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## On rigidity theory, strong solidity, Coxeter groups, graph products and commutator estimates

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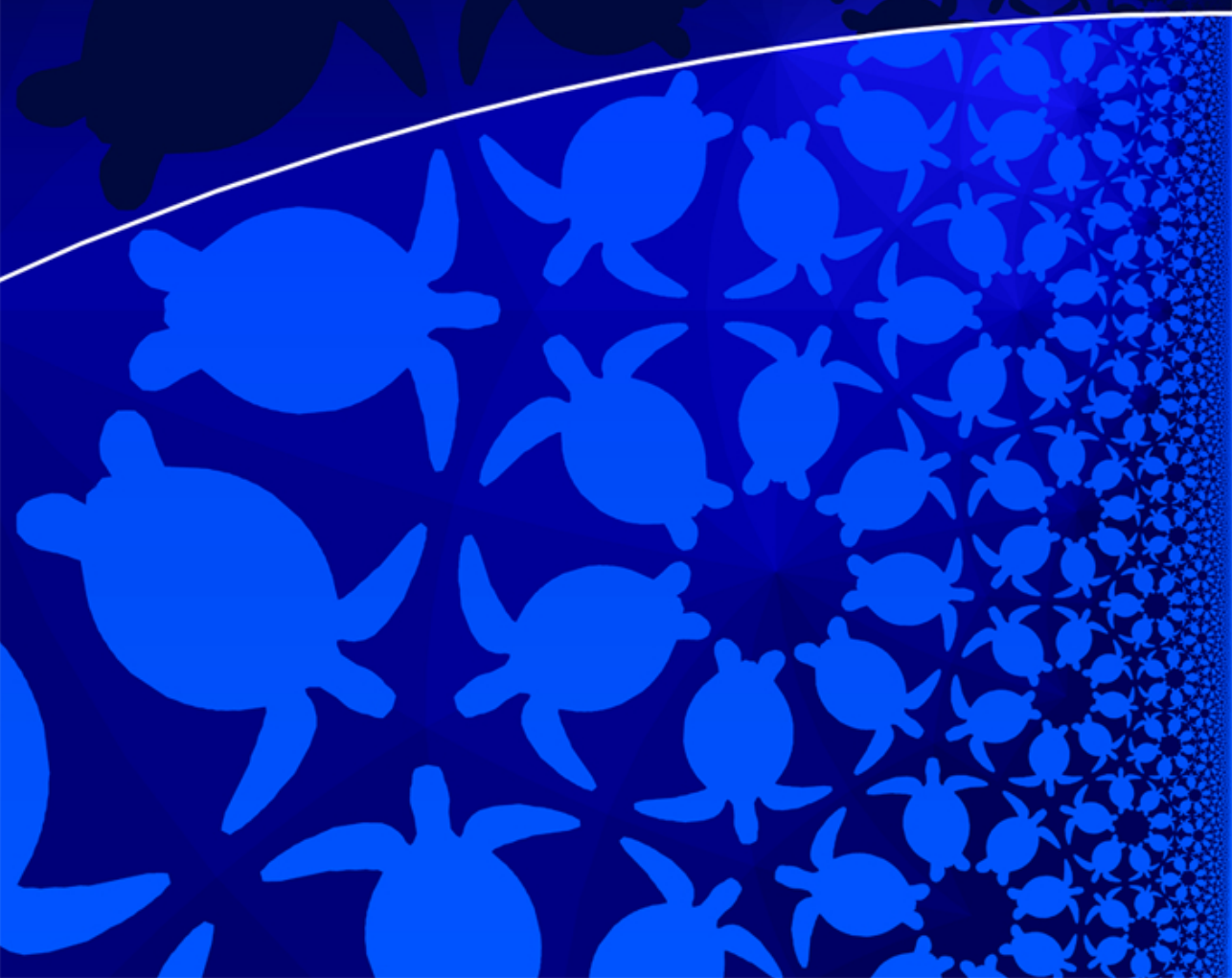
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# On Rigidity Theory, Strong Solidity, Coxeter Groups, Graph Products and Commutator Estimates

Matthijs Borst





**ON RIGIDITY THEORY, STRONG SOLIDITY,  
COXETER GROUPS, GRAPH PRODUCTS AND  
COMMUTATOR ESTIMATES**



# **ON RIGIDITY THEORY, STRONG SOLIDITY, COXETER GROUPS, GRAPH PRODUCTS AND COMMUTATOR ESTIMATES**

## **Proefschrift**

ter verkrijging van de graad van doctor  
aan de Technische Universiteit Delft,  
op gezag van de Rector Magnificus dr. ir. T.H.J.J. van der Hagen,  
voorzitter van het College voor Promoties,  
in het openbaar te verdedigen op maandag 24 maart 2025 om 15:00 uur

door

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*Front & Back:* The cover of this thesis depicts a pattern of turtles in the hyperbolic upper half plane. Its symmetry group is given by the Coxeter group

$$\mathcal{W} = \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^2 = (st)^3 = (tr)^7 = e \rangle.$$

This image is inspired by the work of the Dutch artist M.C. Escher who constructed several tilings of (hyperbolic) space using animals such as birds, fish and lizards (see [Cox79] for a discussion of Coxeter on an artwork of Escher).

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Voor mijn ouders,



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# SUMMARY

This thesis studies the field of operator algebras, non-commutative functional analysis and rigidity theory. We study structural properties of  $C^*$ -algebras and von Neumann algebras, with a focus on the latter. These mathematical structures were introduced by von Neumann in [Neu30] motivated by the need for a non-commutative framework to describe quantum systems. The theory was further developed by Murray and von Neumann in several papers: [MN36], [MN37], [Neu39], [Neu40], [MN43], [Neu43] and [Neu49]. Nowadays, the study of these operator algebras forms its own field in mathematics. Over the years effort has been made in trying to classify von Neumann algebras. Many structural properties of von Neumann algebras have been introduced and studied. In this thesis we study such properties including: absence of Cartan subalgebras, primeness, the (weak-\*) CCAP, the Akemann-Ostrand property and strong solidity. Furthermore we study operator estimates for commutators.

For a discrete group  $G$  we study the group von Neumann algebra  $\mathcal{L}(G)$ . The aim is to establish connections between the group  $G$  and its von Neumann algebra  $\mathcal{L}(G)$ . In particular, we study *rigidity theory*, which concerns the question what information of  $G$  can be retrieved from  $\mathcal{L}(G)$ . We are particularly interested in *Coxeter groups*. Such a group  $\mathcal{W}$  can be seen as an abstract reflection group. For a Coxeter group  $\mathcal{W}$  we will not only study the group von Neumann algebra  $\mathcal{L}(\mathcal{W})$ , but also its  $\mathbf{q}$ -deformations:  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  called *Hecke-von Neumann algebras*. The focus is on Coxeter groups that are right-angled. These Coxeter groups naturally decompose as graph product  $\mathcal{W} = *_{v \in \Gamma} \mathcal{W}_v$  of the groups  $\mathcal{W}_v = \mathbb{Z}/2\mathbb{Z}$ . The construction of graph products of groups was introduced by Green in [Gre90] as a generalization of both direct sums  $G_1 \oplus G_2$  and free products  $G_1 * G_2$ . Later graph products have also been defined in the setting of  $C^*$ -algebras and von Neumann algebras in [Mlo04] and [CF17]. In this setting graph products generalize both tensor products and free products. This notion of graph products interacts nicely with the notion for groups since  $\mathcal{L}(*_{v \in \Gamma} G_v) = *_{v \in \Gamma} \mathcal{L}(G_v)$ . In the case of right-angled Coxeter groups, a similar decomposition holds true for Hecke-von Neumann algebras.

This thesis consists of 7 chapters, including the introduction (Chapter 1) and the preliminaries (Chapter 2). In Chapter 3 we perform calculations in graph products that we need in later chapters. In Chapter 4 the study is focused on (right-angled) Coxeter groups, their group von Neumann algebra  $\mathcal{L}(\mathcal{W})$  and Hecke-von Neumann algebras  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$ . For these von Neumann algebras we study when they are *strongly solid* and when they possess the Akemann-Ostrand property  $(AO)^+$ . Strong solidity is a strengthened version of Ozawa's property solidity [Oza04] and can be seen as a strong indecomposability property. Indeed, this property implies that the von Neumann algebra does not decompose as tensor product  $M = M_1 \bar{\otimes} M_2$  (primeness) nor as a group measure space  $M = L^\infty(0, 1) \rtimes G$  (absence of Cartan). Using quantum Markov semigroups and the

non-commutative Riesz transform we prove new strong solidity results for right-angled Hecke-algebras.

In Chapter 5 we study strong solidity for general graph products  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  of von Neumann algebras. We use Popa's intertwining-by-bimodule theory to obtain a full characterization of strong solidity for graph products. In particular, we complete the characterization for right-angled Hecke-algebra. For right-angled Coxeter groups this provides a simple characterization of when the group von Neumann algebra is strongly solid. We also study other aspects of graph products. Indeed, we give sufficient conditions for the (reduced) graph product to be nuclear. Moreover we fully characterize primeness and free indecomposability for graph products. We also study rigidity theory for graph products. The aim is to retrieve the graph  $\Gamma$  and the vertex von Neumann algebras  $(M_v)_{v \in \Gamma}$  from the von Neumann algebra  $M_\Gamma$ . We introduce in this thesis a class  $\mathcal{C}_{\text{Vertex}}$  of von Neumann algebras and a class of graphs that we call *rigid* and show that from  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  we can retrieve the rigid graph  $\Gamma$  and the vertex von Neumann algebras  $M_v \in \mathcal{C}_{\text{Vertex}}$  up to amplification. In particular, we obtain unique prime factorization and unique free product decompositions for new classes of von Neumann algebras. We also show that, without imposing strong conditions on the vertex von Neumann algebras  $M_v$ , it is possible to retrieve the radius of the graph  $\Gamma$ , up to a constant, from the graph product  $M_\Gamma$ .

