Integrability properties of quasi-regular representations of N A groups

van Velthoven, Jordy Timo

DOI
10.5802/crmath.372

Publication date
2022

Document Version
Final published version

Published in
Comptes Rendus Mathematique

INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Jordy Timo van Velthoven

Integrability properties of quasi-regular representations of NA groups

Volume 360 (2022), p. 1125-1134

https://doi.org/10.5802/crmath.372

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2022, Vol. 360, p. 1125-1134
https://doi.org/10.5802/crmath.372

Harmonic analysis / *Analyse harmonique*

Integrability properties of quasi-regular representations of NA groups

Jordy Timo van Velthoven [a]

[a] Delft University of Technology, Mekelweg 4, Building 36, 2628 CD Delft, The Netherlands
E-mail: j.t.vanvelthoven@tudelft.nl

Abstract. Let $G = N \rtimes A$, where N is a graded Lie group and $A = \mathbb{R}^+$ acts on N via homogeneous dilations. The quasi-regular representation $\pi = \operatorname{ind}_A^G(1)$ of G can be realised to act on $L^2(N)$. It is shown that for a class of analysing vectors the associated wavelet transform defines an isometry from $L^2(N)$ into $L^2(G)$ and that the integral kernel of the corresponding orthogonal projector has polynomial off-diagonal decay. The obtained reproducing formula is instrumental for obtaining decomposition theorems for function spaces on nilpotent groups.

2020 Mathematics Subject Classification. 22E15, 22E27, 43A80, 44A35.

Funding. J.v.V. gratefully acknowledges support from the Research Foundation - Flanders (FWO) Odysseus 1 grant G.0H94.18N and the Austrian Science Fund (FWF) project J-4445.

Manuscript received 7 January 2022, accepted 27 April 2022.

1. Introduction

Let N be a connected, simply connected nilpotent Lie group and let $A = \mathbb{R}^+$ act on N via automorphic dilations. The semi-direct product $G = N \rtimes A$ acts unitarily on $L^2(N)$ via the quasi-regular representation $\pi = \operatorname{ind}_A^G(1)$ of G. For $g \in L^2(N)$, the associated wavelet transform $V_g : L^2(N) \to L^\infty(G)$ is defined as

$$V_g f(x,t) = \langle f, \pi(x,t)g \rangle, \quad (x,t) \in G.$$

A vector $g \in L^2(N)$ is said to be *admissible* if V_g is an isometry from $L^2(N)$ into $L^2(G)$.

Given an admissible vector $g \in L^2(N)$, the orthogonal projector P from $L^2(G)$ onto the closed subspace $V_g(L^2(N)) \subset L^2(G)$ is given by right convolution $P(F) = F * V_g g$. In particular, an element $F \in V_g(L^2(N))$, i.e., $F = V_g f$ for some $f \in L^2(N)$, satisfies the reproducing formula

$$V_g f = V_g f * V_g g. \tag{1}$$

The existence of admissible vectors for irreducible, square-integrable representations π is automatic by the orthogonality relations [10], but a non-trivial problem for reducible representations. For $N = \mathbb{R}^d$ and general dilation groups $A \leq \mathrm{GL}(d,\mathbb{R})$, the admissibility of quasi-regular representations is well-studied, see, e.g. [2, 20, 34] and the references therein. For non-commutative groups N, the admissibility problem is considered in, e.g. [7, 9, 19, 37].

This note is concerned with admissible vectors that are also integrable: A vector $g \in L^2(N)$ is said to be *integrable* if $\Delta_G^{-1/2} V_g g \in L^1(G)$, where $\Delta_G : G \to \mathbb{R}^+$ denotes the modular function on G. The significance of integrably admissible vectors is that $F := \Delta_G^{-1/2} V_g g$ forms a *projection* in $L^1(G)$ by (1), that is, $F = F * F = F^*$, with $F^* := \Delta_G^{-1} \overline{F(\cdot^{-1})}$.

The construction of projections in $L^1(G)$ arising from matrix coefficients is an ongoing research topic, and such projections provide (if they exist) a powerful tool for studying problems in non-commutative harmonic analysis. Among others, they play a vital role in the theory of atomic decompositions in Banach spaces [12, 27].

For the affine group $G = \mathbb{R} \rtimes \mathbb{R}^+$, the construction of projections in $L^1(G)$ goes back to [11]. The papers [8, 28, 32] consider groups $G = \mathbb{R}^d \rtimes A$ and provide criteria for the explicit construction of projections in $L^1(G)$ based on the dual action of A on $\mathbb{R}^d$; see also [21, 23]. The techniques of [28, 32] were used in [40] for the Heisenberg group $N = \mathbb{H}_1$ acted upon by automorphic dilations. For a stratified group N with canonical dilations, the existence of smooth admissible vectors was investigated in [25], although not linked to integrability.

The main concern of this note is the integrability of $\pi = \mathrm{ind}_A^{N \rtimes A}$ when N is a (possibly, non-stratified) graded Lie group. The main result obtained is the following:

Theorem 1. *Let $G = N \rtimes A$, where N is a graded Lie group and $A = \mathbb{R}^+$ acts on N via automorphic dilations. The quasi-regular representation $\pi = \mathrm{ind}_A^G(1)$ admits integrably admissible vectors, i.e., there exist vectors $g \in L^2(N)$ satisfying $\Delta_G^{-1/2} V_g g \in L^1(G)$ and*

$$\int_G |\langle f, \pi(x,t) g\rangle|^2 \, d\mu_G(x,t) = \|f\|_2^2, \qquad \text{for all } f \in L^2(N).$$

The integrably admissible vector g can be chosen to be Schwartz with all moments vanishing, in which case $V_g g \in L^1_w(G)$ for any polynomially bounded weight $w : G \to [1, \infty)$.

