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LIMIT THEOREMS FOR β-LAGUERRE AND β-JACOBI ENSEMBLES*

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Abstract We use tridiagonal models to study the limiting behavior of β-Laguerre and β-Jacobi ensembles, focusing on the limiting behavior of the extremal eigenvalues and the central limit theorem for the two ensembles. For the central limit theorem of β-Laguerre ensembles, we follow the idea in [1] while giving a modified version for the generalized case. Then we use the total variation distance between the two sorts of ensembles to obtain the limiting behavior of β-Jacobi ensembles.

Key words beta-ensembles; largest and smallest eigenvalues; central limit theorem; total variation distance

2010 MR Subject Classification 15B52; 60B20; 60F10

1 Introduction

A β-Laguerre ensemble, also called a Wishart ensemble, is a set of non-negative random variables $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \cdots, \lambda_n)$ with the joint density function

$$f_{\beta,a_1}(x_1, x_2, \cdots, x_n) = C_{\mathrm{L}}^{\beta,a_1} \prod_{1\le i<j\le n} |x_i - x_j|^{\beta} \prod_{i=1}^{n} x_i^{a_1-r} e^{-\frac{1}{2}x_i}, \tag{1.1}$$

where $a_1 > \frac{\beta}{2}(n-1), \beta > 0$, $r = 1 + \frac{\beta}{2}(n-1)$, and

$$C_L^{\beta,a_1} = 2^{-na_1} \prod_{j=1}^{n} \frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}j\right)\Gamma\left(a_1-\frac{\beta}{2}(n-j)\right)}.$$

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A β-Jacobi ensemble, also called the β-MANOVA ensemble, is a set of random variables $\boldsymbol{\mu} := (\mu_1, \mu_2, \cdots, \mu_n) \in [0,1]^n$ with a joint probability density function

$$f_{\beta,a_1,a_2}(x_1, x_2, \cdots, x_n) = C_J^{\beta,a_1,a_2} \prod_{1\le i<j\le n} |x_i - x_j|^\beta \prod_{i=1}^n x_i^{a_1-r}(1-x_i)^{a_2-r}, \tag{1.2}$$

where $a_1, a_2 > \frac{\beta}{2}(n-1)$ and $r := 1 + \frac{\beta}{2}(n-1)$, and

$$C_J^{\beta,a_1,a_2} = \prod_{j=1}^n \frac{\Gamma(1+\frac{\beta}{2})\Gamma(a_1+a_2-\beta(n-j)/2)}{\Gamma(1+\beta j/2)\Gamma(a_1-\beta(n-j)/2)\Gamma(a_2-\beta(n-j)/2)}.$$

In 2002, Dumitriu and Edelman proved that the β-Laguerre ensembles can be seen as the eigenvalues of real symmetric tridiagonal random matrices, which are distributed as $\boldsymbol{L}_{\beta,n}^{a_1}$ in Table 1 ([3]). The calculations in this note are all based on the tridiagonal model. In fact, in the same paper, Dumitriu and Eldeman also gave the random tridiagonal model of another sort of β-ensemble, the β-Hermite ensemble, which we are not going to discuss here. The tridiagonal model of the β-Jacobi ensembles was achieved in [7] and [10]. Jiang ([9]), Ma and Shen ([12]) used the tridiagonal model to calculate the distances between the two sorts of ensembles; this is quite useful in terms of understanding the limiting behavior of β-Jacobi ensembles through the β-Laguerre ensembles.

Based on the tridiagonal random matrices, Dumitriu and Edelman gave the central limit theorem for the β-Laguerre ensembles in [2]. Let $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ be the beta-Laguerre ensembles of parameter a_1 and size n, whose joint density function is given by (1.1). Assume that $\frac{n\beta}{2a_1} \to \gamma \in (0,1)$, and let $\gamma_{\min} = (1-\sqrt{\gamma})^2$, $\gamma_{\max} = (1+\sqrt{\gamma})^2$. For any $i \ge 1$, let

$$X_i = \sum_{j=1}^n \left(\frac{\gamma}{n\beta}\lambda_j\right)^i - n\sum_{r=0}^{i-1} \frac{C_i^r C_{i-1}^r}{r+1}\gamma^r - \left(\frac{2}{\beta}-1\right)\int_{\gamma_{\min}}^{\gamma_{\max}} t^i \mu_L^\gamma(t)\mathrm{d}t,$$

where

$$\mu_L^\gamma(x) := \begin{cases} \dfrac{1}{4}\delta_{\gamma_{\max}}(x) + \dfrac{1}{4}\delta_{\gamma_{\min}}(x) - \dfrac{1}{2\pi}\dfrac{1}{\sqrt{(x-\gamma_{\min})(\gamma_{\max}-x)}}; & \text{if } x \in [\gamma_{\min}, \gamma_{\max}], \\ 0, & \text{otherwise}. \end{cases} \tag{1.3}$$

Here $C_n^m := \frac{n!}{m!(n-m)!}$ is the combinatorial number, with $0 \le m \le n$. Then, for any integer $k \ge 1$, $(X_1, X_2, \cdots, X_k)$ converges weakly to a centered multivariate Guassian as $n \to \infty$.

Set $\lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_n\}, \lambda_{\min} = \min\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and assume that $\frac{n\beta}{2a_1} \to \gamma \in (0,1]$. Then

$$\frac{\lambda_{\max}}{n} \to \beta(1+\sqrt{\gamma^{-1}})^2 \qquad \text{and} \qquad \frac{\lambda_{\min}}{n} \to \beta(1-\sqrt{\gamma^{-1}})^2,$$

almost surely. This result was first reported without proof in [1], and then in [9], the proof was given by Jiang.

Let $(\mu_1, \mu_2, \cdots, \mu_n)$ be random variables with density function f_{β,a_1,a_2}, defined in (1.2). In [9], Jiang obtained similar results for extremal eigenvalues and the central limit theorem for empirical measures. Setting $\mu_{\max} = \max\{\mu_1, \mu_2, \cdots, \mu_n\}, \mu_{\min} = \min\{\mu_1, \mu_2, \cdots, \mu_n\}$, and assuming that $a_1 = o(\sqrt{a_2})$, $n = o(\sqrt{a_2})$ and $\frac{n\beta}{2a_1} \to \gamma \in (0,1]$, we then have that

$$\frac{a_2}{n}\mu_{\max} \to \frac{\beta(1+\sqrt{\gamma})^2}{2\gamma} \quad \text{and} \quad \frac{a_2}{n}\mu_{\min} \to \frac{\beta(1-\sqrt{\gamma})^2}{2\gamma}$$

in probability as $n \to +\infty$.

Given any integer $i \geq 1$, define

$$X_i = \sum_{j=1}^{n} \left(\frac{2\gamma a_2}{n\beta} \mu_j \right)^i - n \sum_{r=0}^{i-1} \frac{C_i^r C_{i-1}^r}{r+1} \gamma^r$$

for $i \geq 1$. Similarly, assume that $a_1 = o(\sqrt{a_2})$ and that $n = o(\sqrt{a_2})$, $(X_1, \cdots, X_k)$ converges weakly to a multivariate normal distribution for any $k \geq 1$.

A considerable amount of literature has been devoted to the eigenvalue distributions of beta ensembles. Dumitriu and Koev presented explicit formulas for the distributions of the extreme eigenvalues of the β-Jacobi random matrix ensemble in terms of the hypergeometric function of a matrix argument (see [3]), and Edelman and Koev presented the explicit expressions for β-Laguerre ensembles (see [4]). Dumitriu and Paquette studied global fluctuations for linear statistics of the β-Jacobi ensembles (see [5]). It is also worth mentioning that Killip studied the Jacobi ensembles and proved Gaussian fluctuations for the number of points in one or more intervals in the macroscopic scaling limit given in [6]. Trinh in [14] gave a unified way to offer the central limit theorems via spectral measures for beta ensembles and also investigated the Gaussian fluctuation around the limit.

