Core Probability Model : Efficient macroscopic sensitivity modelling
Conceptual design

Calvert, Simeon; Taale, Henk; Snelder, Maaike; Hoogendoorn, Serge

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Core Probability Model: Efficient macroscopic sensitivity modelling

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Authors
Ir. Simeon Calvert, Dr. ir. Henk Taale, Dr. Maaike Snelder, Prof dr. ir. Serge Hoogendoorn
Delft University of Technology, Faculty of Civil Engineering and Geosciences, Department of Transport & Planning, the Netherlands

Contents

Abstract

In this contribution a new development in probability, reliability and sensitivity modelling is presented. The Core Probability Model (CPM) is a full probabilistic model for modelling variations in capacity and traffic demand in macroscopic traffic flow. The CPM extends a base model, such as the Cell Transmission Model (CTM), by considering each traffic variable as a discrete stochastic variable denoted as a probability distribution of values for each traffic variable in time and space. Traffic is propagated along a link using the base model and through a larger network with the application of probability merging algorithms at the nodes. Due to the incorporation of probability in the core of traffic propagation, the necessity for multiple simulations diminishes, as the CPM makes use of a one-shot approach. This leads in theory to a shorter simulation time and computational load. Another major advancement is the explicit handling of spatiotemporal dependence. Furthermore the use of complete probability distributions allows for a detailed overview of variable probabilities at any given time and place in the model with a greater degree of accuracy. In this contribution the conceptual design of the CPM is given along with a description of the main issues it tackles.

Keywords

Stochastic traffic flow modelling, Macroscopic traffic flow models, probabilistic modelling

1 Introduction

Stochastic variations and uncertainty typify human life and equally the world around us, and are no less integral to the world of traffic flow. Application of traffic flow analysis carried out using traffic flow models aims to represent the world of traffic flow in simulation. Traffic models are simplifications of reality and make assumptions to allow for a fast and efficient modelling of situations. These assumptions should however have minimal effect on the deviation of results to what would be expected in reality. In the case of stochastic variations and the inclusion of uncertainty, many assumptions are made that have a greater effect on model outcomes than may be desirable.

In many traffic models stochastic variation is ignored or assumed to be of limited importance to the outcome of simulations. In many cases reducing the input of variables in a traffic model to average or representative values can have detrimental effect of the simulation results, and may lead to biased outcomes in relation to what may be found from empirical data (Calvert et al. 2012; Mahmassani et al. 2012; van Lint et al. 2012). It is argued that the stochasticity in traffic cannot be reduced prior to traffic flow simulation and cannot be expected to give the same outcome as if the reduction had not taken place. Instability in traffic, including network effects in congestion, lead to a non-linear propagation of stochastic variation, especially for the more extreme cases. In turn greater traffic flows and congestion will lead to higher values for travel times and delays than can be derived from averaged or representative input values (Calvert et al. 2012). It is therefore imperative to explicitly consider stochastic variation in traffic flow modelling, when this variation is present in the considered scenarios and networks.

In this contribution a new stochastic macroscopic model is introduced which tackles many challenges in macroscopic modelling and is developed with a view for easy and efficient application in practice. The Core Probability Model (CPM) is a full probabilistic model for modelling multi-dimensional variations in capacity and traffic demand in macroscopic traffic flow. The CPM extends a base model, such as the Cell Transmission Model (CTM), by considering each traffic variable as a stochastic variable denoted as a probability distribution of the chance of values for each traffic variable.

In section 1.1 the existing literature on this topic is reviewed, while in section 2 the main issues concerning stochastic macroscopic traffic flow modelling are described. This is followed in section 3 by a description of the conceptual design of the CPM and in section 4 how the model addresses these issues. Section 5 shows a demonstration case of the model in practice. Finally, section 6 describes the current developments of the model.

1.1 Stochastic macroscopic traffic modelling

Since the 1990’s there has been a gradual increase in effort towards improving traffic flow modelling through the explicit inclusion of stochastic variation. Initially a focus was placed on Monte Carlo simulation to include variation in macroscopic traffic modelling and later the focus shifted more towards internalised stochastics. In Monte

Carlo simulation varied input values for the traffic variables would be sampled and applied in simulation for a N number of simulation to approach a distribution of possible outcomes. Although Monte Carlo simulation has been widely applied, mainly due to its relative simplicity and effectiveness, the method has its drawbacks. Main concerns in traffic modelling in the past have been the computational load of the method (Chang et al. 1994; Chen et al. 2002; Sumalee et al. 2011) and the presence of correlation between input variables. The incorporation of variance reduction methods, such as Importance sampling or Latin Hypercube sampling, have helped to reduce the computational effort of such models as well as the use of more powerful computers (Jonnalagadda et al. 2001; Hess et al. 2006; van Lint et al. 2012; Calvert et al. 2014) . Recent developments in marginal simulation approaches further offer an alternative solution to a heavy computational load in Monte Carlo approaches (Corthout et al. 2011). In marginal simulation a significant overlap between traffic flow in successive simulation iterations is presumed. By only simulating the marginal difference in traffic flow, repetitive network loading with a full dynamic macroscopic model is not required. Therefore the marginal simulation method only requires a single full initial model simulation and thereafter simulates the marginal differences using Monte Carlo simulation with a first-order based kinematic model, leading to a gain in computational efficiency. Correlation between input variables may be considered prior to simulation at the sampling stage (Chen et al. 2002). Variables with dependencies may also have probabilities which rely on the values sampled from other variables. In this way correlation between two or more variables is included and allows for a realistic simulation. However calculating non-bias outcomes in situations in which correlations are more complex and, furthermore, have dependencies on variables in the model, becomes much more difficult (Chang et al. 1994). In many approaches the extent of bias is presumed to be limited and therefore little attention is spent on this difficulty.

More recent developments in stochastic macroscopic modelling are found in stochastic extensions of existing mainstream traffic models. Boel and Mihaylova (2006) proposed an extension to the CTM with stochastic elements. Rather than reconstructing the CTM as piece-wise structure based on traffic states, they defined the sending and receiving functions from the CTM as random variables in which the dynamics of the average speed in each cell is stochastically varied. The purpose was to incorporate stochasticity in the heart of the model at link level, which may propagate through an entire network through cell interaction. However, as their approach only considers a single stochastic scenario at a time, repetitive simulations are required to compose a probability distribution of the outcomes. Similar approaches were proposed by Sun et al. (2003) focussing on the explicit defining of traffic states. A main reason for considering multiple traffic states is the avoidance of nonlinearity in the fundamental relation, which is difficult to quantify otherwise. Jabari and Liu (2012) argued that presuming non-linearity, while being mathematically beneficial, may lead to inconsistency with the original deterministic dynamics. Therefore Jabari and Liu (2012) proposed to include stochasticity as a function of the uncertainty in the driver gap choice, represented by the random vehicle headway. In doing so, they argue that non-linearity is avoided in continuous time as all traffic dynamics may be derived to the longitudinal car following behaviour. (Sumalee et al. 2011) proposed a further extension of the CTM in which traffic states are explicitly defined as a stochastic bilinear system. Their Stochastic CTM (S-CTM) avoids non-linearities in the original CTM and considers variation through propagation of the probability of traffic states and corresponding densities as the likely values and surrounding standard deviation. The S-CTM also demonstrated computational efficiency as a *one shot* model in which multiple iterations using a Monte Carlo routine are avoided. This greater efficiency is however also achieved through the simplification of the probability distributions to the aforementioned Gaussian characteristics. Although a legitimate choice, this reduces the accuracy of the probabilistic estimation by presuming a set distribution, which in many not be the case.

