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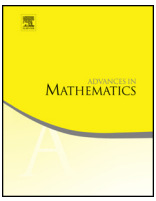

Relative Haagerup property for arbitrary von Neumann algebras

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ABSTRACT

We introduce the relative Haagerup approximation property for a unital, expected inclusion of arbitrary von Neumann algebras and show that if the smaller algebra is finite then the notion only depends on the inclusion itself, and not on the choice of the conditional expectation. Several variations of the definition are shown to be equivalent in this case, and in particular the approximating maps can be chosen to be unital and preserving the reference state. The concept is then applied to amalgamated free products of von Neumann algebras and used to deduce that the standard Haagerup property for a von Neumann algebra is stable under taking free products with amalgamation over finite-dimensional subalgebras. The general results are illustrated by examples coming from q-

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deformed Hecke-von Neumann algebras and von Neumann algebras of quantum orthogonal groups.

1. Introduction

The story of the group-theoretic Haagerup property began in a celebrated article [28] by Haagerup, who noted that the free group admits a sequence of positive-definite functions vanishing at infinity which pointwise converges to a constant function equal to 1; in other words, the free group von Neumann algebra admits a sequence of unital completely positive Herz-Schur multipliers which are in a sense 'small' and yet converge to the identity operator. Soon after that Choda gave in [18] a definition of the Haagerup property for a von Neumann algebra M equipped with a faithful normal trace in terms of existence of abstract approximating maps on M, which behave well with respect to the trace in question. Later Jolissaint proved in [32] that the property does not depend in fact on the choice of a trace as above, and Bannon and Fang showed in [5] that some of the requirements concerning the approximating maps can be weakened. For several years the study focused on finite von Neumann algebras, mainly as the motivating examples came from discrete groups. This changed with the articles [10], [19], which established the Haagerup property for the von Neumann algebras of certain discrete quantum groups, and the paper [21], which introduced and studied the analogous property for quantum groups themselves. Soon after that Okayasu and Tomatsu on one hand, and two authors of this paper on another, gave a definition of the Haagerup property for an arbitrary von Neumann algebra equipped with a faithful normal semifinite weight and proved that in fact the notion does not depend on the choice of the weight in question (see [15], [16], [36] as well as [14] and references therein). In all the cases above the Haagerup property should be thought of as a natural weakening of amenability/injectivity, which permits applying several approximation ideas and techniques beyond the class of amenable groups or algebras. The class of discrete groups enjoying the Haagerup property has good permanence properties, among which we would like to stress the fact that it is closed under taking free products amalgamated over finite subgroups ([17, Section 6]).

In several group-theoretic and operator algebraic contexts it is important to consider also *relative* properties; for example relative Property (T) is key to showing that $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ does not have the Haagerup property – which in turn has several von Neumann algebraic consequences, studied among other places in [30]. In the context of finite von Neumann algebras the relative Haagerup property appeared first in [8] in the study of Jones' towers associated with irreducible finite index subfactors, and was later applied in [41] as a key tool to obtain deep structural results about algebras admitting a certain type of Cartan inclusion (i.e. maximal abelian subalgebras with a 'sufficiently

rich' normalizer). Such Cartan inclusions are deeply related to von Neumann algebras of equivalence relations by the celebrated results of Feldman and Moore in [23]. The case of Cartan subalgebras was also the first in which a definition of a relative Haagerup property was proposed beyond finite von Neumann algebras [48], [1]. Notably the latter developments took place even before the usual Haagerup property for arbitrary von Neumann algebras was well understood.

The study in [48] and [1] was focused on the special case of Cartan subalgebras. In this paper we undertake a systematic investigation of a von Neumann algebraic relative Haagerup property for a unital inclusion $N \subseteq M$ equipped with a faithful normal conditional expectation $\mathbb{E}_N : M \to N$. Again we first define it in terms of a fixed faithful normal state (preserved by $\mathbb{E}_N$) but then quickly show that it depends only on the conditional expectation in question. Much more can be said in the case where N is assumed to be finite; here we obtain the following theorem, which is one of the main results of the paper. Relevant definitions can be found in Section 2; essentially for a triple $(M, N, \mathbb{E}_N)$ to have the relative Haagerup property we require the existence of completely positive, normal, N-bimodular maps on M which are $\mathbb{E}_N$-decreasing, L^2-compact (in the sense determined by the conditional expectation $\mathbb{E}_N$), uniformly bounded and converge point-strongly to the identity, see Definition 3.2.

Theorem A. *Suppose that $N \subseteq M$ is a unital, expected inclusion of von Neumann algebras and assume that N is finite. Then the relative Haagerup property of the triple $(M, N, \mathbb{E}_N)$ does not depend on the choice of a faithful normal conditional expectation $\mathbb{E}_N : M \to N$. Moreover if $(M, N, \mathbb{E}_N)$ has the relative Haagerup property and we are given a fixed state $\tau \in N_*$ we can always assume that the approximating maps are unital and $\tau \circ \mathbb{E}_N$-preserving.*

The key idea of the proof once again, as in [15], uses crossed products by modular actions and the passage to the semifinite setting that (Takai-)Takesaki duality permits. However, the relative context makes the technical details much more demanding and makes adapting the earlier methods – including those developed in [5] – significantly more complicated. On the other hand, allowing non-trivial inclusions allows us to significantly broaden the class of examples fitting into our framework and yields certain facts which are new even in the context of the standard Haagerup property of finite von Neumann algebras. This is exemplified by the next key result of this work and its corollary (which also requires proving a general theorem on the relative Haagerup property of amalgamated free products).

Theorem B. *Suppose that $N \subseteq M$ is a unital inclusion of von Neumann algebras equipped with a faithful normal conditional expectation $\mathbb{E}_N : M \to N$ and assume that N is finite-dimensional. Then the relative Haagerup property of the triple $(M, N, \mathbb{E}_N)$ is equivalent to the usual Haagerup property of M.*

Corollary C. *Suppose that $N \subseteq M_1$ and $N \subseteq M_2$ are unital inclusions of von Neumann algebras equipped with respective faithful normal conditional expectations. If N is finite-dimensional and both M_1, M_2 have the Haagerup property, then the amalgamated free product $M_1 *_N M_2$ also has the Haagerup property.*

We illustrate our results with examples coming on one hand from the class of q-deformed Hecke-von Neumann algebras of (virtually free) Coxeter groups, and on the other hand from discrete quantum groups.

Of particular interest is also the elementary case of $M = \mathcal{B}(\mathcal{H})$ which provides us with both triples $(\mathcal{B}(\mathcal{H}), N, \mathbb{E}_N)$ that do and do not have the relative Haagerup property. This is rather surprising as it gives examples of co-amenable inclusions of von Neumann algebras in the sense of [39] (see also [6]) without the relative Haagerup property.

The detailed plan of the paper is as follows. After this introduction we recall some facts regarding von Neumann algebras, their modular theory and completely positive approximations in Section 2, and introduce (certain variants of) the definition of the relative Haagerup property in Section 3. The latter section also contains the initial discussion of the independence of our notion of various ingredients, mainly in the semifinite setting. Section 4, the most technical part of the paper, introduces the crossed product arguments allowing us to reduce the general problem to the semifinite case. Section 5 contains the main general results of the paper; in particular the above Theorem A is a combination of Theorem 5.4 and Theorem 5.5. In Section 6 we briefly describe the known examples of Haagerup inclusions related to Cartan subalgebras. Here we also study the case $M = \mathcal{B}(\mathcal{H})$ which leads to very interesting counterexamples. In Section 7 we show that in the case of a finite-dimensional subalgebra the relative Haagerup property is equivalent to the Haagerup property of the larger algebra and prove Theorem B above (which is Theorem 7.6). In Section 8 we discuss the behaviour of the relative Haagerup property with respect to the amalgamated free product construction and show Corollary C (i.e. Corollary 8.2). In Section 8 we also discuss briefly a consequence of these results for the Hecke-von Neumann algebras associated to virtually free Coxeter groups. Finally in a short Section 9 we present an example of a Haagerup inclusion coming from quantum groups, which in fact is even strongly of finite index.

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2. Preliminaries and notation

2.1. General notation

We write $\mathbb{N} := \{1, 2, ...\}$, $C(\mathbb{R})$ denotes the continuous functions $\mathbb{R} \to \mathbb{C}$ and let $C_c(\mathbb{R})$ denote the functions in $C(\mathbb{R})$ with compact support. For $f, g \in L^1(\mathbb{R})$ we define the convolution and involution by

$$(f * g)(s) = \int_{\mathbb{R}} f(t)g(s-t)dt, \qquad f^*(s) = \overline{f(-s)}, \qquad s \in \mathbb{R}.$$

The bounded operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$ and $\otimes$ is the von Neumann algebraic tensor product or the tensor product of Hilbert spaces, as will be clear from the context. Hilbert space inner products are linear on the left side.

2.2. General von Neumann algebra theory

For standard results on the theory von Neumann algebras we refer to [44], [45], [46], [2]. For the theory of operator spaces and completely bounded maps we refer to [22], [38]. For several results on approximation properties and the Haagerup property we refer to [11], [17].

Assumption. We will assume throughout that the von Neumann algebras we study are σ-finite, i.e. they admit faithful normal states.

Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A self-adjoint (possibly unbounded) operator h on $\mathcal{H}$ is said to be *affiliated with* M if for all $k \in \mathbb{N}$ the corresponding spectral projection $E_{[-k,k]}(h)$ is an element of M. Equivalently, h is affiliated with M if and only if h commutes with all unitaries in the commutant $M' \subseteq \mathcal{B}(\mathcal{H})$.

We will always assume inclusions of von Neumann algebras $N \subseteq M$ to be unital in the sense that $1_M \in N$, and conditional expectations to be faithful and normal. We will usually repeat these conditions throughout the text. For a functional $\varphi \in M_*$ in the predual of M and elements $a, b \in M$ we denote by $a\varphi b \in M_*$ the normal functional given by $(a\varphi b)(x) := \varphi(bxa)$, $x \in M$, and further write $a\varphi$ for $a\varphi 1$ and φb for $1\varphi b$. If $\varphi \in M_*$ is faithful, normal and positive, we write $L^2(M, \varphi)$ for the GNS Hilbert space associated with φ and Ω_φ for the corresponding cyclic vector. We will usually identify M with its image under the GNS representation, so $M \subseteq \mathcal{B}(L^2(M, \varphi))$. We further write $\|x\|_{2,\varphi} := \varphi(x^*x)^{1/2}$ for $x \in M$ or, if φ is clear from the context, $\|x\|_2 := \|x\|_{2,\varphi}$.

An *action* $\mathbb{R} \overset{\alpha}{\curvearrowright} M$ of the group $\mathbb{R}$ on a von Neumann algebra M is a group homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(M)$, $t \mapsto \alpha_t$ such that for every $x \in M$ the map $t \mapsto \alpha_t(x)$ is strongly continuous. We denote the corresponding *crossed product von Neumann algebra* by $M \rtimes_\alpha \mathbb{R}$.

The following lemma is standard.

Lemma 2.1. *Let φ be a faithful normal state on a von Neumann algebra M. Then, on bounded subsets of M the strong topology coincides with the topology induced by the norm $\|x\|_{2,\varphi} = \varphi(x^*x)^{1/2}$, $x \in M$.*

Proof. Recall that by [45] we may assume $M \subseteq \mathcal{B}(L^2(M,\varphi))$ and that on bounded sets the strong topology of M does not change under the choice of a faithful representation. Now if $(x_i)_{i\in I}$ is a net in M converging strongly to x, then $\|x_i - x\|_{2,\varphi} = \|(x_i - x)\Omega_\varphi\| \to 0$. Conversely, suppose that $(x_i)_{i\in I}$ is a bounded net in M such that $\|x_i - x\|_{2,\varphi} = \|(x_i - x)\Omega_\varphi\| \to 0$. Then for $a \in M$ analytic for the modular group σ^φ (see Section 2.3) we have by [46, Lemma 3.18 (i)] that

$$\|(x_i - x)a\Omega_\varphi\| \leq \|\sigma^\varphi_{i/2}(a)\|\|(x_i - x)\Omega_\varphi\| \to 0.$$

Since by [46, Lemma VIII.2.3] such elements $a\Omega_\varphi$, $a \in M$ are dense in $L^2(M,\varphi)$ and $(x_i)_{i\in I}$ is bounded, we conclude by a 2ϵ-estimate that $x_i \to x$ strongly. □

Note that one implication above naturally does not require the uniform boundedness assumption, provided that we assume that M is represented on its standard Hilbert space $L^2(M,\varphi)$.

2.3. Tomita-Takesaki modular theory

Let M be a von Neumann algebra with a faithful normal positive functional $\varphi \in M_*$. We let S_φ be the closure of the operator

$$L^2(M,\varphi) \to L^2(M,\varphi) : x\Omega_\varphi \mapsto x^*\Omega_\varphi, \qquad x \in M.$$

Let $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$ be the (anti-linear) polar decomposition where J_φ is the modular conjugation and Δ_φ the modular operator. We have the modular automorphism group $\sigma^\varphi_t(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$. Then $(M, L^2(M,\varphi), J_\varphi, L^2(M,\varphi)^+)$ is the standard form of M, where the positive cone is given by

$$L^2(M,\varphi)^+ := \overline{\{x(JxJ)\Omega_\varphi \mid x \in M\}} \subseteq L^2(M,\varphi).$$

The standard form is uniquely determined up to a unique (unitarily implemented) isomorphism. For $x \in M$ and $\xi \in L^2(M,\varphi)$ we write $\xi x := J_\varphi x^* J_\varphi \xi$. An element $x \in M$ is called *analytic* for σ^φ if the function $\mathbb{R} \ni t \mapsto \sigma^\varphi_t(x) \in M$ extends to a (necessarily unique) analytic function on the complex plane $\mathbb{C}$. In this case we write $\sigma^\varphi_z(x)$ for the extension at $z \in \mathbb{C}$.

The *centralizer* of a von Neumann algebra with respect to a faithful normal state φ is the set $M^\varphi := \{x \in M \mid \forall_{y\in M} \varphi(xy) = \varphi(yx)\}$, by [46, Theorem VIII.2.6] equivalently

described as $\{x \in M \mid \sigma_t^\varphi(x) = x, \quad t \in \mathbb{R}\}$. We will often consider the situation where $N \subseteq M$ is a unital embedding, equipped with a faithful normal conditional expectation $\mathbb{E}_N : M \to N$, $\tau \in N_*$ is a faithful tracial state, and $\varphi = \tau \circ \mathbb{E}_N$. Then an easy computation shows that $N \subseteq M^\varphi$.

2.4. Completely positive maps

Let A, B be von Neumann algebras with faithful normal positive functionals φ and ψ respectively. For a linear map $\Phi : A \to B$ we say that its *L^2-implementation* $\Phi^{(2)}$ (with respect to φ and ψ) exists if the map $x\Omega_\varphi \mapsto \Phi(x)\Omega_\psi$ extends to a bounded operator $\Phi^{(2)} : L^2(A, \varphi) \to L^2(B, \psi)$. This is the case if and only if there exists a constant $C > 0$ such that for all $x \in A$,

$$\psi(\Phi(x)^*\Phi(x)) \leq C\varphi(x^*x).$$

In particular, if Φ is 2-positive with $\psi \circ \Phi \leq \varphi$, the Kadison-Schwarz inequality implies that $\Phi^{(2)}$ exists with $\|\Phi^{(2)}\| \leq \|\Phi(1)\|^{1/2}$. Indeed, for $x \in A$

$$\begin{aligned}\|\Phi(x)\Omega_\psi\|_2^2 = \psi(\Phi(x)^*\Phi(x)) &\leq \|\Phi(1)\|\psi(\Phi(x^*x)) \\ &\leq \|\Phi(1)\|\varphi(x^*x) = \|\Phi(1)\|\|x\Omega_\varphi\|_2^2.\end{aligned}$$

This implies that if Φ is contractive then so is $\Phi^{(2)}$.

The general principle of the following lemma was used as part of a proof in [15] and [32] a number of times. Here we present it separately. We will also need several straightforward variations of this lemma. Because they can be proved in a very similar way, we shall not state them here. The essence of the result is that, given two nets of maps with suitable properties that strongly converge to the identity, the composition of these maps gives rise to a net that also converges to the identity in the strong operator topology.

Lemma 2.2. *Let (A, φ) and (B, φ_j), $j \in \mathbb{N}$ be pairs of von Neumann algebras equipped with faithful normal states. Consider a normal completely positive map $\pi : A \to B$, a bounded sequence of normal completely positive maps $(\Psi_j : B \to A)_{j\in\mathbb{N}}$ and for every $j \in \mathbb{N}$ a bounded net of completely positive maps $(\Phi_{j,k} : B \to B)_{k\in K_j}$. Assume that for all $j \in \mathbb{N}$, $k \in K_j$ the inequalities $\varphi_j \circ \pi \leq \varphi$, $\varphi \circ \Psi_j \leq \varphi_j$ and $\varphi_j \circ \Phi_{j,k} \leq \varphi_j$ hold, that $\Psi_j \circ \pi(x) \to x$ strongly in j for every $x \in A$ and that for every $j \in \mathbb{N}$, $x \in B$ we have $\Phi_{j,k}(x) \to x$ strongly in k. Then there exists a directed set $\mathcal{F}$ and a function $(\widetilde{j}, \widetilde{k})$: $\mathcal{F} \to \{(j, k) \mid j \in \mathbb{N}, k \in K_j\}$, $F \mapsto (\widetilde{j}(F), \widetilde{k}(F))$ such that $\Psi_{\widetilde{j}(F)} \circ \Phi_{\widetilde{j}(F),\widetilde{k}(F)} \circ \pi(x) \to x$ strongly in F for every $x \in A$.*

Proof. For $j \in \mathbb{N}$ and $k \in K_j$ write

$$\pi_j^{(2)} : L^2(A, \varphi) \to L^2(B, \varphi_j), x\Omega_\varphi \mapsto \pi(x)\Omega_{\varphi_j},$$

$$\Psi_j^{(2)} : L^2(B,\varphi_j) \to L^2(A,\varphi), x\Omega_{\varphi_j} \mapsto \Psi_{j,k}(x)\Omega_\varphi,$$
$$\Phi_{j,k}^{(2)} : L^2(B,\varphi_j) \to L^2(B,\varphi_j), x\Omega_{\varphi_j} \mapsto \Phi_{j,k}(x)\Omega_{\varphi_j}$$

for the corresponding L^2-implementations with respect to φ and φ_j. Let $C \geq 1$ be a bound for the norms of $(\Psi_j)_{j\in\mathbb{N}}$ and hence for the norms of $(\Psi_j^{(2)})_{j\in\mathbb{N}}$. We shall make use of the fact that on bounded sets the strong topology coincides with the L^2-topology determined by a state, see Lemma 2.1. Therefore we have strong limits $\Psi_j^{(2)}\pi_j^{(2)} \to 1$ in $\mathcal{B}(L^2(A,\varphi))$ and $\Phi_{j,k}^{(2)} \to 1$ in $\mathcal{B}(L^2(B,\varphi_j))$. Now let $F \subseteq L^2(A,\varphi)$ be a finite subset. We may find $j = \widetilde{j}(F) \in \mathbb{N}$ such that for all $\xi \in F$,

$$\|\Psi_j^{(2)}\pi_j^{(2)}\xi - \xi\|_2 < |F|^{-1}.$$

In turn, we may find $k = \widetilde{k}(j,F) = \widetilde{k}(F)$ such that for all $\xi \in F$,

$$\|\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi - \pi_j^{(2)}\xi\|_2 < |F|^{-1}.$$

From the triangle inequality and by using that the operator norm of $\Psi_j^{(2)}$ is bounded by C,

$$\begin{aligned}\|\Psi_j^{(2)}\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi - \xi\|_2 &\leq \|\Psi_j^{(2)}\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi - \Psi_j^{(2)}\pi_j^{(2)}\xi\|_2 + \|\Psi_j^{(2)}\pi_j^{(2)}\xi - \xi\|_2\\ &\leq \|\Psi_j^{(2)}\|\|\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi - \pi_j^{(2)}\xi\|_2 + \|\Psi_j^{(2)}\pi_j^{(2)}\xi - \xi\|_2\\ &< (1+C)|F|^{-1}.\end{aligned}$$

This implies that $\Psi_{\widetilde{j}(F)}^{(2)}\Phi_{\widetilde{j}(F),\widetilde{k}(F)}^{(2)}\pi_{\widetilde{j}(F)}^{(2)} \to 1$ strongly in $\mathcal{B}(L^2(A,\varphi))$ where the net is indexed by all finite subsets of $L^2(A,\varphi)$ partially ordered by inclusion. Using once more Lemma 2.1, one sees that for $x \in A$ we have that $\Psi_{\widetilde{j}(F)} \circ \Phi_{\widetilde{j}(F),\widetilde{k}(F)} \circ \pi(x) \to x$ strongly. The claim follows. □

3. Relative Haagerup property

In this section we introduce the relative Haagerup property for inclusions of general σ-finite von Neumann algebras and consider natural variations of the definition. For this, fix a triple (M,N,φ) where $N \subseteq M$ is a unital inclusion of von Neumann algebras and where φ is a faithful normal positive functional on M whose corresponding modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ satisfies $\sigma_t^\varphi(N) \subseteq N$ for all $t \in \mathbb{R}$. To keep the notation short, we will often just write (M,N,φ) and will implicitly assume that the triple satisfies the mentioned conditions. By [46, Theorem IX.4.2] the assumption $\sigma_t^\varphi(N) \subseteq N$, $t \in \mathbb{R}$ is equivalent to the existence of a (uniquely determined) φ-preserving (necessarily faithful) normal conditional expectation $\mathbb{E}_N^\varphi : M \to N$. If the corresponding functional φ is clear, we will often just write $\mathbb{E}_N$ instead of $\mathbb{E}_N^\varphi$ (compare also with Subsection 3.2).

3.1. First definition of relative Haagerup property

For a triple (M, N, φ) as before define the *Jones projection*

$$e_N^\varphi := \mathbb{E}_N^{(2)} : L^2(M,\varphi) \to L^2(M,\varphi)$$

which is the orthogonal projection onto $L^2(N,\varphi) \subseteq L^2(M,\varphi)$ and let $\langle M, N\rangle \subseteq \mathcal{B}(L^2(M,\varphi))$ be the von Neumann subalgebra generated by e_N^φ and M. This is the *Jones construction*. We will usually write e_N instead of e_N^φ if there is no ambiguity. Further set

$$\mathcal{K}_{00}(M,N,\varphi) := \mathrm{Span}\{xe_N y \mid x,y \in M\} \subseteq \mathcal{B}(L^2(M,\varphi))$$

and

$$\mathcal{K}(M,N,\varphi) := \overline{\mathcal{K}_{00}(M,N,\varphi)}.$$

Then $\mathcal{K}_{00}(M,N,\varphi)$ is a (not necessarily closed) two-sided ideal in $\langle M,N\rangle$ whose elements are called the *finite rank operators* relative to N. Similarly, $\mathcal{K}(M,N,\varphi)$ is a closed two-sided ideal in $\langle M,N\rangle$ whose elements are called the *compact operators* relative to N. Note that if $N = \mathbb{C}1_M$, then e_N is a rank one projection and the operators in $\mathcal{K}(M,N,\varphi)$ are precisely the compact operators on $L^2(M,\varphi)$.

Remark 3.1. In the following it is often convenient to identify a finite rank operator $ae_Nb \in \mathcal{K}_{00}(M,N,\varphi)$, $a,b \in M$ with the map $a\mathbb{E}_N(b\,\cdot\,) : M \to M$. The latter does not depend on φ (but only on the conditional expectation $\mathbb{E}_N$), and the notation is naturally compatible with the inclusion $M \subseteq L^2(M,\varphi)$. We will often write $a\mathbb{E}_N b := a\mathbb{E}_N(b\,\cdot\,)$.

Definition 3.2. Let $N \subseteq M$ be a unital inclusion of von Neumann algebras and let φ be a faithful normal positive functional on M with $\sigma_t^\varphi(N) \subseteq N$ for all $t \in \mathbb{R}$. We say that the triple (M,N,φ) has the *relative Haagerup property* (or just *property (rHAP)*) if there exists a net $(\Phi_i)_{i\in I}$ of normal maps $\Phi_i : M \to M$ such that

(1) Φ_i is completely positive and $\sup_i \|\Phi_i\| < \infty$ for all $i \in I$;
(2) Φ_i is an N-N-bimodule map for all $i \in I$;
(3) $\Phi_i(x) \to x$ strongly for every $x \in M$;
(4) $\varphi \circ \Phi_i \leq \varphi$ for all $i \in I$;
(5) For every $i \in I$ the L^2-implementation

$$\Phi_i^{(2)} : L^2(M,\varphi) \to L^2(M,\varphi), x\Omega_\varphi \mapsto \Phi_i(x)\Omega_\varphi,$$

is contained in $\mathcal{K}(M,N,\varphi)$.

Remark 3.3. (1) In many applications φ will be a faithful normal state, but for notational convenience we shall rather work in the more general setting. Note that we may always normalize φ to be a state and that the definition of the relative Haagerup property does not change under this normalization.
(2) Note that in [41] (see also [33]) a different notion of relative compactness is used to define the relative Haagerup property. It coincides with ours in case $N' \cap M \subseteq N$. However, the alternative notion is not very suitable beyond the tracial situation since it requires the existence of finite projections; we will return to this issue in Subsection 6.1.
(3) In the case where $N = \mathbb{C}1_M$ Definition 3.2 recovers the usual definition of the (non-relative) Haagerup property, see [15, Definition 3.1].

There is a number of immediate variations of Definition 3.2. For instance, one may replace the condition (1) by one of the following stronger conditions:

(1′) For every $i \in I$ the map Φ_i is contractive completely positive.
(1″) For every $i \in I$ the map Φ_i is unital completely positive.

We may also replace the condition (4) by the following condition:

(4′) $\varphi \circ \Phi_i = \varphi$.

One of the results that we shall prove is that if the subalgebra N is finite, then condition (4) is redundant. We will further prove that in this setting the approximating maps Φ_i, $i \in I$ can be chosen to be unital and state-preserving implying that all the variations of the relative Haagerup property from above coincide. To simplify the statements of the following sections, let us introduce the following auxiliary notion, which is a priori weaker (see Section 2).

Definition 3.4. Let $N \subseteq M$ be a unital inclusion of von Neumann algebras and let φ be a faithful normal positive functional on M with $\sigma_t^{\varphi}(N) \subseteq N$ for all $t \in \mathbb{R}$. We say that the triple (M, N, φ) has *property (rHAP)*$^-$ if there exists a net $(\Phi_i)_{i\in I}$ of normal maps $\Phi_i : M \to M$ such that

(1) Φ_i is completely positive for all $i \in I$;
(2) Φ_i is an N-N-bimodule map for all $i \in I$;
(3) $\|\Phi_i(x) - x\|_{2,\varphi} \to 0$ for every $x \in M$;
(4) For every $i \in I$ the L^2-implementation

$$\Phi_i^{(2)} \colon L^2(M, \varphi) \to L^2(M, \varphi), x\Omega_\varphi \mapsto \Phi_i(x)\Omega_\varphi,$$

exists and is contained in $\mathcal{K}(M, N, \varphi)$.

