Delft University of Technology

Packing list-colorings

Cambie, Stijn; Cames van Batenburg, Wouter; Davies, Ewan; Kang, Ross J.

Important note
To cite this publication, please use the final published version (if applicable).
Please check the document version above.

WILEY

Packing list-colorings

Stijn Cambie[1] | **Wouter Cames van Batenburg[2]** | **Ewan Davies[3]** |
Ross J. Kang[4]

[1]Extremal Combinatorics and Probability Group (ECOPRO), Institute for Basic Science (IBS), Daejeon, South Korea

[2]Delft Institute of Applied Mathematics, Delft University of Technology, Delft, Netherlands

[3]Department of Computer Science, Colorado State University, Fort Collins, Colorado, USA

[4]Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, Netherlands

Correspondence

Ross J. Kang, Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, Netherlands.
Email: r.kang@uva.nl

Abstract

List coloring is an influential and classic topic in graph theory. We initiate the study of a natural strengthening of this problem, where instead of one list-coloring, we seek many in parallel. Our explorations have uncovered a potentially rich seam of interesting problems spanning chromatic graph theory. Given a k-list-assignment L of a graph G, which is the assignment of a list $L(v)$ of k colors to each vertex $v \in V(G)$, we study the existence of k pairwise-disjoint proper colorings of G using colors from these lists. We may refer to this as a *list-packing*. Using a mix of combinatorial and probabilistic methods, we set out some basic upper bounds on the smallest k for which such a list-packing is always guaranteed, in terms of the number of vertices, the degeneracy, the maximum degree, or the (list) chromatic number of G. (The reader might already find it interesting that such a minimal k is well defined.) We also pursue a more focused study of the case when G is a bipartite graph. Our results do not yet rule out the tantalising prospect that the minimal k above is not too much larger than the list chromatic number. Our study has taken inspiration from study of the strong chromatic number, and we also explore generalizations of the problem above in the same spirit.

KEYWORDS

graph colouring, graph packing, independent transversals, list colouring, strong chromatic number

---

1 | INTRODUCTION

Some resource allocation problems may be framed in terms of graph coloring, for example if resources are colors to be allocated to vertices of a graph such that vertices which cannot make simultaneous use of any resource are connected by edges. In such a situation, conceivably one might desire not just one coloring, but several in parallel. One might even want that collectively they cover all possible resource usage. What general conditions guarantee this? It is this strengthened graph coloring problem that we systematically study in this work.

A particularly natural way to frame this is with respect to *list coloring* [19, 41]. Given a graph G, a *list-assignment* L of G is a mapping such that $L(v)$ is a subset of natural numbers (list of colors) associated to the vertex $v \in V(G)$. By considering the colors as resources, one can interpret each list as reflecting the specific availability of resources at some site based on some external restrictions. Given a positive integer k, a *k-list-assignment* is a list-assignment for which each list has cardinality k. A *proper L-coloring* is a mapping $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for any $v \in V(G)$ and such that whenever uv is an edge of G, $c(u) \neq c(v)$. That is, c encodes a proper coloring of the vertices of G such that each vertex is colored by a color of its list. The *list chromatic number* (or *choosability*) $\chi_\ell(G)$ is the least k such that G admits a proper L-coloring for *any* k-list-assignment L of G. Note that $\chi_\ell(G)$ is always at least the chromatic number $\chi(G)$, as one of the list-assignments one must consider gives the same list to every vertex.

We formulate the main question above concretely within the framework of list coloring. Given a list-assignment L of G, an *L-packing of G of size k* is a collection of k mutually disjoint L-colorings $c_1, \ldots, c_k$ of G, that is, $c_i(v) \neq c_j(v)$ for any $i \neq j$ and any $v \in V(G)$. We say that an L-packing is proper if each of the disjoint L-colorings is proper. We define the *list (chromatic) packing number* $\chi_\ell^\star(G)$ of G as the least k such that G admits a proper L-packing of size k for any k-list-assignment L of G. Note that $\chi_\ell^\star(G)$ is necessarily at least $\chi_\ell(G)$. The latter implies monotonicity: for every $k > \chi_\ell^\star(G)$ and any k-list-assignment L of G, iteratively one can find $k - \chi_\ell^\star(G)$ disjoint L-colorings and finally a proper L-packing of G by adding $\chi_\ell^\star(G)$ disjoint L-colorings.

It might not be immediately obvious that the parameter $\chi_\ell^\star$ is always well-defined, but we provide a number of different proofs of this fact in the course of our work. It was Alon, Fellows and Hare [7] who first suggested the study of $\chi_\ell^\star$ right at the end of their paper, but ours is the first work to embrace this suggestion.

Upon encountering the definition of list packing number, a natural course of action is to pursue list packing analogues of the most basic and important results on list coloring. This indeed is our programme, and we present several results in this vein. Our programme would be partly redundant if we were able to prove the following conjecture.

Conjecture 1. *There exists $C > 0$ such that $\chi_\ell^\star(G) \leqslant C \cdot \chi_\ell(G)$ for any graph G.*

It could even be the case, moreover, that, for each $\varepsilon > 0$ there is some χ_0 such that $\chi_\ell^\star(G) \leqslant (1 + \varepsilon) \cdot \chi_\ell(G)$ for all G with $\chi_\ell(G) \geqslant \chi_0$. A resolution of Conjecture 1 or its asymptotically stronger variant, either affirmatively or negatively, would be very interesting.

So far we only know that $\chi_\ell^\star(G)$ is upper bounded by some exponential function of $\chi_\ell(G)$; see Theorem 3 below. Let us observe that $\chi_\ell^\star(G)$ can indeed be strictly larger than $\chi_\ell(G)$ for some G. This is the case when G is an even cycle. For example, consider the cycle of length 4 and lists $\{1, 2\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, listed in cyclic order. Incidentally, this implies that the constant C in Conjecture 1, if it exists, may not be smaller than $3/2$.

Our main contribution is to present a collection of bounds on $\chi_\ell^\star$ within a few basic graph coloring settings. Please consult Subsection 1.1 for definitions of any unfamiliar or ambiguous graph theoretic

terminology. Due to the depth and breadth of chromatic graph theory, there remains a plethora of further possibilities. We summarize what we consider to be the most tempting ones at the end of the paper.

Perhaps the simplest upper bound on the chromatic number of a graph is the number of vertices, and this bound easily carries over to list chromatic number too. In Section 2, we show using an unexpectedly delicate inductive argument that the analogous statement for list packing number also holds.

Theorem 2. $\chi_\ell^\star(G) \leqslant n$ *for any graph G on n vertices. Equality holds if and only if G is* K_n, *the complete graph on n vertices.*

Usually, one considers the bound $\chi_\ell(G) \leqslant n$ as a corollary of a more refined statement about greedy coloring. The degeneracy $\delta^\star(G)$ of a graph G is the least k such that there is an ordering of the vertices of G such that each vertex has at most k neighbors preceding it in the order. A greedy algorithm that colors vertices in this order cannot stall if each list has size $1 + \delta^\star(G)$, as at the time of coloring a vertex v at most $\delta^\star(G)$ colors have been used previously on neighbors of v. The degeneracy of a graph G is at most the maximum degree $\Delta(G)$, and this is at most one less than the number of vertices, giving the well-known bounds $\chi_\ell(G) \leqslant 1 + \delta^\star(G) \leqslant 1 + \Delta(G) \leqslant n$ for an n-vertex graph G. In Sections 3 and 4, using what are essentially greedy arguments, we show the following two upper bounds on list packing number in terms of the degeneracy and maximum degree, respectively.

Theorem 3. $\chi_\ell^\star(G) \leqslant 2\delta^\star(G)$ *for any graph G.*

Theorem 4. $\chi_\ell^\star(G) \leqslant 1 + \Delta(G) + \chi_\ell(G)$ *for any graph G.*

Although neither implies Theorem 2, these two results have a number of interesting consequences. Both imply that $\chi_\ell^\star(G)$ is always at most around $2\Delta(G)$, a bound which is sharp up to the factor 2. They imply that for Conjecture 1, one may restrict attention to graphs having list chromatic number much smaller than the degeneracy or maximum degree. Together with the result proved by Alon [5] that for some constant $C > 0$ $\chi_\ell(G) \geqslant C \log \delta^\star(G)$ for all G, note that Theorem 3 implies that $\chi_\ell^\star(G)$ is bounded by an exponential function of $\chi_\ell(G)$, which serves as modest support for Conjecture 1. Theorem 3 together with Euler's formula further implies that $\chi_\ell^\star(G) \leqslant 10$ for any planar graph G. It is worth noting that the bound in Theorem 3 cannot be improved to $1 + \delta^\star(G)$, see Theorem 25.

Another important way to bound the list chromatic number is to estimate it in terms of the chromatic number. A basic but well-known result of Alon [4] asserts that $\chi_\ell(G)$ is within a factor $\log n$ of the chromatic number $\chi(G)$, where n denotes the order of the graph G. We show the following two list packing versions of this in Section 6. Here $\chi_f(G)$ denotes the fractional chromatic number of G, while $\rho(G)$ denotes the Hall ratio of G.

Theorem 5. $\chi_\ell^\star(G) \leqslant \frac{(5+o(1))\log n}{\log(\chi_f(G)/(\chi_f(G)-1))} \leqslant (5 + o(1))\chi_f(G) \log n$ *for any graph G with at least one edge and with n vertices, as* $n \to \infty$. *If* $\chi_f(G)$ *is bounded as* $n \to \infty$ *then the leading constant can be improved to 1.*

Theorem 6. $\chi_\ell^\star(G) \leqslant (5 + o(1))\rho(G)(\log n)^2$ *for any graph G on n vertices, as* $n \to \infty$.

Our proofs of these results rely on good estimates for the probability of zero permanent in a random binary matrix (Theorem 27). In the case that $\chi_f(G)$ is bounded, Theorem 5 is asymptotically sharp for the complete multipartite graphs (see [19]). And since $\chi_f(G) \leqslant \chi(G)$ and since $1/\log(x/(x-1)) \leqslant x$, Theorem 5 is a stronger form of the last-mentioned result of Alon. Since $\chi_\ell(G) \leqslant \chi_\ell^\star(G)$, Theorem 5 strengthens a recent result in [15], while Theorem 6 partially strengthens another recent result in [34]. It could well be that the extra factor $\log n$ in Theorem 6 is unnecessary, in which case, since $\rho(G) \leqslant \chi_f(G)$ and since $1/\log(x/(x-1)) \sim x$ as $x \to \infty$, we would then essentially have a common strengthening of Theorems 5 and 6.

Note that in the special case of a bipartite graph G on n vertices, Theorem 5 (or see also Theorem 29 below) implies that $\chi_\ell(G) \leqslant (1 + o(1)) \log_2 n$ as $n \to \infty$. Alon and Krivelevich [8] conjectured something more refined in terms of maximum degree, in particular, that for some $C > 0$ we have $\chi_\ell(G) \leqslant C \log \Delta(G)$ for any bipartite G. Since its formulation there has been surprisingly little progress on this essential problem. Already then, it was known that for some $C > 0$ $\chi_\ell(G) \leqslant C \Delta / \log \Delta(G)$ for any bipartite G, a statement which is a corollary of the seminal result of Johansson for triangle-free graphs [27]. Recent related efforts have only affected the asymptotic leading constant C, bringing it down to 1; see [6, 14, 33]. Our work matches these recent efforts, but for qualitatively a much stronger structural parameter.

Theorem 7. $\chi_\ell^\star(G) \leqslant (1 + o(1)) \Delta / \log \Delta$ *for any bipartite graph G with $\Delta(G) \leqslant \Delta$, as $\Delta \to \infty$.*

Combining Conjecture 1 and the Alon–Krivelevich conjecture mentioned above, one could already aspire to a much better bound. (Or to put it in another way, the bipartite case might be where it is most natural to seek counterexamples to Conjecture 1.) On the other hand, meeting the 'Shearer barrier' [38] here is difficult enough. Our proof of Theorem 7 is rather involved. We draw on the 'coupon collector intuition', which we know can be neatly combined with the Lovász local lemma to obtain the analogous list chromatic number bound (see [6, 14]). Unfortunately, this approach does not immediately apply here for Theorem 7 as it is impossible to extend the requisite negative correlation property for packing; however, we managed to circumvent this obstacle for a suitable result on transversals in random matrices (Lemma 36) by using significant further probability estimates. It seems challenging to establish the bound of Theorem 7 for all triangle-free G, even at the expense of the asymptotic leading constant. Already a bound strictly smaller than Δ would be welcome progress.

Although we have presented our results so far purely in terms of the list packing number, many of them can be shown in some natural stronger or more general settings. If one interprets list-colorings as independent transversals of some suitably vertex-partitioned auxiliary graph, then one can similarly reinterpret list-packings. By considering generalizations of that auxiliary graph, we obtain a natural range of parameters/settings that expands from list packing towards the classic notion of strong colorings [3, 21]. We introduce and discuss these in Subsection 1.2. Thereafter we 'resume' the introduction to this paper, by also stating and discussing the strengthened or more general versions of our results. Before that, we set out some of our more basic notation, for clarity and completeness.

1.1 │ Basic terminology

For the convenience of the reader, here we indicate some of the standard notational conventions/definitions we assume throughout this work. Let G be a graph. The *order* of G is $|G| = |V(G)|$. For $v \in V(G)$, the *neighborhood* of v is $N(v) = \{u \in V(G) : uv \in E\}$ and the *degree* of v is $\deg(v) = |N(v)|$.

The *minimum degree* of G is $\delta(G) = \min\{\deg(v) : v \in V(G)\}$ and the *maximum degree* of G is $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$. The *degeneracy* of G is $\delta^\star(G) = \max_{H \subseteq G} \delta(H)$, where the maximum is taken over all subgraphs H of G, and this is equivalent to the definition given above in terms of neighbors preceding vertices in some order. A *clique* in a graph G is a set of vertices $X \subseteq V(G)$ such that every pair in X forms an edge of G, and an *independent set* in G is a set Y such that no pair in Y forms an edge of G. The *clique number* $\omega(G)$ of G is size of a largest clique in G and the *independence number* $\alpha(G)$ of G is size of a largest independent set in G.

The *Hall ratio* of G is $\rho(G) = \max_{H \subseteq G} |H| / \alpha(H)$, where the maximum is taken over subgraphs H of G containing at least one vertex. The *fractional chromatic number* $\chi_f(G)$ of G is the least k such

that there exists a probability distribution over the independent sets of G such that $\mathbb{P}(v \in \mathbf{S}) \geqslant 1/k$ for each $v \in V(G)$ and $\mathbf{S}$ a random independent set drawn according to the distribution. The *chromatic number* $\chi(G)$ of G is the least k such that there is a partition of $V(G)$ into k independent sets. We will usually use n to denote the number of vertices of a graph G.

It is worth keeping in mind that the following simple inequalities always hold:

$$\omega(G) \leqslant \rho(G) \leqslant \chi_f(G) \leqslant \chi(G) \leqslant \chi_\ell(G) \leqslant 1 + \delta^\star(G) \leqslant 1 + \Delta(G) \leqslant n.$$

Given a matrix A, we denote its entries with $A_{i,j}$ and with $A_{I \times J}$ we denote the submatrix formed from rows I and columns J. A *binary matrix* is a matrix with entries in $\{0, 1\}$. A *zero matrix* is a matrix with every entry equal to 0. A binary $k \times k$ matrix is said to have a *1-transversal* (resp. *0-transversal*) if it contains k elements that are each of value 1 (resp. each of value 0), such that no two elements are in the same column and no two are in the same row. The *permanent* of a $k \times k$ matrix A is defined as $\text{Per}(A) = \sum_\sigma \prod_{i=1}^n A_{i,\sigma(i)}$, where the sum ranges over all permutations σ of $[k] = \{1, \dots, k\}$. For our applications, it is important to note that a binary matrix A has no 1-transversal if and only if $\text{Per}(A) = 0$.