In Chapter 6 we study approximation properties for graph products. For a group  $G$ , approximation properties assert that we can approximate the constant function  $1_G$  pointwise by *nice* functions  $m_k : G \rightarrow \mathbb{C}$ . Likewise, for an operator algebra  $M$ , approximation properties assert that we can approximate the identity map  $\text{Id}_M$  pointwise by *nice* maps  $\theta_k : M \rightarrow M$ . For reduced graph products of  $C^*$ -algebras we study the *completely contractive approximation property* (CCAP). Similar, for graph products of von Neumann algebras we study the *weak-\** CCAP. These approximation properties are the operator algebraic counterparts of weak amenability with constant 1. We study stability of these properties under graph products and extend results from [Rec17] and [RX06].

In Chapter 7 we deviate from the main topic of this thesis and study commutators estimates. We extend the operator estimates from [BS12b], [BS12a] and [BHS23] for self-adjoint elements to normal elements in factors. More precisely, for a normal element  $a$  in a factor  $M$  we show the existence of a unitary  $u \in M$  that satisfies a *nice* operator estimate for the commutator  $[a, u] := au - ua$ . In particular, for finite factors this provides a lower estimate on the  $L^1$ -norm of the form

$$\sqrt{3} \min_{z \in \mathbb{C}} \|a - z1_M\|_{L^1(M, \tau)} \leq \|[a, u]\|_{L^1(M, \tau)}.$$

We then use this result to obtain sharp estimates on the norm  $\|\delta_a\|_{M \rightarrow L^1(M, \tau)}$  of the derivation  $\delta_a : M \rightarrow L^1(M, \tau)$  given by  $\delta_a(x) = [a, x]$ .

# 1

## INTRODUCTION

The main topics of this thesis include: von Neumann algebras, Coxeter groups, graph products, approximation properties, rigidity theory and commutator estimates. We discuss these topics in Sections 1.1 to 1.5 at the level of a general mathematical audience. In Section 1.6, we present the main results obtained in this thesis and give an overview of the content of the individual chapters.

### 1.1. VON NEUMANN ALGEBRAS

In 1929, John von Neumann initiated the study of rings of operators, [Neu30]. These rings of operators, now known as *von Neumann algebras*, are of main interest in this thesis. By definition a von Neumann algebra  $M$  is a certain *nice* subalgebra of the space of bounded operators  $B(\mathcal{H})$  on a complex Hilbert space  $\mathcal{H}$ . While the main focus is on von Neumann algebras, we also study the related notion of  $C^*$ -algebras. These operator algebras possess rich algebraic and topological structures. Indeed, for  $a, b \in M$  and  $c \in \mathbb{C}$  there are the algebraic operations of scalar multiplication  $ca$ , addition  $a+b$ , multiplication  $ab$ , and the operation of taking adjoints  $a^*$ . Furthermore, these algebras are equipped with several topologies, including: the norm topology, the strong operator topology (SOT), the weak operator topology (WOT), the  $\sigma$ -weak topology and many more. What makes  $C^*$ -algebras and von Neumann algebras most interesting is their non-commutative nature. This is to say that  $ab$  is generally unequal to  $ba$  for  $a, b \in M$ . Such non-commutative behaviour naturally occurs in quantum physics, where the order in which one performs measurements is of interest. As an example, the operators  $x$  and  $p$  corresponding to measuring position and momentum respectively, satisfy the commutation relation

$$xp - px = -i\hbar.$$

This accounts for the Heisenberg uncertainty that one can not know both the exact position and exact velocity of a particle at the same time. The motivation to introduce  $C^*$ -algebras/von Neumann algebras also came from the need for a non-commutative mathematical framework to describe quantum systems. However, the theory has slightly

deviated from the physical theory and in this thesis we will purely study the mathematical structures of  $C^*$ -algebras and von Neumann algebras. Although all von Neumann algebras are in fact  $C^*$ -algebras, the two structures are studied from different angles. The theory of  $C^*$ -algebras is often thought of as *non-commutative topology*, while the study of von Neumann algebras is thought of as *non-commutative measure theory*. This accounts for the fact that every commutative  $C^*$ -algebra can be described as  $C_0(X)$  for some topological space  $X$ , while every commutative von Neumann algebra is of the form  $L^\infty(\Omega, \mu)$  for some measure space  $(\Omega, \mu)$ . Of interest are those von Neumann algebras that are very far away from the commutative setting (factors) and the non-commutative analogue of integrals (traces).

## FACTORS AND TRACIAL VON NEUMANN ALGEBRAS

The most important von Neumann algebras are those that are *factors*. Factors can be thought of as building blocks for general von Neumann algebras since any von Neumann algebra  $M$  decomposes as a direct sum or direct integral of factors, see [Neu49]. Murray and von Neumann [MN36] and Connes [Con73] classified factors precisely into one of the following types:

$$I_n \ (n \in \mathbb{N}), \quad I_\infty, \quad II_1, \quad II_\infty, \quad III_\lambda \ (0 \leq \lambda \leq 1). \quad (1.1)$$

The simplest examples of factors are the spaces  $\text{Mat}_n(\mathbb{C})$  of  $n \times n$  matrices. These von Neumann algebras form precisely the factors of type  $I_n$  for  $n \geq 1$ . Of interest for these spaces is the matrix trace  $\text{Tr}_n : \text{Mat}_n(\mathbb{C}) \rightarrow \mathbb{C}$ , which for a matrix  $A$  is defined as the sum of its diagonal entries. Recall for matrices  $A, B$  that  $\text{Tr}_n(AB) = \text{Tr}_n(BA)$  and that  $\text{Tr}_n(A^*A) \geq 0$ , with strict inequality when  $A$  is non-zero. Interestingly, for projections  $P \in \text{Mat}_n(\mathbb{C})$  the trace satisfies  $\text{Tr}_n(P) = \dim \text{Range}(P)$ , and thus  $\text{Tr}_n$  can be thought of as measuring the dimension.

In this thesis we mostly encounter von Neumann algebras  $M$  of type  $II_1$ , which contrary to matrix algebras are of infinite dimension. For these algebras there exist a linear map  $\tau : M \rightarrow \mathbb{C}$  which for  $a, b \in M$  satisfies

1.  $\tau(1_M) = 1$
2.  $\tau(ab) = \tau(ba)$
3.  $\tau(a^*a) \geq 0$  with strict inequality when  $a$  is non-zero.

The map  $\tau$ , called a *trace*, is analogous to the normalized matrix trace  $\text{tr}_n := \frac{1}{n} \text{Tr}_n$  but with one important difference. Namely, the trace  $\text{tr}_n(P)$  of a projection  $P \in \text{Mat}_n(\mathbb{C})$  lies in the discrete set  $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ , while the trace  $\tau(p)$  of a projection  $p \in M$  can be any value in the interval  $[0, 1]$ . This means that von Neumann algebras of type  $II_1$  have a sort of *continuous dimension function* which makes them interesting to study.

We convey that, while for  $n \geq 1$  there is only one factor of type  $I_n$ , there are many different (i.e. non-isomorphic) factors of type  $II_1$ . Indeed, already Murray and von Neumann distinguished two different  $II_1$ -factors [MN43], and later McDuff showed the existence of uncountably many different  $II_1$ -factors, [McD69a; McD69b]. The question

remains how to classify all  $\text{II}_1$ -factors. A full classification is far beyond reach. However, over the years many properties for von Neumann algebras have been introduced and studied. A recurring theme in this thesis is that we want to characterize what von Neumann algebras possess a given property. The properties we study, and which will be discussed later in this introduction, include: the weak-\* CCAP, primeness, absence of Cartan subalgebra, the Akemann-Ostrand property (AO), solidity and strong solidity. Furthermore, for a von Neumann algebra  $M$  we study unique prime factorizations, unique free product decompositions and unique graph product decompositions inside a given class  $\mathcal{C}$ . We mostly study these properties and decompositions for von Neumann algebras arising from graph products or arising from discrete groups.

## THE GROUP VON NEUMANN ALGEBRA

There are different ways for constructing examples of von Neumann algebras. One that dates back to Murray and von Neumann is the construction of the group von Neumann algebra [MN36]. The group von Neumann algebra  $\mathcal{L}(G)$  can be constructed for a locally compact group  $G$ . Examples of locally compact groups include the integers  $\mathbb{Z}$ , the torus  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ , the real numbers  $\mathbb{R}$  and the general linear group  $\text{GL}_n(\mathbb{R})$ , each equipped with their natural topology. In this thesis we only encounter *discrete groups*, i.e. groups equipped with the discrete topology. For a discrete group  $G$  the group von Neumann algebra is constructed as follows. For  $s \in G$  define the linear operator  $\lambda_s : \ell^2(G) \rightarrow \ell^2(G)$  by

$$(\lambda_s g)(t) = g(s^{-1}t).$$

The group von Neumann algebra  $\mathcal{L}(G) \subseteq B(\ell^2(G))$  is defined as the closure in the strong operator topology of the linear span of the operators  $\{\lambda_g\}_{g \in G}$ , i.e.

$$\mathcal{L}(G) := \overline{\text{Span}\{\lambda_s : s \in G\}}^{\text{SOT}}.$$

We note that, in a similar fashion, the reduced group  $C^*$ -algebra  $C_r^*(G) \subseteq B(\ell^2(G))$  is defined as the norm-closure of the linear span of  $\{\lambda_g\}_{g \in G}$ . We also note that for countable discrete groups  $G$  the group von Neumann algebra  $\mathcal{L}(G)$  in fact possesses a normal faithful trace  $\tau$  given by

$$\tau(x) = \langle x\delta_e, \delta_e \rangle \tag{1.2}$$

where  $\delta_e$  denotes the dirac delta function corresponding to the unit element  $e$  of  $G$ .

It was shown by Connes in [Con75] that not every von Neumann algebra can be constructed from a group. However, group von Neumann algebras do provide many interesting examples. Moreover, the construction connects the study of von Neumann algebras to the study of groups. One of the most fundamental questions in the theory going back to von Neumann is to study relations between the group  $G$  and the group von Neumann algebra  $\mathcal{L}(G)$ . Some properties of the group are known to carry over to the von Neumann algebra. For example a group  $G$  is finite if and only if the von Neumann algebra  $\mathcal{L}(G)$  is finite-dimensional. Another example is that a group  $G$  is abelian if and only if the von Neumann algebra  $\mathcal{L}(G)$  is commutative. Yet another example is that  $G$  is

infinite-conjugacy class (icc) if and only if  $\mathcal{L}(G)$  is a factor of type II<sub>1</sub>. Some other group properties that have a von Neumann algebraic counterpart include: amenability, weak amenability, the Haagerup property and property (T). One might hope that all information of the group  $G$  can be retrieved from its von Neumann algebra. Generally, this is not the case. In [Con76] Connes showed that the group von Neumann algebras  $\mathcal{L}(G)$  of amenable, infinite-conjugacy class, groups are all isomorphic. Thus information is lost. The question what information of the group  $G$  can be retrieved from its von Neumann algebra is part of *rigidity theory* and is one of the main interests in this thesis.