3.2. Dependence on the positive functional: reduction to the dependence on the conditional expectation

Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N : M \to N$. Recall that for every faithful normal positive functional φ on M with $\varphi \circ \mathbb{E}_N = \varphi$ the corresponding modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ satisfies $\sigma_t^\varphi(N) \subseteq N$ for all $t \in \mathbb{R}$. Note that such a functional always exists, as it suffices to pick a faithful normal state $\omega \in N_*$ (which exists by our standing σ-finiteness assumption) and set $\varphi = \omega \circ \mathbb{E}_N$. In this subsection we will examine the dependence of the relative Haagerup property of (M, N, φ) on the functional φ. We shall prove that the property rather depends on the conditional expectation $\mathbb{E}_N$ than on φ.

Lemma 3.5. *Let $\Phi : M \to M$ be N-N-bimodular. Then the following statements are equivalent:*

(1) $\mathbb{E}_N \circ \Phi \leq \mathbb{E}_N$ (resp. $\mathbb{E}_N \circ \Phi = \mathbb{E}_N$).
(2) For all $\varphi \in M_*^+$ with $\varphi \circ \mathbb{E}_N = \varphi$ we have $\varphi \circ \Phi \leq \varphi$ (resp. $\varphi \circ \Phi = \varphi$).
(3) There exists a faithful functional $\varphi \in M_*^+$ with $\varphi \circ \mathbb{E}_N = \varphi$ such that $\varphi \circ \Phi \leq \varphi$ (resp. $\varphi \circ \Phi = \varphi$).

Further, the following statements are equivalent:

(4) There exists $C > 0$ such that $\mathbb{E}_N(\Phi(x)^*\Phi(x)) \leq C\mathbb{E}_N(x^*x)$ for all $x \in M$.
(5) There exists $C > 0$ such that for all $\varphi \in M_*^+$ with $\varphi \circ \mathbb{E}_N = \varphi$ and $x \in M$ we have $\varphi(\Phi(x)^*\Phi(x)) \leq C\varphi(x^*x)$.
(6) There exists $C > 0$ and a faithful functional $\varphi \in M_*^+$ with $\varphi \circ \mathbb{E}_N = \varphi$ such that for all $x \in M$ we have $\varphi(\Phi(x)^*\Phi(x)) \leq C\varphi(x^*x)$.

In particular, if the L^2-implementation of Φ with respect to φ exists, then it exists with respect to any other ψ with $\psi \circ \mathbb{E}_N = \psi$.

Proof. We prove the statements for the inequalities; the respective cases with equalities follow similarly. The implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) of the first three statements are trivial. For the implication (3) $\Rightarrow$ (1) take φ as in (3). For $x \in N$ consider the positive functional $x^*\varphi x \in M_*^+$ which again satisfies $(x^*\varphi x) \circ \mathbb{E}_N = x^*\varphi x$. Then, for $y \in M^+$

$$(x^*\varphi x) \circ \mathbb{E}_N \circ \Phi(y) = \varphi \circ \mathbb{E}_N \circ \Phi(xyx^*) \leq \varphi \circ \mathbb{E}_N(xyx^*) = (x^*\varphi x) \circ \mathbb{E}_N(y).$$

Since the restrictions of functionals $x^*\varphi x$, $x \in N$ to N are dense in N_*^+ we conclude that $\mathbb{E}_N \circ \Phi \leq \mathbb{E}_N$.

The equivalence of the statements (4), (5) and (6) follows in a similar way. □

The following lemma shows that in good circumstances compactness of the L^2-implementations does not depend on the choice of the state.

Lemma 3.6. *Let $\varphi, \psi \in M_*^+$ be faithful with $\varphi \circ \mathbb{E}_N = \varphi$ and $\psi \circ \mathbb{E}_N = \psi$. Let further $\Phi : M \to M$ be a completely positive N-N-bimodule map whose L^2-implementation $\Phi_\varphi^{(2)}$ with respect to φ exists (hence, by Lemma 3.5, the L^2-implementation $\Phi_\psi^{(2)}$ of Φ with respect to ψ exists as well). Then, $\Phi_\varphi^{(2)} \in \mathcal{K}(M, N, \varphi)$ if and only if $\Phi_\psi^{(2)} \in \mathcal{K}(M, N, \psi)$.*

Proof. Let U be the unique unitary mapping the standard form $(M, L^2(M, \varphi), J_\varphi, P_\varphi)$ to the standard form $(M, L^2(M, \psi), J_\psi, P_\psi)$, see [26, Theorem 2.3]. It restricts to the unique unitary map between the standard forms $(N, L^2(N, \varphi), J_{\varphi|_N}, P_{\varphi|_N})$ and $(N, L^2(N, \psi), J_{\psi|_N}, P_{\psi|_N})$. Indeed, for all $x \in M$, $\varphi(x) = \langle xU\Omega_\varphi, U\Omega_\varphi\rangle$ and by [26, Lemma 2.10], $U\Omega_\varphi$ is the unique element in $L^2(M, \psi)$ satisfying this equation. On the other hand, applying [26, Lemma 2.10] to $\varphi|_N$ implies the existence of a unique vector $\xi \in L^2(N, \psi)$ such that $\varphi(x) = \langle x\xi, \xi\rangle$ for all $x \in N$. By approximating ξ by elements in $N\Omega_\psi$ and by using the assumptions $\varphi \circ \mathbb{E}_N = \varphi$ and $\psi \circ \mathbb{E}_N = \psi$ one deduces that for all $x \in M$,

$$\varphi(x) = \varphi \circ \mathbb{E}_N(x) = \langle \mathbb{E}_N(x)\xi, \xi\rangle = \langle x\xi, \xi\rangle$$

and hence $U\Omega_\varphi = \xi \in L^2(N, \psi)$. This implies $U(N\Omega_\varphi) \subseteq L^2(N, \psi)$ and therefore (by density and symmetry) $U(L^2(N, \psi)) = L^2(N, \psi)$. Finally, using that $\sigma_t^\varphi(N) = N$ and hence $J_{\varphi|_N} = (J_\varphi)|_{L^2(N,\varphi)}$ (and similarly $J_{\psi|_N} = (J_\psi)|_{L^2(N,\psi)}$) it is straightforward to check that the restriction of U satisfies the other properties of the unique unitary mapping between the standard forms $(N, L^2(N, \varphi), J_{\varphi|_N}, P_{\varphi|_N})$ and $(N, L^2(N, \psi), J_{\psi|_N}, P_{\psi|_N})$.

Since $e_N^\varphi = (\mathbb{E}_N^\varphi)^{(2)}$ is the orthogonal projection of $L^2(M, \varphi)$ onto $L^2(N, \varphi)$ and $e_N^\psi = (\mathbb{E}_N^\psi)^{(2)}$ is the orthogonal projection of $L^2(M, \psi)$ onto $L^2(N, \psi)$, we see that $U^* e_N^\psi U = e_N^\varphi$. Hence, for every map Λ of the form $\Lambda = a\mathbb{E}_N b$ with $a, b \in M$ the L^2-implementation $\Lambda_\varphi^{(2)}$ with respect to φ and the L^2-implementation $\Lambda_\psi^{(2)}$ with respect to ψ exist with $\Lambda_\varphi^{(2)} = ae_N^\varphi b = U^* ae_N^\psi bU = U^*\Lambda_\psi^{(2)}U$.

Now assume that $\Phi_\varphi^{(2)} \in \mathcal{K}(M, N, \varphi)$. Then there exists a sequence $(\Phi_k : M \to M)_{k\in\mathbb{N}}$ of maps of the form $\Phi_k = \sum_{i=1}^{N_k} a_{i,k}\mathbb{E}_N b_{i,k}$ with $N_k \in \mathbb{N}$ and $a_{1,k}, b_{1,k}, ..., a_{N_k,k}, b_{N_k,k} \in M$ whose L^2-implementations $\Phi_{k,\varphi}^{(2)} \in \mathcal{K}_{00}(M, N, \varphi)$ (with respect to φ) norm-converge to $\Phi_\varphi^{(2)}$. By the above, $U\Phi_{k,\varphi}^{(2)}U^* \in \mathcal{K}_{00}(M, N, \psi)$ is given by $x\Omega_\psi \mapsto \Phi_k(x)\Omega_\psi$ for $x \in M$. We claim that the sequence $(U\Phi_{k,\varphi}^{(2)}U^*)_{k\in\mathbb{N}} \subseteq \mathcal{K}_{00}(M, N, \psi)$ norm-converges to $\Phi_\psi^{(2)}$. Indeed, by the density of the set of all elements of the form $x(\varphi|_N)x^*$, $x \in N$ in N_*^+ we find a net $(x_i)_{i\in I} \subseteq N$ such that $x_i(\varphi|_N)x_i^* \to \psi|_N$. In combination with $\varphi \circ \mathbb{E}_N = \varphi$ and $\psi \circ \mathbb{E}_N = \psi$ this also implies $x_i\varphi x_i^* \to \psi$. For $y \in M$ and $k \in \mathbb{N}$,

$$\begin{aligned}\|(\Phi_\psi^{(2)} - U\Phi_{k,\varphi}^{(2)}U^*)y\Omega_\psi\|_{2,\psi}^2 &= \|(\Phi(y) - \Phi_k(y))\Omega_\psi\|_{2,\psi}^2 \\ &= \lim_i \|(\Phi(y) - \Phi_k(y))x_i\Omega_\varphi\|_{2,\varphi}^2\end{aligned}$$

$$
\begin{aligned}
&= \lim_i \|(\Phi_\varphi^{(2)} - \Phi_{k,\varphi}^{(2)})yx_i\Omega_\varphi\|_{2,\varphi}^2 \\
&\leq \lim_i \|\Phi_\varphi^{(2)} - \Phi_{k,\varphi}^{(2)}\|^2 \varphi(x_i^* y^* y x_i) \\
&= \|\Phi_\varphi^{(2)} - \Phi_{k,\varphi}^{(2)}\|^2 \psi(y^* y),
\end{aligned}
$$

where in the third step we used the N-N-bimodularity of Φ and the right N-modularity of Φ_k. Now, $\Phi_{k,\varphi}^{(2)} \to \Phi_\varphi^{(2)}$ and $(U\Phi_{k,\varphi}^{(2)}U^*)_{k\in\mathbb{N}}$ is a Cauchy sequence, hence the above inequality leads to $U\Phi_{k,\varphi}^{(2)}U^* \to \Phi_\psi^{(2)}$ as claimed. In particular, $\Phi_\psi^{(2)} \in \mathcal{K}(M,N,\psi)$ which finishes the proof. □

Theorem 3.7. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N : M \to N$. Further, let $\varphi, \psi \in M_*^+$ be faithful normal positive functionals with $\varphi \circ \mathbb{E}_N = \varphi$ and $\psi \circ \mathbb{E}_N = \psi$. Then the triple (M,N,φ) has property (rHAP) (resp. property (rHAP)$^-$) if and only if the triple (M,N,ψ) has property (rHAP) (resp. property (rHAP)$^-$). In particular, property (rHAP) (resp. property (rHAP)$^-$) only depends on the triple $(M,N,\mathbb{E}_N)$.*

Proof. It follows from Lemma 3.5 and Lemma 3.6 that if $(\Phi_j)_{j\in J}$ is a net of approximating maps witnessing the relative Haagerup property of (M,N,φ) (resp. property (rHAP)$^-$ of (M,N,φ)), then it also witnesses the Haagerup property of (M,N,ψ) (resp. property (rHAP)$^-$ of (M,N,ψ)) and vice versa. □

We will later see that in the case where the von Neumann subalgebra N is finite the statement in Theorem 3.7 can be strengthened: in this case property (rHAP) (and equivalently property (rHAP)$^-$) does not even depend on the choice of the conditional expectation $\mathbb{E}_N$.

Motivated by Theorem 3.7 we introduce the following natural definition.

Definition 3.8. Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N : M \to N$. We say that the triple $(M,N,\mathbb{E}_N)$ has the *relative Haagerup property* (or just *property (rHAP)*) if (M,N,φ) has the relative Haagerup property for some (equivalently any) faithful normal positive functional $\varphi \in M_*^+$ with $\varphi \circ \mathbb{E}_N = \varphi$. The same terminology shall be adopted for property (rHAP)$^-$.

3.3. State preservation, contractivity and unitality of the approximating maps in a special case

In this subsection we will prove that the relative Haagerup property of certain triples $(M,N,\mathbb{E}_N)$ may be witnessed by approximating maps that satisfy extra conditions, such as state-preservation, contractivity and unitality. This will play a crucial role in Section 5. The approach is inspired by [5, Section 2], where ideas from [32] were used.

Lemma 3.9. *Let M be a von Neumann algebra, $\varphi \in M_*$ a faithful normal state and $y \in M$. If $y\varphi = \varphi y$ (i.e. $y \in M^\varphi$), then $y\Omega_\varphi = \Omega_\varphi y$.*

Proof. As mentioned in Section 2.3, by [46, Theorem VIII.2.6] we have that $\sigma_t^\varphi(y) = y$ for all $t \in \mathbb{R}$. But then y is analytic and moreover $\sigma_{-i/2}^\varphi(y) = y$. Hence

$$\Omega_\varphi y = J_\varphi y^* J_\varphi \Omega_\varphi = J_\varphi \sigma_{-i/2}^\varphi(y^*) J_\varphi \Omega_\varphi = J_\varphi \Delta_\varphi^{1/2} y^* \Delta_\varphi^{-1/2} J_\varphi \Omega_\varphi = S_\varphi y^* S_\varphi \Omega_\varphi = y\Omega_\varphi.$$

The claim follows. □

Proposition 3.10. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras that admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite and let $\tau \in N_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M. Let further $\Phi : M \to M$ be a normal, completely positive, N-N-bimodular map for which there exists $\delta > 0$ such that $c := \Phi(1) \leq 1 - \delta$ and $\varphi \circ \Phi \leq (1-\delta)\varphi$. Then one can find $a, b \in N' \cap M$ such that $a \geq 0$, $\mathbb{E}_N(a) = 1$, $a\mathbb{E}_N(b^*b) = \mathbb{E}_N(b^*b)a = 1 - c$ and $b\varphi b^* = \varphi - \varphi \circ \Phi$.*

Proof. The complete positivity of Φ implies that $0 \leq \|\Phi\| = \|\Phi(1)\| = \|c\| \leq 1 - \delta$, hence the map Φ must be contractive. It is clear that $c = \Phi(1) \geq 0$. Further, since $\mathbb{E}_N(1-c) > \mathbb{E}_N(\delta) = \delta$, the element $\mathbb{E}_N(1-c) \in N$ is boundedly invertible. Additionally, the N-N-bimodularity of Φ implies that for every $n \in N$,

$$nc = n\Phi(1) = \Phi(n) = \Phi(1)n = cn,$$

so $c \in N' \cap M$. The latter two observations imply that for

$$a := (1-c)(\mathbb{E}_N(1-c))^{-1}$$

we have $a \in N' \cap M$, $a \geq 0$ and $\mathbb{E}_N(a) = 1$.

Consider the positive normal functional $\varphi - \varphi \circ \Phi \in M_*$. By [26, Lemma 2.10] there exists a unique vector $\xi \in L^2(M, \varphi)^+$ such that $(\varphi - \varphi \circ \Phi)(x) = \langle x\xi, \xi \rangle$ for all $x \in M$. Note that $\{J_\varphi x \Omega_\varphi \mid x \in M\}$ is dense in $L^2(M, \varphi)$ and define the linear map

$$b : L^2(M, \varphi) \to L^2(M, \varphi),\ J_\varphi x \Omega_\varphi \mapsto J_\varphi x \xi.$$

It is contractive since

$$\|b(J_\varphi x \Omega_\varphi)\|_2^2 = \|J_\varphi x \xi\|_2^2 = \|x\xi\|_2^2 = (\varphi - \varphi \circ \Phi)(x^*x) \leq \varphi(x^*x) = \|x\Omega_\varphi\|_2^2 = \|J_\varphi x \Omega_\varphi\|_2^2$$

for all $x \in M$. Further, for $x, y \in M$,

$$bJ_\varphi x J_\varphi(J_\varphi y \Omega_\varphi) = bJ_\varphi xy\Omega_\varphi = J_\varphi xy\xi = J_\varphi x J_\varphi J_\varphi y \xi = J_\varphi x J_\varphi b(J_\varphi y \Omega_\varphi).$$

It hence follows that b and $J_\varphi x J_\varphi$ commute and therefore that $b \in (J_\varphi M J_\varphi)' = M'' = M$.

We claim that a and b from above satisfy the required conditions. It remains to show that $b \in N' \cap M$, $b\varphi b^* = \varphi - \varphi \circ \Phi$ and $a\mathbb{E}_N(b^*b) = \mathbb{E}_N(b^*b)a = 1 - c$.

- $b \in N' \cap M$*:* By the assumption we have $\varphi - \varphi \circ \Phi \geq \delta\varphi$ and therefore $\varphi - \varphi \circ \Phi$ is a faithful normal functional. For $x \in M$, $n \in N$ the N-N-bimodularity of Φ and the traciality of φ on N (implying that n is contained in the centralizer of φ) imply that $\varphi \circ \Phi(xn) = \varphi(\Phi(x)n) = \varphi(n\Phi(x)) = \varphi \circ \Phi(nx)$, hence $n(\varphi - \varphi \circ \Phi) = (\varphi - \varphi \circ \phi)n$. The unique isomorphism between the standard forms induced by φ and $\varphi - \varphi \circ \Phi$ maps ξ to the canonical cyclic vector in $L^2(M, \varphi - \varphi \circ \Phi)$. Hence, from Lemma 3.9 applied to $\varphi - \varphi \circ \Phi$ we get $n\xi = \xi n$ for all $n \in N$, which, together with the fact that $J_\varphi n \Omega_\varphi = n^* \Omega_\varphi$, implies that for $x \in M$

$$\begin{aligned} bn(J_\varphi x \Omega_\varphi) &= bJ_\varphi x J_\varphi n \Omega_\varphi = bJ_\varphi x n^* \Omega_\varphi = J_\varphi x n^* \xi \\ &= J_\varphi x \xi n^* = J_\varphi x J_\varphi n J_\varphi \xi = nJ_\varphi x \xi = nb(J_\varphi x \Omega_\varphi), \end{aligned}$$

  so $b \in N' \cap M$ by the density of $\{J_\varphi x \Omega_\varphi \mid x \in M\}$ in $L^2(M, \varphi)$.
- $b\varphi b^* = \varphi - \varphi \circ \Phi$*:* For every $x \in M$ the equality

$$(b\varphi b^*)(x) = \langle xb\Omega_\varphi, b\Omega_\varphi \rangle = \langle x\xi, \xi \rangle = (\varphi - \varphi \circ \Phi)(x)$$

  holds, i.e. $b\varphi b^* = \varphi - \varphi \circ \Phi$.
- $a\mathbb{E}_N(b^*b) = \mathbb{E}_N(b^*b)a = 1 - c$*:* For $x \in M$ we find by $b \in N' \cap M$ and $b\varphi b^* = \varphi - \varphi \circ \Phi$ that

$$\begin{aligned} &\varphi(x\mathbb{E}_N(b^*b)) = \varphi(\mathbb{E}_N(x)b^*b) = \varphi(b^*\mathbb{E}_N(x)b) = (\varphi - \varphi \circ \Phi)(\mathbb{E}_N(x)) \\ &= \varphi(\mathbb{E}_N(x)) - \varphi(\mathbb{E}_N(x)\Phi(1)) = \varphi(x) - \varphi(x\mathbb{E}_N(\Phi(1))) = \varphi(x\mathbb{E}_N(1 - c)) \end{aligned}$$

  and hence $\mathbb{E}_N(1 - c) = \mathbb{E}_N(b^*b)$. It follows by the definition of a that $a\mathbb{E}_N(b^*b) = a\mathbb{E}_N(1 - c) = 1 - c$ and similarly, as $a \in N' \cap M$, we have $\mathbb{E}_N(b^*b)a = 1 - c$. □

Lemma 3.11. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite and let $\tau \in N_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M. Let further $x \in N' \cap M$ be an element which is analytic for σ^φ. Then $\mathbb{E}_N(yx) = \mathbb{E}_N(\sigma_i^\varphi(x)y)$ for all $y \in M$.*

Proof. For $n \in N$ we have by the traciality of φ on N (implying that n is contained in the centralizer of φ) that

$$n\sigma_z^\varphi(x) = \sigma_z^\varphi(n)\sigma_z^\varphi(x) = \sigma_z^\varphi(nx) = \sigma_z^\varphi(xn) = \sigma_z^\varphi(x)\sigma_z^\varphi(n) = \sigma_z^\varphi(x)n,$$

for all $z \in \mathbb{C}$. Therefore, $\sigma_z^\varphi(x) \in N' \cap M$ and in particular $\sigma_i^\varphi(x) \in N' \cap M$. One further calculates that for $y \in M$,

$$(\varphi n)(\mathbb{E}_N(yx)) = \varphi(\mathbb{E}_N(nyx)) = \varphi(nyx) = \varphi(\sigma_i^{\varphi}(x)ny)$$
$$= \varphi(n\sigma_i^{\varphi}(x)y) = (\varphi n)(\mathbb{E}_N(\sigma_i^{\varphi}(x)y)).$$

Since the set of all functionals of the form φn, $n \in N$ is dense in N_* we find that $\mathbb{E}_N(yx) = \mathbb{E}_N(\sigma_i^{\varphi}(x)y)$, as claimed. □

Lemma 3.12. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras that admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite and let $\tau \in N_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M. Let $h_1, h_2 \in M$ and let $h_3, h_4 \in N' \cap M$ be analytic for σ^{φ}. Suppose that $\Phi : M \to M$ is a normal map such that $\Phi_{\varphi}^{(2)} \in \mathcal{K}(M, N, \varphi)$ and define the map $\tilde{\Phi} := h_1\Phi(h_2 \cdot h_3)h_4$. Then we also have that $\tilde{\Phi}_{\varphi}^{(2)} \in \mathcal{K}(M, N, \varphi)$.*

Proof. Note first that by [46, VIII.3.18(i)] (and its proof) $\tilde{\Phi}$ has a bounded L^2-implementation, with $\|\tilde{\Phi}_{\varphi}^{(2)}\| \leq C\|\Phi_{\varphi}^{(2)}\|$, with the constant $C > 0$ depending on h_1, h_2, h_3, h_4. It thus suffices to show that the passage $\Phi \to \tilde{\Phi}$ preserves the property of having a finite-rank implementation. Let then $a, b \in M$ so that ae_Nb is in $\mathcal{K}_{00}(M, N, \varphi)$. So for $x \in M$ we have by Lemma 3.11,

$$h_1(ae_Nb)(h_2xh_3)h_4\Omega_{\varphi} = h_1ah_4e_N(\sigma_i^{\varphi}(h_3)bh_2x)\Omega_{\varphi},$$

and so this map is in $\mathcal{K}_{00}(M, N, \varphi)$. □

We are now ready to formulate the main result of this subsection. In combination with Lemma 3.14 it will later allow us to deduce that the relative Haagerup property of a triple $(M, N, \mathbb{E}_N)$ with finite N may be witnessed by unital and state-preserving maps. Its proof is inspired by [5, Section 2].

Theorem 3.13. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite, let $\tau \in N_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M and suppose that the triple (M, N, φ) has property (rHAP) witnessed by contractive approximating maps. Then, if all the elements of M are analytic with respect to the modular automorphism group of φ – for example if there exists a boundedly invertible element $h \in M^+$ with $\sigma_t^{\varphi}(x) = h^{it}xh^{-it}$ for all $t \in \mathbb{R}$, $x \in M$, property (rHAP) of (M, N, φ) may be witnessed by unital and state-preserving approximating maps, i.e. we may assume that (1″) and (4′) hold.*

Proof. Let $(\Phi_j)_{j \in J_1}$ be a net of contractive approximating maps witnessing property (rHAP) of the triple (M, N, φ) and choose a net $(\delta_j)_{j \in J_2}$ with $\delta_j \to 0$. We now set $J = J_1 \times J_2$ with the product partial order and for $j = (j_1, j_2) \in J$ we set $\Phi_j = \Phi_{j_1}$ and $\delta_j = \delta_{j_2}$. Then for all $j \in J$,

$$c_j := (1-\delta_j)\Phi_j(1) \leq 1-\delta_j \qquad \text{and} \qquad (1-\delta_j)(\varphi\circ\Phi_j) \leq (1-\delta_j)\varphi.$$

In particular, we may apply Proposition 3.10 to $(1-\delta_j)\Phi_j$ to find elements $a_j, b_j \in N'\cap M$ with $a_j \geq 0$, $\mathbb{E}_N(a_j) = 1$, $a_j\mathbb{E}_N(b_j^*b_j) = \mathbb{E}_N(b_j^*b_j)a_j = 1-c_j$ and $b_j\varphi b_j^* = \varphi - (1-\delta_j)(\varphi\circ\Phi_j)$. For $j\in J$ define

$$\Psi_j : M\to M,\ \Psi_j(x) := (1-\delta_j)\Phi_j(x) + a_j\mathbb{E}_N(b_j^*xb_j).$$

It is clear that Ψ_j is normal completely positive and N-N-bimodular. Further,

$$\Psi_j(1) = (1-\delta_j)\Phi_j(1) + a_j\mathbb{E}_N(b_j^*b_j) = c_j + (1-c_j) = 1$$

and for any $x\in M$

$$\begin{aligned}\varphi\circ\Psi_j(x) &= (1-\delta_j)\varphi(\Phi_j(x)) + \varphi(a_j\mathbb{E}_N(b_j^*xb_j))\\ &= (1-\delta_j)\varphi(\Phi_j(x)) + \varphi(\mathbb{E}_N(a_j)b_j^*xb_j)\\ &= (1-\delta_j)\varphi(\Phi_j(x)) + (b_j\varphi b_j^*)(x)\\ &= (1-\delta_j)\varphi(\Phi_j(x)) + \varphi(x) - (1-\delta_j)\varphi(\Phi_j(x))\\ &= \varphi(x),\end{aligned}$$

so the Ψ_j are unital and φ-preserving.