1.2 | From list packing to strong coloring

As mentioned, although we have chosen to introduce our results in terms of the parameter $\chi_\ell^\star$, many of our results can be established (and otherwise pursued) in stronger or more general settings. We define and discuss these now.

Let G and H be graphs. We say that H is a *cover* of G with respect to a mapping $L : V(G) \to 2^{V(H)}$ if L induces a partition of $V(H)$ and the subgraph induced between $L(v)$ and $L(v')$ is empty whenever $vv' \notin E(G)$. Under various basic conditions on G, H and L, we are interested in the existence of an *independent transversal* of H with respect to the partition, which is a vertex subset with exactly one vertex chosen from each part that simultaneously forms an independent set. Moreover, we say that a cover H of G with respect to L is *k-fold* if $|L(v)| = k$ for all $v \in V(G)$. In this case, we will also be interested in the existence of some *(k-fold) IT packing* of H with respect to L, that is, a collection of k mutually disjoint independent transversals of H with respect to L. These settings are motivated in a light-hearted way in a well-known survey paper of Haxell [26].

To illustrate, we first reformulate list coloring and list packing in this terminology. From a list-assignment L of G, we derive the *list-cover* $H_\ell(G, L)$ for G via L as follows. For every $v \in V(G)$, we let $L_\ell(v) = \{(v, c)\}_{c \in L(v)}$ and define $V(H_\ell) = \cup_{v \in V(G)} L_\ell(v)$. We define $E(H_\ell)$ by including $(v, c)(v', c') \in E(H_\ell)$ if and only if $vv' \in E(G)$ and $c = c' \in L(v) \cap L(v')$. Note that H_ℓ is a cover of G with respect to L_ℓ, and that H_ℓ is k-fold if L is a k-list-assignment of G. The independent transversals of H_ℓ with respect to L_ℓ are in one-to-one correspondence with the proper L-colorings of G. If L is a k-list-assignment of G, then the k-fold IT packings of H_ℓ with respect to L_ℓ are in one-to-one correspondence with the proper L-packings of G of size k. Thus $\chi_\ell(G)$ and $\chi_\ell^\star(G)$ can equivalently be defined in terms of list-covers.

We can somewhat relax the setting as follows. A *correspondence-assignment* for G via L is a cover H for G via L such that for each edge $vv' \in E(G)$, the subgraph induced between $L(v)$ and $L(v')$ is a matching. We call such an H a *correspondence-cover* for G with respect to L. A *correspondence L-coloring* of H is an independent transversal of H with respect to L. The *correspondence chromatic number* (or *DP-chromatic number*) $\chi_c(G)$ is the least k such that any k-fold correspondence-cover H of G via L admits a correspondence L-coloring [17]. Given a k-fold correspondence-cover, a *(k-fold) correspondence L-packing of H* is a k-fold IT packing of H with respect to L. We define the *correspondence (chromatic) packing number* $\chi_c^\star(G)$ of G as the least k such that any k-fold correspondence-cover

H of G via L admits a k-fold correspondence L-packing. Every list-cover is a correspondence-cover, and moreover, for any list-cover H of G via L, the correspondence L-colorings of H are in one-to-one correspondence with the proper L-colorings of G. In this sense correspondence coloring (or packing) is a relaxation of list coloring (or packing, respectively). Note that $\chi_c(G)$ is necessarily at least $\chi_\ell(G)$, $\chi_c^\star(G)$ is necessarily at least $\chi_\ell^\star(G)$, and $\chi_c^\star(G)$ is necessarily at least $\chi_c(G)$. It is easily observed that $\chi_c(G) \leqslant 1 + \delta^\star(G)$ always.

In the study of $\chi_\ell(G)$ and $\chi_c(G)$ (and indeed also of $\chi_\ell^\star(G)$ and $\chi_c^\star(G)$), it is natural to explicitly take into account structural properties of the graph G, and this in fact is a large and ever-expanding subfield of graph theory; however, given how we may define the parameters as in this subsection, it is also valid and interesting to incorporate the properties of the cover graph H instead, without any explicit assumptions on G. With respect to list-covers, correspondence-covers, or general covers, the existence of independent transversals in these terms has already attracted quite some interest, particularly under the condition that the cover graph has bounded maximum degree.

For general covers, such a problem was investigated by Bollobás, Erdős and Szemerédi [13] in the mid 1970s. For each D, they essentially asked for the least integer $\Lambda(D)$ for which the following holds: for any $\Lambda(D)$-fold cover H of some graph G via L such that $\Delta(H) \leqslant D$, there is an independent transversal of H with respect to L. Using topological methods, Haxell [24] proved that $\Lambda(D) \leqslant 2D$, a result which is sharp for every D [40].

For list-covers, such a problem was proposed by Reed [36]. For each D, he essentially asked for the least integer $\Lambda_\ell(D)$ for which the following holds: for any $\Lambda_\ell(D)$-fold list-cover H of some graph G via L such that $\Delta(H) \leqslant D$, there is an independent transversal of H with respect to L. In fact, he conjectured that $\Lambda_\ell(D) = D + 1$. This was disproved by Bohman and Holzman [12]; however, with nibble methodology Reed and Sudakov [37] proved that $\Lambda_\ell(D) \leqslant D + o(D)$ as $D \to \infty$.

For correspondence-covers, such a problem was proposed by Aharoni and Holzman, see [30]. For each D, they essentially asked for the least integer $\Lambda_c(D)$ for which the following holds: for any $\Lambda_c(D)$-fold correspondence-cover H of some graph G via L such that $\Delta(H) \leqslant D$, there is an independent transversal of H with respect to L. Again with nibble methods, Loh and Sudakov [30] proved that $\Lambda_c(D) \leqslant D + o(D)$ as $D \to \infty$. We note that very recently two independent works [22, 28] showed a substantial strengthening of this last result in terms of a part-averaged degree condition for the correspondence-cover.

For each of the three parameters $\Lambda, \Lambda_\ell, \Lambda_c$, one can define the analogous packing variants. That is, for each D, what is the least integer $\Lambda^\star(D)$ (or $\Lambda_\ell^\star(D)$ or $\Lambda_c^\star(D)$) for which the following holds: for any $\Lambda^\star(D)$-fold cover (or $\Lambda_\ell^\star(D)$-fold list-cover or $\Lambda_c^\star(D)$-fold correspondence-cover, respectively) H of some graph G via L such that $\Delta(H) \leqslant D$, there is an IT packing of H with respect to L. Indeed, under the guise of *strong coloring*, $\Lambda^\star(D)$ has already (long) been investigated: independently, Alon [3] and Fellows [21] first considered this parameter, and in particular Alon showed using the Lóvasz local lemma that $\Lambda^\star(D) = O(D)$. Haxell [25] proved that $\Lambda^\star(D) \leqslant 2.75D + o(D)$ as $D \to \infty$, which is the best known bound to date. The folkloric *strong coloring conjecture* (see [1]) asserts that $\Lambda^\star(D) \leqslant 2D$ for all D. Note that this longstanding conjecture has a hint of (and indeed inspired) our Conjecture 1.

1.3 | Introduction continued

In Subsection 1.2, we have presented essentially three additional problem settings 'between' list packing and strong coloring. Equipped with these notions, we can further discuss our results and trajectory in this context.

Let us resume the introduction by first considering how the results on the list packing number extend (or not) to the correspondence packing number, and contrast the respective behaviors. Note that, in this realm, we can maintain our view on the 'covered' graph G, and so the panoply of results and problems in graph coloring can still guide our explorations for correspondence packing. We note that, just as before, these explorations would be made partly redundant if we were able to prove the following conjecture.

Conjecture 8. *There exists $C > 0$ such that $\chi_c^\star(G) \leqslant C \cdot \chi_c(G)$ for any graph G.*

We will see later for this assertion both that it is true up to a logarithmic factor and that it cannot be true with C taken smaller than 2.

In terms of the number of vertices, the problem of correspondence packing has recently been highlighted (in different terminology) by Yuster [42]. Casting it as a specialization of earlier conjectures related to the Hajnal–Szemerédi theorem, he stated the conjecture, rephrased in our terminology, that $\chi_c^\star(K_n) = n$ if n is even and $\chi_c^\star(K_n) = n + 1$ if n is odd. This assertion would be best possible if true, due to a construction of Catlin [16]. Note that Catlin's construction also (barely) precludes a direct correspondence packing analogue of Theorem 2. Yuster used semirandom methods to prove that $\chi_c^\star(K_n) \leqslant 1.78n$ for all sufficiently large n.

In terms of degree parameters, we indeed have the following generalizations of Theorems 3 and 4 for correspondence packing.

Theorem 9. $\chi_c^\star(G) \leqslant 2\delta^\star(G)$ *for any graph G.*

Theorem 10. $\chi_c^\star(G) \leqslant 1 + \Delta(G) + \chi_c(G)$ *for any graph G.*

Note first that since $\chi_\ell^\star(G) \leqslant \chi_c^\star(G)$, Theorem 9 implies Theorem 3. Theorem 10 does not immediately imply Theorem 4, but their proofs are almost identical. Again, both results imply that $\chi_c^\star(G)$ is always at most around $2\Delta(G)$, a bound which is sharp up to the factor 2. Together with a result noticed independently by Bernshteyn [11] and Kráľ, Pangrác and Voss [29] that for some constant $C > 0$ $\chi_c(G) \geqslant C\delta^\star(G)/\log \delta^\star(G)$ for all G, note that Theorem 9 implies that $\chi_c^\star(G)$ is at most a logarithmic factor larger than $\chi_c(G)$, in support of Conjecture 8. Theorem 9 together with Euler's formula further implies that $\chi_c^\star(G) \leqslant 10$ for any planar graph G. In Section 3, we exhibit an example (see Proposition 24 below) to show that the bound in Theorem 9 is best possible. Since $\chi_c(G) \leqslant 1 + \delta^\star(G)$ always, this example also shows that C in Conjecture 8 (if true) cannot be taken smaller than 2.

In terms of direct extensions of Theorems 5 and 6 (in terms of fractional chromatic number and Hall ratio) to correspondence packing, the just-mentioned result in [11] and [29] proves it impossible, as the corresponding statements already fail for complete bipartite graphs. To be sure, complete bipartite graphs are where the difference between list packing and correspondence packing is clearest. We are able to establish the correspondence packing strengthening of Theorem 7.

Theorem 11. $\chi_c^\star(G) \leqslant (1 + o(1))\Delta/\log \Delta$ *for any bipartite graph G with $\Delta(G) \leqslant \Delta$, as* $\Delta \to \infty$.

However, the complete bipartite graph $K_{\Delta,\Delta}$ certifies this statement to be sharp up to an asymptotic multiple of 2, due to the observation from [11] and [29]. This gives a sharp contrast with Theorem 5. This contrast might appear to follow mainly from the known difference in behavior between list and correspondence chromatic numbers. On the other hand, we also demonstrate that there are complete bipartite graphs having list chromatic, list packing and correspondence chromatic numbers all equal, but correspondence packing number almost twice as large (Corollary 34).

Just as for list packing, we suspect that a more general result holds, namely that Theorem 11 could be extended to all triangle-free G.

Conjecture 12. $\chi_c^{\star}(G) \leqslant (1 + o(1))\Delta / \log \Delta$ *for any triangle-free graph G with $\Delta(G) \leqslant \Delta$, as $\Delta \to \infty$.*

If true, this conjecture would be a striking finding following in a long and acclaimed sequence of results, including the work of Johansson [27] and Molloy [33], among others.

As we alluded to in Subsection 1.2, our trajectory tends naturally towards a couple of problem settings interpolating between list/correspondence packing and strong coloring. For example, analogously to the strong coloring conjecture, it would be interesting to investigate the following possibilities for the two packing parameters introduced at the end of Subsection 1.2.

Conjecture 13. $\Lambda_{\ell}^{\star}(D) \leqslant D + o(D)$ *and* $\Lambda_c^{\star}(D) \leqslant D + o(D)$ *as $D \to \infty$.*

Note that this conjecture implies the following conjectural statements for list and correspondence packing number: for each $\varepsilon > 0$ there is some Δ_0 such that $\chi_{\ell}^{\star}(G) \leqslant (1 + \varepsilon)\Delta$ (respectively, $\chi_c^{\star}(G) \leqslant (1 + \varepsilon)\Delta$) for all G with $\Delta(G) \geqslant \Delta_0$. We also note that the argument outlined in Section 4 for Theorems 4 and 10 also shows that $\Lambda_{\ell}^{\star}(D) \leqslant \Lambda_c^{\star}(D) \leqslant 2D + o(D)$ as $D \to \infty$; see both Section 4 and a remark in [30].

In considering problems such as Conjecture 13, it is also sensible to explore what holds under some basic structural conditions on the cover graph. Along these lines, we would like to emphasize the possibilities related to Theorems 7 and 11 and Conjecture 12 in particular. Let us reformulate Theorem 11: there is some $k = k(\Delta)$ satisfying $k \leqslant (1+o(1))\Delta / \log \Delta$ as $\Delta \to \infty$ such that for any bipartite graph G with $\Delta(G) \leqslant \Delta$, it holds that any k-fold correspondence-cover H of G via L admits a correspondence L-packing. In a conservative manner, we could propose the following stronger statement.

Conjecture 14. *There is some $k = k(\Delta)$ satisfying $k \leqslant (1 + o(1))\Delta / \log \Delta$ as $\Delta \to \infty$ such that the following holds. Suppose that G and H are bipartite graphs such that H is a k-fold correspondence-cover of G via some L, and that $\Delta(H) \leqslant \Delta$. Then H admits a correspondence L-packing.*

If true, this would constitute a correspondence packing analogue of a recent result due to two of the authors [14]. Beyond a common strengthening of Conjectures 12 and 14, one could more boldly speculate the following.

Conjecture 15. *For every $r \geqslant 3$, there is some $C_r > 0$ such that the following holds for $k = C_r \Delta / \log \Delta$. Suppose that G and H are graphs such that H is a k-fold correspondence-cover of G via some L, H contains no copy of K_r, and $\Delta(H) \leqslant \Delta$. Then H admits a correspondence L-packing.*

If true, this would imply a recent conjecture of Anderson, Bernshteyn and Dhawan [10], which in turn would imply a conjecture of Alon, Krivelevich and Sudakov [9], which in turn would imply a conjecture of Ajtai, Erdős, Komlós and Szemerédi [2]. Due to the connections with these last two central and longstanding conjectures, even partial progress on Conjecture 15, in either direction, would be intriguing.

1.4 | Outline of the paper

See Tables 1 and 2 for an overview of our results that refers to where we state and prove them in the paper. In Section 2 we prove Theorem 2 by showing that $\chi_{\ell}^{\star}(K_n) = n$ and $\chi_{\ell}^{\star}(K_n^-) = n - 1$, where K_n^-

TABLE 1 An overview of our results on the list packing number $\chi_\ell^\star(G)$ and the correspondence packing number $\chi_c^\star(G)$ of a graph G.

