+2vu_{xt}+v^2u_{xx}+au_x+a)+c_2(u_t+vu_x)-EAz_x=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \beta_2\cos(\omega_2 t)<x<l(t),\ t>0, \\ [m(u_{tt}+2vu_{xt}+v^2u_{xx}+au_x+a)+c_u(u_t+vu_x)+EAz]|_{x=l(t)}=0, \quad t\geq 0, \\ u(\beta_2\cos(\omega_2 t),t)=\beta_2\cos(\omega_2 t), \quad t\geq 0, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad \beta_2<x<l_0, \end{cases} \tag{4.2}$$

where $z=u_x+\frac{1}{2}w_x^2$ and

$$T(x,t)=[m+\rho(l(t)-x)]g, \quad \beta_2\cos(\omega_2 t)\leq x\leq l(t). \tag{4.3}$$

4

In this paper, we use the following assumptions for the parameters and functions:

- The longitudinal velocity v is small compared to the wave velocities $\sqrt{\frac{EA}{\rho}}$ and $\sqrt{\frac{mg}{\rho}}$, that is, $v=\varepsilon v_0$;
- The nominal wave velocities $\sqrt{\frac{EA}{\rho}}$ and $\sqrt{\frac{mg}{\rho}}$ are of the same order of magnitude, that is, $\frac{EA}{mg}=O(1)$, $\sqrt{\frac{EA}{mg}}>1$, and $\frac{EA}{mg}$ is not near 1, i.e., $\sqrt{\frac{EA}{mg}}-1>>O(\varepsilon)$;
- The cable mass ρL is small compared to the car mass m (L is the maximum length of the cable), that is, $\mu=\frac{\rho L}{m}=\varepsilon\mu_0$;
- The viscous damping parameters c_1, c_2, and c_u are small, that is, $c_1=\varepsilon c_{1,0}, c_2=\varepsilon c_{2,0}, c_u=\varepsilon c_{u,0}$;
- The fundamental excitations at the top of the elevator rope are small, and the longitudinal excitation is smaller than the transverse excitation, that is, $\beta_1=\varepsilon\beta_{1,0}, \beta_2=\varepsilon^2\beta_{2,0}$;
- The initial conditions $w_0(x)=\varepsilon h_0(x), w_1(x)=\varepsilon h_1(x)$, $u_0(x)=\varepsilon^2 h_2(x)$ and $u_1(x)=\varepsilon^2 h_3(x)$;
- For convenience we only consider a non-accelerating cable, that is, the cable length $l(t)=l_0+vt$ and $a=0$, where l_0 is the initial string length.

In the above assumptions, v_0, μ_0, $c_{1,0}$, $c_{2,0}$, $c_{u,0}$, $\beta_{1,0}$, $\beta_{2,0}$, α m, ρ, ω_1, ω_2, L and l_0 are positive constants and are of order 1, the functions $h_0(x), h_1(x), h_2(x), h_3(x)$ are of order 1, and ε is a small parameter with $0<\varepsilon<<1$.

To put the equations (4.1) and (4.2) into non-dimensional forms, the following dimensionless variables and parameters are used:

$$w^*=\frac{w}{L},\ u^*=\frac{u}{L},\ x^*=\frac{x}{L},\ t^*=\frac{t}{L}\sqrt{\frac{mg}{\rho}},\ v^*=v\sqrt{\frac{\rho}{mg}},\ \beta_1^*=\frac{\beta_1}{L},\ \beta_2^*=\frac{\beta_2}{L},$$

$$c_1^*=c_1\frac{L}{\sqrt{mg\rho}},\ c_u^*=c_u\frac{L}{m}\sqrt{\frac{\rho}{mg}},\ \omega_1^*=L\omega_1\sqrt{\frac{\rho}{mg}},\ u_0^*=\frac{u_0}{L},\ u_1^*=\sqrt{\frac{\rho}{mg}}u_1,$$

$$l^* = \frac{l}{L},\ \mu = \frac{\rho L}{m},\ c_2^* = c_2\frac{L}{\sqrt{mg\rho}},\ \omega_2^* = L\omega_2\sqrt{\frac{\rho}{mg}},\ w_0^* = \frac{w_0}{L},\ w_1^* = \sqrt{\frac{\rho}{mg}}w_1.$$

The initial boundary value problem for the transverse motion in non-dimensional form becomes:

$$\begin{cases} w_{tt} + 2vw_{xt} + v^2w_{xx} - w_{xx} - \mu(l(t)-x)w_{xx} + \mu w_x + c_1(w_t + vw_x) - \frac{EA}{mg}(zw_x)_x = 0, \\ \qquad\qquad \beta_2\cos(\omega_2 t) < x < l(t),\ t > 0, \\ w(l(t),t) = 0, \quad t \geq 0, \\ w(\beta_2\cos(\omega_2 t), t) = \beta_1\cos(\omega_1 t + \alpha), \quad t \geq 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad \beta_2 < x < l_0, \end{cases} \tag{4.4}$$

and the initial boundary value problem for the longitudinal motion in non-dimensional form becomes:

$$\begin{cases} u_{tt} + 2vu_{xt} + v^2u_{xx} + c_2(u_t + vu_x) - \frac{EA}{mg}u_{xx} - \frac{EA}{mg}(\frac{1}{2}w_x^2)_x = 0, \\ \qquad\qquad \beta_2\cos(\omega_2 t) < x < l(t),\ t > 0, \\ [u_{tt} + 2vu_{xt} + v^2u_{xx} + c_u(u_t + vu_x) + \frac{\mu EA}{mg}z]|_{x=l(t)} = 0, \quad t \geq 0, \\ u(\beta_2\cos(\omega_2 t), t) = \beta_2\cos(\omega_2 t), \quad t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \beta_2 < x < l_0, \end{cases} \tag{4.5}$$

where the asterisks (indicating the dimensionless variables and parameters) are omitted in the problems (4.4) and (4.5) for convenience.

In order to convert the time-varying spatial domain $[\beta_1\cos(\omega_1 t), l(t)]$ for x to a fixed domain [0,1] for ξ, a new independent spatial coordinate $\xi = \frac{x-\beta_2\cos(\omega_2 t)}{h(t)}$, in which $h(t) = l(t) - \beta_2\cos(\omega_2 t)$, is introduced. After this spatial transformation, the equation for the transverse motion becomes:

$$\begin{cases} \bar{w}_{tt} + \frac{2v}{h(t)}\bar{w}_{\xi t} - \frac{1}{h^2(t)}\bar{w}_{\xi\xi} - \frac{\mu}{h(t)}(1-\xi)\bar{w}_{\xi\xi} + \frac{\mu}{h(t)}\bar{w}_\xi + c_1\bar{w}_t - \frac{EA}{mgh^3(t)}(\bar{u}_\xi\bar{w}_\xi)_\xi \\ \qquad - \frac{EA}{mgh^4(t)}(\frac{1}{2}\bar{w}_\xi^3)_\xi - \frac{2v\xi}{h(t)}\bar{w}_{\xi t} = O(\varepsilon^2\bar{w}), \quad 0 < \xi < 1,\ t > 0, \\ \bar{w}(1,t) = 0, \quad t \geq 0, \\ \bar{w}(0,t) = \beta_1\cos(\omega_1 t + \alpha), \quad t \geq 0, \\ \bar{w}(\xi,0) = \bar{w}_0(\xi), \quad \bar{w}_t(\xi,0) = \bar{w}_1(\xi), \quad 0 < \xi < 1, \end{cases} \tag{4.6}$$

where $\bar{w}_0(\xi) = w_0(\xi l_0 + \beta_2(1-\xi))$ and $\bar{w}_1(\xi) = w_1(\xi l_0 + \beta_2(1-\xi))$. It should be observed that the order of the term $-\frac{EA}{mgh^3(t)}(\bar{u}_\xi\bar{w}_\xi)_\xi$ in (4.6) is unknown a priori due to possibly occurring resonances. So, we keep this term explicitly in the equation, and analyse it later. The equation for the longitudinal motion then becomes:

$$\begin{cases} \bar{u}_{tt} + \frac{2v}{h(t)}\bar{u}_{\xi t} + c_2\bar{u}_t - \frac{EA}{mgh^2(t)}\bar{u}_{\xi\xi} - \frac{EA}{mgh^3(t)}\bar{w}_\xi\bar{w}_{\xi\xi} - \frac{2v\xi}{h(t)}\bar{u}_{\xi t} = O(\varepsilon^2\bar{u}), \quad 0 < \xi < 1,\ t > 0, \\ [\bar{u}_{tt} + c_u\bar{u}_t + \frac{\mu EA}{mgh(t)}\bar{u}_\xi + \frac{\mu EA}{2mgh^2(t)}\bar{w}_\xi^2]|_{\xi=1} = O(\varepsilon^2\bar{u}), \quad t \geq 0, \\ \bar{u}(0,t) = \beta_2\cos(\omega_2 t), \quad t \geq 0, \\ \bar{u}(\xi,0) = \bar{u}_0(\xi), \quad \bar{u}_t(\xi,0) = \bar{u}_1(\xi), \quad 0 < \xi < 1, \end{cases} \tag{4.7}$$

where $\bar{u}_0(\xi) = u_0(\xi l_0 + \beta_2(1-\xi))$ and $\bar{u}_1(\xi) = u_1(\xi l_0 + \beta_2(1-\xi))$. The orders of the term $-\frac{EA}{mgh^3(t)}\bar{w}_\xi \bar{w}_{\xi\xi}$ and the term $\frac{\mu EA}{2mgh^2(t)}\bar{w}_\xi^2(1,t)$ in (4.7) are unknown a priori due to possible resonances. So, we keep these terms in the equation, and analyse them later.

Further, we also introduce the Liouville-Green transformation (as used in the WKBJ method) with $\frac{ds}{dt} = \frac{1}{l(t)}$, and a transformation to eliminate the non-homogenous terms in the boundary conditions (see Appendix C.2 for details). Then, the initial boundary value problem in the transverse direction becomes:

$$\begin{cases} \hat{w}_{ss} - \hat{w}_{\xi\xi} = v\hat{w}_s - 2v\hat{w}_{\xi s} - c_1\hat{l}\hat{w}_s - \mu\hat{l}\hat{w}_\xi + \mu\hat{l}(1-\xi)\hat{w}_{\xi\xi} + \frac{EA}{mg\hat{l}}(\hat{u}_\xi\hat{w}_\xi)_\xi + \frac{EA}{mg\hat{l}^2}(\frac{1}{2}\hat{w}_\xi^3)_\xi \\ \qquad +2v\xi\hat{w}_{\xi s} + \beta_1(1-\xi)\omega_1^2\hat{l}^2\cos(\omega_2\chi(s)+\alpha) \\ \qquad -\frac{EA}{mg\hat{l}^2}\hat{w}_\xi(1,s)\hat{w}_{\xi\xi}(1,s)[\hat{w}_\xi + \xi\hat{w}_{\xi\xi}] + h.o.t., \quad 0<\xi<1,\ s>0, \\ \hat{w}(1,s) = 0, \quad s\geq 0, \\ \hat{w}(0,s) = 0, \quad s\geq 0, \\ \hat{w}(\xi,0) = \tilde{w}(\xi,0) - \beta_1(1-\xi)\cos(\alpha),\ \hat{w}_s(\xi,0) = \tilde{w}_s(\xi,0) + \beta_1\omega_1 l_0(1-\xi)\sin(\alpha),\ 0<\xi<1, \end{cases} \tag{4.8}$$

where according to the initial condition assumptions, the terms in "h.o.t.", consisting of $O(\varepsilon\hat{u})$, $O(\varepsilon^2\hat{w})$, $O(\varepsilon\hat{w}^2)$, $O(\varepsilon\hat{u}\hat{w})$ and $O(\varepsilon\hat{w}^3)$, can not influence the lowest order of the solution of problem (4.8) on time-scales of $O(\frac{1}{\varepsilon})$.
The initial boundary value problem in the longitudinal direction then becomes:

$$\begin{cases} \hat{u}_{ss} - \frac{EA}{mg}\hat{u}_{\xi\xi} = v\hat{u}_s - 2v\hat{u}_{\xi s} - c_2\hat{l}\hat{u}_s + 2v\xi\hat{u}_{\xi s} + (c_2-c_u)\hat{l}\hat{u}_s(1,s) - \frac{EA}{mg}\mu\hat{l}\hat{u}_\xi(1,s) \\ \qquad +\frac{\mu\hat{l}\xi^2}{2}\frac{EA}{mg}\hat{u}_{\xi\xi\xi}(1,s) - \frac{\xi^2}{2}(c_2-c_u)\hat{l}\hat{u}_{s\xi\xi}(1,s) + O(\varepsilon^2\hat{u}) \\ \qquad +\frac{EA}{mg\hat{l}}\hat{w}_\xi\hat{w}_{\xi\xi} - \frac{EA}{mg\hat{l}}\hat{w}_\xi(1,s)\hat{w}_{\xi\xi}(1,s) \\ \qquad -\frac{\xi^2}{2\hat{l}}[\hat{w}_{\xi\xi\xi}(1,s)\hat{w}_{\xi\xi}(1,s) + \hat{w}_\xi\hat{w}_{\xi\xi\xi\xi}(1,s)] - \frac{\xi^2}{\hat{l}}\hat{w}_{\xi s}(1,s)\hat{w}_{\xi\xi s}(1,s) + O(\varepsilon\hat{w}^2) \\ \qquad -\frac{EA}{mg\hat{l}}\beta_1\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi} + \frac{EA}{mg\hat{l}}\beta_1\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi}(1,s) \\ \qquad +\frac{\xi^2}{2\hat{l}}\frac{EA}{mg}\beta_1\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi\xi\xi}(1,s) - \xi^2\beta_1\omega_1\sin(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi s}(1,s) \\ \qquad +O(\varepsilon^2\hat{w}) + \beta_2\omega_2^2\hat{l}^2\cos(\omega_2\chi(s)) + O(\varepsilon^3), \quad 0<\xi<1,\ s>0, \end{cases} \tag{4.9}$$

and

$$\begin{cases} \hat{u}_{\xi\xi}(1,s) = O(\varepsilon^2\hat{u}), \quad s\geq 0, \\ \hat{u}(0,s) = 0, \quad s\geq 0, \\ \hat{u}(\xi,0) = \tilde{u}(\xi,0) - \frac{\xi^2}{2}[-\mu\hat{l}\tilde{u}_\xi(1,0) + \frac{mg}{EA}(c_2-c_u)\hat{l}\tilde{u}_s(1,0) - \frac{\mu}{2}\tilde{w}_\xi^2(1,0) - \frac{1}{\hat{l}}\tilde{w}_\xi(1,0)\tilde{w}_{\xi\xi}(1,0)] \\ \qquad +O(\varepsilon^3), \\ \hat{u}_s(\xi,0) = \tilde{u}_s(\xi,0) - \frac{\xi^2}{2}[-\mu\hat{l}\tilde{u}_{\xi s}(1,0) + \frac{mg}{EA}(c_2-c_u)\hat{l}\tilde{u}_{\xi\xi}(1,0) - \mu\tilde{w}_\xi(1,0)\tilde{w}_{\xi s}(1,0) \\ \qquad -\frac{1}{\hat{l}}\tilde{w}_{\xi s}(1,0)\tilde{w}_{\xi\xi}(1,0) - \frac{1}{\hat{l}}\tilde{w}_\xi(1,0)\tilde{w}_{\xi\xi s}(1,0)] + O(\varepsilon^3), \quad 0<\xi<1, \end{cases} \tag{4.10}$$

where according to the initial condition assumptions, the terms in $O(\varepsilon\hat{w}^2)$, $O(\varepsilon^2\hat{u})$, $O(\varepsilon^2\hat{w})$, and $O(\varepsilon^3)$ can not influence the lowest order of the solution $\hat{u}(\xi,s)$ in problem (4.9) on time-scales of $O(\frac{1}{\varepsilon})$, so they can be neglected in the further analysis. In the following sections, the solutions of $\hat{w}(\xi,s)$, $\hat{u}(\xi,s)$ in problem (4.8) and (4.9) will be approximated by using an interior layer analysis and a three time-scales perturbation method.

4.3. INNER LAYER ANALYSIS

It will be shown that an interior layer analysis (including a rescaling and balancing procedure) leads to a description of an (un-)expected resonance manifold and leads to time-scales which describe the solutions of the partial differential equations (4.8) and (4.9) sufficiently accurately. To derive the solutions $\hat{w}(\xi,s)$ and $\hat{u}(\xi,s)$ in problem (4.8) and (4.9), firstly the method of separation of variables is employed. In accordance with the method of separation of variables, the general solution of the transverse problem (4.8) can be expanded in the following form:

$$\hat{w}(\xi,s) = \sum_{n=1}^{\infty} T_n(s)\sin(n\pi\xi), \tag{4.11}$$

and the general solution of the longitudinal problem (4.9) can be expanded in the following form:

$$\hat{u}(\xi,s) = \sum_{n=1}^{\infty} Y_n(s)\sin(n\pi\xi). \tag{4.12}$$

Substituting (4.11) into the initial boundary value problem (4.8), and substituting (4.12) into problem (4.9), further by multiplying the obtained equations with $\sin(k\pi\xi)$, and by integrating with respect to ξ from $\xi = 0$ to $\xi = 1$, and by using the orthogonality properties of the sin-functions on $0 < \xi < 1$, we obtain the following ordinary differential equations for $T_k(s)$ (with $k = 1,2,3,\ldots$) in the transverse direction:

$$T_{k,ss} + k^2\pi^2 T_k = \hat{\chi}(s), \tag{4.13}$$

where

$$\begin{aligned}
&\hat{\chi}(s)\\
= \;&\varepsilon v_0 T_{k,s} - \varepsilon c_{1,0}\hat{l}T_{k,s} - \sum_{n=1}^{\infty}\varepsilon c^1_{n,k}\mu_0\hat{l}(n\pi)^2 T_n + \sum_{n=1}^{\infty} c^2_{n,k}(-2v_0 n\pi T_{n,s} - \mu_0\hat{l}n\pi T_n)\\
&+\varepsilon\sum_{n=1}^{\infty} c^3_{n,k} 2v_0 n\pi T_{n,s} + \frac{EA\pi^3}{2mg\hat{l}}[\sum_{p=k+1}^{\infty} kp(k-p)Y_p T_{p-k} - \sum_{p=1}^{k-1} kp(k-p)Y_p T_{k-p}]\\
&-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{\infty} kp(k+p)Y_p T_{k+p} + \frac{3EA}{8mg\hat{l}^2}\sum_{n+p=k+1}^{n+p=\infty} np\pi^3(k-n-p)T_nT_pT_{n+p-k}\\
&+\frac{3EA}{8mg\hat{l}^2}[\sum_{n-p=-\infty}^{n-p=k-1} np\pi^3(k-n+p)T_nT_pT_{k-n+p}\\
&+\sum_{n,p=1}^{\infty} np\pi^3(k+n+p)T_nT_pT_{k+n+p}]\\
&+\frac{3EA}{8mg\hat{l}^2}[\sum_{p-n=-\infty}^{p-n=k-1} np\pi^3(k+n-p)T_nT_pT_{k+n-p}\\
&-\sum_{p-n=k+1}^{p-n=\infty} np\pi^3(p-k-n)T_nT_pT_{p-k-n}]\\
&+\frac{3EA}{8mg\hat{l}^2}[\sum_{n+p=2}^{n+p=k-1} np\pi^3(k-n-p)T_nT_pT_{k-n-p}
\end{aligned}$$

4

$$
\begin{aligned}
&+\sum_{n-p=k+1}^{n-p=\infty} np\pi^3(k-n+p)T_nT_pT_{n-p-k}] \\
&+\varepsilon\beta_{2,0}d_k\omega_2^2\hat{l}^2\cos(\omega_2\chi(s)+\alpha)+h.o.t., \\
&T_k(0)=\frac{\int_0^1 \hat{w}_0(\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin(k\pi\xi)\sin(k\pi\xi)d\xi}=F_k, \\
&T_{k,s}(0)=\frac{\int_0^1 \hat{w}_1(\xi)\sin(k\pi\xi)d\xi}{\int_0^1 \sin(k\pi\xi)\sin(k\pi\xi)d\xi}=G_k,
\end{aligned}
\tag{4.14}
$$

where $F_k=O(\varepsilon)$ and $G_k=O(\varepsilon)$. $c^1_{n,k}$, $c^2_{n,k}$, $c^3_{n,k}$ and d_k are given by:

$$
\begin{aligned}
c^1_{n,k} &= \frac{\int_0^1(1-\xi)\sin(n\pi\xi)\sin(k\pi\xi)d\xi}{\int_0^1\sin^2(k\pi\xi)d\xi}, \\
c^2_{n,k} &= \frac{\int_0^1\cos(n\pi\xi)\sin(k\pi\xi)d\xi}{\int_0^1\sin^2(k\pi\xi)d\xi}, \\
c^3_{n,k} &= \frac{\int_0^1\xi\cos(n\pi\xi)\sin(k\pi\xi)d\xi}{\int_0^1\sin^2(k\pi\xi)d\xi}, \quad d_k=\frac{\int_0^1(1-\xi)\sin(k\pi\xi)d\xi}{\int_0^1\sin^2(k\pi\xi)d\xi}.
\end{aligned}
\tag{4.15}
$$

Further, the differential equation (4.13) can be written as:

$$
\begin{aligned}
&T_{k,ss}+k^2\pi^2T_k \\
=\;&\varepsilon[v_0T_{k,s}-c_{1,0}\hat{l}T_{k,s}-\sum_{n=1}^{\infty}c^1_{n,k}\mu_0\hat{l}(n\pi)^2T_n+\sum_{n=1}^{\infty}c^2_{n,k}(-2v_0n\pi T_{n,s}-\mu_0\hat{l}n\pi T_n) \\
&+\sum_{n=1}^{\infty}c^3_{n,k}2v_0n\pi T_{n,s}+\frac{EA\pi^3}{2\varepsilon mg\hat{l}}\sum_{p=k+1}^{\infty}kp(k-p)Y_pT_{p-k} \\
&-\frac{EA\pi^3}{2\varepsilon mg\hat{l}}\sum_{p=1}^{k-1}kp(k-p)Y_pT_{k-p} \\
&-\frac{EA\pi^3}{2\varepsilon mg\hat{l}}\sum_{p=1}^{\infty}kp(k+p)Y_pT_{k+p}+\beta_{1,0}d_k\omega_1^2\hat{l}^2\cos(\omega_1\chi(s)+\alpha)]+h.o.t.,
\end{aligned}
\tag{4.16}
$$

where $T_k(0)$ and $T_{k,s}(0)$ are given by (4.14), $c^1_{n,k}$, $c^2_{n,k}$, $c^3_{n,k}$ and d_k are given by (4.15). Note that the term "h.o.t."(including $T_nT_pT_j$ in (4.13)) can not influence the lowest order of the solution of the differential equation (4.16) on time-scales of $O(\frac{1}{\varepsilon})$. This can be seen as follows. When the addition or subtraction of the three subscripts in $T_nT_pT_j$ equals to k or $-k$, then for the given initial conditions of $O(\varepsilon)$, $T_nT_pT_j$ leads to $O(\varepsilon^2)$ contributions in the solution of the differential equation (4.16) on time-scales of $O(\frac{1}{\varepsilon})$; otherwise, $T_nT_pT_j$ will leads to contributions of $O(\varepsilon^3)$ in the solution of the differential equation (4.16) on time-scales of $O(\frac{1}{\varepsilon})$.