Analysis of stochastic variation due to randomness in driver behaviour has led to developments in stochastic modelling for both microscopic and macroscopic models. Variations in traffic flow are easily viewed empirically from fundamental diagram plots. Therefore it is unsurprising that stochastics are also being included in (macroscopic) traffic models by means of a stochastic fundamental diagram. Li et al. (2009) make a strong argument that a simple, but effective manner of probabilistic modelling is to make use of a stochastic fundamental diagram. Such a diagram is constructed through a flux function obtained from random elements observed from speed-density data. Kim and Zhang (2008) also previously described stochasticity in the fundamental diagram by defining the growth and delay of perturbations from random fluctuations in both the gap time and transitions between traffic states. In their work they closely examined fluctuations in car following to derive their defined *gap time*. Boel and Mihaylova (2006) also make use of similar fundamental diagrams in their stochastic switching traffic state model, previously mentioned. While these models address the incorporation of variation in the model, this is performed in a simplified fashion, such that traffic states are not all well defined (Sumalee et al. 2011), or fail to fully deal other stochastic modelling challenges, such as spatiotemporal correlations.

Other models involving stochastic variation relate to a wide number of analytical approaches that have been suggested, especially in relation to travel time reliability (Du and Nicholson 1997; Clark and Watling 2005), however these are not purely considered as stochastic traffic flow models and are therefore not considered here.

Despite recent developments, challenges remain for the development of stochastic macroscopic traffic flow models and more so for their practical application. The important issues still facing stochastic macroscopic modelling, and yet not completely addressed in a single model, are discussed in the following section.

2 Important issues

In the theoretical development, but also for the practical application of dynamic stochastic macroscopic traffic flow models, there remain a number of issues that have not been solved in full or in combination with each other. It may be that one issue is addressed at the expense of another. In this section four important issues are discussed:

1. Computational efficiency
2. Spatiotemporal dependency
3. Stochastic propagation of probability
4. Generality

2.1 Computational efficiency

Originally the issue of computational efficiency arose with the application of Monte Carlo simulation in traffic models. Often performing hundreds of simulations was time consuming and acted as a deterrent to apply stochastic variation. Even through the application of variance reduction techniques and faster and more powerful computers, this remains an issue. A trend that counteracts such advancements originates from a desire to apply more complex traffic models on larger and more detailed networks. Also an increasing number of (stochastic) variables demand a greater computational effort that somewhat undermines hardware and software advancement.

The development of *one shot* models, which largely do away with the necessity for repetitive simulations have a great potential to allow for stochastic simulation at a lesser computational cost. Such models as the S-CTM (Sumalee et al. 2011) and that of Jabari and Liu (2012) are at the forefront of these developments. A danger however is that a simplification of the stochastic input or propagation may be required to allow one shot models to be effective. The opposite effect may be an over-complicated model without simplification, but at a cost computational efficiency and even possible application. Therefore the challenge is not just in reducing computational load, but doing so in a way that a model is not reduced in stochastic and modelling accuracy. This is a balance that is still in the process of being optimised for stochastic modelling.

2.2 Spatiotemporal dependency

Incorporation of spatial and temporal dependant variation from different sources brings a further issue of correlation on a number of levels. On a temporal plane it is clear that a stochastic element will affect traffic during a certain time frame, possibly with differing severity. A basic example is that of an accident that reduces road capacity. At the time an accident occurs, the capacity is affected differently than during the aftermath and the clean-up, but nevertheless the capacity reduction is correlated in time, as a natural consequence of a chain of events. In the same way there is also a spatial correlation. The capacity reduction affects the location of the accident, but due to congestion propagation, also affects both upstream capacity and traffic flow. A further complexity in dependence comes from not only considering a single stochastic influence variable, such as the capacity, but also the traffic demand. In the case of an accident, drivers may reroute, shift departure time, etc. This does not only affect traffic flow in time, but also in space. Furthermore, correlation effects also exist between the traffic demand and road capacity in some instances. When considering a greater number of variables, the dependency relations explode.

In many cases some of these dependencies are presumed non-existent for ease of modelling (Clark and Watling 2005; Sumalee et al. 2011). Especially for the interdependent correlations between variables this is readily the case, while spatiotemporal dependencies must be considered on some level to avoid disutility of a model. Even then, these correlations may be simplified by means of presumptions or transformations (Clark and Watling 2005; Jabari and Liu 2012). It should not immediately be presumed that a less than full consideration of dependency will have large detrimental effects on model outcomes, as there are cases in which this is clearly the case (Calvert et al. 2012), however the possibility thereof should always be considered.

2.3 Stochastic propagation of probability

In traffic flow models it is commonplace for traffic to propagate through a link and network. However upon including stochastic probability in traffic flow modelling, these probabilities also propagate in time and space with traffic (Lebacque et al. 2007; Hoogendoorn et al. 2008). For Monte Carlo simulation this is not an issue, as each simulation is a single probability value. For one shot models there is a challenge to allow as much information and as full as possible probability range without compromising model accuracy or one of the other important issues, such as computational efficiency.
In models which apply stochastic effects through the fundamental diagram, traffic flow is presumed to propagate in an identical fashion to that of the base model, however application of a stochastic fundamental diagram, probabilities are derived in accordance with the stochastic form of the diagram. In the S-CTM, median and standard deviations of traffic variables are propagated through time and space, dependent on the relevant traffic state. It is not uncommon to only consider a median and standard deviation, as this requires the least computational effort and still gives a good estimation of variational spread. However more in-depth analysis is harder as the underlying distribution is not preserved. Furthermore, such an approach often presumes probability distributions to be symmetrical according to a presumed shape, which is not always the case. In such a case biases are allowed, which may not accurately represent the underlying distribution. It should however be noted that these biases may be small compared to the overall error level.

2.4 Generality

Inclusion of stochastic variation does not only demand solid and accurate modelling, but also realistic and correct model input. The level of stochastic input depends on which variables are considered stochastic. These may be the time headway (or gap time) between vehicles, capacity values, traffic demand values, or even ‘lower level’ variables, such as vehicle population or probability of accidents. Depending on how a model processes the stochastic variables, these may be offered to the model as a complete distribution, either of a specific form or empirical, or as a description of variations, such as median, standard deviation and possibly a shape parameter. The difficulty with this issue is that of generality. A set distribution of probable values for a set variable may not be valid for every location on a network or under certain other conditions. Furthermore, such variables may not pertain to a set distribution type. Often presumptions are made to how general distributions or variations are. In many instances white noise may be applied to known representative values to imitate variation (Helbing et al. 2001; Jabari and Liu 2012). The validity of such approaches is not often considered and is taken as a model assumption. However here there is also room for improvement, when applying stochastic variation to traffic flow models. In the case of stochastic fundamental diagrams, the difficulty of generality may also arise. In some cases allowing specific local data to influence the extent of stochastic variation can help solve this.