The case $\gamma := \lim\limits_{n\to\infty} \frac{n\beta}{2a_1} = 0$ was excluded in [1, 2] and [9]. In this note, we will prove the limit for extremal eigenvalues when $\gamma = 0$ for both β-Laguerre and β-Jacobi ensembles, modify the central limit for β-Laguerre in [2], and generalize the central limit theorem for β-Jacobi from $a_1 = o(\sqrt{a_2}), n = o(\sqrt{a_2})$ to $a_1 n = o(a_2)$.

Recently, high-dimensional data has appeared in many fields, and analysis of this data has become increasingly important in modern statistics. However, it has long been observed that several well-known methods in multivariate analysis become inefficient, or even misleading, when the data dimension p, which is equivalent to the parameter $2a_1/\beta$ in this note, is much larger than the sample size n. Here a high dimensional scenario $n/p \to 0$ is considered.

1.1 Main results for β-Laguerre ensembles

The main results for β-Laguerre ensembles in this note are as follows:

Theorem 1.1 Let $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ be the β-Laguerre ensembles of parameter a_1 and size n, whose joint density function is given by (1.1). Assume that $\lim\limits_{n\to\infty} \frac{n\beta}{2a_1} = \gamma \in [0, 1]$. Then as $n \to \infty$,

$$\frac{\lambda_{\max}}{2a_1} \to (1+\sqrt{\gamma})^2, \quad \frac{\lambda_{\min}}{2a_1} \to (1-\sqrt{\gamma})^2,$$

almost surely.

In particular, for $\gamma = 0$, $\frac{\lambda_i}{2a_1} \to 1$ almost surely for all $1 \leq i \leq n$.

The following theorem is an extension of Theorem 1.5 in [2]:

Theorem 1.2 Let $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and γ be defined as Theorem 1.1. For any $k \geq 1$, set

$$X_k = \sum_{i=1}^{n} \left(\frac{\lambda_i}{2a_1} \right)^k - n \sum_{r=0}^{k-1} \left(\frac{n\beta}{2a_1} \right)^r \frac{C_k^r C_{k-1}^r}{r+1} - \left(\frac{2}{\beta} - 1 \right) \int_{(1-\sqrt{\gamma})^2}^{(1+\sqrt{\gamma})^2} t^k \mu_L^\gamma(t) \mathrm{d}t, \tag{1.4}$$

where $\mu_L^\gamma(t)$ is defined as in (1.3). Then, for any integer $m \geq 1$, as $n \to \infty$,

$$\sqrt{\frac{a_1}{n}} (X_1, X_2, \cdots, X_m) \xrightarrow{w} (Y_1, Y_2, \cdots, Y_m).$$

Here $(Y_1, Y_2, \cdots, Y_m)$ is a centered multivariate Guassian with the covariance matrix

$$\mathrm{Cov}\,(Y_i, Y_j) = S_1(i,j) + S_2(i,j), \tag{1.5}$$

where

$$S_1(i,j) = \sum_{q=1}^{i+j-1} (-1)^{q+1} \gamma^{i+j-q-1} \frac{C_{i+j}^q}{i+j} \sum_{l=q+1}^{i+j} \frac{(-1)^l}{C_{i+j-1}^{l-1}} \sum_{\substack{r+s=l \\ 1\le r\le i \\ 1\le s\le j}} rs(C_i^r)^2 (C_j^s)^2, \tag{1.6}$$

$$S_2(i,j) = \sum_{q=0}^{i+j-2} (-1)^{q+1} \gamma^{i+j-q-1} \frac{C_{i+j}^q}{i+j} \sum_{l=q}^{i+j-2} \frac{(-1)^l}{C_{i+j-1}^{l-1}} \sum_{\substack{r+s=l \\ 1\le r\le i \\ 1\le s\le j}} (i-r)(j-s)(C_i^r)^2 (C_j^s)^2. \tag{1.7}$$

Remark 1.3 There is an inconsistency in Theorem 1.5 in [2], which we found is established only for $\frac{n\beta}{2a} = \gamma$, but not for the genereal case $\frac{n\beta}{2a} \to \gamma (0 < \gamma \le 1)$. Take the following case as an example:

Suppose that $(\lambda_1, \lambda_2 \cdots, \lambda_n)$ are the eigenvalues of $\boldsymbol{L}_{\beta,n}^{a_1}$, then

$$\mathbb{E}\left(\sum_{i=1}^n \lambda_i\right) = \mathbb{E}(\mathrm{tr}(\boldsymbol{L}_{\beta,n}^{a_1})) = \mathbb{E}\left(\sum_{l=0}^{n-1} \chi^2_{2a_1-\beta l} + \sum_{l=1}^{n-1} \chi^2_{\beta l}\right) = 2a_1 n.$$

Taking scaling into account, we have that

$$\mathbb{E}\left(\sum_{i=1}^n \frac{\gamma\lambda_i}{n\beta}\right) = \frac{\gamma\cdot 2a_1}{\beta} = n\cdot\frac{\gamma\cdot 2a_1}{\beta n} = n(1+o(1)) = n + o(n). \tag{1.8}$$

According to the central limit theorem from Dumitriu (Theorem 1.5 in [2]),

$$\mathbb{E}\left(\sum_{i=1}^n \frac{\gamma\lambda_i}{n\beta}\right) = n + \left(\frac{2}{\beta} - 2\right) c + o(\frac{1}{n}),$$

where c is a constant; this contradicts (1.8), therefore, we need to make a slight modification and rewrite the formula as (1.4).

Remark 1.4 It is easy to see that when $\gamma = 0$,

$$S_1(i,j) = ij; \quad S_2(i,j) = 0.$$

Therefore, for $\forall i,\ j \ge 1$,

$$\mathrm{Cov}\,(Y_i, Y_j) = S_1(i,j) + S_2(i,j) = ij.$$

The improvement from the result of Dumitriu and Edelman in [2] is reflected in the power of γ in (1.6) and (1.7). In [2], the central limit theorem is only adaptable for $0 < \gamma \le 1$. Here, we obtain a more general consequence adaptable for $0 \le \gamma \le 1$.

1.2 Main results for β-Jacobi ensembles

For Jacobi ensembles, we utilize the conclusions of β-Laguerre to yield the relative results of β-Jacobi through the connection between the two ensembles. The following lemma was first proved by Jiang in [9] for $a_1 = o(\sqrt{a_2}), n = o(\sqrt{a_2})$, and then generalized by Ma and Shen in [12] for $a_1 n = o(a_2)$:

Lemma 1.5 Let $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_n)$ and let $\boldsymbol{\mu}=(\mu_1,\mu_2,\cdots,\mu_n)$ be the random variables with density functions (1.1) and (1.2), respectively. Denote with $||\mathcal{L}(2(a_1+a_2)\boldsymbol{\mu})-\mathcal{L}(\boldsymbol{\lambda})||$ the total variation distance between the probability distributions of $2(a_1+a_2)\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$. Assume that $a_1n=o(a_2)$. Then

$$\lim_{n\to\infty}||\mathcal{L}(2(a_1+a_2)\boldsymbol{\mu})-\mathcal{L}(\boldsymbol{\lambda})||=0.$$

Following the method in [9], and based on Lemma 1.5 and Theorem 1.1, we can easily have

Theorem 1.6 Let $(\mu_1,\mu_2\cdots,\mu_n)$ be the β-Jacobi ensembles with density function f_{β,a_1,a_2} given by (1.2). Recall that $\mu_{\max}=\max\{\mu_1,\mu_2,\cdots,\mu_n\},\mu_{\min}=\min\{\mu_1,\mu_2,\cdots,\mu_n\}$. Suppose that $a_1n=o(a_2)$ and $\lim\limits_{a_1\to\infty}\frac{\beta n}{2a_1}=\gamma\in[0,1]$. Then, as $n\to\infty$,

$$\frac{a_2}{a_1}\mu_{\max}\to(1+\sqrt{\gamma})^2 \quad\text{and}\quad \frac{a_2}{a_1}\mu_{\min}\to(1-\sqrt{\gamma})^2$$

in probability.

