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The interior of randomly perturbed self-similar sets on the line

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ABSTRACT

Can we find a self-similar set on the line with positive Lebesgue measure and empty interior? Currently, we do not have the answer for this question for deterministic self-similar sets. In this paper we answer this question negatively for random self-similar sets which are defined with the construction introduced in the paper by Jordan et al. (2007) [6]. For the same type of random self-similar sets we prove the Palis-Takens conjecture which asserts that at least typically the algebraic difference of dynamically defined Cantor sets is either large in the sense that it contains an interval or small in the sense that it is a set of zero Lebesgue measure.

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1. Introduction

In this paper we consider only Random self-similar Iterated Function Systems (RIFS) which are defined on the line and which can be obtained as a small random perturbation of a deterministic self-similar Iterated Function System (IFS) on the line. First we give a short description of our results for the expert, and then we provide a more detailed introduction. We do not write "self-similar" in the abbreviation since all iterated function systems considered in this paper are self-similar (random or deterministic).

1.1. Informal description of the main result for experts

Using the construction introduced by Jordan, Pollicott and Simon [6, p. 521], we define self-similar Random Iterated Function Systems (RIFS) $\mathcal{F}$ on the line as follows: We start with a self similar IFS $\mathcal{S}$ on the line and we add a small random additive error to every map in every step of the iterative construction of the attractor (see Definition 2.1 for the precise definition of RIFSs). The scaling parts of the similarities of the deterministic IFS $\mathcal{S}$ are left unchanged. So, the similarity dimension $s(\mathcal{F})$ of the RIFS $\mathcal{F}$ is the same as the similarity dimension of the deterministic self-similar IFS $\mathcal{S}$. Our main result is that

$$s(\mathcal{F}) > 1 \Longrightarrow \mathrm{int}(C_{\mathcal{F}}) \neq \emptyset, \quad \text{almost surely,} \tag{1}$$

where $C_{\mathcal{F}}$ is the attractor of the RIFS $\mathcal{F}$. This implies that whenever $C_1,\ C_2$ are two independent copies of the attractor of the RIFS $\mathcal{F}$ then the algebraic difference set $C_2 - C_1 := \{c_2 - c_1 : c_1 \in C_1, c_2 \in C_2\}$ satisfies

$$s(\mathcal{F}) > \frac{1}{2} \Longrightarrow \mathrm{int}(C_2 - C_1) \neq \emptyset, \quad \text{almost surely.} \tag{2}$$

1.2. A gentle introduction

A deterministic self-similar Iterated Function System (IFS) on $\mathbb{R}$ is a finite list of contracting similarities of $\mathbb{R}$:

$$\mathcal{S} := \{S_i : S_i(x) = r_i x + t_i,\ x \in \mathbb{R}\}_{i=1}^{L}, \tag{3}$$

with contractions $r_i \in (-1,1) \setminus \{0\}$ and translations[1] $t_i \in \mathbb{R}$ for all $i \in [L] := \{1, \ldots, L\}$. The attractor $C_{\mathcal{S}}$ of the IFS $\mathcal{S}$ is what we are left with after infinitely many iterations of the system. More formally, it is easy to see that we can find a non-degenerate compact interval I such that $S_i(I) \subset I$ holds for all $i \in [L]$. For all $\mathbf{i} \in [L]^n$ we consider the n-fold iterate

[1] We do not assume that the t_i are distinct.

$$S_{\mathbf{i}} := S_{i_1} \circ \cdots \circ S_{i_n} \tag{4}$$

and form the corresponding n-cylinder interval $I_{\mathbf{i}} := S_{\mathbf{i}}(I)$. Then the union of all n-cylinders $\Big\{ \bigcup_{\mathbf{i}\in[L]^n} I_{\mathbf{i}} \Big\}_{n=1}^{\infty}$ is a nested sequence of non-empty compact sets. The attractor is their intersection:

$$C_{\mathcal{S}} := \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i}\in[L]^n} I_{\mathbf{i}}. \tag{5}$$

In this paper, we consider random IFSs (RIFS) on $\mathbb{R}$, which are small (translational) perturbations of a self-similar IFS of the form (3).

Informally, the attractor of an RIFS is obtained by a formula similar to (5) with the following difference: Instead of the deterministic n-cylinder intervals $I_{\mathbf{i}}$ in (5), we work with the random intervals $\widehat{I_{\mathbf{i}}} = f_{i_1} \circ \cdots \circ f_{i_n}(I)$, where the random mappings f_{i_k} are small translational perturbations of S_{i_k}. Namely, $f_{i_k} = S_{i_k} + Y_{i_k}$ for small random translations Y_{i_k}. The precise description of the distribution of these random translations is given in Definition 2.1. The attractor $C_{\mathcal{F}}$ of the RIFS $\mathcal{F} := \{f_i\}_{i=1}^{L}$ is defined by a formula analogous to (5): we just replace $I_{\mathbf{i}}$ with $\widehat{I_{\mathbf{i}}}$ in (5).

Jordan, Pollicott, and Simon [6] studied this kind of RIFSs in the more general self-affine case. As an immediate consequence of the results in [6], we get that the Hausdorff dimension $\dim_{\mathrm{H}} C_{\mathcal{F}}$ is the minimum of 1 and the similarity dimension $s_{\mathcal{F}}$ (solution of (12)) almost surely. Moreover, if $s_{\mathcal{F}} > 1$ then the Lebesgue measure of $C_{\mathcal{F}}$ is positive almost surely.

1.3. Motivation: the interior of the difference of random Cantor sets

In 1987, Palis and Takens [10] studying the dynamical behavior of diffeomorphisms presented a conjecture about the size of the algebraic difference of two Cantor sets. Informally, the conjecture states that if the size of the Cantor sets is large (see Equation (6)), then the difference contains an interval. More precisely, if C_1 and C_2 are two Cantor sets then the algebraic difference

$$C_2 - C_1 = \{y - x : x \in C_1, y \in C_2\}$$

contains an interval if

$$\dim_{\mathrm{H}} C_1 + \dim_{\mathrm{H}} C_2 > 1, \tag{6}$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension.

In 2001, De Moreira and Yoccoz ([9]) proved the conjecture for generic dynamically generated *non-linear* Cantor sets. The conjecture has not been proven for generic linear Cantor sets. See also [14].

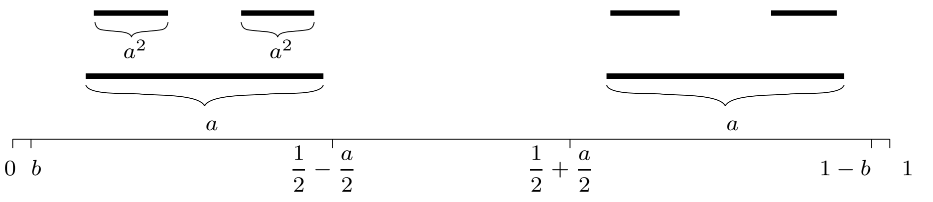

Fig. 1. The construction of the Cantor set $C_{a,b}$. The figure shows the level 1 and level 2 cylinder intervals $C^1_{a,b}$ and $C^2_{a,b}$.

In 1990, Per Larsson put the problem into a probabilistic context in [8], (see also [7]). He considered a very special family of two parameter random Cantor sets $C_{a,b}$ and proved the conjecture for a certain subset of a's and b's. Although the main idea of Larsson's argument is brilliant, unfortunately the proof contains significant gaps and incorrect reasonings. In 2011, three out of the four authors of the present paper gave a precise proof for Larsson's family in [1]. We briefly recall the Larsson family from [1]: let

$$a > \frac{1}{4} \quad \text{and} \quad 3a + 2b < 1.$$

Since

$$\dim_{\mathrm{H}} C_{a,b} = -\frac{\log 2}{\log a},$$

the first condition is equivalent to $\dim_{\mathrm{H}} C_{a,b} > 1/2$, which is equivalent to equation (6).

Larsson's construction is as follows (see also Fig. 1): first remove the interval

$$\left[\frac{1}{2} - \frac{a}{2}, \frac{1}{2} + \frac{a}{2}\right]$$

from the middle of $[0, 1]$, then the length b parts from both the beginning and the end of the unit interval. Next, put intervals of length a according to a uniform distribution in the remaining two gaps $\left[b, \frac{1}{2} - \frac{a}{2}\right]$ and $\left[\frac{1}{2} + \frac{a}{2}, 1 - b\right]$. These two randomly chosen intervals of length a are called the level one intervals of the random Cantor set $C_{a,b}$.

We write $C^1_{a,b}$ for their union. In both of the two level one intervals we repeat the same construction independently of each other and of the previous step. In this way we obtain four disjoint intervals of length a^2. We emphasize that, because of independence, the relative positions of these second level intervals in the first level ones are in general completely different. Similarly, we construct the 2^n level n intervals of length a^n. We call their union $C^n_{a,b}$. Then Larsson's random Cantor set is defined by

$$C_{a,b} := \bigcap_{n=1}^{\infty} C^n_{a,b}.$$

As a corollary of the main result of this paper, we prove that the conjecture by Palis and Takens holds for a very broad class of *random* linear Cantor sets, including the Larsson family.

The following result is a generalization of the result in [1]:

Theorem 1.1. *Let $\mathcal{F}$ be an RIFS (see Definition 2.1) with similarity dimension larger than $\frac{1}{2}$. Let C_1 and C_2 be two independent copies of the attractor $C_{\mathcal{F}}$. Then*

$$C_2 - C_1 \text{ contains an interval a.s.}$$

The proof is presented in Section 5.

It is important to note that in our setting the Hausdorff dimension equals the (the minimum of 1 and) the similarity dimension given by the unique solution of equation (12).

We remark that if the Hausdorff dimension of $C_{\mathcal{F}}$ is smaller than $\frac{1}{2}$ then the set $C_2 - C_1$ has Hausdorff dimension less than 1 so it cannot possibly contain any intervals.

The essential part of the proof of this theorem is completely different of that of the main result in [1]. The proof in [1] was tailored for the Larsson's family, and does not have the potential for generalizations. However, we show that it is possible to prove a much more general result with a shorter proof. For this we combine the ideas of the proof in [1] with the method introduced in Rams, Simon [12], and invoke an observation from Peres, Shmerkin [11]. Namely, both in [1] and the present paper we have to verify that the associated multi-type branching processes are uniformly supercritical, where uniformity is meant in the type of the ancestor. In both papers this is stated as the Main Lemma and their proofs follow the same path. However, the step where using the Main Lemma one proves the existence of intervals in the arithmetic difference of the random Cantor sets, is where we use the method introduced in [12] to obtain our powerful Theorem 9, and this makes our present proof much more efficient.

We remark that if *also* the scalings are random, with a uniform distribution over the parameters, then the problem becomes easier ([13]), and can be treated by a method introduced by Hochman and Shmerkin ([4]).

2. RIFS

2.1. The formal description of our random Cantor set

First, we give the formal definition of the Random Iterated Functions System (RIFS) $\mathcal{F}$, whose attractor $C_{\mathcal{F}}$ is the random Cantor set which is the object of our investigation in this paper. It is convenient to identify the collection of all finite words over the alphabet $[L] = \{1, 2, \ldots, L\}$ with the nodes of the L-ary tree $\mathcal{T}$. The empty word is identified with the root of $\mathcal{T}$, and denoted as $\emptyset$. For any $n \geq 1$ the level n sets $\mathcal{L}_n$ of $\mathcal{T}$ are defined by

$$\mathcal{L}_0 = \{\emptyset\}, \quad \mathcal{L}_n = \{i_1 \ldots i_n : i_j \in [L], 1 \leq j \leq n\}.$$

Definition 2.1 *(RIFS).* Let

$$\mathcal{F} = \{f_i : f_i(x) = r_i x + D_i\}_{i=1}^{L}. \tag{7}$$

The contraction ratios $r_1, \dots, r_L \in (-1,1) \setminus \{0\}$ are deterministic. We assume the following about the random translations $(D_1, \dots, D_L)$, of the functions in $\mathcal{F}$ in (7):

(a) *$(D_1, \dots, D_L)$ is an L-dimensional random vector such that for any $i = 1, \dots, L$, the random variable D_i is absolutely continuous w.r.t. the Lebesgue measure, with a density function φ_i which is strictly positive, bounded and continuous on $(t_i - \theta_i, t_i + \theta_i)$* and φ_i is zero outside $(t_i - \theta_i, t_i + \theta_i)$, where the t_i and $\theta_i > 0$ are real numbers.

(b) To define the random translations of the iterates of this system we introduce

$$\left\{D^{(\mathbf{i})} = \left(D_1^{(\mathbf{i})}, \dots, D_L^{(\mathbf{i})}\right)\right\}_{\mathbf{i} \in \mathcal{T}}$$

as a set of i.i.d. random vectors having the same distribution as that of $(D_1, \dots, D_L)$. The iterates $f_{\mathbf{i}}$ for $\mathbf{i} \in \mathcal{L}_n$ are defined for $n \geq 1$ by:

$$\begin{aligned} f_{\mathbf{i}}(x) &= f_{i_1} \circ \cdots \circ f_{i_n}(x) \\ &= r_{i_1}\left(r_{i_2}\left(\dots\left(r_{i_{n-1}}(r_{i_n} x + D_{i_n}^{(i_1 \dots i_{n-1})}) + D_{i_{n-1}}^{(i_1 \dots i_{n-2})}\right)\dots\right) + D_{i_2}^{(i_1)}\right) + D_{i_1}^{(\emptyset)} \\ &= r_{\mathbf{i}} x + T_{\mathbf{i}} \end{aligned} \tag{8}$$

where $r_{\mathbf{i}} = r_{i_1} \cdots r_{i_n}$ and

$$T_{\mathbf{i}} = D_{i_1}^{(\emptyset)} + r_{i_1} D_{i_2}^{(i_1)} + r_{i_1} r_{i_2} D_{i_3}^{(i_1 i_2)} + \dots + r_{i_1} \cdots r_{i_{n-1}} D_{i_n}^{(i_1 \dots i_{n-1})}. \quad \square \tag{9}$$

In addition, we define $T_\emptyset := 0$.

Using that the random mappings f_i, $i \in [L]$ are contractions and the supports of the D_i are bounded, we immediately obtain that there exists a deterministic interval $[\alpha, \beta]$ such that

$$f_i([\alpha, \beta]) \subset [\alpha, \beta], \quad \text{for all } i \in [L]. \tag{10}$$

We call $[\alpha, \beta]$ a *supporting interval* for $\mathcal{F}$.

The attractor $C_{\mathcal{F}}$ of the RIFS $\mathcal{F}$ is defined by (see Fig. 2)

$$C_{\mathcal{F}} = \bigcap_{n=1}^{\infty} \bigcup_{|\mathbf{i}|=n} f_{\mathbf{i}}([\alpha, \beta]). \tag{11}$$

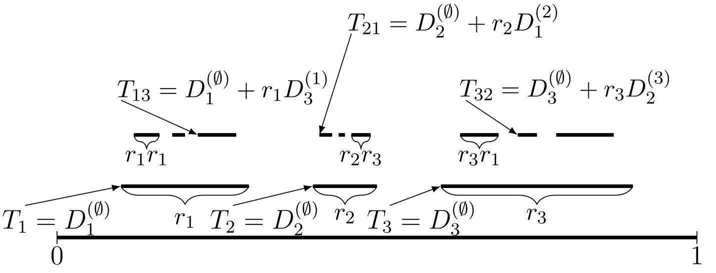

Fig. 2. Level 1 and 2 cylinder intervals of our Cantor set when $L = 3$ and $[\alpha, \beta] = [0, 1]$. The randomly chosen left endpoints T_i and some of the T_{ij}, $i, j = 1, 2, 3$ are indicated.

2.2. The ambient probability space

The sample space is $\Omega := [\mathbb{R}^L]^{\mathcal{T}}$. The corresponding σ-algebra $\mathcal{B}$ is the generated Borel σ-algebra. The probability measure of our Cantor set is

$$\mathbb{P} = \prod_{\mathbf{i} \in \mathcal{T}} d\left(D^{(\mathbf{i})}\right),$$

where $d(X)$ denotes the probability distribution of a random vector X. Then the ambient probability space is $(\Omega, \mathcal{B}, \mathbb{P})$.

A realization $\boldsymbol{\omega} \in \Omega$ is a labeled tree, $\boldsymbol{\omega} = \left\{D^{(\mathbf{i})}\right\}_{\mathbf{i} \in \mathcal{T}}$, where the i.i.d. collection of random vectors $\left\{D^{(\mathbf{i})}\right\}_{\mathbf{i} \in \mathcal{T}}$ was defined in Definition 2.1.

The dimension theory of the RIFS described above is well understood. The following theorem is a direct consequence of the results in [6].