3 Basic Design CPM

In this section the basic design of the Core Probability Model is explained. The Core Probability Model (CPM) extends existing macroscopic traffic flow models to allow stochastic behaviour in traffic to be internalised in the traffic flow model which it extends. Internalisation here refers to the manner in which stochasticity is present in the model, where Monte Carlo simulation is a clear example of external stochastic influence. Initial application of the CPM makes use of the Cell Transmission Model (CTM) as base model. The basic concept entails replacing single traffic variables in time and space, such as the density, in a model with a discrete distribution of that same traffic variable, also in space and time. The distribution, denoted as a vector, consists of equally spaced probabilities of various possible values of the considered traffic variable at a certain time and location. The general dynamics of the base model are kept the same as the deterministic version of the model. In such a way, traffic is propagated through a link (or network) considering possible valid values of each traffic variable with a set probability, using already validated traffic flow dynamics from the base model. The input distributions are empirically determined for specific locations and/or scenarios or from generic empirical analysis (Calvert et al. 2014; van Stralen et al. 2014).

A more detailed description of the CPM is given in the subsequent subsections. This begins with a short explanation of the applied base model (3.1), explains the manner in which probability is included in the model and is propagated (3.2 & 3.3), and how congestion and traffic states are dealt with (3.4). The applied fundamental diagram, as used in the test case, is given in 3.5 and a simple numerical example is shown to conclude the section (3.6).

3.1 Base model

The Core-probability approach considers stochastic probabilities in the core of a macroscopic traffic model. The base model for this is the first order Cell Transmission Model (CTM) model (Daganzo 1994; Daganzo 1995). The CTM describes traffic using the law of conservation of vehicles (eq.(1)), and the fundamental relation (eq.(2)):

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \qquad (1)$$

$$q(x,t) = Q_E(k(x,t)) \qquad (2)$$

Here $\partial k(x,t)$ denotes the change in density in time, t, and space, x. $\partial q(x,t)$ denotes the same for the intensity, while Q_E is the fundamental relation between the density and flow, which is explained in more detail later on.
The traffic flow at the interfaces between two cells, q, is determined by a sending and receiving function, denoted here as the demand, D, and supply, S, which closely represent the available capacity in a cell and the desired traffic flow into a cell:

$$q^{x_m \to x_{m+1}}(k(x,t)) = \min(D_m(k(x,t)), S_{m+1}(k(x,t))) \qquad (3)$$

The demand function D is calculated by the largest flow or capacity of cell m in relation to eq. (2), and the supply function S by the desired outflow from the previous cell according to the fundamental traffic characteristics of the preceding cell. The base model is applied in its discrete form for use in the Core Probability Model and governs the main dynamics of traffic flow.

3.2 Inclusion of probability

In classical first order models, each variable is represented by a single value for each point in time, t, and space, x. In the core-probability approach a further variable is added, which represents the probability of the density occurring, and sequentially the traffic flow, $\boldsymbol{q}$, and the speed, $\boldsymbol{v}$. This further transforms the variables from a single value in time and space into a probability distribution in the same time and space, represented by their corresponding vector. Each value in the probability vector has identical probability by definition to agree with the element-to-element propagation, explained in section 3.3.
The random variable $K(x,t)$ denotes the random variable that reflects the density on a cell [x, x+dx] and during time interval [t, t+dt]. Let $p_i(x,t)$ denote the accompanying probabilities. Normally such a relation is given as:

$$P(K(x,t)=k)=p_i(x,t) \tag{4}$$

Note that the values of k are discrete and hence a discrete probability function can be used. However such a notation indicates a variable probability as a function of given densities. The CPM presumes set probability elements, and therefore the random density variable $K(x,t)$ is defined as a function of set probabilities instead.

So for example $K(x,t)$, now written as vector $\boldsymbol{k}(x,t|\boldsymbol{p})$, denotes all possible values of the density for a moment in time and a location, given the probabilities of these densities. The density vector can also be written as:

$$\boldsymbol{k}(x,t|\boldsymbol{p})=\begin{Bmatrix} k_1(x,t)\ with\ probability\ p_{d.1} \\ k_2(x,t)\ with\ probability\ p_{d.2} \\ \dots \\ k_i(x,t)\ with\ probability\ p_{d.i} \end{Bmatrix} \tag{5}$$

From now on we will only use the short form for the density vector. The addition of the vector $\boldsymbol{p}$ includes all possible values of the appropriate variable with identical probabilities of each value in time and space, so that:

$$\boldsymbol{p}=p_1+p_2+\ldots+p_i=1 \tag{6}$$

Here, i is further limited to a finite value, which is applied as an input parameter of the model.

The equations for the conversation of vehicles (eq. (1)) and the fundamental relation (eq. (2)) now incorporate a further dimension for the probability in time and space, and become dependent on the probability of their value:

$$\frac{\partial \boldsymbol{k}(x,t|\boldsymbol{p})}{\partial t} + \frac{\partial \boldsymbol{q}(x,t|\boldsymbol{p})}{\partial x} = 0 \tag{7}$$

$$\boldsymbol{q}(x,t|\boldsymbol{p}) = Q_E(\boldsymbol{k}(x,t|\boldsymbol{p})) \tag{8}$$

The conservation of vehicles therefore remains intact by definition, as each considered element in the probability distribution vector acts as an individual case of the CTM for which conservation has been proven (Daganzo 1994).

3.3 Application of stochastic demand and traffic propagation

External stochastic traffic demand is applied in the model at the peripherals of a network on the inflowing cells. From there on traffic may propagate applying eq (7) and (8) according to the dynamics of the base model. The initial traffic demand contains j_d times j_c number of elements in the probability vector $\boldsymbol{p}$, where j_d is the number of probability elements in the vector for the demand and j_c is the number of probability elements for the capacity, such that each probability vector $\boldsymbol{p}$ is constructed of all possible combinations of p_{jd} and p_{jd}. The initial flow at the network peripheral is therefore:

$$\boldsymbol{q}(x_0,t_1|\boldsymbol{p}) = \{q_{p1}, q_{p2}, \dots q_{p(jd\cdot jc)}\} \tag{9}$$

Where the probability vector $\boldsymbol{p}$ exists of j_d times j_c elements. This multiplication is performed to accommodate a position in the probability distribution for the outcomes of all combinations of each traffic demand and capacity variation.