Similarly, we obtain the following differential equations for Y_k (with $k=1,2,3,...$) in the longitudinal direction:

$$
\begin{aligned}
&Y_{k,ss}+\frac{EA}{mg}k^2\pi^2Y_k \\
=\;&\varepsilon(v_0-c_{2,0}\hat{l})Y_{k,s}+\sum_{n=1}^{\infty}2\varepsilon v_0n\pi d^1_{n,k}Y_{n,s}-\sum_{n=1}^{\infty}\frac{EA}{mg}\mu\hat{l}n\pi d^3_{n,k}Y_n
\end{aligned}
$$

$$+\sum_{j=1}^{k-1} d^4_{k-j,j} T_{k-j} T_j + \sum_{j=k+1}^{\infty} d^4_{j-k,j} T_{j-k} T_j - \sum_{j=1}^{\infty} d^4_{j+k,j} T_{j+k} T_j$$
$$+\beta_1 \tilde{d}^4_{k,k} T_k \cos(\omega_1 \chi(s) + \alpha) + \beta_2 \omega_2^2 \hat{l}^2 d_{1,k} \cos(\omega_2 \chi(s)) + h.o.t.,$$
$$Y_k(0) = \frac{\int_0^1 \hat{u}_0(\xi) \sin(k\pi\xi) d\xi}{\int_0^1 \sin(k\pi\xi) sin(k\pi\xi) d\xi} = f_k,$$
$$Y_{k,s}(0) = \frac{\int_0^1 \hat{u}_1(\xi) \sin(k\pi\xi) d\xi}{\int_0^1 \sin(k\pi\xi) sin(k\pi\xi) d\xi} = g_k, \tag{4.17}$$

where $f_k = O(\varepsilon^2)$ and $g_k = O(\varepsilon^2)$. $d^1_{n,k}$, $d^3_{n,k}$, $d^4_{n,j}$, $d_{1,k}$, $\tilde{d}^4_{k,k}$ are given by:

$$d^1_{n,k} = \frac{\int_0^1 (\xi - 1) \cos(n\pi\xi) \sin(k\pi\xi) d\xi}{\int_0^1 \sin^2(k\pi\xi) d\xi}, \quad d^3_{n,k} = \frac{\int_0^1 (1 + \frac{(n\pi\xi)^2}{2}) \cos(n\pi) \sin(k\pi\xi) d\xi}{\int_0^1 \sin^2(k\pi\xi) d\xi},$$
$$d^4_{n,j} = \frac{EAnj^2\pi^3}{2mg\hat{l}}, \quad d_{1,k} = \frac{\int_0^1 \sin(k\pi\xi) d\xi}{\int_0^1 \sin^2(k\pi\xi) d\xi}, \quad \tilde{d}^4_{k,k} = \frac{EA}{mg\hat{l}} k^2\pi^2. \tag{4.18}$$

Before approximately solving the ordinary differential equations (4.16) and (4.17), according to an inner layer analysis process (see also chapter 3), we can make the following remarks beforehand. For the given initial conditions for Y_k (which are of $O(\varepsilon^2)$), the terms in the right side of equation (4.17) can lead to different contributions in the solution Y_k on time-scales of $O(\frac{1}{\varepsilon})$. The first three terms in the right side of equation (4.17) only lead to contributions of $O(\varepsilon^2)$. The coupled, nonlinear terms including $T_p T_j$ can lead to contributions up to $O(\frac{1}{\varepsilon} T_p T_j)$, and the term with frequency ω_1 lead to contributions up to $O(\sqrt{\varepsilon} T_k)$. The term with frequency ω_2 can lead to contributions up to $O(\varepsilon\sqrt{\varepsilon})$. Since T_k may increase from the initial state order of $O(\varepsilon)$ to lower orders, the orders of the terms including $T_p T_j$ determine that of the solution Y_k. Therefore, the solution of equation (4.17) can be approximated as $Y_k = O(\frac{1}{\varepsilon} T_p T_j)$. Similarly, we obtain that for the given initial conditions for T_k (which are of $O(\varepsilon)$), terms in the right side of equation (4.16) can also have different contributions to the solution T_k on time-scales of $O(\frac{1}{\varepsilon})$. The first five terms in the right side of equation (4.16) only can lead to contributions of $O(\varepsilon)$. Based on the fact that $Y_k = O(\frac{1}{\varepsilon} T_p T_j)$, terms including $Y_p T_j$ can lead to contributions up to $O(\varepsilon)$, and the last term with frequency ω_1 can lead to contributions up to $O(\sqrt{\varepsilon})$. This implies that in equation (4.16) only the external forcing with frequency ω_1 produces resonance, and leads to a jump in the solution T_k from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. Further, it follows from equation (4.17) that the coupled terms including $T_p T_j$ produce maximum amplitude responses, and the amplitude responses are depend on the solution T_k of equation (4.16).

After the above made observations, to obtain the (un-)expected resonance manifolds which describe the solutions of ordinary differential equations (4.16) and (4.17) sufficiently accurately, the following standard transformations are introduced:

$$\begin{aligned} T_k(s) &= A_{1,k}(s) \sin(k\pi s) + B_{1,k}(s) \cos(k\pi s), \\ T_{k,s}(s) &= k\pi A_{1,k}(s) \cos(k\pi s) - k\pi B_{1,k}(s) \sin(k\pi s), \\ Y_k(s) &= C_{1,k}(s) \sin(\lambda_k s) + D_{1,k}(s) \cos(\lambda_k s), \\ Y_{k,s}(s) &= \lambda_k C_{1,k}(s) \cos(\lambda_k s) - \lambda_k D_{1,k}(s) \sin(\lambda_k s), \end{aligned}$$

where $\lambda_k = \sqrt{\frac{EA}{mg}}k\pi$. The transverse problem (4.16) can now be rewritten in the following form (where the dot $\cdot$ represents differentiation with respect to s):

$$
\begin{aligned}
\dot{A}_{1,k} \quad = \quad & \varepsilon(v_0 - c_{1,0}\hat{l})A_{1,k}\cos^2(k\pi s) - \varepsilon(v_0 - c_{1,0}\hat{l})B_{1,k}\sin(k\pi s)\cos(k\pi s) \\
& +\varepsilon\sum_{n=1}^{\infty}[(-c_{n,k}^1\mu_0\hat{l}\frac{n^2\pi}{k})A_{1,n} + (2c_{n,k}^2 v_0\frac{n^2\pi}{k})B_{1,n} \\
& -(c_{n,k}^2\mu_0\hat{l}\frac{n}{k})A_{1,n} - (2c_{n,k}^3 v_0\frac{n^2\pi}{k})B_{1,n}]\sin(n\pi s)\cos(k\pi s) \\
& +\varepsilon\sum_{n=1}^{\infty}[(-c_{n,k}^1\mu_0\hat{l}\frac{n^2\pi}{k})B_{1,n} + (2c_{n,k}^2 v_0\frac{n^2\pi}{k})A_{1,n} \\
& -(c_{n,k}^2\mu_0\hat{l}\frac{n}{k})B_{1,n} + (2c_{n,k}^3 v_0\frac{n^2\pi}{k})A_{1,n}]\cos(n\pi s)\cos(k\pi s) \\
& +\frac{EA\pi^2}{2mg\hat{l}}\sum_{p=k+1}^{\infty} p(k-p)[C_{1,p}A_{1,p-k}\sin(\lambda_p s)\sin((p-k)\pi s) \\
& +C_{1,p}B_{1,p-k}\sin(\lambda_p s)\cos((p-k)\pi s) \\
& +D_{1,p}A_{1,p-k}\cos(\lambda_p s)\sin((p-k)\pi s) \\
& +D_{1,p}B_{1,p-k}\cos(\lambda_p s)\cos((p-k)\pi s)]\cos(k\pi s) \\
& -\frac{EA\pi^2}{2mg\hat{l}}\sum_{p=1}^{k-1} p(k-p)[C_{1,p}A_{1,k-p}\sin(\lambda_p s)\sin((k-p)\pi s) \\
& +C_{1,p}B_{1,k-p}\sin(\lambda_p s)\cos((k-p)\pi s) \\
& +D_{1,p}A_{1,k-p}\cos(\lambda_p s)\sin((k-p)\pi s) \\
& +D_{1,p}B_{1,k-p}\cos(\lambda_p s)\cos((k-p)\pi s)]\cos(k\pi s) \\
& -\frac{EA\pi^2}{2mg\hat{l}}\sum_{p=1}^{\infty} p(k+p)[C_{1,p}A_{1,p+k}\sin(\lambda_p s)\sin((p+k)\pi s) \\
& +C_{1,p}B_{1,p+k}\sin(\lambda_p s)\cos((p+k)\pi s) \\
& +D_{1,p}A_{1,p+k}\cos(\lambda_p s)\sin((p+k)\pi s) \\
& +D_{1,p}B_{1,p+k}\cos(\lambda_p s)\cos((p+k)\pi s)]\cos(k\pi s) \\
& +\beta_1\frac{d_k\omega_1^2\hat{l}^2}{2}[\cos(k\pi s + \omega_1\chi(s) + \alpha) + \cos(k\pi s - \omega_1\chi(s) - \alpha)] + h.o.t, \qquad (4.19) \\
\dot{B}_{1,k} \quad = \quad & -\varepsilon(v_0 - c_{1,0}\hat{l})A_{1,k}\cos(k\pi s)\sin(k\pi s) + \varepsilon(v_0 - c_{1,0}\hat{l})B_{1,k}\sin^2(k\pi s) \\
& -\varepsilon\sum_{n=1}^{\infty}[(-c_{n,k}^1\mu_0\hat{l}\frac{n^2\pi}{k})A_{1,n} + (2c_{n,k}^2 v_0\frac{n^2\pi}{k})B_{1,n} \\
& -(c_{n,k}^2\mu_0\hat{l}\frac{n}{k})A_{1,n} - (2c_{n,k}^3 v_0\frac{n^2\pi}{k})B_{1,n}]\sin(n\pi s)\sin(k\pi s) \\
& -\varepsilon\sum_{n=1}^{\infty}[(-c_{n,k}^1\mu_0\hat{l}\frac{n^2\pi}{k})B_{1,n} + (2c_{n,k}^2 v_0\frac{n^2\pi}{k})A_{1,n} \\
& -(c_{n,k}^2\mu_0\hat{l}\frac{n}{k})B_{1,n} + (2c_{n,k}^3 v_0\frac{n^2\pi}{k})B_{1,n}]\cos(n\pi s)\sin(k\pi s) \\
& -\frac{EA\pi^2}{2mg\hat{l}}\sum_{p=k+1}^{\infty} p(k-p)[C_{1,p}A_{1,p-k}\sin(\lambda_p s)\sin((p-k)\pi s) \\
& +C_{1,p}B_{1,p-k}\sin(\lambda_p s)\cos((p-k)\pi s) \\
& +D_{1,p}A_{1,p-k}\cos(\lambda_p s)\sin((p-k)\pi s) \\
& +D_{1,p}B_{1,p-k}\cos(\lambda_p s)\cos((p-k)\pi s)]\sin(k\pi s)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{EA\pi^2}{2mg\hat{l}}\sum_{p=1}^{k-1} p(k-p)[C_{1,p}A_{1,k-p}\sin(\lambda_p s)\sin((k-p)\pi s)\\
&+C_{1,p}B_{1,k-p}\sin(\lambda_p s)\cos((k-p)\pi s)\\
&+D_{1,p}A_{1,k-p}\cos(\lambda_p s)\sin((k-p)\pi s)\\
&+D_{1,p}B_{1,k-p}\cos(\lambda_p s)\cos((k-p)\pi s)]\sin(k\pi s)\\
&+\frac{EA\pi^2}{2mg\hat{l}}\sum_{p=1}^{\infty} p(k+p)[C_{1,p}A_{1,p+k}\sin(\lambda_p s)\cos((p+k)\pi s)\\
&+C_{1,p}B_{1,p+k}\sin(\lambda_p s)\cos((p+k)\pi s)\\
&+D_{1,p}A_{1,p+k}\cos(\lambda_p s)\sin((p+k)\pi s)\\
&+D_{1,p}B_{1,p+k}\cos(\lambda_p s)\cos((p+k)\pi s)]\sin(k\pi s)\\
&-\beta_1\frac{d_k\omega_1^2\hat{l}^2}{2}[\sin(k\pi s+\omega_1\chi(s)+\alpha)+\sin(k\pi s-\omega_1\chi(s)-\alpha)]+h.o.t. \qquad (4.20)
\end{aligned}
$$

4

Large transverse amplitude responses in (4.19) and (4.20), due to the external forcing with frequency ω_1, can be expected when $k\pi-\omega_1\dot{\chi}(s)\approx 0$, or $k\pi+\omega_1\dot{\chi}(s)\approx 0$. But since $k\pi>0$ and $\omega_1\dot{\chi}(s)>0$, resonance only will occur when

$$\omega_1 l_0 e^{\varepsilon v_0 s}\approx k\pi. \qquad (4.21)$$

So, transverse resonances are expected for times s around $s^{(k)}$ with

$$s^{(k)}=\frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0}),\quad k\pi\geq\omega_1 l_0,\quad k=1,2,\ldots \qquad (4.22)$$

To study the situation in the transverse resonance zone, we introduce time-like variables

$$\tau=\varepsilon s,\quad \phi_k(s)=k\pi s,\quad \varphi(s)=\omega_1\chi(s)+\alpha,\quad \psi_k(s)=\phi_k(s)-\varphi(s),$$

and rescale $\tau-\tau^{(k)}=\delta(\varepsilon)\hat{\tau}$ with $\hat{\tau}=O(1)$ and $\tau^{(k)}=\varepsilon s^{(k)}=\frac{1}{v_0}\ln(\frac{k\pi}{\omega_1 l_0})$. Then

$$
\begin{cases}
\dot{\tau}=\varepsilon,\quad \dot{\hat{\tau}}=\frac{\varepsilon}{\delta(\varepsilon)},\\
\dot{\phi}_k=k\pi,\\
\dot{\varphi}=\omega_1 l_0 e^{v_0(\tau^{(k)}+\delta(\varepsilon)\hat{\tau})},\\
\dot{\psi}_k=k\pi-\omega_1 l_0 e^{v_0(\tau^{(k)}+\delta(\varepsilon)\hat{\tau})}=\delta(\varepsilon)\omega_1 l_0 v_0 e^{v_0\tau^{(k)}}\hat{\tau},
\end{cases} \qquad (4.23)
$$

and $\dot{A}_{1,k}(s)$, $\dot{B}_{1,k}(s)$ are given by (4.19). It now follows that a balance in system (4.23) occurs when $\frac{\varepsilon}{\delta(\varepsilon)}=\delta(\varepsilon)$, and this implies that in the transverse resonance zone that $\delta(\varepsilon)=\sqrt{\varepsilon}$, i.e., the size of transverse resonance zone is $O(\frac{1}{\sqrt{\varepsilon}})$ for times s. So, together with $\tau-\tau^{(k)}=\delta(\varepsilon)\hat{\tau}$, it follows from (4.23) that

$$\hat{\tau}=\sqrt{\varepsilon}(s-s^{(k)}). \qquad (4.24)$$

Further, from (4.23), we obtain $\psi_k(s)=\psi_k(s^{(k)})+\frac{1}{2}\omega_1 l_0 v_0 e^{v_0\tau^{(k)}}\varepsilon(s-s^{(k)})^2$. Hence, in the transverse resonance zone, we can write

$$\sin(\psi_k(s)) \;=\; \sin(\frac{1}{2}\omega_1 l_0 v_0 e^{v_0\tau^{(k)}}\varepsilon(s-s^{(k)})^2+\psi_k(s^{(k)})),$$

$$\cos(\psi_k(s)) \quad = \quad \cos(\frac{1}{2}\omega_1 l_0 v_0 e^{v_0\tau^{(k)}}\varepsilon(s-s^{(k)})^2+\psi_k(s^{(k)})), \tag{4.25}$$

where $\psi_k(s^{(k)}) = k\pi s^{(k)} - \frac{k\pi-\omega_1 l_0}{\varepsilon v_0} - \alpha$.

It was analysed that the resonance responses for Y_k in (4.17) are depend on the terms including T_pT_j, and the resonance responses for T_k in (4.16) are depend on the terms with frequency ω_1. So, based on the inner layer analysis, the size of the emerged resonance zones has been established, and this size will also be used as a new asymptotic scale to introduce a three-timescale perturbation method in the next section to study problems (4.16) and (4.17) in detail and construct asymptotic approximations of the solutions of the initial-boundary value problems (4.8) and (4.9).

4

4.4. Three-timescales perturbation method

In the previous section, it was shown that (under certain condition on the external frequency ω_1) resonances in the transverse direction can occur around time $s = \frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})$, and that resonances in the longitudinal direction depend on the solutions T_k of equation (4.16). For this reason, we rescale s by defining $s = \tilde{s} + \frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})$. Thus, problem (4.16) can be rewritten in $\tilde{s}$ as follows:

$$\begin{aligned}
&T_{k,\tilde{s}\tilde{s}} + k^2\pi^2 T_k \\
= \;& \varepsilon[v_0 T_{k,\tilde{s}} - c_{1,0}\hat{l}T_{k,\tilde{s}} - \sum_{n=1}^{\infty} c^1_{n,k}\mu_0\hat{l}(n\pi)^2 T_n + \sum_{n=1}^{\infty} c^2_{n,k}(-2v_0 n\pi T_{n,\tilde{s}} - \mu_0\hat{l}n\pi T_n) \\
&+ \sum_{n=1}^{\infty} c^3_{n,k} 2v_0 n\pi T_{n,\tilde{s}} + \frac{EA\pi^3}{2\varepsilon mg\hat{l}}\sum_{p=k+1}^{\infty} kp(k-p)Y_pT_{p-k} \\
&- \frac{EA\pi^3}{2\varepsilon mg\hat{l}}\sum_{p=1}^{k-1} kp(k-p)Y_pT_{k-p} - \frac{EA\pi^3}{2\varepsilon mg\hat{l}}\sum_{p=1}^{\infty} kp(k+p)Y_pT_{k+p} \\
&+ \beta_{1,0}d_k\omega_1^2\hat{l}^2\cos(\omega_1\chi(\tilde{s}+\frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})) + \alpha)] + h.o.t., \\
&T_k(-\frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})) = F_k, \quad T_{k,\tilde{s}}(-\frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})) = G_k,
\end{aligned} \tag{4.26}$$

and problem (4.17) can be rewritten in $\tilde{s}$ as follows:

$$\begin{aligned}
&Y_{k,\tilde{s}\tilde{s}} + \lambda_k^2 Y_k \\
= \;& \varepsilon(v_0 - c_{2,0}\hat{l})Y_{k,\tilde{s}} + \sum_{n=1}^{\infty} 2\varepsilon v_0 n\pi d^1_{n,k}Y_{n,\tilde{s}} - \sum_{n=1}^{\infty}\frac{EA}{mg}\mu\hat{l}n\pi d^3_{n,k}Y_n \\
&+ \sum_{j=1}^{k-1} d^4_{k-j,j}T_{k-j}T_j + \sum_{j=k+1}^{\infty} d^4_{j-k,j}T_{j-k}T_j - \sum_{j=1}^{\infty} d^4_{j+k,j}T_{j+k}T_j \\
&+ \beta_1\tilde{d}^4_{k,k}T_k\cos(\omega_1\chi(\tilde{s}+\frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})) + \alpha) \\
&+ \beta_2\omega_2^2\hat{l}^2 d_{1,k}\cos(\omega_2\chi(\tilde{s}+\frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})) + \alpha) \\
&+ h.o.t., \\
&Y_k(-\frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})) = f_k, \quad Y_{k,\tilde{s}}(-\frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0})) = g_k.
\end{aligned} \tag{4.27}$$

Next, we study problems (4.26) and (4.27) in detail under the assumption that ω_1 is such that a resonance zone exits for the k^{th} oscillation mode. The application of the straightforward expansion method to solve (4.26) and (4.27) will result in the occurrence of so-called secular terms which cause the approximations of the solutions to become unbounded on long timescales. For this reason, to remove secular terms, and to obtain approximations which are valid on long timescales, we introduce three timescales $s_0 = \tilde{s}$, $s_1 = \sqrt{\varepsilon}\tilde{s}$, $s_2 = \varepsilon\tilde{s}$. The time-scale $s_1 = \sqrt{\varepsilon}\tilde{s}$ is introduced because of the size of the resonance zone which has been found in the previous section, and the other two time-scales are the natural scalings for nonlinear equations such as (4.26) and (4.27). By using the three time-scales perturbation method, the functions $T_k(\tilde{s};\sqrt{\varepsilon})$ and $Y_k(\tilde{s};\sqrt{\varepsilon})$ are supposed to be functions of s_0, s_1 and s_2,

$$T_k(\tilde{s};\sqrt{\varepsilon}) = \widetilde{T}_k(s_0,s_1,s_2), \quad Y_k(\tilde{s};\sqrt{\varepsilon}) = \widetilde{Y}_k(s_0,s_1,s_2).$$

4

By substituting $\widetilde{T}_k(s_0,s_1,s_2)$ and $\widetilde{Y}_k(s_0,s_1,s_2)$ into the differential equation (4.16), we obtain the following equations up to $O(\varepsilon\sqrt{\varepsilon})$:

$$\begin{aligned}
&\frac{\partial^2\widetilde{T}_k}{\partial s_0^2} + k^2\pi^2\widetilde{T}_k + 2\sqrt{\varepsilon}\frac{\partial^2\widetilde{T}_k}{\partial s_0\partial s_1} + \varepsilon\Big(2\frac{\partial^2\widetilde{T}_k}{\partial s_0\partial s_2} + \frac{\partial^2\widetilde{T}_k}{\partial s_1^2}\Big) + 2\varepsilon\sqrt{\varepsilon}\frac{\partial^2\widetilde{T}_k}{\partial s_1\partial s_2}\\
= \;& \varepsilon\Big[(v_0 - c_{1,0}\hat{l})\frac{\partial\widetilde{T}_k}{\partial s_0} - \sum_{n=1}^{\infty} c^1_{n,k}\mu_0\hat{l}(n\pi)^2\widetilde{T}_n + \sum_{n=1}^{\infty} c^2_{n,k}\Big(-2v_0 n\pi\frac{\partial\widetilde{T}_n}{\partial s_0} - \mu_0\hat{l}n\pi\widetilde{T}_n\Big)\\
&+\sum_{n=1}^{\infty} c^3_{n,k}2v_0 n\pi\frac{\partial\widetilde{T}_n}{\partial s_0}\Big]\\
&+\varepsilon\sqrt{\varepsilon}\Big[(v_0 - c_{1,0}\hat{l})\frac{\partial\widetilde{T}_k}{\partial s_1} + \sum_{n=1}^{\infty} c^2_{n,k}\Big(-2v_0 n\pi\frac{\partial\widetilde{T}_n}{\partial s_1} + \sum_{n=1}^{\infty} c^3_{n,k}2v_0 n\pi\frac{\partial\widetilde{T}_n}{\partial s_1}\Big]\\
&+\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=k+1}^{\infty} kp(k-p)Y_pT_{p-k} - \frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{k-1} kp(k-p)Y_pT_{k-p}\\
&-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{\infty} kp(k+p)Y_pT_{k+p}\\
&+\beta_1 d_k\omega_1^2\hat{l}^2\cos(\omega_1\chi(s_0-a)+\alpha),\\
&\widetilde{T}_k(a,b,c;\sqrt{\varepsilon}) = F_k = \varepsilon\tilde{F}_k,\\
&\frac{\partial\widetilde{T}_k}{\partial s_0}(a,b,c;\sqrt{\varepsilon}) + \sqrt{\varepsilon}\frac{\partial\widetilde{T}_k}{\partial s_1}(a,b,c;\sqrt{\varepsilon}) + \varepsilon\frac{\partial\widetilde{T}_k}{\partial s_2}(a,b,c;\sqrt{\varepsilon}) = G_k = \varepsilon\tilde{G}_k,
\end{aligned} \tag{4.28}$$

where $\tilde{F}_k = O(1)$ and $\tilde{G}_k = O(1)$. Similarly, by substituting $\widetilde{T}_k(s_0,s_1,s_2)$, $\widetilde{Y}_k(s_0,s_1,s_2)$ into the differential equation (4.17), we obtain the following equations up to $O(\varepsilon\sqrt{\varepsilon})$:

$$\begin{aligned}
&\frac{\partial^2\widetilde{Y}_k}{\partial s_0^2} + \lambda_k^2\widetilde{Y}_k + 2\sqrt{\varepsilon}\frac{\partial^2\widetilde{Y}_k}{\partial s_0\partial s_1} + \varepsilon\Big(2\frac{\partial^2\widetilde{Y}_k}{\partial s_0\partial s_2} + \frac{\partial^2\widetilde{Y}_k}{\partial s_1^2}\Big) + 2\varepsilon\sqrt{\varepsilon}\frac{\partial^2\widetilde{Y}_k}{\partial s_1\partial s_2}\\
= \;& \varepsilon\Big[(v_0 - c_{2,0}\hat{l})\frac{\partial\widetilde{Y}_k}{\partial s_0} + \sum_{n=1}^{\infty} 2v_0 n\pi d^1_{n,k}\frac{\partial\widetilde{Y}_k}{\partial s_0} - \sum_{n=1}^{\infty}\frac{EA}{mg}\mu_0\hat{l}n\pi d^3_{n,k}\widetilde{Y}_n\\
&+\beta_{1,0}d^4_{k,k}\widetilde{T}_k\cos(\omega_1\chi(s_0-a)+\alpha)\Big]\\
&\varepsilon\sqrt{\varepsilon}\Big[(v_0 - c_{2,0}\hat{l})\frac{\partial\widetilde{Y}_k}{\partial s_1} + \sum_{n=1}^{\infty} 2v_0 n\pi d^1_{n,k}\frac{\partial\widetilde{Y}_k}{\partial s_1}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{k-1} d^4_{k-j,j}\tilde{T}_{k-j}\tilde{T}_j+\sum_{j=k+1}^{\infty} d^4_{j-k,j}\tilde{T}_{j-k}\tilde{T}_j-\sum_{j=1}^{\infty} d^4_{j+k,j}\tilde{T}_{j+k}\tilde{T}_j\\
&\tilde{Y}_k(a,b,c;\sqrt{\varepsilon})=f_k=\varepsilon^2\tilde{f}_k,\\
&\frac{\partial\tilde{Y}_k}{\partial s_0}(a,b,c;\sqrt{\varepsilon})+\sqrt{\varepsilon}\frac{\partial\tilde{Y}_k}{\partial s_1}(a,b,c;\sqrt{\varepsilon})+\varepsilon\frac{\partial\tilde{Y}_k}{\partial s_2}(a,b,c;\sqrt{\varepsilon})=g_k=\varepsilon^2\tilde{g}_k,
\end{aligned}
\quad (4.29)
$$

where $\lambda_k=\sqrt{\frac{EA}{mg}}k\pi$, $\tilde{f}_k=O(1)$, $\tilde{g}_k=O(1)$, and

$$
a=-\frac{1}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0}),\quad b=-\frac{\sqrt{\varepsilon}}{\varepsilon v_0}\ln(\frac{k\pi}{\omega_1 l_0}),\quad c=-\frac{1}{v_0}\ln(\frac{k\pi}{\omega_1 l_0}). \quad (4.30)
$$