Rigidity theory in more generality concerns the question what properties of an object can be retrieved when passing to another object. Several rigidity results were obtained by Connes in [Con80] and [CJ85] for icc property (T) groups. Moreover, Connes conjectured in [Con82] that these groups satisfy a very strong rigidity property called  $W^*$ -superrigid. This property asserts for a discrete group  $G$  that if  $\mathcal{L}(G) \simeq \mathcal{L}(H)$  for any other discrete group  $H$ , then  $G \simeq H$ . Some groups, such as the lamplighter group  $L = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ , have been shown to satisfy this property, see [IPV13; CDD23b]. Moreover, [Chi+23] obtained the first examples of icc property (T) groups that are  $W^*$ -superrigid. However, Connes rigidity conjecture still remains open.

In this thesis we study Coxeter groups. These groups are in some sense opposite to property (T) groups and are often not  $W^*$ -superrigid. However, we can still obtain rigidity results for these groups. Moreover, we also study rigidity theory in a broader sense for graph products.

## 1.2. COXETER GROUPS AND GRAPH PRODUCTS

The study of group forms a vast field in mathematics. This originated from the study of solutions of polynomial equations and was formalized by Galois and Cauchy, see [Kle86]. In this thesis we focus on a specific class of (discrete) groups called *Coxeter groups*. For such groups  $\mathcal{W}$  we study their von Neumann algebra  $\mathcal{L}(\mathcal{W})$ , and more generally their Hecke-von Neumann algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$ . We mostly encounter a specific type of Coxeter groups called *right-angled Coxeter groups*. Such groups  $\mathcal{W}$  naturally decompose as graph products, as was defined by Green in [Gre90]. In this thesis we also study more general graph products in the setting of  $C^*$ -algebras and von Neumann algebras as was introduced in [Mlo04] and [CF17].

### COXETER GROUPS

In geometry, a finite reflection group is a finite group generated by orthogonal linear reflections on  $\mathbb{R}^d$ . Examples of such groups include the dihedral groups  $\mathbb{D}_n$  of symmetries of the regular polygon with  $n$  vertices. Finite reflection groups are important for the classification of Lie groups and Lie algebras and for the classification of regular polytopes (see discussion in [Dav08, Appendix B]). We note that finite reflection groups act isometrically on the unit sphere  $\mathbb{S}^{d-1}$ . In a similar fashion, one can study groups generated by reflections in Euclidean space  $\mathbb{E}^d$  and in hyperbolic space  $\mathbb{H}^d$ . As an example, the cover of this thesis depicts a pattern of turtles in hyperbolic space whose symmetry group is an infinite group generated by reflections. The study of reflection groups has led to a classification of regular tessellations of  $\mathbb{S}^d$ ,  $\mathbb{E}^d$  and  $\mathbb{H}^d$ , see [Dav08, Appendix B].

In this thesis we study Coxeter groups which can be seen as abstract reflection groups. These groups were formally introduced by Tits [Tit13] as a group  $\mathcal{W}$  universally generated by a set  $S$  subject to relations of the form

$$(st)^{m_{s,t}} = e \quad \text{for } s, t \in S \quad (1.3)$$

where  $e$  denotes the group unit, and where  $m_{s,t} \in \mathbb{N} \cup \{\infty\}$  satisfies  $m_{s,t} = 1$  whenever  $s = t$  and  $m_{s,t} = m_{t,s} \geq 2$  whenever  $s \neq t$ . Here,  $m_{s,t} = \infty$  means that no relation of the form  $(st)^k = e$  is imposed for any  $k \geq 1$ . Such a group  $\mathcal{W}$  is denoted by  $\mathcal{W} = \langle S|M \rangle$  where  $M$  denotes the Coxeter matrix  $M = (m_{s,t})_{s,t \in S}$ . However, we note that a Coxeter group  $\mathcal{W}$  may be represented by different pairs  $S, M$ , for example through *diagram twisting* [Bra+02]. The pair  $(\mathcal{W}, S)$  is called a *Coxeter system*, to emphasise that we fix a generating set  $S$ . In the study of Coxeter groups it is often sufficient to study Coxeter systems that are *irreducible*. Indeed, any Coxeter group can be written as a direct sum of irreducible ones. All finite irreducible Coxeter systems have been classified by Coxeter in [Cox35] in terms of the Coxeter Matrix  $(m_{s,t})_{s,t \in S}$ . Furthermore, [Cox34] and [Cox35] show that finite Coxeter groups are precisely the finite reflection groups.

To a Coxeter system  $(\mathcal{W}, S)$  one can associate the Cayley graph, which gives rise to the word length function  $|\cdot|_S : \mathcal{W} \rightarrow \mathbb{Z}_{\geq 0}$ . Coxeter groups and other discrete groups can be studied using their Cayley graphs and this has led to interesting notions such as that of hyperbolic groups. For hyperbolic groups one can construct the Gromov boundary of its Cayley graph, see [Gro87]. In this thesis we encounter the notion of *smallness at infinity* (see [BO08]) which requires such boundary to be well-behaved algebraically (Theorem A).

In this thesis we mostly study right-angled Coxeter groups, which are Coxeter groups of the form  $\mathcal{W} = \langle S|M \rangle$  with  $m_{s,t} \in \{1, 2, \infty\}$  for  $s, t \in S$ . We note that, since all elements  $s, t \in S$  satisfy  $s^2 = t^2 = e$ , a relation of the form  $m_{s,t} = 2$  simply asserts that  $s$  and  $t$  commute. This also explains the name ‘right-angled’ since two orthogonal reflections commute if and only if they intersect at a right-angle. All information of a right-angled Coxeter system  $\mathcal{W} = \langle S|M \rangle$  can be encoded in a graph  $\Gamma$  whose vertex set is  $S$  and whose edge set consists of the pairs  $\{s, t\}$  that satisfy  $m_{s,t} = 2$ . In fact there is a unique correspondence

$$\Gamma \longleftrightarrow \mathcal{W}_\Gamma$$

between graphs and right-angled Coxeter groups.

## HECKE-ALGEBRAS

For a Coxeter group  $\mathcal{W}$  one can study the group von Neumann algebra  $\mathcal{L}(\mathcal{W})$ . More generally, one can study Hecke-von Neumann algebras associated to  $\mathcal{W}$ . These are certain  $\mathbf{q}$ -deformations of the group von Neumann algebra  $\mathcal{L}(\mathcal{W})$  and were first introduced in [Dym06]. These structures are of interest in the study of weighted  $L^2$ -cohomology of Coxeter groups, see also [Dav+07]. Given a Coxeter system  $\mathcal{W} = \langle S|M \rangle$  and a tuple  $\mathbf{q} = (q_s)_{s \in S} \in \mathbb{R}_{>0}^S$  satisfying  $q_s = q_t$  whenever  $s$  and  $t$  are conjugate in  $\mathcal{W}$ , the Hecke-von Neu-

mann algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  is defined as follows. For  $s \in S$  define the operator  $T_s \in B(\ell^2(\mathcal{W}))$  by

$$T_s \delta_{\mathbf{v}} = \begin{cases} \delta_{s\mathbf{v}} & |s\mathbf{v}|_S > |\mathbf{v}|_S \\ \delta_{s\mathbf{v}} + p_s(\mathbf{q})\delta_{\mathbf{v}} & |s\mathbf{v}|_S < |\mathbf{v}|_S \end{cases}.$$

Interestingly, for  $\mathbf{w} = w_1 \cdots w_n \in \mathcal{W}$  with  $w_i \in S$  the operator given by

$$T_{\mathbf{w}} := T_{w_1} \cdots T_{w_n}$$

is well-defined, i.e. does not depend on the representative of  $\mathbf{w}$ . Furthermore, for  $\mathbf{w} \in \mathcal{W}$  these operators satisfy

$$T_{\mathbf{w}}^* = T_{\mathbf{w}^{-1}}.$$

The Hecke-von Neumann algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  is defined, similar to the group von Neumann algebra, as the closure in the strong operator topology of the linear span of the operators  $\{T_{\mathbf{w}}\}_{\mathbf{w} \in \mathcal{W}}$ , i.e.

$$\mathcal{N}_{\mathbf{q}}(\mathcal{W}) = \overline{\text{Span}\{T_{\mathbf{w}} : \mathbf{w} \in \mathcal{W}\}}^{SOT}.$$

Similar also the reduced Hecke  $C^*$ -algebra can be defined.

From a single parameter  $q > 0$  and a Coxeter group  $\mathcal{W}$  one can construct the single parameter Hecke-algebra  $\mathcal{N}_q(\mathcal{W})$  ( $= \mathcal{N}_{\mathbf{q}}(\mathcal{W})$  where  $q_s = q$  for  $s \in S$ ) and we observe that  $\mathcal{L}(\mathcal{W}) = \mathcal{N}_1(\mathcal{W})$ . In [Gar16] Garncarek characterized for  $q > 0$  precisely when the single parameter Hecke algebra  $\mathcal{N}_q(\mathcal{W})$  is a factor. This was extended by Raum and Skalski in [RS23] to the multiparameter case. In this thesis we study general Hecke-algebras  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  in the multiparameter setting. Oftentimes the Coxeter group  $\mathcal{W}$  that we consider is right-angled. In such case the Hecke-algebra naturally decompose as a graph product  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma}) = *_{v \in \Gamma} \mathcal{N}_{q_v}(\mathbb{Z}/2\mathbb{Z})$  as we will discuss now.