Admissible vectors that are Schwartz with all vanishing moments are known to exist already for stratified Lie groups [25, Corollary 1]. Theorem 1 provides a modest extension of this result to general graded Lie groups, and complements it with integrability properties of the associated matrix coefficients. More explicit (point-wise) localisation estimates for the matrix coefficients on homogeneous groups are also obtained; see Section 3 below for details.

The proof method for Theorem 1 resembles the construction of Littlewood–Paley functions and Calderón-type reproducing formulae. Most techniques can already be found in some antecedent form in [17] as pointed out throughout the text. Particular use is made of the (non-stratified) Taylor inequality and Hulanicki's theorem for Rockland operators. The use of a Rockland operator instead of a sub-Laplacian is essential for the proof method as the latter are no longer always homogeneous for non-stratified groups. The exploitation of homogeneity is the reason that the strategy fails for non-graded homogeneous groups (see Remark 8).

The motivation for Theorem 1 stems from the study of function spaces, and is twofold:

(i) The question whether there exist vectors yielding a reproducing kernel with suitable off-diagonal decay on homogeneous groups was posed in [27, Remark 6.6 (a)], where it was mentioned that this is a representation-theoretic problem rather than one of function spaces. The use of such vectors for function space theory, however, is due to the fact that the techniques [27] yield frames and atomic decompositions for Besov–Triebel–Lizorkin spaces. The same holds true for the recent sampling theorems in [38]. The admissible vectors provided by Theorem 1 satisfy the integrability conditions assumed in [27, 38] (see Section 3.3), and Theorem 1 solves the problem mentioned in [27, Remark 6.6 (a)] for graded Lie groups.

(ii) The differentiability properties of functions in terms of Banach spaces are well-studied on stratified Lie groups for several classes of spaces, including Lipschitz spaces [16, 33], Sobolev spaces [15, 39], Besov spaces [6, 22, 39] and Triebel–Lizorkin spaces [17, 30]. More recently, there has been an interest in such spaces on possibly non-stratified graded Lie groups, see, e.g. [1,3,5,14]. This was a motivation to obtain Theorem 1 for graded groups, as it allows to apply the techniques [27,38] discussed in (i) to these new classes of spaces. Moreover, even for stratified groups, the integrability properties provided by Theorem 1 allow to apply the techniques [38] and bridge a gap between what has been established on the locality of the sampling expansions for stratified groups in [6, 22, 25, 27] and for the classical setting $N = \mathbb{R}^d$ in [18, 26]; see [26, 38] for more details on the discrepancy between [27] and [18,26,38].

The details on the applications of Theorem 1 to various functional spaces are beyond the scope of the present paper, and will be deferred to subsequent work.

Notation

The open and closed positive half-lines in $\mathbb{R}$ are denoted by $\mathbb{R}^+ = (0,\infty)$ and $\mathbb{R}^+_0 = [0,\infty)$, respectively. For functions $f_1, f_2 : X \to \mathbb{R}^+_0$, it is written $f_1 \lesssim f_2$ if there exists a constant $C > 0$ such that $f_1(x) \le C f_2(x)$ for all $x \in X$. The space of smooth functions on a Lie group G is denoted by $C^\infty(G)$ and the space of test functions by $C^\infty_c(G)$.

2. Preliminaries on homogeneous Lie groups

This section provides background on homogeneous groups. Standard references for the theory are the books [13, 17].

2.1. *Dilations*

Let $\mathfrak{n}$ be a real d-dimensional Lie algebra. A *family of dilations* on $\mathfrak{n}$ is a one-parameter family $\{D_t\}_{t>0}$ of automorphisms $D_t : \mathfrak{n} \to \mathfrak{n}$ of the form $D_t := \exp(A \ln t)$, where $A : \mathfrak{n} \to \mathfrak{n}$ is a diagonalisable linear map with positive eigenvalues $v_1, \ldots, v_d$. If a Lie algebra $\mathfrak{n}$ is endowed with a family of dilations, then it is nilpotent.

A *homogeneous group* is a connected, simply connected nilpotent Lie group N whose Lie algebra $\mathfrak{n}$ admits a family of dilations. The number $Q := v_1 + \cdots + v_d$ is the *homogeneous dimension* of N. The exponential map $\exp_N : \mathfrak{n} \to N$ is a diffeomorphism, providing a global coordinate system on N. Dilations $\{D_t\}_{t>0}$ can be transported to a one-parameter group of automorphisms of N, which will be denoted by $\{\delta_t\}_{t>0}$. The associated action of $t \in \mathbb{R}^+$ on $x \in N$ will often simply be written as $tx = \delta_t(x)$.

A *graded group* is a connected, simply connected nilpotent Lie group N whose Lie algebra $\mathfrak{n}$ admits an $\mathbb{N}$-gradation $\mathfrak{n} = \bigoplus_{j=1}^\infty \mathfrak{n}_j$, where $\mathfrak{n}_j$, $j = 1,2,\ldots$, are vector subspaces of $\mathfrak{n}$, almost all equal to $\{0\}$, and satisfying $[\mathfrak{n}_j, \mathfrak{n}_{j'}] \subset \mathfrak{n}_{j+j'}$ for $j, j' \in \mathbb{N}$. If, in addition, $\mathfrak{n}_1$ generates $\mathfrak{n}$, the group N is *stratified*. Canonical dilations $D_t : \mathfrak{n} \to \mathfrak{n}$, $t > 0$, can be defined through a gradation as $D_t(X) = t^j X$ for $X \in \mathfrak{n}_j$, $j \in \mathbb{N}$.

Henceforth, a homogeneous group N will be fixed with dilations $D_t := \exp(A \ln t)$. Haar measure will be denoted by μ_N. The eigenvalues $v_1, \ldots, v_d$ of A will be listed in increasing order and it will be assumed (without loss of generality) that $v_1 \ge 1$. In addition, a basis $X_1, \ldots, X_d$ of $\mathfrak{n}$ such that $AX_j = v_j X_j$ for $j = 1, \ldots, d$ will be fixed throughout.