Theorem 2.1 *(Dimension of an RIFS). Let $\mathcal{F}$ be an RIFS of size L and let $s(\mathcal{F})$ denote the solution to the equation*

$$\sum_{i=1}^{L} |r_i|^{s(\mathcal{F})} = 1. \tag{12}$$

We say that $s(\mathcal{F})$ is the similarity dimension of $\mathcal{F}$. Then we have for almost all realizations:

$$\dim_{\mathrm{H}} C_{\mathcal{F}} = \overline{\dim}_B C_{\mathcal{F}} = \underline{\dim}_B C_{\mathcal{F}} = \min\{1, s(\mathcal{F})\}$$

Moreover, if $s(\mathcal{F}) > 1$ then for almost all realizations:

$$\textsc{Leb}\left(C_{\mathcal{F}}\right) > 0, \tag{13}$$

where $\textsc{Leb}(\cdot)$ is the 1-dimensional Lebesgue measure.

With these definitions we can state our Main Theorem:

Theorem 2.2 *(Main Theorem). Let $C_{\mathcal{F}}$ be the attractor of an RIFS $\mathcal{F}$ with $s(\mathcal{F}) > 1$. Then*

$$\mathrm{int}(C_{\mathcal{F}}) \neq \emptyset. \tag{14}$$

The proof is presented in Section 5.

2.3. The n-th order RIFS

To handle the notion of an n-th order RIFS we consider the height n subtrees of $\mathcal{T}$ defined by

$$\mathcal{T}_n(\emptyset) = \bigcup_{j=0,\dots,n-1} \mathcal{L}_j, \quad \mathcal{T}_n(\mathbf{k}) = \bigcup_{j=0,\dots,n-1} \{\mathbf{i} \in \mathcal{L}_{\ell+j} : i_1 \dots i_\ell = \mathbf{k}\},$$

for each $\mathbf{k} = k_1 \dots k_\ell$.

Definition 2.2. Let $\mathcal{F} = \{f_i\}_{i=1}^L$ be an RIFS. The *n-th order of* $\mathcal{F}$, written as $\mathcal{F}^n$, is defined for $n = 1, 2, \dots$ by

$$\mathcal{F}^n = \{f_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{L}_n}.$$

Actually $\mathcal{F}^n$ is itself an RIFS.

Lemma 2.1. *Let $\mathcal{F}$ be an RIFS and let $\mathcal{F}^n$ be its n-th order, for some $n \geq 1$. Then $\mathcal{F}^n$ is an RIFS.*

Proof. Recall that the elements of $\mathcal{F}^n$ indeed have the form $f_{\mathbf{i}}(x) = r_{\mathbf{i}}x + T_{\mathbf{i}}$, and (see (9)) that the random translations $T_{\mathbf{i}}$ satisfy

$$T_{\mathbf{i}} = D_{i_1}^{(\emptyset)} + r_{i_1} D_{i_2}^{(i_1)} + r_{i_1} r_{i_2} D_{i_3}^{(i_1 i_2)} + \dots + r_{i_1} \cdots r_{i_1 \dots i_{n-1}} D_{i_n}^{(i_1 \dots i_{n-1})}. \tag{15}$$

Note first that $(T_{\mathbf{i}} : \mathbf{i} \in \mathcal{L}_n)$ is an L^n-dimensional random vector such that for any $\mathbf{i}$, the random vector $T_{\mathbf{i}}$ is bounded, and absolutely continuous w.r.t. the Lebesgue measure, supported on an interval with strictly positive and continuous density on its interior.

For a set of nodes $\mathcal{N}$, let us write $D^{\mathcal{N}}$ for the random vector with elements $D^{(\mathbf{i})}$ with $\mathbf{i} \in \mathcal{N}$.

We see from Equation (15) that the translations of the $f_{\mathbf{i}}$'s with $\mathbf{i} \in \mathcal{L}_n$ are completely determined by $D^{\mathcal{T}_n(\emptyset)}$. More generally, for all $k \geq 1$ the translations of the $f_{\mathbf{i}}$'s with $\mathbf{i} \in \mathcal{L}_{kn}$ are completely determined by $D^{\mathcal{T}_n(\mathbf{j})}$, where $\mathbf{j}$ is the unique ancestor of $\mathbf{i}$ in $\mathcal{L}_{(k-1)n}$. Note that $D^{\mathcal{T}_n(\mathbf{j})}$ has the same distribution as $D^{\mathcal{T}_n(\emptyset)}$ for all $\mathbf{j} \in \mathcal{T}$. Since the

$D^{\mathcal{T}_n(\mathbf{j})}$ are independent for all $k \geq 1$ and $\mathbf{j} \in \mathcal{L}_{(k-1)n}$, by disjointness of the height n subtrees rooted at the levels that are multiples of n, it follows that we have the required independent iteration scheme for $\mathcal{F}^n$. □

At the cost of lowering the similarity dimension with an arbitrary small amount, we can assume that all the scalings in an RIFS are positive: see [2, Lemma 2.10] and a remark in the proof of Proposition 6 in [11]. For completeness, we combine the results of these references in the following proposition and its proof.

Proposition 2.1. *Let $\mathcal{F} = \{f_i : f_i(x) = r_i x + D_i\}_{i=1}^L$ be an RIFS with attractor $C_{\mathcal{F}}$. Then there exists for every $\varepsilon > 0$ an RIFS $\widetilde{\mathcal{F}}$ with $C_{\widetilde{\mathcal{F}}} \subset C_{\mathcal{F}}$, such that all the contraction ratios of the similarities in $\widetilde{\mathcal{F}}$ are positive, and $s(\widetilde{\mathcal{F}}) > s(\mathcal{F}) - \varepsilon$.*

Proof. In case all r_i are positive, there is nothing to prove. Otherwise we may assume w.l.o.g. that $r_1 < 0$. For a natural number n, which will be chosen conveniently large later in the proof, we define for all $\mathbf{i} = i_1 \dots i_n \in \mathcal{L}_n$

$$\widetilde{r_{\mathbf{i}}} = r_1 r_{\mathbf{i}} \quad \text{if } r_{\mathbf{i}} < 0, \qquad \widetilde{r_{\mathbf{i}}} = r_{\mathbf{i}} \quad \text{if } r_{\mathbf{i}} > 0.$$

Here the $r_{\mathbf{i}}$ are the contraction ratios of the $f_{\mathbf{i}}(x) = r_{\mathbf{i}} x + T_{\mathbf{i}}$ mappings from $\mathcal{F}^n$. Since $s := s(\mathcal{F})$ is the solution to $\sum_1^L |r_i|^s = 1$, we must have $\sum_1^L |r_i|^{s-\varepsilon} > 1$ for all $\varepsilon > 0$. So

$$\begin{aligned}\sum_{\mathbf{i}\in\mathcal{L}_n} |\widetilde{r_{\mathbf{i}}}|^{s-\varepsilon} &= \sum_{\widetilde{r_{\mathbf{i}}}<0} |\widetilde{r_{\mathbf{i}}}|^{s-\varepsilon} + \sum_{\widetilde{r_{\mathbf{i}}}>0} |\widetilde{r_{\mathbf{i}}}|^{s-\varepsilon} = \sum_{r_{\mathbf{i}}<0} |r_1|^{s-\varepsilon}|r_{\mathbf{i}}|^{s-\varepsilon} + \sum_{r_{\mathbf{i}}>0} |r_{\mathbf{i}}|^{s-\varepsilon} \\ &\geq |r_1|^{s-\varepsilon} \sum_{\mathbf{i}\in\mathcal{L}_n} |r_{\mathbf{i}}|^{s-\varepsilon} = |r_1|^{s-\varepsilon}\Big(\sum_1^L |r_i|^{s-\varepsilon}\Big)^n > 1,\end{aligned}$$

where we have taken n such that $\left(\sum_1^L |r_i|^{s-\varepsilon}\right)^n > |r_1|^{-s+\varepsilon}$. Conclusion: if we choose $\widetilde{\mathcal{F}}$ with functions $\widetilde{f_{\mathbf{i}}}$ defined by $\widetilde{f_{\mathbf{i}}}(x) = \widetilde{r_{\mathbf{i}}}(x) + T_{\mathbf{i}}$, then $C_{\widetilde{\mathcal{F}}} \subset C_{\mathcal{F}}$ and $s(\widetilde{\mathcal{F}}) > s(\mathcal{F}) - \varepsilon$. □

3. It is enough to consider homogeneous systems

We call an RIFS *homogeneous* if all contraction ratios are the same. Using a simple combination of [2, Lemma 2.8] and [11, Proposition 6] it appears that any RIFS can be well-approximated by a homogeneous RIFS. This is Proposition 3.1.

Given an RIFS $\mathcal{F}$ of the form (7). Let $U \subset \mathcal{L}_n$, $\#U \geq 2$ for an $n \geq 1$. We define

$$\mathcal{F}_U = \{f_{\mathbf{i}}\}_{\mathbf{i}\in U} = \{f_{\mathbf{i}} : f_{\mathbf{i}}(x) = r_{\mathbf{i}} x + T_{\mathbf{i}}\}_{\mathbf{i}\in U}\,. \tag{16}$$

Here the random vectors $T_{\mathbf{i}}$ are defined in (9). According to Lemma 2.1, $\mathcal{F}^n$ is an RIFS. Then $\mathcal{F}_U$ is also an RIFS since $\mathcal{F}_U$ is a subsystem of $\mathcal{F}^n$. For all realizations, the random attractor $C_{\mathcal{F}_U}$ of $\mathcal{F}_U$ is a subset of the random attractor $\mathcal{C}_{\mathcal{F}}$ of $\mathcal{F}$, for the same realization.

Proposition 3.1. *Let* $\mathcal{F} = \{f_i : f_i(x) = r_i x + D_i\}_{i=1}^L$ *be an RIFS as in Definition 2.1.*

Then there exists for every $\varepsilon > 0$ *a number* n*, a set* $U \subset \mathcal{L}_n$*, and an* $a \in (0,1)$ *such that the RIFS* $\mathcal{F}_U$ *has the form*

$$\mathcal{F}_U = \{f_{\mathbf{i}} : f_{\mathbf{i}}(x) = ax + T_{\mathbf{i}}\}_{\mathbf{i}\in U}, \tag{17}$$

and satisfies $C_{\mathcal{F}_{\mathcal{U}}} \subset C_{\mathcal{F}}$, $s(\mathcal{F}_{\mathcal{U}}) > s(\mathcal{F}) - \varepsilon$.

For the convenience of the reader, we give a detailed proof of this result in the Appendix.

4. Lemma 4.1, the Main Lemma

A homogeneous system $\mathcal{H}$ has the form

$$\mathcal{H} := \{H_i : H_i(x) = ax + D_i\}_{i=1}^L, \quad a \in (0,1). \tag{18}$$

Here we are motivated by Proposition 2.1: with an arbitrary small loss in similarity dimension we may assume that $a \in (0,1)$ instead of $a \in (-1,1) \setminus \{0\}$.

It is convenient to introduce a slightly unusual notation. The *support of a function* $f : X \to \mathbb{R}$, for an arbitrary set X, is the set-theoretical support. That is

$$\operatorname{supp}(f) := \{x \in X : f(x) \neq 0\}. \tag{19}$$

This is slightly unusual since $\operatorname{supp}(f)$ most commonly means the closure of the set in (19).

Theorem 4.1. *Let* $\mathcal{H}$ *be a homogeneous RIFS and let* C_1 *and* C_2 *be two independent copies of the attractor of the RIFS. Then the algebraic difference* $C_2 - C_1$ *is the attractor of a homogeneous RIFS* $\mathcal{H}^\ominus$ *with similarity dimension* $s(\mathcal{H}^\ominus) = 2s(\mathcal{H})$.

Proof. Let the homogeneous RIFS $\mathcal{H}$ be given by

$$\mathcal{H} := \{H_i(x) : H_i(x) = ax + D_i\}_{i=1}^L, \quad a \in (0,1), \tag{20}$$

where $D_i = (D_1, \ldots, D_L)$ is an L-dimensional random vector such that for every $i \in [L]$ the random variable D_i is absolutely continuous w.r.t. the Lebesgue measure, with a density φ_i which is bounded, continuous with $\operatorname{supp}(\varphi_i) = (t_i - \theta_i, t_i + \theta_i)$.

For every $i \in [L]$ let $\widehat{D}_i \stackrel{d}{=} D_i$ and $\widetilde{D}_i \stackrel{d}{=} D_i$. Moreover, we require that $\widehat{D}_i$ and $\widetilde{D}_i$ are independent. We define

$$D_{i,j} := \widetilde{D}_i - \widehat{D}_j, \qquad (i,j) \in [L] \times [L]. \tag{21}$$

Then $D_{i,j}$ is absolutely continuous w.r.t. the Lebesgue measure, with a density function $\varphi_{i,j}$ which is continuous, bounded with

$$\mathrm{supp}(\varphi_{i,j}) = (t_i - t_j - \theta_i - \theta_j, t_i - t_j + \theta_i + \theta_j).$$

That is, $\varphi_{i,j}$ satisfies all the requirements we set for the density function in Part (a) of Definition 2.1. Using that, we can define the homogeneous RIFS which consists of a number of L^2 functions $\mathcal{H}^{\ominus} := \{ax + D_{i,j}\}_{i,j=1}^{L}$. It is straightforward to check that the attractor $\Lambda^{\ominus}$ of $\mathcal{H}^{\ominus}$ is the algebraic difference of two independent copies of the attractor of $\mathcal{H}$. □

In the rest of the paper, we always assume that the following two assumptions hold:

A1 $\mathcal{H}$ is a homogeneous RIFS $\{H_i : H_i(x) = ax + D_i\}_{i=1}^{L}$ with supporting interval $I = [\alpha, \beta]$, and $s(\mathcal{H}) > 1$.

A2 $\mathcal{H}$ is a random perturbation of the deterministic IFS $\mathcal{S} := \{ax + t_i\}_{i=1}^{L}$, i.e., for all $i \in [L]$

$$D_i \overset{d}{=} t_i + Y_i, \tag{22}$$

where the absolutely continuous random variable Y_i has a continuous, bounded probability density function $\widetilde{f}_i$ with $\mathrm{supp}(\widetilde{f}_i) = (-\theta_i, \theta_i)$, for some $\theta_i > 0$.

The self-similarity property of an RIFS is expressed by the position of a point $x \in J_i := H_i(I)$ relative to the endpoints of J_i, for $i \in [L]$. This corresponds to the position of a point that we call $\Phi_i(x)$ relative to the endpoints of the supporting interval $I = [\alpha, \beta]$. This leads to the following definition.

Definition 4.1.

(a) For an $i \in [L]$, we write $J_i = H_i(I) =: [A_i, B_i]$, and we define the random variable

$$\Phi_i(x) := \frac{x - A_i}{a} + \alpha \quad \text{if } x \in J_i, \qquad \Phi_i(x) := \Theta \quad \text{if } x \notin J_i, \tag{23}$$

where the symbol Θ represents that $x \notin J_i$.

(b) For an $x \in \mathrm{int}(I)$ let $\phi_i(x, \cdot)$ be the density function of $\Phi_i(x)$. Then an easy calculation yields that

$$\phi_i(x, y) = a\widetilde{f}_i\left(x - t_i - ay\right), \tag{24}$$

where $\widetilde{f}_i$ is the density of the random variables Y_i defined by (22).

The following function plays a crucial role in our argument

$$m_I(x,y) := \sum_{i=1}^{L} \phi_i(x,y), \quad (x,y) \in I^2. \quad \square \tag{25}$$

Lemma 4.1 *(Main Lemma). There exists a set $T(0) \subset I = [\alpha, \beta]$ (the* pre-type space*) which is composed of a finite number of disjoint open intervals, and there exists a real number $\varepsilon_{\text{MAIN}} > 0$ such that for every $\varepsilon \in (0, \varepsilon_{\text{MAIN}})$, the so-called* type space

$$T(\varepsilon) := T(0) \setminus B(\partial T(0), \varepsilon), \tag{26}$$

where $B(E, \varepsilon) := \bigcup\{(x - \varepsilon, x + \varepsilon) : x \in E\}$ for an $E \subset \mathbb{R}$, satisfies

(i) *The compact set $T(\varepsilon)$ consists of as many intervals as $T(0)$.*
(ii) *Let $m^\varepsilon := m_1^\varepsilon := m_I \cdot \mathbf{1}_{T(\varepsilon)\times T(\varepsilon)}$ and for $n \geq 1$ let*

$$m_{n+1}^\varepsilon(x,y) := \int\limits_{T(\varepsilon)} m_n^\varepsilon(x,z) \cdot m_1^\varepsilon(z,y)\,\mathrm{d}z. \tag{27}$$

Then there is an index Q for which the function m_Q^ε is uniformly positive and bounded on $T(\varepsilon) \times T(\varepsilon)$.
(iii) *The Perron-Frobenius eigenvalue of the operator*

$$F^\varepsilon: \; h(x) \mapsto \int\limits_T m_1^\varepsilon(x,y) \cdot h(y)\;\mathrm{d}y, \quad x \in T(\varepsilon)$$

acting on $L^2(T(\varepsilon))$ is larger than 1.
(iv) *The corresponding eigenfunction $f_\varepsilon(x)$ is continuous on $T(\varepsilon)$.*
(v) *$H_{\mathbf{i}}(T(0)) \subset T(0)$ for every $\mathbf{i} \in \mathcal{T}$.*

The proof is given at the end of Section 8.