The variation in the capacity of the network is applied for each cell corresponding to the probability of the capacity of that cell in a similar way to the traffic flow $\boldsymbol{q}$. In a simplified case only bottleneck cells will have varied capacity values, with the other cells yielding identical capacity values for each element in $\boldsymbol{p}$. The capacity contains j_c probability elements for the capacity in both time and space, although in many cases the capacity will not vary in time:

$$\boldsymbol{q_{cap}}(x_m,t_n|\boldsymbol{p}) = \begin{Bmatrix} q_{cap.1}(x,t)\ with\ probability\ p_{c.1} \\ q_{cap.2}(x,t)\ with\ probability\ p_{c.2} \\ \dots \\ q_{cap.i}(x,t)\ with\ probability\ p_{c.i} \end{Bmatrix} \tag{10}$$

Once the stochastic traffic is on the network, the traffic propagates through the network dependent on the corresponding demand and following the dynamics as previously shown in eq. (7) and eq. (8).

Spatial dependence is applied as a conditional probability at the entrance of a network. This spatial dependence entails that each element in the probability vector of the density corresponds to the same place in the probability vector of the density of the following time step. This ensures an identical number of elements in the resulting probability vector for propagation through the network, and therefore avoids an explosion of marginal probability elements. Basically, this creates a set of values which can be seen as scenarios of unique traffic demand and capacity combinations.

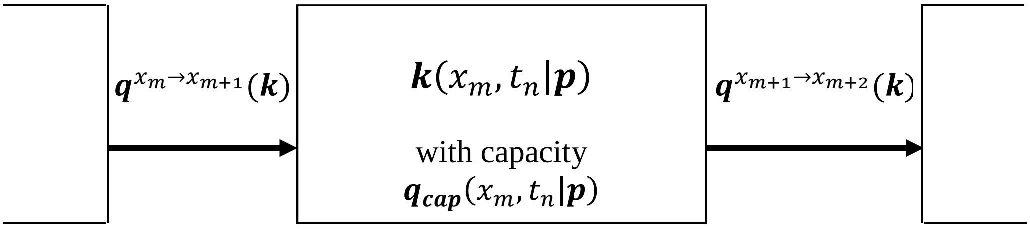

Figure 1 Traffic propagation in the CPM

The process is explained as such: there is a traffic demand $q(x_1, t_1)$ with a set of possible values, $\boldsymbol{q}_p$, corresponding to certain probabilities:

$$\boldsymbol{q}(x_1, t_1|\boldsymbol{p}) = \{q_{p1}, q_{p2}, \dots q_{pi}\} \tag{11}$$

Calculations in the model are performed using the density, therefore q is transformed using (2) to:

$$\boldsymbol{k}(x_1, t_1|\boldsymbol{p}) = \{k_{p1}, k_{p2}, \dots k_{pi}\} \tag{12}$$

In the following time step there is a new q and k at location x_1, in line with traffic flow in and out of the cell and in keeping with the conservation of vehicles (eq(1)) :

$$\boldsymbol{k}(x_1, t_2|\boldsymbol{p}) = \{k_{p1}, k_{p2}, \dots k_{pi}\} \tag{13}$$

However the position of each element in the $\boldsymbol{k}(x_1, t_2|\boldsymbol{p})$ corresponds only to that of the element in the same position in the following time step in $\boldsymbol{k}(x_1, t_2|\boldsymbol{p})$, so that for each element, i, applies:

$$q(x_1, t_2|p_i)\ follows\ \ q(x_1, t_1|p_i) \tag{14}$$

This strict 'chain' requirement, that for each location in consecutive time steps the same probability must apply, protects the validity of the initial conditional dependence between the capacity and traffic demand in both time and space.

Although the base model, and therefore also CPM, calculates traffic using the density, it is often required to translate this to the traffic flow $\boldsymbol{q}(x, t|\boldsymbol{p})$, for determination of the flux for example. This is performed using the fundamental relation shown in eq. (2), in which each value of q is transformed using a deterministic fundamental diagram. The resulting values of $\boldsymbol{q}(x, t|\boldsymbol{p})$ from $\boldsymbol{k}(x, t|\boldsymbol{p})$ maintain the same probabilities for each time step and cell in space.

In the same way, the traffic flow on the subsequent cells is also calculated. The only difference is that the supply and demand refer to those of the following cells, x_j. In such a way, one can speak of multiple scenarios in a single procedure, as each element of the marginal probabilities are considered individually for a single variable.

3.4 Determination of Congestion

The sending and receiving functions, or rather demand and supply, $\boldsymbol{d}$ and $\boldsymbol{s}$, are in part determined by the traffic state. Traffic states are in turn determined by the density of traffic in a cell at a specific time. Under congestion, the demand function is equal to the capacity, and the supply function of the outgoing traffic flow:

$$\boldsymbol{d}(x_m, t_n|\boldsymbol{p}) = \boldsymbol{q_{cap}}(x_{m-1}, t_n|\boldsymbol{p}) \tag{15}$$

$$\boldsymbol{s}(x_m, t_n|\boldsymbol{p}) = \boldsymbol{q}(x_m, t_n|\boldsymbol{p}) \tag{16}$$

For uncongested states, the demand function is the incoming traffic flow, and the supply function is the available capacity:

$$\boldsymbol{d}(x_m, t_n|\boldsymbol{p}) = \boldsymbol{q}(x_{m-1}, t_n|\boldsymbol{p}) \tag{17}$$

$$\boldsymbol{s}(x_m, t_n|\boldsymbol{p}) = \boldsymbol{q_{cap}}(x, t|\boldsymbol{p}) \tag{18}$$

For the Core Probability model without capacity variation, congestion is determined by comparison between the probable density and the critical density of a cell:

$$\boldsymbol{Cong}(x, t|\boldsymbol{p}) = \boldsymbol{k}(x, t|\boldsymbol{p}) \geq k_{crit}(x, t) \tag{19}$$

However, when capacity is also varied, the congestion equation states a distribution vector on either side of the operator:

$$\boldsymbol{Cong}(x, t|\boldsymbol{p}) = \boldsymbol{k}(x, t|\boldsymbol{p}) \geq \boldsymbol{k_{crit}}(x, t|\boldsymbol{p}) \tag{20}$$

3.5 Fundamental relation

The fundamental relation, as given in eq. (8), defines the relationship between the density and the flow of traffic. The fundamental diagram in the CPM is a dynamic diagram dependant on a static maximum speed and critical density values on a link, and a dynamic capacity in time for a set location. The applied fundamental relation in the CPM is a piecewise diagram with a minor capacity discontinuity. The exact form of the diagram does not necessarily need to follow the applied piecewise fundamental diagram shown here. It is defined as such:

$$q(x,t) = v_{max} \cdot k(x,t) \qquad \textit{for} \quad k(x,t) \leq \frac{{}^{2}/_{3} \cdot q_{cap.ff}}{v_{max}} \tag{21}$$

$$q(x,t) = \frac{2}{3} \cdot q_{cap.ff} + \left(\frac{{}^{1}/_{3} \cdot q_{cap.ff}}{k_{crit}(x,t) - \frac{{}^{2}/_{3} \cdot q_{cap.ff}}{v_{max}}} \right) \cdot k(x,t) \qquad \textit{for} \quad \frac{{}^{2}/_{3} \cdot q_{cap.ff}}{v_{max}} < k(x,t) \leq k_{crit}(x,t) \tag{22}$$

$$q(x,t) = q_{cap.cong} \cdot \left(\frac{1 - (k(x,t) - k_{crit}(x,t))}{k_{jam}(x,t) - k_{crit}(x,t)} \right) k(x,t) \qquad \textit{for} \quad k(x,t) > k_{crit}(x,t) \tag{23}$$

where v_{max} is the maximum speed, $q_{cap.ff}$ is the free-flow capacity, $q_{cap.cong}$ is the congested capacity, $\mathrm{k_{crit}(x,t)}$ is the critical density, and $\mathrm{k_{jam}(x,t)}$ is the jam density.