Since the functions $\tilde{T}_k$ and $\tilde{Y}_k$ can increase in s from the initial state orders to $O(\sqrt{\varepsilon})$ as has been shown in the previous section, a three-timescales perturbation method will be used, and $\tilde{T}_k(s_0,s_1,s_2)$ and $\tilde{Y}_k(s_0,s_1,s_2)$ will be approximated by the following formal asymptotic expansions:

$$
\tilde{T}_k(s_0,s_1,s_2) = \sqrt{\varepsilon}\tilde{T}_{k,0}(s_0,s_1,s_2)+\varepsilon\tilde{T}_{k,1}(s_0,s_1,s_2)+\varepsilon\sqrt{\varepsilon}\tilde{T}_{k,2}(s_0,s_1,s_2)+O(\varepsilon^2), \quad (4.31)
$$

$$
\tilde{Y}_k(s_0,s_1,s_2) = \sqrt{\varepsilon}\tilde{Y}_{k,0}(s_0,s_1,s_2)+\varepsilon\tilde{Y}_{k,1}(s_0,s_1,s_2)+\varepsilon\sqrt{\varepsilon}\tilde{Y}_{k,2}(s_0,s_1,s_2)+O(\varepsilon^2), \quad (4.32)
$$

where $\tilde{T}_{k,0},\tilde{T}_{k,1},\tilde{T}_{k,2},\tilde{Y}_{k,0},\tilde{Y}_{k,1},\tilde{Y}_{k,2}$ are all functions of $O(1)$.In the transverse direction, by substituting (4.31) and (4.32) into problem (4.28), and after equating the coefficients of like powers in $\sqrt{\varepsilon}$, we obtain :
the $O(\sqrt{\varepsilon})$-problem:

$$
\begin{aligned}
&\frac{\partial^2\tilde{T}_{k,0}}{\partial s_0^2}+k^2\pi^2\tilde{T}_{k,0}=0,\\
&\tilde{T}_{k,0}(a,b,c)=0,\qquad \frac{\partial\tilde{T}_{k,0}}{\partial s_0}(a,b,c)=0,
\end{aligned}
\quad (4.33)
$$

the $O(\varepsilon)$-problem:

$$
\begin{aligned}
&\frac{\partial^2\tilde{T}_{k,1}}{\partial s_0^2}+k^2\pi^2\tilde{T}_{k,1}+2\frac{\partial^2\tilde{T}_{k,0}}{\partial s_0\partial s_1}\\
=&\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=k+1}^{\infty}kp(k-p)\tilde{Y}_{p,0}\tilde{T}_{p-k,0}-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{k-1}kp(k-p)\tilde{Y}_{p,0}\tilde{T}_{k-p,0}\\
&-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{\infty}kp(k+p)\tilde{Y}_{p,0}\tilde{T}_{k+p,0}+\beta_{1,0}d_k\omega_1^2\hat{l}^2\cos(\omega_1\chi(s_0-a)+\alpha),\\
&\tilde{T}_{k,1}(a,b,c)=\tilde{F}_k,\qquad \frac{\partial\tilde{T}_{k,1}}{\partial s_0}(a,b,c)+\frac{\partial\tilde{T}_{k,0}}{\partial s_1}(a,b,c)=\tilde{G}_k,
\end{aligned}
\quad (4.34)
$$

the $O(\varepsilon\sqrt{\varepsilon})$-problem:

$$
\begin{aligned}
&\frac{\partial^2\tilde{T}_{k,2}}{\partial s_0^2}+k^2\pi^2\tilde{T}_{k,2}+2\frac{\partial^2\tilde{T}_{k,1}}{\partial s_0\partial s_1}+2\frac{\partial^2\tilde{T}_{k,0}}{\partial s_0\partial s_2}+\frac{\partial^2\tilde{T}_{k,0}}{\partial s_1^2}\\
=&(v_0-c_{1,0}\hat{l})\frac{\partial\tilde{T}_{k,0}}{\partial s_0}-\sum_{n=1}^{\infty}c^1_{n,k}\mu_0\hat{l}(n\pi)^2\tilde{T}_{n,0}+\sum_{n=1}^{\infty}c^2_{n,k}(-2v_0n\pi\frac{\partial\tilde{T}_{n,0}}{\partial s_0}-\mu_0\hat{l}n\pi\tilde{T}_{n,0})
\end{aligned}
$$

$$+\sum_{n=1}^{\infty} c_{n,k}^3 2v_0 n\pi \frac{\partial \tilde{T}_{n,0}}{\partial s_0},$$

$$\tilde{T}_{k,2}(a,b,c)=0, \qquad \frac{\partial \tilde{T}_{k,2}}{\partial s_0}(a,b,c)+\frac{\partial \tilde{T}_{k,1}}{\partial s_1}(a,b,c)+\frac{\partial \tilde{T}_{k,0}}{\partial s_2}(a,b,c)=0. \tag{4.35}$$

The solution of the $O(\sqrt{\varepsilon})$-problem (4.33) can be written as:

$$\tilde{T}_{k,0}(s_0,s_1,s_2)=A_k(s_1,s_2)\cos(k\pi s_0)+B_k(s_1,s_2)\sin(k\pi s_0), \tag{4.36}$$

where $A_k(b,c)=0$, $B_k(b,c)=0$, and where $A_k(s_1,s_2)$, $B_k(s_1,s_2)$ can be obtained explicitly by solving the $O(\varepsilon)$-problem (4.34) and the $O(\varepsilon\sqrt{\varepsilon})$-problem (4.35). We will study these problems later in this section.

In the longitudinal direction, by substituting (4.31) and (4.32) into problem (4.29), and after equating the coefficients of like powers in ε, we obtain :
the $O(\sqrt{\varepsilon})$-problem:

$$\frac{\partial^2 \tilde{Y}_{k,0}}{\partial s_0^2}+\lambda_k^2\tilde{Y}_{k,0}=0,$$

$$\tilde{Y}_{k,0}(a,b,c)=0, \qquad \frac{\partial \tilde{Y}_{k,0}}{\partial s_0}(a,b,c)=0, \tag{4.37}$$

the $O(\varepsilon)$-problem:

$$\frac{\partial^2 \tilde{Y}_{k,1}}{\partial s_0^2}+\lambda_k^2\tilde{Y}_{k,1}+2\frac{\partial^2 \tilde{Y}_{k,0}}{\partial s_0\partial s_1}$$

$$= [\sum_{j=1}^{k-1} d_{k-j,j}^4 T_{k-j,0}T_{j,0}+\sum_{j=k+1}^{\infty} d_{j-k,j}^4 T_{j-k,0}T_{j,0}-\sum_{j=1}^{\infty} d_{j+k,j}^4 T_{j+k,0}T_{j,0}],$$

$$\tilde{Y}_{k,1}(a,b,c)=0, \qquad \frac{\partial \tilde{Y}_{k,1}}{\partial s_0}(a,b,c)+\frac{\partial \tilde{Y}_{k,0}}{\partial s_1}(a,b,c)=0, \tag{4.38}$$

and the $O(\varepsilon\sqrt{\varepsilon})$-problem:

$$\frac{\partial^2 \tilde{Y}_{k,2}}{\partial s_0^2}+\lambda_k^2\tilde{Y}_{k,2}+2\frac{\partial^2 \tilde{Y}_{k,1}}{\partial s_0\partial s_1}+2\frac{\partial^2 \tilde{Y}_{k,0}}{\partial s_0\partial s_2}+\frac{\partial^2 \tilde{Y}_{k,0}}{\partial s_1^2}$$

$$= (v_0-c_{2,0}\hat{l})\frac{\partial \tilde{Y}_{k,0}}{\partial s_0}+\sum_{n=1}^{\infty} 2v_0 n\pi d_{n,k}^1\frac{\partial \tilde{Y}_{n,0}}{\partial s_0}-\sum_{n=1}^{\infty}\frac{EA}{mg}\mu_0\hat{l}n\pi d_{n,k}^3\tilde{Y}_{n,0}$$

$$+\beta_{1,0}\tilde{d}_{k,k}^4\tilde{T}_{k,0}\cos(\omega_1\chi(s_0-a)+\alpha),$$

$$\tilde{Y}_{k,2}(a,b,c)=0, \qquad \frac{\partial \tilde{Y}_{k,2}}{\partial s_0}(a,b,c)+\frac{\partial \tilde{Y}_{k,1}}{\partial s_1}(a,b,c)+\frac{\partial \tilde{Y}_{k,0}}{\partial s_2}(a,b,c)=0, \tag{4.39}$$

where $\lambda_k=\sqrt{\frac{EA}{mg}}k\pi$.

The solution of the $O(\sqrt{\varepsilon})$-problem (4.37) can be written as:

$$\tilde{Y}_{k,0}(s_0,s_1,s_2) \quad = \quad C_k(s_1,s_2)\cos(\lambda_k s_0)+D_k(s_1,s_2)\sin(\lambda_k s_0), \tag{4.40}$$

where $C_k(s_1,s_2)$ and $D_k(s_1,s_2)$ are still unknown functions in the slow variables s_1 and s_2, and these functions can be determined by avoiding secular terms in the $O(\varepsilon)-$ problem

(4.38) and in the $O(\varepsilon\sqrt{\varepsilon})-$ problem (4.39). By using the initial conditions in (4.37), it follows that $C_k(b,c) = D_k(b,c) = 0$. Now, we shall solve the $O(\varepsilon)-$ problem (4.38). By using (4.36) for $\tilde{T}_{k,0}$, and by using $d^1_{k,k} = -\frac{1}{2k\pi}$, which is given in (4.18), problem (4.38) can be written as:

$$\begin{aligned}
&\frac{\partial^2 \tilde{Y}_{k,1}}{\partial s_0^2} + \lambda_k^2 \tilde{Y}_{k,1} \\
= \;& 2\lambda_k \frac{\partial C_k}{\partial s_1}\sin(\lambda_k s_0) - 2\lambda_k \frac{\partial D_k}{\partial s_1}\cos(\lambda_k s_0) \\
&+\frac{1}{2}\sum_{j=1}^{k-1} d^4_{k-j,j}[(A_{k-j}A_j - B_{k-j}B_j)\cos(k\pi s_0) + (A_{k-j}B_j + B_{k-j}A_j)\sin(k\pi s_0) \\
&+(A_{k-j}A_j + B_{k-j}B_j)\cos((k-2j)\pi s_0) + (A_{k-j}B_j - B_{k-j}A_j)\sin((k-2j)\pi s_0)] \\
&+\frac{1}{2}\sum_{j=k+1}^{\infty} d^4_{j-k,j}[(A_{j-k}A_j + B_{j-k}B_j)\cos(k\pi s_0) + (A_{j-k}B_j - B_{j-k}A_j)\sin(k\pi s_0) \\
&+(A_{j-k}A_j - B_{j-k}B_j)\cos((2j-k)\pi s_0) + (A_{j-k}B_j + B_{j-k}A_j)\sin((2j-k)\pi s_0)] \\
&-\frac{1}{2}\sum_{j=1}^{\infty} d^4_{k+j,j}[(A_{k+j}A_j + B_{k+j}B_j)\cos(k\pi s_0) + (A_jB_{j+k} - B_jA_{j+k})\sin(k\pi s_0) \\
&+(A_{k+j}A_j - B_{k+j}B_j)\cos((k+2j)\pi s_0) + (A_jB_{j+k} + B_jA_{j+k})\sin((k+2j)\pi s_0)], \\
&\tilde{Y}_{k,1}(0,0,0) = 0, \qquad \frac{\partial \tilde{Y}_{k,2}}{\partial s_0}(0,0,0) + \frac{\partial \tilde{Y}_{k,1}}{\partial s_1}(0,0,0) + \frac{\partial \tilde{Y}_{k,0}}{\partial s_2}(0,0,0) = 0.
\end{aligned} \tag{4.41}$$

It is obvious that the right-hand side of (4.41) contains resonant terms, such that $\sin(\lambda_k s_0)$ and $\cos(\lambda_k s_0)$, the term in the right-hand side of (4.41) involving $\sin((2j-k)\pi s_0), \cos((2j-k)\pi s_0), \sin((k+2j)\pi s_0), \cos((k+2j)\pi s_0)$ is also a resonant term when there exist k, j_1, j_2, s.t., $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$. Actually, for any fixed parameter value of $\sqrt{\frac{EA}{mg}}$ with assumptions $\sqrt{\frac{EA}{mg}} = O(1)$ and $\sqrt{\frac{EA}{mg}} - 1 > O(\varepsilon)$, there always exist k s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$ with $j_1 = \frac{(1+\sqrt{\frac{EA}{mg}})k}{2}$ and $j_2 = \frac{(\sqrt{\frac{EA}{mg}}-1)k}{2}$. Therefore, to avoid secular terms in (4.41) the functions of $C_k(s_1,s_2)$ and $D_k(s_1,s_2)$ have to satisfy the following:

- When k does not satisfy the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$\frac{\partial C_k}{\partial s_1} = 0, \quad \frac{\partial D_k}{\partial s_1} = 0, \tag{4.42}$$

and $C_k(s_1,s_2)$ and $D_k(s_1,s_2)$ are given by:

$$C_k(s_1,s_2) = \overline{C}_k(s_2), \quad D_k(s_1,s_2) = \overline{D}_k(s_2). \tag{4.43}$$

- When k satisfies the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$\begin{aligned}\frac{\partial C_k}{\partial s_1} &= -\frac{d^4_{j_1-k,j_1}}{4\lambda_k}(A_{j_1-k}B_{j_1}+B_{j_1-k}A_{j_1}) \\ &+\frac{d^4_{k+j_2,j_2}}{4\lambda_k}(A_{j_2}B_{j_2+k}+B_{j_2}A_{j_2+k}) = \tilde{P}_2(s_1,s_2), \\ \frac{\partial D_k}{\partial s_1} &= \frac{d^4_{j_1-k,j_1}}{4\lambda_k}(A_{j_1-k}A_{j_1}-B_{j_1-k}B_{j_1}) \\ &-\frac{d^4_{k+j_2,j_2}}{4\lambda_k}(A_{k+j_2}A_{j_2}-B_{k+j_2}B_{j_2}) = \tilde{Q}_2(s_1,s_2), \end{aligned} \tag{4.44}$$

and $C_k(s_1,s_2)$ and $D_k(s_1,s_2)$ can be obtained as:

$$C_k(s_1,s_2) = \int_b^{s_1} \tilde{P}_2(\bar{\tau},s_2)d\bar{\tau} + \overline{C}_k(s_2), \quad D_k(s_1,s_2) = \int_b^{s_1} \tilde{Q}_2(\bar{\tau},s_2)d\bar{\tau} + \overline{D}_k(s_2), \tag{4.45}$$

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where $\overline{C}_k(s_2)$ and $\overline{D}_k(s_2)$ in (4.43) and (4.45) are still unknown functions in the slow variable s_2. By $C_k(b,c)=0$ and $D_k(b,c)=0$, we obtain that $\overline{C}_k(c)=0$ and $\overline{D}_k(c)=0$. The undetermined behaviour with respect to s_2 can be used to avoid secular terms in the solution of the $O(\varepsilon\sqrt{\varepsilon})-$ problem (4.39).

According to (4.41), taking into account the secularity conditions, the general solution of the $O(\varepsilon)-$ problem (4.38) is given by

$$Y_{k,1}(s_0,s_1,s_2;\sqrt{\varepsilon}) = E_k(s_0,s_1,s_2)\cos(\lambda_k s_0) + H_k(s_0,s_1,s_2)\sin(\lambda_k s_0), \tag{4.46}$$

where

$$E_k(a,b,c)=0, \quad H_k(a,b,c) = -\frac{\partial Y_{k,0}}{\partial s_1}(a,b,c). \tag{4.47}$$

Then the $O(\varepsilon\sqrt{\varepsilon})-$ problem (4.39) can be written as:

$$\begin{aligned} &\frac{\partial^2 \tilde{Y}_{k,2}}{\partial s^2} + \lambda_k^2 \tilde{Y}_{k,2} \\ = \; &[-2\frac{\partial^2 E_k}{\partial s_0 \partial s_1} - 2\lambda_k\frac{\partial H_k}{\partial s_1} - 2\lambda_k\frac{\partial D_k}{\partial s_2} - \frac{\partial^2 C_k}{\partial s_1^2} \\ &+(v_0 - c_{2,0}\hat{l})\lambda_k D_k + 2v_0 k\pi d^1_{k,k}\lambda_k D_k - \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k} C_k]\cos(\lambda_k s_0) \\ &+[-2\frac{\partial^2 H_k}{\partial s_0 \partial s_1} + 2\lambda_k\frac{\partial E_k}{\partial s_1} + 2\lambda_k\frac{\partial C_k}{\partial s_2} - \frac{\partial^2 D_k}{\partial s_1^2} \\ &-(v_0 - c_{2,0}\hat{l})\lambda_k C_k - 2v_0 n\pi d^1_{k,k}\lambda_k C_k - \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k} D_k]\sin(\lambda_k s_0), \\ &+\beta_{1,0}\tilde{d}^4_{k,k}[A_k\cos(k\pi s_0) + B_k\sin(k\pi s_0)]\cos(\omega_1\chi(s_0-a)+\alpha), \\ &\tilde{Y}_{k,2}(a,b,c)=0, \qquad \frac{\partial \tilde{Y}_{k,2}}{\partial s_0}(a,b,c) = -\frac{\partial \tilde{Y}_{k,1}}{\partial s_1}(a,b,c) - \frac{\partial \tilde{Y}_{k,0}}{\partial s_2}(a,b,c). \end{aligned} \tag{4.48}$$

Note that in the analysis of section 3, the last term including $\cos(\omega_1\chi(s_0-a)+\alpha)$ in (4.48) can not affect the function $\tilde{Y}_{k,0}$. So, to avoid secular terms in the solution $\tilde{Y}_{k,2}$ in equation (4.48), the following different cases have to be considered:

- When k does not satisfy the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$\begin{aligned}
&-2\frac{\partial^2 E_k}{\partial s_0 \partial s_1} - 2\lambda_k \frac{\partial H_k}{\partial s_1} \\
= \;& 2\lambda_k \frac{\partial \overline{D}_k}{\partial s_2} - (v_0 - c_{2,0}\hat{l})\lambda_k \overline{D}_k - 2v_0 k\pi d^1_{k,k}\lambda_k \overline{D}_k + \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k}\overline{C}_k, \\
&-2\frac{\partial^2 H_k}{\partial s_0 \partial s_1} + 2\lambda_k \frac{\partial E_k}{\partial s_1} \\
= \;& -2\lambda_k \frac{\partial \overline{C}_k}{\partial s_2} + (v_0 - c_{2,0}\hat{l})\lambda_k \overline{C}_k + 2v_0 k\pi d^1_{k,k}\lambda_k \overline{C}_k + \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k}\overline{D}_k.
\end{aligned} \tag{4.49}$$

- When k satisfies the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$\begin{aligned}
&-2\frac{\partial^2 E_k}{\partial s_0 \partial s_1} - 2\lambda_k \frac{\partial H_k}{\partial s_1} - 2\lambda_k \frac{\partial \int_0^{s_1} \tilde{Q}_2(\bar{\tau}, s_2) d\bar{\tau}}{\partial s_2} - \frac{\partial \tilde{P}_2}{\partial s_1} + (v_0 - c_{2,0}\hat{l})\lambda_k \int_0^{s_1} \tilde{Q}_2(\bar{\tau}, s_2) d\bar{\tau} \\
&+2v_0 k\pi d^1_{k,k}\lambda_k \int_0^{s_1} \tilde{Q}_2(\bar{\tau}, s_2) d\bar{\tau} - \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k} \int_0^{s_1} \tilde{P}_2(\bar{\tau}, s_2) d\bar{\tau} \\
= \;& 2\lambda_k \frac{\partial \overline{D}_k}{\partial s_2} - (v_0 - c_{2,0}\hat{l})\lambda_k \overline{D}_k - 2v_0 k\pi d^1_{k,k}\lambda_k \overline{D}_k + \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k}\overline{C}_k, \\
&-2\frac{\partial^2 H_k}{\partial s_0 \partial s_1} + 2\lambda_k \frac{\partial E_k}{\partial s_1} + 2\lambda_k \frac{\partial \int_0^{s_1} \tilde{P}_2(\bar{\tau}, s_2) d\bar{\tau}}{\partial s_2} - \frac{\partial \tilde{Q}_2}{\partial s_1} - (v_0 - c_{2,0}\hat{l})\lambda_k \int_0^{s_1} \tilde{P}_2(\bar{\tau}, s_2) d\bar{\tau} \\
&-2v_0 k\pi d^1_{k,k}\lambda_k \int_0^{s_1} \tilde{P}_2(\bar{\tau}, s_2) d\bar{\tau} - \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k} \int_0^{s_1} \tilde{Q}_2(\bar{\tau}, s_2) d\bar{\tau} \\
= \;& -2\lambda_k \frac{\partial \overline{C}_k}{\partial s_2} + (v_0 - c_{2,0}\hat{l})\lambda_k \overline{C}_k + 2v_0 k\pi d^1_{k,k}\lambda_k \overline{C}_k + \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k}\overline{D}_k.
\end{aligned} \tag{4.50}$$

Solving (4.49) and (4.50) for E_k and H_k, we observe that the solution will be unbounded in s_0 and s_1, due to terms which are only depending on s_2. Therefore, to have secular-free solutions for E_k and H_k, the following conditions have to be imposed independently:

$$\begin{aligned}
2\lambda_k \frac{d\overline{D}_k}{ds_2} - (v_0 - c_{2,0}\hat{l})\lambda_k \overline{D}_k - 2v_0 k\pi d^1_{k,k}\lambda_k \overline{D}_k + \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k}\overline{C}_k &= 0, \\
-2\lambda_k \frac{d\overline{C}_k}{ds_2} + (v_0 - c_{2,0}\hat{l})\lambda_k \overline{C}_k + 2v_0 k\pi d^1_{k,k}\lambda_k \overline{C}_k + \frac{EA}{mg}\mu_0 \hat{l} k\pi d^3_{k,k}\overline{D}_k &= 0.
\end{aligned} \tag{4.51}$$

Due to $d^2_{k,k} = -\frac{1}{2k\pi}$, we then obtain from (4.61):

$$\begin{aligned}
\overline{C}_k(s_2) = \;& e^{-\frac{1}{2}c_{2,0}\hat{l}(s_2 - c)}[\overline{C}_k(c)\cos(\frac{EA\mu_0 \hat{l} k\pi d^3_{k,k}}{2mg\lambda_k}(s_2 - c)) \\
&-\overline{D}_k(c)\sin(\frac{EA\mu_0 \hat{l} k\pi d^3_{k,k}}{2mg\lambda_k}(s_2 - c))],
\end{aligned}$$

$$\overline{D}_k(s_2) = e^{-\frac{1}{2}c_{2,0}\hat{l}(s_2-c)}[\overline{C}_k(c)\sin(\frac{EA\mu_0\hat{l}k\pi d^3_{k,k}}{2mg\lambda_k}(s_2-c)) + \overline{D}_k(c)\cos(\frac{EA\mu_0\hat{l}k\pi d^3_{k,k}}{2mg\lambda_k}(s_2-c))].$$

Since $\overline{C}_k(c) = 0$ and $\overline{D}_k(c) = 0$, this implies that

$$\overline{C}_k(s_2) = 0, \quad \overline{D}_k(s_2) = 0. \tag{4.52}$$

Now, all unknown functions in (4.40) can be determined, and the solution of the $O(\sqrt{\varepsilon})$-problem (4.37) can be written as:

$$\tilde{Y}_{k,0}(s_0,s_1,s_2) = C_k(s_1,s_2)\cos(\lambda_k s_0) + D_k(s_1,s_2)\sin(\lambda_k s_0), \tag{4.53}$$

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where $C_k(s_1,s_2)$ and $D_k(s_1,s_2)$ are given by (4.43), (4.45) and (4.52).

Now, substituting (4.36) and (4.53) into the $O(\varepsilon)$-problem (4.34) for $\tilde{T}_{k,1}$, together with $c^1_{k,k} = \frac{1}{2}$, $c^2_{k,k} = 0$ and $c^3_{k,k} = -\frac{1}{2k\pi}$ in (4.15), problem (4.34)becomes a nonlinear ordinary differential equation without coupling term:

$$\begin{aligned}
&\frac{\partial^2 \tilde{T}_{k,1}}{\partial s_0^2} + k^2\pi^2\tilde{T}_{k,1} \\
= \; & 2k\pi\frac{\partial A_k}{\partial s_1}\cos(k\pi s_0) - 2k\pi\frac{\partial B_k}{\partial s_1}\cos(k\pi s_0) \\
&\overbrace{+\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=k+1}^{\infty}} kp(k-p)[\frac{A_{p-k}D_p + B_{p-k}C_p}{2}\sin((\lambda_p+(p-k)\pi)s_0) \\
&+\frac{A_{p-k}C_p - B_{p-k}D_p}{2}\cos((\lambda_p+(p-k)\pi)s_0) \\
&+\frac{A_{p-k}D_p - B_{p-k}C_p}{2}\sin((\lambda_p-(p-k)\pi)s_0) \\
&+\frac{A_{p-k}C_p + B_{p-k}D_p}{2}\cos((\lambda_p-(p-k)\pi)s_0)] \\
&-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{k-1} kp(k-p)[\frac{A_{k-p}D_p + B_{k-p}C_p}{2}\sin((\lambda_p+(k-p)\pi)s_0) \\
&+\frac{A_{k-p}C_p - B_{k-p}D_p}{2}\cos((\lambda_p+(k-p)\pi)s_0) \\
&+\frac{A_{k-p}D_p - B_{k-p}C_p}{2}\sin((\lambda_p-(k-p)\pi)s_0) \\
&+\frac{A_{k-p}C_p + B_{k-p}D_p}{2}\cos((\lambda_p-(k-p)\pi)s_0)] \\
&-\frac{EA\pi^3}{2mg\hat{l}}\sum_{p=1}^{\infty} kp(k+p)\frac{A_{k+p}D_p + B_{k+p}C_p}{2}\sin((\lambda_p+(k+p)\pi)s_0) \\
&+\frac{A_{k+p}C_p - B_{k+p}D_p}{2}\cos((\lambda_p+(k+p)\pi)s_0) \\
&+\frac{A_{k+p}D_p - B_{k+p}C_p}{2}\sin((\lambda_p-(k+p)\pi)s_0)
\end{aligned}$$

$$\underbrace{+\frac{A_{k+p}C_p + B_{k+p}D_p}{2}\cos((\lambda_p - (k+p)\pi)s_0)]}_{I}$$
$$\underbrace{+\beta_{1,0}d_k\omega_1^2\hat{l}^2 cos(\omega_1\chi(s_0 - a) + \alpha),}_{II} \tag{4.54}$$

where C_p and D_p are given by (4.43) and (4.45). The right-hand side of equation (4.54) contains resonant terms: for instance, at least one of the I terms is a resonant term when there exist k, p_1, p_2 s.t. $\frac{2k}{p_1} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2k}{p_2} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$. The II term with ω_1 can be resonant when $k\pi - \omega_1\dot{\chi}(s) \approx 0$ or $k\pi + \omega_1\dot{\chi}(s) \approx 0$. Obviously, the terms in (4.54) involving $\sin(k\pi s_0)$ or $\cos(k\pi s_0)$ are resonant.