The contents of the following theorem form the essential part of the proof of our main result Theorem 2.2.

Theorem 4.2. *Let $\mathcal{H}$ be a homogeneous RIFS with $s(\mathcal{H}) > 1$. Then int$(C_{\mathcal{H}}) \neq \emptyset$ almost surely.*

Theorem 4.2 will be proved in Section 7, as a consequence of Lemma 4.1 (the Main Lemma).

5. Proof of Theorem 1.1 and Theorem 2.2 assuming Theorem 4.2

Since the proof of Theorem 1.1 uses Theorem 2.2, we first prove Theorem 2.2.

Proof of Theorem 2.2. Given an RIFS $\mathcal{F}$ with similarity dimension $s(\mathcal{F}) > 1$. Then according to Proposition 3.1, we can find a homogeneous RIFS $\mathcal{H} := \mathcal{F}_U$, with $s(\mathcal{H}) > 1$. It follows from Theorem 4.2 that the attractor $C_{\mathcal{H}}$ of the homogeneous RIFS $\mathcal{H}$ contains an interval almost surely. Then this interval is also contained in the attractor $C_{\mathcal{F}}$ of the RIFS $\mathcal{F}$. □

Proof of Theorem 1.1. Given is an RIFS $\mathcal{F}$ with similarity dimension larger than $\frac{1}{2}$. Let $\varepsilon > 0$ so that $\varepsilon < s(\mathcal{F}) - \frac{1}{2}$. Using Proposition 3.1 there exists a homogeneous RIFS $\mathcal{H} := \mathcal{F}_U$, with $C_{\mathcal{H}} \subseteq C_{\mathcal{F}}$ and $s(\mathcal{H}) > s(\mathcal{F}) - \varepsilon > \frac{1}{2}$. Let C_1, C_2 be two independent copies of the attractor of $\mathcal{F}$. Then we also have two independent copies $C^1_{\mathcal{H}} \subseteq C_1$ and $C^2_{\mathcal{H}} \subseteq C_2$ where $s(\mathcal{H}) > \frac{1}{2}$. According to Theorem 4.1 we can find another homogeneous RIFS $\mathcal{H}^{\ominus}$ whose attractor is $C^1_{\mathcal{H}} - C^2_{\mathcal{H}}$, and $s(\mathcal{H}^{\ominus}) = 2s(\mathcal{H}) > 1$. So by Theorem 2.2 $C^2_{\mathcal{H}} - C^1_{\mathcal{H}}$ contains an interval almost surely. But then also $C_2 - C_1 \supseteq C^1_{\mathcal{H}} - C^2_{\mathcal{H}}$ contains an interval almost surely. □

6. The multi type branching process $\mathcal{Z}$

On the probability space Ω we define a multi type branching process $\mathcal{Z} = (\mathcal{Z}_n)_{n=0}^{\infty}$. For a fixed $0 < \varepsilon < \varepsilon_{\text{MAIN}}$ let $T := T(\varepsilon)$ be the type space from Equation (26). Let $\mathcal{Z}_0 = \{x\}$, where $x \in T$, and for $i \in [L]$ let $Z_i := \Phi_i(x)$. Then

$$\mathcal{Z}_1 := \{Z_i : i = 1, \dots, L\}.$$

Note that Θ is a state of the branching process, but not an element of the type space $T \subset [\alpha, \beta]$.

To define the $\mathcal{Z}_n$, we need some preparations. We follow the definition of an RIFS given in Definition 2.1, but now from the viewpoint of the Y-vectors, instead of the D-vectors. So we are given

$$\left\{Y^{(\mathbf{i})} = \left(Y_1^{(\mathbf{i})}, \dots, Y_L^{(\mathbf{i})}\right)\right\}_{\mathbf{i} \in \mathcal{T}}$$

as a set of i.i.d. random vectors having the same distribution as that of $(Y_1, \dots, Y_L)$.

For an $\mathbf{i} = i_1 \dots i_n$ we define

$$t_{\mathbf{i}} := \sum_{k=1}^{n} a^{k-1} \cdot t_{i_k}, \;\; Y_{\mathbf{i}} := \sum_{k=1}^{n} a^{k-1} \cdot Y_{i_k}^{(i_1 i_2 \dots i_{k-1})} \text{ and } \theta_{\mathbf{i}} := \sum_{k=1}^{n} a^{k-1} \cdot \theta_{i_k}. \tag{28}$$

Here we interpret by convention $i_1 i_2 \dots i_{k-1}$ as $\emptyset$ when $k = 1$.

Clearly, the iterates from (8) take the following form for a homogeneous RIFS

$$H_{\mathbf{i}}(x) = a^n x + T_{\mathbf{i}}, \text{ where } T_{\mathbf{i}} := t_{\mathbf{i}} + Y_{\mathbf{i}}. \tag{29}$$

It follows from our assumption on the density $\widetilde{f}_i$ of Y_i that (identifying somewhat carelessly the support of an absolutely continuous random variable with the support of its density function)

$$\text{supp}(Y_{\mathbf{i}}) = (-\theta_{\mathbf{i}}, \theta_{\mathbf{i}}). \tag{30}$$

Let $I = [\alpha, \beta]$ be the supporting interval of $\mathcal{H}$. We define the level-n (random) cylinder intervals

$$J_{\mathbf{i}} := H_{\mathbf{i}}(I) = [a^n\alpha + T_{\mathbf{i}}, a^n\beta + T_{\mathbf{i}}] \subset [a^n\alpha + t_{\mathbf{i}} - \theta_{\mathbf{i}}, a^n\beta + t_{\mathbf{i}} + \theta_{\mathbf{i}}].$$

The collection of all of these random level n intervals is denoted by $\mathfrak{I}_n$. Note that these are intervals of length $(\beta - \alpha)a^n$. The endpoints of the random interval $J_{\mathbf{i}} \in \mathfrak{I}_n$ are $A_{\mathbf{i}} = a^n\alpha + T_{\mathbf{i}}$ and $B_{\mathbf{i}} = A_{\mathbf{i}} + (\beta - \alpha)a^n$. That is, by definition $J_{\mathbf{i}} = [A_{\mathbf{i}}, B_{\mathbf{i}}]$.

We get the level n children of an $x \in T$ with $H_{\mathbf{i}}(x) \in J_{\mathbf{i}}$ as follows:

If $H_{\mathbf{i}}^{-1}(x) = \alpha + \frac{x - A_{\mathbf{i}}}{a^n} \notin T$ then the level n child of x is Θ. Otherwise, the level n child of x is $H_{\mathbf{i}}^{-1}(x) = \alpha + \frac{x - A_{\mathbf{i}}}{a^n}$.

Note that $H_i^{-1}(x) = \Phi_i(x)$, where Φ_i was defined in Definition 4.1. In the sequel we will use the notation with the H_i^{-1} and their iterates.

In general, if $x \in T$ and $\mathcal{Z}_0 = \{x\}$ and $A \subset T$ is a Borel set, then for any $n \geq 1$ the set of level n descendants of x contained in A is denoted by $\mathfrak{D}_n(x, A)$. So,

$$\mathfrak{D}_n(x, A) := \left\{\mathbf{i} \in \mathcal{L}_n : \; H_{\mathbf{i}}^{-1}(x) \in A\right\} \tag{31}$$

and

$$\mathcal{Z}_n(x, A) = \#\mathfrak{D}_n(x, A). \tag{32}$$

We remark that the process $(\mathcal{Z}_n)_{n=0}^{\infty}$ is a Markov chain since an individual in $\mathcal{Z}_n$ gives birth to descendants independently of the individuals of the same generation if $\mathcal{Z}_{n-1}$ is given.

A major role in our analysis is played by the expectations $\mathbb{E}[\mathcal{Z}_n(x, A)]$, for $A \subset T$, $n \geq 1$. For $n = 1$ we have

$$\begin{aligned}\mathbb{E}[\mathcal{Z}_1(x, A)] &= \int_{\Omega} \mathcal{Z}_1(x, A)\, \mathrm{d}\mathbb{P} = \int_{\Omega} \sum_{i=1}^{L} \mathbf{1}_{\{\Phi_i(x) \in A\}}\, \mathrm{d}\mathbb{P} \\ &= \sum_{i=1}^{L} \mathbb{P}\left(\Phi_i(x) \in A\right) = \sum_{i=1}^{L} \int_{A} \phi_i(x, y)\, \mathrm{d}y,\end{aligned} \tag{33}$$

where $\phi_i(x,y)$ was defined in part (b) of Definition 4.1. It follows that $M_1(x,\cdot) := \mathbb{E}[\mathcal{Z}_1(x,\cdot)]$ has a kernel, given by

$$m(x,y) := m_1(x,y) = \sum_{i=1}^{L} \phi_i(x,y), \quad (x,y) \in T \times T. \tag{34}$$

Let for $n \geq 1$ and $A \subset T$ and $x \in T$,

$$M_n(x,A) := \mathbb{E}[\mathcal{Z}_n(x,A)]$$

We remark that if M_1 has a kernel then M_n also has a kernel. Let us write $m_n(x,\cdot)$ for the kernel of $M_n(x,\cdot)$. That is

$$M_n(x,A) = \int\limits_{y\in A} m_n(x,y)\,\mathrm{d}y. \tag{35}$$

The branching structure of $\mathcal{Z}$ yields (see [3, p. 67])

$$m_{n+1}(x,y) = \int\limits_{T} m_n(x,z)m_1(z,y)\,\mathrm{d}z,$$

which was already introduced in (27), where one has to realize that in the notation we suppressed the dependence on ε of the kernel function $m(\cdot,\cdot)$ in Section 6 and 7.

6.1. Supercritical branching processes with uniformly positive kernel

Harris in his book [3, Condition 10.1] considers the following condition on the kernel function.

There exist $a_{\min}$, $a_{\max}$ and N_0, such that for all $x,y \in T$ we have

$$0 < a_{\min} \leq m_{N_0}(x,y) \leq a_{\max} < \infty. \tag{36}$$

We next consider the following two operators:

$$F: \ \varphi(x) \mapsto \int\limits_{T} m_1(x,y)\cdot\varphi(y)\ \mathrm{d}y, \quad x \in T \tag{37}$$

$$G: \ \psi(y) \mapsto \int\limits_{T} \psi(x)\cdot m_1(x,y)\ \mathrm{d}x, \quad y \in T.$$

These operators are closely related to the expectations of the branching process. Note in particular that

$$\mathbb{E}[\mathcal{Z}_n(x,A)] = \int_T m_n(x,y)\mathbf{1}_A(y)\,\mathrm{d}y = F^n(\mathbf{1}_A(x)). \tag{38}$$

We cite the following theorem from [3, Theorem 10.1]:

Theorem 6.1 *(Harris). It follows from the condition in (36) that the operators in (37) have a common dominant eigenvalue ρ. Let f and g be the corresponding eigenfunctions of the first and second operator in (37) respectively. Then the functions f and g are bounded and uniformly positive on T. Moreover, apart from a scaling, f and g are the only non-negative eigenfunctions of these operators. Further, if we normalize f and g so that $\int f(x)g(x)\,\mathrm{d}x = 1$, then for all $x, y \in T$ as $n \to \infty$*

$$\Big|\frac{m_n(x,y)}{\rho^n} - f(x)g(y)\Big| \le C_1\, f(x)g(y)\Delta^n,$$

where the bound $\Delta < 1$ can be taken independently of x and y, and the constant C_1 is independent of x, y and n.

7. The proof of Theorem 4.2 assuming Lemma 4.1, the Main Lemma

For ε with $0 < \varepsilon < \varepsilon_{\text{MAIN}}$ let $T := T(\varepsilon)$ be the type space defined in (26), and let be f an F-eigenfunction. We define for $0 < \eta < \varepsilon_{\text{MAIN}} - \varepsilon$

$$W := T(\eta+\varepsilon), \text{ and } f_W := f\cdot \mathbf{1}_W. \tag{39}$$

Lemma 7.1. *Let f be an F-eigenfunction. Then there exist $0 < \varepsilon, \eta < \frac{\varepsilon_{\text{MAIN}}}{2}$ and $C_0 > 0$ such that*

$$F f_W(x) > C_0 f(x) \qquad \textit{holds for all } x \in T. \tag{40}$$

Proof. By the fact that f is a right eigenfunction of F corresponding to the Perron-Frobenius eigenvalue ρ we obtain that for every $x \in T$ we have

$$\int_T m(x,y)\,\mathrm{d}y \ge \rho\,\frac{f(x)}{\max\limits_{z\in T} f(z)} \ge \rho\,\frac{\min\limits_{z\in T} f(z)}{\max\limits_{z\in T} f(z)}.$$

Hence, for all $x \in T$,

$$\begin{aligned} Ff(x) &= \int_T m(x,y)f(y)\,\mathrm{d}y \ge \min_{z\in T} f(z)\int_T m(x,y)\,\mathrm{d}y \\ &\ge \rho\big(\min_{z\in T} f(z)\big)^2/\max_{z\in T} f(z) =: C^* > 0. \end{aligned} \tag{41}$$

The fact that $C^* > 0$ follows from Harris' Theorem (Theorem 6.1).

Using the definition of m in (34), (24), and the properties of $\widetilde{f_i}$ in **A2** we obtain that there exists a number U such that

$$0 \leq m(x, y) \leq U, \qquad \text{for all } (x, y) \in T \times T. \tag{42}$$

We choose $\eta > 0$ so small that the second inequality below holds:

$$\int\limits_{T \setminus W} m(x, y) f(y)\, \mathrm{d}y \leq \textsc{Leb}(T \setminus W) \cdot U \cdot \max_{z \in T} f(z) < C^*/2. \tag{43}$$

Putting together (41) and (43) we obtain that

$$F f_W(x) = \int\limits_W m(x, y) f(y)\, \mathrm{d}y = F f(x) - \int\limits_{T \setminus W} m(x, y) f(y)\, \mathrm{d}y > C^*/2.$$

Hence for all $x \in T$

$$F f_W(x) > \frac{C^*}{2 \max\limits_{z \in T} f(z)} f(x).$$

That is, (40) holds with $C_0 := C^*/2 \max\limits_{z \in T} f(z)$. □

Corollary 7.1. *For any $n \geq 1$ we have $\forall x \in T, \quad F^n \mathbf{1}_W(x) > C_1 \rho^{n-1}$, where $C_1 := C_0 \cdot \min\limits_{z \in T} f(z) / \max\limits_{z \in T} f(z)$.*

Consequently, there exists a positive integer r such that

$$\forall x \in T, \quad \mathbb{E}[\mathcal{Z}_n(x, W)] = F^n \mathbf{1}_W(x) > 6 \qquad \textit{for each } n \geq r. \tag{44}$$

Proof. Let $x \in T$. Since $\mathbf{1}_W(x) = f_W(x)/f(x)$, Equation (40) of Lemma 7.1 implies

$$F \mathbf{1}_W(x) \geq \int\limits_T m(x, y) \frac{f_W(x)}{\max\limits_{z \in T} f(z)}\, \mathrm{d}y = F f_W(x) / \max_{z \in T} f(z) > C_0 f(x) / \max_{z \in T} f(z) > C_1.$$

Using that F^{n-1} is a monotone operator for all integers $n \geq 2$, we obtain from this that

$$\begin{aligned} F^{n-1} F \mathbf{1}_W(x) &> C_0 F^{n-1} f(x) / \max_{z \in T} f(z) = C_0 \rho^{n-1} f(x) / \max_{z \in T} f(z) \\ &\geq C_0 \rho^{n-1} \min_{z \in T} f(z) / \max_{z \in T} f(z). \end{aligned}$$

Hence for all $n \geq 1$

$$F^n \mathbf{1}_W(x) > C_0 \frac{\min_{z\in T} f(z)}{\max_{z\in T} f(z)} \rho^{n-1} = C_1 \rho^{n-1}. \tag{45}$$

So we can take

$$r := \Big\lceil (\log(6) - \log(C_1))/\log(\rho) \Big\rceil. \tag{46}$$

To finish the proof, note that $\mathbb{E}[\mathcal{Z}_n(x,W)] = F^n \mathbf{1}_W(x)$, according to (38). □

Our next lemma is a corollary to the Hoeffding inequality [5]:

Lemma 7.2 *(Hoeffding). Assume $Y_1,\dots,Y_C$ are independent random variables such that for any $i = 1,\dots,C$ we have $a_i \le Y_i \le b_i$ for some real numbers a_i, b_i. Let $S_C = \sum_{i=1}^C Y_i$ and let t be a positive real number. Then we have*

$$\mathbb{P}(S_C - \mathbb{E}\, S_C > t) \le \exp\Big\{ - \frac{2t^2}{\sum_{i=1}^C (b_i - a_i)^2} \Big\}.$$

The following statement, Lemma 7.3, is an assertion similar to the easy part of the Cramér theorem.