Similarly, utilizing Lemma 1.5 and Theorem 1.2, we have

Theorem 1.7 Define

$$Z_k=\sum_{i=1}^n\left(\frac{a_2}{a_1}\mu_i\right)^k-n\sum_{r=0}^{k-1}\left(\frac{n\beta}{2a_1}\right)^r\frac{C_k^rC_{k-1}^r}{r+1}-\left(\frac{2}{\beta}-1\right)\int_{(1-\sqrt{\gamma})^2}^{(1+\sqrt{\gamma})^2}t^k\mu_L^\gamma(t)\mathrm{d}t. \tag{1.9}$$

Suppose that $a_1n=o(a_2)$. Then for any given $m\geq 1$,

$$\sqrt{\frac{a_1}{n}}(Z_1,Z_2,\cdots,Z_m)\xrightarrow{w}(Y_1,Y_2,\cdots,Y_m),\quad\text{as } n\to\infty,$$

where $(Y_1,Y_2,\cdots,Y_m)$ is the same as in Theorem 1.2.

2 The Proof of Theorem 1.1: the Extremal Eigenvalues for β-Laguerre Ensembles

We can see a set of random variables from the β-Laguerre ensembles as the eigenvalues of the corresponding tridiagonal model, which is $\boldsymbol{L}_{\beta,n}^{a_1}=\boldsymbol{B}_{\beta,n}^{a_1}(\boldsymbol{B}_{\beta,n}^{a_1})^T$, where

$$\boldsymbol{B}_{\beta,n}^{a_1}\sim\begin{pmatrix}\chi_{2a_1} & & & \\ \chi_{\beta(n-1)} & \chi_{2a_1-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a_1-\beta(n-1)}\end{pmatrix}, \tag{2.1}$$

and entries of the $\boldsymbol{B}_{\beta,n}^{a_1}$ are mutually independent ([3]). Denote by $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_n)$ the eigenvalues of $\boldsymbol{L}_{\beta,n}^{a_1}$. Through simple calculation, we have

Lemma 2.1

$$\boldsymbol{L}_{\beta,n}^{a_1}\sim\begin{pmatrix}\chi_{2a_1}^2 & \chi_{2a_1}\chi_{\beta(n-1)} & & \\ \chi_{2a_1}\chi_{\beta(n-1)} & \chi_{2a_1-\beta}^2+\chi_{\beta(n-1)-\beta}^2 & \ddots & \\ & \ddots & \ddots & \chi_{2a_1-\beta(n-2)}\chi_\beta \\ & & \chi_{2a_1-\beta(n-2)}\chi_\beta & \chi_{2a_1-\beta(n-1)}^2+\chi_\beta^2\end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned}(L^{a_1}_{\beta,n})_{11} &= \chi^2_{2a_1},\\ (L^{a_1}_{\beta,n})_{jj} &= \chi^2_{2a_1-\beta(j-1)} + \chi^2_{\beta(n-j+1)},\\ (L^{a_1}_{\beta,n})_{j-1,j} &= \chi_{2a_1-\beta(j-2)}\chi_{\beta(n-j+1)}\end{aligned}$$

for $j = 2, \cdots, n$. Note that $\boldsymbol{L}^{a_1}_{\beta,n}$ is a tridiagonal symmetric matrix.

In this section, we will prove the limiting behaviour of the extremal eigenvalues of $\boldsymbol{L}^{a_1}_{\beta,n}$. The idea is borrowed from [13] (Silverstein 1985). To prove Theorem 1.1, we first introduce two lemmas.

Lemma 2.2 χ_n is a chi distribution with n degrees of freedom, so $\frac{\chi_n}{\sqrt{n}} \to 1$ almost surely as $n \to \infty$.

Proof As χ^2_n can be seen as the sum of the squares of n independent normal random variables, according to Kolmogorov strong law of large numbers, we have that

$$\frac{\chi^2_n}{n} \longrightarrow 1$$

almost surely as $n \to \infty$. Note that the property of convergence is maintained under the transformation of continuous functions, thus

$$\frac{\chi_n}{\sqrt{n}} = \sqrt{\frac{\chi^2_n}{n}} \longrightarrow 1$$

almost surely as $n \to \infty$. □

The proof of Theorem 1.1 From Theorem 6.0.5 in [1], we know that

$$F^{a_1}_{\beta,n}(x) \to E_\gamma(x)$$

almost surely, where $F^{a_1}_{\beta,n}(x) := \frac{1}{n}\sum\limits_{i=1}^{n} 1_{\frac{\lambda_i}{2a_1}\le x}$ and E_γ is a cumulative distribution function with density function

$$e_\gamma(x) = \begin{cases} \frac{1}{2\pi\gamma}\frac{\sqrt{(x-\gamma_{\min})(\gamma_{\max}-x)}}{x}, & \text{if } x \in [\gamma_{\min}, \gamma_{\max}],\\ 0, & \text{otherwise.}\end{cases}$$

Here $\gamma_{\max} = (1+\sqrt{\gamma})^2$ and $\gamma_{\min} = (1-\sqrt{\gamma})^2$. Thus, similarly to the proof in [13], we can conclude that

$$\begin{aligned}\overline{\lim_{n\to\infty}} \tfrac{\lambda_{\min}}{2a_1} &\le (1-\sqrt{\gamma})^2,\\ \underline{\lim}_{n\to\infty} \tfrac{\lambda_{\max}}{2a_1} &\ge (1+\sqrt{\gamma})^2,\end{aligned} \tag{2.3}$$

almost surely.

Next, we will prove that $\underline{\lim}_{n\to\infty} \frac{\lambda_{\min}}{2a_1} \ge (1-\sqrt{\gamma})^2$ and $\overline{\lim_{n\to\infty}} \frac{\lambda_{\max}}{2a_1} \le (1+\sqrt{\gamma})^2$, almost surely. Gerŝgorin's theorem from [8] says that each eigenvalue z of an $n\times n$ complex matrix $\boldsymbol{A} = (a_{ij})$ lies in at least one of the disks $|z - a_{jj}| \le \sum\limits_{i\ne j}|a_{ij}|, j = 1, 2, \cdots, n$ in the complex plane. Thus, combined with Lemma 2.1, Gerŝgorin's theorem leads to

$$\frac{\lambda_{\min}}{2a_1} \ge \frac{1}{2a_1}\min\left\{\chi^2_{2a_1} - \chi_{2a_1}\chi_{\beta(n-1)}; \chi^2_{2a_1-\beta(n-1)} + \chi^2_\beta - \chi_{2a_1-\beta(n-2)}\chi_\beta; \min_{2\le j\le n-1} A^n_j\right\},$$

with

$$\begin{aligned} A_j^n &:= (L_{\beta,n}^{a_1})_{j,j} - (L_{\beta,n}^{a_1})_{j-1,j} - (L_{\beta,n}^{a_1})_{j+1,j} \\ &= \chi_{2a_1-\beta(j-1)}^2 + \chi_{\beta(n-j+1)}^2 - \chi_{2a_1-\beta(j-2)}\chi_{\beta(n-j+1)} - \chi_{2a_1-\beta(j-1)}\chi_{\beta(n-j)}. \end{aligned}$$