2.2. *Homogeneity*

A function $f : N \to \mathbb{C}$ is called ν-*homogeneous* ($\nu \in \mathbb{C}$) if $f \circ \delta_t = t^\nu f$ for $t > 0$. For all measurable functions $f_1, f_2 : N \to \mathbb{C}$,

$$\int_N f_1(x)(f_2 \circ \delta_t)(x) \, \mathrm{d}\mu_N(x) = t^{-Q} \int_N (f_1 \circ \delta_{1/t})(x) f_2(x) \, \mathrm{d}\mu_N(x)$$

provided the integral is convergent. The map $f \mapsto f \circ \delta_t$ is naturally extended to distributions.

A linear operator $T : C_c^\infty(N) \to (C_c^\infty(N))'$ is said to be homogeneous of degree $\nu \in \mathbb{C}$ if $T(f \circ \delta_t) = t^\nu (Tf) \circ \delta_t$ for all $f \in C_c^\infty(N)$ and $t > 0$.

A *homogeneous quasi-norm* on N is a continuous function $|\cdot| : N \to [0,\infty)$ that is symmetric, 1-homogeneous and definite. If $|\cdot|$ is a homogeneous quasi-norm on N, there is a constant $C > 0$ such that $|xy| \leq C(|x| + |y|)$ for all $x, y \in N$.

2.3. *Derivatives and polynomials*

A basis element $X_j \in \mathfrak{n}$ acts as a left-invariant vector field on $\mathfrak{n}$ by

$$X_j f(x) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f(x \exp_N(sX_j))$$

for $f \in C^\infty(N)$ and $x \in N$. The first-order left-invariant differential operator X_j is homogeneous of degree v_j. For a multi-index $\alpha \in \mathbb{N}_0^d$, higher-order differential operators are defined by $X^\alpha := X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_d^{\alpha_d}$. The algebra of all left-invariant differential operators on N is denoted by $\mathscr{D}(N)$.

A function $P : N \to \mathbb{C}$ is a *polynomial* if $P \circ \exp_N$ is a polynomial on $\mathfrak{n}$. Denoting by $\xi_1, \ldots, \xi_d$ a dual basis of $X_1, \ldots, X_d$, the system $\eta_j = \xi_j \circ \exp_N^{-1}$, $j = 1, \ldots, d$, forms a global coordinate system on N. Each $\eta_j : N \to \mathbb{C}$ forms a polynomial on N, and any polynomial P on N can be written uniquely as

$$P = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \eta^\alpha, \tag{2}$$

where all but finitely many $c_\alpha \in \mathbb{C}$ vanish and $\eta^\alpha := \eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_d^{\alpha_d}$ for a multi-index $\alpha \in \mathbb{N}_0^d$. The homogeneous degree of $\alpha \in \mathbb{N}_0^d$ is defined as $[\alpha] := v_1 \alpha + \cdots + v_d \alpha_d$ and the homogeneous degree of a polynomial P written as (2) is $d(P) := \max\{[\alpha] : \alpha \in \mathbb{N}_0^d \text{ with } c_\alpha \neq 0\}$.

For any $k \geq 0$, the set of polynomials P on N such that $d(P) \leq k$ is denoted by $\mathscr{P}_k$.

2.4. *Schwartz space*

A function $f : N \to \mathbb{C}$ belongs to the Schwartz space $\mathscr{S}(N)$ if $f \circ \exp_N$ is a Schwartz function on $\mathfrak{n}$. A family of semi-norms on $\mathscr{S}(N)$ is given by

$$\|f\|_{\mathscr{S},K} = \sup_{|\alpha| \leq K, x \in N} (1 + |x|)^K |X^\alpha f(x)|, \quad K \in \mathbb{N}_0.$$

For simplicity, the parameter K is sometimes suppressed from the notation $\|\cdot\|_{\mathscr{S},K}$ and it is simply written $\|\cdot\|_{\mathscr{S}}$. The closed subspace of $\mathscr{S}(N)$ of functions with all moments vanishing is defined by

$$\mathscr{S}_0(N) = \left\{ f \in \mathscr{S}(N) : \int_N x^\alpha f(x) \mathrm{d}\mu_N(x) = 0, \quad \forall\, \alpha \in \mathbb{N}_0^d \right\}.$$

For arbitrary $f \in \mathscr{S}(N)$, it will be written $\check{f}(x) := \overline{f(x^{-1})}$ and $f_t(x) := t^{-Q} f(t^{-1}x)$ for $t > 0$.

The dual space $\mathscr{S}'(N)$ of $\mathscr{S}(N)$ is the space of tempered distributions on N. If $f \in \mathscr{S}'(N)$ and $\varphi \in \mathscr{S}(N)$, the conjugate-linear evaluation is denoted by $\langle f, \varphi \rangle$. If well-defined, the evaluation is also written as $\langle f, \varphi \rangle = \int_N f(x) \overline{\varphi(x)}\, \mathrm{d}\mu_N(x)$ and extends the L^2-inner product. Convolution is defined by $f * \varphi(x) := \langle f, \check{\varphi}(x^{-1} \cdot) \rangle$ and $\varphi * f(x) := \langle f, \check{\varphi}(\cdot x^{-1}) \rangle$ for $x \in N$.

3. Matrix coefficients of quasi-regular representations

This section is devoted to point-wise estimates and integability properties of the matrix coefficients of a quasi-regular representation.