Note that eq. (21), (22) and (23) are given in the scalar notation as would correspond to eq. (2). For the use as in eq. (8), the quantities $k(x,t)$ and $q(x,t)$ would be represented as vectors. A graphical representation of this relation is given in Figure 1.

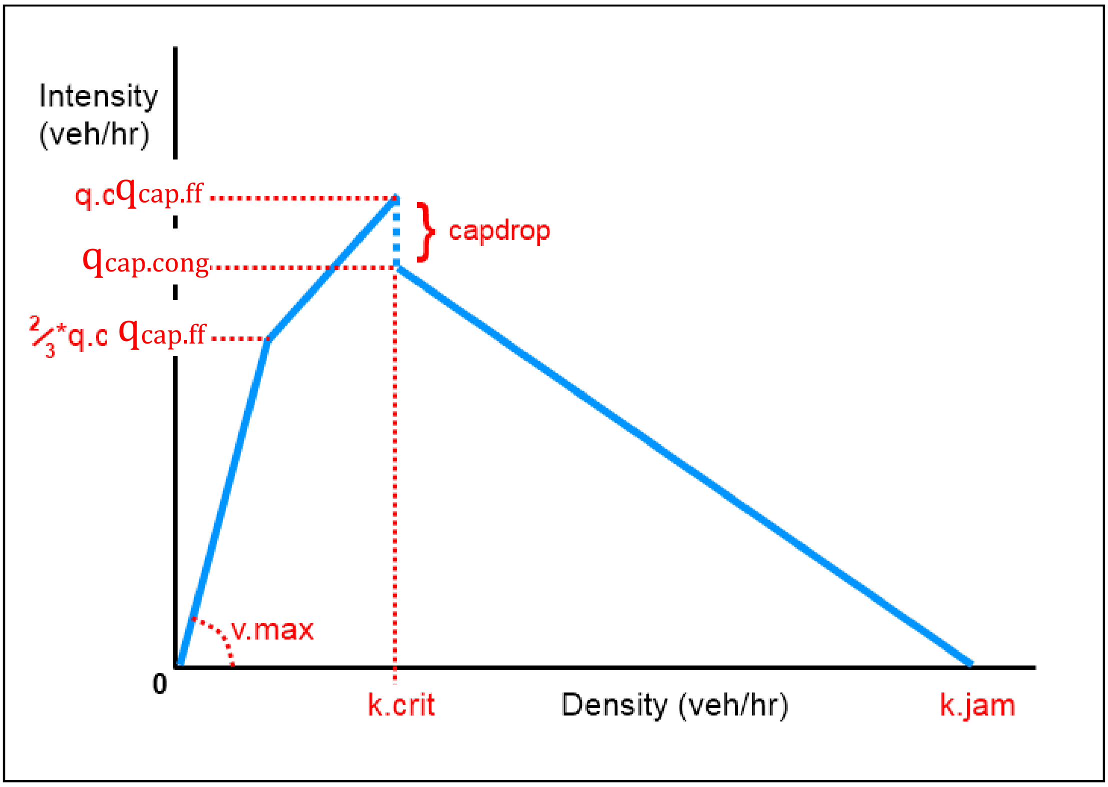

Figure 2 Applied fundamental diagram for the initial tests of the CPM

Uncongested traffic flow
The two piece construction of the fundamental relation for the uncongested part describes accurately the initial conditions of traffic flow for low densities and flows. The speed for low densities and flows remains near the maximum speed of a road, which is defined in traffic theory as:

$$v(x,t) = \frac{q(x,t)}{k(x,t)} \tag{24}$$

And therefore the speed is represented by the slope of the fundamental relation through the origin. The first piece of the fundamental relation follows this line according to the maximum speed limit. As traffic approaches critical density, the speed decreases and the fundamental relation will near the capacity flow. This is represented in the second uncongested piece.

Congested traffic flow
The fundamental relation for congested traffic is presumed to follow a line from the discharge capacity flow and critical density to the jam density value at a zero flow. This is a simplification, as in reality the congested flow will tend to be an area along this line rather than specific points on the line itself. Nevertheless this forms a good representation of reality. The congested piece does not commence from the capacity flow, but rather from the discharge capacity flow (or congested capacity flow). The

difference between the capacity and the discharge capacity is known as the capacity drop. The capacity drop is included in the CPM at two points. Firstly, in this difference between the free flow capacity and the discharge capacity. The discharge capacity is defined as a set factor of the free flow capacity:

$$\boldsymbol{q}_{\boldsymbol{cap.cong}} = c \cdot \boldsymbol{q}_{\boldsymbol{cap}} \tag{25}$$

where c is defined as the capacity drop factor, for which a value of 1.0 denotes not drop in capacity.

The second point of inclusion of the capacity drop is in the sending and receiving functions. When congestion occurs, it is the discharge capacity that is relevant to the traffic flux, rather than free flow capacity. Therefore the $\boldsymbol{q}_{\boldsymbol{cap}}$ term in the sending and receiving functions is not the free-flow capacity, $\boldsymbol{q}_{\boldsymbol{cap.ff}}$, but rather the discharge capacity, $\boldsymbol{q}_{\boldsymbol{cap.cong}}$, so that eq. (18) becomes:

$$\boldsymbol{S}(x, t|\boldsymbol{p}) = \boldsymbol{q}_{\boldsymbol{cap.cong}}(x, t|\boldsymbol{p}) \tag{26}$$

3.6 Simple numerical example (both capacity and demand varied)

To demonstrate the manner in which the CPM works, a simple numerical example is given as demonstration. The traffic demand at the network peripherals is given as an intensity with a set probability. In this example there is a 50% chance of two different inflow values, and there is 50% of two different capacity values. Therefore there are 4 elements in the demand vector, because the size of $\boldsymbol{q}(x, t|\boldsymbol{p})$ is equal to j_d times j_c (see eq. (9)):

$$\boldsymbol{q}\left(x_1, t_1, \boldsymbol{p} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{Bmatrix}\right) = \begin{Bmatrix} 1900 \\ 1900 \\ 2200 \\ 2200 \end{Bmatrix} \tag{A1}$$

The capacity values of the cell are also given in the, j_d times j_c number of elements, capacity flow vector:

$$\boldsymbol{q}_{\boldsymbol{cap}}\left(x_1, t_1, \boldsymbol{p} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{Bmatrix}\right) = \begin{Bmatrix} 2100 \\ 2300 \\ 2100 \\ 2300 \end{Bmatrix} \tag{A2}$$

Note that the sequences for the values of the flow in the demand vector (Eq A1) are differently arranged over the j_d times j_c elements in comparison to the capacity flow vector (Eq A2).