Outside the resonance zone (or equivalently the resonance manifold), the corresponding timescales are $s_0 = \tilde{s}$ and $s_2 = \varepsilon\tilde{s}$ (without $s_1 = \sqrt{\varepsilon}\tilde{s}$), so to avoid secular terms in (4.54), A_k and B_k have to satisfy the following equations depending on the parameter values:

$$\frac{\partial A_k}{\partial s_1} = 0, \quad \frac{\partial B_k}{\partial s_1} = 0, \tag{4.55}$$

which implies that:

$$A_k(s_1, s_2) = \overline{A}_k(s_2), \quad B_k(s_1, s_2) = \overline{B}_k(s_2), \tag{4.56}$$

where $\overline{A}_k(s_2)$ and $\overline{B}_k(s_2)$ are still unknown functions in the slow variable s_2. Since $A_k(b,c)=0$ and $B_k(b,c) = 0$, we obtain that $\overline{A}_k(c) = 0$ and $\overline{B}_k(c) = 0$. The undetermined behaviour with respect to s_2 can be used to avoid secular terms in the $O(\varepsilon\sqrt{\varepsilon})-$ problem (4.35). According to (4.54), taking into account the secularity conditions, the general solution of the $O(\varepsilon)-$ problem (4.34) can be written as

$$T_{k,1}(s_0, s_1, s_2; \sqrt{\varepsilon}) = L_k(s_0, s_1, s_2)\cos(k\pi s_0) + M_k(s_0, s_1, s_2)\sin(k\pi s_0), \tag{4.57}$$

where

$$L_k(a,b,c) = F_k, \quad M_k(a,b,c) = -\frac{\partial T_{k,0}}{\partial s_1}(a,b,c) + G_k. \tag{4.58}$$

Then, together with $c^1_{k,k} = \frac{1}{2}$, $c^2_{k,k} = 0$ and $c^3_{k,k} = -\frac{1}{2k\pi}$ in (4.15), the $O(\varepsilon\sqrt{\varepsilon})-$ problem (4.35) can be written as

$$\begin{aligned}
&\frac{\partial^2 \tilde{T}_{k,2}}{\partial s^2} + (k\pi)^2\tilde{T}_{k,2} \\
= \; &[-2\frac{\partial^2 L_k}{\partial s_0 \partial s_1} - 2k\pi\frac{\partial M_k}{\partial s_1} - 2k\pi\frac{\partial B_k}{\partial s_2} - \frac{\partial^2 A_k}{\partial s_1^2} - c_{1,0}\hat{l}k\pi B_k - \frac{\mu_0\hat{l}(k\pi)^2}{2}A_k]\cos(k\pi s_0) \\
&+[-2\frac{\partial^2 M_k}{\partial s_0 \partial s_1} + 2k\pi\frac{\partial L_k}{\partial s_1} + 2k\pi\frac{\partial A_k}{\partial s_2} - \frac{\partial^2 B_k}{\partial s_1^2} + c_{1,0}\hat{l})k\pi A_k - \frac{\mu_0\hat{l}(k\pi)^2}{2}B_k]\sin(k\pi s_0), \\
&\tilde{Y}_{k,2}(0,0,0) = 0, \qquad \frac{\partial \tilde{Y}_{k,2}}{\partial s}(0,0,0) = 0.
\end{aligned} \tag{4.59}$$

To avoid secular terms in $\tilde{T}_{k,2}$ in equation (4.59), the following conditions have to be imposed

$$\begin{aligned}
-2\frac{\partial^2 L_k}{\partial s_0 \partial s_1} - 2k\pi\frac{\partial M_k}{\partial s_1} &= 2k\pi\frac{\partial \overline{B}_k(s_2)}{\partial s_2} + c_{1,0}\hat{l}k\pi\overline{B}_k(s_2) + \frac{\mu_0\hat{l}(k\pi)^2}{2}\overline{A}_k(s_2), \\
-2\frac{\partial^2 M_k}{\partial s_0 \partial s_1} + 2k\pi\frac{\partial L_k}{\partial s_1} &= -2k\pi\frac{\partial \overline{A}_k(s_2)}{\partial s_2} - c_{1,0}\hat{l}k\pi\overline{A}_k(s_2) + \frac{\mu_0\hat{l}(k\pi)^2}{2}\overline{B}_k(s_2). \quad (4.60)
\end{aligned}$$

By solving (4.60) for L_k and M_k, we observe that the solution will be unbounded in s_0 and s_1, due to terms which are only depending on s_2. Therefore, to have secular-free solutions for L_k and M_k, the following conditions have to be imposed independently

$$\begin{aligned}
2k\pi\frac{d\overline{B}_k(s_2)}{ds_2} + c_{1,0}\hat{l}\lambda_k\overline{B}_k(s_2) + \frac{\mu_0\hat{l}(k\pi)^2}{2}\overline{A}_k(s_2) &= 0, \\
-2k\pi\frac{d\overline{A}_k(s_2)}{ds_2} - c_{1,0}\hat{l}\lambda_k\overline{A}_k(s_2) + \frac{\mu_0\hat{l}(k\pi)^2}{2}\overline{B}_k(s_2) &= 0, \quad (4.61)
\end{aligned}$$

we then obtain

$$\begin{aligned}
\overline{A}_k(s_2) &= e^{-\frac{1}{2}c_{2,0}\hat{l}(s_2-c)}[\overline{A}_k(c)\cos(\frac{k\pi\mu_0\hat{l}}{4}(s_2-c)) - \overline{B}_k(c)\sin(\frac{k\pi\mu_0\hat{l}}{4}(s_2-c))], \\
\overline{B}_k(s_2) &= e^{-\frac{1}{2}c_{2,0}\hat{l}(s_2-c)}[\overline{A}_k(c)\sin(\frac{k\pi\mu_0\hat{l}}{4}(s_2-c)) + \overline{B}_k(c)\cos(\frac{k\pi\mu_0\hat{l}}{4}(s_2-c))]. \quad (4.62)
\end{aligned}$$

Since $\overline{A}_k(c) = 0$ and $\overline{B}_k(c) = 0$, together with (4.62), this implies that

$$\overline{A}_k(s_2) = 0, \quad \overline{B}_k(s_2) = 0. \quad (4.63)$$

Now, outside the resonance zone, all these unknown functions in (4.36) have been determined in (4.56), so the solution of the $O(\sqrt{\varepsilon})$-problem (4.33) is $\tilde{T}_{k,0}(s_0, s_1, s_2) \equiv 0$.

Inside the resonance zone around $s = s^{(k)}$ (or equivalently, in the resonance manifold), according to the inner analysis in section 4.3, and to avoid secular terms in (4.54), A_k, B_k have to satisfy the following equations:

- When k does not satisfy the conditions that there always exist p_1, p_2 s.t. $\frac{2k}{p_1} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2k}{p_2} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$\begin{aligned}
\frac{\partial A_k}{\partial s_1} &= \frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\sin(\frac{1}{2}\omega_1 l_0 v_0 e^{v_0\tau^{(k)}} s_1^2 + \psi_k(s^{(k)})), \\
\frac{\partial B_k}{\partial s_1} &= -\frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\cos(\frac{1}{2}\omega_1 l_0 v_0 e^{v_0\tau^{(k)}} s_1^2 + \psi_k(s^{(k)})), \quad (4.64)
\end{aligned}$$

which implies that

$$\begin{aligned}
A_k(s_1, s_2) &= \frac{\sqrt{2}\beta_{1,0}d_k\omega_1^2\hat{l}^2}{\sqrt{\tilde{\alpha}}}\sin(\psi_k(s^{(k)}))\bar{C}_{Fr}(s_1) \\
&+ \frac{\sqrt{2}\beta_{1,0}d_k\omega_1^2\hat{l}^2}{\sqrt{\tilde{\alpha}}}\cos(\psi_k(s^{(k)}))\bar{S}_{Fr}(s_1) + \overline{A}_k(s_2),
\end{aligned}$$

$$\begin{aligned} B_k(s_1,s_2) &= -\frac{\sqrt{2}\beta_{1,0}d_k\omega_1^2\hat{l}^2}{\sqrt{\tilde{\alpha}}}\cos(\psi_k(s^{(k)}))\bar{C}_{Fr}(s_1) \\ &\quad+\frac{\sqrt{2}\beta_{1,0}d_k\omega_1^2\hat{l}^2}{\sqrt{\tilde{\alpha}}}\sin(\psi_k(s^{(k)}))\bar{S}_{Fr}(s_1)+\overline{B}_k(s_2), \end{aligned} \tag{4.65}$$

where $\tilde{\alpha}=\omega_1 l_0 v_0 \mathrm{e}^{v_0\tau^{(k)}}$, and

$$\bar{C}_{Fr}(s)=\int_{\sqrt{\frac{\tilde{\alpha}}{2}}b}^{\sqrt{\frac{\tilde{\alpha}}{2}}s}\cos(x^2)dx, \quad \text{and} \quad \bar{S}_{Fr}(s)=\int_{\sqrt{\frac{\tilde{\alpha}}{2}}b}^{\sqrt{\frac{\tilde{\alpha}}{2}}s}\sin(x^2)dx, \tag{4.66}$$

which are the well-known Fresnel integrals. The presence of Fresnel functions $C_{Fr}(s_1)$ and $S_{Fr}(s_1)$ cause resonance jumps in the system. In (4.65) , $\overline{A}_k(s_2)$ and $\overline{B}_k(s_2)$ are still unknown functions in the slow variable s_2. Since $A_k(b,c)=0$ and $B_k(b,c)=0$, we obtain that $\overline{A}_k(c)=0$ and $\overline{B}_k(c)=0$. The undetermined behaviour with respect to s_2 can be used to avoid secular terms in the solutions of the $O(\varepsilon\sqrt{\varepsilon})-$ problem (4.35). Following the derivation of (4.57)-(4.52) together with (4.65), we obtain

$$\overline{A}_k(s_2)=0, \quad \overline{B}_k(s_2)=0. \tag{4.67}$$

- When k satisfies the conditions that there always exist p_1,p_2 s.t. $\frac{2k}{p_1}=\sqrt{\frac{EA}{mg}}+1+O(\varepsilon)$ or $\frac{2k}{p_2}=\sqrt{\frac{EA}{mg}}-1+O(\varepsilon)$, then there exist $j_1=k$, $j_2=\frac{kp_1}{p_2}=\theta k$ s.t. $\frac{2j_1}{p_1}=\sqrt{\frac{EA}{mg}}+1+O(\varepsilon)$ or $\frac{2j_2}{p_1}=\sqrt{\frac{EA}{mg}}-1+O(\varepsilon)$; and there exist $j_1=\frac{kp_2}{p_1}=\vartheta k$, $j_2=k$ s.t. $\frac{2j_1}{p_2}=\sqrt{\frac{EA}{mg}}+1+O(\varepsilon)$ or $\frac{2j_2}{p_2}=\sqrt{\frac{EA}{mg}}-1+O(\varepsilon)$. Then, the functions of $A_k(s_1,s_2)$ and $B_k(s_1,s_2)$ have to satisfy:

$$\begin{aligned} \frac{\partial A_k}{\partial s_1} &= \frac{EA\pi^3}{4mg\hat{l}}(1-\theta)\theta k^3\frac{A_{\theta k}D_{p_1(k)}-B_{\theta k}C_{p_1(k)}}{2} \\ &\quad+\frac{EA\pi^3}{4mg\hat{l}}(\vartheta-1)\vartheta k^3\frac{A_{\vartheta k}D_{p_2(k)}-B_{\vartheta k}C_{p_2(k)}}{2} \\ &\quad+\frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\sin(\frac{1}{2}\omega_1 l_0 v_0 \mathrm{e}^{v_0\tau^{(k)}}s_1^2+\psi_k(s^{(k)})), \\ \frac{\partial B_k}{\partial s_1} &= -\frac{EA\pi^3}{4mg\hat{l}}(1-\theta)\theta k^3\frac{A_{\theta k}C_{p_1(k)}+B_{\theta k}(\tau)D_{p_1(k)}}{2} \\ &\quad-\frac{EA\pi^3}{4mg\hat{l}}(\vartheta-1)\vartheta k^3\frac{A_{\vartheta k}C_{p_2(k)}+B_{\vartheta k}D_{p_2(k)}}{2} \\ &\quad-\frac{\beta_{1,0}d_k\omega_1^2\hat{l}^2}{2}\cos(\frac{1}{2}\omega_1 l_0 v_0 \mathrm{e}^{v_0\tau^{(k)}}s_1^2+\psi_k(s^{(k)})), \end{aligned} \tag{4.68}$$

where

$$\begin{aligned} C_{p_1(k)} &= \frac{1}{4\lambda_{p_1}}(d_{k,\theta k}^4-d_{\theta k,k}^4)\int_b^{s_1}(A_{\theta k}(\overline{\tau})B_k(\overline{\tau})+B_{\theta k}(\overline{\tau})A_k(\overline{\tau}))d\overline{\tau}, \\ D_{p_1(k)} &= \frac{1}{4\lambda_{p_1}}(d_{k,\theta k}^4-d_{\theta k,k}^4)\int_b^{s_1}(A_{\theta k}(\overline{\tau})A_k(\overline{\tau})-B_{\theta k}(\overline{\tau})B_k(\overline{\tau}))d\overline{\tau}, \end{aligned}$$

$$
\begin{aligned}
C_{p_2(k)} &= \frac{1}{4\lambda_{p_2}}(d^4_{\vartheta k,k} - d^4_{k,\vartheta k}) \int_b^{s_1} (A_k(\overline{\tau}) B_{\vartheta k}(\overline{\tau}) + B_k(\overline{\tau}) A_{\vartheta k}(\overline{\tau})) d\overline{\tau}, \\
D_{p_2(k)} &= \frac{1}{4\lambda_{p_2}}(d^4_{\vartheta k,k} - d^4_{k,\vartheta k}) \int_b^{s_1} (A_k(\overline{\tau}) A_{\vartheta k}(\overline{\tau}) - B_k(\overline{\tau}) B_{\vartheta k}(\overline{\tau})) d\overline{\tau}, \qquad (4.69)
\end{aligned}
$$

and $p_1(k) = \frac{2k}{1+\sqrt{\frac{EA}{mg}}}$, $p_2(k) = \frac{2k}{\sqrt{\frac{EA}{mg}}-1}$, $\theta = \frac{\sqrt{\frac{EA}{mg}}-1}{1+\sqrt{\frac{EA}{mg}}}$, $\vartheta = \frac{\sqrt{\frac{EA}{mg}}+1}{\sqrt{\frac{EA}{mg}}-1}$.

By noting that $A_{\vartheta k} = 0$ and $B_{\vartheta k} = 0$ inside the resonance zone around $s^{(k)}$, it follows that system (4.68) can be written as

$$
\begin{aligned}
\frac{\partial A_k}{\partial s_1} &= \frac{EA\pi^3}{4mg\hat{l}}(1-\theta)\theta k^3 \frac{A_{\theta k} D_{p_1(k)} - B_{\theta k} C_{p_1(k)}}{2} \\
&\quad + \frac{\beta_{1,0} d_k \omega_1^2 \hat{l}^2}{2} \sin(\frac{\omega_1 l_0 v_0 e^{v_0 \tau_k}}{2} s_1^2 + \psi_k(s_k)), \\
\frac{\partial B_k}{\partial s_1} &= -\frac{EA\pi^3}{4mg\hat{l}}(1-\theta)\theta k^3 \frac{A_{\theta k} C_{p_1(k)} + B_{\theta k} D_{p_1(k)}}{2} \\
&\quad - \frac{\beta_{1,0} d_k \omega_1^2 \hat{l}^2}{2} \cos(\frac{\omega_1 l_0 v_0 e^{v_0 \tau_k}}{2} s_1^2 + \psi_k(s_k)), \qquad (4.70)
\end{aligned}
$$

where $C_{p_1(k)}(s_1, s_2)$ and $D_{p_1(k)}(s_1, s_2)$ are given by (4.69). For any mode k satisfying the conditions that there exist p_1, p_2 s.t. $\frac{2k}{p_1} = \sqrt{\frac{EA}{mg}} + 1$ or $\frac{2k}{p_2} = \sqrt{\frac{EA}{mg}} - 1$, we can always find k_1 (k_1 is an integer), s.t. $\theta^{n-1} k = k_1$, and $\theta^n k$ is not an integer, $n = 1, 2, \ldots$ From that, we get a mode sequence $(k_1, \vartheta k_1, \vartheta^2 k_1, \ldots, k, \vartheta^n k_1, \ldots)$. We firstly solve the ordinary differential equations (4.70) for mode k_1, which can be rewritten as (4.64) (here the mode k_1 is denoted by k), and it can be solved as in (4.65). For the mode k_2 in (4.70), $k = k_2$, $A_{\theta k} = A_{k_1}$ and $B_{\theta k} = B_{k_1}$, thereby inside the resonance zone around $s^{(k_2)}$, we can obtain the solutions A_{k_2} and B_{k_2} from (4.70). Next, by using an iterative method we can predict and obtain the functions A_k and B_k. Note that (4.70) is a nonlinear perturbation problem. It is hard to obtain the analytical, explicit solution, but we can find properties of A_k and B_k by the above analysis, which can be used to describe the behaviors of the solution $\tilde{T}_{k,0}(s_0, s_1, s_2)$ of $O(\sqrt{\varepsilon})$-problem (4.33). Moreover, the solution of (4.70) can be obtained by numerical calculations. Now, inside the resonance zone around $s^{(k)}$ in (4.22), the solution of the $O(\sqrt{\varepsilon})$-problem (4.33) is given by (4.65) and (4.70).

To summarize, the solution $\hat{w}(\xi, s)$ of equation (4.8) readily follows:

$$
\begin{aligned}
\hat{w}(\xi, s) &= \sum_{n=1}^{\infty} [A_n(\sqrt{\varepsilon}(s - s^{(n)}), \varepsilon(s - s^{(n)})) \cos(n\pi(s - s^{(n)})) \\
&\quad + B_n(\sqrt{\varepsilon}(s - s^{(n)}), \varepsilon(s - s^{(n)})) \sin(n\pi(s - s^{(n)}))] sin(n\pi\xi) + O(\varepsilon), \quad (4.71)
\end{aligned}
$$

where $s^{(n)}$ is given by (4.22), and A_n and B_n are given by (4.56), (4.63), (4.65) and (4.70).

In the longitudinal direction, according to the analysis of (4.40)-(4.53), $\tilde{Y}_{k,0}$ in equation (4.40) can be approximated as:

$$
\tilde{Y}_{k,0}(s_0, s_1, s_2) = C_k(s_1, s_2) \cos(\lambda_k s_0) + D_k(s_1, s_2) \sin(\lambda_k s_0). \qquad (4.72)
$$

- When k does not satisfy the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$C_k(s_1,s_2) = D_k(s_1,s_2) = 0, \tag{4.73}$$

which follows from (4.43) and (4.52).

- When k satisfies the conditions that there always exist j_1, j_2 s.t. $\frac{2j_1}{k} = \sqrt{\frac{EA}{mg}} + 1 + O(\varepsilon)$ or $\frac{2j_2}{k} = \sqrt{\frac{EA}{mg}} - 1 + O(\varepsilon)$, then:

$$\begin{aligned} C_k(s_1,s_2) &= \int_b^{s_1} -\frac{d^4_{j_1-k,j_1}}{4\lambda_k}(A_{j_1-k}B_{j_1} + B_{j_1-k}A_{j_1}) \\ &\quad + \frac{d^4_{k+j_2,j_2}}{4\lambda_k}(A_{j_2}B_{j_2+k} + B_{j_2}A_{j_2+k})d\bar{\tau}, \\ D_k(s_1,s_2) &= \int_b^{s_1} \frac{d^4_{j_1-k,j_1}}{4\lambda_k}(A_{j_1-k}A_{j_1} - B_{j_1-k}B_{j_1}) \\ &\quad - \frac{d^4_{k+j_2,j_2}}{4\lambda_k}(A_{k+j_2}A_{j_2} - B_{k+j_2}B_{j_2})d\bar{\tau}, \end{aligned} \tag{4.74}$$

which follows from (4.45) and (4.52). And inside the resonance zone around $s^{(k)}$, A_k and B_k are given by (4.65) and (4.70); outside the resonance zone, A_k and B_k are given by (4.56) and (4.63).

The solution $\hat{u}(\xi,s)$ of equation (4.9) readily follows:

$$\begin{aligned} \hat{u}(\xi,s) &= \sum_{n=1}^{\infty}[C_n(\sqrt{\varepsilon}(s-s^{(n)}),\varepsilon(s-s^{(n)}))\cos(n\pi(s-s^{(n)})) \\ &\quad + D_n(\sqrt{\varepsilon}(s-s^{(n)}),\varepsilon(s-s^{(n)}))\sin(n\pi(s-s^{(n)}))]sin(n\pi\xi) + O(\varepsilon), \end{aligned} \tag{4.75}$$

where $s^{(n)}$ is given by (4.22), C_n and D_n are given by (4.73) and (4.74).

By the three time-scales perturbation method, we obtained that for special frequencies in the boundary excitations and for certain parameter values of the longitudinal stiffness and the conveyance mass, the transverse solution $\hat{w}(\xi,s)$ of equation (4.8) jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$, and the longitudinal solution $\hat{u}(\xi,s)$ of equation (4.9) jump up from $O(\varepsilon^2)$ to $O(\sqrt{\varepsilon})$. We can not (always) construct formal approximations of the solutions but we can get properties and predictions of solution behaviors analytically on time-scales of $O(\frac{1}{\varepsilon})$. Based on the properties and equations in the analysis, the approximated solutions for transverse and longitudinal motions will be computed by using an iterative method as well as by using a numerical method in the next section. Also the approximations will be computed by using a central finite difference scheme in the next section to verify the analytical results in this section.

4.5. NUMERICAL RESULTS

Since the nonlinear initial-boundary value problems for the transverse motion (4.8) and for the longitudinal motion (4.9) are coupled, we can not construct formal explicit approximations of the solutions, but we can reduce the problems to ordinary differential equations in the transverse and in the longitudinal directions. So, on the one hand, we can compute the transverse and the longitudinal motions of the cable for (4.8) and (4.9) by computing numerically the solutions of the ordinary differential equation (4.71) and (4.75). On the other hand, we can compute numerically the solutions of the problem (4.8) and the problem (4.9) straight-forwardly by applying a finite difference method.

4.5.1. ANALYTICAL APPROXIMATIONS

The numerical results simulating the transverse and the longitudinal vibration responses are computed based on the analytical expression (4.71) for $\hat{w}(\xi, s)$ and the expression (4.75) for $\hat{u}(\xi, s)$. The computations are performed by using the following parameters:

$$\begin{aligned} &v = 0.01,\ \frac{EA}{mg} = 9,\ \mu = 0.01,\ c_u = 0.01,\ c_1 = 0.01,\ c_2 = 0.01, \\ &\beta_1 = 0.0001,\ \beta_2 = 0.01,\ \omega_1 = 0.6\pi,\ \omega_2 = 0.5\pi,\ l_0 = 1, \varepsilon = 0.01, \end{aligned} \tag{4.76}$$

and the initial conditions are taken to be:

$$\begin{aligned} \hat{w}(\xi,0) &= 0.01\sin(1.5\xi),\quad \hat{w}_s(\xi,0) = 0, \\ \hat{u}(\xi,0) &= 0.0001\sin(1.5\xi),\quad \hat{u}_s(\xi,0) = 0,\quad 0 \le \xi \le 1. \end{aligned} \tag{4.77}$$

By using the Liouville-Green transformation with $\frac{ds}{dt} = \frac{1}{l(t)}$, we obtain that the resonance zones (in the transverse direction) are located around the times

$$t_k = \frac{l_0 e^{v s_k} - l_0}{v} = \frac{k\pi}{\omega_2 v} - \frac{l_0}{v}, \tag{4.78}$$

where the resonance time depends on the mode number k. For the first three oscillation modes, resonance emerges at times $t_1 \approx 100$, $t_2 \approx 300$, $t_3 \approx 500$. The displacements of the first and third mode are given by (4.65) and (4.71), the displacements of the second mode are given by (4.70) and (4.71). They are all illustrated in Figure 4.2(a). The displacements

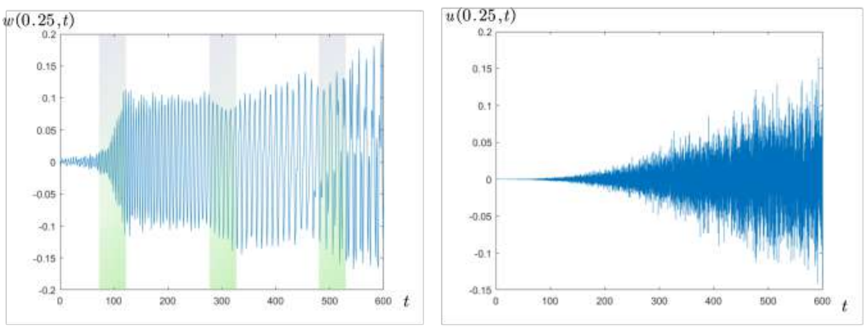

Figure 4.2: (a) Transverse displacements $w(0.25, t)$. (b) Longitudinal displacements $u(0.25, t)$.

of the longitudinal motion are given by (4.75), which are illustrated in Figure 4.2(b).