3.1. *Quasi-regular representation*

Let N be a homogeneous Lie group and let $A = \mathbb{R}^+$ be the multiplicative group. Then A acts on N via automorphic dilations $A \ni t \mapsto \delta_t \in \mathrm{Aut}(N)$. The semi-direct product $G = N \rtimes A$ is defined via the operations

$$(x,t)(y,u) = (x\delta_t(y), tu), \quad (x,t)^{-1} = (\delta_{t^{-1}}(x^{-1}), t^{-1}).$$

Identity element in G is $e_G = (e_N, 1)$. The group G is an exponential Lie group, that is, the exponential map $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism, see, e.g. [19, Proposition 5.27].

The quasi-regular representation $\pi = \mathrm{ind}_A^G(1)$ of G acts unitarily on $L^2(N)$ by

$$\pi(x,t)f = t^{-Q/2} f(t^{-1}(x^{-1}\cdot)), \quad (x,t) \in N \times A,$$

for $f \in L^2(N)$. Note that $\pi(x,t) = L_x D_t$, where $L_x f = f(x^{-1}\cdot)$ and $D_t f = t^{-Q/2} f(t^{-1}(\cdot))$.

A detailed account on the representation theory of quasi-regular representations of exponential groups can be found in [7, 35, 37], but these results will not be used in this paper.

3.2. *Point-wise estimates*

For $f_1, f_2 \in L^2(N)$, denote the associated matrix coefficient by

$$V_{f_2} f_1(x,t) = \langle f_1, \pi(x,t) f_2 \rangle, \quad (x,t) \in N \rtimes A.$$

The following result provides point-wise estimates for a class of matrix coefficients.

Proposition 2. *Let $f_1, f_2 \in \mathscr{S}_0(N)$ and $K, M \in \mathbb{N}$ be arbitrary.*

(i) *For all $(x,t) \in N \rtimes A$ with $t \le 1$,*

$$|V_{f_2} f_1(x,t)| \lesssim t^{Q/2+M} (1+|x|)^{-K} \|f_1\|_{\mathscr{S}} \|f_2\|_{\mathscr{S}}. \tag{3}$$

(ii) *For all $(x,t) \in N \rtimes A$ with $t \ge 1$,*

$$|V_{f_2} f_1(x,t)| \lesssim t^{-(Q/2+M)} (1+|x|)^{-K} \|f_1\|_{\mathscr{S}} \|f_2\|_{\mathscr{S}}. \tag{4}$$

The implicit constants in (3) *and* (4) *are group constants that depend further only on M, K.*

Proof. Throughout the proof, a Schwartz semi-norm $\|\cdot\|_{\mathscr{S},N}$ is simply denoted by $\|\cdot\|_N$.

Let $K, M \in \mathbb{N}$ and let $P = P_{x,M} \in \mathscr{P}_M$ denote the Taylor polynomial of $f \in \mathscr{S}(N)$ at $x \in N$ of homogeneous degree M. By Taylor's inequality [13, Theorem 3.1.51], there exist constants $c, C > 0$ such that for all $x, y \in N$,

$$|f(xy) - P(y)| \le C \sum_{\substack{|\alpha| \le M'+1 \\ [\alpha] > M}} |y|^{[\alpha]} \sup_{|z| \le c^{M'+1}|y|} |(X^\alpha f)(xz)|,$$

where $M' := \max\{|\alpha| : \alpha \in \mathbb{N}_0^d \text{ with } [\alpha] \le M\}$. For $|\alpha| \le M'+1$ and $x, y \in N$,

$$\begin{aligned}
\sup_{|z| \le c^{M'+1}|y|} |(X^\alpha f)(xz)| &\le \|f\|_{K+M'+1} \sup_{|z| \le c^{M'+1}|y|} (1+|xz|)^{-K} \\
&\lesssim \|f\|_{K+M'+1} \sup_{|z| \le c^{M'+1}|y|} (1+|x|)^{-K}(1+|z|)^K \\
&\lesssim \|f\|_{K+M'+1} (1+|x|)^{-K}(1+|y|)^K,
\end{aligned}$$

where the second line follows from the Peetre-type inequality [17, Lemma 1.10]. Thus,

$$|f(xy) - P(y)| \lesssim \|f\|_{K+M'+1} (1+|x|)^{-K} \sum_{\substack{|\alpha| \le M'+1 \\ [\alpha] > M}} |y|^{[\alpha]} (1+|y|)^K \tag{5}$$

for all $x, y \in N$.

(i) Let $(x,t) \in N \rtimes A$ with $t \le 1$. Then, using that $f_2 \in \mathscr{S}_0(N)$,

$$|V_{f_2} f_1(x,t)| = \left| \int_N f_1(xy) D_t \check{f}_2(y^{-1}) \, \mathrm{d}\mu_N(y) \right| \le \int_N |f_1(xy) - P(y)| \left| D_t \check{f}_2(y^{-1}) \right| \mathrm{d}\mu_N(y).$$

Applying (5) thus gives

$$\begin{aligned} |V_{f_2} f_1(x,t)| &\lesssim \|f_1\|_{K+M'+1} (1+|x|)^{-K} t^{-Q/2} \sum_{\substack{|\alpha| \le M'+1 \\ [\alpha] > M}} \int_N |y|^{[\alpha]} |\check{f}_2(t^{-1} y^{-1})| (1+|y|)^K \, \mathrm{d}\mu_N(y) \\ &= \|f_1\|_{K+M'+1} (1+|x|)^{-K} t^{Q/2} \sum_{\substack{|\alpha| \le M'+1 \\ [\alpha] > M}} \int_N |ty|^{[\alpha]} |\check{f}_2(y^{-1})| (1+|ty|)^K \, \mathrm{d}\mu_N(y) \\ &\lesssim \|f_1\|_{K+M'+1} (1+|x|)^{-K} t^{Q/2+M} \int_N |f_2(y)| (1+|y|)^{K+Q(M'+1)} \, \mathrm{d}\mu_N(y), \end{aligned} \tag{6}$$

where the last inequality uses $[\alpha] \le Q|\alpha| \le Q(M'+1)$. The integral in (6) can be estimated by

$$\begin{aligned} \int_N |f_2(y)| (1+|y|)^{K+Q(M'+1)} \, \mathrm{d}\mu_N(y) &\le \|f_2\|_{K+Q(M'+1)+Q+1} \int_N (1+|y|)^{-Q-1} \, \mathrm{d}\mu_N(y) \\ &\lesssim \|f_2\|_{K+Q(M'+1)+Q+1}, \end{aligned} \tag{7}$$

where convergence of the integral follows by using polar coordinates [17, Proposition 1.15]; see also [17, Corollary 1.17]. A combination of (7) and (6) yields the desired claim (3).