Result	Reference
$\chi_\ell^\star(G) \leqslant n$ with equality if and only if G is the complete graph K_n.	Theorem 2; Section 2
$\chi_c^\star(G) \leqslant 2d$ if G is d-degenerate.	Theorem 9, Proposition 24
For all d, there is a d-degenerate G with $\chi_c^\star(G) = 2d$.	Theorem 25; Section 3
For all d, there is a d-degenerate G with $\chi_\ell^\star(G) \geqslant d + 2$.	
$\chi_\ell^\star(G) \leqslant 1 + \Delta + \chi_\ell(G)$	Theorems 4 and 10
$\chi_c^\star(G) \leqslant 1 + \Delta + \chi_c(G)$	Section 4
$\chi_\ell^\star(G) \leqslant (5 + o(1)) \cdot \chi_f(G) \cdot \log n$.	Theorems 5, 29, 30
$\chi_\ell^\star(G) \leqslant (1 + o(1)) \cdot \chi_f(G) \cdot \log n$, if $\chi_f(G)$ is bounded.	Section 6
$\chi_\ell^\star(G) \leqslant (1 + o(1)) \log_2 n$, if G is bipartite.	
$\chi_\ell^\star(G) \leqslant \min(\Delta_A, \Delta_B) + 1$	Lemma 33; Section 7
if $G = (A \cup B, E)$ is bipartite with maximum degrees Δ_A, Δ_B.	
$\chi_\ell^\star(K_{a,b}) = b + 1$ and $\chi_c^\star(K_{a,b}) = 2b$ for all b and $a = a(b)$ large enough.	Corollary 34; Section 7
$\chi_c^\star(G) \leqslant (1 + o(1))\Delta / \log \Delta$ if G is bipartite.	Theorem 11; Section 7

Note: Here n denotes the number of vertices and Δ is the maximum degree of G. Recall that $\chi_\ell^\star(G) \leqslant \chi_c^\star(G)$ for every graph G.

TABLE 2 Two of our auxiliary results on random matrices, perhaps of independent interest.

Result	Reference
A random $k \times k$ binary matrix with independent Bernoulli$(1 - p)$ distributed entries has no 1-transversal with probability $2kp^k(1 + o(1))$, as $k \to \infty$ provided $k^5 p^k \to 0$.	Theorem 27; Section 5
If $k \geqslant \lceil(1+\epsilon)n/\log n\rceil$, then a sum of n independent uniformly random $k \times k$ permutation matrices has a 0-transversal with probability at least $1 - 3k^2 \cdot \exp(-n^{\epsilon/3})$.	Lemma 36; Section 7

equals K_n minus one edge. In Section 3 we prove Theorem 3 by doing this for the correspondence version, Theorem 9, through an application of Hall's marriage theorem. In Section 4 we prove Theorem 10 by showing that a partial coloring that does not use colors from every list is not a maximum partial coloring. Theorem 4 is proved with exactly the same method. In Section 5 we generalize a theorem of Everett and Stein [20] on the permanent of random $k \times k$ matrices. This essentially says that (under certain conditions) the probability that a random binary matrix A has a permanent equal to zero is approximately the probability that it contains a row or column of zeros. This is applied in Section 6 to prove Theorems 5 and 6. Here we construct some matrices for every $M^\star(v)$ for every vertex v in which a 1-transversal corresponds with an ordering of the colors in the list, resulting in different L-colorings. In Section 7 we prove that the list packing number of a bipartite graph with parts having maximum degrees Δ_A and Δ_B is at most $\min\{\Delta_A, \Delta_B\} + 1$. We show sharpness of this result and prove that there are complete bipartite graphs for which the correspondence packing number is nearly twice the correspondence chromatic number. At the end, we prove Theorem 11, and thus also Theorem 7.

1.5 | Further preliminaries

In this subsection, we collect some basic results we require for the proofs. We will use the following formulation of Hall's marriage theorem.

Theorem 16 ([23])**.** *Given a family $\mathcal{F}$ of finite subsets of some ground set X, where the subsets are counted with multiplicity, suppose $\mathcal{F}$ satisfies the marriage condition, that is that for each subfamily $\mathcal{F}' \subseteq \mathcal{F}$*

$$|\mathcal{F}'| \leqslant \left| \bigcup_{A \in \mathcal{F}'} A \right|.$$

Then there is an injective function $f : \mathcal{F} \to X$ such that $f(A)$ is an element of the set A for every $A \in \mathcal{F}$, that is, the image $f(\mathcal{F})$ is a system of distinct representatives of $\mathcal{F}$.

The following elementary criterion, also known as the Frobenius–König theorem, allows us to efficiently count the matrices that do not have a 1-transversal. It is a consequence of Hall's theorem, but we include a proof for completeness.

Lemma 17 (see [32])**.** *Let $A = (A_{i,j}), i \in I, j \in J$ be a $k \times k$ binary matrix. Then A has no 1-transversal if and only if there exist $S \subseteq I$ and $T \subseteq J$ with $|S| + |T| > k$ such that the submatrix $A_{S \times T}$ is a zero matrix.*

Proof. For each column $j \in J$, let $R_j \subseteq I$ denote the set of rows i such that $A_{ij} = 1$. Observe that A has no 1-transversal if and only if $\{R_1, \dots, R_{|J|}\}$ has no system of distinct representatives. Furthermore, by Hall's marriage theorem, this is true if and only if there exists $T \subseteq J$ such that $|\bigcup_{j \in T} R_j| < |T|$.

Assuming the existence of such a set T, let $S := I \setminus \bigcup_{j \in T} R_j$, which is the set of rows that do not contain any 1 on any column of T. Then $A_{S \times T}$ is a zero matrix. Furthermore, $|S| + |T| > |I| - |T| + |T| = k$, as desired. Conversely, if $A_{S \times T}$ is a zero submatrix with $|S| + |T| > k$, then $|\bigcup_{j \in T} R_j| \leqslant |I \setminus S| = k - |S| < |T|$, as required. ∎

Given a probability space, the $\{0, 1\}$-valued random variables $X_1, \dots, X_n$ are *negatively correlated* if for each subset S of $\{1, \dots, n\}$,

$$\mathbb{P}(X_i = 1, \forall i \in S) \leqslant \prod_{i \in S} \mathbb{P}(X_i = 1).$$

Correspondingly, we say that a collection of events is negatively correlated if the random variables given by their indicator functions are negatively correlated. We will use the following Chernoff bound.

Theorem 18 (see [35])**.** *Let $X_1, \dots, X_n$ be $\{0, 1\}$-valued random variables, and let $X = \sum_{i=1}^{n} X_i$. If the variables $X_1, \dots, X_n$ are negatively correlated, then*

$$\forall \delta \in [0, 1], \quad \mathbb{P}(X \geqslant (1 + \delta)\mathbb{E}X) \leqslant e^{-\delta^2 \mathbb{E}X / 3}.$$

If the variables $1 - X_1, \dots, 1 - X_n$ are negatively correlated, then

$$\forall \eta \in [0, 1], \quad \mathbb{P}(X \leqslant (1 - \eta)\mathbb{E}X) \leqslant e^{-\eta^2 \mathbb{E}X / 2}.$$

In our applications of Theorem 18, we will sometimes use the following simple lemma.

Lemma 19. *Let $X_1, \dots, X_n$ be $\{0, 1\}$-valued random variables such that $\mathbb{P}(X_i = X_j = 1) = 0$ for all distinct $i, j \in \{1, \dots, n\}$. Then $X_1, \dots, X_n$ are negatively correlated and $1 - X_1, \dots, 1 - X_n$ are negatively correlated.*

Proof. It is immediate that $X_1, \dots, X_n$ are negatively correlated, so we only show that $1 - X_1, \dots, 1 - X_n$ are negatively correlated. Let S be a subset of $\{1, \dots, n\}$. Then for any subset $T \subseteq S$ of size at least two, $\mathbb{E}\left(\prod_{i \in T} X_i\right) = \mathbb{P}(X_i = 1, \forall i \in T) = 0$. Therefore

$$
\begin{aligned}
\mathbb{P}(1 - X_i = 1, \forall i \in S) &= \mathbb{E}\left[\prod_{i \in S}(1 - X_i)\right] \\
&= 1 - \sum_{i \in S} \mathbb{E} X_i + \sum_{i < j \in S} \mathbb{E}\left[X_i X_j\right] - \dots + (-1)^{|S|} \cdot \mathbb{E}\left[\prod_{i \in S} X_i\right] \\
&= 1 - \sum_{i \in S} \mathbb{E} X_i.
\end{aligned}
$$

As $\mathbb{E}(X_i) \in [0, 1]$ for all i, the right-hand side is a lower bound on $\prod_{i \in S}(1 - \mathbb{E} X_i)$, which is easily proved by induction as in the simpler case of Bernoulli's inequality. We conclude that $1 - X_1, \dots, 1 - X_n$ are negatively correlated because $1 - \mathbb{E} X_i = \mathbb{P}(1 - X_i = 1)$. ∎

We will use the following symmetric form of the Lovász local lemma due to Shearer.

Theorem 20 ([18, 39]). *Consider a set $\mathcal{E}$ of (bad) events such that for each $A \in \mathcal{E}$*

1 $\mathbb{P}(A) \leqslant p < 1$, *and*
2 A *is mutually independent of a set of all but at most d of the other events.*

If $epd \leqslant 1$, then with positive probability none of the events in $\mathcal{E}$ occur.

2 | ORDER

In this section we prove Theorem 2. We first prove a lemma that implies $\chi_\ell^\star(K_n) = n$.

Lemma 21. *Given $1 \leqslant k \leqslant n$, if L is a k-list-assignment of K_n such that every color belongs to at most k lists, then there is a proper L-packing.*

Proof of Lemma 21. We proceed by induction on k. The statement is trivially true in the base case $k = 1$. Assume $k > 1$ and that the statement holds with k replaced by $k - 1$.

Let V be the vertex set of K_n. Let us call a color *rich* if it belongs to exactly k lists, and write R for the set of rich colors. If there is a proper L-coloring c of K_n that uses all the rich colors then we are done, as then the $(k - 1)$-fold list-cover L' such that $L'(v) = L(v) \setminus \{c(v)\}$ for all $v \in V$ has the property that every color is used on at most $k - 1$ lists. By induction, there is a proper L'-packing of K_n, which we may combine with c to form a proper L-packing. So it suffices to construct the desired coloring c.

We start by showing first that some proper L-coloring c' of K_n exists. Such a coloring is precisely a system of distinct representatives for the family $\mathcal{F} = \{L(v) : v \in V\}$, and hence it suffices to show that the marriage condition holds. This entails showing that $|V'| \leqslant \left|\bigcup_{v \in V'} L(v)\right|$ for every $V' \subseteq V$. This holds by a simple counting argument: every list has size k, and so counting colors with multiplicity on lists of $v \in V'$ gives precisely $k|V'|$. But every color appears on at most k lists by assumption, so appears with multiplicity at most k, and hence the total number of colors counted without multiplicity is at least $|V'|$, as required to guarantee the desired c' by Hall's theorem.

If c' uses all the rich colors already then we are done, but if not we swap colors in c' to ensure this property. First, note that there is a partial proper L-coloring which uses every rich color by the following argument. For a rich color $r \in R$, let $V_r = \{v \in V : r \in L(v)\}$, and let $\mathcal{F}' = \{V_r : r \in R\}$. Now every set in $\mathcal{F}'$ has size k by the definition of R, and each $L(v)$ has size k. The marriage condition holds for $\mathcal{F}'$ by the same counting argument as above: given a subset $R' \subseteq R$, with multiplicity we count at least $k|R'|$ vertices whose lists contain colors in R', but the multiplicities are at most the list size k. Hall's theorem thus guarantees that there is an injective function $f : R \to V$ such that $r \in L(f(r))$ for each $r \in R$.

The desired coloring c is now obtained by doing the following. Initialize $c = c'$. Then set $c(f(r)) = r$ for every unused rich color $r \in R \setminus c(V)$. In doing so we removed the original color from $f(r)$, so this modification may lead to a new (and disjoint) set of unused rich colors. Therefore we repeat the process as long as necessary. That is, while there is a rich color r unused by c, we modify c by setting $c(f(r)) = r$. When this process terminates, c is a proper L-coloring of K_n which uses all the rich colors, as desired. ∎

For our next result, recall that K_n^- is the graph formed from K_n by removing an arbitrary edge.

Proof of Theorem 2. First observe that $\chi_\ell^\star$ is monotone under edge-deletion. That is, if G' is a subgraph of G, then $\chi_\ell^\star(G') \leqslant \chi_\ell^\star(G)$. Thus, for the bound it suffices to prove that $\chi_\ell^\star(K_n) \leqslant n$. In turn, this follows from Lemma 21 with $k = n$. For the equality statement, it suffices to show that $\chi_\ell^\star(K_n^-) \leqslant n - 1$, where $n \geqslant 3$.

Let V denote the vertex set of K_n^- and let u_1 and u_2 denote the two nonadjacent vertices in V. Let L be an $(n-1)$-list-assignment of K_n^-. If $L(u_1) = L(u_2)$, then we can treat u_1 and u_2 as the same vertex (since they may take the same color), and conclude by Lemma 21 for K_{n-1}. If there is no color that belongs to all n lists, then we are also done by Lemma 21.

Let there be exactly i colors belonging to all n lists. Without loss of generality, we may assume these are $[i]$, that is, $[i] \subseteq L(v)$ for all $v \in V$. We seek to define an i-list-assignment L_1 of K_n^- satisfying $L_1(v) \subseteq L(v)$ for all $v \in V$ such that the following properties hold: $L_1(u_1) = L_1(u_2) = [i]$; every color in $[i]$ belongs to at most $i+1$ lists $L_1(v)$ where $v \in V$; every color not in $[i]$ belongs to at most i lists $L_1(v)$ where $v \in V$; and the $(n-i-1)$-list-assignment $L \setminus L_1$ of K_n^- satisfies the hypothesis of Lemma 21 for $k = n-i-1$. If we can do this, then by treating u_1 and u_2 as the same vertex, we are guaranteed an L_1-packing of K_n^- by Lemma 21. We are also guaranteed a disjoint $L \setminus L_1$-packing of K_n^- by Lemma 21. Then these packings combine to form an L-packing of K_n^-, which completes the proof.

We prove that such an L_1 exists with two claims. First, we explain some additional terminology. We call a color $j \notin [i]$ *rich* if it belongs to more than $n-i-1$ lists $L(v)$ where $v \in V$. We write $\alpha(j)$ for the number of lists to which a color j belongs, and we call $s(j) = \max\{0, \alpha(j) - (n-i-1)\}$ the surplus of j. We write R for the set of rich colors, which by definition are the colors with positive surplus.

Claim 22. *We have $\sum_{j \in R} s(j) \leqslant i(n-i-1)$.*

Proof. Note that for $j \in R$ we have $s(j) \leqslant n - 1 - (n-i-1) = i$, as by assumption every $j \notin [i]$ belongs to fewer than n lists. So if $|R| \leqslant n-i-1$ the claim is immediate.

In any case, a simple color-counting argument gives

$$\sum_{j \in R} \alpha(j) \leqslant \sum_{j \notin [i]} \alpha(j) = n(n-1) - \sum_{j \in [i]} \alpha(j) = n(n-i-1),$$

because $[i] \subset L(v)$ for all $v \in V$. So in the case $|R| \geqslant n - i$, we have

$$\sum_{j \in R} s(j) = \sum_{j \in R} \alpha(j) - |R|(n-i-1) \leqslant n(n-i-1) - (n-i)(n-i-1) = i(n-i-1),$$

completing the proof of the claim. ∎

Note that in order for $L \setminus L_1$ to fulfil the condition of Lemma 21 for $k = n - i - 1$, we need that the color j belongs to at least $s(j)$ lists of L_1 for every rich color j. Thus the following claim proves the existence of the desired L_1.