4.5.2. NUMERICAL APPROXIMATIONS

In this subsection, the finite difference method is applied in both the time and the space domain for both PDEs and boundary conditions in (4.6) and (4.7) with space grid size $d\xi = 5 \times 10^{-2}$, and time step $dt = 5 \times 10^{-3}$. We rewrite the so-obtained discretized equation (4.6) and (4.7) in matrix forms and use as numerical time integration method, the Crank-Nicolson method (see Appendix C.3). Note that the same parameter values as for the analytic approximations in section 4.5.1 are used here for the computations. In Fig-

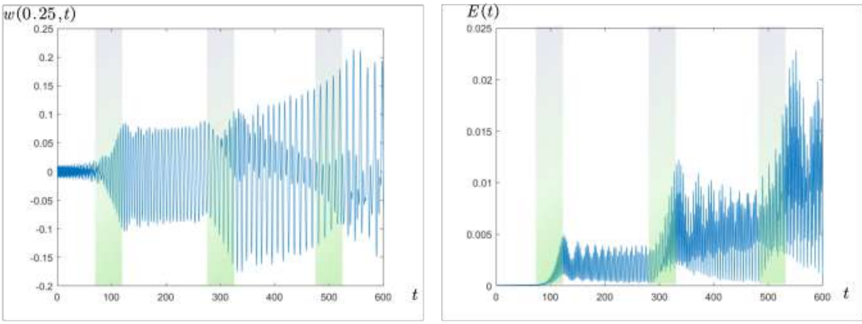

Figure 4.3: (a) Transverse displacements $w(0.25, t)$. (b) Transverse vibratory energy.

ure 4.3 the transverse displacements and the vibratory energy of the cable up to the first three oscillation modes on timescales up to t = 600 are presented. In Figure 4.3, one can see that the transverse resonances emerge around times $t_1 = 100$, $t_2 = 300$ and $t_3 = 500$. In the resonance zones the displacements and the energy increase, and in between these zones, stay constant (approximately). Around the first resonance time t_1, the displacement amplitudes jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. Around the second resonance time t_2 and the third resonance time t_3, the amplitudes change again at the $O(\sqrt{\varepsilon})$ level, where ε is a small parameter with $\varepsilon = 0.01$. In Figure 5.2 the longitudinal displacements and

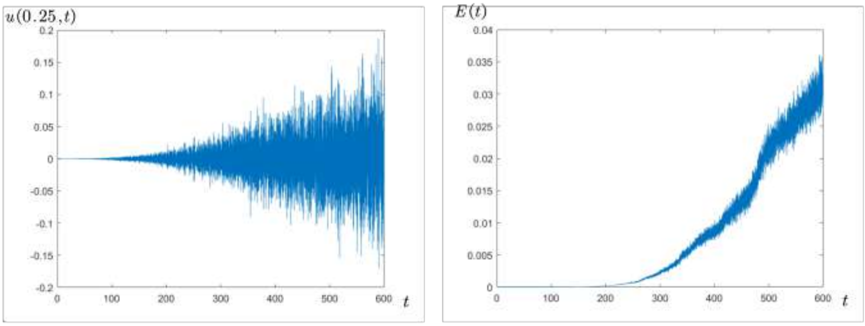

Figure 4.4: (a) Longitudinal displacements $u(0.25, t)$. (b) Longitudinal vibratory energy.

the vibratory energy of the cable on timescales up to t = 600 are given. In Figure 5.2, one

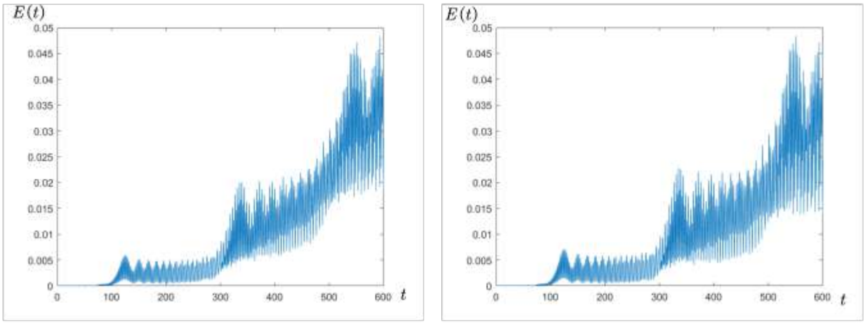

Figure 4.5: (a) The total mechanical energy based on analytical results. (b) The total mechanical energy based on numerical results.

can see that the longitudinal displacements increase from $O(\varepsilon^2)$ to $O(\sqrt{\varepsilon})$, and that the vibratory energy increases from $O(\varepsilon^4)$ to $O(\varepsilon)$. In Figure 4.5 compare the total mechanical energy (see also Appendix C.4 for definations) based on the analytical results and the total energy based on the numerical results can be compared. Based on the Figures 4.2, 4.3, 5.2, and 4.5, we can draw the conclusion that the general dynamic behavior of the solution as approximated by direct numerical integration of the problem is in agreement with the analytic approximations as obtained by applying perturbation methods.

4.6. CONCLUSIONS

In this chapter, we studied the coupled transverse and longitudinal vibrations and associated resonances induced by boundary excitations in a elevator system. The problem is described by nonlinear coupled partial differential equations on a time-varying spatial interval with small harmonic disturbances at one end and a moving nonclassical boundary condition at the other end. Assuming that the transverse harmonic boundary disturbances and the corresponding initial values are of order ε, and the longitudinal harmonic boundary disturbances and the corresponding initial values are of order ε^2, it is shown in this chapter that for special frequencies in the boundary excitations and that for certain parameter values of the longitudinal stiffness and the conveyance mass, many large oscillations arise in transverse and longitudinal directions. The oscillation modes for transverse motion jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$, and the oscillation modes for longitudinal motion jump up from $O(\varepsilon^2)$ to $O(\sqrt{\varepsilon})$. To obtain these results the method of separation of variables is presented, and perturbation methods, (such as averaging methods, and singular perturbation techniques) are used. Furthermore, since the initial-boundary value problems for the transverse motion and the longitudinal motion are nonlinearly coupled, we can not (always) construct formal approximations of the solutions but we can get properties and predictions of solution behaviors analytically on time-scales of order ε^{-1}. Furthermore, approximations of the solutions are computed by using an iterative method as well as by using a numerical method. Also approximations of the solutions of the initial-boundary value problems are computed by using a central finite difference scheme. The numerical approximations are in agreement with the analytically obtained approximations.

APPENDIX C

APPENDIX C.1 THE DERIVATION OF THE EQUATIONS (4.1) AND (4.2)

According to Figure 1, the partial differential equation (PDE) can be derived by the extended
Hamilton's principle:

$$\int_{t_1}^{t_2} (\delta E_k(t) - \delta E_p(t) + \delta W_c(t))dt = 0. \tag{4.79}$$

The kinetic energy $E_k(t)$ can be represented as

$$E_k(t) = \frac{1}{2}\rho[\int_{e_u(t)}^{l(t)} (\frac{Du}{Dt} + v)^2 dx + \int_{e_u(t)}^{l(t)} (\frac{Dw}{Dt})^2 dx] + \frac{1}{2}m[(\frac{Du}{Dt} + v)^2|_{x=l(t)} + (\frac{Dw}{Dt})^2|_{x=l(t)}], \tag{4.80}$$

where the operator $\frac{Du}{Dt}$ is defined as $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = u_t + vu_x$, and the operator $\frac{Dw}{Dt}$ is defined as $\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + v\frac{\partial w}{\partial x} = w_t + vw_x$. The potential energy $E_p(t)$ can be expressed as

$$E_p(t) = \frac{1}{2}EA\int_{e_u(t)}^{l(t)} z^2 dx + \int_{e_u(t)}^{l(t)} Tz dx + E_{gs} - \int_{e_u(t)}^{l(t)} \rho g u dx - mgu|_{x=l(t)}, \tag{4.81}$$

where $z = u_x + \frac{1}{2}w_x^2$, and

$$\begin{aligned}\delta E_k(t) - \delta E_p(t) \quad = \quad & \rho\int_{e_u(t)}^{l(t)} (\frac{Du}{Dt} + v)\delta\frac{Du}{Dt} dx + m(\frac{Du}{Dt} + v)\delta\frac{Du}{Dt}|_{x=l(t)} \\ & +\rho\int_{e_u(t)}^{l(t)} (\frac{Dw}{Dt})\delta\frac{Dw}{Dt} dx + m(\frac{Dw}{Dt})\delta\frac{Dw}{Dt}|_{x=l(t)} \\ & -[EA\int_{e_u(t)}^{l(t)} z\delta z dx + \int_{e_u(t)}^{l(t)} T\delta z dx - \int_{e_u(t)}^{l(t)} \rho g\delta u dx \\ & -mg\delta u|_{x=l(t)}].\end{aligned} \tag{4.82}$$

The virtual work δW_c done by the distributed and the lumped damping force is given as

$$\delta W_c(t) = -\int_{e_u(t)}^{l(t)} c_2\frac{Du}{Dt}\delta u dx - \int_{e_u(t)}^{l(t)} c_1\frac{Dw}{Dt}\delta w dx - c_u\frac{Du}{Dt}\delta u|_{x=l(t)}. \tag{4.83}$$

By substituting equation (4.80)-(4.83) into (4.79), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} \rho(\frac{Du}{Dt} + v)\delta\frac{Du}{Dt} dxdt + \int_{t_1}^{t_2} m(\frac{Du}{Dt} + v)\delta\frac{Du}{Dt}|_{x=l(t)} dt \\ &+ \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} \rho(\frac{Dw}{Dt})\delta\frac{Dw}{Dt} dxdt + \int_{t_1}^{t_2} m(\frac{Dw}{Dt})\delta\frac{Dw}{Dt}|_{x=l(t)} dt \\ &-EA\int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} z\delta z dxdt - \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} T\delta z dxdt + \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} \rho g\delta u dxdt \\ &+ \int_{t_1}^{t_2} mg\delta u|_{x=l(t)} dt - \int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} c_2\frac{Du}{Dt}\delta u dxdt - \int_{t_1}^{t_2} c_u\frac{Du}{Dt}\delta u|_{x=l(t)} dt\end{aligned}$$

$$-\int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} c_1 \frac{Dw}{Dt}\delta w dx dt \quad = \quad 0. \quad (4.84)$$

By integrating by parts it follows from (4.84) that

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} [-\rho(u_{tt}+2vu_{xt}+v^2u_{xx}+au_x+a)+EAz_x \\
&\quad +T_x+\rho g-c_2(u_t+vu_x)]\delta u dx dt \\
&+\int_{t_1}^{t_2}\int_{e_u(t)}^{l(t)} [-\rho(w_{tt}+2vw_{xt}+v^2w_{xx}+aw_x) \\
&\quad +EA(zw_x)_x+(Tw_x)_x-c_1(w_t+vw_x)]\delta w dx dt \\
&+\int_{t_1}^{t_2} [-m(u_{tt}+2vu_{xt}+v^2u_{xx}+au_x+a) \\
&\quad -EAz-T+mg-c_u(u_t+vu_x)]\delta u|_{x=l(t)} dt \\
&+\int_{t_1}^{t_2} [-m(w_{tt}+2vw_{xt}+v^2w_{xx}+aw_x)-EAzw_x-Tw_x]\delta w|_{x=l(t)} dt \\
&+\int_{t_1}^{t_2} [-\rho v(u_t+vu_x+v)+EAz+T]\delta u|_{x=e_u(t)} dt \\
&+\int_{t_1}^{t_2} \dot{e}_u(t)\rho(u_t+vu_x+v)\delta u|_{x=e_u(t)} dt \\
&+\int_{t_1}^{t_2} [-\rho v(w_t+vw_x)+EAzw_x+Tw_x]\delta w|_{x=e_u(t)} dt \\
&+\int_{t_1}^{t_2} \dot{e}_u(t)\rho(w_t+vw_x)\delta w|_{x=e_u(t)} dt \quad = \quad 0.
\end{aligned}$$

So, the governing equations of motion are given by

$$\rho(u_{tt}+2vu_{xt}+v^2u_{xx}+au_x+a)-EAz_x-T_x-\rho g+c_2(u_t+vu_x) \quad = \quad 0,$$
$$e_u(t)<x<l(t),\ t>0, \quad (4.85)$$
$$\rho(w_{tt}+2vw_{xt}+v^2w_{xx}+aw_x)-EA(zw_x)_x-(Tw_x)_x+c_1(w_t+vw_x) \quad = \quad 0,$$
$$e_u(t)<x<l(t),\ t>0. \quad (4.86)$$

The corresponding boundary conditions on the upper end at $x=e_u(t)$ are given by:

$$[-\rho v(u_t+vu_x+v)+EAz+T+\dot{e}_u(t)\rho(u_t+vu_x+v)]|_{x=e_u(t)} \quad = \quad 0,\ t\geq 0,$$
$$[-\rho v(w_t+vw_x)+EAzw_x+Tw_x+\dot{e}_u(t)\rho(w_t+vw_x)]|_{x=e_u(t)} \quad = \quad 0,\ t\geq 0, \quad (4.87)$$

and the boundary conditions at $x=l(t)$ are given by:

$$[m(u_{tt}+2vu_{xt}+v^2u_{xx}+au_x+a)+EAz+T-mg+c_u(u_t+vu_x)]|_{x=l(t)}=0,$$
$$[m(w_{tt}+2vw_{xt}+v^2w_{xx}+aw_x)+EAzw_x+Tw_x]|_{x=l(t)}=0,\ t\geq 0. \quad (4.88)$$

Note that (4.87) and (4.88) are the natural boundary conditions. But (4.87) is not appropriate for our problem, since the string is excited at the top boundary with the fundamental excitations $e_u(t)$ and $e_w(t)$. Thus, the correct boundary condition at the upper end are:

$$u(e_u(t),t)=e_u(t), \quad w(e_u(t),t)=e_w(t), \quad t\geq 0, \quad (4.89)$$

where $e_u(t)$ and $e_w(t)$ are given in Nomenclature in section 4.2, and at the bottom boundary, the string is assumed to be fixed in horizontal direction. Thus, the corresponding transverse boundary condition at $x = l(t)$ is:

$$w(l(t), t) = 0, \quad t \geq 0. \tag{4.90}$$

Considering

$$T(x, t) = [m + \rho(l(t) - x)]g, \quad e_u(t) \leq x \leq l(t), \tag{4.91}$$

together with the governing equations given in (4.85) and in (4.86), the boundary excitations conditions given in (4.88), (4.89) and (4.90), we obtain the initial boundary value problem (4.1) for the transverse vibration and (4.2) for the longitudinal vibration.

4

APPENDIX C.2 SIMPLIFICATION OF THE PROBLEMS (4.1) AND (4.5)

In order to convert the time-varying spatial domain $[\beta_2 cos(\omega_2 t), l(t)]$ for x to a fixed domain [0,1] for ξ, a new independent spatial coordinate $\xi = \frac{x-\beta_2 cos(\omega_2 t)}{h(t)}$, where $h(t) = l(t) - \beta_2 cos(\omega_2 t)$, is introduced. After this spatial transformation, new dependent variables $\bar{w}(\xi, t) = w(x, t)$, $\bar{u}(\xi, t) = u(x, t)$, and all the partial derivatives have to be rewritten as follows:

$$\begin{aligned}
\xi_t &= -\frac{v\xi}{h(t)} + \beta_2 \frac{\omega_2(1-\xi)sin(\omega_2 t)}{h(t)}, \\
\xi_{tt} &= \frac{v^2\xi}{h^2(t)} + \beta_2\left[\frac{\omega_2^2(1-\xi)cos(\omega_2 t)}{h(t)} - \frac{v\omega_2(1-2\xi)sin(\omega_2 t)}{h^2(t)}\right] - \beta_2^2 \frac{\omega_2^2(1-\xi)sin^2(\omega_2 t)}{h^2(t)}, \\
w_t &= \bar{w}_\xi \xi_t + \bar{w}_t, \quad w_{tt} = \bar{w}_{\xi\xi}(\xi_t)^2 + 2\bar{w}_{\xi t}\xi_t + \bar{w}_\xi \xi_{tt} + \bar{w}_{tt}, \\
w_x &= \frac{1}{h(t)}\bar{w}_\xi, \quad w_{xx} = \frac{1}{h^2(t)}\bar{w}_{\xi\xi}, \\
u_t &= \bar{u}_\xi \xi_t + \bar{u}_t, \quad u_{tt} = \bar{u}_{\xi\xi}(\xi_t)^2 + 2\bar{u}_{\xi t}\xi_t + \bar{u}_\xi \xi_{tt} + \bar{u}_{tt}, \\
u_x &= \frac{1}{h(t)}\bar{u}_\xi, \quad u_{xx} = \frac{1}{h^2(t)}\bar{u}_{\xi\xi}.
\end{aligned}$$

So, the initial boundary value problem for the transverse motion is given by (4.6), and the initial boundary value problem for the longitudinal motion is given by (4.7).

In order to eliminate the time-variable coefficients in $\frac{1}{h^2(t)}\bar{w}_{\xi\xi}$ and in $\frac{EA}{mgh^2(t)}\bar{u}_{\xi\xi}$ in the initial boundary problems (4.6) and (4.7), the Liouville-Green transformation (see also the WKBJ method) is introduced with $\frac{ds}{dt} = \frac{1}{l(t)}$. In accordance with a new time variable s, all the partial derivatives have to be rewritten as follows:

$$\begin{aligned}
s &= \frac{1}{\varepsilon v_0} ln(\frac{l(t)}{l_0}), \quad l(t) = \hat{l}(s) = l_0 e^{\varepsilon v_0 s}, \quad \chi(s) = \frac{l_0(e^{\varepsilon v_0 s} - 1)}{\varepsilon v_0}, \\
\bar{w}_t &= \frac{1}{\hat{l}}\tilde{w}_s, \quad \bar{w}_{\xi t} = \frac{1}{\hat{l}}\tilde{w}_{\xi s}, \quad \bar{w}_{tt} = \frac{1}{\hat{l}^2}\tilde{w}_{ss} - \frac{v}{\hat{l}^2}\tilde{w}_s, \\
\bar{u}_t &= \frac{1}{\hat{l}}\tilde{u}_s, \quad \bar{u}_{\xi t} = \frac{1}{\hat{l}}\tilde{u}_{\xi s}, \quad \bar{u}_{tt} = \frac{1}{\hat{l}^2}\tilde{u}_{ss} - \frac{v}{\hat{l}^2}\tilde{u}_s.
\end{aligned}$$

Substituting these derivatives into the problem (4.6), the initial boundary value problem for the transverse motion becomes:

$$\begin{cases} \tilde{w}_{ss} - \tilde{w}_{\xi\xi} = \nu\tilde{w}_s - 2\nu\tilde{w}_{\xi s} - c_1\hat{l}\tilde{w}_s - \mu\hat{l}\tilde{w}_\xi + \mu\hat{l}(1-\xi)\tilde{w}_{\xi\xi} + \frac{EA}{mg\hat{l}}(\tilde{u}_\xi\tilde{w}_\xi)_\xi + \frac{EA}{mg\hat{l}^2}(\frac{1}{2}\tilde{w}_\xi^3)_\xi \\ \qquad +2\nu\xi\tilde{w}_{\xi s} + O(\varepsilon^2\tilde{w}), \quad 0<\xi<1,\ s>0, \\ \tilde{w}(1,s) = 0, \quad s\geq 0, \\ \tilde{w}(0,s) = \beta_1 cos(\omega_1\chi(s)+\alpha), \quad s\geq 0, \\ \tilde{w}(\xi,0) = \bar{w}_0(\xi), \quad \tilde{w}_s(\xi,0) = l_0\bar{w}_1(\xi), \quad 0<\xi<1. \end{cases} \tag{4.92}$$

The initial boundary value problem for the longitudinal motion becomes:

$$\begin{cases} \tilde{u}_{ss} - \frac{EA}{mg}\tilde{u}_{\xi\xi} = \nu\tilde{u}_s - 2\nu\tilde{u}_{\xi s} - c_2\hat{l}\tilde{u}_s + \frac{EA}{mg\hat{l}}\tilde{w}_\xi\tilde{w}_{\xi\xi} + 2\nu\xi\tilde{u}_{\xi s} + O(\varepsilon^2\tilde{u}), \quad 0<\xi<1,\ s>0, \\ \tilde{u}_{ss}(1,s) = [-\frac{\mu EA\hat{l}}{mg}\tilde{u}_\xi + \nu\tilde{u}_s - c_u\hat{l}\tilde{u}_s - \frac{\mu EA}{2mg}\tilde{w}_\xi^2]|_{\xi=1} + O(\varepsilon^2\tilde{u}), \quad s\geq 0, \\ \tilde{u}(0,s) = \beta_2 cos(\omega_2\chi(s)), \quad s\geq 0, \\ \tilde{u}(\xi,0) = \bar{u}_0(\xi), \quad \tilde{u}_s(\xi,0) = l_0\bar{u}_1(\xi), \quad 0<\xi<1. \end{cases} \tag{4.93}$$

The initial boundary value problem (4.93) can further be rewritten as

$$\begin{cases} \tilde{u}_{ss} - \frac{EA}{mg}\tilde{u}_{\xi\xi} = \nu\tilde{u}_s - 2\nu\tilde{u}_{\xi s} - c_2\hat{l}\tilde{u}_s + \frac{EA}{mg\hat{l}}\tilde{w}_\xi\tilde{w}_{\xi\xi} + 2\nu\xi\tilde{u}_{\xi s} + O(\varepsilon^2\tilde{u}), \quad 0<\xi<1,\ s>0, \\ \tilde{u}_{\xi\xi}(1,s) = [-\mu\hat{l}\tilde{u}_\xi + \frac{mg}{EA}(c_2-c_u)\hat{l}\tilde{u}_s - \frac{\mu}{2}\tilde{w}_\xi^2 - \frac{1}{\hat{l}}\tilde{w}_\xi\tilde{w}_{\xi\xi}]|_{\xi=1} + O(\varepsilon^2\tilde{u}), \quad s\geq 0, \\ \tilde{u}(0,s) = \beta_2 cos(\omega_2\chi(s)), \quad s\geq 0, \\ \tilde{u}(\xi,0) = \bar{u}_0(\xi), \quad \tilde{u}_s(\xi,0) = l_0\bar{u}_1(\xi), \quad 0<\xi<1. \end{cases} \tag{4.94}$$

In order to eliminate the non-homogenous terms in the boundary conditions in (4.92) and in (4.94), the following transformations are used:

$$\tilde{w}(\xi,s) = \beta_1(1-\xi)\cos(\omega_1\chi(s)+\alpha) + \hat{w}(\xi,s), \tag{4.95}$$

$$\begin{aligned} \tilde{u}(\xi,s) = {} & \hat{u}(\xi,s) + \frac{\xi^2}{2}[-\mu\hat{l}\hat{u}_\xi + \frac{mg}{EA}(c_2-c_u)\hat{l}\hat{u}_s - \frac{\mu}{2}\hat{w}_\xi^2]|_{\xi=1} + \frac{\xi^2}{2}[-\frac{1}{\hat{l}}\hat{w}_\xi\hat{w}_{\xi\xi}]|_{\xi=1} \\ & + \frac{\xi^2}{4}\mu[\hat{w}_{\xi\xi}^2 + \hat{w}_\xi\hat{w}_{\xi\xi\xi}]|_{\xi=1} - \frac{\xi^2}{4}\frac{mg}{EA}(c_2-c_u)[\hat{w}_{\xi s}\hat{w}_{\xi\xi} + \hat{w}_\xi\hat{w}_{\xi\xi s}]|_{\xi=1} \\ & + \frac{\xi^2\mu\beta_1}{2}\cos(\omega_1\chi(s)+\alpha)\hat{w}_\xi(1,s) + \frac{\xi^2\beta_1}{2\hat{l}}\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi}(1,s) \\ & - \frac{\xi^2}{4}\frac{mg}{EA}(c_2-c_u)\beta_1[\omega_1\hat{l}sin(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi}(1,s) - \cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi s}(1,s)] \\ & - \frac{\xi^2\mu\beta_1}{4}\cos(\omega_1\chi(s)+\alpha)\hat{w}_{\xi\xi\xi}(1,s) + \beta_2\cos(\omega_2\chi(s)), \end{aligned} \tag{4.96}$$

Thus, in the transverse direction, we obtain the initial boundary value problem (4.8), and in the longitudinal direction, we obtain the initial boundary value problem (4.9).