(ii) Note that $|V_{f_2} f_1(x,t)| = |V_{f_1} f_2((x,t)^{-1})|$ for $(x,t) \in N \rtimes A$. Hence, if $t \ge 1$, then it follows by part (i) with $M_0 := M + K$ that

$$\begin{aligned} |V_{f_2} f_1(x,t)| &\lesssim t^{-(Q/2+M_0)} (1+t^{-1}|x|)^{-K} \|f_1\|_{K+M'_0+1} \|f_2\|_{K+Q(M'_0+1)+Q+1} \\ &\le t^{-Q/2-M} t^{-K} t^K (1+|x|)^{-K} \|f_1\|_{K+M'_0+1} \|f_2\|_{K+Q(M'_0+1)+Q+1}, \end{aligned}$$

showing (4). This completes the proof. □

The estimates provided by Proposition 2 recover the well-known polynomial localisation for wavelet transforms when $N = \mathbb{R}$, see, e.g. [29, Section 11-12]. A similar use of the Taylor inequality for (compactly supported) atoms can be found in [17, Theorem 2.9].

3.3. *Analysing vectors*

Left Haar measure on G is given by $\mu_G(x,t) = t^{-(Q+1)} \mathrm{d}\mu_N(x) \mathrm{d}t$ and the modular function is given by $\Delta_G(x,t) = t^{-Q}$. The measure μ_G is used to define the Lebesgue space $L^p(G) = L^p(G, \mu_G)$ for $p \in [1,\infty]$, and $\|\cdot\|_p$ will denote the p-norm.

A measurable function $w : G \to [1,\infty)$ is said to be a *weight* if it is submultiplicative, i.e., $w((x,t)(y,u)) \lesssim w(x,t) w(y,u)$ for $(x,t),(y,u) \in G$. A weight w is called *polynomially bounded* if

$$w(x,t) \lesssim (1+|x|)^k (t^m + t^{-m'}), \quad (x,t) \in G, \tag{8}$$

for some $k, m, m' \ge 0$. Given such a weight w, the weighted Lebesgue space $L^1_w(G)$ consists of all $F \in L^1(G)$ satisfying $\|F\|_{L^1_w} := \|Fw\|_1 < \infty$.

In [12, 27, 38], the space of *w-analysing vectors* of π, defined by

$$\mathscr{A}_w := \left\{ g \in L^2(N) : V_g g \in L^1_w(G) \right\},$$

plays a prominent role.

The following result provides a simple criterion for analysing vectors:

Lemma 3. *Suppose $g \in \mathscr{S}_0(N)$. Then $g \in \mathscr{A}_w$ for any polynomially bounded weight function $w : G \to [1,\infty)$. In particular, the representation $\pi = \mathrm{ind}_A^G(1)$ is integrable.*

Proof. Let $k, m, m' \geq 0$ be such that $w(x,t) \lesssim (1+|x|)^k (t^m + t^{-m'})$ for all $(x,t) \in G$. Then, choosing $K, M, M' \in \mathbb{N}$ sufficiently large, it follows by Proposition 2 that

$$\begin{aligned}\|V_g g\|_{L^1_w} &\lesssim \int_0^\infty \int_N V_g g(x,t)(1+|x|)^k (t^m + t^{-m'})\, \mathrm{d}\mu_N(x) \frac{\mathrm{d}t}{t^{Q+1}} \\ &\lesssim \int_0^1 t^{Q/2+M'-m'} t^{-(Q+1)} \mathrm{d}t + \int_1^\infty t^{-(Q/2+M)+m} t^{-(Q+1)} \mathrm{d}t < \infty.\end{aligned}$$

This shows that $g \in \mathscr{A}_w$, and thus π is w-integrable. □

4. Admissible vectors

A vector $g \in L^2(N)$ is said to be *admissible* for the quasi-regular representation $(\pi, L^2(N))$ if the map

$$V_g : L^2(N) \to L^\infty(G), \quad f \mapsto \langle f, \pi(\cdot) g \rangle$$

is an isometry into $L^2(G)$.

4.1. *Reproducing formulae*

The following observation relates admissibility to a Calderón-type reproducing formula.

Lemma 4. *Let $g \in \mathscr{S}(N)$ with $\int_N g(x)\, \mathrm{d}\mu_N(x) = 0$. Then g is admissible if, and only if,*

$$f = \int_0^\infty f * \check{g}_t * g_t \frac{\mathrm{d}t}{t} \equiv \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho f * \check{g}_t * g_t \frac{\mathrm{d}t}{t}, \quad f \in \mathscr{S}(N), \tag{9}$$

with convergence in $\mathscr{S}'(N)$.