## GRAPH PRODUCTS

In mathematics it is often useful to construct a new object  $A$  from two smaller objects  $A_1$  and  $A_2$ . For groups common operations are those of the direct sum  $G_1 \oplus G_2$  and of the free product  $G_1 * G_2$ . In [Gre90] Green introduced a new construction, called *graph products*, which generalizes both these constructions. Given a simple graph  $\Gamma$  and groups  $G_v$  for every vertex  $v$  of  $\Gamma$ , the graph product  $G_{\Gamma} := *_{v \in \Gamma} G_v$  is the group defined by  $G_{\Gamma} = G/H$  where  $G$  is the free product of the groups  $(G_v)_{v \in \Gamma}$  and where  $H \subseteq G$  is the normal subgroup generated by the set

$$\{sts^{-1}t^{-1} : s \in G_v, t \in G_w, \text{ such that } v \text{ and } w \text{ share an edge}\}.$$

Edges in the graph  $\Gamma$  correspond to direct sums, and absence of edges correspond to free products. Indeed, when the graph  $\Gamma$  has no edges, then  $G_{\Gamma}$  equals the free product  $G_{\Gamma} = G$  and when  $\Gamma$  is a complete graph then  $G_{\Gamma}$  equals the direct sum  $G_{\Gamma} = \bigoplus_{v \in \Gamma} G_v$ . Some groups, such as right-angled Coxeter groups, naturally decompose as graph products. Indeed, any right-angled Coxeter group  $\mathcal{W}_{\Gamma}$  can be written as  $\mathcal{W}_{\Gamma} = *_{v \in \Gamma} \mathcal{W}_v$  where  $\mathcal{W}_v = \mathbb{Z}/2\mathbb{Z}$  for each vertex  $v$ . Many group properties are preserved under graph products, including approximation properties such as the Haagerup property [AD14; DG23]

and weak amenability with constant 1 [Rec17].

Graph product have also been defined in the setting of operator algebras in [Mto04] and [CF17] as a generalization of both tensor products  $M_1 \bar{\otimes} M_2$  and (tracial/statial) free products  $M_1 * M_2$ . The von Neumann algebraic graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \varphi_v)$  is constructed such that  $M_\Gamma$  equals the tensor product  $\bar{\otimes}_{v \in \Gamma} M_v$  whenever  $\Gamma$  is a complete graph and such that  $M_\Gamma$  equals the free product  $*_{v \in \Gamma} (M_v, \tau_v)$  whenever  $\Gamma$  is a graph with no edges. The reduced graph product  $A_\Gamma = *_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  of  $C^*$ -algebras satisfies similar relations. The notions of operator algebraic graph products agree with that for groups in the sense that for discrete groups  $G_\nu$  one has

$$C_r^*(*_{v \in \Gamma} G_\nu) = *_{v \in \Gamma}^{\min} C_r^*(G_\nu) \quad \text{and} \quad \mathcal{L}(*_{v \in \Gamma} G_\nu) = *_{v \in \Gamma} \mathcal{L}(G_\nu). \quad (1.4)$$

Similar to the case of groups, many properties of  $C^*$ -algebras/von Neumann algebras are preserved under graph products. Indeed, in [CF17] they showed stability of exactness (for  $C^*$ -algebras), Haagerup property and  $\text{II}_1$ -factoriality (for von Neumann algebras). In this thesis we will study stability of the CCAP and the weak- $*$  CCAP under graph products. We will now discuss these and other approximation properties.

### 1.3. APPROXIMATION PROPERTIES

In this thesis we study and apply approximation properties including: amenability, weak-amenability and the Haagerup property. For a discrete group  $G$ , these properties assert that we can approximate the constant function 1 on  $G$  pointwise by certain *nice* functions  $m_k : G \rightarrow \mathbb{C}$ . These properties also have their counterparts for (unital)  $C^*$ -algebras and von Neumann algebras. For a  $C^*$ -algebra/von Neumann algebra  $M$  it asserts that we can approximate the identity map  $\text{Id}_M$  pointwise by *nice* maps  $\theta_k : M \rightarrow M$ . These approximation properties play an important role in group theory, in operator algebras, in functional analysis and in harmonic analysis.

We show how approximation properties appear in harmonic analysis. Of main interest in harmonic analysis is the Fourier transform  $\mathcal{F} : L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{Z})$ , which is the unitary satisfying

$$\mathcal{F}(g)(n) := \int_{\mathbb{T}} g(z) z^n dz \quad \mathcal{F}^{-1}(\hat{g})(z) = \sum_{n \in \mathbb{Z}} \hat{g}(n) z^n.$$

In many practical applications it is important to approximate functions by functions with finite Fourier series. For this we can use approximation properties of the group  $\mathbb{Z}$ . Given a sequence  $(m_k)_{k \geq 1}$  of bounded functions  $m_k : \mathbb{Z} \rightarrow \mathbb{C}$  converging to 1 pointwise, we can consider their Fourier multiplier  $T_{m_k} : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$  given by

$$\sum_{n \in \mathbb{Z}} f(n) z^n \mapsto \sum_{n \in \mathbb{Z}} m_k(n) f(n) z^n. \quad (1.5)$$

When the function  $m_k$  are chosen appropriately (finitely supported and positive definite), then the maps  $T_{m_k}$  can be used to approximate any continuous function  $g \in C(\mathbb{T})$  uniformly by the continuous functions  $(T_{m_k} g)_{k \geq 1}$  that satisfy  $\|T_{m_k} g\|_{C(\mathbb{T})} \leq \|g\|_{C(\mathbb{T})}$  and have finite Fourier series. Examples of such appropriate functions  $(m_k)_{k \geq 1}$  are given by

$m_k(n) := \max\{0, 1 - \frac{|n|}{k}\}$  which corresponds to approximation by Cesàro sums.

The above result is precisely Fejér's theorem. While for a discrete group  $G$ , the Fourier transform is only defined when  $G$  is abelian, it is still possible to study such approximations when the group  $G$  is non-abelian by considering the multipliers  $T_{m_k} : \mathcal{L}(G) \rightarrow \mathcal{L}(G)$  given by

$$\sum_{g \in G} f(g) \lambda_g \mapsto \sum_{g \in G} m_k(g) f(g) \lambda_g. \quad (1.6)$$

The multipliers  $T_{m_k}$  can also be regarded as maps on the reduced  $C^*$ -algebra  $C_r^*(G)$  (note for the group  $\mathbb{Z}$  that  $C_r^*(\mathbb{Z}) \simeq C(\mathbb{T})$  and  $\mathcal{L}(\mathbb{Z}) \simeq L^\infty(\mathbb{T})$ ). When the functions  $m_k$  are chosen appropriately, then the maps  $\theta_k := T_{m_k}$  provide approximations of the identity map on the  $C^*$ -algebra/von Neumann algebra.

### AMENABILITY, NUCLEARITY AND SEMIDISCRETENESS

The property of amenability appears in different equivalent forms. One of the many characterizations is that a discrete group  $G$  is amenable if the approximating functions  $m_k : G \rightarrow \mathbb{C}$  can be chosen to be finitely supported and positive definite. The original definition, which was given by von Neumann in [Neu29], involved the existence of invariant means. The motivation came from the Banach-Tarski paradox [BT24] which loosely states that the 3-dimensional unit ball can be cut into several pieces which can be rearranged in such a way that they will form two unit balls. More precisely, it asserts for  $d \geq 3$  that any two bounded subsets  $A, B \subseteq \mathbb{R}^d$  with non-empty interior can be decomposed in finitely many mutually disjoint pieces  $A = A_1 \cup \dots \cup A_n$  and  $B = B_1 \cup \dots \cup B_n$  such that for  $1 \leq i \leq n$  there is a Euclidean transformation  $T_i \in E(d)$  for which  $T_i(A_i) = B_i$ . This paradoxical decomposition has to do with the non-amenability of the group  $E(d)$  (consisting of isometries of  $d$ -dimensional Euclidean space). Examples of amenable groups include abelian groups, groups with polynomial growth and solvable groups. The standard example of a non-amenable group is  $\mathbb{F}_2 := \mathbb{Z} * \mathbb{Z}$ , the free group on 2 generators. In fact, in [Neu29] von Neumann conjectured that any non-amenable group must contain  $\mathbb{F}_2$  as a subgroup, but this was disproven by Ol'shanski [Ols80] who gave an example of a non-amenable group whose proper subgroups are all cyclic.

The operator algebraic counterparts of amenability are nuclearity and semidiscreteness. Indeed, a discrete group  $G$  is amenable if and only if the reduced  $C^*$ -algebra  $C_r^*(G)$  is nuclear, if and only if the group von Neumann algebra  $\mathcal{L}(G)$  is semidiscrete. It was shown by the fundamental work of Connes [Con76] and others, that for von Neumann algebras (with separable predual) semidiscreteness coincides with other known von Neumann algebraic notions such as: hyperfiniteness, the extension property, injectivity and amenability. This property, now often referred to as amenability, is well understood and plays a central role in the theory of von Neumann algebras. The notion of amenability also plays an important role in Popa's deformation/rigidity theory which arose from [Pop06b; Pop06a]. In this thesis we encounter amenability due to its connection to strong solidity, as well as in various other places. Furthermore, we also study the notion of amenability in the relative setting as was introduced by Ozawa and Popa in [OP10a],

see Theorem N.

### WEAK AMENABILITY AND THE CBAP/CCAP

Weak amenability is a weakened version of amenability that for a discrete group  $G$  asserts that the approximating functions  $m_k : G \rightarrow \mathbb{C}$  can be chosen to be finitely supported and such that  $\sup_k \|T_{m_k}\|_{\text{cb}} \leq \Lambda$  for some finite constant  $\Lambda$ . If the constant  $\Lambda$  may be chosen to be 1, then  $G$  is said to be weakly amenable with constant 1. Examples of such groups include all amenable groups but also, for instance, the free group  $\mathbb{F}_2$  [Haa78]. The notion of weak amenability for groups originates from the work of Haagerup [Haa78], De Cannière-Haagerup [CH85] and Cowling-Haagerup [CH89]. The corresponding notion for unital  $C^*$ -algebras and von Neumann algebras is given by the completely bounded approximation property (CBAP) and the weak- $*$  CBAP in the sense that a discrete group is weakly amenable if and only if its reduced group  $C^*$ -algebra possesses the CBAP if and only if its group von Neumann algebra possess the weak- $*$  CBAP. The notions corresponding to weak amenability with constant 1 are the completely contractive approximation property (CCAP) and the weak- $*$  CCAP. We study these properties for graph products, see Theorem O and Theorem P.