Since $\frac{\chi_n}{\sqrt{n}} \to 1$ almost surely from Lemma 2.2, we have that

$$\begin{aligned} &\frac{1}{2a_1}(\chi_{2a_1}^2 - \chi_{2a_1}\chi_{\beta(n-1)}) \to (1-\sqrt{\gamma}), \\ &\frac{1}{2a_1}(\chi_{2a_1-\beta(n-1)}^2 + \chi_\beta^2 - \chi_{2a_1-\beta(n-2)}\chi_\beta) \to 1-\gamma \end{aligned}$$

almost surely as $n \to \infty$, and

$$\lim_{n\to\infty} \frac{A_j^n}{2a_1} = 1 + (1-2c)\gamma - 2\sqrt{(1-c\gamma)(1-c)\gamma} \geq (1-\sqrt{\gamma})^2$$

almost surely for j satisfying $\lim\limits_{n\to\infty} \frac{j}{n} = c \in [0,1]$. Note that $1-\gamma \geq 1-\sqrt{\gamma} \geq (1-\sqrt{\gamma})^2$, and thus

$$\varliminf_{n\to\infty} \frac{\lambda_{\min}}{2a_1} \geq (1-\sqrt{\gamma})^2. \tag{2.4}$$

Combining (2.3) and (2.4), we conclude that

$$\lim_{n\to\infty} \frac{\lambda_{\min}}{2a_1} = (1-\sqrt{\gamma})^2,$$

almost surely. In a similar way, we have that

$$\lim_{n\to\infty} \frac{\lambda_{\max}}{2a_1} = (1+\sqrt{\gamma})^2,$$

almost surely. □

3 The Proof of Theorem 1.2: CLT for β-Laguerre Ensembles

In this section, we are going to prove Theorem 1.2. The proof is divided into two parts.

Lemma 3.1 (the fluctuation) Recall, for $k \geq 1$, that

$$X_k = \sum_{i=1}^n \left(\frac{\lambda_i}{2a_1}\right)^k - \mathbb{E}\left[\sum_{i=1}^n \left(\frac{\lambda_i}{2a_1}\right)^k\right].$$

Assuming that $\lim\limits_{a_1\to\infty} \frac{\beta n}{2a_1} = \gamma \in [0,1]$, we have, for any $m \geq 1$, that

$$\sqrt{\frac{a_1}{n}}(X_1, X_2, \cdots, X_m) \xrightarrow{w} (Y_1, Y_2, \cdots, Y_m),$$

where $(Y_1, Y_2, \cdots, Y_m)$ is a centered multivariate Guassian defined as in Theorem 1.2.

Lemma 3.2 (The deviation) With the same assumption as in Lemma 3.1, for n large enough, we have that

$$\mathbb{E}\sum_{i=1}^n \left(\frac{\lambda_i}{2a_1}\right)^k = n\sum_{r=0}^{k-1}\left(\frac{n\beta}{2a_1}\right)^r \frac{C_k^r C_{k-1}^r}{r+1} - \left(\frac{2}{\beta}-1\right)\int_{(1-\sqrt{\gamma})^2}^{(1+\sqrt{\gamma})^2} t^k \mu_L^\gamma(t)\mathrm{d}t + o\left(\sqrt{\frac{n}{a_1}}\right), \tag{3.1}$$

where $\mu_L^\gamma(t)$ is as defined by (1.3).

Remark 3.3 In particular, if $\gamma = 0$,

$$\mathbb{E}\sum_{i=1}^{n}\left(\frac{\lambda_i}{2a_1}\right)^k = n\sum_{r=0}^{k-1}\left(\frac{n\beta}{2a_1}\right)^r \frac{C_k^r C_{k-1}^r}{r+1} + o\left(\sqrt{\frac{n}{a_1}}\right)$$

for n large enough.

3.1 The proof of Lemma 3.1: the fluctuation

To prove Lemma 3.1, we are going to prove the following claims first (all the notations are defined as above):

Claim 1 Set $\tilde{X}_k = \sqrt{\frac{a_1}{n}} X_k$ for any $k \geq 1$. For any positive integers k and l,

$$\lim_{n\to\infty} \mathbb{E}[(\tilde{X}_k)^l] = \begin{cases} (l-1)!!\,(S_1(k)+S_2(k))^{\frac{l}{2}}, & \text{if l is even;} \\ 0, & \text{if l is odd,} \end{cases} \tag{3.2}$$

where

$$S_1(k) = \sum_{q=1}^{2k-1}(-1)^{q+1}\gamma^{2k-q-1}\frac{C_{2k}^q}{2k}\sum_{l=q+1}^{2k}\frac{(-1)^l}{C_{2k-1}^{l-1}}\sum_{r+s=l,1\leq r,s\leq k} rs(C_k^r C_k^s)^2 \tag{3.3}$$

and

$$S_2(k) = \sum_{q=0}^{2k-2}(-1)^{q+1}\gamma^{2k-q-1}\frac{C_{2k}^q}{2k}\sum_{l=q}^{2k-2}\frac{(-1)^l}{C_{2k-1}^{l}}\sum_{r+s=l,1\leq r,s\leq k}(k-r)(k-s)(C_k^r C_k^s)^2. \tag{3.4}$$

Claim 2 For any fixed positive integers i and j,

$$\lim_{n\to\infty} \mathrm{Cov}\left(\tilde{X}_i, \tilde{X}_j\right) = S_1(i,j) + S_2(i,j), \tag{3.5}$$

where $S_1(i,j), S_2(i,j)$ are defined as in (1.6) and (1.7).

Claim 3 For any $\tilde{X}_{k_1}, \tilde{X}_{k_2}, \cdots, \tilde{X}_{k_m} (1 \leq k_i \leq n)$, and $\forall t_i \in \mathbb{R} (i = 1, \cdots, m)$,

$$\lim_{n\to\infty}\mathbb{E}\left(\sum_{i=1}^{m} t_i\tilde{X}_{k_i}\right)^l = \begin{cases} (l-1)!!\left(\sum_{1\leq i,j\leq m} t_i t_j (S_1(k_i,k_j)+S_2(k_i,k_j))\right)^{\frac{l}{2}}, & \text{if } l \text{ is even;} \\ 0, & \text{if } l \text{ is odd.} \end{cases} \tag{3.6}$$

The proof of the three claims are basically the same, and the idea also comes from [2]. Before the proof, we first introduce some notations.

Table 1 Unscaled and scaled tridiagonal models for beta-Laguerre ensembles ($\beta > 0$, $n \in \mathbb{N}$, $a_1 \in \mathbb{R}^+$, and $a_1 > \frac{\beta}{2}(n-1)$)

unscaled	$\boldsymbol{B}_{\beta,n}^{a_1} \sim \begin{pmatrix} \chi_{2a_1} & & & \\ \chi_{\beta(n-1)} & \chi_{2a_1-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a_1-\beta(n-1)} \end{pmatrix}$
	$\boldsymbol{L}_{\beta,n}^{a_1}$
scaled	$\tilde{\boldsymbol{B}}_{\beta,n}^{a_1} = \frac{1}{\sqrt{2a_1}}\boldsymbol{B}_{\beta,n}^{a_1}$
	$\tilde{\boldsymbol{L}}_{\beta,n}^{a_1} = \frac{1}{2a_1}\boldsymbol{L}_{\beta,n}^{a_1}$