Proof. Under the assumptions on g, it follows by [17, Theorem 1.65] that

$$H_{\varepsilon,\rho}(z) := \int_\varepsilon^\rho \check{g}_t * g_t(z) \frac{\mathrm{d}t}{t}, \quad z \in N,$$

converges in $\mathscr{S}'(N)$ to a distribution $H := \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} H_{\varepsilon,\rho}$ which is smooth on $N \setminus \{e_N\}$ and homogeneous of degree $-Q$. Let $f \in \mathscr{S}(N)$. Then

$$\begin{aligned}\|V_g f\|_2^2 &= \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho \int_N |f * D_t \check{g}(x)|^2\, \mathrm{d}\mu_G(x,t) \\ &= \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho \int_N \int_N \int_N f(y) \check{g}_t(y^{-1}x) \overline{\check{g}_t(z^{-1}x) f(z)}\, \mathrm{d}\mu_N(z) \mathrm{d}\mu_N(y) \mathrm{d}\mu_N(x) \frac{\mathrm{d}t}{t} \\ &= \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho \int_N \int_N f(y) \check{g}_t * g_t(y^{-1}z) \overline{f(z)}\, \mathrm{d}\mu_N(y) \mathrm{d}\mu_N(z) \frac{\mathrm{d}t}{t} \\ &= \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_N f * H_{\varepsilon,\rho}(z) \overline{f(z)}\, \mathrm{d}\mu_N(z) \\ &= \int_N f * H(z) \overline{f(z)}\, \mathrm{d}\mu_N(z),\end{aligned}$$

where the last equality used that $f * H_{\varepsilon,\rho} \to f * H$ in $\mathscr{S}'(N)$ as $\varepsilon \to 0$ and $\rho \to \infty$.

The map $f \mapsto f * H$ is bounded on $L^2(N)$ by [17, Theorem 6.19]. Hence $V_g : \mathscr{S}(N) \to L^2(G)$ is well-defined, and it follows that

$$\int_G |\langle f, \pi(x,t) g \rangle|^2\, \mathrm{d}\mu_G(x,t) = \langle f * H, f \rangle, \quad f \in L^2(N). \tag{10}$$

Thus g is admissible if, and only if, $\langle f * H, f \rangle = \langle f, f \rangle$ for all $f \in L^2(N)$. Polarisation yields that this is equivalent to (9), which completes the proof. □

The calculations in the proof of Lemma 4 are classical, see, e.g. [17, Theorem 7.7].

4.2. *Rockland operators*

This section provides background on spectral multipliers for Rockland operators, see, e.g. [13, Chapter 4] for a detailed account. The stated results will be used in Section 4.3 below for the construction of admissible vectors.

Let $\mathscr{L} \in \mathscr{D}(N)$ be positive and formally self-adjoint. Then $\mathscr{L}$ is essentially self-adjoint on $L^2(N)$, and $\mathscr{L}$ will also denote its self-adjoint extension. Let $E_{\mathscr{L}}$ be the spectral measure of $\mathscr{L}$. For $m \in L^\infty(\mathbb{R}_0^+)$, the operator

$$m(\mathscr{L}) := \int_{\mathbb{R}_0^+} m(\lambda)\, \mathrm{d}E_{\mathscr{L}}(\lambda)$$

is a left-invariant bounded linear operator on $L^2(N)$. By the Schwartz kernel theorem, the action of $m(\mathscr{L})$ on $\mathscr{S}(N)$ is given by

$$m(\mathscr{L})f = f * K_{m(\mathscr{L})}, \quad f \in \mathscr{S}(N),$$

where $K_{m(\mathscr{L})} \in \mathscr{S}'(N)$ is the associated convolution kernel.

A *Rockland operator* is a homogeneous differential operator $\mathscr{L} \in \mathscr{D}(N)$ of positive degree that is hypoelliptic, i.e. for every distribution $f \in (C_c^\infty(N))'$ and every open set $U \subseteq N$, the condition $(\mathscr{L}f)|_U \in C^\infty(U)$ implies that $f|_U \in C^\infty(U)$. Positive Rockland operators are well-known to exist on any graded Lie group.

The following theorem is the key result used to construct admissible Schwartz functions.

Theorem 5 (Hulanicki [31]). *Let N be a graded Lie group. Let $\mathscr{L} \in \mathscr{D}(N)$ be a positive Rockland operator and let $|\cdot| : N \to [0,\infty)$ be a fixed homogeneous quasi-norm on N.*

For any $M_1 \in \mathbb{N}$, $M_2 \geq 0$, there exist $C = C(M_1, M_2) > 0$ and $k = k(M_1, M_2), k' = k'(M_1, M_2) \in \mathbb{N}_0$ such that, for any $m \in C^k(\mathbb{R}_0^+)$, the kernel $K_{m(\mathscr{L})}$ of $m(\mathscr{L})$ satisfies

$$\sum_{[\alpha] \leq M_1} \int_G |X^\alpha K_{m(\mathscr{L})}(x)|(1+|x|)^{M_2}\, \mathrm{d}\mu_N(x) \leq C \sup_{\substack{\lambda>0 \\ \ell=0,\dots,k \\ \ell'=0,\dots,k'}} (1+\lambda)^{\ell'} |\partial_\lambda^\ell m(\lambda)|.$$

Corollary 6. *Let $\mathscr{L} \in \mathscr{D}(N)$ be a positive Rockland operator.*

(i) *If $m \in \mathscr{S}(\mathbb{R}_0^+)$, then $K_{m(\mathscr{L})} \in \mathscr{S}(N)$.*
(ii) *If $m \in \mathscr{S}(\mathbb{R}_0^+)$ vanishes near the origin, then $K_{m(\mathscr{L})} \in \mathscr{S}_0(N)$.*

4.3. *Existence of admissible vectors*

The following result yields a class of Schwartz vectors that are admissible.

Proposition 7. *Let N be a graded Lie group and let $\mathscr{L} \in \mathscr{D}(N)$ be a positive Rockland operator of degree ν. Let $K_{m(\mathscr{L})}$ be the convolution kernel of a multiplier $m \in \mathscr{S}(\mathbb{R}_0^+)$ satisfying*

$$\int_0^\infty |m(t)|^2 \frac{\mathrm{d}t}{t} = \nu. \tag{11}$$

Then $g := K_{m(\mathscr{L})} \in \mathscr{S}(N)$ is an admissible vector for $\pi = \mathrm{ind}_A^{N \rtimes A}(1)$.