For the relative compactness note that by the assumption that every element in M is analytic for σ^φ, Lemma 3.11 implies that for all $x\in M$

$$\Psi_j(x) = (1-\delta_j)\Phi_j(x) + a_j\mathbb{E}_N(\sigma_i^\varphi(b_j)b_j^*x),$$

hence,

$$\Psi_j^{(2)} = (1-\delta_j)\Phi_j^{(2)} + a_je_N\sigma_i^\varphi(b_j)b_j^* \in \mathcal{K}(M,N,\varphi).$$

It remains to show that for every $x\in M$, $\Psi_j(x)\to x$ strongly. For this, estimate for $x\geq 0$,

$$\begin{aligned}(\Psi_j - (1-\delta_j)\Phi_j)(x) &= a_j^{1/2}\mathbb{E}_N(b_j^*xb_j)a_j^{1/2}\\ &\leq \|x\|a_j^{1/2}\mathbb{E}_N(b_j^*b_j)a_j^{1/2}\\ &= \|x\|(1-c_j).\end{aligned}$$

Since $c_j = (1-\delta_j)\Phi_j(1)\to 1$ and $(1-\delta_j)\Phi_j(x)\to x$ strongly it then follows that

$$\Psi_j(x) = (\Psi_j(x) - (1-\delta_j)\Phi_j(x)) + (1-\delta_j)\Phi_j(x)\to x$$

strongly for every $x\in M$. This completes the proof. □

Another important statement that was proved in [16] in case of the usual (non-relative) Haagerup property is the following lemma. It will later ensure the contractivity of certain approximating maps and allow us to apply Theorem 3.13 in a suitable setting.

Lemma 3.14. *Let M be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau \in M_*$ and let $N \subseteq M$ be a unital inclusion. Assume that $h \in N' \cap M$ is a boundedly invertible self-adjoint element and define $\varphi \in M_*$ by $\varphi(x) := \tau(hxh)$ for $x \in M$. Then, if (M, N, φ) has property (rHAP), the approximating maps $(\Phi_i)_{i\in I}$ witnessing property (rHAP) may be chosen contractively, i.e. we may assume that (1′) holds.*

Proof. The proof is given in [16, Lemma 4.3]. One only needs to check that the condition $h \in N' \cap M^+$ ensures that the maps Φ'_k, Φ^l_k and Ψ_j defined there are N-N-bimodule maps that are compact relative to N. Let us comment on this.

In Step 1 of the proof of [16, Lemma 4.3] it is shown that the approximating maps Φ_k witnessing the Haagerup property may be chosen such that $\sup_k \|\Phi_k\| < \infty$. In the current setup of (rHAP) this is automatic (see Definition 3.2) and so we may skip this step.

We now turn to Step 2 in the proof of [16, Lemma 4.3]. Let Φ_k be the approximating maps witnessing the (rHAP) for (M, N, φ). In particular Φ_k is N-N-bimodular and $\Phi_k^{(2)} \in \mathcal{K}(M, N, \varphi)$. By [46, Theorem VIII.2.211] we have $\sigma^\varphi_t(x) = h^{it}xh^{-it}$, $t \in \mathbb{R}, x \in M$. Now recall the map defined in [16, Lemma 4.3] given by

$$\begin{aligned}\Phi^l_k(x) =& \sqrt{\frac{1}{l\pi}} \int_{-\infty}^{\infty} e^{-t^2/l} \sigma^\varphi_t(\Phi_k(\sigma^\varphi_{-t}(x)))dt \\ =& \sqrt{\frac{1}{l\pi}} \int_{-\infty}^{\infty} e^{-t^2/l} h^{it}\Phi_k(h^{-it}xh^{it})h^{-it}dt.\end{aligned} \tag{3.1}$$

Since $h \in N' \cap M$ this map is N-N-bimodular. Since $\sigma^\varphi_t(h^{is}) = h^{is}, s, t \in \mathbb{R}$ it follows from Lemma 3.12 that the L^2-implementation of

$$x \mapsto \sigma^\varphi_t(\Phi_k(\sigma^\varphi_{-t}(x))) = h^{it}\Phi_k(h^{-it}xh^{it})h^{-it}, \qquad t \in \mathbb{R}, \tag{3.2}$$

exists and is compact, i.e. contained in $\mathcal{K}(M, N, \varphi)$. By assumption h is boundedly invertible and so $t \mapsto h^{it}$ depends continuously (in norm) on t. Hence the map (3.2) depends continuously on t and it follows that (3.1) is compact.

Next, in the proof of [16, Lemma 4.3] the following operators were defined:

$$g^l_k = \Phi^l_k(1), \qquad f^{n,l}_k = F_n(g^l_k),$$

where $F_n(z) = e^{-n(z-1)^2}, z \in \mathbb{C}, n \in \mathbb{N}$. Since Φ^l_k is N-N-bimodular it follows that $g^l_k \in N' \cap M$. Therefore also $f^{n,l}_k \in N' \cap M$. Then the proof of [16, Lemma 4.3] defines

for suitable $n(j), k(j), l(j) \in \mathbb{N}, \epsilon_j > 0$ depending on some j in a directed set the map $\Psi_j : M \to M$ via the formula:

$$\Psi_j(\,\cdot\,) = \frac{1}{(1+\epsilon_j)^2} f_{k(j)}^{n(j),l(j)} \Phi_{k(j)}^{l(j)}(\,\cdot\,) f_{k(j)}^{n(j),l(j)}.$$

Since $f_{k(j)}^{n(j),l(j)} \in N' \cap M$ it follows that Ψ_j is both N-N-bimodular and compact, i.e. $\Psi_j^{(2)} \in \mathcal{K}(M, N, \varphi)$. The last part of the proof of [16, Lemma 4.3] shows that $\Psi_j^{(2)} \to 1$ strongly and this holds true here as well with the same proof. By Lemma 2.1 this shows that for every $x \in M$ we have $\Psi_j(x) \to x$ strongly. □

4. For finite N: translation into the finite setting

Let again $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume moreover that N is a general σ-finite von Neumann algebra, though in many of the statements below we shall add the assumption that N is finite. The aim of this section is to characterise the relative Haagerup property (resp. property (rHAP)$^-$) of the triple $(M, N, \mathbb{E}_N)$ in terms of the structure of certain corners of crossed product von Neumann algebras associated with the modular automorphism group of some faithful $\varphi \in M_*^+$ with $\varphi \circ \mathbb{E}_N = \varphi$. These statements will play a crucial role in Section 5.

4.1. Crossed products

Let us first recall some of the theory of crossed product von Neumann algebras and their duality for which we refer to [46, Section X.2]. For this, fix an action $\mathbb{R} \curvearrowright^{\alpha} M$ on $M \subseteq \mathcal{B}(\mathcal{H})$, define the corresponding *fixed point algebra*

$$M^{\alpha} := \{x \in M \mid \alpha_t(x) = x \text{ for all } t \in \mathbb{R}\}$$

and let $M \rtimes_{\alpha} \mathbb{R} \subseteq \mathcal{B}(\mathcal{H} \otimes L^2(\mathbb{R})) \cong \mathcal{B}(L^2(\mathbb{R}, \mathcal{H}))$ be the corresponding *crossed product von Neumann algebra*. It is generated by the operators $\pi_{\alpha}(x)$, $x \in M$ and $\lambda_t := \lambda_t^{\alpha}$, $t \in \mathbb{R}$ where

$$(\pi_{\alpha}(x)\xi)(t) = \alpha_{-t}(x)\xi(t) \qquad \text{and} \qquad (\lambda_t \xi)(s) = \xi(s-t)$$

for $s, t \in \mathbb{R}$, $x \in M$, $\xi \in \mathcal{H} \otimes L^2(\mathbb{R})$; we will also occasionally use λ to denote the left regular representation on $L^2(\mathbb{R})$, which should not cause any confusion. Recall that this construction does not depend on the choice of the embedding $M \subseteq \mathcal{B}(\mathcal{H})$ and that $M \cong \pi_{\alpha}(M)$. For notational convenience we will therefore omit the faithful normal representation π_{α} in our notation and identify M with $\pi_{\alpha}(M)$ and N with $\pi_{\alpha}(N)$. Note that $\pi_{\alpha}(x) = x \otimes 1$ for all $x \in M^{\alpha}$. Set further $\lambda(f) := \int_{\mathbb{R}} f(t)\lambda_t dt$ for $f \in L^1(\mathbb{R})$ and

$$\mathcal{L}(\mathbb{R}) := \{\lambda(f) \mid f \in L^1(\mathbb{R})\}'' = \{\lambda_s \mid s \in \mathbb{R}\}'' \subseteq \mathcal{B}(\mathcal{H} \otimes L^2(\mathbb{R})).$$

Remark 4.1. For $f \in L^1(\mathbb{R})$ we denote by

$$\widehat{f}(s) = \int_{\mathbb{R}} f(t)e^{ist}dt \in L^\infty(\mathbb{R}),$$

its Fourier transform. Let $\mathcal{F}_2 : L^2(\mathbb{R}) \to L^2(\mathbb{R}) : f \mapsto (2\pi)^{-\frac{1}{2}}\widehat{f}$ be the unitary Fourier transform operator on $L^2(\mathbb{R})$. Then $\mathcal{F}_2\lambda(f)\mathcal{F}_2^*$ is the multiplication operator with $\widehat{f}$. We shall occasionally extend our notation in the following way. Let $f \in L^2(\mathbb{R})$ be such that its Fourier transform $\widehat{f}$ is in $L^\infty(\mathbb{R})$. We shall write $\lambda(f)$ for $\mathcal{F}_2^*\widehat{f}\mathcal{F}_2$ where we view $\widehat{f}$ as a multiplication operator. This is naturally compatible with the earlier notation for $f \in L^1(\mathbb{R})$

Let $\mathbb{R} \overset{\widehat{\alpha}}{\curvearrowright} M \rtimes_\alpha \mathbb{R}$ be the *dual action* determined by

$$\widehat{\alpha}_t(x) = x, \qquad \text{and} \qquad \widehat{\alpha}_t(\lambda_s) = \exp(-ist)\lambda_s, \tag{4.1}$$

for $x \in M$, $s, t \in \mathbb{R}$ and recall that its fixed point algebra is given by

$$M = (M \rtimes_\alpha \mathbb{R})^{\widehat{\alpha}}. \tag{4.2}$$

The expression

$$T_{\widehat{\alpha}}(x) := \int_{\mathbb{R}} \widehat{\alpha}_s(x)ds, \qquad x \in (M \rtimes_\alpha \mathbb{R})^+,$$

defines a faithful normal semi-finite operator valued weight on $M \rtimes_\alpha \mathbb{R}$ which takes values in the extended positive part of M. Choose $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\|f\|_2 = 1$ such that the support of the Fourier transform $\widehat{f}$ equals $\mathbb{R}$. We keep f fixed throughout the whole section. One has $T_{\widehat{\alpha}}(\lambda(f)^*\lambda(f)) = \|f\|_2^2 = 1$, hence we may define the unital normal completely positive map

$$T_f := T_{f,\widehat{\alpha}} : M \rtimes_\alpha \mathbb{R} \to M,\ x \mapsto T_{\widehat{\alpha}}(\lambda(f)^* x \lambda(f)).$$

By Lemma 2.1 T_f is strongly continuous on the unit ball. For a given map $\Phi : M \rtimes_\alpha \mathbb{R} \to M \rtimes_\alpha \mathbb{R}$ and a positive normal functional $\varphi \in M_*$ we further define

$$\widetilde{\Phi}_f : M \to M, \quad \widetilde{\Phi}_f(x) = T_f(\Phi(x)),$$

and

$$\widehat{\varphi}_f : M \rtimes_\alpha \mathbb{R} \to \mathbb{C}, \quad \widehat{\varphi}_f(x) = \varphi(T_f(x)).$$

The functional $\widehat{\varphi}_f$ is normal and positive. It is moreover a state if φ is a state. Since we assumed the support of $\widehat{f}$ to be equal to $\mathbb{R}$, by Remark 4.1 the support projection of $\lambda(f)$ equals 1. It follows that $\widehat{\varphi}_f$ is faithful if and only if φ is faithful.

Lemma 4.2. *Assume that $N \subseteq M^\alpha$. Then T_f is N-N-bimodular, meaning that for $x, y \in N$, $a \in M \rtimes_\alpha \mathbb{R}$ we have $T_f(xay) = xT_f(a)y$.*

Proof. As $N \subseteq M^\alpha$ we have that N and $\lambda(f)$ commute. From the definition of $T_{\widehat{\alpha}}$ and (4.2) we get that for $x, y \in N$ and $a \in M \rtimes_\alpha \mathbb{R}$,

$$T_{\widehat{\alpha}}(\lambda(f)^* xay \lambda(f)) = T_{\widehat{\alpha}}(x\lambda(f)^* a\lambda(f)y) = xT_{\widehat{\alpha}}(\lambda(f)^* a\lambda(f))y.$$

This concludes the proof. □

We recall the following formula which was proved in [15, Lemma 5.2] (which extends [27, Theorem 3.1 (c)]) in case $k = g$; the general case then follows from the polarization identity. For $k, g \in L^2(\mathbb{R})$ such that $\widehat{k}, \widehat{g} \in L^\infty(\mathbb{R})$ we have:

$$T_{\widehat{\alpha}}(\lambda(k)^* x\lambda(g)) = \int_{\mathbb{R}} \overline{k(t)} g(t)\alpha_{-t}(x)dt, \qquad x \in M. \tag{4.3}$$

We shall need the following consequence of it. For $g \in L^1(\mathbb{R})$ define $g^*(t) := \overline{g(-t)}$, which is the involution for the convolution algebra $L^1(\mathbb{R})$.

Lemma 4.3. *Let $h \in C_c(\mathbb{R})$ and let $x \in M$. Then, for $k, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $a := \lambda(h)x$,*

$$T_{\widehat{\alpha}}(\lambda(k)^* a\lambda(g)) = \int_{\mathbb{R}} \int_{\mathbb{R}} k^*(s)g(t)h(-s-t)\alpha_{-t}(x)dsdt,$$

and

$$T_{\widehat{\alpha}}(\lambda(k)^* \lambda(g)a) = \int_{\mathbb{R}} \int_{\mathbb{R}} k^*(s)g(t)h(-s-t)xdsdt.$$

Proof. We have $\lambda(k)^* a = \lambda(h^* * k)^* x$. The equality (4.3) then implies

$$\begin{aligned} T_{\widehat{\alpha}}(\lambda(k)^* a\lambda(g)) &= T_{\widehat{\alpha}}(\lambda(h^* * k)^* x\lambda(g)) \\ &= \int_{\mathbb{R}} \overline{(h^* * k)}(t)g(t)\alpha_{-t}(x)dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} k^*(s)g(t)h(-s-t)\alpha_{-t}(x)dsdt. \end{aligned}$$

This concludes the proof of the first formula. The second formula follows from the first after observing that $T_{\widehat{\alpha}}(\lambda(k)^*\lambda(g)a) = T_{\widehat{\alpha}}(\lambda(k)^*\lambda(g)\lambda(h))x$. □

4.2. Passage to crossed products

Let us now study the stability of the relative Haagerup property with respect to certain crossed products. The setting is the same as in Subsection 4.1.

Proposition 4.4. *Let $\Phi : M \rtimes_\alpha \mathbb{R} \to M \rtimes_\alpha \mathbb{R}$ be a linear map and fix $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as before. Then the following statements hold:*

(1) If Φ is completely positive then so is $\widetilde{\Phi}_f$.
(2) Assume that $N \subseteq M^\alpha$. If Φ is an N-N-bimodule map then $\widetilde{\Phi}_f$ is an N-N-bimodule map.

In the remaining statements let $\varphi \in M_*^+$ be a faithful normal positive functional with $\varphi \circ \mathbb{E}_N = \varphi$ and $\varphi \circ \alpha_t = \varphi$ for all $t \in \mathbb{R}$. Then:

(3) If $\widehat{\varphi}_f \circ \Phi \leq \widehat{\varphi}_f$ (resp. $\widehat{\varphi}_f \circ \Phi = \widehat{\varphi}_f$) then $\varphi \circ \widetilde{\Phi}_f \leq \varphi$ (resp. $\varphi \circ \widetilde{\Phi}_f = \varphi$).
(4) If the L^2-implementation of Φ with respect to $\widehat{\varphi}_f$ exists, then the L^2-implementation of $\widetilde{\Phi}_f$ with respect to φ exists as well.

Now, if $N \subseteq M^\alpha$, $\mathbb{E}_N \circ \alpha_t = \mathbb{E}_N$ for all $t \in \mathbb{R}$ and f is continuous, then:

(5) If $\Phi \in \mathcal{K}_{00}(M \rtimes_\alpha \mathbb{R}, N, \widehat{\varphi}_f)$, then $\widetilde{\Phi}_f \in \mathcal{K}_{00}(M, N, \varphi)$.
(6) If $\Phi \in \mathcal{K}(M \rtimes_\alpha \mathbb{R}, N, \widehat{\varphi}_f)$, then $\widetilde{\Phi}_f \in \mathcal{K}(M, N, \varphi)$.

Proof. (1) is straightforward from the constructions and (2) follows from Lemma 4.2.

(3): If $\widehat{\varphi}_f \circ \Phi \leq \widehat{\varphi}_f$ we have for $x \in M^+$, using (4.3) and the α-invariance of φ,

$$\begin{aligned}\varphi \circ \widetilde{\Phi}_f(x) &= \varphi(T_f(\Phi(x))) = \widehat{\varphi}_f(\Phi(x)) \leq \widehat{\varphi}_f(x) \\ &= \varphi(T_{\widehat{\alpha}}(\lambda(f)^* x \lambda(f))) = \int_{\mathbb{R}} |f(t)|^2 \varphi(\alpha_{-t}(x))dt = \varphi(x).\end{aligned}$$

Moreover, if $\widehat{\varphi}_f \circ \Phi = \widehat{\varphi}_f$ then the inequality above is actually an equality.

(4): Assume that there exists a constant $C > 0$ such that $\widehat{\varphi}_f(\Phi(x)^*\Phi(x)) \leq C\widehat{\varphi}_f(x^*x)$ for all $x \in M$. Then, by the Kadison-Schwarz inequality and (4.3),

$$\varphi(\widetilde{\Phi}_f(x)^*\widetilde{\Phi}_f(x)) = \varphi(T_f(\Phi(x))^* T_f(\Phi(x))) \leq \widehat{\varphi}_f(\Phi(x)^*\Phi(x)) \leq C\widehat{\varphi}_f(x^*x) = C\varphi(x^*x)$$

for all $x \in M$, where we use the fact (proved above) that φ and $\hat{\varphi}_f$ coincide on M_+. This implies that the L^2-implementation of $\widetilde{\Phi}_f$ with respect to φ exists.

(5): By Lemma 4.2 and the discussion before, $\mathbb{F}_N = \mathbb{E}_N \circ T_f$ is the unique faithful normal $\widehat{\varphi}_f$-preserving conditional expectation of $M \rtimes_\alpha \mathbb{R}$ onto N. Let $a, b \in M \rtimes_\alpha \mathbb{R}$. By $N \subseteq M^\alpha$ we have for $x \in M$,

$$\widetilde{(a\mathbb{F}_N b)}_f(x) := T_f(a\mathbb{F}_N(bx)) = T_f(a)\mathbb{F}_N(bx) \tag{4.4}$$

We shall show that $\mathbb{F}_N(bx) = \mathbb{E}_N(\widetilde{b}x)$ for all $x \in M$, where $\widetilde{b} := T_{\widehat{\alpha}}(\lambda(f)^*\lambda(f)b)$. For this it suffices to consider the case where $b = \lambda(h)y$ for some compactly supported function $h \in C_c(\mathbb{R})$ and $y \in M$, since such elements span a σ-weakly dense subset of $M \rtimes_\alpha \mathbb{R}$ and the map $b \mapsto \widetilde{b}$ is σ-weakly continuous. Using Lemma 4.3 twice and the fact that $\mathbb{E}_N \circ \alpha_t = \mathbb{E}_N$ for all $t \in \mathbb{R}$ one has

$$\begin{aligned}
\mathbb{F}_N(bx) &= \mathbb{E}_N \circ T_f(bx) \\
&= \mathbb{E}_N\left(\int_{\mathbb{R}}\int_{\mathbb{R}} f^*(s)f(t)h(-s-t)\alpha_{-t}(yx)dsdt\right) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f^*(s)f(t)h(-s-t)\mathbb{E}_N(yx)dsdt \\
&= \mathbb{E}_N\left(\int_{\mathbb{R}}\int_{\mathbb{R}} f^*(s)f(t)h(-s-t)ydsdt\ x\right) \\
&= \mathbb{E}_N(\widetilde{b}x),
\end{aligned}$$

as claimed. Combining this equality and (4.4) we get that $\widetilde{(a\mathbb{E}_N b)}_f = T_f(a)\mathbb{F}_N\widetilde{b}$. By considering linear combinations of such expressions one gets that if $\Phi \in \mathcal{K}_{00}(M \rtimes_\alpha \mathbb{R}, N, \widehat{\varphi}_f)$ then also $\widetilde{\Phi}_f \in \mathcal{K}_{00}(M, N, \varphi)$. This proves (5).

(6): The statement follows directly from (5) by approximation and the fact that $\|\widetilde{\Phi}_f\| \leq \|\Phi\|$. □

In the following we will direct our attention to certain choices of functions $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\|f\|_2 = 1$ whose support of the Fourier transform $\widehat{f}$ equals $\mathbb{R}$. For this, define for $j \in \mathbb{N}$ the L^2-normalized *Gaussian*

$$f_j : \mathbb{R} \to \mathbb{R},\ f_j(s) := \left(\frac{j}{\pi}\right)^{1/4} \exp(-js^2/2).$$

Further set for a given map $\Phi : M \rtimes_\alpha \mathbb{R} \to M \rtimes_\alpha \mathbb{R}$ and a positive normal functional $\varphi \in M_*$

$$\widehat{\varphi}_j := \widehat{\varphi}_{f_j} \qquad \text{and} \qquad \widetilde{\Phi}_j := \widetilde{\Phi}_{f_j}.$$

Theorem 4.5. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathbb{E}_N : M \to N$. Let further $\varphi \in M_*^+$ be a faithful normal positive functional with $\varphi \circ \mathbb{E}_N = \varphi$ and $\mathbb{R} \curvearrowright^{\alpha} M$ be an action such that $N \subseteq M^{\alpha}$. Finally assume that $\mathbb{E}_N \circ \alpha_t = \mathbb{E}_N$ (or, equivalently under the earlier assumptions, that $\varphi = \varphi \circ \alpha_t$) for all $t \in \mathbb{R}$. Then the following statements hold:*

(1) If for all $j \in \mathbb{N}$ the triple $(M \rtimes_{\alpha} \mathbb{R}, N, \widehat{\varphi}_j)$ has property (rHAP) (resp. property (rHAP)$^-$), then (M, N, φ) has property (rHAP) (resp. property (rHAP)$^-$).
(2) If for all $j \in \mathbb{N}$ property (rHAP) of $(M \rtimes_{\alpha} \mathbb{R}, N, \widehat{\varphi}_j)$ is witnessed by unital (resp. $\widehat{\varphi}_j$-preserving) approximating maps (see (1″) and (4′) in Section 3), then also property (rHAP) of (M, N, φ) may be witnessed by unital (resp. φ-preserving) approximating maps.

Proof. (1): For fixed $j \in \mathbb{N}$ let $(\Phi_{j,k})_{k \in K_j}$ be a bounded net of normal completely positive maps witnessing the relative Haagerup property of $(M \rtimes_{\alpha} \mathbb{R}, N, \widehat{\varphi}_j)$. In particular, $\Phi_{j,k} \to 1$ in the point-strong topology in k. Set $\widetilde{\Phi}_{j,k} := T_{f_j} \circ \Phi_{j,k}$. As $s \mapsto \alpha_s(x)$ is strongly continuous for $x \in M$ and f_j is L^2-normalized with mass concentrated around 0, Lemma 4.3 shows that for $x \in M$,

$$T_{f_j}(x) = \int_{\mathbb{R}} |f_j(s)|^2 \alpha_s(x) ds \to x$$

as $j \to \infty$ in the strong topology. Lemma 2.2 then shows that we may find a directed set $\mathcal{F}$ and a function $(\widetilde{j}, \widetilde{k}) : \mathcal{F} \to \{(j,k) \mid j \in \mathbb{N}, k \in K_j\}$, $F \mapsto (\widetilde{j}(F), \widetilde{k}(F))$ such that the net $(\widetilde{\Phi}_{\widetilde{j}(F),\widetilde{k}(F)})_{F \in \mathcal{F}}$ converges to the identity in the point-strong topology. By Proposition 4.4 these maps then witness the relative Haagerup property for (M, N, φ). In the same way, using a variant of Lemma 2.2, we can deduce that if $(M \rtimes_{\alpha} \mathbb{R}, N, \widehat{\varphi}_j)$ has property (rHAP)$^-$, then (M, N, φ) has property (rHAP)$^-$ as well.

(2): Note that if $\Phi_{j,k}$ is unital for all $k \in \mathbb{N}$, then $\widetilde{\Phi}_{j,k}$ is unital as well and if $\Phi_{j,k}$ is $\widehat{\varphi}_j$-preserving for all $k \in \mathbb{N}$, then $\widetilde{\Phi}_{j,k}$ is φ-preserving, cf. Proposition 4.4. □

We will now apply this theorem to the modular automorphism group σ^{φ} of φ as well as its dual action.

Theorem 4.6. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathbb{E}_N : M \to N$. Assume that N is finite and let $\tau \in N_*$ be a faithful normal tracial state. Further define the faithful normal (possibly non-tracial) state $\varphi := \tau \circ \mathbb{E}_N$ on M. Then the following statements hold:*

(1) If for all $j \in \mathbb{N}$ the triple $(M \rtimes_{\sigma^{\varphi}} \mathbb{R}, N, \widehat{\varphi}_j)$ has property (rHAP) (resp. property (rHAP)$^-$), then (M, N, φ) has property (rHAP) (resp. property (rHAP)$^-$).

(2) If for all $j \in \mathbb{N}$ property (rHAP) of $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ is witnessed by unital (resp. $\widehat{\varphi}_j$-preserving) approximating maps, then also property (rHAP) of (M, N, φ) may be witnessed by unital (resp. φ-preserving) approximating maps.

Proof. This is Theorem 4.5 for $\alpha = \sigma^\varphi$; the assumptions are satisfied, as follows from the fact that $N \subseteq M^\varphi$. □

We will also prove the converse of Theorem 4.6 by using crossed product duality. We first recall the following lemma which is well-known. We will use the fact that every function $g \in L^\infty(\mathbb{R})$ may be viewed as a multiplication operator on $L^2(\mathbb{R})$.

Lemma 4.7. *For $g, h \in C_b(\mathbb{R}) \cap L^2(\mathbb{R})$ we have that $g\lambda(h) \in \mathcal{B}(L^2(\mathbb{R}))$ is Hilbert-Schmidt with*

$$\mathrm{Tr}((g\lambda(h))^* g\lambda(h)) = \|g\|_2^2 \|h\|_2^2.$$

Proof. Let $\mathcal{S}_2(\mathcal{H})$ denote the Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$. We have linear identifications $\mathcal{H} \otimes \overline{\mathcal{H}} \simeq \mathcal{S}_2(\mathcal{H})$ where $\xi \otimes \overline{\eta}$ corresponds to the rank 1 operator $v \mapsto \xi\eta^*(v)$. We identify $L^2(\mathbb{R})$ with $\overline{L^2(\mathbb{R})}$ linearly and isometrically through the pairing $\langle \xi, \eta \rangle = \int_{\mathbb{R}} \xi(s)\eta(s)ds$. Therefore we have isometric linear identifications

$$\mathcal{S}_2(L^2(\mathbb{R})) \simeq L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \simeq L^2(\mathbb{R}^2), \tag{4.5}$$

where the rank 1 operator $\xi\eta^*$ corresponds to the function $(s,t) \mapsto \xi(s)\eta(t)$.