Write the scaled matrix $\tilde{\boldsymbol{B}}^{a_1}_{\beta,n}$ (see Table 1) as

$$\tilde{\boldsymbol{B}}^{a_1}_{\beta,n} = \boldsymbol{D} + \frac{1}{\sqrt{2a_1}}\boldsymbol{Z}, \tag{3.7}$$

where

$$\boldsymbol{D} = \begin{pmatrix} \frac{1}{\sqrt{2a_1}}\mathbb{E}(\chi_{2a_1}) & & & \\ \frac{1}{\sqrt{2a_1}}\mathbb{E}(\chi_{\beta(n-1)}) & \frac{1}{\sqrt{2a_1}}\mathbb{E}(\chi_{2a_1-\beta}) & & \\ & \ddots & \ddots & \\ & & \frac{1}{\sqrt{2a_1}}\mathbb{E}(\chi_{\beta}) & \frac{1}{\sqrt{2a_1}}\mathbb{E}((\chi_{2a_1-\beta(n-1)})) \end{pmatrix} \tag{3.8}$$

and

$$\boldsymbol{Z} = \begin{pmatrix} \chi_{2a_1} - \mathbb{E}(\chi_{2a_1}) & & & \\ \chi_{\beta(n-1)} - \mathbb{E}(\chi_{\beta(n-1)}) & \chi_{2a_1-\beta} - \mathbb{E}(\chi_{2a_1-\beta}) & & \\ & \ddots & \ddots & \\ & & \chi_{\beta} - \mathbb{E}(\chi_{\beta}) & \begin{matrix}\chi_{2a_1-\beta(n-1)} \\ -\mathbb{E}(\chi_{2a_1-\beta(n-1)})\end{matrix} \end{pmatrix}. \tag{3.9}$$

Remark 3.4 As in [2], the entries of D are bounded and for any finite k and l there exists a constant M such that

$$\mathbb{E}\left[\prod_{i=1}^{kl} Z^{c_i}_{j_i j_i'}\right] \leqslant M$$

for all $0 \leqslant c_i \leqslant kl$ and for all $j_1, \cdots, j_{kl}$ and $j_1', \cdots, j_{kl}'$ such that $|j_i - j_i'| \leqslant 1$.

For the expression of $\operatorname{tr}\left((\boldsymbol{B}\boldsymbol{B}^T)^k\right)$, it holds that

$$\operatorname{tr}\left(\left(\boldsymbol{B}\boldsymbol{B}^T\right)^k\right) = \sum_{1\leqslant i_1,i_2,\cdots,i_{2k}\leqslant n} B_{i_1,i_2}B^T_{i_2,i_3}\cdots B_{i_{2k-1},i_{2k}}B^T_{i_{2k},i_1},$$

where the sum is taken over sequences $(i_1, \cdots, i_{2k})$ with the property that $i_{2j-1} - i_{2j} \in \{0,1\}$ (for all $1\leqslant j \leqslant k$), $i_{2j} - i_{2j+1} \in \{0,-1\}$ (for all $1 \leqslant j \leqslant k-1$), and also $i_{2k} - i_1 \in \{0,-1\}$.

Definition 3.5 Denote by $S_{n,k} \in \{1, \cdots, n\}^{2k}$ the set of sequences of integers $i_1, \cdots, i_{2k}$ such that $i_{2j-1} - i_{2j} \in \{0,1\}$ for all $1\leqslant j \leqslant k$, and $i_{2j} - i_{2j+1} \in \{0,-1\}$ for all $1 \leqslant j \leqslant k-1$, and also $i_{2k} - i_1 \in \{0,-1\}$.

For each $\mathcal{I} \in S^k_n$, we consider all the ways in which we can break $\mathcal{I} := (i_1, \cdots, i_{2k})$ up into overlapping J and R; i.e.,

$$J = \left((i_{p_0}, \cdots, i_{p_1}), (i_{p_2}, \cdots, i_{p_3}), (i_{p_4}, \cdots, i_{p_5}), \cdots, (i_{p_{2q}}, \cdots, i_{p_{2q+1}})\right)$$

and

$$R = \left((i_{p_1}, \cdots, i_{p_2}), (i_{p_3}, \cdots, i_{p_4}), \cdots, (i_{p_{2q-1}}, \cdots, i_{p_{2q}})\right),$$

with

$$(i_1, \cdots, i_{2k}) = \left(i_{p_0}, \cdots, i_{p_1}, i_{p_1+1}, \cdots, i_{p_2}, i_{p_2+1}, \cdots, i_{p_3}, \cdots, i_{p_{2q+1}}\right).$$

We allow for the possibility of having empty sequences $i_{p_0}, \cdots, i_{p_1}$ in the beginning and/or $i_{p_{2q}}, \cdots, i_{p_{2q+1}}$ in the end of J.

Definition 3.6 For any $\mathcal{I} \in S_n^k$, we introduce the set $\mathcal{J} = \mathcal{J}(\mathcal{I})$ of pairs (J, R) corresponding to the sequence $\mathcal{I}$. For a bidiagonal matrix B, donote that $\boldsymbol{B}^* = \boldsymbol{B}^T$ if $i_j - i_{j+1} = -1$, otherwise, $\boldsymbol{B}^* = \boldsymbol{B}$. Define

$$\left(\boldsymbol{B}\boldsymbol{B}^T\right)_{\mathcal{I}} = B^*_{i_1,i_2} B^*_{i_2,i_3} \cdots B^*_{i_{2k-1} i_{2k}} B^*_{i_{2k} i_1},$$

$$\left(\boldsymbol{B}\boldsymbol{B}^T\right)_J = B^*_{i_{p_0},i_{p0+1}} \cdots B^*_{i_{p_1-1},i_{p_1}} B^*_{i_{p_2} i_{p_2+1}} \cdots B^*_{i_{p_3-1},i_{p_3}} \cdots B^*_{i_{p_{2q}},i_{p_{2q}+1}} \cdots B^*_{i_{p_{2q+1}-1},i_{p_{2q+1}}},$$

$$\left(\boldsymbol{B}\boldsymbol{B}^T\right)_R = B^*_{i_{p_1},i_{p_1+1}} \cdots B^*_{i_{p_2-1},i_{p_2}} B^*_{i_{p_3} i_{p_3+1}} \cdots B^*_{i_{p_4-1},i_{p_4}} \cdots B^*_{i_{p_{2q-1}},i_{p_{2q-1}+1}} \cdots B^*_{i_{p_{2q}-1},i_{p_{2q}}}.$$

Remark 3.7 Note that any term in $\operatorname{tr}\left((\boldsymbol{L}^{a_1}_{\beta,n})^k\right)$ will consist of terms in $\boldsymbol{D}$ and terms in $\boldsymbol{Z}$, with a sequence of runs J recording the former, and a sequence of runs R recording the latter.

The proof of Claim 1 For simplicity, set $\tilde{\lambda}_i := \frac{\lambda_i}{2a_1}$. According to the definition above, we have that

$$\begin{aligned}
\mathbb{E}(X_k)^l &= \mathbb{E}\left[\sum_{i=1}^n \tilde{\lambda}_i^k - \mathbb{E}(\sum_{i=1}^n \tilde{\lambda}_i^k)\right]^l \\
&= \mathbb{E}\left[\operatorname{tr}(\tilde{\boldsymbol{L}}^{a_1}_{\beta,n})^k - \mathbb{E}\left(\operatorname{tr}(\tilde{\boldsymbol{L}}^{a_1}_{\beta,n})^k\right)\right]^l \\
&= \mathbb{E}\left[\sum_{\mathcal{I}\in S_{n,k}} (\tilde{L}^{a_1}_{\beta,n})_{\mathcal{I}} - \mathbb{E}\left(\sum_{\mathcal{I}\in S_{n,k}} (\tilde{L}^{a_1}_{\beta,n})_{\mathcal{I}}\right)\right]^l.
\end{aligned}$$