Claim 23. For every $v \in V$, we can select a subset $L_1(v) \subseteq L(v)$ of size i in such a way that $L_1(u_1) = L_1(u_2) = [i]$, every color in $[i]$ belongs to $i + 1$ lists, every color not in $[i]$ belongs to at most i lists, and every $j \in R$ belongs to at least $s(j)$ lists $L_1(v)$ where $v \in V$.

Proof. The main part of this proof is to make a partial selection for L_1 from the colors not in $[i]$. In particular, we will make a total of $i(n - i - 1)$ selections of colors not in $[i]$ from the lists $L(v)$ where $v \in V \setminus \{u_1, u_2\}$, such that no color is selected more than i times, no vertex has more than i selected colors, and every $j \in R$ is selected at least $s(j)$ times. Once we have done this, since $[i] \subseteq L(v)$ for all $v \in V$, the remaining selection for L_1 of the colors in $[i]$ is straightforward: we take $L(u_1) = L(u_2) = [i]$, and for the remaining vertices we greedily/arbitrarily add colors of $[i]$ to the selection until $|L_1(v)| = i$ for all $v \in V$ and every color of $[i]$ has been added precisely $i - 1$ times. The resulting selection satisfies the desired conclusion. This means that it suffices to make such a selection for L_1 of the colors not in $[i]$. For this, we consider two cases.

First, suppose that $i \geqslant n - i - 1$. We make a somewhat arbitrary partial selection and then modify it. We start by taking an arbitary set A of i vertices in $V \setminus \{u_1, u_2\}$, letting $L_1(v) = L(v) \setminus [i]$ for $v \in A$, and letting $L_1(v) = \emptyset$ for all $v \in V \setminus \{u_1, u_2\} \setminus A$. Note that $i(n - i - 1)$ selections have been made, no color has been selected more than i times, and no vertex has more than i selected colors thus far. While there is a color $j \in R$ that belongs to fewer than $s(j)$ lists of L_1, we can select j for some $v \in V \setminus \{u_1, u_2\} \setminus A$ and deselect for some $v \in A$ some *unnecessary* color, that is, some selected color j' such that either $j' \in R$ and j' already appears more than $s(j')$ times or $j' \notin R$. Note that at each step of this process, the total number of selections is preserved, the selected color j is still selected at most $s(j) \leqslant i$ times in total, and still each vertex has at most $|L(v) \setminus [i]| = n - 1 - i \leqslant i$ selected colors. Moreover, Claim 22 ensures that this process will terminate, at which point we have the desired partial selection for L_1.

Second, suppose that $i < n - i - 1$. Here we make a more careful selection before modifying it. We start by taking some set A of $n - i - 1$ vertices in $V \setminus \{u_1, u_2\}$ and consider the $(n - i - 1)$-list-assignment L' of the complete graph on vertex set A that is defined by $L'(v) = L(v) \setminus [i]$ for all $v \in A$. By Lemma 21, there is a proper L'-packing of the complete graph on A. Define a partial selection for L_1 by arbitrarily taking i of the proper L'-colorings from this L'-packing and taking their union. Again $i(n - i - 1)$ color

selections have been made, no vertex has more than i selected colors and, since we have taken i disjoint proper L'-colorings of a complete graph, no color has been selected more than i times so far.

In what follows, we will modify the partial selection L_1 iteratively, by selecting and deselecting some vertices, while at each step maintaining the above three constraints on the number of selections. The goal is of course to make sure that L_1 in the end also satisfies the fourth constraint, that every $j \in R$ is selected at least $s(j)$ times.

At each step, we distinguish between colors in L_1 that are merely selected for their vertex, and colors that are also *fixed*. If a color has been fixed for a vertex, we do not allow it to be deselected from that vertex later in the process. Throughout the process, we will only fix rich colors, and each rich color $j \in R$ will be fixed at most $s(j)$ times, thus ensuring that the total number of times we fix a color does not exceed $\sum_{j \in R} s(j)$.

For the first round of modifications, we first focus our attention on the set R_1 of rich colors that belong to at most $n - i - 2$ lists $L(v)$ where $v \in V \setminus \{u_1, u_2\}$. A color j only belongs to R_1 when $\alpha(j) = n - i$ (so $s(j) = 1$) and $j \in L(u_1) \cap L(u_2)$. Since $L(u_1) \neq L(u_2)$, $|R_1| \leqslant n - i - 2$. But since all of these colors in fact belong to exactly $n - i - 2$ lists $L(v)$ for $v \in V \setminus \{u_1, u_2\}$, there exists some injective function $w : R_1 \to V \setminus \{u_1, u_2\}$ such that $L(w(j)) \ni j$ for all $j \in R_1$. We now use the function w to iteratively modify L_1 so that every color in R_1 is selected (at least) once. More fully, let us consider every $j \in R_1$ and do the following in order, where we write $j \in L_1$ if the color j has been selected under (the current) L_1 for at least one vertex.

1. For all $j \in R_1$ such that $w(j) \in A$ and $j \in L_1(w(j))$, we fix color j for $L_1(w(j))$.
2. Iteratively, for all $j \in R_1$ such that $w(j) \in A$, but $j \notin L_1$, we select and fix j for $w(j)$, and deselect an arbitrary color from $L_1(w(j))$.
3. For each remaining $j \in R_1$ such that $w(j) \in A$ we have that $j \in L_1$ but not $j \in L_1(w(j))$. We fix j for one arbitrary selection under L_1.
4. Each remaining $j \in R_1$ satisfies $w(j) \notin A$. We (select and) fix color j for vertex $w(j)$. If as a result of this, there are now more than i vertices for which color j has been selected, we deselect j for one of them (for which j is not fixed). Otherwise we deselect an arbitrary nonfixed color.

After this first round of modifications, every color in R_1 has exactly one fixed selection under L_1. Note that the four modification steps above are rather delicate because we need to preserve the total number of selections, and the properties that no vertex has more than i selected colors and that no color has been selected more than i times.

Now we consider the set $R \setminus R_1$ of rich colors that belong to at least $n - i - 1$ lists $L(v)$ where $v \in V \setminus \{u_1, u_2\}$. We repeat the following until we can no longer do so. Take some $j \in R \setminus R_1$ that currently has been selected by L_1 for fewer than $s(j)$ vertices. Let us *fix* all of its current selections. Let $w'(j) \in V \setminus \{u_1, u_2\}$ be some vertex such that $L(w'(j)) \ni j$ and $w'(j)$ does not have i selected colors fixed. The existence of $w'(j)$ is guaranteed by the fact that j belongs to at least $n - i - 1$ lists and Claim 22.

(Indeed, suppose such $w'(j)$ does not exist. Then there must be at least $n - i - 1$ vertices $w \in V \setminus \{u_1, u_2\}$ with $j \in L(w)$ and with at least i colors fixed for w. So there are at least $i(n - i - 1)$ fixed colors in total. On the other hand, by assumption fewer than $s(j)$ vertices have been fixed for j, so in total we have fixed at most $-1 + \sum_{h \in R} s(h)$ colors, which is strictly less than $i(n - i - 1)$ by Claim 22, a contradiction.)

As before, we select j for $w'(j)$, *fixing* this choice, and deselect some other (nonfixed) color, possibly within the list of $w'(j)$ (if we need to ensure that $w'(j)$ has no more than i selected colors). This procedure will terminate due to Claim 22, at which point we will have the desired partial selection for L_1. ∎

This completes the proof that there exists such an L_1, thus guaranteeing the desired L-packing. ∎

3 | DEGENERACY

In this section we give the argument for Theorem 9, which also yields Theorem 3. This greedy approach with Hall's theorem is given by MacKeigan [31] and discussed by Yuster [42] as a way of proving that $\chi_c^\star(K_n) \leqslant 2(n-1)$. Although Yuster proved that the leading constant '2' there is not tight, we show with Proposition 24 that when the method is refined and stated in terms of degeneracy it is indeed tight. We also give an example showing that the bound in Theorem 3 cannot be improved to $\delta^\star(G) + 1$.

Proof of Theorem 9. Let G be a graph with degeneracy $\delta^\star(G) = d$, and order the vertices $V(G)$ as $v_1, \dots, v_n$ such that no vertex has more than d neighbors preceding it in the order. That is, in each of the induced subgraphs $G_i := G[\{v_1, \dots, v_i\}]$, the degree of v_i is at most d. Let (L, H) be a k-fold correspondence-cover of G, where $k = 2d$. We prove by induction and Hall's theorem that G admits a proper L-packing. Write H_i and L_i for the correspondence-cover and associated correspondence-assignment induced on G_i by H and L, respectively. Note that trivially G_1 admits a correspondence L_1-packing. So for the induction let $i > 1$ and assume that G_{i-1} admits a correspondence L_{i-1}-packing. This packing is equivalent to an ordered list $(c_1, \dots, c_k)$ of pairwise-disjoint proper L_{i-1}-colorings of G_{i-1}. We want to extend each of these colorings to the vertex v_i so that they remain disjoint and are proper L_i-colorings of G_i. Since v_i has degree at most d in G_i, each color $x \in L(v_i)$ is a possible valid extension for at least $|L(v_i)| - d = d$ of the colorings $c_1, \dots, c_k$ to the vertex v_i. For each $x \in L(v_i)$, let $\Xi(x)$ be the set of such colorings (so $|\Xi(x)| \geqslant d$ for all x). Conversely, for each $1 \leqslant j \leqslant k$ there are also at least d colors in $L(v_i)$ that can be used to extend the coloring c_j, so $|\Xi^{-1}(c_j)| \geqslant d$ for all j. Consider the family $\mathcal{F} = \{\Xi(x) \mid x \in L(v_i)\}$. Using the facts that $|\Xi(x)| \geqslant d$ for all x and $|\Xi^{-1}(c_j)| \geqslant d$ for all j, one can verify that the marriage condition holds. Let f be the injective function guaranteed by Hall's theorem. This gives a one-to-one correspondence between the colorings $c_1, \dots, c_k$ and colors in $L(v_i)$ that may extend them to the vertex v_i. We obtain the claimed correspondence L_i-packing of G_i, as desired. In particular, this holds for $i = n$. ∎

Proposition 24. *For every $d \geqslant 2$, there exists some graph G satisfying $\delta^\star(G) = d$ and $\chi_c^\star(G) = 2d$. In particular, this is true for the complete bipartite graph $K_{d,((2d-1)!)^{d-1}}$.*

Proof. Let G be $K_{d,((2d-1)!)^{d-1}}$, with parts A and B of size d and $((2d-1)!)^{d-1}$, respectively. This graph has degeneracy d, so by Theorem 9 we need to show that it satisfies $\chi_c^\star(G) \geqslant 2d$. We form a $(2d-1)$-fold correspondence-cover of G with respect to L by including every possible combination of d matchings running between d copies of $[2d-1]$ and a single copy of $[2d-1]$. Then, whatever the partial L-packing of size $2d-1$ induced on A, there is guaranteed to be a vertex $b \in B$ and d of the proper L-colorings such that

d of the colors of $L(b)$ are each matched to d colors of $\cup_{a \in A} L(a)$ that belong to each of the d proper L-colorings. And thus to extend the partial L-packing induced on A, those d colors of $L(b)$ may only be used to extend $d - 1$ of the proper L-colorings, which is clearly impossible. ∎

Having established this tight characterization of $\chi_c^{\star}$ in terms of degeneracy, it is natural to wonder what the right bound for list packing could be. We conclude this section with a modest lower bound on the extremal list packing number in terms of degeneracy. In light of Proposition 24, we note on the other hand that (complete) bipartite graphs cannot certify a better lower bound for $\chi_\ell^{\star}(G)$, see Lemma 33 below.

Theorem 25. *For every $d \geqslant 2$, there exists a graph G with $\delta^{\star}(G) = d$ and $\chi_\ell^{\star}(G) \geqslant d + 2$.*

Proof. We will iteratively construct a graph G with $\delta^{\star}(G) = d$ and a $(d + 1)$-list-assignment L such that G does not admit a proper L-packing. We will do so by constructing a sequence of subgraphs $G_1, G_2, \ldots, G$ such that $V(G_1) \subset V(G_2) \subset \ldots \subset V(G)$. We will start with a list-assignment and arbitrary list-packing of G_1 (see below for more details). Next, for each G_{j+1} apart from the final graph G, we consider the list-packing $C := (c_i)_{1 \leqslant i \leqslant d+1}$ of G_j and we choose the lists of $V(G_{j+1}) \setminus V(G_j)$ such that there is a unique extension of C to G_{j+1}. For the final graph G, we choose the remaining lists such that C cannot be extended.

We start with choosing $G_1 = (V_1, E_1) = K_{d+1}$ and the associated lists being equal to $[d + 1]$ for all its vertices. Note that for every $v \in G_1$, the vector of colors $C(v) = (c_i(v))_{1 \leqslant i \leqslant d+1}$ is a permutation of $[d + 1]$. We now construct G_2 by adding a copy v' for every $v \in V_1$ that is connected to all vertices in $V_1 \setminus v$. Let $V_2 = V(G_2) \setminus V_1$. We let $L(v') = [d + 1] \setminus \{1\} \cup \{d + 2\}$. Note that when extending the L-packing, the vector $C(v')$ will be equal to $C(v)$ where 1 is replaced by $d + 2$, as every color $2 \leqslant j \leqslant d + 1$ can only be used for one of the L-colorings c_i.

We will repeat this procedure. In step m, we add copies v' for every $v \in V_m$ and connect it to all vertices in $V_m \setminus v$. we call the set of added vertices V_{m+1}. For $v' \in V_{m+1}$, we let $L(v') = s_{ij}(L(v)) = L(v) \setminus \{i\} \cup \{j\}$ for some $i, j \in [d + 2]$, that is, an (i, j)-shift is applied to the lists. Here we set $V_m = \{v_1^m, \ldots, v_{d+1}^m\}$, where v_i^{m+1} denotes the copy of v_i^m.

By choosing the shifts to be $s_{1,d+2}, s_{2,1}, s_{d+2,2}$ in the first three steps, we see that in the L-packing, the permutations $C(v_i^1)$ and $C(v_i^4)$ are equal up to an interchange of 2 and 1, that is, the permutation (12) has been applied to them. We can repeat this procedure and form the permutation $(123 \ldots d)$ as it is equal to $(12), (13), \ldots, (1d)$, a sequence of $d - 1$ transpositions. So in the L-packing, $C(v_i^1)$ and $C(v_i^{3(d-1)+1})$ are two permutations that are equal up to performing the cyclic permutation $(123 \ldots d)$.

Now continue doing this another $d - 2$ times. Now $C(v_i^{3p(d-1)+1})$ for $0 \leqslant p \leqslant d - 1$ are all permutations of $[d + 1]$ and every color in $[d]$ has been used at all spots except one (where $d + 1$ is placed every time). Now add a final vertex x and connect it to $v_1^{3p(d-1)+1}$ for every $0 \leqslant p \leqslant d - 1$, and let $L(x) = [d + 1]$ as well. A proper L-packing of this final graph is impossible.

In all steps, we connected new vertices to exactly d existing vertices and so the degeneracy of the construction satisfies $\delta^{\star}(G) = d$. ∎

4 | MAXIMUM DEGREE

In this section, we give the argument that proves Theorems 4 and 10. The proof closely follows an argument of Aharoni, Berger and Ziv for a result on strong coloring [1]; see also the survey of Haxell [26]. We only give the argument for correspondence packing as the proof for list packing is essentially the same. We note that Loh and Sudakov [30] outlined this same argument in a slightly more general context, namely, in showing that $\Lambda_c^\star(D) \leqslant 2D + o(D)$ as $D \to \infty$.