APPENDIX C.3 DISCRETIZATION AND TIME INTEGRATION

To solve (4.8) numerically, it is convenient to rewrite the second order partial differential equation as a system of two coupled first-order partial differential equations:

$$\check{w}_t = \check{\zeta},$$

$$\check{\zeta}_t = \frac{1}{l^2}[1+\mu l(1-\xi)]\check{w}_{\xi\xi} + \frac{2v}{l}(\xi-1)\check{\zeta}_\xi - c_1\check{\zeta} - \frac{\mu}{l}\check{w}_\xi + (1-\xi)\omega_2^2\beta_2\cos(\omega_2 t+\alpha). \tag{4.97}$$

Next, let us use the mesh grids $\xi_j = (j-1)\Delta\xi$ for $j = 1,2,\ldots,n,n+1$ with $n\Delta\xi = 1$. By introducing the differences, $\check{w}_{\xi\xi}(\xi_j,t) = \frac{\check{w}_{j+1}-2\check{w}_j+\check{w}_{j-1}}{(\Delta\xi)^2} + O((\Delta\xi)^2)$, $\check{\zeta}_\xi(\xi_j,t) = \frac{\check{\zeta}_{j+1}-\check{\zeta}_{j-1}}{2\Delta\xi} + O((\Delta\xi)^2)$, it follows how system (4.97) can be discretized, yielding:

$$\begin{cases} \frac{d\check{w}}{dt}(\xi_j,t) = \check{\zeta}_j, \\ \frac{d\check{\zeta}}{dt}(\xi_j,t) = r_j(\check{w}_{j+1}-2\check{w}_j+\check{w}_{j-1}) + q_j(\check{\zeta}_{j+1}-\check{\zeta}_{j-1}) - c_1\zeta_j - p(\check{w}_{j+1}-\check{w}_{j-1}) \\ \qquad +(1-\xi)\omega_2^2\beta_2\cos(\omega_2 t+\alpha), \end{cases}$$

where $r_j = \frac{1+\mu l(1-\xi_j)}{l^2(\Delta\xi)^2}$, $q_j = \frac{v(\xi_j-1)}{l\Delta\xi}$, $p = \frac{\mu}{2l\Delta\xi}$ for j=1,2,..,n. Further,

$$R = \begin{pmatrix} -2r_1 & r_1-p & 0 & \cdots & \cdots & 0 \\ r_2+p & -2r_2 & r_2-p & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & r_{n-1}+p & -2r_{n-1} & r_{n-1}-p \\ 0 & \cdots & \cdots & 0 & r_n+p & -2r_n \end{pmatrix} \in \mathbb{R}^{n\times n}, \text{ and}$$

$$P = \begin{pmatrix} -c_1 & q_1 & 0 & \cdots & \cdots & 0 \\ -q_2 & -c_1 & q_2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -q_{n-1} & -c_1 & q_{n-1} \\ 0 & \cdots & \cdots & 0 & -q_n & -c_1 \end{pmatrix} \in \mathbb{R}^{n\times n},$$

The four matrices $\emptyset$, I, R, and P compose the system matrix M:

$$M = \begin{pmatrix} \emptyset & I \\ R & P \end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

where $\emptyset$ is the zero matrix, and I is the identity matrix. In addition, let us introduce the following
vectors: $w = (w_1(\xi_1,t), w_2(\xi_2,t),\ldots, w_n(\xi_n,t), \zeta_1(\xi_1,t), \zeta_2(\xi_2,t),\ldots,\zeta_n(\xi_n,t))^T$,
$s = (\underbrace{0,0,\ldots,0}_{\text{n times}}, \underbrace{\bar{s}_1,\bar{s}_2,\ldots,\bar{s}_n}_{\text{n times}})^T$, where $\bar{s}_i = (1-\xi_i)\omega_2^2\beta_2\cos(\omega_2 t+\alpha)$. So, system (4.97) can be written in the following matrix
form: $\frac{dw}{dt} = Mw + s$. In order to perform a time integration, we apply the Crank-Nicolson method. Introducing the mesh grid in time, $t_k = k\Delta t$ for k=1,2,...,n, we obtain

$$w^{k+1} = Dw^k + \frac{\Delta t}{2}(I - \frac{\Delta t}{2}M^{k+1})^{-1}(s^{k+1}+s^k), \tag{4.98}$$

where $w^k = (w_1(\xi_1,t_k), w_2(\xi_2,t_k),\ldots, w_n(\xi_n,t_k), \zeta_1(\xi_1,t_k), \zeta_2(\xi_2,t_k),\ldots,\zeta_n(\xi_n,t_k))^T$,
$s^k = (\underbrace{0,0,\ldots,0}_{\text{n times}}, \underbrace{\bar{s}_1^k,\bar{s}_2^k,\ldots,\bar{s}_n^k}_{\text{n times}})^T$ with $\bar{s}_i^k = (1-\xi_i)\omega_2^2\beta_2\cos(\omega_2 t_k+\alpha)$, I is the identity matrix and $I \in \mathbb{R}^{2n\times 2n}$, and $D = (I - \frac{\Delta t}{2}M^{k+1})^{-1}(I + \frac{\Delta t}{2}M^k)$. Similarly, we can also directly integrate problem (4.7) with the above numerical scheme.

APPENDIX C.4 ENERGY

The mechanical energy of the initial-boundary value problem (4.2) related to the transverse motion is given by

$$E_1(t) = \frac{1}{2}\int_0^{l(t)} [\rho(w_t + v w_x)^2 + T w_x^2] dx,$$

where T is given by (4.3). Using the dimensionless quantities, we rewrite the energy in a dimensionless form:

$$E_1(t) = \frac{1}{2}\int_0^{l(t)} [(w_t + v w_x)^2 + (1 + \mu(l(t) - x)) w_x^2] dx.$$

In order to define the energy on the interval (0,1), we change the variables by using the following transformation $\xi = \frac{x}{l(t)}$:

$$E_1(t) = \frac{1}{2l(t)}\int_0^{1} [(l(t)\tilde{w}_t + (1-\xi) v \tilde{w}_\xi)^2 + (1 + l(t)\mu(1-\xi)) \tilde{w}_\xi^2] d\xi. \tag{4.99}$$

The mechanical energy of the initial-boundary value problem (4.1) related to the longitudinal motion is given by

$$E_2(t) = \frac{1}{2}\int_0^{l(t)} [\rho(u_t + v u_x)^2 + EA u_x^2] dx + \frac{m}{2}[u_t(l(t),t) + v u_x(l(t),t)]^2.$$

Using the dimensionless quantities, we rewrite the energy in a dimensionless form:

$$E_2(t) = \frac{1}{2}EAL\int_0^{l(t)} [(u_t + v u_x)^2 + u_x^2] dx + \frac{EAm}{2\rho}[u_t(l(t),t) + v u_x(l(t),t)]^2.$$

In order to define the energy on the interval (0,1), we change the variables by using the following transformation $\xi = \frac{x}{l(t)}$:

$$\begin{aligned} E_2(t) &= \frac{EAL}{2l(t)}\int_0^{1} [(l(t)\tilde{u}_t + (1-\xi) v \tilde{u}_\xi)^2 + \tilde{u}_\xi^2] d\xi \\ &\quad + \frac{EAm}{2\rho l^2(t)}[l(t)\tilde{u}_t(1,t) + (1-\xi) v \tilde{u}_\xi(1,t)]^2. \end{aligned} \tag{4.100}$$

The total mechanical energy is now given by

$$\begin{aligned} E(t) &= \frac{1}{2l(t)}\int_0^{1} [(l(t)\tilde{w}_t + (1-\xi) v \tilde{w}_\xi)^2 + (1 + l(t)\mu(1-\xi)) \tilde{w}_\xi^2] d\xi \\ &\quad + \frac{EAL}{2l(t)}\int_0^{1} [(l(t)\tilde{u}_t + (1-\xi) v \tilde{u}_\xi)^2 + \tilde{u}_\xi^2] d\xi \\ &\quad + \frac{EAm}{2\rho l^2(t)}[l(t)\tilde{u}_t(1,t) + (1-\xi) v \tilde{u}_\xi(1,t)]^2. \end{aligned} \tag{4.101}$$

5

OUTPUT FEEDBACK STABILIZATION OF AN AXIALLY MOVING STRING

5.1. INTRODUCTION

THe previous chapters showed resonances and vibrations in moving cables. In this chapter we study, the vibration stabilization of an axially moving string with constant speed on a finite spatial domain subject to a spring-mass-dashpot attached at one end, which is shown in Figure 5.1. This model arises from conveyor belts, cranes or elevators devices for suppressing large vibrations. For more information on this model, the reader is referred to [71, 72, 73, 74, 75]. The objective of the chapter is to design an observer-based output feedback controller at the nature boundary to stabilize the system. There are many methods to achieve the vibration stabilization of axially moving strings or beams. One of the most useful methods for boundary controller is based on Lyapunov method, by which control laws to reduce vibration energy to zero are derived using Lyapunov function candidates constructed by the total mechanical energy of the moving system. Nguyen and Hong [76] investigated an adaptive boundary control based on Lyapunov's method for an nonlinear axially moving string. Nguyen and Hong [77] presented simultaneous controls of longitudinal and transverse vibrations of an axially moving string with velocity tracking. Tebou [78] studied the boundary stabilization of an axially moving Euler-Bernoulli beam. In the literature, the controllers are required to follow the end causing vibration excitation, which is sometimes difficult to achieve in the practical implementation due to the inconvenient installation. Hence, the control system where control is applied at the end opposite to the instability is necessary to study. This is a more challenging task than the classical collocated "boundary damper" feedback control (Krstic *et al.*, [79]). Backstepping approach, which is proposed by Krstic, can deal with the proposed non-collocated stabilization problem. Ren *et al.*[80] analysed boundary stability of an ODE-Schr*ö*dinger cascade. Krstic [81] provided an explicit

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feedback law that compensates the wave PDE dynamics at the input of an LTI ODE and stabilizes the overall system. In Susto and Krstic [82], a ODE-PDE cascade system was extended from the Dirichlet type interconnections to Neumann type interconnections. Wang *et al.*[83] designed an obsever-based output-feedback control law for the stability of the axial vibration in the ascending mining cable elevator. For more information on vibration suppression problems of axially moving strings, the reader is referred to (Zhu *et al.*[84], He *et al.*[85] and He *et al.*[86]).

The remaining part of this chapter is organized as follows. Section 5.2 formulates the problem by extended Hamilton's principle. Section 5.3 designs a controller based state feedback to stabilize the system exponentially. Section 5.4 concludes the output feedback law based observer. Section 5.5 presents some numerical approximations by using a central finite difference scheme to validate the theoretical results, and in the last section we draw some conclusions.

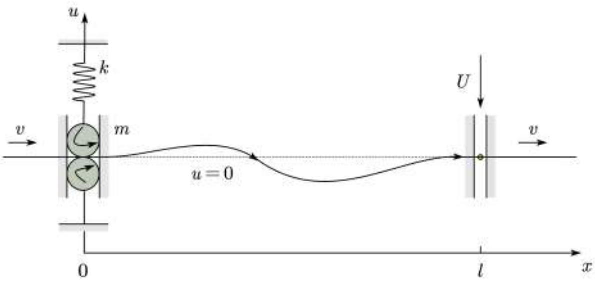

Figure 5.1: An axially moving string with a spring-mass-dashpot boundary.

5.2. FORMULATION OF THE PROBLEM

5.2.1. MODELING OF THE PHYSICAL SYSTEM

Nomenclature:	
$u(x,t)$	the transverse displacement of the string at the coordinate x and the time t
l	the distance between two boundary ends
v	the traveling speed of the moving string
ρ	the mass density of the string
m	the mass of the spring-mass
T	the uniform tension of the string
k	the stiffness of the spring

According to Figure 5.1, we can obtain the partial differential equation (PDE) for the moving string by applying Hamilton's principle in the following form :

$$\int_{t_1}^{t_2} (\delta E_k(t) - \delta E_p(t) + \delta W(t))\mathrm{d}t = 0. \tag{5.1}$$

The Kinetic energy $E_k(t)$ is given by

$$E_k(t) = \frac{1}{2}\rho\int_0^l (u_t + vu_x)^2 \mathrm{d}x + \frac{1}{2}mu_t^2(0,t), \tag{5.2}$$

where $u_t + vu_x$ is the instantaneous transverse velocity of a material particle. The potential energy $E_p(t)$ is given by

$$E_p(t) = \frac{1}{2}\int_0^l Tu_x^2 \mathrm{d}x + \frac{1}{2}ku^2(0,t), \tag{5.3}$$

and the difference of $\delta E_k(t)$ and $\delta E_p(t)$ is

$$\begin{aligned} \delta E_k(t) - \delta E_p(t) &= \rho\int_0^l (u_t + vu_x)\delta(u_t + vu_x)\mathrm{d}x + mu_t(0,t)\delta u_t(0,t) \\ &\quad -[\int_0^l Tu_x\delta u_x \mathrm{d}x + ku(0,t)\delta u(0,t)]. \end{aligned} \tag{5.4}$$

The virtual work $\delta W(t)$ is written as

$$\delta W(t) = U(t)\delta u(l,t). \tag{5.5}$$

Substituting the equations (5.4)-(5.5) into (5.1) yields:

$$\begin{aligned} &\int_{t_1}^{t_2}\int_0^l \rho(u_t + vu_x)\delta(u_t + vu_x)\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} mu_t(0,t)\delta u_t(0,t)\mathrm{d}t \\ -&\int_{t_1}^{t_2}\int_0^l Tu_x\delta u_x \mathrm{d}x\mathrm{d}t - \int_{t_1}^{t_2} ku(0,t)\delta u(0,t)\mathrm{d}t + \int_{t_1}^{t_2} U(t)\delta u(l,t)\mathrm{d}t = 0. \end{aligned} \tag{5.6}$$

Integrating (5.6) by parts with respect to the spatial variable (refer to Chen *et al.*, [16]) yields:

$$\begin{cases} \rho(u_{tt} + 2vu_{xt} + v^2u_{xx}) - Tu_{xx} = 0,\ 0 \le x \le l,\ t > 0, \\ mu_{tt}(0,t) + Tu_x(0,t) + ku(0,t) + \rho vu_t(0,t) - \rho v^2u_x(0,t) = 0,\ t > 0, \\ Tu_x(l,t) + \rho vu_t(l,t) - \rho v^2u_x(l,t) = U(t),\ t > 0. \end{cases} \tag{5.7}$$

To simplicity, we introduce the following dimensionless parameters: $u^* = \frac{u}{l}$, $x^* = \frac{x}{l}$, $t^* = \frac{t}{l}\sqrt{\frac{T}{\rho}}$, $v^* = v\sqrt{\frac{\rho}{T}}$, $m^* = \frac{m}{\rho l}$, $k^* = \frac{kl}{T}$, $U^* = \frac{U}{T}$. The problem (5.7) then becomes:

$$\begin{cases} u_{tt} + 2vu_{xt} + (v^2 - 1)u_{xx} = 0, \quad 0 \le x \le 1,\ t > 0, \\ mu_{tt}(0,t) - (v^2 - 1)u_x(0,t) + ku(0,t) + vu_t(0,t) = 0, \quad t > 0, \\ (1 - v^2)u_x(1,t) + vu_t(1,t) = U(t), \quad t > 0, \end{cases} \tag{5.8}$$

where the asterisks are omitted in problem (5.8) for convenience, and $0 < v < 1$.

5.2.2. SIMPLIFIED MODEL FOR CONTROLLER DESIGN

Define the control force as

$$U(t) = vu_t(1,t) + (1-v^2)U_2(t), \tag{5.9}$$

where $U_2(t)$ is a new control. Then, problem (5.8) can be rewritten as

$$\begin{cases} u_{tt} + 2vu_{xt} + (v^2-1)u_{xx} = 0, \quad 0 \le x \le 1, \ t > 0, \\ mu_{tt}(0,t) - (v^2-1)u_x(0,t) + ku(0,t) + vu_t(0,t) = 0, \quad t > 0, \\ u_x(1,t) = U_2(t), \quad t > 0, \end{cases} \tag{5.10}$$

Notice that the axially moving problem (5.10) is a wave PDE with a second-order derivative in time boundary condition, we introduce new variables $x_1(t)$ and $x_2(t)$:

$$x_1(t) = u(0,t), \quad x_2(t) = u_t(0,t), \tag{5.11}$$

5

Substituting (5.11) into the boundary condition at $x = 0$ in problem (5.10), we have

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \frac{v^2-1}{m}u_x(0,t) - \frac{k}{m}u(0,t) + \frac{v}{m}u_t(0,t). \end{aligned} \tag{5.12}$$

Let $X(t) \in \mathbb{R}^{2\times 1}$ be a state variable:

$$X(t) = [x_1(t), x_2(t)]^T, \tag{5.13}$$

then we rewrite problem (5.10) as the following coupled ODE-PDE system:

$$\begin{cases} \dot{X}(t) = AX(t) + Bu_x(0,t), \ t > 0, \\ u_{tt} + 2vu_{xt} + (v^2-1)u_{xx} = 0, \quad 0 \le x \le 1, \ t > 0, \\ u(0,t) = CX(t), \ t > 0, \\ u_x(1,t) = U_2(t), \quad t > 0, \end{cases} \tag{5.14}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & \frac{v}{m} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{v^2-1}{m} \end{pmatrix}, \quad C = \begin{pmatrix} 1, 0 \end{pmatrix}. \tag{5.15}$$

5.3. STATE FEEDBACK CONTROL

In this section, we construct an invertible transformation to make the system (5.14) equivalent to a ODE-PDE cascade target system. For the target system, we present the well-posedness and stability results in a suitable space.

First, we consider the backstepping transformation of the form (Krstic *et al.*, [81, 87]):

$$w(x,t) = u(x,t) - \int_0^x b(x,y)u(y,t)dy - \int_0^x c(x,y)u_t(y,t)dy - \gamma(x)X(t), \tag{5.16}$$

where the kernel functions $b(x,y)\in\mathbb{R}$, $c(x,y)\in\mathbb{R}$ and $\gamma(x)\in\mathbb{R}^{1\times 2}$ need to be chosen to transform the system (5.14) into the system of ODE-PDE cascade

$$\begin{cases} \dot{X}(t)=(A+BK)X(t)+Bw_x(0,t), \quad t>0, \\ w_{tt}+2vw_{xt}+(v^2-1)w_{xx}=0, \quad 0\le x\le 1, \ t>0, \\ w(0,t)=0, \quad t>0, \\ w_x(1,t)=0, \quad t>0, \end{cases} \tag{5.17}$$

where $K=(k_1,k_2)$ is chosen to make $A+BK$ Hurwitz (which implies that system $\dot{X}(t)=(A+BK)X(t)$ is asymptotically stable), and

$$\begin{aligned} U_2(t) &= b(1,1)u(1,t)+c(1,1)u_t(1,t)+\gamma'(1)X(t) \\ &\quad +\int_0^1 b_x(1,y)u(y,t)dy+\int_0^1 c_x(1,y)u_t(y,t)dy. \end{aligned} \tag{5.18}$$

5.3.1. KERNELS OF $b(x,y), c(x,y)$ AND $\gamma(x)$

In this subsection, we compute the kernels of $b(x,y), c(x,y)$ and $\gamma(x)$. Differentiate (5.16) with respect to t and to x, we get

$$\begin{aligned} & w_{tt}+2vw_{xt}+(v^2-1)w_{xx} \\ = \ & 2(1-v^2)(\frac{d}{dx}b(x,x))u(x,t)+(1-v^2)\int_0^x (b_{xx}(x,y)-b_{yy}(x,y))u(y,t)dy \\ & -2v\int_0^x (b_x(x,y)+b_y(x,y))u_t(y,t)dy \\ & +2(1-v^2)(\frac{d}{dx}c(x,x))u_t(x,t)+(1-v^2)\int_0^x (c_{xx}(x,y)-c_{yy}(x,y))u_t(y,t)dy \\ & -2v\int_0^x (c_x(x,y)+c_y(x,y))u_{tt}(y,t)dy \\ & +[-2v\gamma'(x)B+(1-v^2)b(x,0)-2vb(x,0)CB-\gamma(x)AB \\ & -(1-v^2)c_y(x,0)CB-2vc(x,0)CAB]u_x(0,t) \\ & +[-\gamma(x)B-2vc(x,0)CB+(1-v^2)c(x,0)]u_{xt}(0,t) \\ & +[-\gamma(x)A^2+(1-v^2)\gamma''(x)-2v\gamma'(x)A-2vb(x,0)CA-(1-v^2)b_y(x,0)C \\ & -(1-v^2)c_y(x,0)CA-2vc(x,0)CA^2]X(t)=0, \end{aligned} \tag{5.19}$$

which together with $CB=0$ yields

$$\begin{cases} \frac{d}{dx}b(x,x)=0, \quad \frac{d}{dx}c(x,x)=0, \\ b_{xx}(x,y)-b_{yy}(x,y)=0, \quad c_{xx}(x,y)-c_{yy}(x,y)=0, \\ b_x(x,y)+b_y(x,y)=0, \quad c_x(x,y)+c_y(x,y)=0, \\ -2v\gamma'(x)B+(1-v^2)b(x,0)-\gamma(x)AB-2vc(x,0)CAB=0, \\ -\gamma(x)B-2vc(x,0)CB+(1-v^2)c(x,0)=0, \\ -\gamma(x)A^2+(1-v^2)\gamma''(x)-2v\gamma'(x)A-2vb(x,0)CA \\ -(1-v^2)b_y(x,0)C-(1-v^2)c_y(x,0)CA-2vc(x,0)CA^2=0. \end{cases} \tag{5.20}$$

Substitute the transformation (5.16) into the first and third equations of system (5.17), we derive

$$\gamma(0) \ = \ C,$$

$$\gamma'(0) \quad = \quad K - b(0,0)C - c(0,0)CA, \tag{5.21}$$

for which, the solutions $b(x,y), c(x,y)$ and $\gamma(x)$ of (5.20) can be presented as follows

$$\begin{aligned}
\gamma(x) &= [\gamma(0),\gamma'(0)]e^{Dx}\begin{pmatrix} I^{2\times2} \\ 0^{2\times2}\end{pmatrix}, \\
b(x,y) &= \frac{2v}{1-v^2}\gamma'(x-y)B + \frac{1}{1-v^2}\gamma(x-y)AB + \frac{2v}{(1-v^2)^2}\gamma(x-y)BCAB, \\
c(x,y) &= \frac{1}{1-v^2}\gamma(x-y)B
\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}
\gamma(0) &= (1,0), \\
\gamma'(0) &= (k_1 + \frac{1+2vk_2}{m}, k_2), \\
D &= \begin{pmatrix} 0 & 0 & \frac{1+3v^2}{m(1-v^2)} & \frac{2v}{1-v^2} \\ 0 & 0 & \frac{2kmv+v(1-5v^2)}{m^2(1-v^2)} & \frac{1-3v^2}{m(1-v^2)} \\ 1 & 0 & -\frac{km+2v^2}{m^2(1-v^2)} & -\frac{v}{m(1-v^2)} \\ 0 & 1 & \frac{-kmv+2v^3}{m^3(1-v^2)} & \frac{v^2-mk}{m^2(1-v^2)} \end{pmatrix}.
\end{aligned}$$

In the same deduction, we seek the inverse transformation $w(x,t) \to u(x,t)$:

$$u(x,t) = w(x,t) - \int_0^x \varphi(x,y)w(y,t)dy - \int_0^x \lambda(x,y)w_t(y,t)dy - \alpha(x)X(t), \tag{5.23}$$

with

$$\begin{aligned}
\alpha(x) &= [-C,-K]e^{Zx}\begin{pmatrix} I^{2\times2} \\ 0^{2\times2}\end{pmatrix}, \\
\varphi(x,y) &= \frac{1}{1-v^2}\alpha(x-y)(A+BK)B + \frac{2v}{1-v^2}\alpha'(x-y)B, \\
\lambda(x,y) &= \alpha(x-y)B, \\
Z &= \begin{pmatrix} 0 & 0 & \frac{k_1v^2-k_1-k}{m(1-v^2)} & \frac{k_2v^2-k_2+v}{m(1-v^2)} \\ 0 & 0 & \frac{(k_1v^2-k_1-k)(k_2v^2-k_2+v)}{m^2(1-v^2)} & \frac{(k_2v^2-k_2+v)^2}{m^2(1-v^2)} \\ 1 & 0 & 0 & \frac{2v}{1-v^2} \\ 0 & 1 & \frac{2v(k_1v^2-k_1-k)}{m(1-v^2)} & \frac{2v(k_2v^2-k_2+v)}{m(1-v^2)} \end{pmatrix}.
\end{aligned}$$

5.3.2. STABILITY OF TARGET SYSTEM

Firstly, let us reformulate target system (5.17) in an appropriate Hilbert state space $\mathscr{H}$. Let $\mathscr{H}$ be the following space:

$$\mathscr{H} = \mathbb{R}^2 \times V^1(0,1) \times L^2(0,1), \quad V^k(0,1) = \{\xi \in H^k(0,1) | \xi(0) = 0\}, \tag{5.24}$$

equipped with an inner product, for $(X_1, w_1, w_2), (X_2, \bar{w}_1, \bar{w}_2) \in \mathscr{H}$:

$$\langle (X_1, w_1, w_2), (X_2, \bar{w}_1, \bar{w}_2) \rangle$$

$$= \quad X_1^T X_2 + \int_0^1 (w_2 + v w_{1,x})(\overline{\bar{w}_2} + v\overline{\bar{w}_{1,x}})dx + \int_0^1 w_{1,x}\overline{\bar{w}_{1,x}}dx, \tag{5.25}$$

Define a linear operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ as follows:

$$\begin{cases} \mathscr{A}(X, f_1, f_2) = ((A+BK)X(t) + Bf_1'(0), f_2, -2vf_2' + (1-v^2)f_1''), \\ D(\mathscr{A}) = \{(X, f_1, f_2) \in \mathbb{R}^2 \times H^2(0,1) \times V^1(0,1) | f_1'(1) = 0\}. \end{cases} \tag{5.26}$$

Then, system (5.17) can be written as an evolution equation in $\mathscr{H}$:

$$\frac{d}{dt}(X(t), w(\cdot,t), w_t(\cdot,t)) = \mathscr{A}(X(t), w(\cdot,t), w_t(\cdot,t)). \tag{5.27}$$

Lemma 5.3.1 *Let $\mathscr{A}$ and $\mathscr{H}$ be defined as before. $\mathscr{A}$ generates a C_0 semigroup of contractions on $\mathscr{H}$.*

Proof. Define an equivalent inner product:

$$\begin{aligned} &\langle (X_1, w_1, w_2), (X_2, \bar{w}_1, \bar{w}_2)\rangle_1 \\ = \quad &\mu X_1^T P_1 X_2 + \int_0^1 (w_2 + v w_{1,x})(\overline{\bar{w}_2} + v\overline{\bar{w}_{1,x}})dx + \int_0^1 w_{1,x}\overline{\bar{w}_{1,x}}dx, \end{aligned} \tag{5.28}$$

where

$$0 < \mu \le \frac{v(1-v^2)\lambda_{\min}(Q_1)}{2|P_1B|^2}, \tag{5.29}$$

and the matrix $P_1 = P_1^T > 0$ is the solution to the equation:

$$P_1(A+BK) + (A+BK)^T P_1 = -Q_1, \tag{5.30}$$

for some $Q_1 = Q_1^T > 0$. For any $z = (X(t), w(\cdot,t), w_t(\cdot,t))^T \in D(\mathscr{A})$, a straight forward calculation yields

$$\begin{aligned} &\Re\langle \mathscr{A}z, z\rangle_1 \\ = \quad & -\frac{\mu}{2}X(t)^T Q_1 X(t) + \mu X(t)^T P_1 B w_x(0,t) - \frac{v}{2}w_t^2(1,t) - \frac{v(1-v^2)}{2}w_x^2(0,t). \end{aligned}$$

According to Young's inequality $ab \le \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$, we obtain

$$\begin{aligned} &\Re\langle \mathscr{A}z, z\rangle_1 \\ \le \quad & -\frac{\mu\lambda_{\min}(Q_1)}{4}|X(t)|^2 - \left[-\frac{\mu|P_1B|^2}{\lambda_{\min}(Q_1)} + \frac{v(1-v^2)}{2}\right] w_x^2(0,t) - \frac{v}{2}w_t^2(1,t) \le 0, \end{aligned}$$

where μ is given by (5.29). Hence, $\mathscr{A}$ is dissipative in $\mathscr{H}$. Moreover, let $(Y, g_1, g_2) \in \mathscr{H}$, and solve $\mathscr{A}(X, f_1, f_2) = (Y, g_1, g_2)$ for $(X, f_1, f_2) \in D(\mathscr{A})$, that is,

$$\begin{cases} (A+BK)X(t) + Bf_1'(0) = Y, \\ f_2 = g_1, \\ -2vf_2' + (1-v^2)f_1'' = g_2, \\ f_1(0) = 0, \ f_1'(1) = 0. \end{cases} \tag{5.31}$$

A direct computation gives the unique solution

$$\begin{cases} f_2 = g_1, \\ f_1 = -\int_0^x \int_\xi^1 \frac{1}{1-v^2}(g_2(\varsigma)+2vg_1'(\varsigma))d\varsigma d\xi, \\ X(t) = (A+BK)^{-1}Y - \left[-\int_0^1 \frac{1}{1-v^2}(g_2(\varsigma)+2vg_1'(\varsigma))d\varsigma\right](A+BK)^{-1}B. \end{cases} \tag{5.32}$$

Hence, we get the unique solution $(X, f_1, f_2) \in D(\mathscr{A})$ and $\mathscr{A}^{-1}$ exists. The Sobolev embedding theorem (Adams *et al.*, [88]) implies that $\mathscr{A}^{-1}$ is compact on $\mathscr{H}$. Therefore, the Lumer-Phillips theorem asserts that $\mathscr{A}$ generates a C_0 semigroup of contractions on $\mathscr{H}$. The proof is complete.