## 1.4. RIGIDITY THEORY AND INDECOMPOSABILITY

One of the most important series of von Neumann algebras are the free group factors, which for  $n \geq 2$  are the group von Neumann algebras  $\mathcal{L}(\mathbb{F}_n)$  corresponding to the free group  $\mathbb{F}_n$  on  $n$  generators. The free group factors satisfy many interesting indecomposability properties like: primeness, absence of Cartan subalgebra, solidity and strong solidity. We will discuss these properties as well as unique prime factorizations (UPF), unique free product decompositions and unique rigid graph product decompositions.

### PRIMENESS AND UPF

It is a simple fact that a matrix algebra  $\text{Mat}_k(\mathbb{C})$  factorizes as the tensor product

$$\text{Mat}_k(\mathbb{C}) = \text{Mat}_n(\mathbb{C}) \overline{\otimes} \text{Mat}_m(\mathbb{C})$$

if and only if  $k = nm$ . In particular, the matrix algebra  $\text{Mat}_k(\mathbb{C})$  does not have any non-trivial tensor product decomposition if and only if  $k$  is a prime number. Similarly, for a  $\text{II}_1$ -factor  $M$  one can ask whether it decomposes as a tensor product  $M = M_1 \overline{\otimes} M_2$  of factors  $M_1$  and  $M_2$ . As it turns out for  $n \geq 1$  every  $\text{II}_1$ -factor  $M$  decomposes as tensor product  $M = N \overline{\otimes} \text{Mat}_n(\mathbb{C})$  for some unique  $\text{II}_1$ -factor  $N$  (denoted  $M^{1/n}$ ) called an amplification of  $M$ . In fact, amplifications  $M^t$  are defined more generally for any  $t \in (0, \infty)$ . The question that we are interested in however, is what factors  $M$  decompose as

$$M = M_1 \overline{\otimes} M_2$$

for some *infinite*-dimensional von Neumann algebras  $M_1$  and  $M_2$ . A factor that does not decompose in this way is called *prime*. The first known example of a prime factor was the group von Neumann algebra of the free group with uncountable many generators  $\mathcal{L}(\mathbb{F}_{\mathbb{R}})$  as was shown Popa in [Pop83]. Thereafter, Ge showed in [Ge96] that  $\mathcal{L}(\mathbb{F}_n)$  is a prime

factor for  $n \geq 2$  by computing Voiculescu's free entropy. Later, in [Oza04] Ozawa introduced a new property, called solidity, which for non-amenable factors implies primeness. He showed that all  $\text{II}_1$ -factors satisfying the Akemann-Ostrand property (AO) are solid. For a discrete hyperbolic group  $G$  it had already been shown in [HG04] by using the Gromov boundary that  $\mathcal{L}(G)$  possesses (AO). Thus  $\mathcal{L}(G)$  is a prime factor for any icc, non-amenable, hyperbolic group  $G$ . There are many more examples of prime factors, see e.g. [BHV18; CSS18; CKP16; CSU13; DHI19; Pet08; Sak09; SW13]. In this thesis we will present a characterization of primeness for von Neumann algebras coming from graph products, see Theorem H.

Given a class  $\mathcal{C}$  of von Neumann algebras, a natural question is whether any von Neumann algebra  $M \in \mathcal{C}$  has a tensor product decomposition  $M = M_1 \bar{\otimes} \cdots \bar{\otimes} M_m$  for some  $m \geq 1$  and prime factors  $M_1, \dots, M_m \in \mathcal{C}$  and whether this prime factorization is unique. Generally, it holds true that if  $M = M_1 \bar{\otimes} M_2$  is a prime factorization, then  $M = M_1^t \bar{\otimes} M_2^{1/t}$  also is a prime factorization for any  $t \in (0, \infty)$ . Hence, uniqueness of prime factorizations is always studied up to amplifications. The first unique prime factorization (UPF) results were established by Ozawa and Popa in [OP04] for tensor products of certain group von Neumann algebras. Later, UPF results were obtained in [Iso17], [HI17] for other classes of von Neumann algebras. In the setting of graph products, UPF results have been obtained in [CSS18, Theorem 6.16] under the condition that the vertex von Neumann algebras are group von Neumann algebras. In this thesis, we present in Theorem I new UPF results for von Neumann algebras in the class  $\mathcal{C}_{\text{Rigid}}$  coming from (rigid) von Neumann algebraic graph products.

## FREE-INDECOMPOSABILITY AND KUROSH TYPE THEOREMS

Similar to unique prime factorizations one can ask whether a tracial von Neumann algebra  $(M, \tau)$  decomposes as a (reduced) free product  $M = M_1 * \cdots * M_m$  in a unique way. In [Oza06] Ozawa extended the results [OP04] for tensor products to the setting of free products. In particular, he showed for  $M = M_1 * \cdots * M_m$  a von Neumann algebraic free product of non-prime, non-amenable, semiexact  $\text{II}_1$ -factors  $M_1, \dots, M_m$  that if  $M = N_1 * \cdots * N_n$  is another free product decomposition into non-prime, non-amenable, semiexact  $\text{II}_1$ -factors  $N_1, \dots, N_n$ , then  $m = n$  and, up to permutation of the indices,  $M_i$  unitarily conjugates to  $N_i$  inside  $M$  for each  $1 \leq i \leq m$ . This result can be seen as a von Neumann algebraic version of the Kurosh isomorphism theorem [Kur34], which states that any discrete group uniquely decomposes as a free product of freely indecomposable groups. Versions of Ozawa's result were later shown for other classes of von Neumann algebras, see [Ash09; IPP08; Pet08]. In [HU16] these results were then extended by Houdayer and Ueda to a single, large class of von Neumann algebras. Other Kurosh type theorems have recently been obtained in [Dri23, Corollary 8.1], [DE24b, Corollary 1.8]. In this thesis we obtain Kurosh type results for graph product in the class  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ , see Theorem K.

## RIGID GRAPH PRODUCTS

In rigidity theory, the famous free factor problem [Kad67] asks whether one can retrieve the number  $n \geq 2$  of generators from the von Neumann algebra  $\mathcal{L}(\mathbb{F}_n)$ , i.e. is it true that

$\mathcal{L}(\mathbb{F}_n) \neq \mathcal{L}(\mathbb{F}_m)$  for  $n \neq m$ ? This question is related to Kurosh type results since the free group factor  $\mathcal{L}(\mathbb{F}_n)$  decomposes as the tracial free product

$$\mathcal{L}(\mathbb{F}_n) = R * \cdots * R \quad (n \text{ times})$$

of hyperfinite  $\text{II}_1$ -factors  $R$ , see [Dyk94]. On a level of  $C^*$ -algebras the question has been answered by Pimsner and Voiculescu in [PV82]. Indeed, they showed that the reduced group  $C^*$ -algebras  $C_r^*(\mathbb{F}_n)$  for  $n \geq 2$  are pairwise non-isomorphic by computing their K-theory. However, the question for von Neumann algebras is still open.

It was shown independently by Rădulescu [Răd94] and by Dykema [Dyk94] that it is possible to extend the definition of the free group factors  $\mathcal{L}(\mathbb{F}_n)$  and more generally, for  $r \in (1, \infty]$ , construct the interpolated free group factors  $\mathcal{L}(\mathbb{F}_r)$  which satisfy

$$\mathcal{L}(\mathbb{F}_{r+s}) = \mathcal{L}(\mathbb{F}_r) * \mathcal{L}(\mathbb{F}_s) \quad r, s \in (1, \infty] \quad (1.7)$$

$$\mathcal{L}(\mathbb{F}_r)^t = \mathcal{L}(\mathbb{F}_{1+\frac{r-1}{t^2}}) \quad r \in (1, \infty], t \in (0, \infty) \quad (1.8)$$

(observe that the group  $\mathbb{F}_r$  is only defined when  $r \in \mathbb{N}$ ). Rădulescu moreover showed in [Răd94] that the free factors  $\mathcal{L}(\mathbb{F}_r)$  for  $r \in (1, \infty]$  are either all isomorphic or pairwise non-isomorphic. It is widely believed that the free factor problem is true, but the problem is considered very hard.

A natural generalization of the free factor problem is to ask what information of the graph  $\Gamma$  we can retrieve from the graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$ . For the case of Hecke algebras, Garncarek showed in [Gar16] using [Dyk93] that when  $\Gamma$  is a non-complete graph whose connected components are complete and when  $q \in [0, 1]$  is close enough to 1 then  $\mathcal{N}_q(\mathcal{W}_\Gamma)$  is equal to an interpolated free group factor. In particular, as stated in [CSW19] when  $\Gamma$  is a graph of size  $N = |\Gamma| \geq 3$  and with no edges, then for  $q \in [\frac{1}{N+1}, 1]$  it holds true that

$$\mathcal{N}_q(\mathcal{W}_\Gamma) = \mathcal{L}(\mathbb{F}_{2Nq/(1+q)^2}). \quad (1.9)$$

This shows the connection with the free factor problem.

The theory of graph products becomes somewhat more elegant when the vertex von Neumann algebras  $M_v$  are all taken to be  $\text{II}_1$ -factors. In this setting rigidity results were obtained for graph products  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  in [CDD22; CDD23a] when  $M_v$  comes from a class of group von Neumann algebras of certain property (T) groups. In this thesis we study rigidity results for other classes of von Neumann algebraic graph products of  $\text{II}_1$ -factors. We will obtain several rigidity results that allow us, in some cases, to fully or partially retrieve the graph  $\Gamma$  and the von Neumann algebras  $M_v$  from the graph product  $M_\Gamma$ , see Theorem F and Theorem G.