Lemma 7.3. *Let $C \in \mathbb{N}$, $K > 0$ and $Z_1,\dots,Z_C$ be a sequence of independent random variables such that Z_i takes values in an interval of length K, and $m_i = \mathbb{E}[Z_i] > 6$ for all $i = 1,\dots,C$. Then there exists $0 < \tau = \tau(K) < 1$ such that*

$$\mathbb{P}\Big(Z_1 + \cdots + Z_C < 2C\Big) \le \tau^C.$$

Proof. Let $m_i = \mathbb{E}\, Z_i$. Observe that

$$\mathbb{P}\Big(Z_1 + \cdots + Z_C < 2C\Big) = \mathbb{P}\Big(m_1 - Z_1 + \cdots + m_C - Z_C > \sum_{i=1}^C (m_i - 2)\Big). \tag{47}$$

Motivated by this, we introduce

$$Y_i = m_i - Z_i, \quad \text{and} \quad t = \sum_{i=1}^C (m_i - 2).$$

Then $Y_1,\dots,Y_C$ are independent, and they are contained in an interval of length K. So, we can apply Lemma 7.2 with the choice of $b_i - a_i = K$. Clearly, $\mathbb{E}[S_C] = 0$. Now we use (47), and the fact that by $m_i > 6$ we have $t^2 \ge 16C^2$. We get

$$\mathbb{P}(Z_1 + \cdots Z_C < 2C) = \mathbb{P}(S_C - \mathbb{E}[S_C] > t)$$

$$\le \exp\left(-\frac{2t^2}{\sum_{i=1}^{C}(b_i-a_i)^2}\right) \le \exp\left(-\frac{2\cdot 16C^2}{C\cdot K^2}\right) = \left(\exp\left(-\frac{32}{K^2}\right)\right)^C = \tau(K)^C,$$

where $\tau(K) := \exp\left(-\frac{32}{K^2}\right) < 1$. $\square$

Definition 7.1.

(a) In Lemma 7.3 we choose the constant K equal to $K := L^r$, where r is defined in (46). So $\tau = \tau(r)$ from now on.
(b) Let c_1 be the length of the smallest interval in $T(0)$. We choose n_1 such that $a^{n_1}\text{LEB}(W) \approx c_1$, specifically, $n_1 := \left\lceil \log_a \frac{c_1}{\text{LEB}(W)} \right\rceil$.
(c) Let $N(n) := \frac{\text{LEB}(W)}{2(\beta-\alpha)}(La)^n$. We will show in Lemma 7.4 that $N(n)$ can serve as a lower bound for the growth of the number of intervals. We remark that it will be very important for us that $N(n)$ tends to infinity exponentially, which follows from our assumption that $La > 1$ (equivalently, the similarity dimension of the IFS $\mathcal{H}$ is greater than one).
(d) We define the sequence $(a_0(n), a_1(n), a_2(n), \dots)$ by

$$a_k(n) := L^{n+kr}\tau^{2^{k-1}\cdot N(n)}.$$

(e) Let ℓ_1 be the length of the smallest interval in W.
(f) We fix a small $\xi > 0$. Let $n_2 \ge n_1$ be chosen such that for any $n \ge n_2$ we have $\frac{\ell_1}{\eta} \le (La)^n$ (where η was defined in Lemma 7.1), and such that for any $n \ge n_2$

$$\sum_{k=0}^{\infty} a_k(n) < \xi. \quad \square \tag{48}$$

For an $A \subset T$ we defined $\mathcal{Z}_n(x, A)$ in (31) and (32) by

$$\mathcal{Z}_n(x, A) := \#\mathcal{D}_n(x, A) := \#\left\{\mathbf{i} \in \mathcal{L}_n : H_{\mathbf{i}}^{-1}(x) \in W\right\}$$

only for $x \in T$. Using this formula we extend these definitions to all $x \in T(0)$.

The sample space Ω was defined in Section 2.2. A realization $\boldsymbol{\omega} \in \Omega$ was given by the i.i.d. collection of L-dimensional vectors $\left\{D^{(\mathbf{i})}\right\}_{\mathbf{i}\in\mathcal{T}}$. Using (29) we get for all $\mathbf{i} = i_1 \dots i_n$

$$H_{\mathbf{i}}(x)(\boldsymbol{\omega}) = a^n x + T_{\mathbf{i}}(\boldsymbol{\omega}) = a^n x + t_{\mathbf{i}} + Y_{\mathbf{i}}(\boldsymbol{\omega}) = a^n x + \sum_{k=1}^{n} a^{k-1} t_{i_k} + \sum_{k=1}^{n} a^{k-1} Y_{i_k}^{(i_1 i_2 \dots i_{k-1})}(\boldsymbol{\omega}), \tag{49}$$

where $T_{\mathbf{i}}$ was defined in (9), $t_{\mathbf{i}}$ was defined in (28), and according to (22)

$$Y_{i_k}^{(i_1 i_2 \dots i_{k-1})}(\boldsymbol{\omega}) = D_{i_k}^{(i_1 i_2 \dots i_{k-1})}(\boldsymbol{\omega}) - t_{i_k}. \tag{50}$$

So, we can, and from now on we will, identify a realization $\boldsymbol{\omega} \in \Omega$ with the labeled tree $\left\{Y^{(\mathbf{i})} = (Y_1^{(\mathbf{i})}, \dots, Y_L^{(\mathbf{i})})\right\}_{\mathbf{i}\in\mathcal{T}}$. Clearly, by (49) we get

$$H_{\mathbf{i}}^{-1}(x) = \frac{x}{a^n} - \sum_{p=1}^{n} \frac{1}{a^{n+1-p}} t_{i_p} - \sum_{p=1}^{n} \frac{1}{a^{n+1-p}} Y_{i_p}^{(i_1 i_2 \dots i_{p-1})}. \tag{51}$$

Lemma 7.4. *For every $\boldsymbol{\omega} \in \Omega$ there exists a (random) interval $\mathrm{J} = \mathrm{J}(\boldsymbol{\omega}) \subset T(0)$ of length $\textsc{Leb}(\mathrm{J}) = \frac{1}{2}\ell_1 a^n$ such that for any $x \in \mathrm{J}$*

$$\mathcal{Z}_n(x, W)(\boldsymbol{\omega}) \geq \frac{\textsc{Leb}(W)}{2(\beta - \alpha)}(La)^n = N(n).$$

Proof. The proof uses the observation that for any bounded integrable function h

$$\int_{T(0)} h(x)\,\mathrm{d}x \leq \textsc{Leb}(T(0)) \cdot ||h||_\infty \leq (\beta - \alpha)||h||_\infty.$$

By the definition of $\mathcal{Z}_n(x, W)$ (see (31)) we have

$$\begin{aligned}
\int_{T(0)} \mathcal{Z}_n(x, W)\,\mathrm{d}x &= \int_{T(0)} \sum_{|\mathbf{i}|=n} \mathbf{1}_{H_{\mathbf{i}}^{-1}(x)\in W}\,\mathrm{d}x \\
&= \sum_{|\mathbf{i}|=n} \int_{T(0)} \mathbf{1}_{H_{\mathbf{i}}^{-1}(x)\in W}\,\mathrm{d}x \\
&= \sum_{|\mathbf{i}|=n} \textsc{Leb}(\{x \in T(0) : H_{\mathbf{i}}^{-1}(x) \in W\}) \\
&= \sum_{|\mathbf{i}|=n} \textsc{Leb}\left(T(0) \cap H_{\mathbf{i}}(W)\right) \\
&= L^n \textsc{Leb}(W) a^n = 2(\beta - \alpha)N(n),
\end{aligned}$$

where we use in the one but last step that $H_{\mathbf{i}}(W) \subset T(0)$. This follows from part (v) of the Lemma 4.1 (Main Lemma) since $W \subset T(0)$.

In this way, for every $\boldsymbol{\omega} \in \Omega$ there exists an $x_{\max} = x_{\max}(\boldsymbol{\omega}) \in T(0)$ such that

$$\mathcal{Z}_n(x_{\max}, W) \geq 2N(n).$$

Let $G_n := \left\{\mathbf{i} \in \mathcal{L}_n : H_{\mathbf{i}}^{-1}(x_{\max}) \in W\right\}$. Then by definition $\# G_n = \mathcal{Z}_n(x_{\max}, W)$. For each $\mathbf{i} \in G_n$, $H_{\mathbf{i}}^{-1}(x_{\max})$ is contained in a connected component $C_{\mathbf{i}}$ of W. By definition, $\textsc{Leb}(C_{\mathbf{i}}) \geq \ell_1$.

We write G_n^l for the collection of those $\mathbf{i} \in G_n$ for which the center of $C_{\mathbf{i}}$ is to the left from $H_{\mathbf{i}}^{-1}(x_{\max})$. So, for an $\mathbf{i} \in G_n^l$ there is an interval of length at least $\ell_1/2$, contained in W with right endpoint $H_{\mathbf{i}}^{-1}(x_{\max})$. Let $G_n^r := G_n \setminus G_n^l$. Then at least one of the sets G_n^l or G_n^r — say G_n^l — has cardinality at least $N(n)$. This means that for

$$\mathrm{J} := (x_{\max} - a^n \ell_1/2,\, x_{\max})\,, \tag{52}$$

and for every $x \in \mathrm{J}$ and for every $\mathbf{i} \in G_n^l$ we have $H_{\mathbf{i}}^{-1}(x) \in W$. So, by (31), we have $\mathcal{Z}_n(x, W) \geq \#G_n^l \geq N(n)$ for all $x \in \mathrm{J}$. To verify that $\mathrm{J} \subset T(0)$, pick an $\mathbf{i} \in G_n^l$. Then $\mathrm{J} \subset H_{\mathbf{i}}(C_{\mathbf{i}})$. Using part (v) of Lemma 4.1 (the Main Lemma) we obtain $H_{\mathbf{i}}(C_{\mathbf{i}}) \subset H_{\mathbf{i}}(W) \subset T(0)$. Hence, $\mathrm{J} \subset T(0)$. □

Recall that n is fixed. We partition each interval of $T(0)$ into intervals of length $\ell_1 a^n/6$. In general there will of course be one interval in this partition that is shorter. We take this to be always the rightmost interval. Let $\widetilde{L} = \widetilde{L}(n)$ be the total number of intervals in $T(0)$ obtained in this way. Let $J_1, \ldots, J_{\widetilde{L}}$ be the intervals in increasing order from left to right in $T(0)$.

Let J be the random interval defined in (52). Since $\mathrm{J} = \mathrm{J}(\boldsymbol{\omega}) \subset T(0)$, one of the intervals J_ℓ is starting exactly at the left border of J, and since J is at least three times as long, this is in fact one of the intervals of length $\ell_1 a^n/6$ from the partitioning intervals. We denote the index of this unique interval by the number $\ell_{\max} = \ell_{\max}(\boldsymbol{\omega})$. Let

$$\Omega_l = \{\boldsymbol{\omega} \in \Omega :\ \ell_{\max}(\boldsymbol{\omega}) = l\} \quad \text{for } l = 1, \ldots \widetilde{L}.$$

It follows from (51) that

$$\text{for an } \boldsymbol{\omega} = \{Y^{(\mathbf{i})}\}_{\mathbf{i} \in \mathcal{T}} \text{ the event } \{\boldsymbol{\omega} \in \Omega_l\} \ \text{ depends only on } Y^{(\mathbf{i})} \text{ with } \mathbf{i} \in \bigcup_{k=0}^{n-1} \mathcal{L}_k. \tag{53}$$

Clearly, we also have

$$\Omega = \bigcup_{l=1}^{\widetilde{L}} \Omega_l.$$

For two finite words $\mathbf{i}, \mathbf{j}$, in general, $H_{\mathbf{ij}} \neq H_{\mathbf{i}} \circ H_{\mathbf{j}}$. To get equality in this formula, in $H_{\mathbf{j}} = a^{|\mathbf{j}|}x + t_{\mathbf{j}} + Y_{\mathbf{j}}$ we need to replace the random part $Y_{\mathbf{j}} = \sum_{p=1}^{|\mathbf{j}|} a^{p-1} \cdot Y_{j_p}^{(j_1 j_2 \ldots j_{p-1})}$ by its shifted version $Y_{\mathbf{j}}^{(\mathbf{i})} := \sum_{p=1}^{|\mathbf{j}|} a^{p-1} \cdot Y_{j_p}^{(\mathbf{i} j_1 j \ldots j_{p-1})}$, where $\mathbf{i} j_1 j \ldots j_{p-1}$ is the concatenation of the words $\mathbf{i}$ and $j_1 \ldots j_{p-1}$, and where by convention we interpret $\mathbf{i} j_1 j \ldots j_{p-1}$ as $\mathbf{i}$ when $p = 1$.

Lemma 7.5. *Let $\mathbf{i} \in \mathcal{L}_n$ and $\mathbf{j} \in \mathcal{L}_m$ for some $n, m > 0$. We define the random function $H_{\mathbf{i} \to \mathbf{ij}}$ on T by*

$$H_{\mathbf{i}\to\mathbf{ij}}(x) := a^m x + t_{\mathbf{j}} + Y_{\mathbf{j}}^{\mathbf{i}} = a^m x + \sum_{p=1}^{m} a^{p-1}\, t_{i_p} + \sum_{p=1}^{m} a^{p-1}\, Y_{j_p}^{(\mathbf{i}j_1 j_2 \dots j_{p-1})}. \tag{54}$$

Then $H_{\mathbf{ij}}(x) = H_{\mathbf{i}} \circ H_{\mathbf{i}\to\mathbf{ij}}(x)$*. Consequently,* $H_{\mathbf{ij}}^{-1}(x) = H_{\mathbf{i}\to\mathbf{ij}}^{-1} \circ H_{\mathbf{i}}^{-1}(x)$*, and*

$$H_{\mathbf{i}\to\mathbf{ij}}^{-1}(x) = \frac{x}{a^m} - \sum_{p=1}^{m} \frac{1}{a^{m+1-p}}\, t_{i_p} - \sum_{p=1}^{m} \frac{1}{a^{m+1-p}} Y_{j_p}^{(\mathbf{i}j_1 j_2 \dots j_{p-1})}. \tag{55}$$

Proof. Immediate from the definitions. □

Let $x \in T(0)$ and $U \subset T$. Recall that $\mathcal{D}_n(x,U) = \left\{\mathbf{i} \in \mathcal{L}_n : H_{\mathbf{i}}^{-1}(x) \in U\right\}$. For an $\mathbf{i} \in \mathcal{D}_n(x,T)$ we define $x_{\mathbf{i}} := H_{\mathbf{i}}^{-1}(x)$, and

$$\mathcal{D}_{r,\mathbf{i}}(x,U) := \left\{\mathbf{j} \in \mathcal{L}_r : H_{\mathbf{ij}}^{-1}(x) \in U\right\} = \left\{\mathbf{j} \in \mathcal{L}_r : H_{\mathbf{i}\to\mathbf{ij}}^{-1}(x_{\mathbf{i}}) \in U\right\}.$$

Finally, for an $\mathbf{i} \in \mathcal{D}_n(x,T)$ we write

$$V_{\mathbf{i}} := \#\mathcal{D}_{r,\mathbf{i}}(x,W).$$

Now we observe that, since the $H_{\mathbf{ij}}^{-1}$ have the same distribution as the $H_{\mathbf{j}}^{-1}$, by Corollary 7.1 we have

$$\mathbb{E}\left[V_{\mathbf{i}}\right] > 6 \qquad \text{holds for all } \mathbf{i} \in \mathcal{D}_n(x,T). \tag{56}$$

Moreover,

$$\{V_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{D}_n(x,T)} \text{ are independent and conditionally independent on } \Omega_l. \tag{57}$$

Namely, the independence is obvious from the definition. To verify the conditional independence, recall that by (53) for an $\boldsymbol{\omega} = \left\{Y^{(\mathbf{j})}\right\}_{\mathbf{j}\in\mathcal{T}} \in \Omega$, the event $\{\boldsymbol{\omega} \in \Omega_l\}$ depends only on $Y^{(\mathbf{j})}$ for which $\mathbf{j} \in \bigcup_{k=0}^{n-1} \mathcal{L}_k$.

On the other hand, it follows from (55) that the value of $V_{\mathbf{i}}(\boldsymbol{\omega})$ depends only on $Y^{(\mathbf{j})}$ for $\mathbf{j} \in \bigcup_{k=n}^{n+r-1} \mathcal{L}_k$.