Lemma 5.3.2 *For any initial values $(X(t), w(x,0), w_t(x,0))$, which belong to $\mathscr{H}$, the target system (5.17) is exponentially stable in $\mathscr{H}$.*

Proof. Define

$$\Xi_1(t) \quad = \quad \|w_t(t)+vw_x(t)\|^2 + \|w_x(t)\|^2 + |X(t)|^2. \tag{5.33}$$

5

Let V_1 be a Lyapunov function written as

$$V_1(t) = X(t)^T P_1 X(t) + a_1 E_1(t), \tag{5.34}$$

where the matrix P_1 is given by (5.30). The positive parameter a_1 is to be chosen later and function $E_1(t)$ is defined by

$$\begin{aligned} E_1(t) \quad = \quad & \frac{1}{2}\left[\|w_t(t)+vw_x(t)\|^2 + \|w_x(t)\|^2\right] \\ & +\delta_1 \int_0^1 (1+y)w_x(y,t)\left[w_t(y,t)+vw_x(y,t)\right]dy, \end{aligned} \tag{5.35}$$

We observe that

$$\theta_{11}\Xi_1(t) \le V_1(t) \le \theta_{12}\Xi_1(t), \tag{5.36}$$

where

$$\begin{aligned} \theta_{11} &= \min\left\{\lambda_{min}(P_1), \frac{a_1}{2}[1-2\delta_1]\right\}, \\ \theta_{12} &= \max\left\{\lambda_{max}(P_1), \frac{a_1}{2}[1+2\delta_1]\right\}. \end{aligned} \tag{5.37}$$

We choose $0 < \delta_1 < \frac{1}{2}$.

$$\begin{aligned} \dot V_1(t) \quad &= \quad -X(t)^T Q_1 X(t) + 2X(t)^T P_1 B w_x(0,t) + a_1 \dot E_1(t) \\ &= \quad -X(t)^T Q_1 X(t) + 2X(t)^T P_1 B w_x(0,t) \\ & \quad +a_1\Big[-\frac{\delta_1}{2}((1-v^2)\|w_x\|^2 + \|w_t\|^2 + (1-v^2)|w_x(0,t)|^2) \\ & \quad -(\frac{v}{2}-\delta_1)|w_t(1,t)|^2 - \frac{v(1-v^2)}{2}|w_x(0,t)|^2\Big] \end{aligned}$$

$$\leq -\frac{\lambda_{\min}(Q_1)}{2}|X(t)|^2 - \left[-\frac{2|P_1B|^2}{\lambda_{\min}(Q_1)} + \frac{a_1 v(1-v^2)}{2} + \frac{a_1\delta_1(1-v^2)}{2}\right]|w_x(0,t)|^2$$
$$-a_1(\frac{v}{2}-\delta_1)|w_t(1,t)|^2 - \frac{a_1\delta_1}{2}((1-v^2)\|w_x\|^2 + \|w_t\|^2). \tag{5.38}$$

To have $\dot{V}_1(t) < 0$ we choose

$$a_1 \geq \frac{4|P_1B|^2}{\left[v(1-v^2)+\delta_1(1-v^2)\right]\lambda_{min}(Q_1)}, \quad 0 < \delta_1 \leq \frac{v}{2}. \tag{5.39}$$

We now have

$$\dot{V}_1(t) \leq -\frac{\lambda_{\min}(Q_1)}{2}|X(t)|^2 - \frac{a_1\delta_1(1-v^2)}{2(1+v^2+v)}\left(\|w_x\|^2 + \|w_t + vw_x\|^2\right) \leq -\eta_1 V_1(t), \tag{5.40}$$

where

$$\eta_1 = \frac{\min\left\{\frac{\lambda_{min}(Q_1)}{2}, \frac{a_1\delta_1(1-v^2)}{2(1+v^2+v)}\right\}}{\theta_{12}}. \tag{5.41}$$

Thus, we arrive at

$$V_1(t) \leq e^{-\eta_1 t}V_1(0). \tag{5.42}$$

The proof is complete.

Theorem 5.3.3 *For initial value $(X(0), u(x,0), u_t(x,0))$, which belongs to $\mathcal{H}$, the closed-loop system (5.14) with state feedback control law $U_2(t)$ in (5.18) admits a unique solution $(X(t), u(x,t), u_t(x,t))$ that decays to zero exponentially in $\mathcal{H}$ as time t goes to infinity.*

Proof. The equivalent well-posedness and stability property between the target system (5.17) and the closed-loop system (5.14) are ensured due to the invertibility of the backstepping transformation. Then by Lemma 5.3.1 and Lemma 5.3.2, the proof is complete.

5.4. OBSERVER AND OUTPUT FEEDBACK CONTROL

In this section we consider an observer-based output feedback control law, and the observation output is given as

$$y_{out}(t) = CX(t), \tag{5.43}$$

where C and $\mathscr{A}$ are given by (5.15) and (5.26), and $(\mathscr{A}, C)$ is observable.

5.4.1. OBSERVER DESIGN

Design the observer of system (5.14):

$$\begin{cases} \dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}_x(0,t) + \bar{L}C(X(t)-\hat{X}(t)), \quad t>0, \\ \hat{u}_{tt} = -2v\hat{u}_{xt} + (1-v^2)\hat{u}_{xx}, \quad 0\leq x\leq 1, \quad t>0, \\ \hat{u}(0,t) = CX(t), \quad t>0, \\ \hat{u}_x(1,t) = U_2(t), \quad t>0. \end{cases} \tag{5.44}$$

The observer gain $\bar{L} = (\bar{l}_1, \bar{l}_2)^T$ is chosen to make $A - \bar{L}C$ Hurwitz. Define the observer error as

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t), \quad \tilde{X}(t) = X(t) - \hat{X}(t). \tag{5.45}$$

Then the observer error system can be written as

$$\begin{cases} \dot{\tilde{X}}(t) = (A - \bar{L}C)\tilde{X}(t) + B\tilde{u}_x(0,t), \ \ t > 0, \\ \tilde{u}_{tt} = -2v\tilde{u}_{xt} + (1 - v^2)\tilde{u}_{xx}, \quad 0 \le x \le 1, \ \ t > 0, \\ \tilde{u}(0,t) = 0, \ \ t > 0, \\ \tilde{u}_x(1,t) = 0, \quad t > 0. \end{cases} \tag{5.46}$$

Let us reformulate error system (5.46) in Hilbert state space $\mathscr{H}$, equipped with inner product in (5.25). Define a linear operator $\tilde{\mathscr{A}} : D(\tilde{\mathscr{A}}) \subset \mathscr{H} \to \mathscr{H}$ as follows:

$$\begin{cases} \tilde{\mathscr{A}}(X, f_1, f_2) = \left((A - \bar{L}C)X(t) + Bf_1'(0), f_2, -2vf_2' + (1 - v^2)f_1''\right), \\ D(\tilde{\mathscr{A}}) = \left\{(X, f_1, f_2) \in \mathbb{R}^2 \times H^2(0,1) \times V^1(0,1) | f_1'(1) = 0\right\}. \end{cases} \tag{5.47}$$

Then, system (5.46) can be written as an evolution equation in $\mathscr{H}$:

$$\frac{d}{dt}(\tilde{X}(t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t)) = \tilde{\mathscr{A}}(\tilde{X}(t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t)). \tag{5.48}$$

Theorem 5.4.1 *For initial value $(\tilde{X}(0), \tilde{u}(x,0), \tilde{u}_t(x,0))$, which belongs to $\mathscr{H}$, the error system (5.46) admits a unique solution $(\tilde{X}(t), \tilde{u}(x,t), \tilde{u}_t(x,t))$ that decays to zero exponentially in $\mathscr{H}$ as time t goes to infinity.*

Proof. The proofs are similar to the proofs for Lemma 5.3.1 and 5.3.2, so we omit the details here.

5.4.2. OUTPUT FEEDBACK CONTROL

Based on the state feedback controller (5.18) and observer (5.44), we can naturally design the following output-feedback controller:

$$U_2(t) = b(1,1)\hat{u}(1,t) + \gamma'(1)\hat{X}(t) + \int_0^1 b_x(1,y)\hat{u}(y,t)dy + \int_0^1 c_x(1,y)\hat{u}_t(y,t)dy, \tag{5.49}$$

which leads to the closed-loop system of (5.14):

$$\begin{cases} \dot{X}(t) = AX(t) + Bu_x(0,t), \\ u_{tt} + 2vu_{xt} + (v^2 - 1)u_{xx} = 0, \\ u(0,t) = CX(t), \\ u_x(1,t) = b(1,1)\hat{u}(1,t) + \gamma'(1)\hat{X}(t) + \int_0^1 b_x(1,y)\hat{u}(y,t)dy + \int_0^1 c_x(1,y)\hat{u}_t(y,t)dy, \\ \dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}_x(0,t) + \bar{L}C(X(t) - \hat{X}(t)), \\ \hat{u}_{tt} = -2v\hat{u}_{xt} + (1 - v^2)\hat{u}_{xx}, \quad 0 \le x \le 1, \\ \hat{u}(0,t) = CX(t), \\ \hat{u}_x(1,t) = b(1,1)\hat{u}(1,t) + \gamma'(1)\hat{X}(t) + \int_0^1 b_x(1,y)\hat{u}(y,t)dy + \int_0^1 c_x(1,y)\hat{u}_t(y,t)dy. \end{cases} \tag{5.50}$$

Theorem 5.4.2 *For any initial state* $(X(0), u(x,0), u_t(x,0), \hat{X}(0), \hat{u}(x,0), \hat{u}_t(x,0)) \in \mathscr{H}^2$, *the closed-loop system (5.50) admits a unique solution* $(X(t), u(x,t), u_t(x,t), \hat{X}(t), \hat{u}(x,t), \hat{u}_t(x,t))$ *that decays to zero exponentially in* $\mathscr{H}$ *as time t goes to infinity.*

Proof. By using the transformation (5.16), the closed-loop system (5.50) can be converted to the following equivalent system:

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bw_x(0,t), \quad t>0, \\ w_{tt} + 2vw_{xt} + (v^2-1)w_{xx} = 0, \quad 0 \le x \le 1, \ t>0, \\ w(0,t) = 0, \quad t>0, \\ w_x(1,t) = \mathscr{F}(\tilde{X}, \tilde{u}, \tilde{u}_t), \\ \dot{\tilde{X}}(t) = (A-\bar{L}C)\tilde{X}(t) + B\tilde{u}_x(0,t), \ t>0, \\ \tilde{u}_{tt} = -2v\tilde{u}_{xt} + (1-v^2)\tilde{u}_{xx}, \quad 0 \le x \le 1, \ t>0, \\ \tilde{u}(0,t) = 0, \ t>0, \\ \tilde{u}_x(1,t) = 0, \quad t>0, \end{cases} \tag{5.51}$$

where operator $\mathscr{F}: \mathscr{H} \to \mathbb{R}$ defined by $\mathscr{F}(\tilde{X}(t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t)) = b(1,1)\tilde{u}(1,t) + \gamma'(1)\tilde{X}(t) + \int_0^1 b_x(1,y)\tilde{u}(y,t)dy + \int_0^1 c_x(1,y)\tilde{u}_t(y,t)dy$. The proof will be completed if we can prove that (5.51) has a unique solution and is exponentially stable in $\mathscr{H}$.

The closed-loop system (5.51) can be written as the following evolution equations:

$$\frac{d}{dt}Y(t) = \mathscr{A}Y(t) + \bar{B}\mathscr{F}\tilde{Y}(t), \tag{5.52}$$

$$\frac{d}{dt}\tilde{Y}(t) = \tilde{\mathscr{A}}\tilde{Y}(t), \tag{5.53}$$

where $\mathscr{A}$ and $\tilde{\mathscr{A}}$ are given by (5.26) and (5.47), $Y(t) = (X(t), w(\cdot,t), w_t(\cdot,t)) \in \mathscr{H}$, $\tilde{Y}(t) = (\tilde{X}(t), \tilde{u}(\cdot,t), \tilde{u}_t(\cdot,t)) \in \mathscr{H}$, and

$$\bar{B}\mathscr{F}\tilde{Y}(t) = \left[0, 0, \delta(x-1)\mathscr{F}\tilde{Y}(t)\right] \tag{5.54}$$

with δ is Dirac function. The operator $\mathscr{A}$ and $\tilde{\mathscr{A}}$ generate C_0 semigroup of contractions $e^{\mathscr{A}t}$ and $e^{\tilde{\mathscr{A}}t}$ on $\mathscr{H}$ respectively. Notice that $\bar{B}$ is an unbounded operator, we will show that $\bar{B}$ is admissible to the C_0 semigroup $e^{\mathscr{A}t}$.

Lemma 5.4.3 *$\bar{B}$ is admissible to the C_0 semigroup $e^{\mathscr{A}t}$.*

Proof. As $\mathscr{A}^*$ is defined by

$$\begin{cases} \mathscr{A}^*(X, f_1, f_2) = \left((A+BK)X(t), -f_2, 2vf_2' - (1-v^2)f_1''\right), \\ D(\mathscr{A}^*) = \left\{ \begin{array}{l} (X, f_1, f_2) \in \mathbb{R}^2 \times H^2(0,1) \times H^1(0,1), \\ (1-v^2)f_1'(1) - vf_2(1) = 0, \\ (1-v^2)(vf_1'(0) + f_2(0)) - B^T X^* = 0. \end{array} \right\}, \end{cases} \tag{5.55}$$

the dual system of (5.52) can be written as

$$\begin{cases} \frac{d}{dt}(X^*(t), -w^*(\cdot,t), w_t^*(\cdot,t)) = \mathscr{A}^*(X^*(t), -w^*(\cdot,t), w_t^*(\cdot,t)), \\ y(t) = \bar{B}^*(X^*(t), -w^*(\cdot,t), w_t^*(\cdot,t)) = \frac{w_t^*(1,t)}{1-v^2}, \end{cases} \tag{5.56}$$

which means

$$\begin{cases} \dot{X}^*(t) = (A+BK)X^*(t), \quad t>0, \\ w^*_{tt} - 2v w^*_{xt} + (v^2-1) w^*_{xx} = 0, \quad 0 \le x \le 1, \ t>0, \\ (1-v^2) w^*_x(1,t) + v w^*_t(1,t) = 0, \quad t>0, \\ (1-v^2)(-v w^*_x(0,t) + w^*_t(0,t)) - B^T X^* = 0, \quad t>0. \end{cases} \tag{5.57}$$

The energy function of system is defined by

$$E^*(t) = \frac{1}{2} X^*(t)^T X^*(t) + \frac{1}{2}\int_0^1 (w^*_t - v w^*_x)^2 dx + \frac{1}{2}\int_0^1 w^{*2}_x dx. \tag{5.58}$$

Let

$$E^*_1(t) = \frac{\chi}{2} X^*(t)^T P_1 X^*(t) + \frac{1}{2}\int_0^1 (w^*_t - v w^*_x)^2 dx + \frac{1}{2}\int_0^1 w^{*2}_x dx, \tag{5.59}$$

where P_1 is given by (5.30), and β is given by

$$\chi \ge \frac{2|B|^2}{v(1-v^2)\lambda_{\min}(Q_1)}. \tag{5.60}$$

A simple computation for the derivative of $E^*_1(t)$ with respect to t along the solution to (5.57) gives

$$\begin{aligned} \dot{E}^*_1(t) &= -\frac{\chi}{2} X^*(t)^T Q_1 X^*(t) - B^T X^*(t) w^*_x(0,t) - \frac{v}{2}|w^*_t(1,t)|^2 \\ &\quad -\frac{v(1-v^2)}{2}|w^*_x(1,t)|^2 - \frac{v}{2}|w^*_t(0,t)|^2 - \frac{v(1-v^2)}{2}|w^*_x(0,t)|^2 \\ &\le -\frac{\chi\lambda_{\min}(Q_1)}{4}|X^*(t)|^2 + \left[\frac{|B|^2}{\chi\lambda_{\min}(Q_1)} - \frac{v(1-v^2)}{2}\right]|w^*_x(0,t)|^2 \\ &\quad -\frac{v}{2}|w^*_t(1,t)|^2 - \frac{v(1-v^2)}{2}|w^*_x(1,t)|^2 - \frac{v}{2}|w^*_t(0,t)|^2. \end{aligned} \tag{5.61}$$

Hence, $E^*_1(t) \le E^*_1(0)$. Define

$$\rho(t) = \int_0^1 x(w^*_t - v w^*_x) w^*_x dx. \tag{5.62}$$

Then $\rho(t) \le E^*_1(t)$ for $\forall\ t \ge 0$. Noticing that

$$\dot{\rho}(t) = \frac{1}{2(1-v^2)}|w^*_t(1,t)|^2 - \frac{1}{2}\int_0^1 (1-v^2) w^{*2}_x dx - \frac{1}{2}\int_0^1 w^{*2}_t dx, \tag{5.63}$$

we have that

$$\begin{aligned} \int_0^T w^{*2}_t(1,t) dt &= 2(1-v^2)\left[\rho(T) - \rho(0)\right] + (1-v^2)\int_0^T \left[\int_0^1 (1-v^2) w^{*2}_x + w^{*2}_t dx\right] dt \\ &\le \left[4(1-v^2) + \frac{T(1-v^2)}{1-v}\right] E^*_1(0) \end{aligned}$$

$$\leq \left[4(1-v^2) + \frac{T(1-v^2)}{1-v}\right]\eta^* E^*(0), \tag{5.64}$$

where $\eta^* = \max\{\chi\lambda_{max}(P_1), 1\}$. A direct calculation shows that

$$\bar{B}^*\mathscr{A}^{*-1}(Y, g_1, g_2) = -\frac{g_1(1)}{1-v^2}, \quad \forall\,(Y, g_1, g_2) \in \mathscr{H}, \tag{5.65}$$

which tells us that $\bar{B}^*\mathscr{A}^{*-1}$ is bounded. This together with (5.64) yields that $\bar{B}^*$ is admissible for $e^{\mathscr{A}^* t}$, which means that $\bar{B}$ is admissible for $e^{\mathscr{A}t}$. The proof is completed.

To prove the stability of the closed-loop system (5.51), define

$$\tilde{V}(t) = \tilde{X}(t)^T P_2 \tilde{X}(t) + a_2 E_2(t), \tag{5.66}$$

The matrix $P_2 = P_2^T > 0$ is the solution to the equation:

$$P_2(A - \bar{L}C) + (A - \bar{L}C)^T P_2 = -Q_2, \tag{5.67}$$

for some $Q_2 = Q_2^T > 0$. Function $E_2(t)$ is defined by

$$\begin{aligned} E_2(t) \quad &= \quad \frac{1}{2}\left(\|\tilde{u}_t(t) + v\tilde{u}_x(t)\|^2 + \|\tilde{u}_x(t)\|^2\right) \\ &\quad +\delta_2 \int_0^1 (1+y)\tilde{u}_x(y,t)\left(\tilde{u}_t(y,t) + v\tilde{u}_x(y,t)\right) dy. \end{aligned} \tag{5.68}$$

By choosing

$$a_2 \geq \frac{4|P_2B|^2}{\left[v(1-v^2) + \delta_2(1-v^2)\right]\lambda_{min}(Q_2)}, \quad 0 < \delta_2 \leq \frac{v}{2}, \tag{5.69}$$

and according to the proof the stability of the target system in subsection 3.2, we arrive at

$$\tilde{V}(t) \leq e^{-\eta_2 t}\tilde{V}(0), \tag{5.70}$$

where

$$\eta_2 = \frac{\min\left\{\frac{\lambda_{min}(Q_2)}{2}, \frac{a_2\delta_2(1-v^2)}{2(1-v^2+v)}\right\}}{\theta_{22}}. \tag{5.71}$$

Let V be a Lyapunov function written as

$$V(t) = V_1(t) + \beta\tilde{V}(t), \tag{5.72}$$

$$\begin{aligned} \dot{V}(t) \quad &= \quad -X(t)^T Q_1 X(t) + 2X(t)^T P_1 B w_x(0,t) + a_1\dot{E}_1(t) + \beta\dot{\tilde{V}}(t) \\ &= \quad -X(t)^T Q_1 X(t) + 2X(t)^T P_1 B w_x(0,t) \\ &\quad +a_1\Big[-\frac{\delta_1}{2}((1-v^2)\|w_x\|^2 + \|w_t\|^2 + (1-v^2)|w_x(0,t)|^2) \\ &\quad -(\frac{v}{2} - \delta_1)|w_t(1,t)|^2 - \frac{v(1-v^2)}{2}|w_x(0,t)|^2\Big] \end{aligned}$$

$$
\begin{aligned}
& +a_1(\delta_1+\frac{v}{2})(1-v^2)|w_x(1,t)|^2+a_1(1-v^2)w_t(1,t)w_x(1,t)+\beta\dot{\tilde{V}}(t) \\
\leq\; & -\frac{\lambda_{\min}(Q_1)}{2}|X(t)|^2-\left[-\frac{2|P_1B|^2}{\lambda_{\min}(Q_1)}+\frac{a_1v(1-v^2)}{2}+\frac{a_1\delta_1(1-v^2)}{2}\right]|w_x(0,t)|^2 \\
& +a_1(1-v^2)(\delta_1+\frac{v}{2}+\frac{1}{v})|w_x(1,t)|^2 \\
& -a_1(\frac{v}{4}-\delta_1)|w_t(1,t)|^2-\frac{a_1\delta_1}{2}\left[(1-v^2)\|w_x\|^2+\|w_t\|^2\right]-\beta\eta_2\tilde{V}(t). \qquad (5.73)
\end{aligned}
$$

From the second boundary condition in the closed-loop system (5.51), we obtain

$$
\begin{aligned}
|w_x(1,t)|^2 \leq\; & b^2(1,1)|\tilde{u}(1,t)|^2+(\gamma'(1))^2|\tilde{X}(t)|^2 \\
& +\left(\int_0^1 b_x(1,y)\tilde{u}(y,t)dy\right)^2+\left(\int_0^1 c_x(1,y)\tilde{u}_t(y,t)dy\right)^2. \qquad (5.74)
\end{aligned}
$$

According to $Agmon's$ inequality and $Poincaré$ inequality, we obtain

$$
\|\tilde{u}(t)\|^2\leq 2|\tilde{u}(0,t)|^2+4\|\tilde{u}_x(t)\|^2, \quad |\tilde{u}(1,t)|^2\leq 3|\tilde{u}(0,t)|^2+5\|\tilde{u}_x(t)\|^2. \qquad (5.75)
$$

Thus, by choosing β big enough and

$$
a_1\geq\frac{4|P_1B|^2}{[v(1-v^2)+\delta_1(1-v^2)]\lambda_{min}(Q_1)}, \quad 0<\delta_1\leq\frac{v}{4}, \qquad (5.76)
$$

there exits η_3 such that

$$
\dot{V}(t)\leq-\eta_3V(t), \qquad (5.77)
$$

Thus, we arrive at

$$
V(t)\leq e^{-\eta_3 t}V(0). \qquad (5.78)
$$

Thus, the closed-loop system of (5.51) admits a unique solution and decays to zero exponentially in $\mathscr{H}$ as time t goes to infinity. The proof is complete.