Proof. Let $m \in \mathscr{S}(\mathbb{R}_0^+)$ be as in the statement, so that

$$\int_0^\infty |m(\lambda t^\nu)|^2 \frac{\mathrm{d}t}{t} = \frac{1}{\nu} \int_0^\infty |m(t)|^2 \frac{\mathrm{d}t}{t} = 1, \quad \text{for all } \lambda > 0. \tag{12}$$

By Corollary 6, $g := K_{m(\mathscr{L})} \in \mathscr{S}(N)$, and it suffices to show the reproducing formula (9). Define $H_{\varepsilon,\rho} := \int_\varepsilon^\rho \check{g}_t * g_t \, t^{-1} dt$ for $0 < \varepsilon < \rho < \infty$. Let $f_1, f_2 \in \mathscr{S}(N)$. Then

$$\langle f_1 * H_{\varepsilon,\rho}, f_2 \rangle = \int_\varepsilon^\rho \langle f_1 * \check{g}_t * g_t, f_2 \rangle \frac{\mathrm{d}t}{t} = \int_\varepsilon^\rho \langle f_1 * (\check{g} * g)_t, f_2 \rangle \frac{\mathrm{d}t}{t}. \tag{13}$$

The spectral theorem implies that $\check{g} * g = K_{\overline{m}(\mathscr{L})} * K_{m(\mathscr{L})} = K_{|m|^2(\mathscr{L})}$. In addition, the homogeneity of $\mathscr{L}$ yields that $(\check{g} * g)_t = K_{|m|^2(t^\nu \mathscr{L})}$ for all $t > 0$, see, e.g. [13, Corollary 4.1.16]. Combining this with (13) gives

$$\begin{aligned} \langle f_1 * H_{\varepsilon,\rho}, f_2 \rangle &= \int_\varepsilon^\rho \left\langle |m|^2(t^\nu \mathscr{L}) f_1, f_2 \right\rangle \frac{\mathrm{d}t}{t} = \int_\varepsilon^\rho \int_0^\infty |m(t^\nu \lambda)|^2 \, \mathrm{d}\langle E_{\mathscr{L}}(\lambda) f_1, f_2 \rangle \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \int_\varepsilon^\rho |m(t^\nu \lambda)|^2 \frac{\mathrm{d}t}{t} \mathrm{d}\langle E_{\mathscr{L}}(\lambda) f_1, f_2 \rangle. \end{aligned}$$

Hence, by the identity (12),

$$\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \langle f_1 * H_{\varepsilon,\rho}, f_2 \rangle = \int_0^\infty \int_0^\infty |m(t^\nu \lambda)|^2 \frac{\mathrm{d}t}{t} \mathrm{d}\langle E_{\mathscr{L}}(\lambda) f_1, f_2 \rangle = \langle f_1, f_2 \rangle.$$

An application of Lemma 4 therefore yields that g is admissible. □

Spectral multipliers for sub-Laplacians on stratified groups were used for constructing admissible vectors in [25]. See also [24] for similar discrete Littlewood–Paley decompositions.

Remark 8. The use of a *homogeneous* operator is essential in the proof of Proposition 7 to guarantee that the spectral dilates $m(t\,\cdot)$, $t > 0$, of a multiplier $m \in \mathscr{S}(\mathbb{R}_0^+)$ yield a convolution kernel $K_{m(t\mathscr{L})}$ that is compatible with automorphic dilations $\{\delta_t\}_{t>0}$. For non-homogeneous operators, other techniques seem required, see, e.g. [4, 36].

4.4. *Proof of Theorem 1*

Theorem 1 follows from combining Lemma 3, Corollary 6 and Proposition 7.

Acknowledgment

Thanks are due to T. Bruno, M. Calzi and D. Rottensteiner for helpful discussions.

References

[1] H. Bahouri, C. Fermanian-Kammerer, I. Gallagher, "Refined inequalities on graded Lie groups", *C. R. Math. Acad. Sci. Paris* **350** (2012), no. 7-8, p. 393-397.

[2] J. Bruna, J. Cufí, H. Führ, M. L. Miró, "Characterizing abelian admissible groups", *J. Geom. Anal.* **25** (2015), no. 2, p. 1045-1074.

[3] T. Bruno, "Homogeneous algebras via heat kernel estimates", https://arxiv.org/abs/2102.11613, to appear in *Trans. Am. Math. Soc.*, 2021.

[4] M. Calzi, F. Ricci, "Functional calculus on non-homogeneous operators on nilpotent groups", *Ann. Mat. Pura Appl.* **200** (2021), no. 4, p. 1517-1571.

[5] D. Cardona, M. Ruzhansky, "Multipliers for Besov spaces on graded Lie groups.", *C. R. Math. Acad. Sci. Paris* **355** (2017), no. 4, p. 400-405.

[6] J. G. Christensen, A. Mayeli, G. Ólafsson, "Coorbit description and atomic decomposition of Besov spaces", *Numer. Funct. Anal. Optim.* **33** (2012), no. 7-9, p. 847-871.

[7] B. N. Currey, "Admissibility for a class of quasiregular representations", *Can. J. Math.* **59** (2007), no. 5, p. 917-942.

[8] B. N. Currey, H. Führ, K. F. Taylor, "Integrable wavelet transforms with abelian dilation groups", *J. Lie Theory* **26** (2016), no. 2, p. 567-595.

[9] B. N. Currey, V. Oussa, "Admissibility for monomial representations of exponential Lie groups", *J. Lie Theory* **22** (2012), no. 2, p. 481-487.

[10] M. Duflo, C. C. Moore, "On the regular representation of a nonunimodular locally compact group", *J. Funct. Anal.* **21** (1976), p. 209-243.