## CARTAN SUBALGEBRAS AND STRONG SOLIDITY

A construction more general than that of the group von Neumann algebra is that of the crossed product  $M \rtimes_\alpha G$ , which can be built from a trace preserving group action  $\alpha : G \rightarrow \text{Aut}(M)$  on a von Neumann algebra  $(M, \tau)$ . Group von Neumann algebras are a special case of crossed products since  $\mathcal{L}(G) = \mathbb{C} \rtimes G$ . A question that has been studied is what von Neumann algebras decompose as a group measure space  $L^\infty(0, 1) \rtimes_\alpha G$

for some group  $G$  and some group action  $\alpha$  on  $L^\infty(0, 1)$ . This question reduces to the question what von Neumann algebras possess a *Cartan subalgebra*. Indeed, the theorem of [FM77a; FM77b] shows that a von Neumann algebra decomposes as a (generalized) group measure space von Neumann algebra if and only if  $M$  possesses a Cartan subalgebra. It was first shown by Voiculescu in [Voi96] that the free group factors  $\mathcal{L}(\mathbb{F}_t)$  for  $t \in (1, \infty)$  do not possess a Cartan subalgebra. Later, Popa and Ozawa introduced in [OP10a] the property *strong solidity* which is a strengthened version of Ozawa's property *solidity* that moreover implies absence of Cartan subalgebras. A von Neumann algebra  $M$  is called *strongly solid* if for every amenable von Neumann subalgebra  $A \subseteq M$  that is diffuse (i.e. that contains no minimal projections) the set of normalizers

$$\text{Nor}_M(A) := \{u \in M \text{ unitary} : u^* A u = A\}$$

generates a von Neumann algebra that is amenable again. Popa and Ozawa showed that the free group factors possess this property which moreover retrieved Voiculescu's result. Many examples of strongly solid von Neumann algebras have been obtained, see [Cas22; CS13; DP23; Iso15a; PV14b]. In this thesis we fully characterize when a graph product of von Neumann algebras is strongly solid (see Theorem C, Theorem D and Theorem E).

## 1.5. DERIVATIONS AND QUANTUM MARKOV SEMIGROUPS

Derivations are linear maps  $\delta$  that satisfy the Leibniz rule  $\delta(xy) = \delta(x)y + x\delta(y)$ . They play an essential role in the theory of Lie algebras, cohomology, and in quantum physics, see [KL14; SS95]. Derivations are also of interest in the study of semigroups since derivations are square-roots of generators of quantum Markov semi-groups (QMS). In this thesis we will study commutator estimates to obtain estimates on the norms of derivations. Furthermore, we will study QMS's on Hecke algebras to obtain strong solidity results.

### COMMUTATOR ESTIMATES AND NORMS OF DERIVATIONS

A classical result on derivations is due to Stampfli [Sta70] which asserts that for  $a \in B(\mathcal{H})$  the derivation  $\delta_a : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  defined by the commutator  $\delta_a(x) = [a, x] = ax - xa$  has operator norm  $\|\delta_a\| = 2 \inf_{z \in \mathbb{C}} \|a - z1_M\|$ . Through the work of [KLT20; Gaj72; Zsi73], the result of Stampfli has been extended to derivations on arbitrary von Neumann algebras  $M$  (see also [Mag95] for more in this direction). More precisely, the result of Zsidó [Zsi73, Corollary] asserts that for  $M$  a von Neumann algebra and  $a \in M$ , the derivation  $\delta_a : M \rightarrow M$  associated to  $a$  satisfies the distance formula:

$$\|\delta_a\|_{M \rightarrow M} = 2 \min_{z \in Z(M)} \|a - z\|, \quad (1.10)$$

where  $Z(M)$  denotes the center of  $M$ . Derivations have also been studied as maps from  $M$  to the predual  $M_*$ . Indeed, the predual  $M_*$  is a  $M$ -bimodule (see Section 7.7) and therefore it is possible to consider derivations  $\delta : M \rightarrow M_*$ . Important work on such derivations was done in [BP80; Haa83; BGM12] and particularly the result of [Haa83, Theorem 4.1] showed that all these derivations are inner, i.e. of the form  $\delta = \delta_a$  for some  $a \in M_*$  defined by  $\delta_a(x) = ax - xa$ . These studies arose after Connes proved in [Con78] that all  $C^*$ -algebras that are amenable (as a Banach  $*$ -algebra) are necessarily nuclear. Haagerup proved in [Haa83] that the reverse implication is also true.

In [BHS23] the norms of these derivations were studied and results analogous to (1.10) were found in certain cases: for  $M$  properly infinite it was shown that some form of formula (1.10) holds true and for  $M$  finite the same was proved under the condition that  $a$  is self-adjoint. The proofs of these results were based on improvements of the operator estimates obtained in [BS12b; BS12a]. The estimates in [BHS23] show in particular for a factor  $M$  and a self-adjoint  $a \in M$  that there is a  $\lambda \in \mathbb{C}$  so that for  $\epsilon > 0$  there exists a unitary  $u_\epsilon \in M$  for which

$$|[a, u_\epsilon]| \geq (1 - \epsilon)(|a - \lambda 1_M| + u_\epsilon |a - \lambda 1_M| u_\epsilon). \quad (1.11)$$

In this thesis we show analogous estimates for normal elements (Theorem Q) and obtain sharp estimates on the norm  $\|\delta_a\|_{M \rightarrow L^1(M, \tau)}$  for finite factors  $M$  (Theorem R).

### QUANTUM MARKOV SEMIGROUPS AND GRADIENT- $S_p$

A quantum Markov semigroup (QMS)  $(\Phi_t)_{t \geq 0}$  on a tracial von Neumann algebra  $(M, \tau)$  is a semigroup of *nice* maps  $\Phi_t : M \rightarrow M$ . As already mentioned, QMS's are connected to derivations, since by [CS03] the generator  $\Delta$  of a (symmetric) QMS can be written as  $\Delta = \delta^* \delta$  for some derivation  $\delta$ . Furthermore, QMS's are also connected to approximation properties, since the maps  $(\Phi_{\frac{1}{k}})_{k \geq 1}$  form an approximation of the identity  $\text{Id}_M$ . In fact, as was shown by Jolissaint in [JM04] there exists on a von Neumann algebra  $(M, \tau)$  a (symmetric) QMS whose generator has compact resolvent if and only if  $M$  possesses the Haagerup approximation property. This approximation property is, just as weak amenability, a weakened version of amenability. The Haagerup property first arose for groups in [Haa78] and later for  $C^*$ -algebras/von Neumann algebras [Cho83; Jol02; CS15]. As was shown by [BJS88] (see also [Tit09, p. 2.22]) all Coxeter groups  $\mathcal{W}$  possess the Haagerup property. For a Coxeter group  $\mathcal{W}$  it is thus possible to study QMS's  $(\Phi_t)_{t \geq 0}$  on the group von Neumann algebra  $\mathcal{L}(\mathcal{W})$ . In this thesis we study such semigroups (Theorem A and Theorem B) with the aim to obtain strong solidity results for the group von Neumann algebra  $\mathcal{L}(\mathcal{W})$  and more generally for the Hecke-algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$ . This further develops the connections between QMS's and rigidity theory that were made in [Cas21] and [CIW21].

## 1.6. THESIS RESULTS AND STRUCTURE OVERVIEW

In this section we present the main results obtained in this thesis. Before we list these results, we give a quick overview of the structure of this thesis.

- In Chapter 2 (the preliminaries) we recap general theory and fix notation. In particular, we introduce the notation that we use for simple graphs and for operator algebraic graph products.
- In Chapter 3 we perform some technical calculations in graph products concerning annihilation, diagonal and creation operators. These calculations are used in Chapter 6 and in a few parts of Chapter 5.
- In Chapter 4 we study the gradient- $S_p$  property for QMS's. We apply this study to obtain strong solidity results for group von Neumann algebras of right-angled Coxeter groups and for right-angled Hecke algebras.

- In Chapter 5 we fully characterize strong solidity for von Neumann algebraic graph products. We stress that the techniques we use here are completely different from those in Chapter 4. Furthermore, in this chapter we also study nuclearity, relative amenability, primeness and free indecomposability for graph products. Moreover, we obtain unique rigid graph product decompositions and show unique prime factorizations and unique free product decompositions for new classes of von Neumann algebras. We also show that in many cases the graph radius can be retrieved from the graph product.
- In Chapter 6 we study stability of the CCAP and weak-\* CCAP under graph products. We are able to show that graph products of finite-dimensional von Neumann algebras possess the weak-\* CCAP. Furthermore, we are able to show that a property slightly stronger than the CCAP is preserved under reduced graph products.
- In Chapter 7 we slightly deviate from the main topics of this thesis and study commutator estimates for normal operators in factors. We apply these estimates to obtain optimal norm bounds on derivations.

We stress that the chapters in this thesis are not logically dependent and can be read in any order; the only exception is Chapter 3. We now summarize the main results. For some of the notation we refer to later chapters.

### QUANTUM MARKOV SEMIGROUPS AND GRADIENT- $S_p$

In Chapter 4 we study quantum Markov semigroups (QMS) on  $\mathcal{L}(\mathcal{W})$  for Coxeter groups  $\mathcal{W}$ . Specifically, for a Coxeter system  $\mathcal{W} = \langle S|M \rangle$  we study the QMS on  $\mathcal{L}(\mathcal{W})$  associated with the word length  $|\cdot|_S$ . This is the semigroup  $(\Phi_t)_{t \geq 0}$  of the form  $\Phi_t = e^{-t\Delta}$  where the (unbounded) generator  $\Delta$  on  $\mathcal{L}(\mathcal{W})$  is given by

$$\Delta(\lambda_{\mathbf{w}}) = |\mathbf{w}|_S \lambda_{\mathbf{w}}.$$

For these QMS's we precisely characterize when it possesses the Gradient- $S_p$  property as was defined in [Cas21] (for  $p = 2$ ) and [CIW21] (for general  $p$ ), see Definition 4.3.3.

**Theorem A** (Theorem 4.4.15). *Let  $\mathcal{W} = \langle S|M \rangle$  be a Coxeter system. Fix  $p \in [1, \infty]$ . The following are equivalent:*

1. *The QMS  $(\Phi_t)_{t \geq 0}$  associated with the word length  $|\cdot|_S$  is gradient- $S_p$  on  $\mathcal{L}(\mathcal{W})$ .*
2. *For all  $s \in S$  the set  $\{\mathbf{v} \in \mathcal{W} : s\mathbf{v} = \mathbf{v}s\}$  is finite.*
3. *The Coxeter system  $\mathcal{W} = \langle S|M \rangle$  is small at infinity.*

As we show in Theorem B, in most cases you can characterize the equivalent statements of Theorem A in purely graph theoretical terms (see Definitions 4.4.5 and 4.4.6).