Note that

$$\left(\tilde{L}^{a_1}_{\beta,n}\right)_{\mathcal{I}} = \left(\left(D + \frac{1}{\sqrt{2a_1}} Z\right)\left(D + \frac{1}{\sqrt{2a_1}} Z\right)^T\right)_{\mathcal{I}} = \sum_{(J,R)\in\mathcal{J}} \frac{1}{(2a_1)^{P/2}} (D)_J (Z)_R,$$

with $P = p_2 - p_1 + 1 + \cdots + p_{2q} - p_{2q-1} + 1$. Thus,

$$\begin{aligned}
\mathbb{E}[(X_k)^l] &= \sum_{\mathcal{I}_j\in\mathcal{S}_{n,k}} \sum_{(J_j,R_j)\in\mathcal{J}_j} \frac{1}{(2a_1)^q} \left(\prod_{j=1}^l (D)_{J_j}\right) \mathbb{E}\left[\prod_{j=1}^l \left((Z)_{R_j} - E\left[(Z)_{R_j}\right]\right)\right] \\
&= \sum_{\mathcal{I}_j\in\mathcal{S}_{n,k}} \sum_{(J_j,R_j)\in\mathcal{J}_j} \frac{1}{(2a_1)^{q-\frac{l}{2}} n^{\frac{l}{2}}} \left(\prod_{j=1}^l (D)_{J_j}\right) \mathbb{E}\left[\prod_{j=1}^l \left((Z)_{R_j} - E\left[(Z)_{R_j}\right]\right)\right],
\end{aligned} \tag{3.10}$$

where $q = q(\mathcal{I}_1, \cdots, \mathcal{I}_l) = \sum\limits_{j=1}^l P_j/2$, and P_j is the length of R_j.

In the next lemma, we will present some results from [2] which also hold here.

Lemma 3.8 Assume that $(R_1, R_2, \cdots, R_l)$ involves s independent variables. The only terms of asymptotical significance are those for which $s = q = l/2$; this means that for k and l fixed, if l is odd,

$$\lim_{n\to\infty} \mathbb{E}[(\tilde{X}_k)^l] = 0. \tag{3.11}$$

Therefore, we only need to examine what happens when l is even and $s = q = l/2$.

Lemma 3.9 If l is even, $s = q = l/2$, and we have $|R_j| = 1$. This means that for each $1 \le j_1 \le l$, there exists a unique $1 \le j_2 \le l$ such that $Z_{R_{j_1}} = Z_{R_{j_2}}$. Moreover, given an ordered $l/2$-tuplet of distinct indices $i_1, i_2, \cdots, i_{l/2}$, there are $(l-1)!!$ ways of pairing these indices to the $R_j's$ in this order.

The next two lemmas are about the estimation of $(Z)_{R_j}$ and $(D)_{J_j}$; there are slight differences from [2]. Also we only need to consider the significant terms.

Lemma 3.10 For (Z_{R_j}) we have that

$$\mathbb{E}\Big[\prod_{j=1}^{l}((Z)_{R_j}-\mathbb{E}((Z)_{R_j}))\Big]=\prod_{j=1}^{l/2}\mathbb{E}(((Z)_{R_{i_j}}-\mathbb{E}((Z)_{R_{i_j}}))^2)\to 2^{-\frac{l}{2}},\quad \text{as}\quad n\to\infty. \tag{3.12}$$

Lemma 3.11 For $(D)_{J_j}$, assuming that $\mathcal{I}_{J_j}$ have $2s$ off diagonal terms, there are two cases:

1) when $(Z)_{R_j}$ corresponds to a diagonal term, then there exists an i such that $(Z)_{R_j}=Z_{ii}$ and there are $2(k-s)\binom{k}{s}^2$ corresponding to $\mathcal{I}_{J_j}$, and

$$(D)_{J_j}=\Big(1-\frac{\beta i}{2a_1}\Big)^{k-s-\frac{1}{2}}\Big(\frac{\beta(n-i)}{2a_1}\Big)^{s}+o(1),\quad \text{as}\quad n\to\infty; \tag{3.13}$$

2) when $(Z)_{R_j}$ corresponds to an off diagonal term, then there exists an i such that $(Z)_{R_j}=Z_{i+1,i}$, and there are $2s\binom{k}{s}^2$ corresponding to $\mathcal{I}_{J_j}$, and

$$(D)_{J_j}=\Big(1-\frac{\beta i}{2a_1}\Big)^{k-s}\Big(\frac{\beta(n-i)}{2a_1}\Big)^{s-\frac{1}{2}}+o(1),\quad \text{as}\quad n\to\infty. \tag{3.14}$$

The proof of Lemmas 3.10 and 3.11 By Lemma 7.1.2 in [1], we know that if $\lim\limits_{n\to\infty} r_n=\infty$, then

$$\chi_{r_n}-\sqrt{r_n}\xrightarrow{w}N(0,\frac{1}{2})\quad \text{as } n\to\infty$$

and $\mathbb{E}(\chi_{r_n})=\sqrt{r_n}+O(\frac{1}{\sqrt{r_n}})$ as n large enough. Thus,

$$\begin{aligned}Z_{ii}&=\chi_{2a_1-\beta(i-1)}-\mathbb{E}[\chi_{2a_1-\beta(i-1)}]\\&=\chi_{2a_1-\beta(i-1)}-\sqrt{2a_1-\beta(i-1)}+\sqrt{2a_1-\beta(i-1)}-\mathbb{E}[\chi_{2a_1-\beta(i-1)}],\end{aligned}$$

where $\chi_{2a_1-\beta(i-1)}-\sqrt{2a_1-\beta(i-1)}\xrightarrow{\omega}N(0,\frac{1}{2})$ and

$$\lim_{n\to\infty}\Big(\mathbb{E}[\chi_{2a_1-\beta(i-1)}]-\sqrt{2a_1-\beta(i-1)}\Big)=0$$

for any i satisfying $i<<a_1$. Therefore, (3.12) is established.

In the same way, we know that for the entries of D, as n is large enough, we have that

$$\begin{aligned}D_{ii}&=\sqrt{\frac{1}{2a_1}}\mathbb{E}(\chi_{2a_1-\beta(i-1)})=\sqrt{1-\frac{\beta i}{2a_1}}+o(1),\\D_{i+1,i}&=\sqrt{\frac{1}{2a_1}}\mathbb{E}(\chi_{\beta(n-i)})=\sqrt{\frac{\beta(n-i)}{2a_1}}+o(1)\end{aligned}$$

for any i satisfying $i<<a_1$. Therefore, (3.13) and (3.14) hold. □

Combining the above lemmas, if l is even, we can easily obtain equation:

$$\lim_{n\to\infty}\mathbb{E}[(\tilde{X}_k)^l]=\lim_{n\to\infty}(l-1)!!(S_{1,n}+S_{2,n})^{\frac{l}{2}},$$

where

$$S_{1,n}(k)=\frac{1}{n}\sum_{1\le i\le n}\sum_{j=0}^{2k-2}\Big(1-\frac{\beta i}{2a_1}\Big)^{2k-j-1}\Big(\frac{\beta(n-i)}{2a_1}\Big)^{j}\sum_{\substack{s_1+s_2=j\\0\le s_1,s_2\le k-1}}(k-s_1)(k-s_2)(C_k^{s_1}C_k^{s_2})^2,$$

$$S_{2,n}(k)=\frac{1}{n}\sum_{1\le i\le n}\sum_{j=2}^{2k}\Big(1-\frac{\beta i}{2a_1}\Big)^{2k-j}\Big(\frac{\beta(n-i)}{2a_1}\Big)^{j-1}\sum_{\substack{s_1+s_2=j\\1\le s_1,s_2\le k}}s_1s_2(C_k^{s_1}C_k^{s_2})^2. \tag{3.15}$$