Now, $g\lambda(h)$ is an integral operator on $L^2(\mathbb{R})$ with a square-integrable kernel $K(x,y) := g(x)h(x-y)$. Then $g\lambda(h)$ is Hilbert-Schmidt and corresponds to $K \in L^2(\mathbb{R}^2)$ in (4.5), so that $\|g\lambda(h)\|_{\mathcal{S}_2} = \|K\|_2 = \|g\|_2\|h\|_2$. □

Further recall that for $j \in \mathbb{N}$ the Gaussian f_j was defined by $f_j(s) := j^{1/4}\pi^{-1/4} \times \exp(-js^2/2)$, $s \in \mathbb{R}$ and $\widehat{f_j}$ denotes its Fourier transform. Both these functions are L^2-normalized by definition and the Plancherel identity. Define for $i, j \in \mathbb{N}$ a positive linear functional $\psi_{i,j}$ on $\mathcal{B}(L^2(\mathbb{R}))$ by

$$\psi_{i,j}(x) := \mathrm{Tr}((\widehat{f_i}\lambda(f_j))^* x \widehat{f_i}\lambda(f_j)).$$

It is a state by Lemma 4.7. We will need the following elementary lemma for which we give a short non-explicit proof following from the results in [15].

Lemma 4.8. *For all $i, j \in \mathbb{N}$ the pair $(\mathcal{B}(L^2(\mathbb{R})), \psi_{i,j})$ has the Haagerup property in the sense that the triple $(\mathcal{B}(L^2(\mathbb{R})), \mathbb{C}, \psi_{i,j})$ has the relative Haagerup property, see [15, Definition 3.1]. Moreover, the approximating maps may be chosen to be unital and $\psi_{i,j}$-preserving.*

Proof. According to [15, Proposition 3.4], $(\mathcal{B}(L^2(\mathbb{R}), \mathrm{Tr})$ has the Haagerup property. By [15, Theorem 1.3] the Haagerup property does not depend on the choice of the faithful normal semi-finite weight and hence $(\mathcal{B}(L^2(\mathbb{R})), \psi_{i,j})$ has the Haagerup property for all $i, j \in \mathbb{N}$. In [16, Theorem 5.1] it was proved that the approximating maps may be taken unital and state preserving. This finishes the proof. □

As before, let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$ and fix a faithful normal state φ on M with $\varphi = \varphi \circ \mathbb{E}_N$. Let σ^φ be the corresponding modular automorphism group, $M \rtimes_{\sigma^\varphi} \mathbb{R}$ the crossed product von Neumann algebra and let

$$\theta := \widehat{\sigma^\varphi} : \mathbb{R} \curvearrowright M \rtimes_{\sigma^\varphi} \mathbb{R}$$

be the dual action as defined in (4.1). Define for $j \in \mathbb{N}$ the state $\widehat{\varphi}_j := \varphi \circ T_{f_j,\theta}$ on $M \rtimes_{\sigma^\varphi} \mathbb{R}$ as before and recall that M (hence also N) is invariant under θ. We may in turn consider the double crossed product which admits an isomorphism of von Neumann algebras (i.e. a bijective $*$-homomorphism, which is automatically normal by Sakai [43, Theorem 1.13.2]),

$$(M \rtimes_{\sigma^\varphi} \mathbb{R}) \rtimes_\theta \mathbb{R} \cong M \otimes \mathcal{B}(L^2(\mathbb{R})). \tag{4.6}$$

Let us describe what this isomorphism looks like. For $g \in L^\infty(\mathbb{R})$ write $\mu(g) := 1_M \otimes g \in M \otimes \mathcal{B}(L^2(\mathbb{R}))$ for the multiplication operator acting in the second tensor leg. The double crossed product above is generated by $M \rtimes_{\sigma^\varphi} \mathbb{R}$ and the left regular representation of the second copy of $\mathbb{R}$, denoted here by λ_t^θ, $t \in \mathbb{R}$. Under the isomorphism, $M \rtimes_{\sigma^\varphi} \mathbb{R}$ is identified as a subalgebra of $M \otimes \mathcal{B}(L^2(\mathbb{R}))$ via inclusion. Further, λ_t^θ is identified for every $t \in \mathbb{R}$ with $\mu(e_t) = 1_M \otimes e_t$ where $e_t(s) := \exp(-ist)$ for $s \in \mathbb{R}$. Under this correspondence, $\lambda^\theta(f_j) = \mu(\widehat{f_j})$. We find that for $x \in M \otimes \mathcal{B}(L^2(\mathbb{R}))$,

$$\begin{aligned}(\varphi \circ T_{f_j,\theta} \circ T_{f_i,\widehat{\theta}})(x) &= \varphi\left(T_\theta\left(\lambda(f_j)^* T_{\widehat{\theta}}\left(\mu(\widehat{f_i})^* x \mu(\widehat{f_i})\right)\lambda(f_j)\right)\right) \\ &= \varphi\left(T_\theta\left(T_{\widehat{\theta}}\left(\lambda(f_j)^* \mu(\widehat{f_i})^* x \mu(\widehat{f_i})\lambda(f_j)\right)\right)\right).\end{aligned}$$

By [46, Theorem X.2.3] and the fact that $\varphi \circ \sigma_t^\varphi = \varphi$ we have that (formally, being imprecise about domains) the normal semi-finite faithful weight $\varphi \circ T_\theta \circ T_{\widehat{\theta}}$ coincides with $\varphi \otimes \mathrm{Tr}$. Hence, for $i, j \in \mathbb{N}$ we have equality of states

$$\varphi \circ T_{f_j,\theta} \circ T_{f_i,\widehat{\theta}} = \varphi \otimes \psi_{i,j}.$$

The following theorem now provides a passage to study the relative Haagerup property on the continuous core of a von Neumann algebra, which is semi-finite.

Theorem 4.9. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$ and assume that N is finite with a faithful normal tracial state $\tau \in N_*$. Set $\varphi = \tau \circ \mathbb{E}_N \in M_*$. Then the following two statements hold:*

(1) The triple (M, N, φ) has property (rHAP) (resp. property (rHAP)$^-$) if and only if $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ has property (rHAP) (resp. property (rHAP)$^-$) for all $j \in \mathbb{N}$.
(2) If property (rHAP) of (M, N, φ) is witnessed by unital (resp. φ-preserving) maps, then for all $j \in \mathbb{N}$ property (rHAP) of $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ is witnessed by unital (resp. $\widehat{\varphi}_j$-preserving) maps, and vice versa.

Proof. The if statements were proven in Theorem 4.6. For the converse of (1) assume that (M, N, φ) has the relative Haagerup property. $(\mathcal{B}(L^2(\mathbb{R})), \mathbb{C}, \psi_{i,j})$ has the relative Haagerup property for all $i, j \in \mathbb{N}$, see Lemma 4.8. Therefore by a suitable modification of [15, Lemma 3.5], we see that $(M \otimes \mathcal{B}(L^2(\mathbb{R})), N \otimes \mathbb{C}, \varphi \otimes \psi_{i,j})$ has the relative Haagerup property for all $i, j \in \mathbb{N}$. It follows from Theorem 4.5 and the discussion above that $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ has the relative Haagerup property.[3]

The statements in (2) and the statement about property (rHAP)$^-$ follow in the same way. □

4.3. Passage to corners of crossed products

In the last subsection we characterised the relative Haagerup property of the triple (M, N, φ) for finite N with a faithful normal tracial state $\tau \in N_*$ and $\varphi := \tau \circ \mathbb{E}_N \in M_*$ in terms of the crossed product triples $(M \rtimes_\alpha \mathbb{R}, N, \widehat{\varphi}_j)$, $j \in \mathbb{N}$. In the following we will pass over to suitable corners of these crossed products which allows to translate our investigations into the setting of finite von Neumann algebras. In this setting the following lemma will be useful.

Lemma 4.10. *Let $N \subseteq M$ be a unital inclusion of finite von Neumann algebras, let $\tau \in M_*$ be a faithful normal tracial state and let $\mathbb{E}_N : M \to N$ be the unique τ-preserving faithful normal conditional expectation onto N. Further, let $h \in N' \cap M$ be self-adjoint and boundedly invertible. For a linear completely positive map $\Phi : M \to M$ set*

$$\Phi^h(x) = h^{-1}\Phi(hxh)h^{-1}.$$

Then, the L^2-implementation $\Phi^{(2)}$ of Φ with respect to τ exists if and only if the L^2-implementation $(\Phi^h)^{(2)}$ of Φ^h with respect to $h\tau h$ exists. Further, $\Phi^{(2)} \in \mathcal{K}(M, N, \tau)$ if and only if $(\Phi^h)^{(2)} \in \mathcal{K}(M, N, h\tau h)$.

[3] Note that in the picture above $\pi(x) = x \otimes 1$ and hence $\pi(N) = N \otimes \mathbb{C}$ since φ is tracial on N. This is used implicitly in the identifications of N in the double crossed product isomorphism (4.6).

Proof. Note first that the assumptions on h imply that $\mathbb{E}_N(h^2)$ is a positive boundedly invertible element of the centre $Z(N)$ of N. Indeed, we have for all $n \in N$ the equality $n\mathbb{E}_N(h^2) = \mathbb{E}_N(nh^2) = \mathbb{E}_N(h^2n) = \mathbb{E}_N(h^2)n$, and if h is boundedly invertible, then $h^2 \geq c1_M$ for some $c > 0$, hence $\mathbb{E}_N(h^2) \geq c1_M$.

The map $\mathbb{E}_N^h : x \mapsto \mathbb{E}_N(h^2)^{-1/2}\mathbb{E}_N(hxh)\mathbb{E}_N(h^2)^{-1/2}$ is the unique normal $h\tau h$-preserving conditional expectation onto N. Indeed, we can verify it is an idempotent, normal, ucp map with image equal to N and for any $x \in M$ we have

$$\begin{aligned}(h\tau h)(\mathbb{E}_N^h(x)) &= \tau(h\mathbb{E}_N(h^2)^{-1/2}\mathbb{E}_N(hxh)\mathbb{E}_N(h^2)^{-1/2}h)\\ &= \tau(h^2\mathbb{E}_N(h^2)^{-1}\mathbb{E}_N(hxh))\\ &= \tau(\mathbb{E}_N(h^2\mathbb{E}_N(h^2)^{-1}\mathbb{E}_N(hxh)))\\ &= \tau(\mathbb{E}_N(h^2)\mathbb{E}_N(h^2)^{-1}\mathbb{E}_N(hxh))\\ &= \tau(\mathbb{E}_N(hxh))\\ &= \tau(hxh)\\ &= (h\tau h)(x).\end{aligned}$$

Now assume that the L^2-implementation $\Phi^{(2)}$ of Φ with respect to τ exists, i.e. that there exists a constant $C > 0$ such that $\tau(\Phi(x)^*\Phi(x)) \leq C\tau(x^*x)$ for all $x \in M$. Then

$$\begin{aligned}(h\tau h)(\Phi^h(x)^*\Phi^h(x)) = \tau(\Phi(hx^*h)h^{-2}\Phi(hxh)) \leq \left\|h^{-2}\right\|\tau(\Phi(hx^*h)\Phi(hxh))\\ \leq C\left\|h^{-2}\right\|\tau(hx^*hhxh) \leq C\left\|h^{-2}\right\|\left\|h^2\right\|\tau(hx^*xh) = C\left\|h^{-2}\right\|\left\|h^2\right\|(h\tau h)(x^*x)\end{aligned}$$

for all $x \in M$, so the L^2-implementation $(\Phi^h)^{(2)}$ exists as well.

The converse implication follows, as $\Phi = (\Phi^h)^{h^{-1}}$.

For elements $a, b, x \in M$ the equality

$$\begin{aligned}(a\mathbb{E}_N b)^h(x) &= h^{-1}a\mathbb{E}_N(bhxh)h^{-1}\\ &= h^{-1}a\mathbb{E}_N(h^2)^{1/2}\mathbb{E}_N^h(h^{-1}bhx)\mathbb{E}_N(h^2)^{1/2}h^{-1}\\ &= (h^{-1}a\mathbb{E}_N(h^2)h^{-1})\mathbb{E}_N^h(h^{-1}bhx)\\ &= (h^{-1}a\mathbb{E}_N(h^2)h^{-1}\mathbb{E}_N^h h^{-1}bh)(x)\end{aligned}$$

implies by taking linear combinations and approximation that if $\Phi^{(2)} \in \mathcal{K}(M, N, \tau)$, then $(\Phi^h)^{(2)} \in \mathcal{K}(M, N, h\tau h)$. The converse statement follows as before, which finishes the proof. □

Now, for a triple (M, N, φ) let h be the unique (possibly unbounded) positive self-adjoint operator affiliated with $M \rtimes_{\sigma^\varphi} \mathbb{R}$ such that $h^{it} = \lambda_t$ for all $t \in \mathbb{R}$. If we further assume that $N \subseteq M^{\sigma^\varphi}$ (which implies that N is finite with a tracial state $\varphi|_N$) we have for $x \in N$ that $\lambda_t x \lambda_t^* = \sigma_t^\varphi(x) = x$ and hence $\lambda_t \in N' \cap (M \rtimes_{\sigma^\varphi} \mathbb{R})$. This implies that

h is affiliated with $N' \cap (M \rtimes_{\sigma^\varphi} \mathbb{R})$ and so its finite spectral projections are elements in $N' \cap (M \rtimes_{\sigma^\varphi} \mathbb{R})$. Set for $k \in \mathbb{N}$

$$p_k = \chi_{[k^{-1},k]}(h) \qquad \text{and} \qquad h_k = hp_k.$$

Here $\chi_{[k^{-1},k]}$ denotes the indicator function of $[k^{-1}, k] \subseteq \mathbb{R}$ and p_k is the corresponding spectral projection. Then, for every $k \in \mathbb{N}$, h_k is boundedly invertible in the corner algebra $p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ and we write h_k^{-1} for its inverse which we view as an operator in $M \rtimes_{\sigma^\varphi} \mathbb{R}$.

Denote by $\widehat{\varphi} := \varphi \circ T_\theta$ the dual weight of φ and let $\tau_\rtimes$ be the unique faithful normal semi-finite weight on $M \rtimes_{\sigma^\varphi} \mathbb{R}$ whose Connes cocycle derivative satisfies $(D\widehat{\varphi}/D\tau_\rtimes)_t = h^{it}$ for all $t \in \mathbb{R}$ (we refer to [29, Lemma 5.2]; the proofs below stay within the realm of bounded functionals). It is a trace on $M \rtimes_{\sigma^\varphi} \mathbb{R}$ which is formally given by

$$\tau_\rtimes(x) = \varphi \circ T_\theta(h^{-\frac{1}{2}} x h^{-\frac{1}{2}}), \qquad x \in (M \rtimes_{\sigma^\varphi} \mathbb{R})^+.$$

By construction we have

$$\widehat{\varphi}_j(p_k x p_k) = \tau_\rtimes(h_k^{\frac{1}{2}} \lambda(f_j)^* x \lambda(f_j) h_k^{\frac{1}{2}}), \qquad x \in M \rtimes_{\sigma^\varphi} \mathbb{R}, \tag{4.7}$$

for all $j \in \mathbb{N}$, where $\widehat{\varphi}_j$ and f_j are defined as in Subsection 4.2. Further note that the operators $\lambda(f_j)$ and h_k commute.

Remark 4.11. Following Remark 4.1, for $k \in \mathbb{N}$ the operators p_k and h_k can be described in terms of multiplication operators conjugated with the Fourier unitary $\mathcal{F}_2$. Indeed, $\mathcal{F}_2 \lambda_t \mathcal{F}_2^*$ is the multiplication operator on $L^2(\mathbb{R}, \mathcal{H})$ with the function $(s \mapsto e^{its})$ and therefore (under proper identification of the domains) $\mathcal{F}_2 h \mathcal{F}_2^*$ coincides with the multiplication operator with $(s \mapsto e^s)$. It follows that for all $k \in \mathbb{N}$, $\mathcal{F}_2 p_k \mathcal{F}_2^*$ is the multiplication with $(I_k : s \mapsto \chi_{[-\log(k),\log(k)]}(s))$ and $\mathcal{F}_2 h_k \mathcal{F}_2^*$ is the multiplication with $(J_k : s \mapsto \chi_{[-\log(k),\log(k)]}(s)e^s)$. Therefore, by Remark 4.1,

$$p_k = \lambda(\widehat{I_k}), \qquad h_k = \lambda(\widehat{J_k}), \qquad \text{and} \qquad h_k^{-1} = \lambda(\widehat{J_k^{-1}}),$$

where J_k^{-1} is the function $(s \mapsto \chi_{[-\log(k),\log(k)]} e^{-s})$. We also have that

$$\lambda(f_j) h_k = \lambda(f_j)\lambda(\widehat{J_k}) = \lambda(f_j * \widehat{J_k}) = \mathcal{F}_2^* \widehat{f}_j J_k \mathcal{F}_2, \tag{4.8}$$

where we view the product $\widehat{f}_j J_k$ as a multiplication operator. Since the Fourier transform of f_j is Gaussian we see that $\mathcal{F}_2^* \widehat{f}_j J_k \mathcal{F}_2$ is positive and boundedly invertible in the corner algebra $p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$. Further, by (4.3) and the Plancherel identity,

$$T_\theta(h_k^{-1}) = T_\theta(\lambda(\widehat{J_k^{-1/2}})\lambda(\widehat{J_k^{-1/2}})) = \|\widehat{J_k^{-1/2}}\|_2^2 = \|J_k^{-1/2}\|_2^2 = k - k^{-1}.$$

It follows that

$$\tau_{\rtimes}(p_k) = \varphi(T_\theta(h^{-1/2}p_k h^{-1/2})) = \varphi(T_\theta(h_k^{-1})) = k - k^{-1}.$$

In particular, $\tau_{\rtimes}(p_k) < \infty$. Since $\tau_{\rtimes}$ is tracial we also have for $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$,

$$\tau_{\rtimes}(p_k x p_k) = \varphi \circ T_\theta(h_k^{-1} p_k x p_k). \tag{4.9}$$

In the next statements it is notationally more convenient to work with property (rHAP) (resp. property (rHAP)$^-$) for general faithful normal positive functionals instead of just states, see Remark 3.3. Note that $p_k \widehat{\varphi}_j p_k$, $j \in \mathbb{N}$ is not a state, but a positive scalar multiple of a state.

We shall use the fact that the unique faithful normal $\widehat{\varphi}_j$-preserving conditional expectation $\mathbb{E}_N^{\widehat{\varphi}_j}$ of $M \rtimes_{\sigma^\varphi} \mathbb{R}$ onto N is given by $\mathbb{E}_N^{\widehat{\varphi}_j} = \mathbb{E}_N \circ T_{f_j}$. This fact was used in the proof of Proposition 4.4 already.

Lemma 4.12. *For every $k \in \mathbb{N}$, $j \in \mathbb{N}$ there is a faithful normal $p_k \widehat{\varphi}_j p_k$-preserving conditional expectation of $p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ onto $p_k N p_k$ given by*

$$x \mapsto \mu_k^{-1} p_k \mathbb{E}_N(T_{f_j}(x)) p_k = \mu_k^{-1} p_k \mathbb{E}_N^{\widehat{\varphi}_j}(x) p_k \tag{4.10}$$

where $\mu_k := T_{f_j}(p_k) = \|\widehat{f}_j \chi_{[-\log(k),\log(k)]}\|_2^2$. In particular, $T_{f_j}(p_k)$ is a scalar multiple of the identity.

Proof. First note that by Remark 4.1 and Remark 4.11 the operator $p_k \lambda(f_j)$ coincides with $\lambda(g_{j,k})$ where $g_{j,k}$ is the inverse Fourier transform of the function $\widehat{f}_j \chi_{[-\log(k),\log(k)]}$. The equality (4.3) then implies that

$$T_{f_j}(p_k) = T_\theta(\lambda(f_j)^* p_k \lambda(f_j)) = T_\theta(\lambda(g_{j,k})^* \lambda(g_{j,k})) = \|g_{j,k}\|_2^2 = \mu_k \tag{4.11}$$

is a multiple of the identity.

For $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ expand

$$\begin{aligned}(p_k \widehat{\varphi}_j p_k)(p_k \mathbb{E}_N^{\widehat{\varphi}_j}(x) p_k) &= \widehat{\varphi}_j(p_k \mathbb{E}_N^{\widehat{\varphi}_j}(x) p_k) \\ &= (\varphi \circ T_{f_j})(p_k \mathbb{E}_N(T_{f_j}(x)) p_k) \\ &= (\varphi \circ T_\theta)(\lambda(f_j)^* p_k \mathbb{E}_N(T_{f_j}(x)) p_k \lambda(f_j)).\end{aligned}$$

Since $N \subseteq M^{\sigma^\varphi}$ we see that

$$\begin{aligned}(p_k \widehat{\varphi}_j p_k)(p_k \mathbb{E}_N^{\widehat{\varphi}_j}(x) p_k) &= (\varphi \circ T_\theta)(\mathbb{E}_N(T_{f_j}(x)) \lambda(f_j)^* p_k \lambda(f_j)) \\ &= \varphi(\mathbb{E}_N(T_{f_j}(x)) T_{f_j}(p_k)).\end{aligned}$$

With (4.11) we can continue as follows:

$$(p_k\widehat{\varphi}_j p_k)(p_k\mathbb{E}_N^{\widehat{\varphi}_j}(x)p_k) = \mu_k\varphi(\mathbb{E}_N(T_{f_j}(x))) = \mu_k\varphi(T_{f_j}(x)) = \mu_k\widehat{\varphi}_j(x) = \mu_k\widehat{\varphi}_j(p_kxp_k).$$

This proves that (4.10) is $p_k\widehat{\varphi}_j p_k$-preserving, as claimed. For $x \in N \subseteq M^{\sigma^\varphi}$ we have that x and p_k commute. Therefore, using the N-module property of the maps involved,

$$p_k\mathbb{E}_N(T_{f_j}(p_kxp_k))p_k = p_kxp_k\mathbb{E}_N(T_{f_j}(p_k)) = \mu_k p_kxp_k.$$

This shows that the map $x \mapsto \mu_k^{-1}p_k\mathbb{E}_N(T_{f_j}(x))p_k$ is a unital (the unit being p_k) normal completely positive projection onto p_kNp_k (see [11, Theorem 1.5.10]). □

Lemma 4.13. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite and let $\tau \in N_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M. Then we have $\mathbb{E}_N(T_{f_j}(xa)) = \mathbb{E}_N(T_{f_j}(ax))$ and $\mathbb{E}_N(T_\theta(xa)) = \mathbb{E}_N(T_\theta(ax))$ for every $j \in \mathbb{N}$, $a \in \mathcal{L}(\mathbb{R})$ and $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$.*

Proof. We first prove that $\mathbb{E}_N(T_{f_j}(xa)) = \mathbb{E}_N(T_{f_j}(ax))$. Suppose $a = \lambda(k)$ and $x = y\lambda(g)$ for $y \in M$, $k \in L^1(\mathbb{R})$ and $g \in C_c(\mathbb{R})$. Let us first compute $T_{f_j}(xa)$ and $T_{f_j}(ax)$. By the formula (4.3) we have

$$T_{f_j}(xa) = \int_{\mathbb{R}} f_j^*(-t)(g * k * f_j)(t)\sigma_{-t}^\varphi(y)dt.$$

By a similar computation we get

$$T_{f_j}(ax) = \int_{\mathbb{R}} (f_j^* * k)(-t)(g * f_j)(t)\sigma_{-t}^\varphi(y)dt.$$

We now apply $\mathbb{E}_N$ to these expressions and use the fact that N is contained in the centralizer of φ, so $\mathbb{E}_N(\sigma_{-t}^\varphi(y)) = \mathbb{E}_N(y)$. It therefore suffices to prove the equality of the integrals $\int_{\mathbb{R}} f_j^*(-t)(g * k * f_j)(t)dt$ and $\int_{\mathbb{R}}(f_j^* * k)(-t)(g * f_j)(t)dt$. Using the commutativity of the convolution on $\mathbb{R}$, we can rewrite the first one as

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_j^*(-t)(g * f_j)(t-s)k(s)dsdt$$

and the second one is equal to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_j^*(-t-s)k(s)(g * f_j(t))dsdt.$$

In the second integral we can introduce a new variable $t' := t + s$ and it transforms into

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_j^*(-t')(g * f_j)(t'-s)k(s)dsdt',$$

which is equal to the first one. For arbitrary $a \in \mathcal{L}(\mathbb{R})$ and $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$ we can find bounded nets $(a_i)_{i\in I}$ and $(x_i)_{i\in I}$ formed by linear combinations of elements of the form discussed above that converge strongly to a and x, respectively, as a consequence of Kaplansky's density theorem. As multiplication is strongly continuous on bounded subsets, we have strong limits $\lim_{i\in I} a_i x_i = ax$ and $\lim_{i\in I} x_i a_i$. As both $\mathbb{E}_N$ and T_{f_j} are strongly continuous on bounded subsets, we may conclude.

The equality $\mathbb{E}_N(T_\theta(xa)) = \mathbb{E}_N(T_\theta(ax))$ follows by a similar computation. □

The ideas appearing in the proof of the next statements are of a similar type.

Proposition 4.14. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite and let $\tau \in N_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M. Then, for every $j \in \mathbb{N}$, the following statements hold:*

(1) The triple $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ satisfies property (rHAP) if and only if the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k \widehat{\varphi}_j p_k)$ satisfies property (rHAP) for every $k \in \mathbb{N}$. Moreover, the (rHAP) may be witnessed by contractive maps, i.e. we may assume that (1′) holds.

(2) If for every $k \in \mathbb{N}$ the property (rHAP) of the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k \widehat{\varphi}_j p_k)$ is witnessed by unital $p_k\widehat{\varphi}_j p_k$-preserving approximating maps, then the relative Haagerup property of $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ is witnessed by unital $\widehat{\varphi}_j$-preserving maps.

(3) If the triple $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ satisfies property (rHAP)$^-$ then the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k \widehat{\varphi}_j p_k)$ satisfies property (rHAP)$^-$ as well for every $k \in \mathbb{N}$.