5.5. NUMERICAL SIMULATIONS

In this section, we give some numerical simulation results for the system (5.8). The finite difference method is adopted in both the time and the space domain for both PDEs and boundary conditions in (5.8). In the numerical scheme, we choose the space grid size $N=200$, time step $dt=5\times10^{-2}$. The parameter values are set to be

$$
v=0.1,\ k=1,\ m=1,\ K=[k_1,k_2]=[1,1],\ \bar{L}=[\bar{l}_1,\bar{l}_2]=[-1,0.5], \qquad (5.79)
$$

and the initial conditions are taken to be

$$
u(x,0)=0.1\sin(1.5x), \quad u_t(x,0)=0, \quad 0\leq\xi\leq1. \qquad (5.80)
$$

Figure 5.2 show that the displacement of the open-loop system (5.8) is not convergent to zero. Figure 5.3 show that the displacement of the closed-loop system (5.8) with state feedback controller (5.18) converges to zero. It can be seen that the output feedback controller can make the closed-loop system exponentially stable.

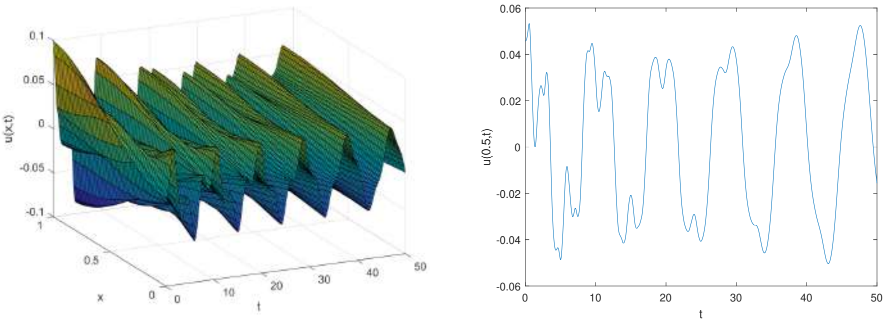

Figure 5.2: The state of system (5.8) when $U(t) = 0$. (a) The responses for the whole space domain $(0, 1)$. (b) The responses at the midpoint.

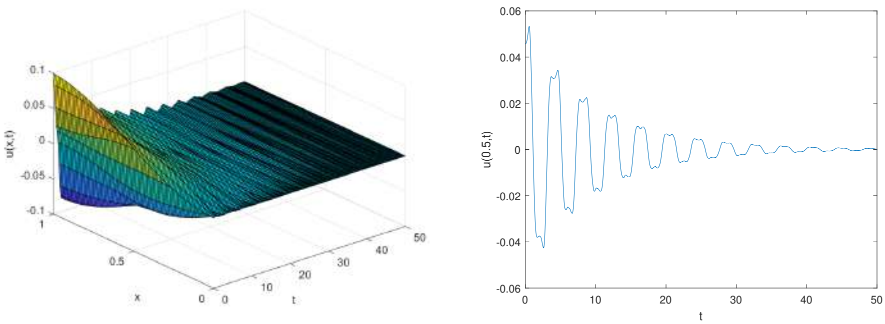

Figure 5.3: The state $u(x,t)$ of closed-loop system (5.8) with full-state feedback controller (5.18). (a) The responses for the whole space domain $(0, 1)$. (b) The responses at the midpoint.

5.6. CONCLUSIONS

In this chapter, we have presented a control design to stabilize an unstable moving string subject to a spring-mass-dashpot boundary, where the control actuator is located at the other nature boundary. Firstly, by constructing an invertible backstepping transformation, we design a state feedback controller to stabilize the system. Next, we present an observer to estimate the states of the system, and based on the estimated states, we design an output-feedback controller. It is shown that by using boundary measurements only, the output feedback can make the closed-loop system exponentially stable. Finally, the simulation results illustrate that the proposed control law can efficiently suppress the axial vibrations of the moving string system.

6

CONCLUSION

In this thesis four initial-boundary value problems are studied. These problems describe the motion of axially moving continua, which may be regarded in reality as models describing the transverse or longitudinal vibrations of mechanical elastic structures such as conveyor belts, elevator cables and hoisting ropes.

Chapter 2 starts with a simple model for transverse vibrations of a string on a bounded, fixed interval with a slowly time-varying Robin boundary condition. We showed how to (approximately) solve initial-boundary value problems with different choices of time-dependent coefficients $k(t)$ in Robin boundary conditions. Different values for $k(t)$ lead to different difficulties in the analysis. So, we tackle these problems by using the method of d'Alembert, averaging and singular perturbation techniques, and by using a three time-scales perturbation method, respectively. It turns out that small order excitations can lead to large responses when the frequency of the external force satisfies certain conditions, and these results are valid on time-scales of order $\frac{1}{\varepsilon}$, where ε is a dimensionless small parameter.

Chapter 3 continues with the longitudinal vibrations and associated resonances in a practical elevator system due to a harmonic excitation at one of its boundaries. We analyse this problem by an adapted version of the method of separation of variables and perturbation methods, (such as averaging methods, singular perturbation techniques, and multiple timescales perturbation methods). Based on the analysis, explicit, and accurate approximations of the solution of the initial-boundary value problem are constructed, and these approximations are valid on time-scales of order $\frac{1}{\varepsilon}$. It turns out that for a given boundary disturbance frequency, many oscillation modes jump up from order ε amplitudes to order $\sqrt{\varepsilon}$ amplitudes.

Chapter 4 extends the study in chapter 3 and analyses transverse as well as longitudinal oscillations and resonances in an elevator system due to boundary excitations. It is shown in this chapter that for special frequencies in boundary excitations and for certain parameter values of the longitudinal stiffness and the conveyance mass, many large oscillations arise in transverse and longitudinal directions. The oscillation modes for transverse motion jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$, and the oscillation modes for longitudinal motion jump up from $O(\varepsilon^2)$ to $O(\sqrt{\varepsilon})$. To obtain these results the method of separation of variables is presented, and perturbation methods, (such as averaging methods, singular perturbation techniques) are used. Furthermore, since the initial-boundary value problems for transverse motion and longitudinal motion are nonlinear, we can not always construct formal approximations of the solutions but we can get properties and predictions of solutions analytically on time-scales of order $\frac{1}{\varepsilon}$. And approximations of the solutions are computed by using an iterative method analytically together with numerical methods.

Chapter 5 presents a control design to stabilize an unstable moving string subject to a spring-mass-dashpot boundary, where the control actuator is located at the other boundary of the string. Firstly, by a transformation for the boundary condition, the problem can be convert to a coupled ODE-PDE system. Secondly, by an invertible backstepping transformation, the coupled ODE-PDE system is equivalent to a target system of ODE-PDE cascades, which is shown to be exponentially stable in a suitable Hilbert space. Thirdly, we design the observer-based output feedback controller. It is shown that by using boundary measurements only, the output feedback can make the closed-loop system exponentially stable.

Also approximations of the solutions of the above four initial-boundary value problems are computed by using central finite difference schemes. The numerical approximations are in agreement with the analytically obtained results.

The analytical schemes in this thesis can be extended to study other, and more complicated types of moving cable systems.

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Bibliography

[1] D. W. Dareing and B. J. Livesay. "Longitudinal and Angular Drill-String Vibrations With Damping". In: *Journal of Engineering for Industry* 90.4 (1968), p. 671.

[2] A. Arakawa and K. Miyata. "Method of Suppressing Elevator Vibration to Improve Ride Quality : Vibration-Proof Structure of Machine-Room-Less Elevator". In: *Transactions of the Japan Society of Mechanical Engineers* 66.651 (2000), pp. 3547–3553.

[3] K. D. Do and J. Pan. "Boundary control of three-dimensional inextensible marine risers". In: *Journal of Sound and Vibration* 327.3 (2009), pp. 299–321.

[4] K. D. Do and J. Pan. "Boundary control of transverse motion of marine risers with actuator dynamics". In: *Journal of Sound and Vibration* 318.4-5 (2008), pp. 768–791.

[5] D. M. Siringoringo and Y. Fujino. "System identification of suspension bridge from ambient vibration response". In: *Engineering Structures* 30.2 (2008), pp. 462–477.

[6] A. J. Chen, Y. L. Xu, and R. C. Zhang. "Modal parameter identification of Tsing Ma suspension bridge under Typhoon Victor: EMD-HT method". In: *Journal of Wind Engineering and Industrial Aerodynamics* 92.10 (2004), pp. 805–827.

[7] E. Ottaviano, M. Ceccarelli, and M. De Ciantis. "A 4–4 cable-based parallel manipulator for an application in hospital environment". In: *2007 Mediterranean Conference on Control & Automation*. IEEE. 2007, pp. 1–6.

[8] S. Kaczmarczyk and W. Ostachowicz. "Transient vibration phenomena in deep mine hoisting cables. Part 1: Mathematical model". In: *Journal of Sound and Vibration* 262.2 (2003), pp. 219–244.

[9] S. Kaczmarczyk and W. Ostachowicz. "Transient vibration phenomena in deep mine hoisting cables. Part 2: Numerical simulation of the dynamic response". In: *Journal of Sound and Vibration* 262.2 (2003), pp. 245–289.

[10] S. Kaczmarczyk. "The passage through resonance in a catenary–vertical cable hoisting system with slowly varying length". In: *J. Sound Vib.* 208.2 (1997), pp. 243–269.

[11] J. Fajans, E. Gilson, and L. Friedland. "Autoresonant (nonstationary) excitation of the diocotron mode in non-neutral plasmas". In: *Phys. Rev. Lett.* 82.22 (1999), p. 4444.

[12] L. Q. Chen and X. D. Yang. "Stability in parametric resonance of axially moving viscoelastic beams with time-dependent speed". In: *J. Sound Vib.* 284.3-5 (2005), pp. 879–891.

[13] L. Friedland, P. Khain, and A. Shagalov. "Autoresonant phase-space holes in plasmas". In: *Phys. Rev. Lett.* 96.22 (2006), p. 225001.

[14] H. Kimura et al. “Forced vibration analysis of an elevator rope with both ends moving”. In: *J. Vib. Acoust.* 129.4 (2007), pp. 471–477.

[15] E. W. Chen et al. “On the reflected wave superposition method for a travelling string with mixed boundary supports”. In: *J. Sound Vib.* 440 (2019), pp. 129–146.

[16] E. W. Chen et al. “A wave solution for energy dissipation and exchange at nonclassical boundaries of a traveling string”. In: *Mechanical Systems and Signal Processing* 150 (2021), pp. 107272.1–107272.16.

[17] J. Wang et al. “Exponential regulation of the anti-collocatedly disturbed cage in a wave PDE-modeled ascending cable elevator”. In: *Automatica* 95 (2018), pp. 122–136.

[18] N. V. Gaiko and W. T. van Horssen. “On transversal oscillations of a vertically translating string with small time-harmonic length variations”. In: *Journal of Sound and Vibration* 383 (2016), pp. 339–348.

[19] W. D. Zhu and K. Wu. “Dynamic stability of a class of second-order distributed structural systems with sinusoidally varying velocities”. In: *J. Appl. Mech.* 80.6 (2013), p. 061008.

[20] R. A. Malookani and W. T. van Horssen. “On the asymptotic approximation of the solution of an equation for a non-constant axially moving string”. In: *J. Sound Vib.* 367 (2016), pp. 203–218.

[21] S. H. Sandilo and W. T. van Horssen. “On variable length induced vibrations of a vertical string”. In: *J. Sound Vib.* 333.11 (2014), pp. 2432–2449.

[22] L. Q. Chen. “Analysis and control of transverse vibrations of axially moving strings”. In: *Appl. Mech. Rev.* 58.2 (2005), pp. 91–116.

[23] G. Suweken and W. T. van Horssen. “On the weakly nonlinear, transversal vibrations of a conveyor belt with a low and time-varying velocity”. In: *Nonlinear Dyn* 31.2 (2003), pp. 197–223.

[24] S. V. Ponomareva and W. T. van Horssen. “On transversal vibrations of an axially moving string with a time-varying velocity”. In: *Nonlinear Dyn* 50.1-2 (2007), pp. 315–323.

[25] H. R. Öz and H. Boyaci. “Transverse vibrations of tensioned pipes conveying fluid with time-dependent velocity”. In: *J. Sound Vib.* 236.2 (2000), pp. 259–276.

[26] D. J. Hoppe. *Impedance boundary conditions in electromagnetics.* CRC Press, 2018.

[27] T. B. A. Senior. “Impedance boundary conditions for imperfectly conducting surfaces”. In: *Applied Scientific Research, Section B* 8.1 (1960), pp. 418–436.

[28] R. Z. Zhdanov. “Separation of variables in the nonlinear wave equation”. In: *Journal of Physics A: Mathematical and General* 27.9 (1994), p. L291.

[29] I. Podlubny. “The Laplace transform method for linear differential equations of the fractional order”. In: *arXiv preprint funct-an/9710005* (1997).

[30] W. T. van Horssen, Y. D. Wang, and G. H. Cao. “On solving wave equations on fixed bounded intervals involving Robin boundary conditions with time-dependent coefficients”. In: *J. Sound Vib.* 424 (2018), pp. 263–271.

[31] A. Pagani, R. Augello, and E. Carrera. “Frequency and mode change in the large deflection and post-buckling of compact and thin-walled beams”. In: *Journal of Sound and Vibration* 432 (2018), pp. 88–104.

[32] C. A. Felippa. “Introduction to finite element methods”. In: *University of Colorado* 885 (2004).

[33] S. C. Brenner, L. R. Scott, and L. R. Scott. *The mathematical theory of finite element methods*. Vol. 3. Springer, 2008.

[34] R. L. Kuhlemeyer and J. Lysmer. “Finite element method accuracy for wave propagation problems”. In: *Journal of the Soil Mechanics and Foundations Division* 99.5 (1973), pp. 421–427.

[35] Y. Waki, B. R. Mace, and M. J. Brennan. “Numerical issues concerning the wave and finite element method for free and forced vibrations of waveguides”. In: *Journal of Sound and Vibration* 327.1-2 (2009), pp. 92–108.

[36] L. Meirovitch. *Principles and techniques of vibrations*. Vol. 1. Prentice Hall Upper Saddle River, NJ, 1997.

[37] J. L. R. d’Alembert. “Recherches sur la courbe que forme une corde tendue mise en vibration”. In: *Hist. I’Académie R. Des Sci. Belles Lettres* 3 (1747), pp. 214–249.

[38] C. A. Tan and S. Ying. “Dynamic Analysis of the Axially Moving String Based on Wave Propagation”. In: *Journal of Applied Mechanics* 64.2 (1997), pp. 394–400.

[39] W. D. Zhu and J. Ni. “Energetics and stability of translating media with an arbitrarily varying length”. In: *J. Vib. Acoust.* 122.3 (2000), pp. 295–304.

[40] S. H. Sandilo and W. T. van Horssen. “On a cascade of autoresonances in an elevator cable system”. In: *Nonlinear Dynamics* 80.3 (2015), pp. 1613–1630.

[41] N. V. Gaiko and W. T. van Horssen. “On the Transverse, Low Frequency Vibrations of a Traveling String with Boundary Damping”. In: *Journal of Vibration & Acoustics* 137.4 (2015), pp. 041004.1–041004.10.

[42] N. V. Gaiko and W. T. van Horssen. “Resonances and vibrations in an elevator cable system due to boundary sway”. In: *Journal of Sound and Vibration* 424 (2018), pp. 272–292.

[43] Y. wang et al. “Longitudinal response of parallel hoisting system with time-varying rope length”. In: *Journal of Vibroengineering* 16.8 (2014), pp. 4088–4101.

[44] J. H. Bao, P. Zhang, and C. M. Zhu. “Modeling and control of longitudinal vibration on flexible hoisting systems with time-varying length”. In: *Procedia Engineering* 15 (2011), pp. 4521–4526.

[45] M. H. Ghayesh. “Coupled longitudinal–transverse dynamics of an axially accelerating beam”. In: *Journal of Sound and Vibration* 331.23 (2012), pp. 5107–5124.

[46] N. Wang et al. “Modelling and passive control of flexible guiding hoisting system with time-varying length”. In: *Math Comp Model Dyn* 26.1 (2020), pp. 31–54.

[47] J. Wang et al. “Axial Vibration Suppression in a Partial Differential Equation Model of Ascending Mining Cable Elevator”. In: *Journal of Dynamic Systems, Measurement, and Control* 140.11 (2018), pp. 111003.1–111003.13.

[48] S. Kaczmarczyk. “The resonance conditions and application of passive and active control strategies in high-rise lifts to mitigate the effects of building sway”. In: *Lift and Escalator Symposium* 10 (2019), pp. 18.1–18.12.

[49] N. Bekiaris-Liberis and M. Krstic. “Compensation of Transport Actuator Dynamics With Input-Dependent Moving Controlled Boundary”. In: *Automatic Control, IEEE Transactions on* 63.11 (2018), pp. 3889–3896.

[50] X. Cai and M. Krstic. “Nonlinear stabilization through wave PDE dynamics with a moving uncontrolled boundary”. In: *Automatica* 68 (2016), pp. 27–38.

[51] M. Gugat. “Optimal Energy Control in Finite Time by varying the Length of the String”. In: *Siam Journal on Control & Optimization* 46.5 (2007), pp. 1705–1725.

[52] E. W. Chen and N. S. Ferguson. “Analysis of energy dissipation in an elastic moving string with a viscous damper at one end”. In: *Journal of Sound and Vibration* 333.9 (2014), pp. 2556–2570.

[53] M. Ferretti and G. Piccardo. “Dynamic modeling of taut strings carrying a traveling mass”. In: *Continuum Mechanics and Thermodynamics* 25.2 (2013), pp. 469–488.

[54] J. K. Kevorkian and J. D. Cole. *Multiple scale and singular perturbation methods*. Vol. 114. Springer Science & Business Media, 2012.

[55] A. H. Nayfeh and P. F. Pai. *Linear and nonlinear structural mechanics*. John Wiley & Sons, 2008.

[56] F. Verhulst. *Nonlinear differential equations and dynamical systems*. Springer Science & Business Media, 2006.

[57] F. Petitjean, A. Ketterlin, and P. Gançarski. “A global averaging method for dynamic time warping, with applications to clustering”. In: *Pattern recognition* 44.3 (2011), pp. 678–693.

[58] Z. Lin and H. Ma. “Modeling and analysis of three-phase inverter based on generalized state space averaging method”. In: *IECON 2013-39th Annual Conference of the IEEE Industrial Electronics Society*. IEEE. 2013, pp. 1007–1012.

[59] R. S. Crespo et al. “Modelling and simulation of a stationary high-rise elevator system to predict the dynamic interactions between its components”. In: *International Journal of Mechanical Sciences* 137 (2018), pp. 24–45.

[60] J. Wang et al. “Modeling and dynamic behavior analysis of a coupled multi-cable double drum winding hoister with flexible guides”. In: *Mechanism and Machine Theory* 108 (2017), pp. 191–208.

[61] G. Cao, J. Wang, and Z. Zhu. “Coupled vibrations of rope-guided hoisting system with tension difference between two guiding ropes”. In: *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science* 232.2 (2018), pp. 231–244.

[62] J. Chung, C. S. Han, and K. Yi. “Vibration of an axially moving string with geometric non-linearity and translating acceleration”. In: *Journal of Sound and Vibration* 240.4 (2001), pp. 733–746.

[63] L. Q. Chen, W. Zhang, and J. W. Zu. “Nonlinear dynamics for transverse motion of axially moving strings”. In: *Chaos, Solitons & Fractals* 40.1 (2009), pp. 78–90.

[64] D. R. Rowland. “The potential energy density in transverse string waves depends critically on longitudinal motion”. In: *European journal of physics* 32.6 (2011), p. 1475.

[65] Q. C. Nguyen and K. S. Hong. “Longitudinal and transverse vibration control of an axially moving string”. In: *2011 IEEE 5th International Conference on Cybernetics and Intelligent Systems (CIS)*. IEEE. 2011, pp. 24–29.

[66] M. H. Ghayesh. “Coupled longitudinal–transverse dynamics of an axially accelerating beam”. In: *Journal of Sound and Vibration* 331.23 (2012), pp. 5107–5124.

[67] N. V. Gaiko and W. T. van Horssen. “On the lateral vibrations of a vertically moving string with a harmonically varying length”. In: *ASME International Mechanical Engineering Congress and Exposition* 57403 (2015), V04BT04A060.

[68] J. Bao et al. “Transverse vibration of flexible hoisting rope with time-varying length”. In: *Journal of Mechanical Science and Technology* 28.2 (2014), pp. 457–466.

[69] C. Chung and I. Kao. “Modeling of axially moving wire with damping: Eigenfunctions, orthogonality and applications in slurry wiresaws”. In: *Journal of Sound and Vibration* 330.12 (2011), pp. 2947–2963.

[70] C. Chung and I. Kao. “Green’s function and forced vibration response of damped axially moving wire”. In: *Journal of Vibration and Control* 18.12 (2012), pp. 1798–1808.

[71] E. W. Chen et al. “On the reflected wave superposition method for a travelling string with mixed boundary supports”. In: *Journal of Sound and Vibration* 440 (2019), pp. 129–146.

[72] E. W. Chen and N. S. Ferguson. “Analysis of energy dissipation in an elastic moving string with a viscous damper at one end”. In: *Journal of Sound and Vibration* 333.9 (2014), pp. 2556–2570.

[73] E. W. Chen et al. “A reflected wave superposition method for vibration and energy of a travelling string”. In: *Journal of Sound and Vibration* 400 (2017), pp. 40–57.

[74] L. Q. Chen. “Analysis and Control of Transverse Vibrations of Axially Moving Strings”. In: *Applied Mechanics Reviews* 58.2 (2005), pp. 91–116.

[75] W. T. van Horssen. “On the influence of lateral vibrations of supports for an axially moving string”. In: *Journal of Sound and Vibration* 268.2 (2003), pp. 323–330.

[76] Q. C. Nguyen and K. S. Hong. “Asymptotic stabilization of a nonlinear axially moving string by adaptive boundary control”. In: *Optics Communications* 329.22 (2010), pp. 4588–4603.

[77] Q. C. Nguyen and K. S. Hong. “Simultaneous control of longitudinal and transverse vibrations of an axially moving string with velocity tracking”. In: *Journal of Sound and Vibration* 331.13 (2012), pp. 3006–3019.

[78] L. Tebou. “A note on the boundary stabilization of an axially moving elastic tape”. In: *Zeitschrift für angewandte Mathematik und Physik* 70.1 (2019), p. 19.

[79] M. Krstic et al. “Output-feedback stabilization of an unstable wave equation”. In: *Automatica* 44.1 (2008), pp. 63–74.

[80] B. Ren, J. M. Wang, and M. Krstic. “Stabilization of an ODE-Schr*ö*dinger Cascade”. In: *Systems & Control Letters* 62.6 (2013), pp. 503–510.

[81] M. Krstic. “Compensating a String PDE in the Actuation or Sensing Path of an Unstable ODE”. In: *IEEE Transactions on Automatic Control* 54.6 (2009), pp. 1362–1368.

[82] G. A. Susto and M. Krstic. “Control of PDE-ODE cascades with Neumann interconnections”. In: *Journal of the Franklin Institute* 347.1 (2010), pp. 284–314.

[83] J. Wang et al. “Axial Vibration Suppression in a Partial Differential Equation Model of Ascending Mining Cable Elevator”. In: *Journal of Dynamic Systems, Measurement, and Control* 140.11 (2018), p. 111003.

[84] W. D. Zhu, J. Ni, and J. Huang. “Active Control of Translating Media With Arbitrarily Varying Length”. In: *Journal of Vibration and Acoustics* 123.3 (2001), pp. 347–358.

[85] W. He, S. S. Ge, and D. Huang. “Modeling and Vibration Control for a Nonlinear Moving String With Output Constraint”. In: *IEEE/ASME Transactions on Mechatronics* 20.4 (2015), pp. 1886–1897.

[86] W. He et al. “Modeling and Vibration Control for a Moving Beam With Application in a Drilling Riser”. In: *IEEE Transactions on Control Systems Technology* 25.3 (2016), pp. 1036–1043.

[87] M. Krstic. “Delay Compensation for Nonlinear, Adaptive, and PDE Systems”. In: *American Journal of Medicine* 124.1 (2009), pp. 20–25.

[88] R. A. Adams and J. F. Fournier. *Sobolev spaces*. Vol. 2nd edn. Amsterdam: Elsevier/Academic Press, 2003.

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List of Publications

1. **J. Wang, W. T. van Horssen & J. M. Wang**, *On resonances in transversally vibrating strings induced by an external force and a time-dependent coefficient in a Robin boundary condition*, Journal of Sound and Vibration, 512, 116356 (2021).

2. **J. Wang & W. T. van Horssen**, *Analysis of longitudinal oscillations in a vertically moving cable subject to nonclassical boundary conditions*, Applied Mathematical Modelling, 111, 44-62 (2022).

3. **J. Wang & J. M. Wang**, *Output feedback stabilization of an axially moving string subject to a spring-mass-dashpot*, International Journal of Control (Published online: 14 Jun 2022).

4. **J. Wang & W. T. van Horssen**, *On transverse and longitudinal oscillations and resonances in a hoisting system due to boundary excitations, (submitted).*