First, calculate $S_{1,n}$. For simplicity, set

$$U_1(k,j):=\sum_{\substack{s_1+s_2=j\\1\le s_1,s_2\le k}}s_1s_2(C_k^{s_1}C_k^{s_2})^2,$$
$$U_2(k,j):=\sum_{\substack{s_1+s_2=j\\0\le s_1,s_2\le k-1}}(k-s_1)(k-s_2)(C_k^{s_1}C_k^{s_2})^2.$$

By binomial expansion and a changing summation order, we have that

$$\begin{aligned}S_{1,n}(k)=&\sum_{j=0}^{2k-2}\sum_{q=0}^{2k-j-1}(-1)^{j+1+q}C_{2k-j-1}^q\frac{1}{n}\sum_{1\le i\le n}\left(\frac{i}{n}\right)^{2k-j-1-q}\\&\times\left(1-\frac{i}{n}\right)^j\left(\frac{n\beta}{2a_1}\right)^{2k-1-q}U_2(k,j).\end{aligned}\tag{3.16}$$

Using the properties of the Riemann integral and the Beta function, we know that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{1\le i\le n}\left(\frac{i}{n}\right)^{2k-j-1-q}\left(1-\frac{i}{n}\right)^j&=\int_0^1x^{2k-j-1-q}(1-x)^j\mathrm{d}x\\&=\frac{(2k-j-1-1)!j!}{(2k-q)!}.\end{aligned}\tag{3.17}$$

Hence, plugging (3.17) into (3.16), we have that

$$\lim_{n\to\infty}S_{1,n}(k)=\sum_{q=1}^{2k-1}(-1)^{q+1}\frac{C_{2k}^q}{2k}\gamma^{2k-1-q}\sum_{j=q+1}^{2k}\frac{(-1)^j}{C_{2k-1}^{j-1}}U_1(k,j).$$

Similarly,

$$\lim_{n\to\infty}S_{2,n}(k)=\sum_{q=1}^{2k-2}(-1)^q\frac{C_{2k}^q}{2k}(\gamma)^{2k-1-q}\sum_{j=q}^{2k-2}\frac{(-1)^j}{C_{2k-1}^{j-1}}U_2(k,j).$$

Therefore, Claim 1 is established. □

The proof of Claim 2 For any $k_1,k_2\ge1$, by the precedent calculations, we have that

$$\mathbb{E}(\tilde X_{k_1}\tilde X_{k_2})=2^{-\frac{l}{2}}\sum_{\substack{\mathcal{I}_1\in\mathcal{S}_{n,k_1}\\(J_1,R_1)\in\mathcal{I}_1}}\sum_{\substack{\mathcal{I}_2\in\mathcal{S}_{n,k_2}\\(J_2,R_2)\in\mathcal{I}_2}}\frac{1}{(2a_1)^{q-1}n}\left(\prod_{j=1}^2(D)_{J_j}\right)\mathbb{E}\left[\prod_{j=1}^2\big((Z)_{R_j}-\mathbb{E}\left[(Z)_{R_j}\right]\big)\right],$$

where q is the total length of R_1 and R_2. The contribution terms are for $|R_1|=|R_2|=1$ and $R_1=R_2$. Similarly, we have that

$$\lim_{n\to\infty}\mathrm{Cov}(\tilde X_{k_1},\tilde X_{k_2})=\lim_{n\to\infty}\mathbb{E}(\tilde X_{k_1}\tilde X_{k_2})=\lim_{n\to\infty}(S_{1,n}(k_1,k_2)+S_{2,n}(k_1,k_2))^{\frac{l}{2}},$$

where

$$\begin{aligned}S_{1,n}(k_1,k_2)&=\frac{1}{n}\sum_{1\le i\le n}\sum_{j=0}^{k_1+k_2-2}(1-\frac{\beta i}{2a_1})^{k_1+k_2-j-1}(\frac{\beta(n-i)}{2a_1})^jU_2(k_1,k_2,j),\\S_{2,n}(k_1,k_2)&=\frac{1}{n}\sum_{1\le i\le n}\sum_{j=2}^{k_1+k_2}\left(1-\frac{\beta i}{2a_1}\right)^{k_1+k_2-j}\left(\frac{\beta(n-i)}{2a_1})^{j-1}U_1(k_1,k_2,j)\right).\end{aligned}\tag{3.18}$$

Denote that

$$U_1(k_1,k_2,j)=\sum_{\substack{s_1+s_2=j\\1\le s_1\le k_1\\1\le s_2\le k_2}} s_1s_2(C_{k_1}^{s_1})^2(C_{k_2}^{s_2})^2,$$

$$U_2(k_1,k_2,j)=\sum_{\substack{s_1+s_2=j\\0\le s_1\le k_1-1\\0\le s_2\le k_2-1}} (k_1-s_1)(k_2-s_2)(C_{k_1}^{s_1})^2(C_{k_1}^{s_2})^2.$$

Repeating a process similar to that of Claim 1, we obtain the equation (3.5). This means that Claim 2 is established. □

The proof of Claim 3 For simplicity, we take $m=2$, for example. For other $m\ge 1$, the process is basically the same. In fact, for any t_1, $t_2\in\mathbb{R}$ and $k_1,k_2\ge 1$, we have that

$$\mathbb{E}\left[(t_1\tilde{X}_{k_1}+t_2\tilde{X}_{k_2})^l\right]=\sum_{\substack{\mathcal{I}_j\\(J_j,R_j)\in\mathcal{I}_j}}(t_1)^p(t_2)^{l-p}\frac{2^{-\frac{l}{2}}}{(2a_1)^{q-\frac{l}{2}}n^{\frac{l}{2}}}\prod_{j=1}^{l}(D)_{J_j}\mathbb{E}\Big[\prod_{j=1}^{l}((Z)_{R_j}-\mathbb{E}(Z)_{R_j})\Big],$$

where p is the number of picking $\tilde{X}_{k_1}$ among the product of l terms, and q is still the total length of R_j.

By Lemmas 3.8, 3.9 and 3.10, if l is odd, it holds that

$$\lim_{n\to\infty}\mathbb{E}(t_1\tilde{X}_{k_1}+t_2\tilde{X}_{k_2})^l=0.$$

In the next calculations, we only need to consider what happens when l is even. The only difference from the proof of Claim 1 is in terms of the estimation of $(D)_{J_j}$.

For the $\frac{l}{2}$-tuplet $(R_{i_1},R_{i_2},\cdots,R_{i_{\frac{l}{2}}})$, every R_{i_j} corresponds to two types of R for any $1\le j\le\frac{l}{2}$; each one of these has two choices: either from X_{k_1} or X_{k_2}. Then, there are three combinations, we denote r_1 as the occurrence number of the combination (k_1,k_1), r_2 as the occurrence number of the combination (k_2,k_2), and r_3 as the occurence number of the combination (k_1,k_2). In addition, $\sum\limits_{i=1}^{3}r_i=l/2$, and there are $\frac{\frac{l}{2}}{r_1!r_2!r_3!}$ choices for each combination (r_1,r_2,r_3).