This flow vector, $\boldsymbol{q}(x, t|\boldsymbol{p})$, in eq. (A1) is transformed to a density vector, $\boldsymbol{k}(x, t|\boldsymbol{p})$, using the fundamental relation $q = Q_E(k)$ in which the critical density is $k_{crit} = 25$. This gives:

$$\boldsymbol{k}\left(x_1, t_1, \boldsymbol{p} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{Bmatrix}\right) = \begin{Bmatrix} 22 \\ 20 \\ 26 \\ 24 \end{Bmatrix} \tag{A3}$$

The probability of congestion is calculated using eq. (20):

$$\boldsymbol{Cong}\left(x_1, t_1, \boldsymbol{p} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{Bmatrix}\right) = \boldsymbol{k}(x, t|\boldsymbol{p}) \geq k_{crit}(x, t) = \left[\begin{Bmatrix} 22 \\ 20 \\ 26 \\ 24 \end{Bmatrix} \geq 25\right] = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \tag{A4}$$

Therefore, based on eq.(15) through eq. (18), the demand *D* and supply *S*, can be calculated as:

$$\boldsymbol{D}\left(x_1, t_1, \boldsymbol{p} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{Bmatrix}\right) = \begin{Bmatrix} 1900 \\ 1900 \\ 2200 \\ 2200 \end{Bmatrix} \quad \text{and} \quad \boldsymbol{S}\left(x_1, t_1, \boldsymbol{p} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{Bmatrix}\right) = \begin{Bmatrix} 2100 \\ 2300 \\ 2100 \\ 2300 \end{Bmatrix} \tag{A5}$$

The flux between two cells is defined and given as:

$$\boldsymbol{q}^{x_i \to x_{i+1}}\left(x_1, t_1, \boldsymbol{p} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{Bmatrix}\right) = \min\big(\boldsymbol{D}(\boldsymbol{k}), \boldsymbol{S}_{i+1}(\boldsymbol{k})\big) = \begin{Bmatrix} 1900 \\ 1900 \\ 2100 \\ 2200 \end{Bmatrix} \tag{A6}$$

The density therefore in the current and following cells in the following time step, t_2, is given by the previous density adjusted by the flux into and out of that cell, during the size of the time step, *h*. Here we presume an identical inflow into cell x_1 for t_2 as in t_1:

$$\boldsymbol{k}\left(x_1, t_2, \boldsymbol{p} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{Bmatrix}\right) = \boldsymbol{k}(x_1, t_1|\boldsymbol{p}) + (\boldsymbol{q}^{x_0 \to x_1} - \boldsymbol{q}^{x_1 \to x_2}) \cdot h$$
$$= \begin{Bmatrix} 22 \\ 20 \\ 26 \\ 24 \end{Bmatrix} + \left(\begin{Bmatrix} 1900 \\ 1900 \\ 2200 \\ 2200 \end{Bmatrix} - \begin{Bmatrix} 1900 \\ 1900 \\ 2100 \\ 2200 \end{Bmatrix}\right) \cdot h$$

Similarly, the flow into the yet unoccupied cell x_1 is calculated:

$$\boldsymbol{k}\left(x_1, t_2, \boldsymbol{p} = \begin{Bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{Bmatrix}\right) = \boldsymbol{k}(x_1, t_1|\boldsymbol{p}) + (\boldsymbol{q}^{x_0 \to x_1} - \boldsymbol{q}^{x_1 \to x_2}) \cdot h$$
$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \left(\begin{Bmatrix} 1900 \\ 1900 \\ 2100 \\ 2200 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\right) \cdot h$$

This same process repeats itself for each cell in each time step and so on.

4 Addressing the main issues

This section describes the manner in which the important issues from section 2 are addressed in the CPM and improve on the current state-of-art.

For **computational efficiency**, the main challenge is to reduce computational load and in doing so, do it in a way that the model is not reduced in stochastic and modelling accuracy. Compared to a Monte Carlo simulation, the CPM does not require multiple repetitive simulations before arriving at a distribution, as the distribution of the traffic variables is explicit to the methodology. Therefore the computational load will be lighter if a single CPM simulation run is quicker than the sum of the required number of Monte Carlo simulations on the same base model. It is hypothesised that this is the case, as the CPM has a single computational overhead for the entire distributions, while a Monte Carlo simulation has a computational overhead for each simulation iteration. Furthermore, a lower detail of discretisation is hypothesised to be required for the CPM as the model calculates using distributions throughout. Monte Carlo simulation makes use of less efficient random process of sampling, which reduces the completeness of a distribution and therefore requires a greater number of simulations to reach the same level of accuracy, therefore increasing the computational load. On simple network or corridors, the efficiency effect will be limited, however for larger networks and for a greater spread of variation the gains should be greater. As mentioned, the hypotheses have still to be extensively tested and may only be attainable for networks rather than for corridors as described in this contribution, however theoretically, gains seem highly plausible.

Spatiotemporal dependency is catered for in the CPM through the explicit consideration of correlations at the peripheral of the model and maintenance thereof in propagation. Reduced to two dependant variables, the traffic demand and road capacity, correlations between possible values of both are explicitly considered in the distributions entering a network at the peripherals. Values in the initial distribution vector of the traffic demand entering the network correspond on an element-to-element bases to that of values of the capacity distribution vector at the same element location. This was explained in section 4. By explicitly maintaining this ‘chain’ connection throughout the traffic propagation, independency between traffic demand and capacity is maintained. Dependency in time for both the demand and capacity is also explicitly dealt with outside the model. Input values for certain elements in the distribution vectors follow those of the preceding time step and therefore already consider a logical and dependant propagation from the input vectors in time. Spatial dependency is dealt with in the same way as in the base model and therefore requires no further attention. Simplified, each element in a distribution vector may be seen as a single input value for a single Monte Carlo simulation, therefore it may also be considered as independent from other elements just as a single Monte Carlo iteration is from another Monte Carlo iteration.

Stochastic propagation of probability in traffic flow is performed as described in section 3.3 and is also touched upon in the previous paragraph on spatiotemporal dependency. A complete distribution of possible values per traffic variable is present as a distribution in the form of a vector. This vector exists of more elements than is necessary, so to allow each possible values of that vector to correspond to the elements of other vectors and therefore to avoid correlation difficulties. As these distribution vectors are propagated in space and time, there is no need to reduce variables to a representation of the distribution using a set distribution type, median, standard deviation, shape parameter or such like. Although this may lead to a higher computational effort, it maintains a guaranteed accuracy of the propagation of the traffic variables and their probabilities, as the distributions remain intact in the process of propagation. Therefore a greater accuracy can be achieved in comparison to methods that do transform distributions to characteristics of the distribution.