**Theorem B** (Theorem 4.4.8 and Theorem 4.4.9). *Let  $\mathcal{W} = \langle S|M \rangle$  be a Coxeter group. If there does not exist a cyclic parity path in  $\text{Graph}_S(\mathcal{W})$  then the semi-group  $(\Phi_t)_{t \geq 0}$  associated to the word length  $|\cdot|_S$  is gradient- $S_p$  for all  $p \in [1, \infty]$ . The converse holds true if  $m_{i,j} \neq 2$  for all  $i, j$ .*

## STRONG SOLIDITY

Continuing, in Chapter 4 we study for right-angled Coxeter groups  $\mathcal{W}_\Gamma$  some QMS's on  $\mathcal{L}(\mathcal{W}_\Gamma)$  that are associated to different kinds of word lengths. Using the non-commutative Riesz transform  $R = \nabla \circ \Delta^{-\frac{1}{2}}$  we show that if  $\mathcal{W}_\Gamma$  is hyperbolic then the group von Neumann algebra  $\mathcal{L}(\mathcal{W}_\Gamma)$  possesses  $AO^+$  and is strongly solid (Theorem 4.6.2). This result was already known using different techniques, see [PV14b]. However, when we apply our results on QMS's on right-angled Hecke von Neumann algebras we are able to obtain the following new result.

**Theorem C** (Theorem 4.7.5). *Let  $\Gamma$  be a finite simple graph and let  $\mathbf{q} = (q_v)_{v \in \Gamma}$  with  $q_v > 0$ . Assume*

$$\Lambda := \{r \in \Gamma : \exists s, t \in \Gamma \text{ such that } r \in \text{Link}_\Gamma(s) \cap \text{Link}_\Gamma(t), s \notin \text{Star}_\Gamma(t)\}$$

*is a clique in  $\Gamma$ . Then the Hecke von Neumann algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_\Gamma)$  satisfies the Akemann-Ostrand property  $AO^+$  and is strongly solid.*

In the next chapter, Chapter 5, we study general von Neumann algebraic graph products. Using completely different techniques (such as Popa's intertwining by bimodule theory) we are able to fully characterize when a graph product of tracial von Neumann algebras is strongly solid.

**Theorem D** (Theorem 5.6.7). *Let  $\Gamma$  be a finite graph, and for each  $v \in \Gamma$  let  $M_v (\neq \mathbb{C})$  be a von Neumann algebra with normal faithful trace  $\tau_v$ . Then  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  is strongly solid if and only if the following conditions are satisfied:*

1. *For each vertex  $v \in \Gamma$  the von Neumann algebra  $M_v$  is strongly solid.*
2. *For each subgraph  $\Lambda \subseteq \Gamma$  with  $M_\Lambda$  non-amenable, we have that  $M_{\text{Link}(\Lambda)}$  is not diffuse.*
3. *For each subgraph  $\Lambda \subseteq \Gamma$  with  $M_\Lambda$  non-amenable and diffuse, we have moreover that  $M_{\text{Link}(\Lambda)}$  is atomic.*

We remark that in most cases the stated conditions can be easily verified from the graph  $\Gamma$  and the vertex von Neumann algebras  $M_v$ . In particular, Theorem D completes the characterization of strong solidity for right-angled Hecke-algebras, see Theorem 5.6.12. Moreover, for group von Neumann algebras of Coxeter groups we obtain the following simple characterization of strong solidity.

**Theorem E** (Theorem 5.6.13). *Let  $\mathcal{W}_\Gamma$  be a right-angled Coxeter group. The following are equivalent:*

1. *The von Neumann algebra  $\mathcal{L}(\mathcal{W}_\Gamma)$  is strongly solid.*
2. *The Coxeter group  $\mathcal{W}_\Gamma$  does not contain  $\mathbb{Z} \times \mathbb{F}_2$  as a subgroup.*
3. *The graph  $\Gamma$  does not contain  $K_{2,3}$  nor  $K_{2,3}^+$  as a subgraph (see Figure 5.1).*

We do remark that a right-angled Coxeter group  $\mathcal{W}_\Gamma$  is hyperbolic if and only if  $\mathcal{W}_\Gamma$  does not contain  $\mathbb{Z} \times \mathbb{Z}$  as a subgroup, if and only if the graph  $\Gamma$  does not contain  $\mathbb{Z}_4$  (the cyclic graph with four vertices) as a (induced) subgraph (see [Dav08]). The result of Theorem E establishes strong solidity of  $\mathcal{L}(\mathcal{W}_\Gamma)$  for a broader class of right-angled Coxeter groups than just those that are hyperbolic.

## RIGID GRAPH PRODUCTS

In Chapter 5 we moreover study other rigidity properties of graph products. Indeed, we introduce a class  $\mathcal{C}_{\text{Vertex}}$  of  $\text{II}_1$ -factors (see Definition 5.5.4) and introduce the notion of a *rigid graph* (see Definition 5.2.1). We then study graph product von Neumann algebras  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  whose vertex algebras  $M_v$  are in  $\mathcal{C}_{\text{Vertex}}$ , and whose graph  $\Gamma$  is rigid. We show that the Neumann algebra  $M_\Gamma$  uniquely decomposes as such a graph product. This is the content of the following theorem.

**Theorem F** (Theorem 5.5.19 and Theorem 5.7.5). *Let  $\Gamma$  be finite rigid graphs and for  $v \in \Gamma$  let  $M_v$  be von Neumann algebras in the class  $\mathcal{C}_{\text{Vertex}}$  with faithful normal trace  $\tau_v$ . Let  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  be their graph product. Suppose there is another graph product decomposition of  $M_\Gamma$  over another rigid graph  $\Lambda$  and other von Neumann algebras  $N_w \in \mathcal{C}_{\text{Vertex}}$ ,  $w \in \Lambda$ , i.e.  $M_\Gamma = *_{w \in \Lambda} (N_w, \tau_w)$ . Then there is a graph isomorphism  $\alpha : \Gamma \rightarrow \Lambda$ , and for each  $v \in \Gamma$  there is a unitary  $u_v \in M_\Gamma$  and a real number  $0 < t_v < \infty$  such that:*

$$M_{\text{Star}(v)} = u_v^* N_{\text{Star}(\alpha(v))} u_v \quad \text{and} \quad M_v \simeq N_{\alpha(v)}^{t_v}. \quad (1.12)$$

Furthermore, for the connected component  $\Gamma_v \subseteq \Gamma$  of any vertex  $v \in \Gamma$ , we have  $M_{\Gamma_v} = u_v^* N_{\alpha(\Gamma_v)} u_v$ ; and for any irreducible component  $\Gamma_0 \subseteq \Gamma$ ,  $\exists t_0 \in (0, \infty)$  such that  $M_{\Gamma_0} \simeq N_{\alpha(\Gamma_0)}^{t_0}$ .

For more general graph products we study what information of the graph  $\Gamma$  can be retrieved from the graph products  $M_\Gamma$ . Indeed, we introduce the notion of the *radius* of a von Neumann algebra (see Definition 5.9.3), and show that in many cases we can retrieve the radius of the graph  $\Gamma$  (up to some constant) from the radius of the von Neumann algebra  $M_\Gamma$  (Remark 5.9.7). In particular, this allows us to distinguish certain graph products of hyperfinite  $\text{II}_1$ -factors.

**Theorem G** (Theorem 5.9.6 and Theorem 5.9.11). *Let  $\Gamma$  be a finite, non-complete graph. For  $v \in \Gamma$  let  $M_v$  be a  $\text{II}_1$ -factor and let  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  be the tracial graph product. Suppose one of the following holds true.*

1. *For all  $v \in \Gamma$  the vertex algebra  $M_v$  possesses strong (AO) and has separable predual.*
2. *For all  $v \in \Gamma$  we have  $M_v = \mathcal{L}(G_v)$  for some countable icc group  $G_v$ .*

Then

$$\text{Radius}(\Gamma) - 2 \leq \text{Radius}(M_\Gamma) \leq \max\{2, \text{Radius}(\Gamma)\}.$$

## UNIQUE PRIME FACTORIZATIONS

In Chapter 5 we also prove the following result which characterizes primeness for graph products.

**Theorem H** (Theorem 5.7.4). *Let  $\Gamma$  be a finite graph of size  $|\Gamma| \geq 2$ . For any  $v \in \Gamma$ , let  $M_v$  be a  $\text{II}_1$ -factor. The graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  is prime if and only if  $\Gamma$  is irreducible.*

By combining Theorem H with Theorem F we are able to obtain the following unique prime factorization result.

**Theorem I** (Theorem 5.7.6). *Any von Neumann algebra  $M \in \mathcal{C}_{\text{Rigid}}$  has a prime factorization inside  $\mathcal{C}_{\text{Rigid}}$ , i.e.*

$$M = M_1 \bar{\otimes} \cdots \bar{\otimes} M_m, \quad (1.13)$$

for some  $m \geq 1$  and prime factors  $M_1, \dots, M_m \in \mathcal{C}_{\text{Rigid}}$ .

Suppose  $M$  has another prime factorization inside  $\mathcal{C}_{\text{Rigid}}$ , i.e.

$$M = N_1 \bar{\otimes} \cdots \bar{\otimes} N_n, \quad (1.14)$$

for some  $n \geq 1$ , and prime factors  $N_1, \dots, N_n \in \mathcal{C}_{\text{Rigid}}$ . Then  $m = n$  and there is a permutation  $\sigma$  of  $\{1, \dots, m\}$  such that  $M_i$  is isomorphic to an amplification of  $N_{\sigma(i)}$  for  $1 \leq i \leq m$ .

## UNIQUE FREE PRODUCT DECOMPOSITION

In Chapter 5 we also study free product decompositions. The following result characterizes when a graph product decomposes as free product of  $\text{II}_1$ -factor.

**Theorem J** (Theorem 5.8.1). *Let  $\Gamma$  be a finite graph of size  $|\Gamma| \geq 2$ , and for each  $v \in \Gamma$  let  $M_v$  be  $\text{II}_1$ -factor with separable predual. Then the graph product  $M_\Gamma := \ast_{v \in \Gamma} (M_v, \tau_v)$  can decompose as a tracial free product  $M_\Gamma = (M_1, \tau_1) \ast (M_2, \tau_2)$  of  $\text{II}_1$ -factors  $M_1, M_2$  if and only if  $\Gamma$  is not connected.*

Combining with Theorem J with Theorem F we obtain the follow unique free product decomposition for von Neumann in the class  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ .

**Theorem K** (Theorem 5.8.2). *Any von Neumann algebra  $M \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  can decompose as a tracial free product inside  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ , i.e.*

$$M = M_1 \ast \cdots \ast M_m, \quad (1.15)$$

for some  $m \geq 1$  and factors  $M_1, \dots, M_m \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  that can not decompose as any tracial free product of  $\text{II}_1$ -factors.