Proof of Theorem 10. Let $k := 1 + \Delta(G) + \chi_c(G)$. Let H be a k-fold correspondence-cover of G, via some correspondence-assignment $L : V(G) \to 2^{V(H)}$. Recall that this means that the vertices of H are partitioned into parts $(L(v))_{v \in V(G)}$ such that $|L(v)| = k$ for all $v \in V(G)$, and if $vw \in E(G)$ there is a (possibly empty) matching between $L(v)$ and $L(w)$, and for every $vw \notin E(G)$ there is no edge between $L(v)$ and $L(w)$.

We need to show that G admits a correspondence L-packing. Equivalently, we need to find a proper k-coloring of H such that for every $v \in V(G)$, no two vertices of H in the same part $L(v)$ receive the same color (recall that each color class of such a k-coloring corresponds to a single L-coloring in the desired correspondence L-packing). To this end, let α be a *maximum partial coloring* of H, that is, a proper k-coloring of the vertices of H using as many vertices of H as possible. If α assigns a color to every vertex of H then we are done. So suppose for contradiction that α does not assign a color to some vertex x of H in some part $L(v_1)$. Then some color in $\{1, \dots, k\}$ is missing from $L(v_1)$; let us call it *red*.

Let $v_1, \dots, v_n$ denote the vertices of G. For each $2 \leqslant i \leqslant n$, let r_i denote the red vertex in $L(v_i)$ if it exists, and define $L'(v_i) := L(v_i) \setminus \{y : \alpha(y) = \alpha(z) \text{ for some } z \in N_H(r_i)\}$. Thus, $L'(v_i)$ consists of those vertices of $L(v_i)$ whose color could be reassigned to r_i without creating a conflict. Also let $L'(v_1) = L(v_1)$. Then $|L'(v_i)| \geqslant \chi_c(G) + 1$ for every i.

Next let $L''(v_i) = L'(v_i) \setminus N_H(x)$. Since x has at most one neighbor in each $L(v_i)$, it follows that $|L''(v_i)| \geqslant \chi_c(G)$ for each $1 \leqslant i \leqslant n$, and thus by definition of correspondence coloring, $(L''(v_i))_{1 \leqslant i \leqslant n}$ has an independent transversal T''. By replacing the representative of T'' in $L''(v_1)$ with x, this in turn yields an independent transversal T' of $(L'(v_i))_{1 \leqslant i \leqslant n}$ which contains x. Next, we modify α by giving color red to every vertex of T', and for each i for which r_i exists and $r_i \notin T'$ we give r_i the color of the element t_i of T' in $L(v_i)$. By definition of $L'(v_i)$, this does not create any conflict, so we obtain a valid k-coloring β. Moreover, β colors x as well as all the vertices that were colored by α, contradicting the maximality of α. ∎

5 | ON THE PERMANENT OF RANDOM BINARY MATRICES

In this section, we describe and derive auxiliary results on transversals and permenants of random binary matrices. These are our key tools for the proofs of Theorems 5 and 6. Related ideas are also useful for Theorem 7, which we elaborate upon in Section 7.

Our starting point is a theorem of Everett and Stein [20] from 1973. For the benefit of the reader, we will use this below in a warm-up to prove the special case for Theorem 5 of list packing bipartite graphs with n vertices (Theorem 29).

Theorem 26 ([20]). *The fraction of binary $k \times k$ matrices A with $\mathrm{Per}(A) = 0$ is $(1 + o(1))k2^{1-k}$, as $k \to \infty$.*

Theorem 26 can be interpreted probabilistically: a uniformly random binary matrix has permanent 0 with probability $(1 + o(1))k2^{1-k}$ as $k \to \infty$. For the proofs of Theorems 5 and 6, we will need a more general result that allows for an asymmetric treatment of the ones and zeroes. The following is a generalization of Theorem 26.

Theorem 27. *Let $0 \leqslant p < 1$ be a real number. Let A be a random binary $k \times k$-matrix for which each entry independently equals 0 with probability p, and 1 with probability $1 - p$. Then*

$$\text{Per}(A) = 0 \quad \text{with probability} \quad 2kp^k(1 + o(1)) \quad \text{as} \quad k \to \infty.$$

Moreover, the same conclusion holds if p is allowed to depend on k, provided $k^5p^k \to 0$ as $k \to \infty$. In particular this holds if $p \leqslant 1 - \frac{5\log k + \omega(1)}{k}$ as $k \to \infty$.

In the proof of Theorem 27, we will use the following technical lemma.

Lemma 28. *Let $0 \leqslant p < 1$ be such that $k^5p^k \to 0$ as $k \to \infty$. Then as $k \to \infty$,*

$$\max_{\substack{s,t \in \mathbb{N} \text{ such that} \\ 2 \leqslant s \leqslant t \leqslant k \text{ and} \\ s+t \geqslant k}} \left\{ \binom{k}{s}\binom{k}{t}p^{st} \right\} = o\left(\frac{1}{k}p^k\right).$$

Proof. Let $2 \leqslant s \leqslant t \leqslant k$ be such that $s + t \geqslant k$.
If $s \geqslant \frac{k}{2}$, we have

$$\binom{k}{s}\binom{k}{t}p^{st} < 4^k p^{k^2/4} = o\left(\frac{1}{k}p^k\right).$$

If $2 \leqslant s < \frac{k}{2}$, then

$$\begin{aligned}
\frac{k}{p^k}\binom{k}{s}\binom{k}{t}p^{st} &\leqslant \frac{k}{p^k}\binom{k}{s}\binom{k}{k-s}p^{s(k-s)} \\
&\leqslant k^{2s+1}p^{(s-1)k-s^2} \\
&= \left(k^5p^k\right)^{\frac{2s+1}{5}} \cdot p^{\frac{3k(s-2)}{5} - s^2} \\
&\leqslant k^5p^k \cdot p^{\frac{3(2s+1)(s-2)}{5} - s^2} \\
&\leqslant k^5p^{k-6} = o(1)
\end{aligned}$$

as $k \to \infty$. ∎

Proof of Theorem 27. By Lemma 17, if $\text{Per}(A) = 0$ then there are sets S and T with s and t elements respectively for which $s + t > k$ and the submatrix $B = A_{S \times T}$ only contains zeros. In particular, we have such a submatrix B with $s + t = k + 1$. Let us call such B a *t-bad submatrix*.

Given s and t, there are $\binom{k}{s}\binom{k}{t}$ choices for the sets S, T. The probability that $A_{i,j} = 0$ for all $i \in S, j \in T$ equals p^{st}. So the probability that there is a *t*-bad submatrix is upper bounded by $\binom{k}{k+1-t}\binom{k}{t}p^{(k+1-t)t}$.

We now take the union bound over all $1 \leqslant t \leqslant k$. The cases $t = 1$ and $t = k$ each contribute kp^k, while each $2 \leqslant t \leqslant k - 1$ contributes $o\left(\frac{1}{k}p^k\right)$ by Lemma 28. Hence the probability that there exists a t-bad submatrix for some $1 \leqslant t \leqslant k$ is at most $2kp^k + (k - 2) \cdot o\left(\frac{1}{k}p^k\right) = (1 + o(1))2kp^k$, as desired.

On the other hand, we know that any matrix with a row or column with only zeros has permanent equal to 0. Using a truncated version of the principle of inclusion–exclusion, that is, subtracting the probabilities that are counted twice, we note that the probability that this happens is at least $2kp^k - (2\binom{k}{2}p^{2k} + k^2p^{2k-1}) = (1 - o(1))2kp^k$, as desired. ∎

Remark. In Theorem 27, the opposite random variables $1 - A_{i,j}$ are independent and Bernoulli(p) distributed. We note that the *upper* bound $2kp^k(1 + o(1))$ in Theorem 27 remains true with the same proof if instead of independence, we only require that the random variables $1 - A_{i,j}$ are negatively correlated. Indeed, in this case we still have for each subset of rows S and columns T of A that the probability that $A_{i,j} = 0$ for all $i \in S, j \in T$ is at most (rather than equal to) p^{st}, which is sufficient.

6 | FRACTIONAL CHROMATIC NUMBER AND HALL RATIO

In this section we prove Theorems 5 and 6, using the results on random binary matrices from Section 5. As a warm-up using the classic result of Everett and Stein, we first derive the special case of Theorem 5 for bipartite graphs.

Theorem 29. *Let G be a bipartite graph on n vertices. Then as $n \to \infty$,*

$$\chi_\ell^\star(G) \leqslant (1 + o(1)) \log_2 n.$$

Proof. Let $\varepsilon > 0$ and set k to be an integer that is at least $(1+\varepsilon) \log_2 n$. Let L be an arbitrary k-list-assignment for G.

We consider the union of all lists, $\mathcal{L} = \bigcup_{v \in V(G)} L(v)$, and color each list element $\ell \in \mathcal{L}$ independently with a uniformly random vector $x_\ell \in \{0, 1\}^k$. We use these vectors to associate a random binary matrix with each vertex of G, as follows. For each $v \in V(G)$, let $M(v)$ be a $k \times k$ matrix formed by concatenating the vectors $(x_\ell)_{\ell \in L(v)}$ in some arbitrary way.

Let $V_0, V_1 \subseteq V(G)$ denote the two parts of the bipartition of G. Now, for each vertex $v \in V(G)$ we define a *bad event* $\mathcal{B}(v)$ as follows: if $v \in V_0$, then $\mathcal{B}(v)$ is the event that $M(v)$ has no 0-transversal, and similarly if $v \in V_1$, then $\mathcal{B}(v)$ is the event that $M(v)$ has no 1-transversal.

If there are no bad events, then for each $v \in V_0$ we can choose a 0-transversal of $M(v)$, and for each $v \in V_1$ we can choose a 1-transversal of $M(v)$. These transversals then determine the desired k disjoint L-colorings $c_1, c_2, \ldots, c_k$ as follows: for $1 \leqslant i \leqslant k$ and $v \in V(G)$, we set $c_i(v)$ to be the element of $L(v)$ that corresponds to the i-th element of the transversal of $M(v)$. Since we choose the colors of v according to a transversal of $M(v)$, we have $c_i(v) \neq c_j(v)$ for any $i \neq j$, so the colorings $c_1, \ldots, c_k$ are indeed disjoint. Furthermore, for each i the coloring c_i is proper because the color sets on V_0 and V_1 are

disjoint; indeed, any vertex $v \in V_0$ (resp. $v \in V_1$) is colored by an element $l \in \mathcal{L}$ such that the i-th element of x_l is 0 (resp. 1).

Thus, it suffices to find a coloring without bad events. For $v \in V_1$, the probability of $\mathcal{B}(v)$ equals the probability that a uniformly random binary matrix has no 1-transversal. By Theorem 26, the number of binary $k \times k$ matrices that have no 1-transversal is at most $(1 + o(1))k2^{k^2-k+1}$ as $n \to \infty$. Hence, $\mathbb{P}(\mathcal{B}(v)) \leqslant (1 + o(1)) \cdot \frac{k \cdot 2^{k^2-k+1}}{2^{k^2}} = (2 + o(1))k2^{-k}$. By symmetry, the same bound holds for every $v \in V_0$. Therefore, by a union bound, the probability that at least one bad event holds is at most $n \cdot (2 + o(1))k2^{-k}$, and for n large enough this is strictly smaller than 1. Thus, there exists a coloring of $\mathcal{L}$ without bad events and hence the desired L-packing. ∎

We now proceed to prove Theorems 5 and 6, thus extending the result on bipartite graphs (Theorem 29) to graphs with bounded fractional chromatic number or bounded Hall ratio. This includes the special case of multipartite graphs and generalizes bounds on χ_ℓ from [15] and [4].

Theorem 30. *The graphs G with n vertices and at least one edge satisfy as $n \to \infty$*

$$\chi_\ell^\star(G) \leqslant (5 + o(1)) \frac{\log n}{\log(\chi_f(G)/(\chi_f(G) - 1))} \leqslant (5 + o(1)) \cdot \chi_f(G) \log n.$$

Moreover, for every fixed rational number m, the graphs G with $\chi_f(G) = m$ and n vertices and at least one edge satisfy as $n \to \infty$

$$\chi_\ell^\star(G) \leqslant (1 + o(1)) \cdot \frac{\log n}{\log(m/(m - 1))} \leqslant (1 + o(1)) \cdot m \log n.$$

Proof. We give the proof for the first bound (where $\chi_f(G)$ is allowed to grow with n), and at the end we detail why almost the same proof also yields the second bound (for $\chi_f(G)$ bounded by a constant independent of n).

We write $m := \chi_f(G)$. Since the clique number is a lower bound for fractional chromatic number and G has at least one edge, we have $m \geqslant 2$.

Let $\varepsilon > 0$ and consider a k-list-assignment L with $k = \lfloor \frac{5+\varepsilon}{\log(m/(m-1))} \cdot \log n \rfloor$. We may assume that $k < n$, since otherwise there is an L-packing due to Theorem 2. We also note that $k \to \infty$ if and only if $n \to \infty$; this is where we use that $m \geqslant 2$.

Let $\mathcal{L} = \bigcup_{u \in V(G)} L(u)$ be the set of all colors. For a positive integer a, recall that $[a]$ is the set $\{1, \ldots, a\}$. Let c be a proper (a, b)-coloring of G with $\frac{a}{b} = m$. By definition, this means that $c : V(G) \to [a]^b$ is a function such that $c(u) \cap c(v) = \emptyset$ for every two adjacent vertices u and v. For every $\ell \in \mathcal{L}$, let x_ℓ be a uniformly random vector in $[a]^k$. For each $v \in V(G)$, let $M(v)$ be a $k \times k$ matrix formed by concatenating the vectors $(x_\ell)_{\ell \in L(v)}$ in some arbitrary way.

For every vertex $v \in V(G)$, we define the bad event $\mathcal{B}(v)$ as the event that the $[a]$-valued $k \times k$ matrix $M(v)$ has no transversal with only elements from $c(v)$. Consider the binary matrix $M^*(v)$ obtained from $M(v)$ by replacing every element that belongs to $c(v)$ with a 1, and every other element with a 0.

Note that $M^*(v)$ is a random binary $k \times k$ matrix for which each element independently equals 0 with probability $p = \frac{a-b}{a} = \frac{m-1}{m}$. Therefore, we wish to apply Theorem 27 to

$M^*(v)$ with $p = \frac{m-1}{m}$. For that, we first need to verify the condition that $k^5 p^k = o(1)$ as $k \to \infty$. This is satisfied because

$$k^5 \cdot p^k < n^5 \cdot \left(\frac{m-1}{m} \right)^{\frac{(5+\varepsilon)\log n}{\log(m/(m-1))} - 1}$$

$$\leqslant n^5 \cdot \exp\left(\frac{(5+\varepsilon)\log n}{\log(m/(m-1))} \cdot \log\left(\frac{m-1}{m} \right) \right) \cdot 2$$

$$= n^5 \cdot n^{-(5+\varepsilon)} \cdot 2$$

goes to 0 as $n \to \infty$.

So we may indeed apply Theorem 27 to $M^*(v)$, which yields that M^* has no 1-transversal with probability $2kp^k(1 + o(1))$. Therefore the probability that $\mathcal{B}(v)$ occurs is $2kp^k(1 + o(1))$, as $k \to \infty$.