By the definitions and the last two sentences we obtain the conditional independence stated in (57).

Lemma 7.6. *For any* $l = 1, \dots, \tilde{L}$*, any* $n \geq r$*, and any* $x \in J_l$

$$\mathbb{P}\left(\mathcal{Z}_{n+r}(x,W) \leq 2N(n) \mid \Omega_l\right) \leq \tau^{N(n)}. \tag{58}$$

Proof. Since $x \in J_l \subset$ J, Lemma 7.4 gives that $\mathcal{Z}_n(x, W)(\boldsymbol{\omega}) \geq N(n)$, i.e., there are at least $\lfloor N(n) \rfloor$ descendants $\mathbf{i}$ at level n. Each of these gives $V_{\mathbf{i}}$ descendants at level $n+r$. Since $n \geq r$, and since $\mathbb{E}[V_{\mathbf{i}}] > 6$ (see (56)), we obtain by (57) that we can apply Lemma 7.3 with $K = L^r$ and $C = \lfloor N(n) \rfloor$. This gives Equation (58). □

Let X_k be a ηa^{2n+kr} dense set in J_l, where η has been set in Lemma 7.1. X_k can be chosen such that

$$\# X_k \leq \frac{\ell_1 a^n}{\eta a^{2n+kr}} = \frac{\ell_1}{\eta} a^{-(n+kr)} < L^{n+kr} \text{ if } n \geq n_2. \tag{59}$$

The most important step towards our goal of proving Theorem 4.2 is to verify the following inequality:

$$\mathbb{P}\left(\mathcal{Z}_{n+kr}(x, T) > 2^k \cdot N(n), \forall 0 \leq k \leq M,\ \forall x \in J_l \mid \Omega_l\right) > \prod_{k=0}^{M} (1 - a_k(n)), \tag{60}$$

holds for any positive integer M if $\mathbb{P}(\Omega_l) > 0$. This will be the key step in the proof of Lemma 7.8 which will easily imply the assertion of Theorem 4.2.

For $M = 1$, by (58) we obtain:

$$\mathbb{P}\left(\exists x \in X_1 :\ \mathcal{Z}_{n+r}(x, W) \leq 2N(n) | \Omega_l\right) \leq \# X_1 \cdot \tau^{N(n)}.$$

Recall that X_1 was defined as an ηa^{2n+r}-dense subset of J_l. Recall that by (59) we have $\# X_1 \leq L^{n+r}$. Hence,

$$\mathbb{P}\left(\forall x \in X_1 : \mathcal{Z}_{n+r}(x, W) > 2N(n) \mid \Omega_l\right) \geq 1 - L^{n+r} \tau^{N(n)}. \tag{61}$$

Next, our purpose is to extend the inequality (61) from all $x \in X_1$ to all $x \in J_l$, and from 1 to all $k \geq 2$.

Lemma 7.7. *For any $k \geq 1$ and $l = 1, \ldots, \widetilde{L}$ we have*

$$\left\{\forall x \in X_k : \mathcal{Z}_{n+kr}(x, W) > 2^k N(n)\right\} \cap \Omega_l \subset \left\{\forall x \in J_l : \mathcal{Z}_{n+kr}(x, T) > 2^k N(n)\right\} \cap \Omega_l, \tag{62}$$

and the set

$$\left\{\forall x \in J_l :\ \mathcal{Z}_{n+kr}(x, T) \geq 2^k N(n)\right\} \tag{63}$$

is measurable.

Proof. Let us fix $k \geq 1$. Since X_k is ηa^{2n+kr} dense in J_l, for any $x \in J_l$ we can find $x' \in X_k$ such that $|x - x'| < \eta a^{2n+kr} < \eta a^{n+kr}$.

We claim that if for some x' and $\boldsymbol{\omega}$ one has $\mathcal{Z}_{n+kr}(x', W) > 2^k N(n)$, then for the same $\boldsymbol{\omega}$ and for any x such that $|x - x'| < \eta a^{n+kr}$, and the larger set T, we also have $\mathcal{Z}_{n+kr}(x, T) > 2^k N(n)$.

The point is here that if x and x' are ηa^{n+kr} close, then their images with the function $H_{\mathbf{i}}^{-1}$ are η-close if $\mathbf{i}$ is a word of length $n + kr$. So if $\mathbf{i}$ is a word of length $n + kr$ such that $H_{\mathbf{i}}^{-1}(x') \in W$ then $H_{\mathbf{i}}^{-1}(x)$ is in the η-neighborhood of W. But T is exactly the η-neighborhood of W since $W = T(0) \setminus B(\partial T(0), \varepsilon + \eta)$ and $T = T(0) \setminus B(\partial T(0), \varepsilon)$, where ∂ means boundary. This proves (62).

It remains to be proved that the set in (63) is measurable which is formally not straightforward since x is running over an interval J_l.

First, we note that it is enough to prove that for any fixed $x' \in X_k$ the set

$$\left\{\forall x \in \left[x' - \eta a^{n+kr}, x' + \eta a^{n+kr}\right] \cap J_l : \ \mathcal{Z}_{n+kr}(x, T) \geq 2^k N(n)\right\}$$

is measurable since X_k is a finite set.

We have to take into consideration two facts. T is a union of finite number of intervals and according to (32) $\mathcal{Z}_{n+kr}(x, T)$ is a sum of a finite number of indicator functions:

$$\mathcal{Z}_{n+kr}(x, T) = \#\mathfrak{D}_{n+k}(x, T)$$

Therefore, the function $\mathcal{Z}_{n+kr}(\cdot, T)$ for any $\boldsymbol{\omega}$ is a jump function on T with a finite number of jumps. Let $\{\iota_i : i \in \mathcal{I}\}$ denote the partition of T into the intervals on which $\mathcal{Z}_{n+kr}(\cdot, T)$ is constant. So, $\mathcal{Z}_{n+kr}(x, T)$ depends on the interval ι_i which x falls into. Therefore,

$$\begin{aligned}&\left\{\forall x \in \left[x' - \eta a^{n+kr}, x' + \eta a^{n+kr}\right] \cap J_l : \ \mathcal{Z}_{n+kr}(x, T) \geq 2^k N(n)\right\} = \\ &\quad \left\{\forall i \in \mathcal{I} \text{ such that } \iota_i \cap \left[x' - \eta a^{n+kr}, x' + \eta a^{n+kr}\right] \cap J_l \neq \emptyset : \ \mathcal{Z}_{n+kr}(\iota_i, T) \geq 2^k N(n)\right\}.\end{aligned}$$

The last set is given by a measurable function of a finite number of random variables hence measurable. □

Lemma 7.8. *Fix an arbitrary $n \geq n_2$ and $l \in \{1, \ldots, \widetilde{L}\}$. Then*

$$\forall x \in J_l \text{ and } \forall \boldsymbol{\omega} \in \Omega_l, \quad \mathcal{Z}_n(x, T)(\boldsymbol{\omega}) > N(n). \tag{64}$$

Further, if $\mathbb{P}(\Omega_l) > 0$ we have

$$\mathbb{P}\left(\mathcal{Z}_{n+Mr}(x, T) > 2^M \cdot N(n), M = 0, 1, \ldots, \ \forall x \in J_l \mid \Omega_l\right) > \prod_{k=0}^{\infty}(1 - a_k(n)). \tag{65}$$

Proof. Equation (64) follows from Lemma 7.4.

Concerning (65), it is enough to verify that (60) holds for every M. To do so, as a consequence of Lemma 7.7 we can exchange X_1 with J_l at a price of replacing the set W with the larger set T in (61). In this way, as we have already pointed out in (61), we obtain

$$\mathbb{P}\left(\mathcal{Z}_{n+r}(x,T) > 2N(n), \forall x \in J_l \mid \Omega_l\right) \geq 1 - L^{n+r}\tau^{N(n)} = 1 - a_1(n).$$

For $k=0$, using Lemma 7.4, the definition of Ω_l and $W \subset T$, we have

$$\mathcal{Z}_n(x,T)(\boldsymbol{\omega}) \geq \mathcal{Z}_n(x,W)(\boldsymbol{\omega}) \geq N(n) \quad \text{for all } \boldsymbol{\omega} \in \Omega_l.$$

Therefore, we have (60) for $M=1$:

$$\mathbb{P}\left(\mathcal{Z}_{n+kr}(x,T) > 2^k N(n), 0 \leq k \leq 1, x \in J_l \mid \Omega_l\right) \geq 1 - L^{n+r}\tau^{N(n)} = \\ 1 - a_1(n) > (1-a_1(n))(1-a_0(n)).$$

Now, assume that we have proved (60) for $M-1$. We will prove it for M. Yet again we use the simple fact that for any three events A, B, D of positive probability we have: $\mathbb{P}(A \cap B | D) = \mathbb{P}(A | B \cap D) \cdot \mathbb{P}(B|D)$ in the following way:
$A := \left\{\mathcal{Z}_{n+Mr}(x,T) > 2^M \cdot N(n),\ \forall x \in J_l\right\}$,
$B := \left\{\mathcal{Z}_{n+kr}(x,T) > 2^k \cdot N(n), \forall 0 \leq k \leq M-1, \forall x \in J_l\right\}$, $D := \Omega_l$. Then we have

$$\mathbb{P}\Bigg(\underbrace{\mathcal{Z}_{n+kr}(x,T) > 2^k \cdot N(n), \forall 0 \leq k \leq M,\ \forall x \in J_l}_{A \cap B} \mid \underbrace{\Omega_l}_{D}\Bigg)$$
$$= \mathbb{P}\Bigg(\underbrace{\mathcal{Z}_{n+Mr}(x,T) > 2^M \cdot N(n),\ \forall x \in J_l}_{A} \mid$$
$$\underbrace{\mathcal{Z}_{n+kr}(x,T) > 2^k \cdot N(n), \forall 0 \leq k \leq M-1, \forall x \in J_l}_{B},\ \underbrace{\Omega_l}_{D}\Bigg) \cdot$$
$$\cdot\, \mathbb{P}\Bigg(\underbrace{\mathcal{Z}_{n+kr}(x,T) > 2^k \cdot N(n), \forall 0 \leq k \leq M-1,\ \forall x \in J_l}_{B} \mid \underbrace{\Omega_l}_{D}\Bigg)$$

By induction, it is known that

$$\mathbb{P}(B \mid D) \geq \prod_{k=0}^{M-1}(1-a_k(n)). \tag{66}$$

Now we prove that

$$\mathbb{P}\left(A \mid B \cap D\right) > 1 - a_M(n). \tag{67}$$

As in the proof of (58) we use Lemma 7.3. Let $n' = n + (M-1)r$, $C = \mathcal{Z}_{n'}(x, T)$. By induction, we know that for $\boldsymbol{\omega} \in \Omega_l$ and $x \in J_l$ we have $C \geq 2^{M-1} \cdot N(n)$.

Then for any $x \in J_l$ we have

$$\mathbb{P}\left(\mathcal{Z}_{n'+r}(x, W) \leq 2 \cdot 2^{M-1} N(n) \mid B \cap D\right) \leq \tau^{2^{M-1} N(n)}.$$

This can be proved in exactly the same way as (58) was proved. The continuation is also similar, we first take a dense set X_M and prove the counterpart of (61), that is,

$$\mathbb{P}\left(\mathcal{Z}_{n'+r}(x, W) > 2 \cdot 2^{M-1} N(n), x \in X_M \mid B \cap D\right) \geq 1 - L^{n+Mr} \tau^{2^{M-1} N(n)} = 1 - a_M(n).$$

Applying Lemma 7.7 yields

$$\mathbb{P}(A \mid B \cap D) = \mathbb{P}\left(\mathcal{Z}_{n+Mr}(x, T) > 2^M \cdot N(n),\ \forall x \in J_l \mid B \cap D\right) \geq 1 - a_M(n),$$

where we used that $\mathcal{Z}_{n+Mr}(x, T) = \mathcal{Z}_{n'+r}(x, T)$. So, we have verified (67). This and (66) together implies that (60) holds for every M. Hence, we get that (65) also holds. This finishes the proof of Lemma 7.8. □

Now, we are ready to present the proof of Theorem 4.2.

Proof of Theorem 4.2 assuming Lemma 4.1, the Main Lemma. Using Lemma 7.8 and Lemma 7.9, we have that whenever $\mathbb{P}(\Omega_l) > 0$

$$\mathbb{P}\left(C_{\mathcal{H}} \text{ contains an interval } \mid \Omega_l\right) \geq \mathbb{P}\left(C_{\mathcal{H}} \text{ contains } J_l \mid \Omega_l\right) \geq$$
$$\mathbb{P}\left(\mathcal{Z}_{n+Mr}(x, T) > 2^M \cdot N(n), \forall M,\ \forall x \in J_l \mid \Omega_l\right) > \prod_{k=0}^{\infty}(1 - a_k(n)) > 1 - \sum_{k=0}^{\infty} a_k(n).$$

Getting rid of the condition, we obtain that

$$\mathbb{P}\left(C_{\mathcal{H}} \text{ contains an interval}\right) = \sum_{l=1}^{\infty} \mathbb{P}\left(C_{\mathcal{H}} \text{ contains an interval } \mid \Omega_l\right) \mathbb{P}(\Omega_l) >$$
$$\left(1 - \sum_{k=0}^{\infty} a_k(n)\right) \sum_{l=1}^{L} \mathbb{P}(\Omega_l) = 1 - \sum_{k=0}^{\infty} a_k(n) > 1 - \xi,$$

where in the last step we used (48). Since ξ can be chosen arbitrarily small this proves Theorem 4.2. □

Lemma 7.9. *Let $(a_k)_{k \geq 0}$ be a sequence of positive real numbers. Then $\prod_{k=0}^{\infty}(1 - a_k) > 1 - \sum_{k=0}^{\infty} a_k$.*

Proof. First note that for two positive numbers x and y one has: $(1-x)(1-y) > 1-(x+y)$, simply because $xy > 0$. So we have $(1 - a_0)(1 - a_1) > 1 - (a_0 + a_1)$.

We continue by induction. Suppose that the formula holds with upper index $n \geq 1$ instead of ∞.

Let $x = 1 - \prod_{k=0}^{n}(1-a_k)$, and $y = a_{n+1}$. Then $1 - x = \prod_{k=0}^{n}(1-a_k) > 1 - \sum_{k=0}^{n} a_k$. So

$$\prod_{k=0}^{n+1}(1-a_k) = (1-x)(1-y) > 1-(x+y) = 1-x-y > 1-\sum_{k=0}^{n} a_k - a_{n+1} = 1-\sum_{k=0}^{n+1} a_k.$$

This finishes the induction proof. Letting $n \to \infty$ we obtain the statement of the lemma. □

8. Construction of the pre-typespace $T(0)$ and the type space $T(\varepsilon)$

In this section we prove Lemma 4.1, the Main Lemma.

We consider the support of m_I:

$$\text{supp}(m_I) = \bigcup_{i=1}^{L} \{(x,y) : y \in \text{supp } \Phi_i(x)\}.$$

It is immediate that

$$\text{supp}(m_I) = \bigcup_{i=1}^{L} \widehat{S}_i, \tag{68}$$

where,

$$\widehat{S}_i := \left\{(x,y) : x \in \widehat{W}_i,\ y \in \text{supp } \Phi_i(x)\right\} \text{ for } \widehat{W}_i := (a\alpha + t_i - \theta_i, a\beta + t_i + \theta_i).$$

It is easy to see that for all $i \in [L]$, $\widehat{S}_i$ is a parallelogram with two horizontal sides: $\{(x,y) : y = \alpha\}$, $\{(x,y) : y = \beta\}$ and the two other sides are the following two lines of slope $1/a$

$$\widehat{\ell}_i^2(x) := \frac{1}{a}x - \frac{t_i}{a} + \frac{1}{a}\theta_i, \quad \widehat{\ell}_i^1(x) := \frac{1}{a}x - \frac{t_i}{a} - \frac{1}{a}\theta_i.$$

That is

$$\begin{aligned} \widehat{S}_k &= \left\{(x,y) : x \in \widehat{W}_k,\ \max\left\{\alpha, \widehat{\ell}_k^1(x)\right\} < y < \min\left\{\beta, \widehat{\ell}_k^2(x)\right\}\right\} \\ &= \{(x,y) : y \in (\alpha,\beta), ay + t_i - \theta_i < x < ay + t_i + \theta_i\}. \end{aligned} \tag{69}$$

Clearly,

$$\text{width}(\widehat{S}_k) = 2\theta_k \quad \text{height}(\widehat{S}_k) = \frac{2\theta_k}{a}. \tag{70}$$

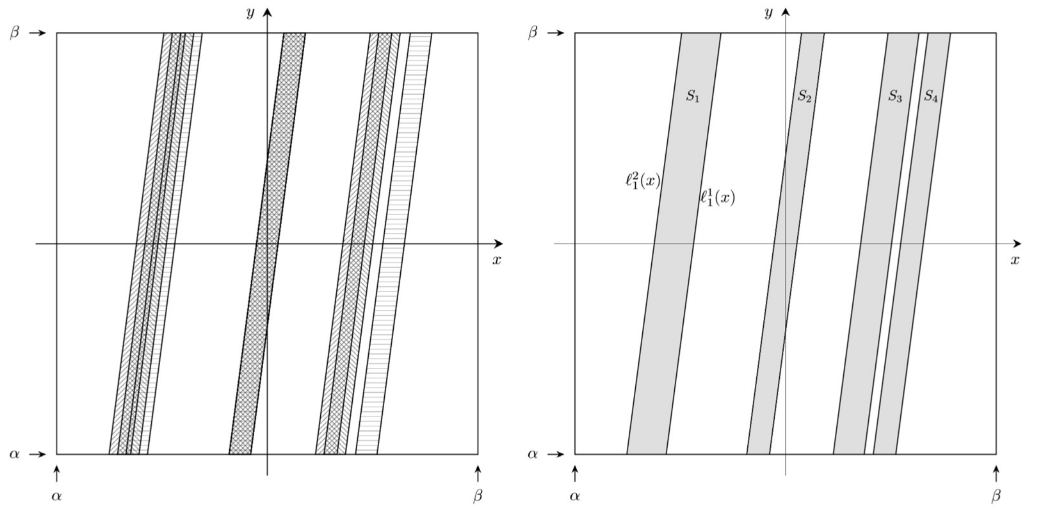

Fig. 3. Parallelograms $\widehat{S}_i$ on the left and parallelograms S_k on the right.