Proof. *First part of* (1): For the "⇒" direction assume that $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ satisfies property (rHAP), that it is witnessed by a net of maps $(\Phi_i)_{i\in I}$ and fix $k \in \mathbb{N}$. We will show that $(p_k\Phi_i(\,\cdot\,)p_k)_{i\in I}$ is a net of approximating maps witnessing the relative Haagerup property of $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k \widehat{\varphi}_j p_k)$:

It is clear that for every $i \in I$ the map $p_k\Phi_i(\,\cdot\,)p_k$ is completely positive, that the net $(p_k\Phi_i(\,\cdot\,)p_k)_{i\in I}$ admits a uniform bound on its norms and that $p_k\Phi_i(\,\cdot\,)p_k \to \mathrm{id}$ in the point-strong topology in i as maps on $p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$.

By our assumptions, $N \subseteq M^{\sigma^\varphi}$ and hence p_k and N commute. Hence for $a, b \in N$, $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have

$$p_k\Phi_i(p_k a p_k x p_k b p_k)p_k = p_k\Phi_i(a p_k x p_k b)p_k = p_k a\Phi_i(p_k x p_k)b p_k = p_k a p_k \Phi_i(p_k x p_k)p_k b p_k,$$

which shows that $p_k\Phi_i(\,\cdot\,)p_k$ is a p_kNp_k-p_kNp_k-bimodule map for every $i \in I$.

We have by [46, Theorem VIII.3.19.(vi)], [46, Theorem X.1.17.(ii)] and the fact that p_k and $\lambda(f_j)$ commute that

$$\sigma_t^{\widehat{\varphi}_j}(p_k) = \lambda(f_j)^{it}\sigma_t^{\widehat{\varphi}}(p_k)\lambda(f_j)^{-it} = \lambda(f_j)^{it}p_k\lambda(f_j)^{-it} = p_k.$$

Therefore by [15, Lemma 2.3], for $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ positive,

$$(p_k\widehat{\varphi}_j p_k)(p_k\Phi_i(x)p_k) = \widehat{\varphi}_j(p_k\Phi_i(x)p_k) \leq \widehat{\varphi}_j(\Phi_i(x)) \leq \widehat{\varphi}_j(x) = (p_k\widehat{\varphi}_j p_k)(x),$$

i.e. $(p_k\widehat{\varphi}_j p_k) \circ (p_k\Phi_i(\,\cdot\,)p_k) \leq p_k\widehat{\varphi}_j p_k$.

Now, for every map Φ on $M \rtimes_{\sigma^\varphi} \mathbb{R}$ of the form $\Phi = a\mathbb{E}_N^{\widehat{\varphi}_j}(\cdot)b$ with $a, b \in M \rtimes_{\sigma^\varphi} \mathbb{R}$ and $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have, using Lemma 4.13 (recalling that $\mathbb{E}_N^{\widehat{\varphi}_j} = \mathbb{E}_N \circ T_{f_j}$) and the fact that p_k commutes with N, that

$$p_k\Phi(x)p_k = p_k a\mathbb{E}_N^{\widehat{\varphi}_j}(bp_kxp_k)p_k = p_k a\mathbb{E}_N^{\widehat{\varphi}_j}(p_kbp_kx)p_k = (p_kap_k)\mathbb{E}_N^{\widehat{\varphi}_j}(p_kbp_kx).$$

Lemma 4.12 then implies that $(p_k\Phi(\,\cdot\,)p_k)^{(2)} \in \mathcal{K}_{00}(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_j p_k)$. By taking linear combinations and approximating we see that if Φ is a map on $M \rtimes_{\sigma^\varphi} \mathbb{R}$ with $\Phi^{(2)} \in \mathcal{K}(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ then $(p_k\Phi(\cdot)p_k)^{(2)} \in \mathcal{K}(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_j p_k)$. Therefore for the approximating maps Φ_i, $i \in I$ we conclude that

$$(p_k\Phi_i(\,\cdot\,)p_k)^{(2)} \in \mathcal{K}(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_j p_k).$$

This shows that $(p_k\Phi_i(\,\cdot\,)p_k)_{i\in I}$ indeed witnesses the relative Haagerup property of the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_j p_k)$.

(3): Note that if $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ has property $(\mathrm{rHAP})^-$ witnessed by the net $(\Phi_i)_{i\in I}$, then property $(\mathrm{rHAP})^-$ of $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_j p_k)$ follows in a very similar way as above. The only condition that remains to be checked is that the L^2-implementation $(p_k\Phi_i(\,\cdot\,)p_k)^{(2)}$ exists. For this, assume that there exists $C > 0$ with $\widehat{\varphi}_j(\Phi_i(x)^*\Phi_i(x)) \leq C\widehat{\varphi}_j(x^*x)$ for all $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$. Then, using again [15, Lemma2.3] for the second inequality,

$$\begin{aligned}(p_k\widehat{\varphi}_j p_k)((p_k\Phi_i(x)p_k)^*(p_k\Phi_i(x)p_k)) &= \widehat{\varphi}_j(p_k\Phi_i(x^*)p_k\Phi_i(x)p_k)\\ &\leq \widehat{\varphi}_j(p_k\Phi_i(x)^*\Phi_i(x)p_k)\\ &\leq \widehat{\varphi}_j(\Phi_i(x)^*\Phi_i(x))\\ &\leq C\widehat{\varphi}_j(x^*x)\\ &= C(p_k\widehat{\varphi}_j p_k)(x^*x)\end{aligned}$$

for all $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$. The claim follows.

Second part of (1): For the "$\Leftarrow$" direction assume that for every $k \in \mathbb{N}$ the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_j p_k)$ satisfies property (rHAP) witnessed by approximating maps $(\Phi_{k,i})_{i\in I_k}$. We wish to apply Lemma 3.14 for which we check the conditions. By $N \subseteq M^{\sigma^\varphi}$ we have that N and λ_t commute for every $t \in \mathbb{R}$ and hence so do N and

h_k. In particular, $h_k \in (p_k N p_k)' \cap p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$. By (4.8) and the remarks after it, it follows that $\lambda(f_j)h_k$ is positive and boundedly invertible. Now, from (4.7) we see that the conditions of Lemma 3.14 are fulfilled and this lemma shows that the maps of the net $(\Phi_{k,i})_{i\in I_k}$ can be chosen contractively, i.e. we may assume that (1′) holds.

We shall prove that $(\Phi_{k,i}(p_k \cdot p_k))_{k\in\mathbb{N},i\in I_k}$ induces a net witnessing property (rHAP) of $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$. This in particular shows that we may assume (1′).

By the contractivity of the $\Phi_{k,i}$ it is clear that the maps $\Phi_{k,i}(p_k \cdot p_k)$ are completely positive with a uniform bound on their norms.

Since N and p_k commute we see that for $a, b \in N$ and $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$

$$\begin{aligned}\Phi_{k,i}(p_k a x b p_k) &= \Phi_{k,i}(p_k a p_k x p_k b p_k)\\ &= p_k a p_k \Phi_{k,i}(p_k x p_k) p_k b p_k\\ &= a p_k \Phi_{k,i}(p_k x p_k) p_k b\\ &= a \Phi_{k,i}(p_k x p_k) b.\end{aligned}$$

Therefore $\Phi_{k,i}(p_k \cdot p_k)$ is an N-N-bimodule map for every $k \in \mathbb{N}$, $i \in I_k$.

We have, using again [15, Lemma 2.3], that for $x \in (M \rtimes_{\sigma^\varphi} \mathbb{R})^+$

$$\begin{aligned}\widehat{\varphi}_j(\Phi_{k,i}(p_k x p_k)) &= \widehat{\varphi}_j(p_k \Phi_{k,i}(p_k x p_k) p_k)\\ &= (p_k \widehat{\varphi}_j p_k)(\Phi_{k,i}(p_k x p_k))\\ &\leq (p_k \widehat{\varphi}_j p_k)(p_k x p_k)\\ &= \widehat{\varphi}_j(p_k x p_k) \leq \widehat{\varphi}_j(x).\end{aligned}$$

i.e. $\widehat{\varphi}_j \circ \Phi_{k,i}(p_k \cdot p_k) \leq \widehat{\varphi}_j$.

We claim that $(\Phi_{k,i}(p_k \cdot p_k))^{(2)} \in \mathcal{K}(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ for all $k \in \mathbb{N}$, $i \in I_k$. Indeed, take an arbitrary map Φ of the form $\Phi(x) = p_k a p_k \mathbb{E}_N(T_{f_j}(p_k b p_k x))$ for $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$ where $a, b \in M \rtimes_{\sigma^\varphi} \mathbb{R}$. The L^2-implementations of such operators span $\mathcal{K}_{00}(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k \widehat{\varphi}_j p_k)$ by Lemma 4.12. Lemma 4.13 and the fact that p_k and N commute show that for $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$,

$$\Phi(p_k x p_k) = p_k a p_k \mathbb{E}_N(T_{f_j}(p_k b p_k x p_k)) = p_k a p_k \mathbb{E}_N(T_{f_j}(p_k b p_k x)).$$

Then, since $\mathbb{E}_N \circ T_{f_j}$ is the faithful normal $\widehat{\varphi}_j$-preserving conditional expectation of $M \rtimes_{\sigma^\varphi} \mathbb{R}$ onto N, this implies that $(\Phi(p_k \cdot p_k))^{(2)} \in \mathcal{K}_{00}(M \rtimes^{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$. By taking linear combinations and approximation we see that if $\Phi^{(2)} \in \mathcal{K}(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k \widehat{\varphi}_j p_k)$, then $(\Phi(p_k \cdot p_k))^{(2)} \in \mathcal{K}(M \rtimes^{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$. We conclude that

$$(\Phi_{k,i}(p_k \cdot p_k))^{(2)} \in \mathcal{K}(M \rtimes^{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j).$$

Now, for $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$ we see that

$$\lim_{k\to\infty} \lim_{i\in I_k} \Phi_{k,i}(p_k x p_k) = x,$$

in the strong topology. Then a variant of Lemma 2.2 shows that there is a directed set $\mathcal{F}$ and a function $(\widetilde{k},\widetilde{i}) : \mathcal{F} \to \{(k,i) \mid k \in \mathbb{N}, i \in I_k\}$, $F \mapsto (\widetilde{k}(F),\widetilde{i}(F))$ such $(\Phi_{\widetilde{k}(F),\widetilde{i}(F)})_{F\in\mathcal{F}}$ witnesses the relative Haagerup property of $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$.

(2): It only remains to show that if for every $k \in \mathbb{N}$ the property (rHAP) of the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k \widehat{\varphi}_j p_k)$ is witnessed by unital $p_k\widehat{\varphi}_j p_k$-preserving approximating maps, then the relative Haagerup property of $(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ is witnessed by unital $\widehat{\varphi}_j$-preserving maps. For this, assume that the maps $(\Phi_{k,i})_{i\in I}$ from before are unital and $p_k\widehat{\varphi}_j p_k$-preserving and choose a sequence $(\epsilon_k)_{k\in\mathbb{N}} \subseteq (0,1)$ with $\epsilon_k \to 0$. Recall that $p_k \in N' \cap (M \rtimes_{\sigma^\varphi} \mathbb{R})$ and note that $\mathbb{E}_N^{\widehat{\varphi}_j}(1-(1-\epsilon_k)p_k) \geq \epsilon_k$. We then have $\mathbb{E}_N^{\widehat{\varphi}_j}(1-(1-\epsilon_k)p_k) \in N \cap N'$, the inverse $(\mathbb{E}_N^{\widehat{\varphi}_j}(1-(1-\epsilon_k)p_k))^{-1} \in N \cap N'$ exists and $a_k := (1-(1-\epsilon_k)p_k)(\mathbb{E}_N^{\widehat{\varphi}_j}(1-(1-\epsilon_k)p_k))^{-1} \in N' \cap (M \rtimes_{\sigma^\varphi} \mathbb{R})$ is positive. Set $b_k := 1-(1-\epsilon_k)p_k \geq 0$. Define the maps

$$\widetilde{\Phi}_{k,i}(\,\cdot\,) := (1-\epsilon_k)\Phi_{k,i}(p_k\ \cdot\ p_k) + a_k\mathbb{E}_N^{\widehat{\varphi}_j}(b_k^{1/2}\ \cdot\ b_k^{1/2}).$$

Obviously $\widetilde{\Phi}_{k,i}$ is normal, completely positive and N-N-bimodular. We may finish the proof as in Theorem 3.13 now; since the statement of that theorem is not directly applicable here we will give the complete proof for the convenience of the reader.

We have

$$\widetilde{\Phi}_{k,i}(1) = (1-\epsilon_k)\Phi_{k,i}(p_k) + a_k\mathbb{E}_N^{\widehat{\varphi}_j}(b_k) = (1-\epsilon_k)p_k + (1-(1-\epsilon_k)p_k) = 1.$$

Now, since $\Phi_{k,i}$ is $p_k\widehat{\varphi}_j p_k$-preserving we have that $\widehat{\varphi}_j \circ \Phi_{k,i}(p_k x p_k) = \widehat{\varphi_j}(p_k x p_k)$ for all $x \in M \rtimes_{\sigma^\varphi} \mathbb{R}$, and hence with Lemma 4.13 we deduce that

$$\begin{aligned}
\widehat{\varphi}_j \circ \widetilde{\Phi}_{k,i}(x) &= (1-\epsilon_k)\widehat{\varphi}_j(\Phi_{k,i}(p_k x p_k)) + \widehat{\varphi}_j(a_k\mathbb{E}_N^{\widehat{\varphi}_j}(b_k^{1/2} x b_k^{1/2})) \\
&= (1-\epsilon_k)\widehat{\varphi}_j(p_k x p_k) + \widehat{\varphi}_j(\mathbb{E}_N^{\widehat{\varphi}_j}(a_k) b_k^{1/2} x b_k^{1/2}) \\
&= (1-\epsilon_k)\widehat{\varphi}_j \circ \mathbb{E}_N^{\widehat{\varphi}_j}(p_k x p_k) + \widehat{\varphi}_j \circ \mathbb{E}_N^{\widehat{\varphi}_j}(b_k^{1/2} x b_k^{1/2}) \\
&= (1-\epsilon_k)\widehat{\varphi}_j \circ \mathbb{E}_N^{\widehat{\varphi}_j}(p_k x) + \widehat{\varphi}_j \circ \mathbb{E}_N^{\widehat{\varphi}_j}(b_k x) \\
&= (1-\epsilon_k)\widehat{\varphi}_j(p_k x) + \widehat{\varphi}_j(b_k x) \\
&= \widehat{\varphi}_j(x).
\end{aligned}$$

By the fact that $(\Phi_{k,i}(p_k\ \cdot\ p_k))^{(2)} \in \mathcal{K}(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j)$ and by Lemma 4.13, we have

$$\widetilde{\Phi}_{k,i}^{(2)} = (1-\epsilon_k)(\Phi_{k,i}(p_k\ \cdot\ p_k))^{(2)} + a_k e_N^{\widehat{\varphi}_j} b_k \in \mathcal{K}(M \rtimes_{\sigma^\varphi} \mathbb{R}, N, \widehat{\varphi}_j).$$

Further, for every $x \in (M \rtimes_{\sigma^\varphi} \mathbb{R})_+$,

$$\begin{aligned}
\widetilde{\Phi}_{k,i}(x) - (1-\epsilon_k)\Phi_{k,i}(p_k x p_k) &= a_k\mathbb{E}_N^{\widehat{\varphi}_j}(b_k^{1/2} x b_k^{1/2}) \\
&\leq \|x\|\, a_k\mathbb{E}_N^{\widehat{\varphi}_j}(b_k) \\
&= \|x\|\,(1-(1-\epsilon_k)p_k),
\end{aligned}$$

from which we deduce that $\lim_{F\in\mathcal{F}} \widetilde{\Phi}_{\widetilde{k}(F),\widetilde{i}(F)} = \mathrm{id}_{M\rtimes_{\sigma^\varphi}\mathbb{R}}$. This implies that the net $(\widetilde{\Phi}_{\widetilde{k}(F),\widetilde{i}(F)})_{F\in\mathcal{F}}$ of unital $\widehat{\varphi}_j$-preserving maps witnesses the relative Haagerup property of $(M\rtimes_{\sigma^\varphi}\mathbb{R}, N, \widehat{\varphi}_j)$. □

We are now ready to formulate the key statement of this section. Note that for every $k\in\mathbb{N}$ the von Neumann algebra $p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ is finite with a faithful normal tracial state $p_k\tau_{\rtimes}p_k$.

Theorem 4.15. *Let $N\subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite and let $\tau\in N_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau\circ\mathbb{E}_N$ on M. Then the following statements are equivalent:*

(1) The triple (M,N,φ) has property (rHAP);
(2) $(M\rtimes_{\sigma^\varphi}\mathbb{R}, N, \widehat{\varphi}_j)$ has property (rHAP) for every $j\in\mathbb{N}$;
(3) $(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_kNp_k, p_k\tau_{\rtimes}p_k)$ has property (rHAP) for every $k\in\mathbb{N}$.

Further, the following statement holds:

(4) If the triple (M,N,φ) has property (rHAP)$^-$, then $(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_kNp_k, p_k\tau_{\rtimes}p_k)$ has property (rHAP)$^-$ for every $k\in\mathbb{N}$.

Proof. The equivalence "(1) ⇔ (2)" was proved in Theorem 4.9.

"(2) ⇒ (3)": Assume that for $j\in\mathbb{N}$ the triple $(M\rtimes_{\sigma^\varphi}\mathbb{R}, N, \widehat{\varphi}_j)$ has property (rHAP) and fix $k\in\mathbb{N}$. Then by Proposition 4.14, the triple $(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_jp_k)$ also has the property (rHAP). Let $(\Phi_i)_{i\in I}$ be a net of suitable approximating maps and define the self-adjoint boundedly invertible operator $A_{j,k} := \lambda(f_j)h_k^{1/2}\in(p_kNp_k)'\cap(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k)$. By (4.7) for every $x\in p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ the equality

$$(p_k\widehat{\varphi}_jp_k)(x) = \tau_{\rtimes}(A_{j,k}^*xA_{j,k}) = (A_{j,k}p_k\tau_{\rtimes}p_kA_{j,k})(x)$$

holds and hence Lemma 4.10 implies that the L^2-implementation of the map $\Phi_i'(\,\cdot\,) := A_{j,k}\Phi_i(A_{j,k}^{-1}\,\cdot\,A_{j,k}^{-1})A_{j,k}$ exists and is contained in $\mathcal{K}(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_kNp_k, p_k\tau_{\rtimes}p_k)$. Similarly to the proof of Proposition 4.14 one checks that the net $(\Phi_i')_{i\in I}$ witnesses property (rHAP) of $(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_kNp_k, p_k\tau_{\rtimes}p_k)$. We omit the details.

"(2) ⇐ (3)" Now assume that the triple $(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_kNp_k, p_k\tau_{\rtimes}p_k)$ has property (rHAP) for every $k\in\mathbb{N}$. It suffices to show that the triple $(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_jp_k)$ has property (rHAP) as it implies the desired statement by Proposition 4.14. So let $(\Phi_i)_{i\in I}$ be a net that witnesses property (rHAP) of $(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_kNp_k, p_k\tau_{\rtimes}p_k)$ and set $\Phi_i' := A_{j,k}^{-1}\Phi_i(A_{j,k}\,\cdot\,A_{j,k})A_{j,k}^{-1}$. Lemma 4.10 and (4.7) imply that for every $i\in I$ the L^2-implementation $(\Phi_i')^{(2)}$ of Φ_i' with respect to the positive functional $p_k\widehat{\varphi}_jp_k$ is contained in $\mathcal{K}(p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_jp_k)$. Again, similarly to the proof of Proposition 4.14 one checks that the net $(\Phi_i')_{i\in I}$ witnesses property (rHAP).

It remains to show (4). The statement easily follows from Proposition 4.9, Proposition 4.14 and the arguments used in the proof of the implication "(2) ⇒ (3)". □

5. Main results

After the main work has been done in Section 4 we can now put the pieces together. This allows us to show that in the case of a finite von Neumann subalgebra the notion of relative Haagerup property is independent of the choice of the corresponding faithful normal conditional expectation, that the approximating maps may be chosen to be unital and state-preserving and that property (rHAP) and property (rHAP)$^-$ are equivalent. The general notation will be the same as in Section 4.

5.1. Independence of the conditional expectation

Let $N \subseteq M$ be a unital inclusion of von Neumann algebras for which N is finite with a faithful normal tracial state $\tau \in N_*$. Let further $\mathbb{E}_N, \mathbb{F}_N : M \to N$ be two faithful normal conditional expectations and extend τ to states $\varphi := \tau \circ \mathbb{E}_N$ and $\psi := \tau \circ \mathbb{F}_N$ on M. In this subsection we will prove that the triple $(M, N, \mathbb{E}_N)$ has property (rHAP) if and only if the triple $(M, N, \mathbb{F}_N)$ does, i.e. the relative Haagerup property is an intrinsic invariant of the inclusion $N \subseteq M$. Let us first introduce some notation.

As in Section 4 consider the crossed product von Neumann algebra $M \rtimes_{\sigma^\varphi} \mathbb{R}$ which contains the projections $p_k \in N' \cap (M \rtimes_{\sigma^\varphi} \mathbb{R})$, $k \in \mathbb{N}$ and carries the canonical normal semi-finite tracial weight $\tau_\rtimes$ which we will from now on denote by $\tau_{\rtimes,1}$. For $t \in \mathbb{R}$ write λ_t^φ for the left regular representation operators in $M \rtimes_{\sigma^\varphi} \mathbb{R}$. Similarly, we write $\tau_{\rtimes,2}$ for the canonical normal semi-finite tracial weight on $M \rtimes_{\sigma^\psi} \mathbb{R}$ and denote the corresponding left regular representation operators by λ_t^ψ, $t \in \mathbb{R}$.

For $t \in \mathbb{R}$ let $u_t := (D\varphi/D\psi)_t \in M$ be the Connes cocycle Radon-Nikodym derivative, so in particular $u_t \sigma_t^\varphi(u_s) = u_{t+s}$ and $\sigma_t^\psi(x) = u_t^* \sigma_t^\varphi(x) u_t$ hold for all $s, t \in \mathbb{R}$. Then (see [46, Proof of Theorem X.1.7]) there exists an isomorphism $\rho : M \rtimes_{\sigma^\psi} \mathbb{R} \to M \rtimes_{\sigma^\varphi} \mathbb{R}$ of von Neumann algebras which restricts to the identity on M and for which $\rho(\lambda_t^\psi) = u_t \lambda_t^\varphi$ for all $t \in \mathbb{R}$. This implies that the dual actions θ^φ and θ^ψ of σ^φ and σ^ψ respectively are related by the equality $\theta_t^\varphi \circ \rho = \rho \circ \theta_t^\psi$, $t \in \mathbb{R}$. Further, $\tau_{\rtimes,1} \circ \rho = \tau_{\rtimes,2}$ (see the footnote[4]). Denote by h_ψ the unique unbounded self-adjoint positive operator affiliated with $M \rtimes_{\sigma^\psi} \mathbb{R}$ such that $h_\psi^{it} = \lambda_t^\psi$ for all $t \in \mathbb{R}$ and set

[4] This is well-known to specialists, but it seems that the statement does not appear explicitly in [46]. The argument goes as follows. Firstly, as ρ intertwines the dual actions on $M \rtimes_{\sigma^\psi} \mathbb{R}$ and $M \rtimes_{\sigma^\varphi} \mathbb{R}$ we find that $\widehat{\varphi} \circ \rho$ is the dual weight of φ in the crossed product $M \rtimes_{\sigma^\psi} \mathbb{R}$. Let $t \in \mathbb{R}$. By [46, Theorem X.1.17] we have Connes cocycle derivative $\left(\frac{D\widehat{\psi}}{D\widehat{\varphi} \circ \rho}\right)_t = u_t = \rho(u_t)$. Then by the chain rule [46, Theorem VIII.3.7],

$$\left(\frac{D\tau_{\rtimes,2}}{D\tau_{\rtimes,1} \circ \rho}\right)_t = \left(\frac{D\tau_{\rtimes,2}}{D\widehat{\psi}}\right)_t \left(\frac{D\widehat{\psi}}{D\widehat{\varphi} \circ \rho}\right)_t \left(\frac{D\widehat{\varphi} \circ \rho}{D\tau_{\rtimes,1} \circ \rho}\right)_t = \lambda_{-t}^\psi \rho^{-1}(u_t \lambda_t^\varphi) = 1.$$

Hence $\tau_{\rtimes,1} \circ \rho = \tau_{\rtimes,2}$.

$$p_{\psi,k} := \chi_{[k^{-1},k]}(h_\psi) \qquad \text{and} \qquad q_k := \rho(p_{\psi,k}).$$

for $k \in \mathbb{N}$. Further, define

$$h_{\psi,k} := \rho(\chi_{[k^{-1},k]}(h_\psi)h_\psi) = \rho(p_{\psi,k}h_\psi).$$

Recall that for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we write $h^{it} = \lambda_t^\varphi$, $p_k := \chi_{[k^{-1},k]}(h)$, and $h_k := p_k h$.

The following statement compares to Lemma 4.12.

Lemma 5.1. *For every $k \in \mathbb{N}$ there is a (unique) faithful normal $p_k\tau_{\rtimes,1}p_k$-preserving conditional expectation $\mathbb{E}_{1,k} : p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k \to p_kNp_k$ given by*

$$x \mapsto \nu_k^{-1} p_k \mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1}x))p_k, \tag{5.1}$$

where $\nu_k := T_{\theta^\varphi}(h_k^{-1}) = k - k^{-1}$. In particular, $T_{\theta^\varphi}(h_k^{-1})$ is a scalar multiple of the identity.

Proof. The proof is essentially the same as that of Lemma 4.12. First note that by Remark 4.11 the operator h_k coincides with $\lambda(\widehat{J_k})$ where $J_k(s) = \chi_{[-\log(k),\log(k)]}e^s$ and that $\nu_k = T_{\theta^\varphi}(h_k^{-1}) = k - k^{-1}$ is a multiple of the identity. For $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ one checks using (4.9) for the second and last equality, that

$$\begin{aligned}
(p_k\tau_{\rtimes,1}p_k)(p_k\mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1}x))p_k) &= \tau_{\rtimes,1}(p_k\mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1}x))p_k) \\
&= \varphi \circ T_{\theta^\varphi}(p_k h_k^{-1}\mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1}x))p_k) \\
&= \varphi \circ T_{\theta^\varphi}(h_k^{-1}\mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1}x))) \\
&= \varphi\left(T_{\theta^\varphi}(h_k^{-1})\mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1}x))\right) \\
&= \nu_k\varphi\left(\mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1}x))\right) \\
&= \nu_k\varphi \circ T_{\theta^\varphi}(h_k^{-1}x)) \\
&= \nu_k\tau_{\rtimes,1}(p_kxp_k),
\end{aligned}$$

hence $\mathbb{E}_{1,k}$ is indeed $p_k\tau_{\rtimes,1}p_k$-preserving. Here we used in the fourth line that N is invariant under the dual action θ^φ and in the fifth line that $T_{\theta^\varphi}(h_k^{-1})$ is a multiple of the identity.