By a union bound, the probability that at least one bad event occurs is at most

$$n \cdot 2k\left(\frac{m-1}{m} \right)^k (1 + o(1)) < 2n^2 \cdot \left(\frac{m-1}{m} \right)^{\frac{(5+\varepsilon)\log(n)}{\log(m/(m-1))} - 1} \cdot (1 + o(1))$$

$$\leqslant 4n^2 \cdot \exp\left(\frac{(5+\varepsilon)\log(n)}{\log(m/(m-1))} \cdot \log\left(\frac{m-1}{m} \right) \right) \cdot (1 + o(1))$$

$$\leqslant \frac{4}{e \cdot n^3} \cdot (1 + o(1)).$$

For n large enough (and hence for k large enough), this is smaller than 1. So with positive probability, every $M(v)$ has a transversal with only elements from $c(v)$.

These transversals then determine the desired k disjoint L-colorings $c_1, c_2, \ldots, c_k$ as follows: for $1 \leqslant i \leqslant k$ and $v \in V(G)$, we set $c_i(v)$ to be the element of $L(v)$ that corresponds to the i-th element of the transversal of $M(v)$. Since we choose the colors of v according to a transversal of $M(v)$, we have $c_i(v) \neq c_j(v)$ for any $i \neq j$, so the colorings $c_1, \ldots, c_k$ are indeed disjoint. Furthermore, for each i the coloring c_i is proper because for any two neighbors $u, v \in V(G)$, they are colored by elements $\ell_u, \ell_v \in \mathcal{L}$ such that the i-th element of x_{ℓ_u}, x_{ℓ_v} belongs to $c(u)$ and $c(v)$ respectively, which are disjoint since c is a proper (a, b)-coloring.

This concludes our proof that $\chi_\ell^\star(G) \leqslant (5 + o(1)) \cdot \frac{\log n}{\log(\chi_f(G)/(\chi_f(G)-1))}$. We remark that the condition of Theorem 27 is the main obstruction for finding a better constant; if it were not for that part of the argument, we could have improved the factor $5 + o(1)$ to $2 + o(1)$. To confirm that $5 + o(1)$ *can* be improved to $1 + o(1)$ in the special case that $\chi_f(G) = m$ is a constant independent of n, we need to check that our two asymptotic estimates survive if we instead choose list size $k = \lfloor \frac{(1+\varepsilon)\log n}{\log(m/(m-1))} \rfloor$: the condition of Theorem 27 indeed still holds because

$$k^5 \cdot p^k \leqslant \left(\frac{(1+\varepsilon)\log n}{\log(m/(m-1))} \right)^5 \cdot n^{-(1+\varepsilon)} \cdot 2 \to 0$$

as $n \to \infty$, and so the probability that a bad event occurs is bounded from above by

$$n \cdot 2k \cdot \left(\frac{m-1}{m} \right)^k (1 + o(1)) \leqslant 4n \cdot \frac{(1+\varepsilon)\log n}{\log(m/(m-1))} \cdot n^{-(1+\varepsilon)}(1 + o(1)) \to 0$$

as $n \to \infty$. $\blacksquare$

As an almost immediate corollary, we can derive the following bound, which is reminiscent of a similar result for χ_ℓ instead of $\chi_\ell^\star$ in [34].

Corollary 31. *Let G be a graph with order n and Hall ratio ρ. Then*

$$\chi_\ell^\star(G) \leqslant (5 + o(1)) \cdot \rho \log^2 n.$$

Proof. Iteratively, one can select an independent set of size at least $1/\rho$ times the order of the remaining graph. Since $n\left(1 - \frac{1}{\rho}\right)^{\rho \log n} < 1$, we have $\chi(G) \leqslant \rho \log n$. Hence by Theorem 30, we can conclude as $\chi_\ell^\star(G) \leqslant (5 + o(1)) \cdot \chi(G) \log n$. ∎

As a corollary of Theorem 1.3 in [15] and Theorem 30, we immediately get the following Ramsey-type result.

Corollary 32. *There exists a constant $C > 0$ such that $\chi_\ell^\star(G) \leqslant C \cdot \min\{\sqrt{n \log n}, \frac{n \log n}{d}\}$ for every triangle-free graph G with n vertices and minimum degree d.*

7 | BIPARTITE GRAPHS

This section is devoted mainly to the proof of Theorem 11, which in turn implies Theorem 7. Before that though, we start with a simple bound on the list packing number of a bipartite graph that resembles the greedy bound (for not-necessarily-bipartite graphs) on list chromatic number but has a somewhat more intricate proof.

Lemma 33. *Let $G = (A \cup B, E)$ be a bipartite graph with parts A and B having maximum degrees Δ_A and Δ_B, respectively, with $\Delta_A \leqslant \Delta_B$. Then $\chi_\ell^\star(G) \leqslant \Delta_A + 1$.*

Proof. Let $k = \Delta_A + 1$, and let L be any k-list-assignment of G. To construct a proper L-packing of G, we first choose the colors of vertices in B for every coloring in the packing. To do this, consider the lists of vertices in B in their numerical orderings and let $c_i(b)$ be the i-th color in the list $L(b)$ for each $1 \leqslant i \leqslant k$ and each $b \in B$. This is the unique proper L-packing of $G[B]$ such that the colorings c_i for $1 \leqslant i \leqslant k$ satisfy for all vertices b in B, $c_1(b) < c_2(b) < \cdots < c_k(b)$.

Next, we prove that we can extend this partial L-packing to A by applying Hall's marriage theorem. Let a be an arbitrary vertex in A. For every color $j \in L(a)$, let I_j be the set of indices $1 \leqslant i \leqslant k$ such that setting $c_i(a) = j$ retains the property that c_i is a proper L-coloring (i.e., no neighbor b of a has $c_i(b) = j$). Consider the family $\mathcal{F} = \{I_j : j \in L(a)\}$. A system $f(\mathcal{F})$ of distinct representatives for $\mathcal{F}$ is precisely an extension of the partial L-packing to a, as we can set $c_{f(I_j)}(a) = j$ for all $j \in L(a)$. For the marriage condition, it suffices to prove that for every $a \in A$ and any subset $J \subseteq L(a)$, we have $\left|\cup_{j \in J} I_j\right| \geqslant |J|$. Suppose for a contradiction that this is not the case. Then there is some J with $\left|\cup_{j \in J} I_j\right| \leqslant |J| - 1$, and hence there is a set Z of $k - (|J| - 1)$ indices i for which $c_i(a)$ cannot be set equal to any color in J. Let $z^\star$ be the largest index in Z and $j^\star$ be the largest color in J. There are at least $k - (|J| - 1)$ neighbors b of a which have $c_z(b) = j^\star$ for some $z \in Z$. Furthermore, by the choice of the colorings c_i these are different from the $|J| - 1$ neighbors b that satisfy $c_{z^\star}(b) = j$ for $j \in J \setminus \{j^\star\}$.

To see that they are indeed different, suppose for a contradiction that there exists $b \in B$ such that both $c_z(b) = j^\star$ and $c_{z^\star}(b) = j$, for some $z \in Z$ and $j \in J \setminus \{j^\star\}$. As $j < j^\star$ we must have $z \neq z^\star$, so by the choice of the colorings c_i it would follow that $c_z(b) < c_{z^\star}(b) = j < j^\star = c_z(b)$, a contradiction.

We conclude that a has at least $k > \Delta_A$ neighbors in B, which is a contradiction. As we can perform this extension for all $a \in A$ independently, this completes the proof. ∎

Corollary 34. *When a is sufficiently large in terms of b, we have $\chi_\ell^\star(K_{a,b}) = b + 1$, while $\chi_c^\star(K_{a,b}) = 2b$.*

Proof. This follows from Theorem 9, Proposition 24 and Lemma 33, together with the classic fact that $\chi_\ell(K_{b^b,b}) \geqslant b + 1$ for every b. To see the latter: assign b disjoint lists $L_1, \ldots, L_b$ of size b to the vertices in the smaller part of the bipartition of $K_{b^b,b}$, and then assign each of the b^b distinct b-tuples in $L_1 \times \ldots \times L_b$ as a list to some vertex in the larger part of the bipartition. This list-assignment does not admit a proper list-coloring. ∎

In very broad terms, the proof of Theorem 11 is similar to the proof of Lemma 33, except that instead of Hall's theorem, we use the Lovász local lemma to complete the extension of a (random) partial (correspondence) packing of $G[B]$. We will establish the result in the following slightly more refined form.

Theorem 35. *For every $\varepsilon > 0$ fixed, there is $\Delta_0 > 0$ such that the following holds. Let $G = (A \cup B, E)$ be a bipartite graph with parts A and B having maximum degrees Δ_A and Δ_B, respectively, with $\Delta_0 \leqslant \Delta_A \leqslant \Delta_B$. Suppose that $k = \left\lceil (1 + \varepsilon) \frac{\Delta_A}{\log \Delta_A} \right\rceil$ satisfies that*

$$3e\Delta_A(\Delta_B - 1) \cdot k^2 \exp\left(-\Delta_A^{\varepsilon/3}\right) < 1.$$

Then $\chi_c^\star(G) \leqslant k$.

Note that Theorem 11 follows easily from this result by checking that the condition holds for $\Delta_A = \Delta_B = \Delta$ taken large enough. The argument to prove Theorem 35 is inspired by a similar method used in [6, 14] and a technical extension of the results in Section 5. Just like in our derivations of Theorems 5 and 6, we prove Theorem 35 with the aid of a suitably strong bound on the probability that some random matrix has no 0-transversal.

Lemma 36. *Fix $\varepsilon > 0$ and let $k = \lceil (1 + \varepsilon) \frac{n}{\log n} \rceil$ for n sufficiently large. Let M be a $k \times k$-matrix which is the sum of n independent uniformly random $k \times k$ permutation matrices. Then the probability that there is no transversal in M containing only zeros is smaller than $3k^2 \exp\left(-n^{\varepsilon/3}\right)$.*

Let us first show how this probability bound is sufficient to derive the main result through a straightforward application of the Lovász local lemma.

Proof of Theorem 35. We may assume that every vertex in A has degree Δ_A as we can embed G in such a graph. Let H be a k-fold correspondence-cover of G, via some correspondence-assignment $L : V(G) \to 2^{V(H)}$. We may assume that for every $uv \in E(G)$, the matching in H between $L(u)$ and $L(v)$ is a perfect matching. To construct a correspondence L-packing of H, we first define a random partial L-packing restricted to the vertices in B. To do this, take a uniform total ordering of $L(b)$ independently at random for each

$b \in B$, and let $c_i(b)$ be the i-th element in $L(b)$ for each $1 \leqslant i \leqslant k$. Note that $c_i(B)$ for each i is a uniformly random maximum independent set in $H[L(B)]$. We now prove that the probability that the random L-packing cannot be extended in a fixed vertex a is very small.

We define a *bad event* $\mathcal{B}(a)$ for $a \in A$ as occurring if it is impossible to order the elements of $L(a)$ in such a way that for all i, the ith element of $L(a)$ has no neighbor in $c_i(B)$. Let $T_{i,c}(a)$ be the event that $c \in L(a)$ is adjacent in H to a vertex in $c_i(B)$. Let $M(a)$ be a $k \times k$ matrix such that the entry (i, c) equals 1 if $T_{i,c}(a)$ occurs and equals 0 otherwise. Then a good ordering of the elements of $L(a)$ exists precisely when there is a transversal only containing zeros in $M(a)$. By applying Lemma 36 with $n = \Delta_A$, we conclude that $q := \mathbb{P}(\mathcal{B}(a)) < 3k^2 \exp\left(-\Delta_A^{\varepsilon/3}\right)$ provided Δ_0 is chosen large enough.

Note also that each event $\mathcal{B}(a)$ is mutually independent of all other events $\mathcal{B}(a')$ apart from those corresponding to vertices $a' \in A$ that have a common neighbor with a in G. There are at most $d := \Delta_A(\Delta_B - 1)$ such vertices a' other than a. With the above choices of q and d, we have by assumption that $eqd < 1$, so that the Lovász local lemma guarantees that with positive probability none of the events $\mathcal{B}(a)$ occur. And thus the partial correspondence L-packing on $L(B)$ can be extended to all of H, as desired. ∎

It only remains to prove Lemma 36. Hypothetically this lemma would follow easily from a certain negative correlation property, and so we first discuss some somewhat surprising situations where this property fails.

Let R be the $k \times k$-matrix which records for each of its elements whether it appears in at least one of n independent uniformly random $k \times k$ permutation matrices $P^1, \ldots, P^n$, that is, $R_{i,j} = 1$ if and only if there is some $1 \leqslant r \leqslant n$ for which $P^r_{i,j} = 1$ and $R_{i,j} = 0$ otherwise. Equivalently, one can consider the sum M of the n permutations matrices, and then replace every nonzero element with a 1 to obtain R. By Lemma 17, we would like to estimate the probability that a fixed $s \times t$ submatrix of R contains no zero, and in fact we would want to aim for p^{st} as an upper bound, where p is the probability that one entry is nonzero. This would be immediate if all entries of R were independent or negatively correlated, but it turns out this is not the case.

Note that within one row of R, the values of its entries are negatively correlated, so the desired upper bound p^{st} indeed holds in the special case $s = 1, t = k$. Intuitively, one might expect this negative correlation to survive in a $s \times t$ submatrix for other values of s and t. Unfortunately, it is already false for $s = t = k = n = 2$.

More surprising is the fact that the probability that the entries in $R_{[2] \times [2]}$ are all nonzero can be larger than $\mathbb{P}(R_{1,1} \neq 0)^4$ for larger values of k and n as well, for example, when $(k, n) = (4, 32)$. This has been computed for certain values with Maple[1].

The obstructions discussed above could help the reader understand how our computations ended up being a bit more complicated than we might have hoped.

We now state the main technical engine in the proof of Theorem 11, in which we compute an upper bound for the probability that a given submatrix of a sum of random permutation matrices is nowhere zero. In the proof it will be useful to denote by $S(M)$ the sum of all entries in a matrix M.

---

[1]The relevant maple code can be accessed in the document CounterexamplesNegativeCorrelationSumPermutationMatrices.mw at https://github.com/StijnCambie/ListPack.

Lemma 37. *Let M be a $k \times k$-matrix which is the sum of n independent uniformly random $k \times k$ permutation matrices P^r, $1 \leqslant r \leqslant n$. For a fixed $s \times t$-submatrix of M, with $s \geqslant t$ and $t \leqslant \frac{k}{4}$, the probability that all its elements are nonzero is at most*

$$2t \cdot \exp\left(-\frac{\delta^2}{3}\frac{sn}{k}\right) + \left[4\exp\left(-\frac{\delta^2}{12}\left(1-\frac{\delta}{3}\right)\frac{sn}{k}\right) + \exp\left(-s\exp\left(-\left(1+\frac{\varepsilon}{3}\right)(1+\delta)\frac{n}{k}\right)\right)\right]^t$$

for every choice of $\delta = \delta(n,k) \in [0,1]$, $\varepsilon > 0$ and k sufficiently large as a function of ε.

Proof. By symmetry and to ease notation, it is sufficient to prove the case where the $s \times t$-submatrix is $[s] \times [t]$. First we fix $j \in [t]$ and estimate $S(M_{[s] \times j})$, the sum of elements in the $[s] \times j$-submatrix of M.