[11] P. Eymard, M. Terp, "La transformation de Fourier et son inverse sur le groupe des ax+b d'un corps local", in *Analyse harmonique sur les groupes de Lie II*, Lecture Notes in Mathematics, vol. 739, Springer, 1979, p. 207-248.

[12] H. G. Feichtinger, K. H. Gröchenig, "Banach spaces related to integrable group representations and their atomic decompositions. I", *J. Funct. Anal.* **86** (1989), no. 2, p. 307-340.

[13] V. Fischer, M. Ruzhansky, *Progress in Mathematics*, vol. 314, Birkhäuser/Springer, 2016, xiii+557 pages.

[14] ———, "Sobolev spaces on graded Lie groups", *Ann. Inst. Fourier* **67** (2017), no. 4, p. 1671-1723.

[15] G. B. Folland, "Subelliptic estimates and function spaces on nilpotent Lie groups", *Ark. Mat.* **13** (1975), p. 161-207.

[16] ———, "Lipschitz classes and Poisson integrals on stratified groups", *Stud. Math.* **66** (1979), p. 37-55.

[17] G. B. Folland, E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, 1982.

[18] M. Frazier, B. Jawerth, G. L. Weiss, *Littlewood-Paley theory and the study of function spaces*, Regional Conference Series in Mathematics, vol. 79, American Mathematical Society, 1991, vii+132 pages.

[19] H. Führ, *Abstract harmonic analysis of continuous wavelet transforms*, Lect. Notes Math., vol. 1863, Springer, 2005, x+193 pages.

[20] ———, "Generalized Calderón conditions and regular orbit spaces", *Colloq. Math.* **120** (2010), no. 1, p. 103-126.

[21] ———, "Coorbit spaces and wavelet coefficient decay over general dilation groups", *Trans. Am. Math. Soc.* **367** (2015), no. 10, p. 7373-7401.

[22] H. Führ, A. Mayeli, "Homogeneous Besov spaces on stratified Lie groups and their wavelet characterization", *J. Funct. Spaces Appl.* **2012** (2012), article no. 523586 (41 pages).

[23] H. Führ, J. T. van Velthoven, "Coorbit spaces associated to integrably admissible dilation groups", *J. Anal. Math.* **144** (2021), no. 1, p. 351-395.

[24] G. Furioli, C. Melzi, A. Veneruso, "Littlewood–Paley decompositions and Besov spaces on Lie groups of polynomial growth", *Math. Nachr.* **279** (2006), no. 9-10, p. 1028-1040.

[25] D. Geller, A. Mayeli, "Continuous wavelets and frames on stratified Lie groups. I", *J. Fourier Anal. Appl.* **12** (2006), no. 5, p. 543-579.

[26] J. E. Gilbert, Y. S. Han, J. A. Hogan, J. D. Lakey, D. Weiland, G. L. Weiss, *Smooth molecular decompositions of functions and singular integral operators*, Mem. Am. Math. Soc., vol. 742, American Mathematical Society, 2002, 74 pages.

[27] K. H. Gröchenig, "Describing functions: Atomic decompositions versus frames", *Monatsh. Math.* **112** (1991), no. 1, p. 1-42.

[28] K. H. Gröchenig, E. Kaniuth, K. F. Taylor, "Compact open sets in duals and projections in L^1-algebras of certain semi-direct product groups", *Math. Proc. Camb. Philos. Soc.* **111** (1992), no. 3, p. 545-556.

[29] M. Holschneider, *Wavelets. An analysis tool*, Oxford Math. Monogr., Clarendon Press, 1995, xiii+423 pages.

[30] G. Hu, "Homogeneous Triebel-Lizorkin spaces on stratified Lie groups", *J. Funct. Spaces Appl.* **2013** (2013), article no. 475103 (16 pages).

[31] A. Hulanicki, "A functional calculus for Rockland operators on nilpotent Lie groups", *Stud. Math.* **78** (1984), p. 253-266.

[32] E. Kaniuth, K. F. Taylor, "Minimal projections in L^1-algebras and open points in the dual spaces of semi-direct product groups", *J. Lond. Math. Soc.* **53** (1996), no. 1, p. 141-157.

[33] S. Krantz, "Lipschitz spaces on stratified groups", *Trans. Am. Math. Soc.* **269** (1982), p. 39-66.

[34] R. S. Laugesen, N. Weaver, G. L. Weiss, E. N. Wilson, "A characterization of the higher dimensional groups associated with continuous wavelets", *J. Geom. Anal.* **12** (2002), no. 1, p. 89-102.

[35] R. L. Lipsman, "Harmonic analysis on exponential solvable homogeneous spaces: The algebraic or symmetric cases", *Pac. J. Math.* **140** (1989), no. 1, p. 117-147.

[36] A. Nagel, F. Ricci, E. M. Stein, "Harmonic analysis and fundamental solutions on nilpotent Lie groups", in *Analysis and partial differential equations*, Lecture Notes in Pure and Applied Mathematics, vol. 122, Marcel Dekker, 1990, p. 249-275.

[37] V. Oussa, "Admissibility for quasiregular representations of exponential solvable Lie groups", *Colloq. Math.* **131** (2013), no. 2, p. 241-264.

[38] J. L. Romero, J. T. van Velthoven, F. Voigtlaender, "On dual molecules and convolution-dominated operators", *J. Funct. Anal.* **280** (2021), no. 10, article no. 108963 (57 pages).

[39] K. Saka, "Besov spaces and Sobolev spaces on a nilpotent Lie group", *Tôhoku Math. J.* **31** (1979), p. 383-437.

[40] E. Schulz, K. F. Taylor, "Extensions of the Heisenberg group and wavelet analysis in the plane", in *Spline functions and the theory of wavelets*, American Mathematical Society, 1999, p. 217-225.