From Lemma 4.13 we see that

$$\nu_k^{-1} p_k \mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1}\,\cdot\,))p_k = \nu_k^{-1} p_k \mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1/2}\,\cdot\,h_k^{-1/2}))p_k,$$

and from the right-hand side of this expression it is clear that (5.1) is completely positive. The remaining statements (i.e. that $\mathbb{E}_{1,k}$ is a unital faithful normal p_kNp_k-p_kNp_k-bimodule map) are then easy to check. □

The following lemma provides the analogous statement for the functional $q_k\tau_{\rtimes,1}q_k$ and the inclusion $q_kNq_k \subseteq q_k(M \rtimes_{\sigma^\varphi} \mathbb{R})q_k$. We omit the proof.

Lemma 5.2. *For every $k \in \mathbb{N}$ there is a (unique) faithful normal $q_k\tau_{\rtimes,1}q_k$-preserving conditional expectation $\mathbb{E}_{2,k} : q_k(M \rtimes_{\sigma^\varphi} \mathbb{R})q_k \to q_kNq_k$ given by*

$$x \mapsto \nu_k^{-1} q_k \mathbb{E}_N(T_{\theta^\varphi}(h_{\psi,k}^{-1}x))q_k,$$

where $\nu_k := k - k^{-1}$ as before.

Proposition 5.3. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras for which N is finite with a faithful normal tracial state $\tau \in N_*$. Let further $\mathbb{E}_N, \mathbb{F}_N : M \to N$ be two faithful normal conditional expectations and extend τ to states $\varphi := \tau \circ \mathbb{E}_N$ and $\psi := \tau \circ \mathbb{F}_N$ on M. Then the following statements are equivalent:*

(1) For every $k \in \mathbb{N}$ the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_kNp_k, p_k\tau_{\rtimes,1}p_k)$ has property (rHAP).
(2) For every $k \in \mathbb{N}$ the triple $(q_k(M \rtimes_{\sigma^\varphi} \mathbb{R})q_k, q_kNq_k, q_k\tau_{\rtimes,1}q_k)$ has property (rHAP).

Proof. By symmetry it suffices to consider the direction "(2) $\Rightarrow$ (1)". For this, fix $k, l \in \mathbb{N}$ and let $(\Phi_{l,i})_{i\in I_l}$ be a net of maps witnessing the relative Haagerup property of the triple $(q_l(M \rtimes_{\sigma^\varphi} \mathbb{R})q_l, q_lNq_l, q_l\tau_{\rtimes,1}q_l)$, which we can assume to be contractive by Lemma 3.14. Define for $i \in I_l$ the normal completely positive contractive map

$$\Phi'_{k,l,i} : p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k \to p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k : x \mapsto p_k\Phi_{l,i}(q_lxq_l)p_k.$$

As $N \subseteq M^{\sigma^\varphi}$ and $N \subseteq M^{\sigma^\psi}$, N commutes with both q_l and p_k. Thus we have that for $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ and $a, b \in N$

$$\begin{aligned}
\Phi'_{k,l,i}(p_kap_kxp_kbp_k) &= p_k\Phi_{l,i}(q_lp_kap_kxp_kbp_kq_l)p_k = p_k\Phi_{l,i}(q_laxbq_l)p_k \\
&= p_k\Phi_{l,i}(q_laq_lxq_lbq_l)p_k = p_kq_laq_l\Phi_{l,i}(q_lxq_l)q_lbq_lp_k \\
&= p_ka\Phi_{l,i}(q_lxq_l)bp_k = p_kap_k\Phi_{l,i}(q_lxq_l)p_kbp_k \\
&= p_kap_k\Phi'_{k,l,i}(x)p_kbp_k,
\end{aligned}$$

i.e. $\Phi'_{k,l,i}$ is p_kNp_k-p_kNp_k-bimodular. Further, $(p_k\tau_{\rtimes,1}p_k) \circ \Phi'_{k,l,i} \leq p_k\tau_{\rtimes,1}p_k$ since for all positive $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have

$$\begin{aligned}
(p_k\tau_{\rtimes,1}p_k) \circ \Phi'_{k,l,i}(x) &= \tau_{\rtimes_1}(p_k\Phi_{l,i}(q_lp_kxp_kq_l)p_k) \leq \tau_{\rtimes,1}(\Phi_{l,i}(q_lp_kxp_kq_l)) \\
&= \tau_{\rtimes,1}(q_l\Phi_{l,i}(q_lp_kxp_kq_l)q_l) \leq \tau_{\rtimes,1}(q_lp_kxp_kq_l) \\
&\leq (p_k\tau_{\rtimes,1}p_k)(x).
\end{aligned}$$

For every map Φ on $q_l(M \rtimes_{\sigma^\varphi} \mathbb{R})q_l$ of the form $\Phi = a\mathbb{E}_{2,l}b$ with $a, b \in q_l(M \rtimes_{\sigma^\varphi} \mathbb{R})q_l$ and $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have by Lemma 5.2 that

$$\begin{aligned} p_k\Phi(q_l x q_l)p_k &= p_k a \mathbb{E}_{2,l}(bq_l x q_l)p_k \\ &= \nu_l^{-1} p_k a q_l \mathbb{E}_N(T_{\theta^\varphi}(h_{\psi,l}^{-1} bq_l x q_l))q_l p_k \\ &= \nu_l^{-1} p_k a p_k \mathbb{E}_N(T_{\theta^\varphi}(h_{\psi,l}^{-1} bq_l x q_l)). \end{aligned}$$

Now we may use the isomorphism ρ and apply Lemma 4.13 to $M \rtimes_{\sigma^\psi} \mathbb{R}$ to get

$$\begin{aligned} p_k\Phi(q_l x q_l)p_k &= \nu_l^{-1} p_k a p_k \mathbb{E}_N(T_{\theta^\psi}(p_{\psi,l} h_\psi^{-1}\rho^{-1}(b)p_{\psi,l}\rho^{-1}(x)p_{\psi,l})) \\ &= \nu_l^{-1} p_k a p_k \mathbb{E}_N(T_{\theta^\psi}(p_{\psi,l} h_\psi^{-1}\rho^{-1}(b)p_{\psi,l}\rho^{-1}(x)) \\ &= \nu_l^{-1} p_k a p_k \mathbb{E}_N(T_{\theta^\varphi}(q_l h_{\psi,l}^{-1} bq_l x)). \end{aligned}$$

Then by Lemma 4.13 applied to $M \rtimes_{\sigma^\varphi} \mathbb{R}$ for the second equality and Lemma 5.1 for the last equality, we find

$$\begin{aligned} p_k\Phi(q_l x q_l)p_k &= \nu_l^{-1} p_k a p_k \mathbb{E}_N(T_{\theta^\varphi}(q_l h_{\psi,l}^{-1} bq_l x p_k)) \\ &= \nu_l^{-1} p_k a p_k \mathbb{E}_N(T_{\theta^\varphi}(h_k^{-1}(h_k q_l h_{\psi,l}^{-1} bq_l x)))p_k \\ &= \nu_k\nu_l^{-1} p_k a \mathbb{E}_{1,k}((h_k q_l h_{\psi,l}^{-1} bq_l)x). \end{aligned}$$

Thus $(p_k\Phi(q_l \cdot q_l)p_k)^{(2)} \in \mathcal{K}_{00}(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k\tau_{\rtimes,1}p_k)$. By taking linear combinations and approximation we see that if $\Phi^{(2)} \in \mathcal{K}(q_l(M \rtimes_{\sigma^\varphi} \mathbb{R})q_l, q_l N q_l, q_l\tau_{\rtimes,1}q_l)$, then also $p_k\Phi(q_l \cdot q_l)p_k)^{(2)} \in \mathcal{K}(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k\tau_{\rtimes,1}p_k)$. In particular, $(\Phi'^{(2)}_{k,l,i}) \in \mathcal{K}(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k\tau_{\rtimes,1}p_k)$ for $k, l \in \mathbb{N}$ and $i \in I_l$.

For every $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have that

$$\lim_{l\to\infty} \lim_{i\in I_l} \Phi'_{k,l,i}(x) = x$$

in the strong topology. A variant of Lemma 2.2 then shows that there is a directed set $\mathcal{F}$ and an increasing function $(\widetilde{l}, \widetilde{i}) : \mathcal{F} \to \{(l,i) \mid k \in \mathbb{N}, i \in I_l\}$, $F \mapsto (\widetilde{l}(F), \widetilde{i}(F))$ such that $(\Phi'_{k,\widetilde{l}(F),\widetilde{i}(F)})_{F\in\mathcal{F}}$ witnesses the relative Haagerup property of $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k\tau_{\rtimes,1}p_k)$. □

Theorem 5.4. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras with N finite. Let $\mathbb{E}_N, \mathbb{F}_N : M \to N$ be two faithful normal conditional expectations. Then the triple $(M, N, \mathbb{E}_N)$ has (rHAP) if and only if the triple $(M, N, \mathbb{F}_N)$ has (rHAP).*

Proof. Assume that the triple $(M, N, \mathbb{E}_N)$ has the relative Haagerup property. Let τ be a faithful normal tracial state on N that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M. Theorem 4.15 implies that for every $k \in \mathbb{N}$ the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k\tau_{\rtimes,1}p_k)$ has the (rHAP). With Proposition 5.3 we get that for every $k \in \mathbb{N}$ the triple $(q_k(M \rtimes_{\sigma^\varphi} \mathbb{R})q_k, q_k N q_k, q_k\tau_{\rtimes,1}q_k)$ has the (rHAP). The isomorphism ρ restricts to an isomorphism

$q_k(M \rtimes_{\sigma^\varphi} \mathbb{R})q_k \cong p_{\psi,k}(M \rtimes_{\sigma^\psi} \mathbb{R})p_{\psi,k}$ which maps q_kNq_k onto $p_{\psi,k}Np_{\psi,k}$ and for which $(q_k\tau_{\rtimes,1}q_k)\circ\rho = p_{\psi,k}\tau_{\rtimes,2}p_{\psi,k}$. Combining this with Theorem 4.15 implies that $(M, N, \mathbb{F}_N)$ has the (rHAP). □

5.2. Unitality and state-preservation of the approximating maps

The following theorem states that for triples (M, N, φ) with N finite the approximating maps may be assumed to be unital and state-preserving. The proof combines the passage to suitable crossed products and corners of crossed products from Section 4 with the case considered in Subsection 3.3.

Theorem 5.5. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite. Let $\tau \in N_*$ be a faithful normal (possibly non-tracial) state that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M and assume that the triple (M, N, φ) has property (rHAP). Then property (rHAP) may be witnessed by a net of unital and φ-preserving approximating maps, i.e. we may assume (1″) and (4′).*

Proof. First assume that τ is tracial. Since the triple (M, N, φ) has property (rHAP) we get with Theorem 4.9 and Proposition 4.14 that for all $j \in \mathbb{N}$, $k \in \mathbb{N}$ the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_j p_k)$ has property (rHAP) as well and that it may be witnessed by a net of contractive approximating maps. As we have seen before, for every $k \in \mathbb{N}$ the element $h_k^{1/2}\lambda(f_j) \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ is positive and boundedly invertible in $p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$. Further, by (4.7) and [46, Theorem VIII.2.11] the equality

$$\sigma_t^{p_k\widehat{\varphi}_j p_k}(x) = (h_k^{1/2}\lambda(f_j))^{it} x (h_k^{1/2}\lambda(f_j))^{-it}$$

holds for all $x \in p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$, $t \in \mathbb{R}$. Theorem 3.13 then implies that property (rHAP) of $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_kNp_k, p_k\widehat{\varphi}_j p_k)$ may for every $j, k \in \mathbb{N}$ be witnessed by a net of unital $(p_k\widehat{\varphi}_j p_k)$-preserving maps. By applying the converse directions of Proposition 4.14 and Theorem 4.9 we deduce the claimed statement.

Now we show that we may replace τ by any non-tracial faithful state in N_*. Let still $\tau \in N_*$ be a faithful tracial state. Let $(\Phi_i)_{i\in I}$ be approximating maps witnessing the (rHAP) for $(M, N, \tau \circ \mathbb{E}_N)$ which by the previous paragraph may be taken unital and $\tau \circ \mathbb{E}_N$-preserving. The proof of Theorem 3.7, exploiting Lemmas 3.5 and 3.6 shows that $(\Phi_i)_{i\in I}$ also witness the (rHAP) for $(M, N, \varphi \circ \mathbb{E}_N)$ for any faithful state $\varphi \in N_*$. Further Lemma 3.5 shows that Φ_i is $\varphi \circ \mathbb{E}_N$-preserving and we are done. □

5.3. Equivalence of (rHAP) and (rHAP)$^-$

In [5] among other things Bannon and Fang prove that for triples (M, N, τ) of finite von Neumann algebras with a tracial state $\tau \in M_*$ the subtraciality condition in Popa's

notion of the relative Haagerup property is redundant. It is easy to check that their proof translates into our setting, which leads to the following variation of [5, Theorem 2.2].

Theorem 5.6 *(Bannon-Fang). Let M be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau \in M_*$ and let $N \subseteq M$ be a unital inclusion of von Neumann algebras. If the triple (M, N, τ) has property (rHAP)$^-$, then it has property (rHAP). Further, property (rHAP) may be witnessed by unital and trace-preserving approximating maps.*

In combination with Theorem 5.5 the following theorem provides a generalisation of Theorem 5.6.

Theorem 5.7. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite. Let $\tau \in N_*$ be a faithful normal state that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M. Then the triple (M, N, φ) has property (rHAP) if and only if it has property (rHAP)$^-$.*

Proof. By Theorem 3.7 we may without loss of generality assume that τ is tracial on N. It is clear that property (rHAP) implies property (rHAP)$^-$. Conversely, if the triple (M, N, φ) has property (rHAP)$^-$, then we deduce from Theorem 4.15 that for every $k \in \mathbb{N}$ the triple $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k \tau_{\rtimes,1} p_k)$ has property (rHAP)$^-$ as well. Recall that $p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ is finite since $p_k \tau_{\rtimes} p_k$ is a faithful normal tracial state. We can hence apply Theorem 5.6 to deduce that $(p_k(M \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k N p_k, p_k \tau_{\rtimes,1} p_k)$ has (rHAP) for every $k \in \mathbb{N}$. In combination with Theorem 4.15 this implies that the triple (M, N, φ) has property (rHAP). □

We finish this subsection with an easy lemma which will be needed later on. It could be formulated in a greater generality, but this is the form we will use in Section 8.

Lemma 5.8. *Let $N \subseteq M_1 \subseteq M$ be a unital inclusion of von Neumann algebras with N finite. Assume that we have faithful normal conditional expectations $\mathbb{E}_1 : M_1 \to N$ and $\mathbb{F}_1 : M \to M_1$ and a faithful tracial state $\tau \in N_*$. Set $\varphi = \tau \circ \mathbb{E}_1 \circ \mathbb{F}_1$. Then if the triple (M, N, φ) has property (rHAP) then the triple $(M_1, N, \varphi|_{M_1})$ also has property (rHAP).*

Proof. Suppose that $(\Phi_i)_{i \in I}$ is a net of approximations (unital, φ-preserving maps on M) satisfying the conditions in the property (rHAP) for the triple (M, N, φ). For each $i \in I$ define $\Psi_i := \mathbb{F}_1 \circ \Phi_i|_{M_1}$. Our conditions guarantee that $\mathbb{F}_1$ is φ-preserving, so Ψ_i is a normal, ucp, N-N-bimodular, $\varphi|_{M_1}$ preserving map on M_1. Due to the last theorem, we need only to check that $(\Psi_i)_{i \in I}$ satisfy the conditions in the property (rHAP)$^-$ (for the triple $(M_1, N, \varphi|_{M_1})$). Condition (iii) holds as for $x \in M_1$ we have $\Psi_i(x) - x = \mathbb{F}_1(\Phi_i(x) - x)$ and $\mathbb{F}_1^{(2)}$ is the orthogonal projection from $L^2(M, \varphi)$ onto $L^2(M_1, \varphi|_{M_1})$.

To verify the last condition we assume first that Φ_i is of the form $a(\mathbb{E}_1 \circ \mathbb{F}_1)b$ for some $a, b \in M$. But then for $x \in M_1$ we have

$$\Psi_i(x) = \mathbb{F}_1(a(\mathbb{E}_1 \circ \mathbb{F}_1)(bx))) = \mathbb{F}_1(a)(\mathbb{E}_1 \circ \mathbb{F}_1)(bx))) = \mathbb{F}_1(a)\mathbb{E}_1(\mathbb{F}_1(b)x),$$

so we get that $\Psi_i^{(2)} \in \mathcal{K}_{00}(M_1, N, \varphi|_{M_1})$. Taking linear combinations and approximation ends the proof. □

6. First examples

In this section we first put our definitions and main results in concrete context, discussing examples of the Haagerup (and non-Haagerup) inclusions arising in the framework of Cartan subalgebras, as studied in [32], [48] and [1], and then present the case of the big algebra being just $\mathcal{B}(\mathcal{H})$. The examples related to the latter situation show that the relative Haagerup property is not implied by coamenability as defined in [39].

6.1. Examples from equivalence relations and groupoids

In this subsection we will discuss examples of inclusions of von Neumann algebras which satisfy the relative Haagerup property and have already appeared in the literature. As mentioned in the introduction, the notion of the Haagerup property regarding the von Neumann inclusions beyond the finite context first appeared in the study of von Neumann algebras associated with groupoids/equivalence relations.

The first result here is due to [32], still in the finite context. Note that Jolissaint uses the definition of the Haagerup inclusion $N \subseteq M$ due to Popa in [41], namely the one using the larger ideal of 'generalised compacts' than the one employed in this paper, but also note that due to [41, Proposition 2.2] both notions coincide if $N' \cap M \subseteq N$, so for example if N is a maximal abelian subalgebra in M, which is the case of interest for the result below.

Theorem 6.1 *([32, Theorem 2.1]). Let $\mathcal{R}$ be a measure preserving standard equivalence relation on a set X (with the measure ν on $\mathcal{R}$ induced by the invariant probability measure μ on X). Then the following are equivalent:*

(i) $\mathcal{R}$ has the Haagerup property, i.e. it admits a sequence of positive-definite functions $(\varphi_n : \mathcal{R} \to \mathbb{C})_{n\in\mathbb{N}}$ which are bounded by 1 on the diagonal, converge to 1 ν-almost everywhere and satisfy the 'vanishing property': for every $n \in \mathbb{N}$ and $\epsilon > 0$ there is

$$\nu(\{(x, y) \in \mathcal{R} : |\varphi_n(x, y)| > \epsilon\}) < \infty;$$

(ii) the von Neumann inclusion (of finite von Neumann algebras)

$$L^\infty(X, \mu) \subseteq \mathcal{L}(\mathcal{R})$$

has the relative Haagerup property.

The definition beyond the finite case has first been considered in [48]; a more detailed study has been conducted by Anantharaman-Delaroche in [1]. Note that both these papers use the notion of the relative Haagerup property for arbitrary (expected) von Neumann inclusions identical to the one studied here. We will now describe the setup.

Let $\mathcal{G}$ be a measured groupoid with countable fibers, equipped with a quasi-invariant probability measure μ on the unit space $\mathcal{G}^{(0)}$ (note that a measure preserving standard equivalence relation as considered above is one source of such examples). Again μ induces a measure ν on $\mathcal{G}$; we further obtain a (not necessarily finite) von Neumann algebra $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{B}(L^2(\mathcal{G}, \nu))$. The following result holds.

Theorem 6.2 *([1, Theorem 1]). Let $\mathcal{G}$ be a measured groupoid with countable fibbers, as above. Then the following conditions are equivalent:*

(i) $\mathcal{G}$ has the Haagerup property, i.e. it admits a sequence of positive-definite functions $(F_n : \mathcal{G} \to \mathbb{C})_{n\in\mathbb{N}}$ which are equal to 1 on $\mathcal{G}^{(0)}$, converge to 1 ν-almost everywhere and satisfy the 'vanishing property': for every $n \in \mathbb{N}$ and $\epsilon > 0$ there is

$$\nu(\{g \in \mathcal{G} : |\varphi_n(g)| > \epsilon\}) < \infty;$$

(ii) the von Neumann inclusion

$$L^\infty(\mathcal{G}^{(0)}, \mu) \subseteq \mathcal{L}(\mathcal{G})$$

has the relative Haagerup property.

Ueda shows in [48, Lemma 5] (and then Anantharaman-Delaroche reproves it in [1, Theorem 3]) that a property of a groupoid as above called *treeability* implies the Haagerup property. [1, Theorem 5] also shows that for ergodic measured groupoid with countable fibbers the Haagerup property is incompatible with Property (T); we are however not aware of explicit examples of such Property (T) groupoids leading to von Neumann algebras which are not finite, and a general intuition regarding Property (T) objects says that these should naturally lead to finite von Neumann algebras (for example discrete property (T) quantum groups are necessarily unimodular, see [24]).

6.2. Examples and counterexamples with $M = \mathcal{B}(\mathcal{H})$

We end this section with the example where $M = \mathcal{B}(\mathcal{H})$ and study which triples $(\mathcal{B}(\mathcal{H}), N, \mathbb{E}_N)$ have (rHAP). Since the conditional expectation $\mathbb{E}_N$ is assumed to be normal it follows by a result of Tomiyama from [47] that N must be a direct sum of type I factors, so $N \simeq \oplus_{i\in I}\mathcal{B}(\mathcal{K}_i)$ for some index set I. Note that each $\mathcal{B}(\mathcal{K}_i)$ may occur in $\mathcal{B}(\mathcal{H})$ with a certain multiplicity $m_i \in \mathbb{N} \cup \{\infty\}$. In general, we have that N is spatially isomorphic to $\oplus_{i\in I}\mathcal{B}(\mathcal{K}_i) \otimes \mathbb{C}1_{m_i}$ where 1_{m_i} is the identity acting on a Hilbert space

of dimension m_i. For simplicity in the examples below we assume that all multiplicities m_i equal 1 and ignore the spatial isomorphism. In that case the normal conditional expectation of $\mathcal{B}(\mathcal{H})$ onto $\oplus_{i\in I}\mathcal{B}(\mathcal{K}_i)$ is unique and determined by $\mathbb{E}_N(x) = \sum_{i\in I} p_i x p_i$ where p_i is the projection onto $\mathcal{K}_i$. Therefore, in this case we can speak not only of the Haagerup property of the inclusion $N \subseteq \mathcal{B}(\mathcal{H})$, but also about maps being compact and of finite index relative to this inclusion.

Theorem 6.3. *Assume that $\mathcal{H}$ is a separable Hilbert space, that $\mathcal{H} = \bigoplus_{i\in I}\mathcal{K}_i$, where I is an index set and that the dimension of $\mathcal{K}_i$ does not depend on $i \in I$. Put $N = \bigoplus_{i\in I}\mathcal{B}(\mathcal{K}_i) \subseteq \mathcal{B}(\mathcal{H})$. Then the triple $(\mathcal{B}(\mathcal{H}), N, \mathbb{E}_N)$ has the property (rHAP).*

Proof. We may assume that $\mathcal{K}_i = \mathcal{K}$ for a single (separable) Hilbert space $\mathcal{K}$. The inclusion $N \subseteq \mathcal{B}(\mathcal{H})$ is then isomorphic to the inclusion $\ell^\infty(I)\otimes\mathcal{B}(\mathcal{K}) \subseteq \mathcal{B}(\ell^2(I))\otimes\mathcal{B}(\mathcal{K})$. In the case where I is finite $\ell^\infty(I) \subseteq \mathcal{B}(\ell^2(I))$ is a finite dimensional inclusion which clearly has (rHAP). In the case where I is infinite we may assume that $I = \mathbb{Z}$ and the inclusion $\ell^\infty(\mathbb{Z}) \subseteq \mathcal{B}(\ell^2(\mathbb{Z}))$ has the (rHAP) with approximating maps given by the (Fejér-)Herz-Schur multipliers T_n with

$$T_n((x_{i,j})_{i,j\in\mathbb{Z}}) = (W(i-j)x_{i,j})_{i,j\in\mathbb{Z}}, \qquad W(k) := \max(1 - \frac{|k|}{n}, 0).$$

Since $W = \frac{1}{n}(\chi_{[0,n]})^* * \chi_{[0,n]}$ is positive definite and converges to the identity pointwise it follows that T_n is completely positive and $T_n^{(2)}$ converges to the identity strongly. Further $T_n^{(2)}$ is finite rank relative to $\ell^\infty(\mathbb{Z})$, so certainly compact. In both cases (I being finite or infinite), we tensor the approximating maps with $\mathrm{Id}_{\mathcal{B}(\mathcal{K})}$ and find that $\ell^\infty(I)\otimes\mathcal{B}(\mathcal{K}) \subseteq \mathcal{B}(\ell^2(I))\otimes\mathcal{B}(\mathcal{K})$ has (rHAP). □

With a bit more work Theorem 6.3 could be proved in larger generality by relaxing the assumption that the multiplicities are trivial and that the dimension is constant (as opposed to say for example uniformly bounded). However, we cannot admit just any subalgebra N as the following counterexample shows.

Theorem 6.4. *Let $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where $\mathcal{K}_1, \mathcal{K}_2$ are Hilbert spaces such that $\dim(\mathcal{K}_1) < \infty$ and $\dim(\mathcal{K}_2) = \infty$. Set $N = \mathcal{B}(\mathcal{K}_1) \oplus \mathcal{B}(\mathcal{K}_2)$. Then the triple $(\mathcal{B}(\mathcal{H}), N, \mathbb{E}_N)$ does not have the property (rHAP).*

Proof. Let p be the projection of $\mathcal{H}$ onto $\mathcal{K}_1$. Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a normal linear map. The proof is based on two claims.

Claim 1: If Φ is an N-N-bimodule map then $\mathcal{B}(\mathcal{H})p$ is an invariant subspace. Moreover, the restriction of Φ to $\mathcal{B}(\mathcal{H})p$ lies in the linear span of the two maps $xp \mapsto pxp$ and $xp \mapsto (1-p)xp$.

Proof of Claim 1. Note that p is contained in N from which the first statement follows. For the second part let $E^i_{k,l}$ be matrix units with respect to some basis of $\mathcal{K}_i$. Then for $x \in \mathcal{B}(\mathcal{H})$ we have $\Phi(E^i_{k,k} x E^i_{l,l}) = E^i_{k,k}\Phi(x)E^i_{k,k}$ so that $E^i_{k,k}\mathcal{B}(\mathcal{H})E^i_{k,k}$ is an eigenspace of Φ (i.e. Φ is a Herz-Schur multiplier). Moreover $\Phi(E^i_{k',k'} x E^i_{l',l'}) = E^i_{k',k}\Phi(E^i_{k,k'} x E^i_{l',l})E^i_{l,l'}$ so that the eigenvalues of these spaces only depend on i. This in particular implies the claim.