In general the open filled parallelograms $\widehat{S}_i$ are not disjoint. Their union $\bigcup_{i=1}^{L} \widehat{S}_i$ has say M connected components $\{S_k\}_{k=1}^{M}$. By elementary geometry, for all $k \in [M]$ the connected component S_k is also an open filled parallelogram having two horizontal sides and the left non-horizontal side is one of the lines from $\left\{\widehat{\ell}_i^2(x)\right\}_{i=1}^{L}$. Let us call it $\ell_k^2(x)$. While the right non-horizontal side of S_k is one of the lines from $\left\{\widehat{\ell}_i^1(x)\right\}_{i=1}^{L}$. Let us call it $\ell_k^1(x)$.

Observe that the open filled parallelograms S_k can be adjacent to each other. That is, it can happen that for a $1 \le k \le M-1$, we have $\ell_k^1(x) \equiv \ell_{k+1}^2(x)$.

We introduce the orthogonal projections to the coordinate axes:

$$\pi_1(x,y) := x \quad \text{and} \quad \pi_2(x,y) := y.$$

We have

$$S_k = \left\{(x,y) : x \in \pi_1(S_k), \quad \max\left\{\alpha, \ell_k^1(x)\right\} < y < \min\left\{\beta, \ell_k^2(x)\right\}\right\}. \tag{71}$$

The collection of parallelograms having two horizontal sides such that the slopes of the other two sides are equal to $1/a$ is denoted by $\mathfrak{P}$. Particular attention will be given to the parallelograms of the form $P_{\langle u,v\rangle}^k \in \mathfrak{P}$, where $u \le v$, and $\langle a,b\rangle$ stands for an interval with endpoints $a \le b$ about which we do not know if it is closed or open or half-closed and half-open.

$$P_{\langle u,v\rangle}^k := \left\{(x,y) \in S_k : y \in \langle u,v\rangle\right\}. \tag{72}$$

When $u = v$ then $P^k_{[u,v]}$ is the horizontal line segment $\{(x, y) \in S_k : y = u = v\}$. Without loss of generality, we may assume that $\bigcup_{k=1}^{M} S_k$ is contained in the region between the left side of S_1 (which is determined by the graph of the function $\ell_1^2(x)$) and the right side of S_M (which is determined by the graph of the function $\ell_M^1(x)$).

We define $\alpha < \widetilde{\alpha} < \widetilde{\beta} < \beta$ by

$$\ell_1^2(\widetilde{\alpha}) = \widetilde{\alpha}, \quad \ell_M^1(\widetilde{\beta}) = \widetilde{\beta} \quad \text{and} \quad \widetilde{I} := [\widetilde{\alpha}, \widetilde{\beta}] \subset I. \tag{73}$$

Definition 8.1. Let $\mathfrak{A}$ be the collection of all finite unions of sub-intervals of $\widetilde{I}$, including all open, closed, half-open and half-closed, and even degenerated intervals. In particular there exists a q such that $H \in \mathfrak{A}$, $H = \bigcup_{j=1}^{q} \langle a_i, b_i \rangle$, where $\widetilde{\alpha} \le a_i \le b_i \le \widetilde{\beta}$ for all $i \in [q]$. We define

$$U_H := \bigcup_{k \in [M]} \bigcup_{i \in [q]} P^k_{\langle a_i, b_i \rangle} = \bigcup_{i \in [q]} \{(x, y) : y \in \langle a_i, b_i \rangle\} \bigcap \bigcup_{k=1}^{M} S_k \tag{74}$$

and

$$\Psi(H) := \pi_1 (U_H) = \bigcup_{k \in [M]} \bigcup_{i \in [q]} \pi_1 \left(P^k_{\langle a_i, b_i \rangle} \right). \tag{75}$$

Lemma 8.1. *The mapping* Ψ *satisfies*

(a) *Let* $y \in \widetilde{I}$. *Then*

$$\Psi(\{y\}) = \bigcup_{i=1}^{L} (ay + t_i - \theta_i, ay + t_i + \theta_i). \tag{76}$$

(b) Ψ *is a self-mapping of* $\mathfrak{A}$. *That is,* $\Psi : \mathfrak{A} \to \mathfrak{A}$.
(c) Ψ *is monotone in the sense that* $A \subset B$ *implies* $\Psi(A) \subset \Psi(B)$.

Proof. Part **(a)** follows from the second part of (69). It is clear that for all $k \in [M]$ and $i \in [L]$, $\pi_1 \left(P^k_{\langle a_i, b_i \rangle} \right)$ is an interval, and so $\Psi(H)$ consists of finitely many intervals. The fact that all of these intervals are contained in $\widetilde{I}$ follows from the definition of $\widetilde{\alpha}$ and $\widetilde{\beta}$. Part **(b)** and **(c)** are obvious from the definition. □

Set

$$V_m := \Psi^m(\operatorname{int}(\widetilde{I})) \quad \text{with} \quad V_0 := \operatorname{int}(\widetilde{I}). \tag{77}$$

It follows from part **(b)** and **(c)** of Lemma 8.1, that $V_{m+1} \subset V_m$. Put

$$N_0 := \inf\{m \in \mathbb{N} \cup \{\infty\} : V_m \setminus V_{m+1} = \emptyset\}. \tag{78}$$

We will prove in Lemma 8.3 that N_0 is finite. A short heuristics for this is as follows: Below, we introduce a construction of level m green intervals and level m red intervals. These are the connected components of V_m and $V_{m-1} \setminus V_m$, respectively. We will prove that the length of a level m red interval is less than or equal to $\widetilde{g}^m(\widetilde{\beta} - \widetilde{\alpha})$, where $\widetilde{g}^m$ is the m-th iterate of the function $\widetilde{g}(x) := ax - \theta_{\min}$. The fixed point of this function is negative and $\{\widetilde{g}(x)\}_{m=1}^{\infty}$ converges to this fixed point for every x. So, for sufficiently large m, we have $\widetilde{g}^m(\widetilde{\beta} - \widetilde{\alpha}) < 0$. For such an m, there are no red intervals since the length of an interval cannot be negative. That is $V_m = V_{m+1}$. Hence, N_0 is finite.

Definition 8.2. The pre-type space is defined by $T(0) := V_{N_0}$.

8.1. Elementary properties of the pre-type space $T(0)$

Using that both $V_m \in \mathfrak{A}$ and $\mathrm{int}(\widetilde{I}) \setminus V_m \in \mathfrak{A}$ for every m we can find an $\widehat{n}_m$, n_m and $\widetilde{\alpha} \le \alpha_i^{(m)} < \beta_i^{(m)} \le \widetilde{\beta}$ and $\widetilde{\alpha} \le u_i^{(m)} \le v_i^{(m)} \le \widetilde{\beta}$ such that

$$V_m = \bigcup_{i \in [\widehat{n}_m]} \left(\alpha_i^{(m)}, \beta_i^{(m)}\right), \quad V_{m-1} \setminus V_m = \bigcup_{i \in [n_m]} \langle u_i^{(m)}, v_i^{(m)} \rangle$$

and $\left\{\left(\alpha_i^{(m)}, \beta_i^{(m)}\right)\right\}_{i=1}^{\widehat{n}_m}$ and $\left\{\langle u_i^{(m)}, v_i^{(m)} \rangle\right\}_{i=1}^{n_m}$ are the connected components of V_m and $V_{m-1} \setminus V_m$ respectively. We say that the intervals in the first union are level m green intervals and the intervals in the second union are the level m red intervals. See Fig. 4. Their collections are denoted by $\mathcal{G}_m$ and $\mathcal{R}_m$ respectively. That is

$$\mathcal{G}_m := \left\{\left(\alpha_i^{(m)}, \beta_i^{(m)}\right)\right\}_{i=1}^{\widehat{n}_m} \text{ and } \mathcal{R}_m := \left\{\langle u_i^{(m)}, v_i^{(m)} \rangle\right\}_{i=1}^{n_m}.$$

For an $m \ge 1$, the level-m green and red areas are:

$$\begin{aligned} G_m &:= U_{V_{m-1}} = \bigcup_{i \in [\widehat{n}_{m-1}], k \in [M]} P^k_{\left(\alpha_i^{(m-1)}, \beta_i^{(m-1)}\right)}, \\ R_m &:= U_{V_{m-1} \setminus V_m} = \bigcup_{i \in [n_m], k \in [M]} P^k_{\langle u_i^{(m)}, v_i^{(m)} \rangle}. \end{aligned} \tag{79}$$

Hence,

$$\pi_1(G_m) = V_m, \quad R_m \subset G_m, \quad \text{and} \quad G_{m+1} = G_m \setminus R_m.$$

In words, we obtain the level $m+1$ green area if we take away the level m red area from the level m green area. This implies that the sets $\{R_i\}_i$ are pairwise disjoint and

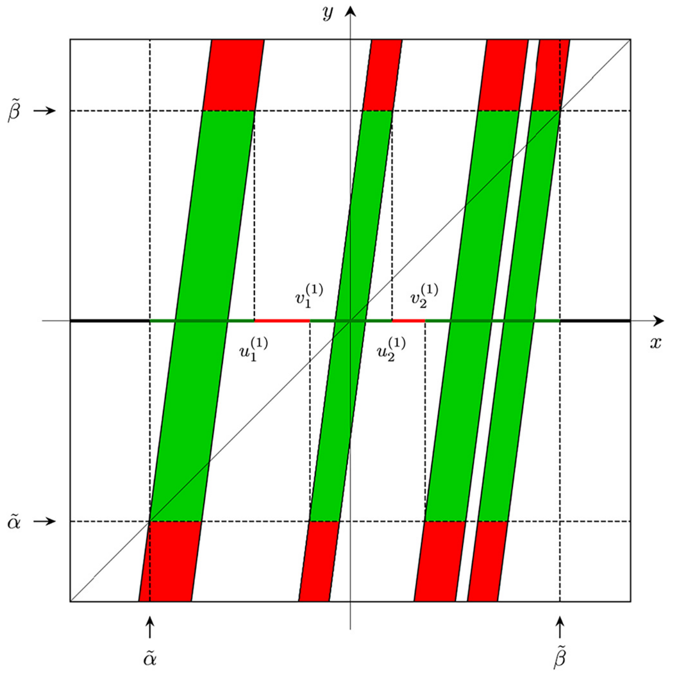

Fig. 4. Level 1 red and green intervals. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$G_{m+1} = G_1 \setminus \bigsqcup_{i=1}^{m} R_\ell, \tag{80}$$

where $\bigsqcup$ means disjoint union. In words: we get the level-$m+1$ green area if we take away from the level 1 green area all the first m level red areas. That is

$$G_{m+1} = G_1 \setminus \bigcup_{\ell=1}^{m} \bigcup_{1 \le i \le n_\ell, k \in [M]} P^k_{\langle u_i^{(\ell)}, v_i^{(\ell)} \rangle}. \tag{81}$$

The following Lemma plays an important role in our proofs.

Lemma 8.2. *The set $\pi_1(G_{m-1}) \setminus \pi_1(G_m)$ is the union of the disjoint closed (possibly degenerated) intervals $\mathcal{R}_m = \left\{[u_i^{(m)}, v_i^{(m)}]\right\}_{i=1}^{n_m}$.*

Proof. This follows from elementary geometry by mathematical induction. As we have mentioned, it is immediate from the construction that for every ℓ the open set $\pi_1(G_\ell) \subset (\widetilde{\alpha}, \widetilde{\beta})$ has finitely many components. It follows from (73) that if $\pi_1(G_1) \setminus \pi_1(G_2)$ is not empty then it is the disjoint union of finitely many compact intervals. Assume that the same holds for an $m \ge 2$ for the non-empty $\pi_1(G_{m-1}) \setminus \pi_1(G_m)$. That is

$$\pi_1(G_{m-1}) \setminus \pi_1(G_m) = V_{m-1} \setminus V_m = \bigcup_{i \in [n_m]} \left[u_i^{(m)}, v_i^{(m)}\right]. \tag{82}$$

Clearly, this is the case if and only if all endpoints of $\pi_1(G_{m-1})$ are also endpoints of $\pi_1(G_m)$. That is, our induction hypothesis yields that for all $i \leq n_m$ and $j \leq \widehat{n}_m$

$$\left(u_i^{(m)}, v_i^{(m)}\right) \subset \left(\alpha_j^{(m-1)}, \beta_j^{(m-1)}\right) \Longrightarrow \left[u_i^{(m)}, v_i^{(m)}\right] \subset \left(\alpha_j^{(m-1)}, \beta_j^{(m-1)}\right). \tag{83}$$

Using this, we prove that whenever $\pi_1(G_m) \setminus \pi_1(G_{m+1})$ is non-empty then it is the union of finitely many disjoint closed intervals. To verify this we assume that

$$x \in \pi_1(G_m) \setminus \pi_1(G_{m+1}) \subset \pi_1\left(G_m \setminus G_{m+1}\right) = \pi_1(R_m). \tag{84}$$

Then by (79) there is a $k \in [M]$ and $i \in [n_m]$, $j \in [\widehat{n}_m]$ and $\left[u_i^{(m)}, v_i^{(m)}\right] \in \mathcal{R}_m$, $\left(\alpha_j^{(m-1)}, \beta_j^{(m-1)}\right) \in \mathcal{G}_{m-1}$ such that by (84) and the induction hypothesis we have

$$x \in \pi_1\big(P^k_{\left[u_i^{(m)}, v_i^{(m)}\right]}\big) \subset \pi_1\big(P^k_{\left(\alpha_j^{(m-1)}, \beta_j^{(m-1)}\right)} \backslash \mathcal{G}_{m+1}\big). \tag{85}$$

Using that $\left[u_i^{(m)}, v_i^{(m)}\right] \subset \left(\alpha_j^{(m-1)}, \beta_j^{(m-1)}\right) \subset V_{m-1}$ is a connected component of $V_{m-1} \setminus V_m$, there exists an $\varepsilon > 0$ such that

$$\left(u_i^{(m)} - \varepsilon, u_i^{(m)}\right) \subset V_m \cap \left(\alpha_j^{(m-1)}, \beta_j^{(m-1)}\right), \left(v_i^{(m)}, v_i^{(m)} + \varepsilon\right) \subset V_m \cap \left(\alpha_j^{(m-1)}, \beta_j^{(m-1)}\right). \tag{86}$$

Putting together this, (85), (84), and (79) we obtain that

$$x \in \pi_1\big(P^k_{[u_i^{(m)}, v_i^{(m)}]}\big) \setminus \left(\pi_1\big(P^k_{(u_i^{(m)} - \varepsilon, u_i^{(m)})}\big) \cup \pi_1\big(P^k_{(v_i^{(m)}, v_i^{(m)} + \varepsilon)}\big)\right). \tag{87}$$

Then by simple elementary geometry this implies that the distance between x and the boundary of the set $\pi_1\left(P^k_{\alpha_j^{(m-1)}, \beta_j^{(m-1)}}\right)$ is at least $\theta_{\min} := \min_{i \in [L]} \theta_i$. Having a look at formula (79) we can see that x cannot be an endpoint of a component of $\pi_1(G_m)$. Hence, any endpoint of any component of $\pi_1(G_m)$ is also an endpoint of a certain component of $\pi_1(G_{m+1})$. $\square$

Actually we proved a little more.