For the CPM, the question of **generality** is one that is less relevant to the model itself, but rather to the quality of the data and distributions that it is fed with. As the CPM performs calculations using discrete distributions, a reduction of the input data may only happen in the case of rediscretisation for the sake of computational efficiency. Therefore the necessity to apply accurate input distributions for the traffic demand and road capacity is applicable for the local circumstances or from a general distribution if the local situation is not known. Construction of generic input distributions for this purpose, taken from empirca, makes it easy to apply the CPM without requiring extensive data analysis for each application of the model (Calvert et al. 2014; van Stralen et al. 2014). Nevertheless, this issue is one that is less explicit to the model, as the quality of input data is relevant and independent to all models. However the manner in which a model deals with accurate input is important. The CPM does not overly simplify input, therefore maintaining high level of accuracy and avoiding additional unnecessary biases, contrary to many other models.

5 Test case

Demonstration of the application and validity of the Core Probability Model (CPM) is given in a test case. The test case aims to show that traffic propagation along a road section in the CPM can accurately resemble traffic flow found from empirical observations.

The test case is carried out for the A12 motorway in The Netherlands between Utrecht and The Hague (see Figure 3). On this motorway in 2009[1], a lane drop was present from four to three lanes, which acted as a structural bottleneck at location A. Daily congestion starting at this location near the town of Woerden would be present, especially during the evening peak period. A section of 11 kilometres is considered, of which 10 km upstream and 1 km downstream of the bottleneck. The CPM is fed with data from 63 afternoon peak period observations of the traffic flow between 2 PM and 9 PM from 2009 as a representation of the probability of certain traffic flows appearing. The input for the model is taken exclusively from the most upstream location. Therefore the validation is that of the stochastic traffic propagation. Each observation is considered as an equal probability of a real traffic demand for this location and is therefore given a 100/63 = 1.6% probability for the input at the inflow of the corridor. These traffic flows are fed into the network at the most upstream location.

[1] Since 2009, this location has been upgraded to four lanes along the entire stretch to eradicate the bottleneck.

Figure 3 Bottleneck location near Woerden at the considered road section on the A12 used in the case study

A comparison is made based on the ability of the model to accurately predict the propagation of the probabilities of traffic flow and corresponding traffic states between the outcome of the CPM simulation and the empirical data. For this, the unfiltered traffic states in time and space are gathered on the entire corridor. The comparison focusses on the time of traffic breakdown, congestion duration, spill-back distance, and the specific speed values in time and space. This is shown for the median probability (most likely traffic situation) and a further demonstration of the results are given in the form of a 3D congestion probability plot. The results of the median probability are shown in the time-space Figure 4.

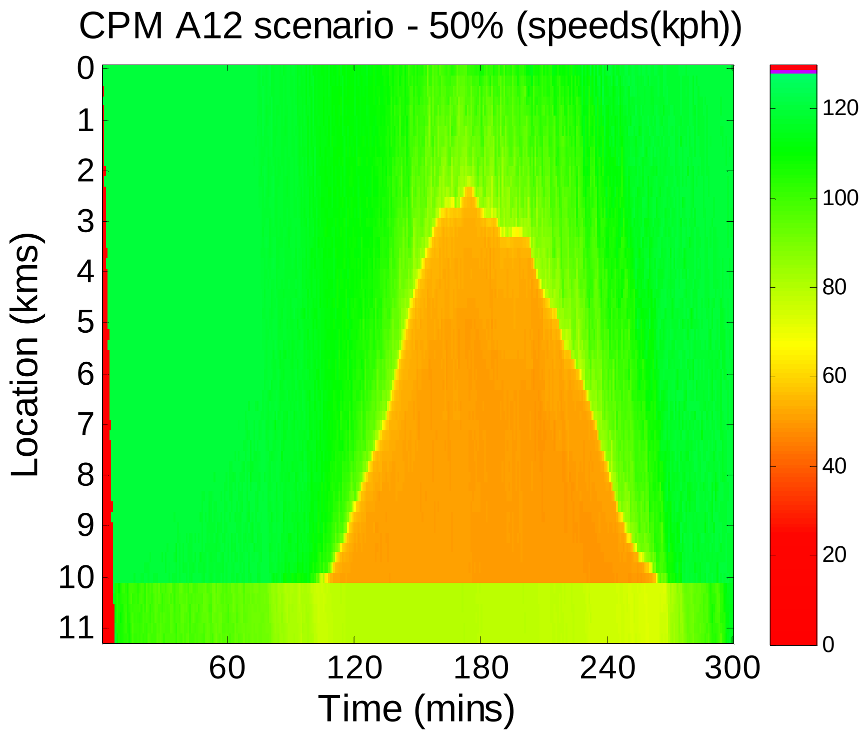

Figure 4 Modelled speed diagram for the median probability in the A12 test case

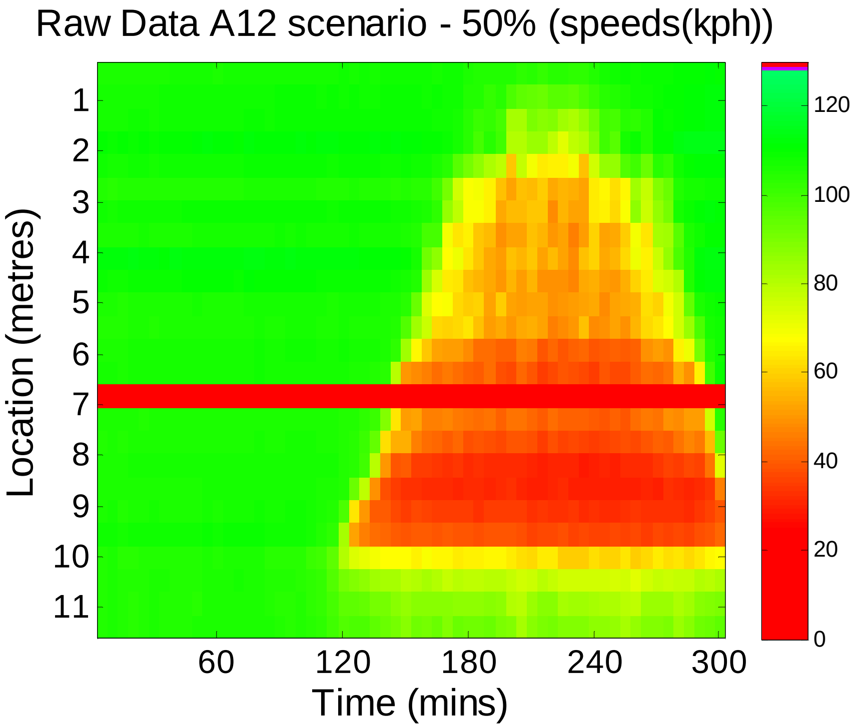

Figure 5 Empirical speed data for the median observation in the A12 test case [2]

The initial results shown in Figure 4 show the simulated median (50%) results from the model, compared to the median from the empirical data shown in Figure 5. The speed values are shown as these give a good indication of where congestion is present, how extreme congestion is and how traffic flow changes the time. Initially the extent of congestion appears to be relatively well modelled. Nevertheless there are certain deviations in comparison to the empirical data. The onset of congestion occurs approximately 10 minutes earlier in the simulation, while congestion lasts for 158 minutes compared to 190 minutes in the data. However, the spillback of congestion in both is of a similar magnitude and deviates no more than 200 meters over a distance of some 9 kilometres. The speed in the heavily congested area of traffic is lower in the empirical data compared to the model (ca. 30 kph versus 40 kph). This may also be a main reason why the duration of congestion differs, as traffic in the simulation may proceed at a slightly higher speed and therefore let congestion disperse earlier. Despite these minor deviations, this initial test case gives cause for optimism. A further fine-tuning of the model parameters when applied in practice may easily compensate for the observed differences.