Claim 2: If Φ is compact relative to the inclusion $N \subseteq \mathcal{B}(\mathcal{H})$ then $\mathcal{B}(\mathcal{H})p$ is an invariant subspace. Moreover, the restriction of Φ to $\mathcal{B}(\mathcal{H})p$ is compact (in the non-relative sense).

Proof of Claim 2. By approximation it suffices to prove Claim 2 with 'compact' replaced by 'finite rank'. So assume that $\Phi = a\mathbb{E}_N b$ with $a, b \in \mathcal{B}(\mathcal{H})$. Note that $p \in N \cap N'$ and therefore $a\mathbb{E}_N(bxp) = ap\mathbb{E}_N(bx)p = a\mathbb{E}_N(pbxp)$. The first of these equalities shows that $\mathcal{B}(\mathcal{H})p$ is invariant. Further $x \mapsto (pxp)$ is finite rank as p projects onto a finite dimensional space. This proves the claim.

Remainder of the proof. Suppose that Φ is both N-N-bimodular and compact relative to N. By Claim 1 we know that there are scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\Phi(xp) = \lambda_1 pxp + \lambda_2(1-p)xp$. If $\lambda_2 \neq 0$ then the associated L^2-map is not compact (in the non-relative sense) since $(1-p)$ projects onto an infinite dimensional Hilbert space. This contradicts Claim 2 because the restriction of Φ to $\mathcal{B}(\mathcal{H})p$ is compact. We conclude that $\lambda_2 = 0$ for any normal map $\Phi : M \to M$ that is N-N-bimodular and compact relative to N. But then we can never find a net of such maps that approximates the identity map on $\mathcal{B}(\mathcal{H})$ in the point-strong topology. Hence the inclusion $N \subseteq \mathcal{B}(\mathcal{H})$ fails to have (rHAP). □

Remark 6.5. Recall that a unital inclusion of von Neumann algebras $N \subseteq M$ is said to be *co-amenable* if there exists a (not necessarily normal) conditional expectation from N' onto M', where the commutants are taken with respect to any Hilbert space realization of M. Theorem 6.4 shows – surprisingly – that a co-amenable inclusion in general need not have (rHAP).

Note that this also means that a naive extension of the definition of relative Haagerup property in terms of correspondences, modelled on the notion of *strictly mixing bimodules* [37, Theorem 9] valid for the non-relative Haagerup property, cannot be equivalent to the definition studied in our paper. Indeed, the last fact, together with the examples above, would contradict [6, Theorem 2.4].

7. Property (RHAP) for finite-dimensional subalgebras

In this section we consider the case of finite-dimensional subalgebras and show equivalence of the relative Haagerup property and the non-relative Haagerup property. For this, we fix a unital inclusion $N \subseteq M$ of von Neumann algebras and assume that it admits a faithful normal conditional expectation $\mathbb{E}_N$. Assume that N is finite-dimensional and let $\tau \in N_*$ be a faithful normal tracial state on N that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on

M. We will prove that the triple (M, N, φ) has property (rHAP) if and only if $(M, \mathbb{C}, \varphi)$ does. Recall that by Theorem 3.7 the Haagerup property of (M, N, φ) does not depend on the choice of the state τ.

Denote by $z_1, ..., z_n \in \mathcal{Z}(N)$ the minimal central projections of N. There exist natural numbers $n_1, ..., n_k \in \mathbb{N}$ such that $z_k N \cong M_{n_k}(\mathbb{C})$ for $k = 1, ..., n$. Let $(f_i^k)_{1 \leq i \leq n_k}$ be an orthonormal basis of $\mathbb{C}^{n_k}$, write $E_{i,j}^k$, $1 \leq i, j \leq n_k$ for the matrix units with respect to this basis and set $E_i^k := E_{i,i}^k$ for the diagonal projections. We have that $E_{i,j}^k f_l^k = \delta_{j,l} f_i^k$ for all $k \in \mathbb{N}$, $1 \leq i, j, l \leq n_k$ and $\sum_{k=1}^n \sum_{i=1}^{n_k} E_i^k = 1$. Set $d := \sum_{k=1}^n n_k$, choose an orthonormal basis $(f_{k,i})_{1 \leq k \leq n,\ 1 \leq i \leq n_k}$ of $\mathbb{C}^d$ with corresponding matrix units $e_{(k,i),(l,j)} \in M_d(\mathbb{C})$ where $1 \leq k, l \leq n$, $1 \leq i \leq n_k$, $1 \leq j \leq n_l$ and define

$$p := \sum_{k=1}^n E_1^k. \tag{7.1}$$

For a general linear map Φ: $pMp \to pMp$ we may define a linear map $\widetilde{\Phi}$: $M \to M$ by

$$\widetilde{\Phi}(E_i^k x E_j^l) := E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l \tag{7.2}$$

for all $1 \leq k, l \leq n$, $1 \leq i \leq n_k$, $1 \leq j \leq n_l$ and $x \in M$.

Let us study the properties of $\widetilde{\Phi}$.

Lemma 7.1. *Let $\Phi : pMp \to pMp$ be a linear map. Define*

$$U := \sum_{k=1}^n \sum_{i=1}^{n_k} e_{(1,1),(k,i)} \otimes E_{i,1}^k \in M_d(\mathbb{C}) \otimes M, \qquad V := \sum_{k=1}^n \sum_{i=1}^{n_k} f_{k,i} \otimes E_{1,i}^k \in \mathbb{C}^d \otimes M.$$

Then,

$$\widetilde{\Phi}(x) = V^* (id_{\mathcal{B}(L^2(N,\tau))} \otimes \Phi) (U^*(1 \otimes x)U) V.$$

Proof. We have for $x \in z_k M z_l$ with $1 \leq k, l \leq n$ that

$$U^*(1 \otimes x)U = \sum_{i=1}^{n_k} \sum_{j=1}^{n_l} e_{(k,i),(l,j)} \otimes E_{1,i}^k x E_{j,1}^l$$

so that

$$V^* (\mathrm{id}_{\mathcal{B}(L^2(N,\tau))} \otimes \Phi) (U^*(1 \otimes x)U) V = \sum_{i=1}^{n_k} \sum_{j=1}^{n_l} E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l.$$

By definition this expression coincides with $\widetilde{\Phi}(x)$. The claim follows. □

Lemma 7.2. *If Φ: $pMp \to pMp$ is a unital normal completely positive map, then $\widetilde{\Phi}$ is contractive, normal and completely positive.*

Proof. The normality and the complete positivity follow from Lemma 7.1. We further have

$$\begin{aligned}\|\widetilde{\Phi}\| = \widetilde{\Phi}(1) = \widetilde{\Phi}\left(\sum_{k=1}^{n}\sum_{i=1}^{n_k} E_i^k\right) &= \sum_{k=1}^{n}\sum_{i=1}^{n_k} E_{i,1}^k \Phi(E_1^k) E_{1,i}^k \\ &\leq \sum_{k=1}^{n}\sum_{i=1}^{n_k} E_{i,1}^k E_{1,i}^k = \sum_{k=1}^{n}\sum_{i=1}^{n_k} E_i^k = 1,\end{aligned}$$

i.e. $\widetilde{\Phi}$ is contractive. □

Lemma 7.3. *Let Φ: $pMp \to pMp$ be a linear map. Then $\widetilde{\Phi}$ is an N-N-bimodule map.*

Proof. Let $x \in M$. For $1 \leq l, k, m \leq n$ and $1 \leq r, s \leq n_l$, $1 \leq i \leq n_k$, $1 \leq j \leq n_m$ we have

$$\begin{aligned}E_{r,s}^l \widetilde{\Phi}(E_i^k x E_j^m) &= E_{r,s}^l E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^m) E_{1,j}^m \\ &= \delta_{s,i}\delta_{l,k} E_{r,1}^k \Phi(E_{1,i}^k x E_{j,1}^m) E_{1,j}^m \\ &= E_{r,1}^k \Phi(E_{1,r}^k E_{r,s}^l E_i^k x E_j^m E_{j,1}^m) E_{1,j}^m.\end{aligned}$$

We hence find that for $y \in E_i^k x E_j^m$, $E_{r,s}^l \widetilde{\Phi}(y) = \widetilde{\Phi}(E_{r,s}^l y)$. The linearity of $\widetilde{\Phi}$ then implies that it is a left N-module map. A similar argument applies to the right-handed case. □

Proposition 7.4. *Define the map*

$$\mathit{Diag}\colon pMp \to pMp,\ x \mapsto \sum_{k=1}^{n} \frac{\varphi(E_1^k x E_1^k)}{\varphi(E_1^k)} E_1^k.$$

Then $\widetilde{\mathit{Diag}} = \mathbb{E}_N$.

Proof. It is clear that the map Diag is linear unital normal and completely positive. Hence, by Lemma 7.2 and Lemma 7.3, $\widetilde{\mathrm{Diag}}$ is contractive normal completely positive and N-N-bimodular. It is easy to check that $\widetilde{\mathrm{Diag}}$ is even unital. In particular, $\widetilde{\mathrm{Diag}}$ restricts to the identity on N. It is further clear that $\widetilde{\mathrm{Diag}}$ is faithful and that it maps M onto N, so $\widetilde{\mathrm{Diag}}$ is a faithful normal conditional expectation. For $x \in M$ and $1 \leq k, l \leq n$, $1 \leq i \leq n_k$, $1 \leq j \leq n_l$ we have

$$\varphi \circ \widetilde{\mathrm{Diag}}(E_i^k x E_j^l) = \varphi\left(E_{i,1}^k \mathrm{Diag}(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l\right)$$

$$
\begin{aligned}
&= \sum_{m=1}^{n} \frac{\varphi(E_1^m E_{1,i}^k x E_{j,1}^l E_1^m)}{\varphi(E_1^m)} \varphi\left(E_{i,1}^k E_1^m E_{1,j}^l\right) \\
&= \frac{\varphi(E_1^l E_{1,i}^k x E_{j,1}^l E_1^l)}{\varphi(E_1^l)} \varphi(E_{i,1}^k E_1^l E_{1,j}^l) \\
&= \delta_{k,l} \frac{\varphi(E_{1,i}^l x E_{j,1}^l)}{\varphi(E_1^l)} \varphi(E_{i,1}^l E_{1,j}^l) \\
&= \delta_{k,l} \frac{\varphi(E_{1,i}^l x E_{j,1}^l)}{\tau(E_1^l)} \tau(E_{i,j}^l).
\end{aligned}
$$

But then, since τ is tracial,

$$
\begin{aligned}
\varphi \circ \widetilde{\mathrm{Diag}}(E_i^k x E_j^l) &= \delta_{i,j}\delta_{k,l}\varphi(E_{1,i}^l x E_{i,1}^l) = \delta_{i,j}\delta_{k,l}\tau(\mathbb{E}_N(E_{1,i}^l x E_{i,1}^l)) \\
&= \delta_{i,j}\delta_{k,l}\tau(E_{1,i}^l \mathbb{E}_N(x) E_{i,1}^l) = \tau(E_i^k \mathbb{E}_N(x) E_j^l) \\
&= \varphi(E_i^k x E_j^l),
\end{aligned}
$$

i.e. $\widetilde{\mathrm{Diag}}$ is φ-preserving. Since $\mathbb{E}_N$ is the unique faithful normal φ-preserving conditional expectation onto N, we get that $\widetilde{\mathrm{Diag}} = \mathbb{E}_N$. □

Lemma 7.5. *Let $\Phi\colon pMp \to pMp$ be a normal completely positive map with $\varphi \circ \Phi \leq \varphi$ and assume that the L^2-implementation $\Phi^{(2)}$ of Φ with respect to $\varphi|_{pMp}$ is a compact operator. Then $\widetilde{\Phi}$ satisfies $\varphi \circ \widetilde{\Phi} \leq \varphi$ and $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(M, N, \varphi)$.*

Proof. For $1 \leq k, l \leq n$, $1 \leq i \leq n_k$, $1 \leq j \leq n_l$ and $x \in M$ positive we have by the traciality of τ,

$$
\begin{aligned}
\varphi \circ \widetilde{\Phi}(E_i^k x E_j^l) &= \varphi\left(E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l\right) \\
&= \tau\left(E_{i,1}^k \mathbb{E}_N\left(\Phi(E_{1,i}^k x E_{j,1}^l)\right) E_{1,j}^l\right) \\
&= \delta_{i,j}\delta_{k,l}\tau\left(E_1^k \mathbb{E}_N(\Phi(E_{1,i}^k x E_{i,1}^k))\right),
\end{aligned}
$$

so in particular $\varphi \circ \widetilde{\Phi}(E_i^k x E_j^l) \geq 0$. We get (as N is contained in the centralizer M^φ)

$$
\begin{aligned}
\varphi \circ \widetilde{\Phi}(E_i^k x E_j^l) &= \delta_{i,j}\delta_{k,l}\tau\left(E_1^k \mathbb{E}_N(\Phi(E_{1,i}^k x E_{i,1}^k))\right) \\
&\leq \delta_{i,j}\delta_{k,l}\tau\left(\mathbb{E}_N(\Phi(E_{1,i}^k x E_{i,1}^k))\right) \\
&\leq \delta_{i,j}\delta_{k,l}\varphi(E_{1,i}^k x E_{i,1}^k) \\
&= \varphi(E_i^k x E_j^l).
\end{aligned}
$$

This implies that $\widetilde{\Phi}$ indeed satisfies $\varphi \circ \widetilde{\Phi} \leq \varphi$. In particular, the L^2-implementation of $\widetilde{\Phi}$ with respect to φ exists.

It remains to show that $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(M,N,\varphi)$. For this, let $\Psi\colon pMp \to pMp$ be a map with $\Psi^{(2)} = ae_{\mathbb{C}}b$ where $a,b \in pMp$ and $e_{\mathbb{C}}$ denotes the rank one projection $(\varphi|_{pMp}(\cdot)p)^{(2)} \in \mathcal{B}(L^2(pMp,\varphi|_{pMp}))$. For $1 \leq k,l \leq n$, $1 \leq i \leq n_k$, $1 \leq j \leq n_l$ and $x \in M$ we then have

$$\widetilde{\Psi}(E_i^k x E_j^l) = E_{i,1}^k a\varphi(bE_{1,i}^k x E_{j,1}^l)E_{1,j}^l = E_{i,1}^k a\varphi(bE_{1,i}^k x E_{j,1}^l)E_1^l E_{1,j}^l.$$

Note that by Proposition 7.4,

$$\begin{aligned}
\mathbb{E}_N(bE_{1,i}^k x E_{j,1}^l) &= \sum_{r=1}^n \mathbb{E}_N(E_1^r bE_{1,i}^k x E_{j,1}^l) = \sum_{r=1}^n \widetilde{\mathrm{Diag}}(E_1^r bE_{1,i}^k x E_{j,1}^l)\\
&= \sum_{r=1}^n E_1^r \mathrm{Diag}(E_1^r bE_{1,i}^k x E_{j,1}^l)E_1^l\\
&= \sum_{r=1}^n \sum_{m=1}^n \frac{\varphi(E_1^m E_1^r bE_{1,i}^k x E_{j,1}^l E_1^m)}{\varphi(E_1^m)} E_1^r E_1^m E_1^l\\
&= \frac{\varphi(E_1^l bE_{1,i}^k x E_{j,1}^l)}{\varphi(E_1^l)} E_1^l = \frac{\varphi(bE_{1,i}^k x E_{j,1}^l)}{\varphi(E_1^l)} E_1^l,
\end{aligned}$$

where in the last equality we again used that τ is tracial. Hence

$$\widetilde{\Psi}(E_i^k x E_j^l) = \varphi(E_1^l)E_{i,1}^k a\mathbb{E}_N(bE_{1,i}^k x E_{j,1}^l)E_{1,j}^l = \varphi(E_1^l)E_{i,1}^k a\mathbb{E}_N(bE_{1,i}^k E_i^k x E_j^l). \tag{7.3}$$

Fix now suitable t_0, j_0, k_0, l_0 and $x \in M$ and compute the following expression:

$$\begin{aligned}
&\left(\sum_{k=1}^n \sum_{l=1}^n \sum_{r=1}^n \sum_{t=1}^{n_k} \varphi(E_1^r)E_{t,1}^k a E_1^l e_N E_1^r bE_{1,t}^k\right)(E_{t_0}^{k_0} x E_{j_0}^{l_0}\Omega_\varphi)\\
&\qquad = \sum_{l=1}^n \sum_{r=1}^n \varphi(E_1^r)E_{t_0,1}^{k_0} a E_1^l \mathbb{E}_N(E_1^r bE_{1,t_0}^{k_0} x E_{j_0}^{l_0})\Omega_\varphi\\
&\qquad = \sum_{l=1}^n \varphi(E_1^l)E_{t_0,1}^{k_0} a E_1^l \mathbb{E}_N(bE_{1,t_0}^{k_0} x E_{j_0,1}^{l_0})E_{1,j_0}^{l_0}\Omega_\varphi
\end{aligned}$$

Now the equality (7.3) implies that the value of the conditional expectation appearing in the last formula is a scalar multiple of $E_1^{l_0}$, so the whole expression equals

$$\varphi(E_1^{l_0})E_{t_0,1}^{k_0} a\mathbb{E}_N(bE_{1,t_0}^{k_0} x E_{j_0}^{l_0})\Omega_\varphi = \widetilde{\Psi}(E_{t_0}^{k_0} x E_{j_0}^{l_0})\Omega_\varphi.$$

Hence we arrive at

$$(\widetilde{\Psi})^{(2)} = \sum_{k=1}^n \sum_{l=1}^n \sum_{r=1}^n \sum_{t=1}^{n_k} \varphi(E_1^r)E_{t,1}^k a E_1^l e_N E_1^r bE_{1,t}^k \in \mathcal{K}_{00}(M,N,\varphi).$$

By taking linear combinations this implies that for every map Ψ with $\Psi^{(2)} \in \mathcal{K}_{00}(pMp, \mathbb{C}, \varphi)$ the L^2-implementation of $\widetilde{\Psi}$ is contained in $\mathcal{K}_{00}(M, N, \varphi)$. Via approximation we then see that $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(M, N, \varphi)$. □

We are now ready to prove the main theorem of this section.

Theorem 7.6. *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras and assume that it admits a faithful normal conditional expectation $\mathbb{E}_N \colon M \to N$. Assume that N is finite-dimensional and let $\tau \in N_*$ be a faithful state on N that we extend to a state $\varphi := \tau \circ \mathbb{E}_N$ on M. Then M has the Haagerup property (in the sense that the triple $(M, \mathbb{C}, \varphi)$ has the relative Haagerup property) if and only if the triple (M, N, φ) has the relative Haagerup property.*

Proof. By Theorem 3.7 we may assume without loss of generality that τ is tracial.

"⇐" Assume that the triple (M, N, φ) has the relative Haagerup property and let $(\Phi_i)_{i \in I}$ be a net of normal completely positive maps witnessing it. Since N is finite dimensional, e_N is a finite rank projection. In particular, $\mathcal{K}_{00}(M, N, \varphi)$ consists of finite rank operators and hence $\mathcal{K}(M, N, \varphi) \subseteq \mathcal{K}(M, \mathbb{C}, \varphi)$. In particular, $\Phi_i^{(2)} \in \mathcal{K}(M, \mathbb{C}, \varphi)$ for every $i \in I$. Further, $\Phi_i(x) \to x$ strongly for every $x \in M$. This implies that the net $(\Phi_i)_{i \in I}$ also witnesses the relative Haagerup property of the triple $(M, \mathbb{C}, \varphi)$.

"⇒" Assume that M has the Haagerup property. Recall that the projection p was defined in (7.1). By [15, Lemma 4.1] the triple $(pMp, \mathbb{C}, \varphi|_{pMp})$ also has the relative Haagerup property and by Theorem 5.5 we find a net $(\Phi_i)_{i \in I}$ of unital normal completely positive φ-preserving maps witnessing it. By Lemma 7.2, Lemma 7.3 and Lemma 7.5 we find that $\widetilde{\Phi}_i$ is a contractive normal completely positive N-N-bimodule map with $\varphi \circ \widetilde{\Phi} \leq \varphi$ and $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(M, N, \varphi)$ for every $i \in I$. It follows directly from the prescription (7.2) that $\widetilde{\Phi}_i(x) \to x$ strongly for every $x \in M$. It follows that the net $(\widetilde{\Phi}_i)_{i \in I}$ witnesses the relative Haagerup property of the triple (M, N, φ). □

8. The relative Haagerup property for amalgamated free products

In this section we study the notion of the relative Haagerup property in the context of amalgamated free products of von Neumann algebras. We will further apply our results to the class of virtually free Hecke-von Neumann algebras.

8.1. Preservation under amalgamated free products

The following theorem demonstrates that in the setting of Section 5 the relative Haagerup property is preserved under taking amalgamated free products (for details on operator algebraic amalgamated free products see [50] or also [11]). For finite inclusions of von Neumann algebras this has been proved in [8, Proposition 3.9].

Theorem 8.1. *Let $N \subseteq M_1$ and $N \subseteq M_2$ be unital embeddings of von Neumann algebras which admit faithful normal conditional expectations $\mathbb{E}_1 : M_1 \to N$, $\mathbb{E}_2 : M_2 \to N$ and for which N is finite. Denote by $M := (M_1, \mathbb{E}_1) *_N (M_2, \mathbb{E}_2)$ the amalgamated free product von Neumann algebra of M_1 and M_2 with respect to the expectations $\mathbb{E}_1$, $\mathbb{E}_2$ and let $\mathbb{E}_N$ be the corresponding conditional expectation of M onto N. Then $(M_1, N, \mathbb{E}_1)$ and $(M_2, N, \mathbb{E}_2)$ have the relative Haagerup property if and only if the triple $(M, N, \mathbb{E}_N)$ has the relative Haagerup property.*

Proof. "⇒" Assume that both $(M_1, N, \mathbb{E}_1)$ and $(M_2, N, \mathbb{E}_2)$ have the relative Haagerup property, let $\tau \in N_*$ be a faithful normal tracial state and set $\varphi_1 := \tau \circ \mathbb{E}_1$, $\varphi_2 := \tau \circ \mathbb{E}_2$. Then the triples (M_1, N, φ_1) and (M_2, N, φ_2) have the relative Haagerup property. Without loss of generality we can assume that the corresponding nets $(\Phi_{i,1})_{i\in I}$ and $(\Phi_{i,2})_{i\in I}$ witnessing the relative Haagerup property are indexed by the same set I. By Theorem 5.5 we can also assume that the maps are unital with $\varphi_1 \circ \Phi_{i,1} = \varphi_1$, $\varphi_2 \circ \Phi_{i,2} = \varphi_2$ for all $i \in I$, which then implies that $\Phi_{i,1}|_N = \Phi_{i,2}|_N = \mathrm{id}_N$ and that $\mathbb{E}_1 \circ \Phi_{i,1} = \mathbb{E}_1$, $\mathbb{E}_2 \circ \Phi_{i,2} = \mathbb{E}_2$. Choose a net $(\epsilon_i)_{i\in I}$ (we can use the same indexing set, modifying it if necessary) with $\epsilon_i \to 0$ and define unital normal completely positive N-N-bimodular maps $\Phi'_{i,1} := \frac{1}{1+\varepsilon_i}(\Phi_{i,1} + \epsilon_i \mathbb{E}_1)$, $\Phi'_{i,2} := \frac{1}{1+\varepsilon_i}(\Phi_{i,2} + \epsilon_i \mathbb{E}_2)$.

In the following we will need to work with certain sets of multi-indices: for each $n \in \mathbb{N}$ set $\mathcal{J}_n = \{\mathbf{j} = (j_1, \ldots, j_n) : j_k \in \{1, 2\} \text{ and } j_k \neq j_{k+1} \text{ for } k = 1, \ldots, n-1\}$; put also $\mathcal{J} = \bigcup_{n\in\mathbb{N}} \mathcal{J}_n$.

Set $\varphi := \tau \circ \mathbb{E}_N$, let $\Psi_i := \Phi_{i,1} * \Phi_{i,2}$: $M \to M$ be the unital normal completely positive map with $\Psi_i|_N = \mathrm{id}_N$ and $\Psi_i(x_1 \ldots x_n) = \Phi_{i,j_1}(x_1) \ldots \Phi_{i,j_n}(x_n)$ for $\mathbf{j} \in \mathcal{J}_n$ and $x_k \in M_{j_k} \cap \ker(\mathbb{E}_{j_k})$ for $k = 1, \ldots, n$ (see [7, Theorem 3.8]) and define $\Psi'_i := \Phi'_{i,1} * \Phi'_{i,2}$ analogously. We claim that the net $(\Psi'_i)_{i\in I}$ witnesses the relative Haagerup property of the triple (M, N, φ). Indeed, it is clear that the maps satisfy the conditions (1), (2) and (4) of Definition 3.2. It remains to show that $\Psi'_i(x) \to x$ strongly for every $x \in M$ and that the L^2-implementations $(\Psi'_i)^{(2)}$ are contained in $\mathcal{K}(M, N, \varphi)$.

Define for $n \in \mathbb{N}$ and $\mathbf{j} \in \mathcal{J}_n$ the Hilbert subspace

$$\mathcal{H}_{\mathbf{j}} := \overline{\mathrm{Span}}\,\{x_1 \ldots x_n \Omega_\varphi \mid x_1 \in \ker(\mathbb{E}_{j_1}), \ldots, x_n \in \ker(\mathbb{E}_{j_n})\} \subseteq L^2(M, \varphi)$$

and let $P_{\mathbf{j}} \in \mathcal{B}(L^2(M, \varphi))$ be the orthogonal projection onto $\mathcal{H}_{\mathbf{j}}$. Note that these Hilbert subspaces are pairwise orthogonal for different multi-indices $\mathbf{j}_1, \mathbf{j}_2 \in \mathcal{J}$, orthogonal to $N\Omega_\varphi \subseteq L^2(M, \varphi)$, one has inclusions $\Psi_i^{(2)} \mathcal{H}_{\mathbf{j}} \subseteq \mathcal{H}_{\mathbf{j}}$, $(\Psi'_i)^{(2)} \mathcal{H}_{\mathbf{j}} \subseteq \mathcal{H}_{\mathbf{j}}$ and the span of the union of all $\mathcal{H}_{\mathbf{j}}$ ($\mathbf{j} \in \mathcal{J}$) with $N\Omega_\varphi$ is dense in $L^2(M, \varphi)$.