Remark 8.1. Note that as a by-product of the proof above, we obtain that the following assertion holds:

If $x \in \pi_1(G_m) \setminus \pi_1(G_{m+1})$ then there exist some $\left[u_i^{(m)}, v_i^{(m)}\right] \in \mathcal{R}_m$ and $k \in [M]$ such that $x \in \pi_1\big(P^k_{\left[u_i^{(m)}, v_i^{(m)}\right]}\big)$. In this case the distance between x and the boundary of $\pi_1\big(P^k_{\left[u_i^{(m)}, v_i^{(m)}\right]}\big)$ is at least $\theta_{\min}$.

Lemma 8.3. *The number N_0, defined in (78), is finite.*

Proof. Assume that $m \geq 2$ and $x \in V_m \setminus V_{m+1}$. Then we can choose $\left[u_i^{(m)}, v_i^{(m)}\right] \in \mathcal{R}_m$ and $k \in [M]$, $x \in \pi_1\big(P^k_{\left[u_i^{(m)}, v_i^{(m)}\right]}\big)$. Let ℓ_k^1 and ℓ_k^2 be the lines defined in and above Fig. 3. Then

$$v' := (\ell_k^2)^{-1}(v_i^{(m)}) \text{ and } u' := (\ell_k^1)^{-1}(u_i^{(m)}).$$

By definition, we can choose a $j \in [n_{m+1}]$ such that $x \in \left[u_j^{(m+1)}, v_j^{(m+1)}\right]$. It follows from Remark 8.1 and elementary geometry that

$$v_j^{(m+1)} - u_j^{(m+1)} \leq v' - u' \leq a\left(v_i^{(m)} - u_i^{(m)}\right) - \theta_{\min}. \tag{88}$$

Let $\widetilde{g}(x) := ax - \theta_{\min}$. Then the length of the maximal interval in $\mathcal{R}_m$ is at most $\widetilde{g}^m(\widetilde{\beta} - \widetilde{\alpha})$, where $\widetilde{g}^m$ is the m fold iterate of $\widetilde{g}$. However, $\widetilde{g}^m(\widetilde{\beta} - \widetilde{\alpha}) < 0$ if m is large enough. The largest m for which $\widetilde{g}^m(\widetilde{\beta} - \widetilde{\alpha}) \geq 0$ is an upper bound on N_0. □

Recall that we defined $T(0)$ as V_{N_0} in Definition 8.2. It is clear that $T(0) \neq \emptyset$, because by the construction $V_m \neq \emptyset$ for all m.

Definition 8.3.

(a) The minimal width and height of the stripes $\{S_k\}_{k=1}^M$ are denoted by w and h. Clearly, $w \geq 2\theta_{\min}$ and $h \geq \frac{2\theta_{\min}}{a}$.

(b) To shorten the notation we write $\tau := \widehat{n}_{N_0}$, and instead of $\left\{(\alpha_i^{(N_0)}, \beta_i^{(N_0)})\right\}_{i=1}^{\widehat{n}_{N_0}}$ we write $\{(\alpha_i, \beta_i)\}_{i=1}^{\tau}$ for the connected components of $T(0) = V_{N_0}$.

(c) The intervals $\{(a_{i,k}, b_{i,k})\}_{k\in[M], i\in[\tau]}$ are defined as follows:

$$(a_{i,k}, b_{i,k}) := \pi_1\left(P^k_{(\alpha_i,\beta_i)}\right).$$

(d) Set $\widehat{d} := \min\left\{|x-y| : x \neq y, x, y \in \bigcup_{k\in[M], i\in[\tau]} \{a_{i,k}, b_{i,k}\}\right\}$. □

Lemma 8.4. *If $x \in T(0) \cap \bigcup_{k\in[M], i\in[\tau]} \{a_{i,k}, b_{i,k}\}$ then there is an $i(x) \in [\tau]$ and $k(x) \in [M]$ such that*

$$\widehat{d} \leq x - a_{i(x),k(x)} \text{ and } \widehat{d} \leq b_{i(x),k(x)} - x. \tag{89}$$

Proof. Without loss of generality we may assume that $x = a_{\widehat{i},\widehat{k}}$. We know that $T(0) = V_{N_0} = V_{N_0+1}$. Hence,

$$x \in T(0) = \Psi(T(0)) = \bigcup_{k\in[M], i\in[\tau]} \pi_1\left(P^k_{(\alpha_i,\beta_i)}\right) = \bigcup_{k\in[M], i\in[\tau]} (a_{i,k}, b_{i,k}). \tag{90}$$

So, there is an $i(x) \in [\tau]$ and a $k(x) \in [M]$ such that $x \in (a_{i(x),k(x)}, b_{i(x),k(x)})$. Then by the definition of $\widehat{d}$, (89) holds. □

Proposition 8.1.

$$y \in T(0) \Longrightarrow \bigcup_{i=1}^{L} (ay + t_i - \theta_i, ay + t_i + \theta_i) \subset T(0). \tag{91}$$

Consequently,

$$H_{\mathbf{i}}(T(0)) \subset T(0), \quad \forall n \geq 1 \text{ and } \mathbf{i} \in \mathcal{L}_n. \tag{92}$$

Proof. The implication in (91) follows from (76) and from the fact that $T(0) = V_{N_0} = V_{N_0+1} = \Psi(V_{N_0})$. Using this, and the definition of H_i we get that the image $H_i(y)$ of an $y \in T(0)$ by the random mapping H_i satisfies $H_i(y) \in (ay + t_i - \theta_i, ay + t_i + \theta_i)$ for all $i \in [L]$. Successive applications of this inclusion yield (92). □

Proposition 8.2. *There is an $\widetilde{\varepsilon} > 0$ such that for all $0 < \varepsilon < \widetilde{\varepsilon}$ the Perron Frobenius eigenvalue of the operator F^{ε} is greater than 1.*

The proof can be obtained by obvious modifications from the proof of [1, Lemma 8A].

Definition 8.4 *(Type space).*

(a) First we define

$$\varepsilon_{\text{MAIN}} := \frac{a}{10} \min\left\{ w, \widehat{d}, \min_{i \in [\tau]} (\beta_i - \alpha_i), \widetilde{\varepsilon} \right\}. \tag{93}$$

(b) Fix an arbitrary $0 < \varepsilon < \varepsilon_{\text{MAIN}}$. Putting together (26) and (93), we obtain that the type space is

$$T(\varepsilon) = \bigcup_{i \in [\tau]} [\alpha_i + \varepsilon, \beta_i - \varepsilon]. \quad \square \tag{94}$$

Lemma 8.5. *For an $x_0 \in T(\varepsilon)$ we define*

$$E_1^{\varepsilon}(x_0) := \{y \in T(\varepsilon) : m_1^{\varepsilon}(x_0, y) > 0\} = \{(x, y) : x = x_0\} \cap \text{supp}\,(m_1^{\varepsilon}). \tag{95}$$

Then there exists a $\kappa = \kappa(\varepsilon) > 0$ such that for all $x_0 \in T(\varepsilon)$ the set $E_1^{\varepsilon}(x_0)$ contains an interval of length κ.

Proof. First recall that $T(0) = V_{N_0} = \pi_1(G_{N_0})$ and $G_{N_0} = \bigcup_{k \in [M], i \in [\tau]} P^k_{(\alpha_i, \beta_i)}$ and $(a_{i,k}, b_{i,k}) = \pi_1\left(P^k_{(\alpha_i,\beta_i)}\right)$. Now we define the hexagon

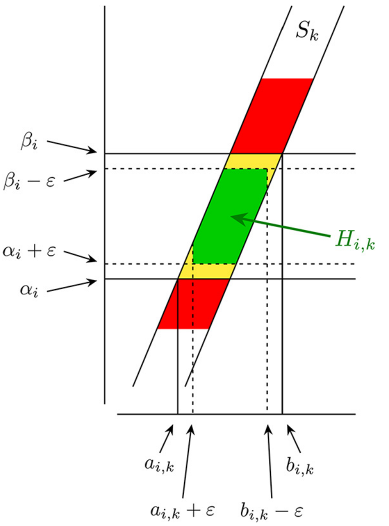

Fig. 5. The definition of $H_{i,k}$.

$$H_{i,k} := P^k_{(\alpha_i,\beta_i)} \cap \{(x,y) : a_{i,k} + \varepsilon \le x \le b_{i,k} - \varepsilon, \alpha_i + \varepsilon \le y \le \beta_i - \varepsilon\}. \tag{96}$$

See Fig. 5. Clearly,

$$\mathrm{supp}(m_1^\varepsilon) = \bigcup_{k\in[M], i\in[\tau]} H_{i,k}. \tag{97}$$

Observe that by elementary geometry

$$\forall k \in [M],\ i \in [\tau],\ x_0 \in \pi_1(H_{i,k}) \text{ we have } |\{(x,y) : x = x_0\} \cap H_{i,k}| \ge \varepsilon\left(\frac{1}{a} - 1\right). \tag{98}$$

Now we verify that

$$T(\varepsilon) \subset \bigcup_{k\in[M], i\in[\tau]} \pi_1(H_{i,k}). \tag{99}$$

Namely, $T(\varepsilon) \subset T(0)$ and in this way for all $x_0 \in T(\varepsilon)$ there exists a $k \in [M]$ and an $i \in [\tau]$ such that $x_0 \in \pi_1(P^k_{(\alpha_i,\beta_i)}) = (a_{i,k}, b_{i,k})$. Observe that

$$\exists i \in [\tau], k \in [M] \text{ such that } a_{i,k} + \varepsilon \le x_0 \le b_{i,k} - \varepsilon \Longrightarrow x_0 \in \pi_1(H_{i,k}). \tag{100}$$

If the condition of (100) does not hold then we may assume, without loss of generality that

$$a_{i,k} < x_0 < a_{i,k} + \varepsilon. \tag{101}$$

This and $x_0 \in T(\varepsilon)$ imply that $a_{i,k} \in T(0)$. We argue by contradiction. If $a_{i,k} \notin T(0)$ then there is a $j \in [\tau]$ such that $a_{i,k} = \alpha_j$. Then $x_0 \in (\alpha_j, \alpha_j + \varepsilon) \subset T(\varepsilon)^c$ which contradicts to our assumption that $x_0 \in T(\varepsilon)$. So, we have verified that $a_{i,k} \in T(0)$. Then we can apply Lemma 8.4 to conclude that there exists an $\ell \in [M]$ and $j \in [\tau]$ such that $a_{j,\ell} + \widehat{d} < a_{i,k} < b_{j,\ell} - \widehat{d}$. Putting together (101) and (93) we obtain that $a_{j,\ell} + \varepsilon < x_0 < b_{j,\ell} - \varepsilon$. That is the condition of (106) holds, which contradicts the assumption that this condition does not hold. This proves that (99) holds. Putting together (99), (98) and (97) we obtain that the assertion of Lemma 8.5 is true, with the choice $\kappa = \varepsilon\left(\frac{1}{a} - 1\right)$. $\square$

As a byproduct of the previous proof we obtain that

$$T(\varepsilon) = \bigcup_{k\in[M],i\in[\tau]} \pi_1(H_{i,k}) = \bigcup_{k\in[M],i\in[\tau]} [a_{i,k} + \varepsilon, b_{i,k} - \varepsilon]. \tag{102}$$

Namely, the non-trivial inclusion was verified above. The opposite inclusion is obvious by the definitions.

Our aim is to prove the following proposition, which is actually Part (ii) of Lemma 4.1 (the Main Lemma).

Proposition 8.3. *Fix an $0 < \varepsilon < \varepsilon_{\text{MAIN}}$. Then there exists an Q such that for every $x_0 \in T(\varepsilon)$ we have*

$$E_Q^\varepsilon(x_0) := \{(x, y) : x = x_0\} \cap \operatorname{supp}(m_Q^\varepsilon) = \left\{y : m_Q^\varepsilon(x_0, y) > 0\right\} = T(\varepsilon). \tag{103}$$

8.1.1. The structure of green and red areas

To prove Proposition 8.3 we need to verify some auxiliary facts about the structure of the green and red areas and intervals. These will be given in Lemma 8.6, Lemma 8.7, and Lemma 8.8.

For an $m \le N_0$, we write $\mathcal{B}_m$ and $\mathcal{J}_m$ for the collection of the left and right endpoints respectively, of the level m red intervals (intervals from $\mathcal{R}_m$), and $\mathcal{B} = \cup_{i=1}^{N_0} \mathcal{B}_m$ and $\mathcal{J} = \cup_{i=1}^{N_0} \mathcal{J}_m$.

Lemma 8.6.

(a) *For all $1 \le m \le N_0$ and for all $(\alpha', \beta') \in \mathcal{G}_m$ there exist $v(\alpha') \in \bigcup_{\ell=1}^{m} \mathcal{J}_\ell$, and $u(\beta') \in \bigcup_{\ell=1}^{m} \mathcal{B}_\ell$ such that $(\alpha', \beta') = (v(\alpha'), u(\beta'))$.*

(b) *All elements of $\bigcup_{\ell=1}^{m} \mathcal{B}_\ell$ are right endpoints of an element of $\mathcal{G}_m$. Similarly, all elements of $\bigcup_{\ell=1}^{m} \mathcal{J}_\ell$ are left endpoints of an element of $\mathcal{G}_m$.*

(c) *Let $(\alpha', \beta') \in \mathcal{G}_m$. Then $(\alpha', \beta') \cap T(\varepsilon) = (\alpha' + \varepsilon, \beta' - \varepsilon) \cap T(\varepsilon)$.*

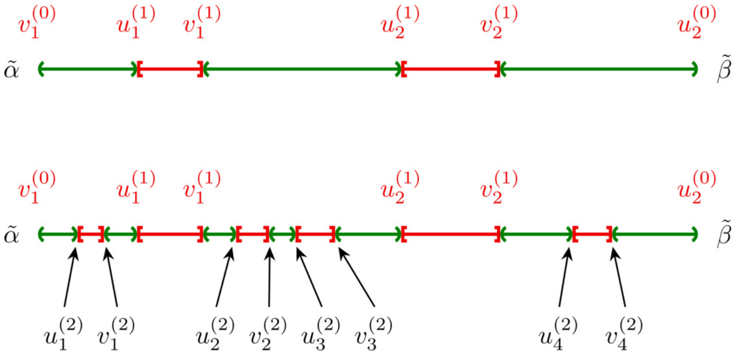

Fig. 6. The red and green intervals.

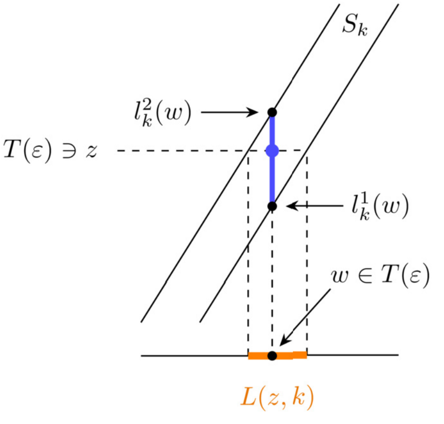

Fig. 7. The notation of Lemma 8.7.

Proof. **(a)** and **(b)** follow immediately from the construction. These imply that every endpoint of a component of V_m is an endpoint of a component of $T(0)$, and so **(c)** follows. See Fig. 6. □

Lemma 8.7. *Let $z \in T(\varepsilon)$ and $k \in [M]$. Set $L(z, k) := \pi_1 (\{(x, y) : y = z\} \cap S_k)$. Then*

$$L(z, k) \cap T(\varepsilon) \neq \emptyset. \tag{104}$$

Proof. Fix a $z \in T(\varepsilon)$ and $k \in [M]$. Observe that $L(z, k) \subset T(0)$. Namely, $z \in T(0) = V_{N_0}$. So $\pi_1(L(z, k)) \subset V_{N_0+1} = V_{N_0} = T(0)$. Then $(\mathcal{J} \cup \mathcal{B}) \cap L(z, k) = \emptyset$. Hence, when we change from $T(0)$ to $T(\varepsilon)$ we can lose in a $1 - 1$ way intervals of length ε each at the two ends of the interval of $L(z, k)$. Using that $|L(z, k)| = w_k \gg 2\varepsilon$ we get that $|L(z, k) \cap T(\varepsilon)| > w - 2\varepsilon$, where w is the minimum width of a stripe S_k. □

This implies that

$$\forall z \in T(\varepsilon), k \in [M] \; \exists w \in T(\varepsilon) \cap L(z, k) \text{ such that } z \in \left(\ell_k^1(w), \ell_k^2(w)\right). \tag{105}$$

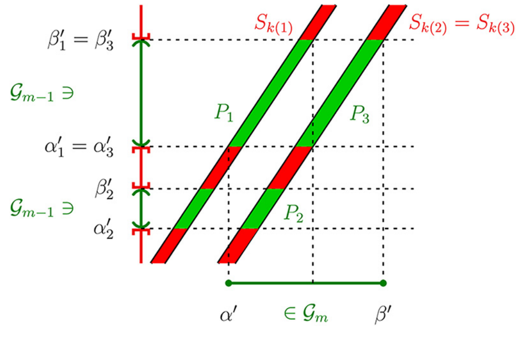

Fig. 8. The definition of the parallelograms P_j.