[2] The red horizontal line indicates a location at which a faulty detector is present. The speed at this location is returned as null.

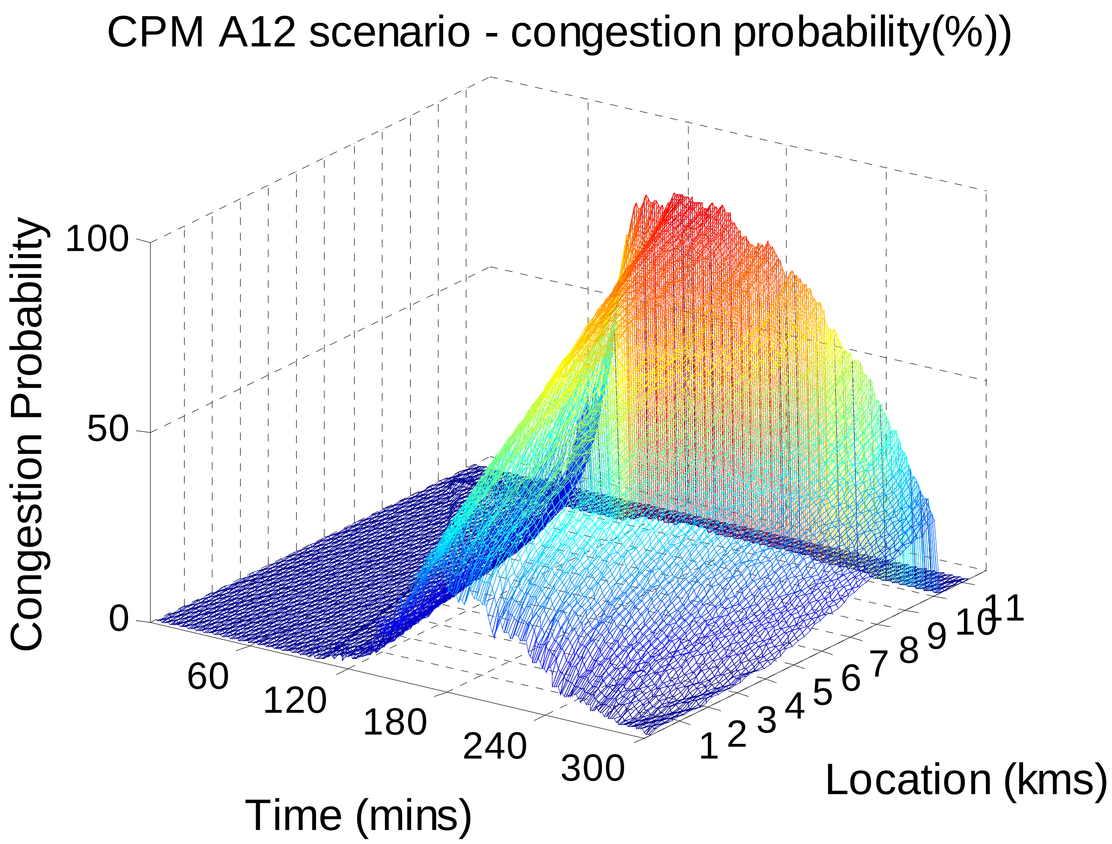

Figure 6 Modelled congestion probability in time and space for the A12 test case

The CPM method allows a vast amount of data to be produced and presented as a probability distribution or in another forms as a direct consequence of the way the CPM works. As each traffic variable is considered as a distribution of possible values, each can therefore be calculated or shown as such at each time step and location. This is demonstrated in Figure 6 in which the congestion probability at each location and for every time step is given. Congestion is defined as such when the critical density is exceeded, while the probability thereof indicates the frequency that congestion is expected to occur for an arbitrary location and time along the corridor. It is possible to show more complex results in a greater number of dimensions, i.e. including the probability as a variable in a diagram, however this leads to difficulties in the interpretation of diagrams. Nevertheless, broad analyses are made much easier and more extensive with the results from the CPM. At this moment there is no reliable estimation available on the possible efficiency gains in terms of computation. This is mainly due to the experimental implementation of the methodology that first requires restructuring before reliable computational efficiency tests can be carried out. Furthermore, significant computational gains are not expected on a single corridor, but rather for networks and for greater variations in stochastic variables.

6 Conclusions and current developments

6.1 Current developments

The basic concept of the Core Probability Model has been explained and an initial validation case has been shown in this paper. Thus far, the CPM has been demonstrated on a single road stretch without additional incoming or outgoing links and traffic. Current developments are focussed on the extension of the CPM for use on networks. This brings greater challenges than addressed thus far in this paper. Propagation of traffic variable probabilities through a network gives arise to convergence and divergence of probability values at each node. For this a methodology has been developed to efficiently cope with these transformation of probabilities at nodes. We are currently in the process of implementing and testing the methodology and plan to disseminate the developments before long. Other developments focus on the application of external stochastic variations of traffic influencing variables, and dealing with correlation effects therein. For traffic propagation, correlation between variables is externalised as described in section 3, however external preparation of variables requires attention prior to application in the CPM.

6.2 Conclusions

In this contribution the Core Probability Model (CPM) is introduced as a new model for dynamic macroscopic modelling of stochastic traffic flow. The CPM extends current deterministic traffic flow models by redefining traffic variables in the core of the model as distribution vectors of probable values for each traffic variable. In such a way stochastic variation in traffic is internalised in the model and does away with the necessity of repetitive Monte Carlo simulation. Furthermore a greater degree of flexibility in analysis is obtained, as each individual traffic variable in time and space may be given as a function of their probability. Moreover, the underlying distribution of each traffic variable in space and time is preserved such that the introduction of distribution fitting errors is limited to a minimum. Important issues facing stochastic traffic flow modelling are given in the contribution, and are identified as *computational efficiency*, *spatiotemporal dependency*, *stochastic propagation of probability*, and *stochastic generality*. The CPM addresses each of these issues and in doing so demonstrates the ability to advance developments in the area of stochastic traffic modelling. In particular the CPM aims to further the possibilities for reliable, accurate, efficient, and most of all, practically applicable stochastic macroscopic traffic flow. At present, reliable calculation of the computational efficiency of the model is not possible, but is expected in the near future.

Future developments of the CPM are focussed on the propagation of stochastic traffic on networks, for which a methodology has been developed and is currently being implemented and tested. Also methodologies for the application of external stochastic variation from multiple variables are currently under development.

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