For the strong convergence it suffices to show that $\|\Psi_i^{(2)} \xi - \xi\|_2 \to 0$ for all $\xi \in \mathcal{H}_{\mathbf{j}}$, $\mathbf{j} \in \mathcal{J}$. So let $n \in \mathbb{N}$, $\mathbf{j} \in \mathcal{J}_n$, $x_1 \in \ker(\mathbb{E}_{j_1})$,..., $x_n \in \ker(\mathbb{E}_{j_n})$. Then,

$$\begin{aligned}
&\|(\Psi'_i)^{(2)}(x_1 \ldots x_n \Omega_\varphi) - x_1 \ldots x_n \Omega_\varphi\|_2 = \|\Phi'_{i,j_1}(x_1) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi - x_1 \ldots x_n \Omega_\varphi\|_2 \\
\leq\ &\|(\Phi'_{i,j_1}(x_1) - x_1)\Phi'_{i,j_2}(x_2) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi\|_2 \\
&\qquad + \|x_1\| \|\Phi'_{i,j_2}(x_2) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi - x_2 \ldots x_n \Omega_\varphi\|_2
\end{aligned}$$

$$\begin{aligned}\leq \ldots \leq \|(\Phi'_{i,j_1}(x_1) - x_1)\Phi'_{i,j_2}(x_2)\ldots\Phi'_{i,j_n}(x_n)\Omega_\varphi\|_2 \\ + \|x_1\|\|(\Phi'_{i,j_2}(x_2) - x_2)\Phi'_{i,j_3}(x_3)\ldots\Phi'_{i,j_n}(x_n)\Omega_\varphi\|_2 \\ + \ldots + \|x_1\|\ldots\|x_{n-1}\|\|\Phi'_{i,j_n}(x_n)\Omega_\varphi - x_n\Omega_\varphi\|_2 \to 0.\end{aligned}$$

This implies that indeed $\Psi_i(x) \to x$ strongly for every $x \in M$.

To treat the relative compactness, express the operators $(\Phi'_{i,1})^{(2)} \in \mathcal{K}(M_1, N, \varphi_1)$, $(\Phi'_{i,2})^{(2)} \in \mathcal{K}(M_2, N, \varphi_2)$ as norm-limits

$$(\Phi'_{i,1})^{(2)} = \lim_{l\to\infty} \sum_{k=1}^{N_l^{(i,1)}} a_{k,l}^{(i,1)} e_N^{\varphi_1} b_{k,j}^{(i,1)} \quad \text{and} \quad (\Phi'_{i,2})^{(2)} = \lim_{l\to\infty} \sum_{k=1}^{N_j^{(i,2)}} a_{k,l}^{(i,2)} e_N^{\varphi_2} b_{k,l}^{(i,2)}$$

for suitable $N_l^{(i,1)}, N_j^{(i,2)} \in \mathbb{N}$, $a_{k,l}^{(i,1)}, b_{k,l}^{(i,1)} \in M_1$ and $a_{k,l}^{(i,2)}, b_{k,l}^{(i,2)} \in M_2$.

Claim. For $n \in \mathbb{N}$, $\mathbf{j} \in \mathcal{J}_n$, we have

$$\left\|(\Psi'_i)^{(2)} P_{\mathbf{j}}\right\| \leq \left(\frac{1}{1+\epsilon_i}\right)^n \tag{8.1}$$

and

$$(\Psi'_i)^{(2)} P_{\mathbf{j}} = \lim_{l_1,\ldots,l_n\to\infty} \sum_{k_1,\ldots,k_n} a_{k_1,l_1}^{(i,j_1)} \ldots a_{k_n,l_n}^{(i,j_n)} e_N b_{k_n,l_n}^{(i,j_n)} \ldots b_{k_1,l_1}^{(i,j_1)}, \tag{8.2}$$

where the convergence is in norm.

Proof of the claim. For $x_1 \in \ker(\mathbb{E}_{j_1})$,..., $x_n \in \ker(\mathbb{E}_{j_n})$ one calculates

$$\begin{aligned}(\Psi'_i)^{(2)} P_{\mathbf{j}}(x_1\ldots x_n\Omega_\varphi) &= \Phi'_{i,j_1}(x_1)\ldots\Phi'_{i,j_n}(x_n)\Omega_\varphi \\ &= \left(\frac{1}{1+\epsilon_i}\right)^n \Phi_{i,j_1}(x_1)\ldots\Phi_{i,j_n}(x_n)\Omega_\varphi \\ &= \left(\frac{1}{1+\epsilon_i}\right)^n \Psi_i^{(2)}(x_1\ldots x_n\Omega_\varphi)\end{aligned}$$

and hence $(\Psi'_i)^{(2)} P_{\mathbf{j}} = (1+\epsilon_i)^{-n} \Psi_i^{(2)} P_{\mathbf{j}}$. By the unitality of $\Phi_{i,1}$ and $\Phi_{i,2}$ the inequality (8.1) then follows from

$$\begin{aligned}\left\|(\Psi'_i)^{(2)} P_{\mathbf{j}}\right\| = \left(\frac{1}{1+\epsilon_i}\right)^n \left\|\Psi_i^{(2)} P_{\mathbf{j}}\right\| \leq \left(\frac{1}{1+\epsilon_i}\right)^n \left\|\Psi_i^{(2)}\right\| \\ \leq \left(\frac{1}{1+\epsilon_i}\right)^n \|\Psi_i\| = \left(\frac{1}{1+\epsilon_i}\right)^n.\end{aligned}$$

We proceed by induction over n. For $n = 1$ the equality (8.2) is clear. Assume that the equality (8.2) holds for $\mathbf{j} \in \mathcal{J}_{n-1}$ and let $j_n \in \{1, 2\}$ with $j_n \neq i_{n-1}$, $\mathbf{j}' = (\mathbf{j}, j_n)$.

One easily checks that the left- and right-hand side of (8.2) both vanish on the orthogonal complement of $\mathcal{H}_{\mathbf{j}'}$. Further, for $x_1 \in \ker(\mathbb{E}_{j_1})$,..., $x_n \in \ker(\mathbb{E}_{j_n})$, we get by the assumption

$$\begin{aligned}(\Psi_i')^{(2)}(x_1...x_n\Omega_\varphi) &= \Psi_i'(x_1...x_{n-1})\Phi_{i,j_n}'(x_n)\Omega_\varphi = \Psi_i'(x_1...x_{n-1})(\Phi_{i,j_n}')^{(2)}(x_n\Omega_\varphi)\\ &= \lim_{l_1,...,l_n\to\infty} \sum_{k_1,...,k_{n-1}} a_{k_1,l_1}^{(i,j_1)}...a_{k_{n-1},l_{n-1}}^{(i,j_{n-1})}\mathbb{E}_N\left(b_{k_{n-1},l_{n-1}}^{(i,j_{n-1})}...b_{k_1,l_1}^{(i,j_1)}x_1...x_{n-1}\right)\\ &\quad\times\left(\sum_{k_n} a_{k_n,l_n}^{(i,j_n)}e_N b_{k_n,l_n}^{(i,j_n)}\right)x_n\Omega_\varphi.\end{aligned}$$

Since the $\Phi_{i,1}'$ and $\Phi_{i,2}'$ are N-N-bimodular, we have $(\Phi_{i,j_n}')^{(2)} \in N' \cap \langle N, M\rangle$ and hence

$$(\Psi_i')^{(2)}(x_1...x_n\Omega_\varphi) = \lim_{l_1,...,l_n\to\infty} \sum_{k_1,...,k_n} a_{k_1,l_1}^{(i,j_1)}...a_{k_n,l_n}^{(i,j_n)}\mathbb{E}_N\left(b_{k_n,l_n}^{(i,j_n)}...b_{k_1,l_1}^{(i,j_1)}x_1...x_n\right)\Omega_\varphi,$$

i.e.

$$\sum_{k_1,...,k_n} a_{k_1,l_1}^{(i,j_1)}...a_{k_n,l_n}^{(i,j_n)}e_N b_{k_n,l_n}^{(i,j_n)}...b_{k_1,l_1}^{(i,j_1)} \to (\Psi_i')^{(2)}$$

strongly in $l_1, ..., l_n$. The second part of the claim, i.e. (8.2), then follows from noticing that

$$\left(\sum_{k_1,...,k_n} a_{k_1,l_1}^{(i,j_1)}...a_{k_n,l_n}^{(i,j_n)}e_N b_{k_n,l_n}^{(i,j_n)}...b_{k_1,l_1}^{(i,j_1)}\right)_{l_1,...,l_n}$$

is a Cauchy sequence (compare with [8, Section 3]).

The (in)equalities (8.1) and (8.2) in particular imply that $(\Psi_i')^{(2)}$ can be expressed as a norm limit

$$(\Psi_i')^{(2)} = e_N + \lim_{n\to\infty} \sum_{\mathbf{j}\in\mathcal{J}_n} \Psi_i^{(2)}P_{\mathbf{j}}$$

and hence $(\Psi_i')^{(2)} \in \mathcal{K}(M, N, \varphi)$ for all $i \in I$. This finishes the direction "⇒".

"⇐" It suffices to prove the result for M_1. Note first that [7, Lemma 3.5] shows that we have a normal conditional expectation $\mathbb{F}_1 : M \to M_1$ such that $\mathbb{E}_1 \circ \mathbb{F}_1 = \mathbb{E}_N$. Hence Lemma 5.8 ends the proof. □

In combination with Theorem 7.6, Theorem 8.1 leads to the following corollary. This generalizes a result by Freslon [25, Theorem 2.3.19] who showed this corollary in the realm of von Neumann algebras of discrete quantum groups, and the analogous property

for classical groups was first shown in [31] (see also [17, Section 6]). To the authors' best knowledge even for inclusions of finite von Neumann algebras the statement of the following corollary is new.

Corollary 8.2. *Let $N \subseteq M_1$ and $N \subseteq M_2$ be unital embeddings of von Neumann algebras which admit faithful normal conditional expectations $\mathbb{E}_1 : M_1 \to N$, $\mathbb{E}_2 : M_2 \to N$ and assume that N is finite-dimensional. Assume moreover that M_1 and M_2 have the Haagerup property. Then the amalgamated free product von Neumann algebra $(M_1, \mathbb{E}_1) *_N (M_2, \mathbb{E}_2)$ has the Haagerup property as well.*

8.2. Haagerup property for Hecke-von Neumann algebras of virtually free Coxeter groups

Let us now demonstrate the application of Corollary 8.2 in the context of virtually free Hecke-von Neumann algebras.

A *Coxeter system* (W, S) consists of a set S and a group W freely generated by S with respect to relations of the form $(st)^{m_{st}} = e$ where $m_{st} \in \{1, 2, ..., \infty\}$ with $m_{ss} = 1$, $m_{st} \geq 2$ for all $s \neq t$ and $m_{st} = m_{ts}$. By $m_{st} = \infty$ we mean that no relation of the form $(st)^m = e$ with $m \in \mathbb{N}$ is imposed, i.e. s and t are free with respect to each other; and the system is said to be *right-angled* if $m_{st} \in \{2, \infty\}$ for all $s, t \in S$, $s \neq t$. The system is of *finite rank* if the generating set S is finite. A subgroup of (W, S) is called *special* if it is generated by a subset of S.

With every Coxeter system one can associate a family of von Neumann algebras, its *Hecke-von Neumann algebras*, which can be viewed as q-deformations of the group von Neumann algebra $\mathcal{L}(W)$ of the Coxeter group W. For this, fix a multi-parameter $q := (q_s)_{s \in S} \in \mathbb{R}^S_{>0}$ with $q_s = q_t$ for all $s, t \in S$ which are conjugate in W. Further, write $p_s(q) := (q_s - 1)/\sqrt{q_s}$ for $s \in S$. Then the corresponding Hecke-von Neumann algebra $\mathcal{N}_q(W)$ is the von Neumann subalgebra of $\mathcal{B}(\ell^2(W))$ generated by the operators $T_s^{(q)}$, $s \in S$ where $T_s^{(q)} : \mathcal{B}(\ell^2(W)) \to \mathcal{B}(\ell^2(W))$ is defined by

$$T_s^{(q)} \delta_{\mathbf{w}} = \begin{cases} \delta_{s\mathbf{w}} & , \text{ if } |s\mathbf{w}| > |\mathbf{w}| \\ \delta_{s\mathbf{w}} + p_s(q)\delta_{\mathbf{w}} & , \text{ if } |s\mathbf{w}| < |\mathbf{w}| \end{cases}.$$

Here $|\cdot|$ denotes the word length function with respect to the generating set S and $(\delta_{\mathbf{w}})_{\mathbf{w} \in W} \subseteq \ell^2(W)$ is the canonical orthonormal basis of $\ell^2(W)$. For a group element $\mathbf{w} \in W$ which can be expressed by a reduced expression of the form $\mathbf{w} = s_1 ... s_n$ with $s_1, ..., s_n \in S$ we set $T_{\mathbf{w}}^{(q)} := T_{s_1}^{(q)} ... T_{s_n}^{(q)} \in \mathcal{N}_q(W)$. This operator does not depend on the choice of the expression $s_1 ... s_n$ and the span of such operators is dense in $\mathcal{N}_q(W)$. Further, the von Neumann algebra $\mathcal{N}_q(W)$ carries a canonical faithful normal tracial state τ_q defined by $\tau_q(x) := \langle x\delta_e, \delta_e \rangle$ for $x \in \mathcal{N}_q(W)$. For more details on Hecke-von Neumann algebras see [20, Chapter 20].

The aim of this subsection is to study the Haagerup property of Hecke-von Neumann algebras of virtually free Coxeter groups. We will approach this by decomposing these Hecke-von Neumann algebras as suitable amalgamated free products over finite-dimensional subalgebras. In the case of right-angled Hecke-von Neumann algebras the Haagerup property has been obtained in [12, Theorem 3.9].

Fix a finite rank Coxeter system (W, S). A subset $T \subseteq S$ is called *spherical* if the *special subgroup* $W_T \subseteq W$ generated by T is finite. (W, S) is called spherical if S is a spherical subset.

If W is an arbitrary group which decomposes as an amalgamated free product $W = W_1 *_{W_0} W_2$ where (W_1, S_1), (W_2, S_2) are Coxeter systems with $W_0 = W_1 \cap W_2$ and $S_0 := S_1 \cap S_2$ generates W_0, then $(W, S_1 \cup S_2)$ is a Coxeter system as well. We may now define the class of *virtually free* Coxeter systems as the smallest class of Coxeter groups containing all spherical Coxeter groups and which is stable under taking amalgamated free products over special spherical subgroups. Note that the original definition of virtually free Coxeter systems is different, but by [20, Proposition 8.8.5] equivalent to the one used here.

Now, for a multi-parameter $q := (q_s)_{s \in S} \in \mathbb{R}^S_{>0}$ as above we have a natural unital embedding $\mathcal{N}_q(W_0) \subseteq \mathcal{N}_q(W)$ (see [20, 19.2.2]). Let $\mathbb{E}_{\mathcal{N}_q(W_0)} : \mathcal{N}_q(W) \to \mathcal{N}_q(W_0)$ be the unique faithful normal trace-preserving conditional expectation onto $\mathcal{N}_q(W_0)$. Then, for $\mathbf{w} \in W$ the equality

$$\mathbb{E}_{\mathcal{N}_q(W_0)}(T^{(q)}_{\mathbf{w}}) = \begin{cases} T^{(q)}_{\mathbf{w}}, & \text{if } \mathbf{w} \in W_0 \\ 0, & \text{if } \mathbf{w} \notin W_0 \end{cases}$$

holds.

Let us show that the amalgamated free product decomposition of a Coxeter group translates into the Hecke-von Neumann algebra setting. Note that the arguments using the (iterated) amalgamated free product description of Hecke-deformed Coxeter group C^*-algebras appear for example in [42], exploiting the earlier work on operator algebraic graph products in [13].

Proposition 8.3. *Let (W, S) be a finite rank Coxeter system that decomposes as $W = W_1 *_{W_0} W_2$ where (W_1, S_1), (W_2, S_2) are Coxeter systems with $S = S_1 \cup S_2$, $W_0 = W_1 \cap W_2$ such that $S_0 := S_1 \cap S_2$ generates W_0. For a multi-parameter $q = (q_s)_{s \in S}$ with $q_s = q_t$ for all $s, t \in S$ which are conjugate in W the corresponding Hecke-von Neumann algebra $\mathcal{N}_q(W)$ decomposes as an amalgamated free product of the form*

$$\mathcal{N}_q(W) = \mathcal{N}_{q_1}(W_1) *_{\mathcal{N}_{q_0}(W_0)} \mathcal{N}_{q_2}(W_2),$$

where $q_0 := (q_s)_{s \in S_0}$, $q_1 := (q_s)_{s \in S_1}$ and $q_2 := (q_s)_{s \in S_2}$. Here the decomposition is taken with respect to the restricted conditional expectations $(\mathbb{E}_{\mathcal{N}_q(W_0)})|_{\mathcal{N}_{q_1}(W_1)}$ and $(\mathbb{E}_{\mathcal{N}_q(W_0)})|_{\mathcal{N}_{q_2}(W_2)}$.

Proof. We will use the multi-index notation from the proof of Theorem 8.1. By the uniqueness of the amalgamated free product construction in combination with our previous discussion, it suffices to show that $\mathbb{E}_{\mathcal{N}_q(W_0)}(a_1...a_n) = 0$ for all $n \in \mathbb{N}$, $\mathbf{j} \in \mathcal{J}_n$ and $a_k \in \mathcal{N}_{q_{j_k}}(W_{j_k})) \cap \ker(\mathbb{E}_{\mathcal{N}_q(W_0)})$. Let $\mathcal{N}_{q_i}(W_i)_1$ denote the unit ball of $\mathcal{N}_{q_i}(W_i)$. Let $\overline{\text{Span}}$ denote the strong closure of the linear span. By Kaplansky's density theorem,

$$\mathcal{N}_{q_1}(W_1)_1 \cap \ker(\mathbb{E}_{\mathcal{N}_q(W_0)}) = \mathcal{N}_{q_1}(W_1)_1 \cap \overline{\text{Span}}\{T_{\mathbf{w}}^{(q)} \mid \mathbf{w} \in W_1 \setminus W_0\},$$

and

$$\mathcal{N}_{q_2}(W_2)_1 \cap \ker(\mathbb{E}_{\mathcal{N}_q(W_0)}) = \mathcal{N}_{q_2}(W_2)_1 \cap \overline{\text{Span}}\{T_{\mathbf{w}}^{(q)} \mid \mathbf{w} \in W_2 \setminus W_0\}.$$

By [34, Remark 4.3.1] the element $a_1...a_n$ can hence be approximated strongly by a bounded net of linear combinations of reduced expressions of the form $T_{\mathbf{w}_1}^{(q)}...T_{\mathbf{w}_n}^{(q)}$ with $\mathbf{w}_k \in W_{j_k} \setminus W_0$. But this expression coincides with $T_{\mathbf{w}_1...\mathbf{w}_n}^{(q)}$ where $\mathbf{w}_1...\mathbf{w}_n \in W \setminus W_0$ is non-trivial, so $\mathbb{E}_{\mathcal{N}_q(W_0)}(a_1...a_n) = 0$ since $\mathbb{E}_{\mathcal{N}_q}$ is normal and hence weakly continuous on bounded sets. □

The following corollary is an example of an application of Corollary 8.2 in a setting which is not covered by the results in [25].

Corollary 8.4. *Let (W, S) be a finite rank Coxeter system, let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^S$ be a multi-parameter with $q_s = q_t$ if $s, t \in S$ are conjugate in W and assume that W is virtually free. Then the corresponding Hecke-von Neumann algebra $\mathcal{N}_q(W)$ has the Haagerup property.*

Proof. This follows from a combination of [12, Theorem 3.9], Proposition 8.3 and Corollary 8.2. □

9. Inclusions of finite index

In this section we will discuss finite index inclusions for not necessarily tracial von Neumann algebras defined in [3]. We will pick one of the (possibly nonequivalent) definitions, which is most suitable in our context, and then we will illustrate this notion using certain compact quantum groups, namely free orthogonal quantum groups.

Definition 9.1. Let $N \subseteq M$ be an inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathbb{E}_N : M \to N$. We say that a family of elements $(m_i)_{i\in I}$ is an *orthonormal basis* of the right N-module $L^2(M)_N$ if

(1) for each $i, j \in I$ we have $\mathbb{E}_N(m_i^* m_j) = \delta_{ij} p_j$, where p_j is a projection in N;
(2) $\overline{\sum_{i\in I} m_i N} = L^2(M)$.

We say that the inclusion $N \subseteq M$ is strongly of finite index if it admits a finite orthonormal basis.

Lemma 9.2. *If an inclusion $N \subseteq M$ is strongly of finite index then it has the Haagerup property.*

Proof. Let $m_1, \dots, m_n$ be a finite orthonormal basis for our inclusion. It suffices to show that $x = \sum_{i=1}^n m_i \mathbb{E}_N(m_i^* x)$ for each $x \in M$. Indeed, this would show that the identity map on $L^2(M)$ is relatively compact with respect to N, so clearly the triple $(M, N, \mathbb{E}_N)$ satisfies the relative Haagerup property. The equality $x = \sum_{i=1}^n m_i \mathbb{E}_N(m_i^* x)$ has been already observed by Popa (see [40, Section 1]) in a more general context. □

9.1. Free orthogonal quantum groups

We will now present a certain inclusion arising in the theory of compact quantum groups that has the relative Haagerup property. For information about compact quantum groups we refer the reader to the excellent book [35].

Definition 9.3 *([49]).* Let $n \geqslant 2$ be an integer and let $F \in M_n(\mathbb{C})$ be a matrix such that $F\overline{F} = c\mathbb{1}$ for some $c \in \mathbb{R} \setminus \{0\}$. Let $\mathrm{Pol}(O_F^+)$ be the universal $*$-algebra generated by the entries of a unitary matrix $U \in M_n(\mathrm{Pol}(O_F^+))$, denoted u_{ij}, subject to the condition $U = F\overline{U}F^{-1}$, where $(\overline{U})_{ij} := (u_{ij})^*$ for all $i, j = 1, \dots, n$. Then the unique $*$-homomorphic extension of the map $\Delta(u_{ij}) := \sum_{k=1}^n u_{ik} \otimes u_{kj}$ makes $\mathrm{Pol}(O_F^+)$ into a Hopf $*$-algebra, whose universal C^*-algebra completion yields a compact quantum group.

Remark 9.4. As every compact quantum group admits a Haar state, we can use the GNS construction to construct a von Neumann algebra $L^\infty(O_F^+)$.

In [4] Banica classified irreducible representations of the compact quantum group O_F^+. He showed that they are indexed by natural numbers, U^k, where U^0 is the trivial representation and $U^1 = U$ is the fundamental representation U. Moreover, the fusion rules satisfied by these representations are the following:

$$U^k \otimes U^l \simeq U^{k+l} \oplus U^{k+l-2} \oplus \cdots \oplus U^{|k-l|}, \quad k, l \in \mathbb{N},$$

just like for the classical compact group $SU(2)$. From the fusion rules one can infer that the coefficients of representations indexed by even numbers form a subalgebra. Further, one can use the defining relation $U = F\overline{U}F^{-1}$ to show that they form a $*$-subalgebra.

Definition 9.5. Let $M := L^\infty(O_F^+)$. We define the *even subalgebra* N to be the von Neumann subalgebra of M generated by the elements $(u_{ij}u_{kl})_{1 \leqslant i,j,k,l \leqslant n}$. It is equal to the von Neumann algebra generated by the coefficients of the even representations; in fact it is related to the *projective version* of O_F^+, usually denoted PO_F^+.

Remark 9.6. It has been shown by Brannan in [9] that $N \subseteq M$ is a subfactor of index 2 in case that $F = \mathbb{1}$ (it is then an inclusion of finite von Neumann algebras).

We now roughly outline Brannan's argument and then mention why it cannot immediately be translated into our setting. There is an automorphism Φ of M such that $\Phi(u_{ij}) = -u_{ij}$; Φ can be first defined on $\mathrm{Pol}(O_F^+)$ by the universal property but it also preserves the Haar state, so can be extended to an automorphism of $L^\infty(O_F^+)$. The fixed point subalgebra of Φ is equal to the even subalgebra N and therefore $\mathbb{E}_N := \frac{1}{2}(\mathrm{Id} + \Phi)$ is a conditional expectation onto N that preserves the Haar state. As a consequence $\mathbb{E}_N - \frac{1}{2}\mathrm{Id}$ is a completely positive map, so one can use the Pimsner-Popa inequality, which works for II_1-factors, to conclude that the index of $N \subseteq M$ is at most 2. On the other hand, any proper inclusion has index at least 2, so the result follows. Unfortunately in the non-tracial case it is not clear if the condition that $\mathbb{E}_N - \frac{1}{2}\mathrm{Id}$ is completely positive implies that the inclusion $N \subseteq M$ is strongly of finite index; so far it is only known that it implies being of finite index in a weaker sense (see [3, Théorème 3.5]). Fortunately in our case it is possible to explicitly define a finite orthonormal basis.

Proposition 9.7. *Let $n \geqslant 2$ be an integer and let $F \in M_n(\mathbb{C})$ be a matrix such that $F\overline{F} = c\mathbb{1}$ for some $c \in \mathbb{R} \setminus \{0\}$. Let $M := L^\infty(O_F^+)$ and let N be the even von Neumann subalgebra of M. Then the inclusion $N \subseteq M$ is strongly of finite index. Moreover, one can find an orthonormal basis consisting of at most $n^2 + 1$ elements.*

Proof. One can verify by an explicit computation that N is left globally invariant by the modular automorphism group of the Haar state h of $L^\infty(O_F^+)$, so we do have a faithful normal h-preserving conditional expectation $\mathbb{E}_N : M \to N$. We start with n^2+1 elements of M, namely $\mathbb{1}$ and all the u_{ij}'s. Since we have all the coefficients of the fundamental representation, it follows from the fusion rules of O_F^+ that $N \oplus \sum_{i,j=1}^n u_{ij} N$ is a dense submodule of $L^2(M)_N$.

Note that all the elements u_{ij} are odd, i.e. $\Phi(u_{ij}) = -u_{ij}$ for $i, j = 1, \ldots, n$. Suppose that we have a family $x_1, \ldots, x_k$ of odd elements. Then we can perform a Gram-Schmidt process to make this set orthonormal. To do it, first notice that $x_i^* x_i$ is an even element, hence so is $|x_i|$ – we conclude that the partial isometry in the polar decomposition $x_i = v_i|x_i|$ is odd as well. Our process works as follows: we first replace x_1 by the corresponding partial isometry v_1. Then we define $\tilde{x}_2 := x_2 - v_1 v_1^* x_2$. Because v_1 is a partial isometry, we get $v_1^* \tilde{x}_2 = v_1^* x_2 - v_1^* v_1 v_1^* x_2 = 0$. We then define v_2 to be the partial isometry appearing in the polar decomposition of $\tilde{x}_2$; it still holds that v_2 is odd and $v_1^* v_2 = 0$. We can continue this process just like the usual Gram-Schmidt process and obtain an orthonormal set of odd partial isometries v_i such that $\sum_{i=1}^k x_i N \subseteq \sum_{i=1}^k v_i N$; note that the projections $v_i^* v_i$ belong to N. If we apply this procedure to the family $(u_{ij})_{1 \leqslant i,j \leqslant n}$, we obtain a finite orthonormal basis for the inclusion $N \subseteq M$. □

Corollary 9.8. *The inclusion $N \subseteq M := L^\infty(O_F^+)$ has the relative Haagerup property.*

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