Lemma 8.8. *For an* $1 \leq m \leq N_0$ *let* $(\alpha', \beta') \in \mathcal{G}_m$. *Consider the maximal collection of* $(\alpha'_j, \beta'_j) \in \mathcal{G}_{m-1}$ *and* $k(j) \in [M]$, $j = 1, \dots, \ell$ *such that for* $P_j := P^{k(j)}_{(\alpha'_j, \beta'_j)}$ *we have* $\bigcup_{j=1}^{\ell} \pi_1(P_j) \subset (\alpha', \beta')$. *See Fig. 8. For* $j \neq j'$ *if* $(\alpha'_j, \beta'_j) = (\alpha'_{j'}, \beta'_{j'})$ *then* $k(j) \neq k(j')$. *We say that the intervals* $(\alpha'_j, \beta'_j) \in \mathcal{G}_{m-1}$, $j = 1, \dots, \ell$ *are the children of the interval* $(\alpha', \beta') \in \mathcal{G}_m$. *Then we have*

$$\bigcup_{j=1}^{\ell} \pi_1(P_j) = (\alpha', \beta'). \tag{106}$$

Moreover, for all $j = 1, \dots, \ell$,

$$\bigcup_{x \in \pi_1(P_j) \cap T(\varepsilon)} \left(\ell^1_{k(j)}(x), \ell^2_{k(j)}(x)\right) \cap T(\varepsilon) \supset (\alpha'_j, \beta'_j) \cap T(\varepsilon). \tag{107}$$

Proof. Equation (106) is immediate from the definitions (75) and (77). To prove Equation (107): Let $z \in (\alpha'_j, \beta'_j) \cap T(\varepsilon)$. Then it follows from (105) that there exists a $w \in L(z, k(j)) \cap T(\varepsilon) \subset \pi_1(P_j) \cap T(\varepsilon)$ such that $z \in (\ell^1_{k(j)}), \ell^2_{k(j)})$. $\square$

Before we start the proof of Proposition 8.3 we recall some definitions: We defined m_I in (25) and we saw that $\mathrm{supp}(m_I) = \bigcup_{k=1}^{M} S_k$. Moreover, in the statement of the Lemma 4.1 [Main Lemma] we defined

$$m^\varepsilon = m_I \cdot \mathbf{1}_{T(\varepsilon) \times T(\varepsilon)} \text{ and } m^\varepsilon_1 := m^\varepsilon. \tag{108}$$

Furthermore, for $n \geq 1$:

$$m^\varepsilon_{n+1}(x, y) := \int_{T(\varepsilon)} m^\varepsilon_n(x, z) \cdot m^\varepsilon_1(z, y)\, \mathrm{d}z. \tag{109}$$

Similarly, for a $k \geq 1$, we defined

$$E_k^\varepsilon(x_0) := \{(x,y) : x = x_0\} \cap \operatorname{supp}(m_k^\varepsilon) = \{y \in T(\varepsilon) : m_k^\varepsilon(x_0,y) > 0\}\,. \tag{110}$$

In particular, $E_1^\varepsilon(z) = \emptyset$ if $z \notin T(\varepsilon)$ and

$$E_1^\varepsilon(z) = \left(\bigcup_{k=1}^{M} \left(\ell_k^1(z), \ell_k^2(z)\right)\right) \cap T \quad \text{if} \quad z \in T(\varepsilon). \tag{111}$$

Proof of Proposition 8.3. We fix an $0 < \varepsilon < \varepsilon_{\text{MAIN}}$ and we write $T := T(\varepsilon)$, $m_k := m_k^\varepsilon$, $E_k := E_k^\varepsilon$ and $\kappa := \kappa(\varepsilon)$ (defined in Lemma 8.5). We divide the proof into two steps. First we recall (see part (a) of Definition 8.3) that the connected components of $T(0) = V_{N_0}$ are $\{(\alpha_i, \beta_i)\}_{i=1}^{\tau}$. So, $(\alpha_i, \beta_i) \in \mathcal{G}_{N_0}$ for all $i \in [\tau]$.

Step 1 There exists an N such that for all $x \in T$ there exists an $n(x) \le N$, such that there exists an i with $1 \le i \le \tau$, such that

$$[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset E_{n(x)}(x). \tag{112}$$

Step 2 For every $x \in T$ we have

$$[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset E_{n(x)}(x) \Longrightarrow E_{n(x)+N_0}(x) = T. \tag{113}$$

Proof of Step 1. It follows from (108), (109), (110), and (111) that

$$E_{n+1}(x) = \bigcup_{z \in E_n(x)} E_1(z) = \bigcup_{z \in E_n(x)} \left(\bigcup_{k=1}^{M} \left(\ell_k^1(z), \ell_k^2(z)\right)\right) \cap T. \tag{114}$$

Let $x \in T$. Using (99) there exists an $i \in [\tau]$, $k \in [M]$ such that $x \in \pi_1(H_{i,k})$. Then either

- **(a)** $(\ell_k^1(x), \ell_k^2(x)) \subset [\alpha_i + \varepsilon, \beta_i - \varepsilon]$ or
- **(b)** either $\alpha_i + \varepsilon$ or $\beta_i - \varepsilon$ is an endpoint of a component of $E_1(x)$ of length at least κ (cf. (98)).

If **(b)** holds, then $E_1(x)$ has a component of length at least κ which has at least one endpoint in T. It is easy to see that if this property holds for $E_k(x)$ then the same property holds for $E_\ell(x)$ for every $\ell > k$.

Now we prove that in case **(a)** the same property holds for $E_n(x)$ for a not too large n. If **(a)** holds, then we consider

$$E_2(x) = \bigcup_{z \in E_1(x)} E_1(z) \supset \bigcup_{z \in (\ell_k^1(x), \ell_k^2(x))} E_1(z).$$

If $J := \bigcup_{z \in (\ell_k^1(x), \ell_k^2(x))} E_1(z)$ is still contained in a component of T then J is an interval of length at least $(1/a)\kappa$. We continue this process and we obtain that if we choose N_1 such

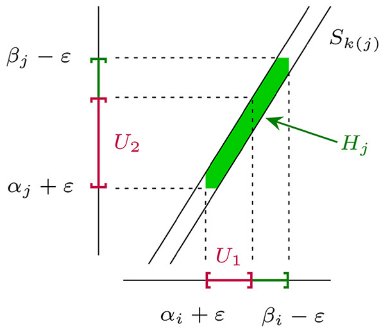

Fig. 9. Definition of U_1 and U_2.

that $\kappa(1/a)^{N_1} > \widetilde{\beta} - \widetilde{\alpha}$ then for an $n = n(x) < N_1$, the set $E_n(x)$ has a component U_1 with length at least κ and with at least one of its endpoints contained in $\bigcup_{i=1}^{\tau} (\alpha_i + \varepsilon, \beta_i - \varepsilon)$. Without loss of generality, we may assume that

$$U_1 = (\alpha_i + \varepsilon, w) \text{ with } w - \alpha_i - \varepsilon > \kappa.$$

Now recall the definition of $a_{j,k}, b_{j,k}$ from Definition 8.3. See Fig. 9. Then there exist $k \in [M]$ and $j \in [\tau]$ such that $\alpha_i = a_{j,k}$. That is

$$\pi_1 \left(P^k_{(\alpha_j, \beta_j)} \right) = (a_{j,k}, b_{j,k}) = (\alpha_i, b_{j,k}). \tag{115}$$

Observe that

$$\bigcup_{x \in (a_{j,k}+\varepsilon, b_{j,k}-\varepsilon)} \left(\left(\ell^1_k(x), \ell^2_k(x)\right) \cap T \right) = [\alpha_j + \varepsilon, \beta_j - \varepsilon]. \tag{116}$$

If $w > b_{j,k} - \varepsilon$ then $U_1 \supset (a_{j,k} + \varepsilon, b_{j,k} - \varepsilon)$. Using this, the fact that $U_1 \subset E_n(x)$, (116) and (114), we get that $E_{n+1}(x) \supset [\alpha_j + \varepsilon, \beta_j - \varepsilon]$. That is, in this case, we are ready. So, from now on, we assume that for the j and k that appear in (115), we have $w \le b_{j,k} - \varepsilon$. Then, by elementary geometry:

$$U_1 \subset \pi_1 \left(P^k_{[\alpha_j + \varepsilon, \beta_j - \varepsilon]} \right). \tag{117}$$

Now we define

$$U_2 := \bigcup_{z \in U_1} \left(\ell^1_k(z), \ell^2_k(z)\right) \cap T.$$

By definition the left endpoint of U_2 is $\alpha_j + \varepsilon$. If $U_2 \supset [\alpha_j + \varepsilon, \beta_j - \varepsilon]$ that is the right endpoint of U_2 is greater than $\beta_j - \varepsilon$, then $E_{n+1}(x) \supset [\alpha_j + \varepsilon, \beta_j - \varepsilon]$. Otherwise, if $U_2 \subset [\alpha_j + \varepsilon, \beta_j - \varepsilon]$, that is the right endpoint of U_2 is not less than $\beta_j - \varepsilon$, then

$|U_2| \subset T(\varepsilon)$ and $|U_2| > (1/a)|U_1|$. So repeating the same for U_2 instead of U_1, we get U_3 and so on until after a uniformly bounded number (not more than N_2) steps U_k contains a component of T. This completes the proof for Step 1 with $N = N_1 + N_2$. □

Proof of Step 2. Now we use the notation of Lemma 8.8. Let $(\alpha', \beta') \in \mathcal{G}_m$ and $(\alpha'_j, \beta'_j) \in \mathcal{G}_{m-1}$ be a child of (α', β'). That is there exists a $k(j) \in [M]$ such that for $P_j = P^{k(j)}_{(\alpha'_j, \beta'_j)}$ we have $\pi_1(P_j) \subset (\alpha', \beta')$. We claim that

$$(\alpha', \beta') \cap T(\varepsilon) \subset E_n(x) \Longrightarrow (\alpha'_j, \beta'_j) \cap T(\varepsilon) \subset E_{n+1}(x). \tag{118}$$

Namely, assume that $(\alpha', \beta') \cap T(\varepsilon) \subset E_n(x)$. Then

$$E_{n+1}(x) = \bigcup_{y \in E_n(x)} \left(\bigcup_{k=1}^{M} \left(\ell^1_k(y), \ell^1_k(y)\right) \right) \cap T \tag{119}$$

$$\supset \bigcup_{y \in \pi_1(P_j) \cap T} \left(\left(\ell^1_{k(j)}(y), \ell^1_{k(j)}(y)\right) \cap T \right) \supset (\alpha'_j, \beta'_j) \cap T(\varepsilon), \tag{120}$$

where in the last step we used Lemma 8.8 ((Equation (107)). In this way we have proved that (118) holds.

Now we fix an $x \in T$ and let $n = n(x)$ be defined as in the proof of Step 1. We start with (α_i, β_i) obtained in the first step. That is $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset E_n(x)$. By Lemma 8.6 this implies that $(\alpha_i, \beta_i) \cap T \subset E_n(x)$. We apply (118) N_0 times. The level N_0-th child of $(\alpha_i, \beta_i) \in \mathcal{G}_{N_0}$ is the only element of $\mathcal{G}_{N_0}$, which is $(\widetilde{\alpha}, \widetilde{\beta})$. So, we get that $E_{n+N_0}(x) \supset (\widetilde{\alpha}, \widetilde{\beta}) \cap T = T$. On the other hand, it is immediate from the definition that $E_{n+N_0}(x) \subset T$. □

The proof of Proposition 8.3 is immediate if we put together what we obtained in Steps 1, 2. □

Proof of Lemma 4.1, the Main Lemma. (i) We take $\varepsilon_{\text{MAIN}}$ as defined in Equation (93). Then the number of intervals remains the same, since $\varepsilon_{\text{MAIN}} < (\beta_i - \alpha_i)/10$ for all i.

(ii) This is immediate from Proposition 8.3 with the Q defined there.

(iii) This is Proposition 8.2.

(iv) This follows easily from the fact that the right eigenfunction f is also an eigenfunction of $(F^\varepsilon)^2 = F^\varepsilon \circ F^\varepsilon$ whose kernel m_2 is continuous.

(v) This is Equation (92) given in Proposition 8.1. □

9. Appendix: Proof of Proposition 3.1

The proof of Proposition 3.1 is a simple combination of ideas of [2, Lemma 2.8] and [11, Proposition 6].

Proof of Proposition 3.1. Given is the RIFS $\mathcal{F} = \{f_i(x) : f_i(x) = r_i x + D_i\}_{i=1}^{L}$ as in Definition 2.1, and we write $s := s(\mathcal{F})$ for the similarity dimension (the solution of the Equation (12)).

The first step is to replace $\mathcal{F}$ by the RIFS $\widetilde{\mathcal{F}}$ defined in Proposition 2.1:

$$\widetilde{\mathcal{F}} = \left\{ \widetilde{f}_i(x) = \widetilde{r}_i x + \widetilde{D}_i \right\}_{i=1}^{\widetilde{L}}, \tag{121}$$

which has the convenient property that all contraction ratios $\widetilde{r}_i > 0$. Further, let $\widetilde{s} := s(\widetilde{\mathcal{F}}) \geq s - \varepsilon/2$ (where we replaced ε by $\varepsilon/2$ in Proposition 2.1).

Let $p_i := \widetilde{r}_i^{s}$. For $k_1, \ldots, k_m \in \mathbb{N}$ and $k := k_1 + \cdots + k_m$ we introduce

$$N(k_1, \ldots, k_m) := \#\{(i_1, \ldots, i_k) : \#\{j \in [k] : i_j = \ell\} = k_\ell, \quad \forall \ell \in [L]\}. \tag{122}$$

Lemma 9.1 *(Farkas and Peres-Shmerkin). There exists a $C > 0$ such that for all $k \in \mathbb{N}$ there exist $k_1, \ldots, k_m \in \mathbb{N}$, such that $\sum_{i=1}^{m} k_i = k$ and*

$$N(k_1, \ldots k_m) \geq C k^{-m/2} p_1^{-k_1} \cdots p_m^{-k_m}. \tag{123}$$

Let $C > 0$ as in Lemma 9.1. For a large k, which will be conveniently chosen at the end of the proof, we can choose $k_1, \ldots, k_{\widetilde{L}} \in \mathbb{N}$ according to Lemma 9.1 such that we have the following

$$N(k_1, \ldots k_{\widetilde{L}}) \geq C k^{-\widetilde{L}/2} p_1^{-k_1} \cdots p_{\widetilde{L}}^{-k_{\widetilde{L}}}, \tag{124}$$

$$\text{Let} \quad J_0 := \left\{ (i_1, \ldots, i_k) \in [\widetilde{L}]^k : \#\{j \in [k] : i_j = \ell\} = k_\ell, \quad \forall \ell \in [\widetilde{L}] \right\}.$$

Note that by definition $\#J_0 = N(k_1, \ldots k_{\widetilde{L}})$. Put $\rho_k := \prod_{\ell=1}^{\widetilde{L}} \widetilde{r}_\ell^{k_\ell}$. Then

$$\mathbf{i} \in J_0 \Longrightarrow r_{\mathbf{i}} = \rho_k \text{ and } \quad \#J_0 \geq C \cdot k^{-\widetilde{L}/2} \rho_k^{-\widetilde{s}}. \tag{125}$$

Let $\mathcal{F}_k := \{f_{\mathbf{i}} : \mathbf{i} \in J_0\}$. Then the similarity dimension $s_k := s(\mathcal{F}_k)$ is the solution of $\#J_0 \cdot \rho_k^{s_k} = 1$. That is, $\log N(k_1, \ldots, k_{\widetilde{L}}) + s_k \log \rho_k = 0$. Hence,

$$s_k = \frac{\log N(k_1, \ldots, k_{\widetilde{L}})}{-\log \rho_k} \geq \frac{\log C - \widetilde{L} \log \sqrt{k} - \widetilde{s} \log \rho_k}{-\log \rho_k} = \frac{\log C - \widetilde{L} \log \sqrt{k}}{-\log \rho_k} + \widetilde{s}.$$

Using that $-\log \rho_k = -\sum_{\ell=1}^{\widetilde{L}} k_\ell \log(r_\ell) \geq -\log(r_{\max}) \sum_{\ell=1}^{\widetilde{L}} k_\ell = -\log(r_{\max}) \cdot k$ we obtain

$$s_k \geq \widetilde{s} - \frac{\widetilde{L} \log \sqrt{k} - \log C}{-\log(r_{\max}) \cdot k} > \widetilde{s} - \frac{\varepsilon}{2} > s - \varepsilon,$$

if k is large enough. □

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