Now, we can obtain the estimation of $(D)_{J_j}$ under the three combinations. Note that for combination (k_1,k_2), there are two choices for the corresponding R. Thus, we have the following formula:

$$\lim_{n\to\infty}\mathbb{E}\left[(t_1\tilde{X}_{k_1}+t_2\tilde{X}_{k_2})^l\right]=\lim_{n\to\infty}(l-1)!!\sum_{r_1+r_2+r_3=\frac{l}{2}}\frac{\frac{l}{2}}{r_1!r_2!r_3!}Q_{1,n}(r_1)Q_{2,n}(r_2)Q_{3,n}(r_3).\tag{3.19}$$

Here

$$Q_{1,n}(r_1)=\Big(t_1^2\big(S_{1,n}(k_1,k_1)+S_{2,n}(k_1,k_1)\big)\Big)^{r_1},$$

$$Q_{2,n}(r_2)=\Big(t_2^2(S_{1,n}(k_2,k_2)+S_{2,n}(k_2,k_2))\Big)^{r_2},$$

$$Q_{3,n}(r_3)=\Big(2t_1t_2(S_{1,n}(k_1,k_2)+S_{2,n}(k_1,k_2))\Big)^{r_3}.$$

Obviously, it holds that

$$\begin{aligned}\lim_{n\to\infty} Q_{1,n}(r_1) &= (t_1^2(S_1(k_1,k_1)+S_2(k_1,k_1)))^{r_1},\\ \lim_{n\to\infty} Q_{2,n}(r_2) &= (t_2^2(S_1(k_2,k_2)+S_2(k_2,k_2)))^{r_2},\\ \lim_{n\to\infty} Q_{3,n}(r_3) &= (2t_1t_2(S_1(k_1,k_2)+S_2(k_1,k_2)))^{r_3},\end{aligned} \tag{3.20}$$

where $S_{1,n}(k_i,k_j)$ and $S_{2,n}(k_i,k_j)(i=1,2)$ are defined as in (3.18); $S_1(k_i,k_j)$ and $S_2(k_i,k_j)$ $(i=1,2)$ are defined as in (1.6) and (1.7). Putting (3.20) back into (3.19), for even l, we have that

$$\lim_{n\to\infty} \mathbb{E}\left[(t_1\tilde{X}_{k_1}+t_2\tilde{X}_{k_2})^l\right] = (l-1)!!\Bigg(\sum_{1\le i,j\le 2} t_it_j(S_1(k_i,k_j)+S_2(k_i,k_j))\Bigg)^{\frac{l}{2}}.$$

Set $m\ge 1$. For any $t_i\in\mathbb{R}$ and any integer $k_i\ge 1$, the following conclusion can be obtained similarly:

$$\lim_{n\to\infty} \mathbb{E}\Bigg(\sum_{i=1}^m t_i\tilde{X}_{k_i}\Bigg)^l = \begin{cases}(l-1)!!\Bigg(\sum\limits_{1\le i,j\le m} t_it_j(S_1(k_i,k_j)+S_2(k_i,k_j))\Bigg)^{\frac{l}{2}}, & \text{if } l \text{ is even;}\\ 0, & \text{if } l \text{ is odd.}\end{cases}$$

□

Lemma 3.1 follows directly from Claim 1, Claim 2 and Claim 3.

3.2 The proof of Lemma 3.2: the deviation

In this section, we are going to prove Lemma 3.2.

The proof of Lemma 3.2 According to Chapter 6.3 in [1], if $0<\gamma\le 1$, for n large enough, we have that

$$\mathbb{E}\sum_{i=1}^n\left(\frac{\lambda_i}{2a_1}\right)^k = n\sum_{r=0}^{k-1}\left(\frac{n\beta}{2a_1}\right)^r\frac{C_k^rC_{k-1}^r}{r+1}+F_k\left(\frac{n\beta}{2a_1}\right)+O(\frac{1}{n}) \tag{3.21}$$

for any $k\ge 1$, where $F_k(\frac{n\beta}{2a_1})$ is a polynomial of $\frac{n\beta}{2a_1}$ with respect to k. More precisely,

$$F_k\left(\frac{n\beta}{2a_1}\right)=\frac{2}{\beta}E_k\left(\frac{n\beta}{2a_1}\right)-D_k\left(\frac{n\beta}{2a_1}\right),$$

where

$$\begin{aligned}D_k(\frac{n\beta}{2a_1}) &= \sum_{r=0}^{k-1}\left(\frac{n\beta}{2a_1}\right)^r\sum_{p\in\mathcal{AGD}_{k,r}}\sum_{i\ge1} i\left(u_i+\frac{n\beta}{2a_1}l_i\right),\\ E_k(\frac{n\beta}{2a_1}) &= \sum_{r=0}^{k-1}\left(\frac{n\beta}{2a_1}\right)^r\sum_{p\in\mathcal{AGD}_{k,r}}\sum_{i\ge1}\left(C_{u_i}^2+\frac{n\beta}{2a_1}C_{l_i}^2\right),\end{aligned}$$

and $\mathcal{AGD}_{k,r}$ denotes the set of alternating Motzkin paths of length $2k$ with r rises, $u_i(p)$ denotes the number of rises between altitutes i and $i+1$ in path p and $l_i(p)$ denotes the number of level steps p taken from altitude i on odd-numbered steps.

Through the proof of Lemma 2.20 in [2], when $\frac{n\beta}{2a_1}=\gamma$,

$$\mathbb{E}\sum_{i=1}^n\left(\frac{\lambda_i}{2a_1}\right)^k = n\sum_{r=0}^{k-1}\gamma^r\frac{C_k^rC_{k-1}^r}{r+1}+\left(\frac{2}{\beta}-1\right)\int_{(1-\sqrt{\gamma})^2}^{(1+\sqrt{\gamma})^2} t^k\mu_L^\gamma(t)\mathrm{d}t+O(\frac{1}{n}). \tag{3.22}$$

Combining (3.21) and (3.22), since $\lim\limits_{n\to\infty}\frac{n\beta}{2a_1}=\gamma$, it is easy to find out, that for n large enough,

$$F_k\left(\frac{n\beta}{2a_1}\right)=\left(\frac{2}{\beta}-1\right)\int_{(1-\sqrt{\gamma})^2}^{(1+\sqrt{\gamma})^2}t^k\mu_L^{\gamma}(t)\mathrm{d}t+o(1).$$

Therefore, returning to (3.21),

$$\mathbb{E}\sum_{i=1}^{n}\left(\frac{\lambda_i}{2a_1}\right)^k=n\sum_{r=0}^{k-1}\left(\frac{n\beta}{2a_1}\right)^r\frac{C_k^rC_{k-1}^r}{r+1}+\left(\frac{2}{\beta}-1\right)\int_{(1-\sqrt{\gamma})^2}^{(1+\sqrt{\gamma})^2}t^k\mu_L^{\gamma}(t)\mathrm{d}t+o(1). \tag{3.23}$$

Naturally, if $0<\gamma\le 1$, (3.23) yields the following result:

$$\begin{aligned}\sqrt{\frac{a_1}{n}}\mathbb{E}\sum_{i=1}^{n}\left(\frac{\lambda_i}{2a_1}\right)^k=&\sqrt{a_1n}\sum_{r=0}^{k-1}\left(\frac{n\beta}{2a_1}\right)^r\frac{C_k^rC_{k-1}^r}{r+1}\\&+\sqrt{\frac{a_1}{n}}\left(\frac{2}{\beta}-1\right)\int_{(1-\sqrt{\gamma})^2}^{(1+\sqrt{\gamma})^2}t^k\mu_L^{\gamma}(t)\mathrm{d}t+o(1).\end{aligned}$$

By definition, $u_i(p)=0$ for any $p\in\mathcal{AGD}_{k,0}$ and any $i\ge 1$. This means, for the case $\lim\limits_{n\to\infty}\frac{n\beta}{2a_1}=0$, that $F_k(\frac{n\beta}{2a_1})=O(\frac{n}{a_1})$ for n large enough. Therefore,

$$\sqrt{\frac{a_1}{n}}\,\mathbb{E}\sum_{i=1}^{n}\left(\frac{\lambda_i}{2a_1}\right)^k=\sqrt{a_1n}\sum_{r=0}^{k-1}\left(\frac{n\beta}{2a_1}\right)^r\frac{C_k^rC_{k-1}^r}{r+1}+o(1)$$

for n large enough. The proof is now complete. □

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