Let T_j, $1 \leqslant j \leqslant t$, be the event that $(1-\delta)\frac{sn}{k} \leqslant S(M_{[s] \times j}) \leqslant (1+\delta)\frac{sn}{k}$. Let $T = \{T_1, T_2, \dots, T_t\}$ be the event that all of them hold, and similarly we write $T \setminus T_j = \{T_1, \dots, T_{j-1}, T_{j+1}, \dots, T_t\}$. With $\overline{T}_j$, we denote the event that T_j does not hold. Note that the random variables $P^r_{i,j}$ and $P^{r'}_{i',j}$ are independent if $r' \neq r$. Moreover, for fixed r and j, we have by Lemma 19 that $P^r_{1,j}, P^r_{2,j}, \dots, P^r_{k,j}$ are negatively correlated, and that the *opposite random variables* $1-P^r_{1,j}, 1-P^r_{2,j}, \dots, 1-P^r_{k,j}$ are negatively correlated as well. Thus

$$S(M_{[s] \times j}) = \sum_{r=1}^{n}\sum_{i=1}^{s} P^r_{i,j}$$

is a sum of $\{0,1\}$-valued random variables that are negatively correlated and whose opposite random variables are negatively correlated as well. So we may apply a Chernoff bound (Theorem 18) to obtain

$$\mathbb{P}\left(S(M_{[s] \times j}) \leqslant (1-\delta)\frac{s}{k}n\right) \leqslant \exp\left(-\frac{\delta^2}{2}\frac{s}{k}n\right);$$

$$\mathbb{P}\left(S(M_{[s] \times j}) \geqslant (1+\delta)\frac{s}{k}n\right) \leqslant \exp\left(-\frac{\delta^2}{3}\frac{s}{k}n\right).$$

By a union bound,

$$\mathbb{P}(\overline{T}) \leqslant \sum_{j=1}^{t}\mathbb{P}(\overline{T}_j) \leqslant 2t \cdot \exp\left(-\frac{\delta^2}{3}\frac{s}{k}n\right).$$

Also consider the event Q_j that all elements in $M_{[s] \times j}$ are nonzero and let $Q_{1..j} = \{Q_1, Q_2 \dots, Q_j\}$ be the event that all of them hold. We now prove two claims on some conditional probabilities.

Claim 38. *For every $1 \leqslant j \leqslant t$,*

$$\mathbb{P}(Q_j | T \setminus T_j, Q_{1..j-1}) \leqslant \exp\left(-s\exp\left(-\left(1+\frac{\varepsilon}{3}\right)(1+\delta)\frac{n}{k}\right)\right).$$

Proof. Fix j between 1 and t. Given a $k \times k$ permutation matrix P, let $\gamma(P)$ be the submatrix of P induced by the columns indexed by $[t] \setminus j$. Likewise, given the random n-tuple $\mathcal{P}$ of $k \times k$ permutation matrices $(P^1, \dots, P^n)$, let $\gamma(\mathcal{P})$ be the pointwise restriction

$(\gamma(P^1), \ldots, \gamma(P^n))$. Let W be the set of all possible tuples $\gamma(\mathcal{P})$ that satisfy $T \setminus T_j, Q_{1..j-1}$. Take any such partial realization $w \in W$. From now on we additionally condition on the event $\gamma(\mathcal{P}) = w$. To shorten notation, we will write $\mathbb{P}_w(\cdot) = \mathbb{P}(\cdot | \gamma(\mathcal{P}) = w)$. Our goal is to upper bound $\mathbb{P}_w(Q_j)$. (At the end of the proof we explain why this is sufficient.)

For each $i \in [s]$, let a_i be equal to the sum $S(M_{i \times ([t] \setminus j)})$ of the entries in $M_{i \times ([t] \setminus j)}$. Let $\overline{a}$ be the mean of a_i over all $i \in [s]$, and note that it is at least $(1 - \delta)\frac{(t-1)n}{k}$ because $T \setminus T_j$ is satisfied. For every $i \in [s]$, let $Q_{i,j}$ be the event that $M_{i,j} \neq 0$.

We first show that the events $Q_{1,j}, \ldots, Q_{s,j}$ are negatively correlated. For this, we have to prove that for any $I \subseteq [s]$, it is true that

$$\mathbb{P}_w(\forall i \in I : Q_{i,j}) \leqslant \prod_{i \in I} \mathbb{P}_w(Q_{i,j}). \tag{1}$$

We prove (1) by induction on $|I|$. When $|I| \leqslant 1$ the statement is trivially true.

Let $I \subseteq [s]$ be a subset for which the statement is true and let $i' \in [s] \setminus I$. We now prove the statement for $I' = I \cup \{i'\}$. We have

$$\mathbb{P}_w(\forall i \in I : Q_{i,j}) \leqslant \mathbb{P}_w(\forall i \in I : Q_{i,j} | \overline{Q_{i',j}}), \tag{2}$$

because (informally) for every permutation P^r and every $i \in I$, the probability of $P^r_{i,j} = 1$ does not decrease if we add the condition that $P^r_{i',j} = 0$.

One can prove this formally as well, as we will now do. Consider the values of r for which $P^r_{i',j} = 1$ is possible for an extension of w, that is, all r for which row i' in $\gamma(P^r)$ is a row with only zeros. Let $R \subseteq [n]$ be the subset containing all such values r.

For any subset $R_1 \subseteq R$, let PP^{R_1} be the set of all possible tuples $\mathcal{P}' = \left(P^r_{[k] \times [t]}\right)_{1 \leqslant r \leqslant n}$ that extend w (i.e., with some abuse of notation, this is precisely when $\gamma(\mathcal{P}') = w$) and satisfy $P^r_{i',j} = 1$ if and only if $r \in R_1$. The set PP^{R_1} helps us to specify the event that $M_{i',j} = |R_1|$.

For a fixed nonempty $R_1 \subseteq R$, consider the map $\varphi : \mathrm{PP}^\emptyset \to \mathrm{PP}^{R_1}$ where $\varphi(\mathcal{P}')$ is obtained from $\mathcal{P}'$ by considering, for each $r \in R_1$, the matrix $P^r_{[k] \times [t]}$ and replacing its jth column $P^r_{[k],j}$ with the unit vector $e_{i'}$, that is, setting $P^r_{i,j} = \mathbf{1}\{i = i'\}$ for every $i \in [k]$. Note that φ is surjective and maps $(k - t)^{|R_1|}$ tuples to 1, that is, for every $\mathcal{P}'' \in \mathrm{PP}^{R_1}$, $|\varphi^{-1}(\mathcal{P}'')| = (k - t)^{|R_1|}$. Moreover, for every $\mathcal{P}'' \in \mathrm{PP}^{R_1}$, if $\mathcal{P}''$ satisfies $Q_{i,j}$ for all $i \in I$, then every preimage $\mathcal{P}' \in \varphi^{-1}(\mathcal{P}'')$ does. This implies that the probability that a random partial realization in $\mathrm{PP}^\emptyset$ satisfies $Q_{i,j}$ for all $i \in I$ is at least as large as the probability that a random partial realization in PP^{R_1} does. To relate this to the probabilities that we are actually interested in, observe that for each $R_1 \subseteq R$, the probability that a random partial realization in PP^{R_1} satisfies $Q_{i,j}$ for all $i \in I$ is exactly equal to $\mathbb{P}^{R_1} := \mathbb{P}_w\left(\forall i \in I : Q_{i,j} | P^r_{i',j} = 1 \Leftrightarrow r \in R_1\right)$. It follows that $\mathbb{P}^\emptyset \geqslant \mathbb{P}^{R_1}$ for all $R_1 \subseteq R$. We conclude that

$$\mathbb{P}_w\left(\forall i \in I : Q_{i,j} | \overline{Q_{i',j}}\right) = \mathbb{P}^\emptyset \geqslant \sum_{R_1 \subseteq R} \mathbb{P}^{R_1} \cdot \mathbb{P}_w\left(P^r_{i',j} = 1 \Leftrightarrow r \in R_1\right) = \mathbb{P}_w\left(\forall i \in I : Q_{i,j}\right).$$

This proves (2).

Now note that

$$
\mathbb{P}_w\left(\forall i \in I : Q_{i,j}\right) \leqslant \mathbb{P}_w\left(\forall i \in I : Q_{i,j} | \overline{Q_{i',j}}\right)
$$
$$
\Leftrightarrow \mathbb{P}_w(\forall i \in I : Q_{i,j}) \geqslant \mathbb{P}_w(\forall i \in I : Q_{i,j} | Q_{i',j})
$$
$$
\Leftrightarrow \mathbb{P}_w(\forall i \in I' : Q_{i,j}) \leqslant \mathbb{P}_w(\forall i \in I : Q_{i,j}) \cdot \mathbb{P}_w(Q_{i',j}).
$$

This last expression is at most $\prod_{i \in I'} \mathbb{P}_w(Q_{i,j})$ by the induction hypothesis, as desired. This concludes the proof of (1).

So given w with $S(M_{i \times ([t] \setminus j)}) = a_i$ for all $i \in [s]$, the probability $\mathbb{P}_w(Q_j)$ that none of the s elements in $M_{[s] \times j}$ equals zero is at most $\prod_{i=1}^{s} \mathbb{P}_w(Q_{i,j})$. We can compute the latter as follows:

$$
\mathbb{P}_w(Q_j) \leqslant \prod_{i=1}^{s} \mathbb{P}_w(Q_{i,j})
$$
$$
= \prod_{i=1}^{s} \left(1 - \mathbb{P}_w\left(P_{i,j}^r = 0 \text{ for all } r \in [n]\right)\right)
$$
$$
= \prod_{i=1}^{s} \left(1 - \prod_{r \in [n]} \mathbb{P}_w\left(P_{i,j}^r = 0\right)\right)
$$
$$
= \prod_{i=1}^{s} \left(1 - \prod_{r \in [n] \text{ s.t. } S\left(P_{i \times ([t] \setminus j)}^r\right) = 0} \mathbb{P}_w\left(P_{i,j}^r = 0\right)\right)
$$
$$
= \prod_{i=1}^{s} \left(1 - \left(1 - \frac{1}{k - (t-1)}\right)^{n - a_i}\right)
$$
$$
\leqslant \exp\left(-\sum_{i=1}^{s} \left(1 - \frac{1}{k - (t-1)}\right)^{n - a_i}\right)
$$
$$
\leqslant \exp\left(-s\left(1 - \frac{1}{k - (t-1)}\right)^{n - \overline{a}}\right)
$$
$$
\leqslant \exp\left(-s\left(1 - \frac{1}{k - (t-1)}\right)^{(1+\delta)\frac{k-(t-1)}{k} n}\right)
$$
$$
\leqslant \exp\left(-s \exp\left(-\left(1 + \frac{\varepsilon}{3}\right)(1 + \delta)\frac{n}{k}\right)\right).
$$

Here we made use of $1 - x \leqslant \exp(-x)$ in the sixth line. We applied Jensen's inequality to the convex function $f(x) = \left(1 - \frac{1}{k-(t-1)}\right)^x$ in the seventh line. In the eighth line we used that $\overline{a} \geqslant (1 - \delta)\frac{(t-1)}{k} n$ and $t - 1 \leqslant \frac{k}{2}$ to derive $n - \overline{a} \leqslant n - (1 - \delta)\frac{(t-1)}{k} n \leqslant (1 + \delta)\frac{k-(t-1)}{k} n$ and in the last line we used that $\left(1 - \frac{1}{x}\right)^x \geqslant \exp\left(-\left(1 + \frac{\varepsilon}{3}\right)\right)$ for $x = k - (t-1) \geqslant \frac{k}{2}$, which is sufficiently large in terms of ε.

Since this upper bound is true for any partial realization $w \in W$, we conclude that

$$\mathbb{P}(Q_j | T \setminus T_j, Q_{1..j-1}) = \sum_{w \in W} \mathbb{P}(\gamma(\mathcal{P}) = w | T \setminus T_j, Q_{1..j-1}) \cdot \mathbb{P}_w(Q_j)$$

$$\leqslant \exp\left(-s \exp\left(-\left(1 + \frac{\varepsilon}{3}\right)(1 + \delta)\frac{n}{k}\right)\right).$$

∎

Claim 39. *For every* $1 \leqslant j \leqslant t$,

$$\mathbb{P}(\overline{T}_j | T \setminus T_j, Q_{1..j-1}) \leqslant 2 \exp\left(-\frac{\delta^2}{12}\left(1 - \frac{\delta}{3}\right)\frac{sn}{k}\right).$$

Proof. As in the proof of Claim 38, we fix j and we condition on the event that the random tuple $\gamma(\mathcal{P})$ of permutation matrices restricted to the columns $[t] \setminus j$ is equal to a certain tuple $w \in W$, that is, such that $T \setminus T_j, Q_{1..j-1}$ are satisfied. We again fix such a realization w and succinctly write $\mathbb{P}_w(\cdot) := \mathbb{P}(\cdot | \gamma(\mathcal{P}) = w)$ for the conditional probability distribution.

We will first estimate $\mathbb{P}_w(\overline{T}_j)$. Let $f_r = S(P^r_{[s] \times ([t] \setminus j)})$ for every $r \in [n]$, and note that we know this value deterministically because $P^r_{[s] \times ([t] \setminus j)}$ is a submatrix of $\gamma(P^r)$ and thus fully determined by w. Furthermore, for each $i \in [s]$, we have that $\mathbb{P}_w(P^r_{i,j} = 1)$ equals $\frac{1}{k-(t-1)}$ if $S(P^r_{i,([t] \setminus j)}) = 0$, and equals 0 otherwise. It follows that $\mathbb{E}_w(S(P^r_{[s] \times j})) = \frac{s - f_r}{k-(t-1)}$ and hence

$$\mathbb{E}_w(S(M_{[s] \times j})) = \sum_{r=1}^{n} \mathbb{E}_w(S(P^r_{[s] \times j})) = \frac{sn - \sum_{r=1}^{n} f_r}{k - (t - 1)}. \tag{3}$$

Since w satisfies $T \setminus T_j$, we know $\sum_{r=1}^{n} f_r = S(M_{[s] \times ([t] \setminus j)})$ is between $(t-1)(1-\delta)\frac{sn}{k}$ and $(t-1)(1+\delta)\frac{sn}{k}$. Evaluating these bounds in (3) and using the assumption $t \leqslant \frac{k}{4}$, it follows that

$$(1 - \delta/3)\frac{sn}{k} \leqslant \mathbb{E}_w\left[S(M_{[s] \times j})\right] \leqslant (1 + \delta/3)\frac{sn}{k}.$$

Viewing $\left(P^r_{[k] \times [t]}\right)_{1 \leqslant r \leqslant n}$ as a random extension of w, we have that $S(P^r_{[s] \times j})$ and $S(P^{r'}_{[s] \times j})$ are independent[2] for $r \neq r'$, with respect to $\mathbb{P}_w(\cdot)$. Thus we can apply Theorem 18 to the $\{0, 1\}$-valued random variables $S(P^1_{[s] \times j}), \ldots, S(P^n_{[s] \times j})$ to obtain

$$\mathbb{P}_w\left(S(M_{[s] \times j}) \leqslant (1 - \delta/2)(1 - \delta/3)\frac{sn}{k}\right) \leqslant \exp\left(-\frac{\delta^2}{8}(1 - \delta/3)\frac{sn}{k}\right);$$

$$\mathbb{P}_w\left(S(M_{[s] \times j}) \geqslant (1 + \delta/2)(1 + \delta/3)\frac{sn}{k}\right) \leqslant \exp\left(-\frac{\delta^2}{12}(1 - \delta/3)\frac{sn}{k}\right).$$

Combining these two inequalities, and noting that $1 - \delta \leqslant (1 - \delta/2)(1 - \delta/3) \leqslant (1 + \delta/2)(1 + \delta/3) \leqslant 1 + \delta$ due to the assumption $\delta \in [0, 1]$, we obtain the desired bound:

$$\mathbb{P}_w(\overline{T}_j) \leqslant 2 \exp\left(-\frac{\delta^2}{12}(1 - \delta/3)\frac{sn}{k}\right).$$

---

[2]This independence is one of the reasons why we condition on $\gamma(\mathcal{P})$ being equal to a fixed realization w, rather than conditioning on the full event $T \setminus T_j, Q_{1..j-1}$.

Finally we can conclude by writing $\mathbb{P}(\overline{T_j}|T \setminus T_j, Q_{1..j-1})$ as the linear combination $\sum_{w \in W} \mathbb{P}(\gamma(\mathcal{P}) = w|T \setminus T_j, Q_{1..j-1}) \cdot \mathbb{P}_w(\overline{T_j})$, which is at most $\max_{w \in W} \mathbb{P}_w(\overline{T_j})$. ∎

We also prove the following claim.

Claim 40. *Let Q, A, T be any three events. Then*

$$\mathbb{P}(Q|A, T) \leqslant \mathbb{P}(Q|A) + 2\mathbb{P}(\overline{T}|A).$$

Proof. If $\mathbb{P}(\overline{T}|A) \geqslant 1/2$, the statement is trivially true. Note that

$$\mathbb{P}(Q|A) = \mathbb{P}(Q|A, T)\mathbb{P}(T|A) + \mathbb{P}(Q|\overline{T}, A)\mathbb{P}(\overline{T}|A) \geqslant \mathbb{P}(Q|A, T)\mathbb{P}(T|A).$$

Since $\frac{1}{1-x} \leqslant 1 + 2x$ for any $x \leqslant \frac{1}{2}$, $\mathbb{P}(\overline{T}|A) < 0.5$ and $\mathbb{P}(Q|A, T) \leqslant \frac{\mathbb{P}(Q|A)}{1-\mathbb{P}(\overline{T}|A)}$ imply that $\mathbb{P}(Q|A, T) \leqslant \mathbb{P}(Q|A) + 2\mathbb{P}(Q|A)\mathbb{P}(\overline{T}|A) \leqslant \mathbb{P}(Q|A) + 2\mathbb{P}(\overline{T}|A)$. ∎

Applying Claim 40 with the formulas in Claims 38 and 39 where (Q, A, T) in Claim 40 is chosen to be $(Q_j, (T \setminus T_j) \cap Q_{1..j-1}, T_j)$, we deduce that for every $1 \leqslant j \leqslant t$

$$\mathbb{P}(Q_j|T \cap Q_{1..j-1}) \leqslant 4\exp\left(-\frac{\delta^2}{12}\left(1 - \frac{\delta}{3}\right)n\frac{s}{k}\right) + \exp\left(-s\exp\left(-\left(1 + \frac{\varepsilon}{3}\right)(1 + \delta)\frac{n}{k}\right)\right).$$

Finally note that $\mathbb{P}(Q_{1..t}|T) = \mathbb{P}(Q_1|T)\mathbb{P}(Q_2|Q_1, T)\cdots\mathbb{P}(Q_t|Q_{1..t-1}, T)$ and so the conclusion follows from $\mathbb{P}(Q_{1..t}) \leqslant \mathbb{P}(Q_{1..t}|T) + \mathbb{P}(\overline{T})$. ∎

Remark. The case $t = 1$ of Lemma 37 is much simpler. In this case the fixed $s \times t$ submatrix of M has only one column, and within that column the events $Q_{1,1}, Q_{1,2}, \ldots, Q_{s,1}$ are negatively correlated, so $\mathbb{P}(Q)$ is at most $\left(1 - \left(1 - \frac{1}{k}\right)^n\right)^s$. In contrast, for $t \geqslant 2$ a more involved argument is required because (as discussed before the proof) even $Q_{1,1}, Q_{1,2}, Q_{2,1}$ and $Q_{2,2}$ are not necessarily negatively correlated.

Proof of Lemma 36. By the Frobenius–König theorem (Lemma 17), no such transversal in M exists if and only there exist s, t with $s + t = k + 1$ and a $s \times t$-submatrix all of whose entries are nonzero. For every such choice of s and t there are $\binom{k}{s}\binom{k}{t}$ possible $s \times t$-submatrices. The case where $s \leqslant t$ is similar to the case where $s \geqslant t$, by switching the two. Note that $t \leqslant s$ and $s + t = k + 1$ imply $s \geqslant \frac{k}{2}$. When $\frac{k}{4} < t$, the probability that a $s \times t$-submatrix has only nonzero entries is obviously at most the probability that a $s \times \frac{k}{4}$-submatrix has only nonzero entries.

We may now use the probability computed in Lemma 37 with the choice $\delta = 1/\sqrt[4]{\log n}$. Noting that $\sum_{s,t}\binom{k}{t}\binom{k}{s} = 4^k$, the sum of the corresponding first terms is bounded by $4^k k \exp\left(-\frac{n}{6\sqrt{\log n}}\right)$. For the second terms, we first compute that for $s \geqslant \frac{k}{2}$,

$$4\exp\left(-\frac{\delta^2}{12}\left(1 - \frac{\delta}{3}\right)n\frac{s}{k}\right) + \exp\left(-s\exp\left(-\left(1 + \frac{\varepsilon}{3}\right)(1 + \delta)\frac{n}{k}\right)\right)$$

$$\leqslant \exp\left(-\frac{k}{2}\exp\left(-\frac{\log n}{1 + \varepsilon/2}\right)\right) + 4\exp\left(-\frac{n}{25\sqrt{\log n}}\right)$$

$$\leqslant \exp\left(-\frac{n^{\frac{\varepsilon/2}{1+\varepsilon/2}}}{2\log n}\right) + 4\exp\left(-\frac{n}{25\sqrt{\log n}}\right)$$

$$\leqslant \exp\left(-\frac{n^{\frac{\varepsilon/2}{1+\varepsilon/2}}}{2\log n}\right)\left(1 + 4\exp\left(-\frac{n}{26\sqrt{\log n}}\right)\right)$$

$$\leqslant \exp\left(-n^{\varepsilon/3}\right).$$

Noting that $\binom{k}{t}\binom{k}{k+1-t} \leqslant k^{2t-1}$, we have the following upper bound for the sum of the corresponding second terms from Lemma 37:

$$2\sum_{t=1}^{k/4}\left(k^2\exp\left(-n^{\varepsilon/3}\right)\right)^t + 4^k\exp\left(-n^{\varepsilon/3}\frac{k}{4}\right) < 2.5k^2\exp\left(-n^{\varepsilon/3}\right).$$

We conclude by noting that

$$2.5k^2\exp\left(-n^{\varepsilon/3}\right) + 4^k k\exp\left(-\frac{n}{6\sqrt{\log n}}\right) < 3k^2\exp\left(-n^{\varepsilon/3}\right).$$

∎

8 | CONCLUDING REMARKS

We have set the stage for this natural fusion between two classic notions in (extremal) graph theory: packing and coloring. We were generous throughout with problems for further investigation, most importantly Conjecture 1, which we might audaciously refer to as the *List Packing Conjecture*. Regardless of its truth in general, Conjecture 1 may be specialized to various graph classes for many research possibilities. We highlight three fundamental classes where it is, in our view, most tempting to push for further progress.

1 *Planar graphs*. We do not yet have any constructions to rule out the possibility that $\chi_\ell^\star(G) \leqslant 5$ for all planar G. What is the optimal value?
2 *Line graphs*. Based on the List Colouring Conjecture, we surmise for every $\varepsilon > 0$ that $\chi_\ell^\star(G) \leqslant (1 + \varepsilon)\omega$ for every line graph G with clique number $\omega \geqslant \omega_0$. Due to its connection to Latin squares, here even the case where G is the line graph of the complete bipartite graph $K_{\omega,\omega}$ is enticing to narrow in on.
3 *Random graphs*. Does it hold that $\chi_\ell^\star(G_{n,1/2}) \leqslant (1 + o(1))n/(2\log_2 n)$ a.a.s.? Related to this question, one might wonder if the $\log n$ factor in Theorem 5 could be improved to one of order $\log(n/\chi_f(G))$, for this would immediately imply an upper bound for $\chi_\ell^\star(G_{n,1/2})$ of order $\frac{n\log\log n}{\log n}$ a.a.s.

We also consider it natural to pursue results along the continuum between list packing and strong coloring as discussed in Subsection 1.3. Here it seems to us that Conjecture 14 is within reach with the methods of Section 7, but we leave this to future investigation.

ACKNOWLEDGMENTS

ORCID

Stijn Cambie https://orcid.org/0000-0002-2385-1137
Wouter Cames van Batenburg https://orcid.org/0000-0001-8631-6222
Ewan Davies https://orcid.org/0000-0002-2699-0976
Ross J. Kang https://orcid.org/0000-0002-4219-593X

REFERENCES

1. R. Aharoni, E. Berger, and R. Ziv, *Independent systems of representatives in weighted graphs*, Combinatorica **27** (2007), no. 3, 253–267.
2. M. Ajtai, P. Erdős, J. Komlós, and E. Szemerédi, *On Turán's theorem for sparse graphs*, Combinatorica **1** (1981), no. 4, 313–317.
3. N. Alon, *The linear arboricity of graphs*, Israel J. Math. **62** (1988), no. 3, 311–325.
4. N. Alon, *Choice numbers of graphs: A probabilistic approach*, Comb. Probab. Comput. **1** (1992), no. 2, 107–114.
5. N. Alon, "*Restricted colorings of graphs*," Surveys in combinatorics, 1993 (Keele), volume 187 of London Math. Soc. Lecture Note Ser, Cambridge Univ. Press, Cambridge, 1993, pp. 1–33.
6. N. Alon, S. Cambie, and R. J. Kang, *Asymmetric list sizes in bipartite graphs*, Ann. Comb. **25** (2021), no. 4, 913–933.
7. N. Alon, M. R. Fellows, and D. R. Hare, *Vertex transversals that dominate*, J. Graph Theory **21** (1996), no. 1, 21–31.
8. N. Alon and M. Krivelevich, *The choice number of random bipartite graphs*, Ann. Comb. **2** (1998), no. 4, 291–297.
9. N. Alon, M. Krivelevich, and B. Sudakov, *Coloring graphs with sparse neighborhoods*, J. Combin. Theory Ser. B **77** (1999), no. 1, 73–82.
10. J. Anderson, A. Bernshteyn, and A. Dhawan, *Coloring graphs with forbidden bipartite subgraphs*, Comb. Probab. Comput. **32** (2023), no. 1, 45–67.
11. A. Bernshteyn, *The asymptotic behavior of the correspondence chromatic number*, Discret. Math. **339** (2016), no. 11, 2680–2692.
12. T. Bohman and R. Holzman, *On a list coloring conjecture of Reed*, J. Graph Theory **41** (2002), no. 2, 106–109.
13. B. Bollobás, P. Erdős, and E. Szemerédi, *On complete subgraphs of r-chromatic graphs*, Discret. Math. **13** (1975), no. 2, 97–107.
14. S. Cambie and R. J. Kang, *Independent transversals in bipartite correspondence-covers*, Can. Math. Bull. **65** (2022), no. 4, 882–894.
15. W. Cames van Batenburg, R. de Joannis de Verclos, R. J. Kang, and F. Pirot, *Bipartite induced density in triangle-free graphs*, Electron. J. Combin **27** (2020), no. 2, 19.
16. P. A. Catlin, *On the Hajnal-Szemerédi theorem on disjoint cliques*, Utilitas Math. **17** (1980), 163–177.
17. Z. Dvořák and L. Postle, *Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8*, J. Combin. Theory Ser. B **129** (2018), 38–54.
18. P. Erdős and L. Lovász, "*Problems and results on 3-chromatic hypergraphs and some related questions*," Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, Vol **10**, Colloq. Math. Soc. János Bolyai, Amsterdam, 1975, pp. 609–627.
19. P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. Proceedings of the west coast conference on combinatorics, graph theory and computing (Humboldt state Univ., Arcata, Calif., 1979), Congress. Numer., XXVI, Utilitas Math., Winnipeg, Man. 1980 125–157.
20. C. J. Everett and P. R. Stein, *The asymptotic number of* $(0, 1)$*-matrices with zero permanent*, Discret. Math. **6** (1973), 29–34.
21. M. R. Fellows, *Transversals of vertex partitions in graphs*, SIAM J. Discret. Math. **3** (1990), no. 2, 206–215.
22. S. Glock and B. Sudakov, *An average degree condition for independent transversals*, J. Combin. Theory Ser. B **154** (2022), 370–391.
23. P. Hall, *On representatives of subsets*, J. London Math. Soc. **10** (1935), no. 1, 26–30.
24. P. E. Haxell, *A note on vertex list colouring*, Comb. Probab. Comput. **10** (2001), no. 4, 345–347.

25. P. E. Haxell, *An improved bound for the strong chromatic number*, J. Graph Theory **58** (2008), no. 2, 148–158.

26. P. E. Haxell, *On forming committees*, Am. Math. Mon. **118** (2011), no. 9, 777.

27. A. Johansson, Asymptotic choice number for triangle-free graphs. Technical Report 91-5, DIMACS, Piscataway, NJ, 1996.

28. R. J. Kang and T. Kelly, *Colorings, transversals, and local sparsity*, Random Struct. Algoritm. **61** (2022), no. 1, 173–192.

29. D. Král, O. Pangrác, and H.-J. Voss, *A note on group colorings*, J. Graph Theory **50** (2005), no. 2, 123–129.

30. P.-S. Loh and B. Sudakov, *Independent transversals in locally sparse graphs*, J. Combin. Theory Ser. B **97** (2007), no. 6, 904–918.

31. K. MacKeigan, *Independent coverings and orthogonal colourings*, Discret. Math. **344** (2021), no. 8, 112431.

32. M. Marcus and H. Minc, A survey of matrix theory and matrix inequalities, Allyn and Bacon, Boston, Mass, 1964.

33. M. Molloy, *The list chromatic number of graphs with small clique number*, J. Combin. Theory Ser. B **134** (2019), 264–284.

34. S. Norin and L. Postle, *Connectivity and choosability of graphs with no K_t minor*, J. Combin. Theory Ser. B **158** (2023), no. 1, 283–300.

35. A. Panconesi and A. Srinivasan, *Randomized distributed edge Coloring via an extension of the Chernoff–Hoeffding bounds*, SIAM J. Comput. **26** (1997), no. 2, 350–368.

36. B. Reed, *The list colouring constants*, J. Graph Theory **31** (1999), no. 2, 149–153.

37. B. Reed and B. Sudakov, *Asymptotically the list colouring constants are 1*, J. Combin. Theory Ser. B **86** (2002), no. 1, 27–37.

38. J. B. Shearer, *A note on the independence number of triangle-free graphs*, Discret. Math. **46** (1983), no. 1, 83–87.

39. J. B. Shearer, *On a problem of Spencer*, Combinatorica **5** (1985), no. 3, 241–245.

40. T. Szabó and G. Tardos, *Extremal problems for transversals in graphs with bounded degree*, Combinatorica **26** (2006), no. 3, 333–351.

41. V. G. Vizing, *Coloring the vertices of a graph in prescribed colors*, Diskret. Analiz **101** (1976), 3–10.

42. R. Yuster, *On factors of independent transversals in k-partite graphs*, Electron. J. Combin **28** (2021), no. 4, 18.