Suppose  $M$  can decompose as another tracial free product inside  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ , i.e.

$$M = N_1 \ast \cdots \ast N_n,$$

for some  $n \geq 1$  and factors  $N_1, \dots, N_n \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  that can not decompose as tracial free product of  $\text{II}_1$ -factors. Then  $m = n$  and there is a permutation  $\sigma$  of  $\{1, \dots, m\}$  such that  $N_i$  unitarily conjugate to  $M_{\sigma(i)}$  in  $M$ .

## NUCLEARITY, (RELATIVE) AMENABILITY AND THE (WEAK-\*) CCAP

We study several approximation properties for graph products.

### GRAPH PRODUCTS AND NUCLEARITY

In Chapter 5 we give sufficient conditions for a reduced graph product of unital  $C^*$ -algebras to be nuclear. This is a generalization of Ozawa's result for free products [Oza02] and is needed in the proof of Theorem F.

**Theorem L** (Theorem 5.3.4). *Let  $A_\Gamma = \ast_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  be the reduced  $C^*$ -algebraic graph product of nuclear, unital  $C^*$ -algebras  $A_v$  with GNS-faithful state  $\varphi_v$ . Let  $\mathcal{H}_v := L^2(A_v, \varphi_v)$  and let  $\pi_v : A_v \rightarrow B(\mathcal{H}_v)$  be the GNS-representation. If for any  $v \in \Gamma$ ,  $\pi_v(A_v)$  contains the space of compact operators  $K(\mathcal{H}_v)$ , then  $A_\Gamma$  is nuclear.*

### GRAPH PRODUCTS AND (RELATIVE) AMENABILITY

The following result from Chapter 5 is the graph product analogue of [HI17, Theorem 5.1] and [Oza06, Theorem 3.3], and is crucial in the proof of Theorem F for establishing the graph isomorphism.

**Theorem M** (Theorem 5.5.15). *Let  $(M_\Gamma, \tau) = *_{v \in \Gamma} (M_v, \tau_v)$  be the graph product of finite von Neumann algebras  $M_v$  that satisfy condition strong (AO) and have separable predual. Let  $Q \subseteq M_\Gamma$  be a diffuse von Neumann subalgebra. At least one of the following holds:*

1. *The relative commutant  $Q' \cap M_\Gamma$  is amenable;*
2. *There exists  $\Gamma_0 \subseteq \Gamma$  such that  $Q <_{M_\Gamma} M_{\Gamma_0}$  and  $\text{Link}(\Gamma_0) \neq \emptyset$ .*

We also study relative amenability for graph products and obtain the following result which is needed in the proof of Theorem D, Theorem G and Theorem H.

**Theorem N** (Theorem 5.4.8). *Let  $\Gamma$  be a graph with subgraphs  $\Gamma_1, \Gamma_2 \subseteq \Gamma$ . For each  $v \in \Gamma$  let  $(M_v, \tau_v)$  be a von Neumann algebra with a normal faithful trace. Let  $P \subset M_\Gamma$  be a von Neumann subalgebra that is amenable relative to  $M_{\Gamma_i}$  inside  $M_\Gamma$  for  $i = 1, 2$ . Then  $P$  is amenable relative to  $M_{\Gamma_1 \cap \Gamma_2}$  inside  $M_\Gamma$ .*

### GRAPH PRODUCTS AND THE CCAP

In the next chapter, Chapter 6, the focus is on showing stability of the CCAP and weak-\* CCAP under graph products. Similar to [RX06] we are able to show for reduced graph products of unital  $C^*$ -algebras that a condition slightly stronger than the CCAP is preserved.

**Theorem O** (Theorem 6.5.2). *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $(A_v, \varphi_v)$  be unital  $C^*$ -algebras that have a u.c.p. extension for the CCAP. Then the reduced graph product  $(A_\Gamma, \varphi) = *_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  has the CCAP.*

For von Neumann algebraic graph products we are able to show the following result.

**Theorem P** (Corollary 6.3.4). *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $M_v$  be a finite-dimensional von Neumann algebra together with a normal faithful state  $\varphi_v$ . Then the von Neumann algebraic graph product  $(M_\Gamma, \varphi) = *_{v \in \Gamma} (M_v, \varphi_v)$  has the weak-\* CCAP.*

### COMMUTATOR ESTIMATES AND DERIVATIONS

The topic of the last chapter, Chapter 7, is slightly different from the other parts of this thesis and does not concern Coxeter groups or graph products. The aim is to generalize commutator estimates from [BS12b], [BS12a] and [BHS23] for self-adjoint elements to normal elements. We obtain the following result; here  $S(M)$  denotes the algebra of measurable operators affiliated with  $M$ . The constants  $\Lambda_n$  and  $\tilde{\Lambda}_n$  are defined in Section 7.3 in (7.12) and (7.13) and estimates on these constants are given in Theorem 7.A.1.

**Theorem Q** (see Theorems 7.5.6, 7.6.4). *Let  $M$  be a factor and let  $a \in S(M)$  be normal. Then there is a  $\lambda_0 \in \mathbb{C}$  and unitaries  $u, v, w \in U(M)$  such that*

$$|[a, u]| \geq C (v|a - \lambda_0 1_M|v^* + w|a - \lambda_0 1_M|w^*) \quad (1.16)$$

for some constant  $C > 0$  independent of  $a$ . Moreover

1. when  $M$  is a  $I_n$ -factor,  $n < \infty$ , the optimal constant satisfies  $\Lambda_n \leq C \leq \frac{1}{2} \tilde{\Lambda}_n$ .
2. when  $M$  is a  $II_1$ -factor, the optimal constant is  $C = \frac{\sqrt{3}}{2}$ .
3. when  $M$  is an infinite factor, we can choose  $C$  arbitrarily close to 1.

We apply the commutator estimates from Theorem Q to obtain for a finite factor  $M$  sharp estimates on the norm  $\|\delta_a\|_{M \rightarrow L^1(M, \tau)}$ , where  $\delta_a : M \rightarrow L^1(M, \tau)$  is the inner derivation given by  $\delta_a(x) = ax - xa$  associated to a normal measurable  $a \in L^1(M, \tau)$ . In this result we denote  $n(M) = n$  if  $M$  is a  $I_n$ -factor and put  $n(M) = \infty$  if  $M$  is a  $II_1$ -factor.

**Theorem R.** *Let  $M$  be a finite factor with a faithful tracial state  $\tau$  and let  $a \in L^1(M, \tau) \setminus Z(M)$  be normal and measurable. Then the derivation  $\delta_a : M \rightarrow L^1(M, \tau)$  satisfies:*

$$2\Lambda_{n(M)} \leq \frac{\|\delta_a\|_{\infty,1}}{\min_{z \in \mathbb{C}} \|a - z1_M\|_1} \leq 2. \quad (1.17)$$

Moreover, when  $M \neq \mathbb{C}$  there exist non-zero derivations  $\delta_a, \delta_b$  corresponding to normal  $a, b \in M$  such that  $\|\delta_a\|_{\infty,1} \leq \tilde{\Lambda}_{n(M)} \min_{z \in \mathbb{C}} \|a - z1_M\|_1$  and  $\|\delta_b\|_{\infty,1} = 2 \min_{z \in \mathbb{C}} \|b - z1_M\|_1$ .

We remark if  $n(M) \notin \{1, 2, 4\}$  then the distance formula  $\|\delta_a\|_{\infty,1} = 2 \min_{z \in \mathbb{C}} \|a - z1_M\|_1$  from [BHS23, Theorem 1.1] (Theorem 7.7.1) for self-adjoint  $a$  does not extend to arbitrary normal measurable  $a \in L^1(M, \tau)$ , since  $\tilde{\Lambda}_{n(M)} < 2$  in these cases. Furthermore, we remark when  $M$  is a  $II_1$ -factor or a  $I_n$ -factor with  $n \equiv 0 \pmod{3}$  then the constant bounds given in (1.17) can not be improved as in these cases  $2\Lambda_{n(M)} = \sqrt{3} = \tilde{\Lambda}_{n(M)}$ .

## DISCUSSION

In the last section of Chapter 4 and of Chapter 5 respectively we discuss some natural open problems related to the topics of these chapters. In the discussion of Chapter 5 we moreover state a conjecture on rigidity of graph products of hyperfinite  $II_1$ -factors (Conjecture 5.10.5).



# 2

## PRELIMINARIES

We recall general theory used in this thesis and establish our notation. In Section 2.1 we recap theory of  $C^*$ -algebras, von Neumann algebras and bimodules. In Section 2.2 we establish the notation we use for (simple) graphs. In Section 2.3 we discuss discrete groups and their  $*$ -algebras, reduced  $C^*$ -algebras and group von Neumann algebra. Furthermore, we establish notation for Coxeter groups and Hecke-algebras and we define graph products of groups. In Section 2.4 we show the construction of (reduced) graph products in the setting of  $C^*$ -algebras/ von Neumann algebras as was introduced in [CF17]. In Section 2.5 we define several properties for groups,  $C^*$ -algebras and von Neumann algebras and discuss how they are connected.

**Conventions and general notation:** We denote  $\mathbb{N} = \{1, 2, \dots\}$  for the set of natural numbers. For a set  $S$  we write  $|S|$  for its cardinality and  $2^S$  for its power set. All Hilbert spaces  $\mathcal{H}$  considered are complex, and with inner products  $\langle \cdot, \cdot \rangle$  that are linear in the first variable.

### 2.1. OPERATOR ALGEBRAS AND BIMODULES

In Section 2.1.1 we discuss the bounded operators, Schatten classes,  $C^*$ -algebras and operator spaces. In Section 2.1.2 we discuss von Neumann algebras and in Section 2.1.3 we discuss bimodules. For a detailed exposition on these topics we refer to [SZ19; Mur90; AP17; Tak02; Tak03a; Tak03b]. Furthermore, we refer to [ER00; Pis03] for general theory of operator spaces, and we refer to [DPS22; FK86] for more theory on locally measurable operators.

#### 2.1.1. THE SPACE OF BOUNDED OPERATORS

Given a complex Hilbert space  $\mathcal{H}$  we denote  $B(\mathcal{H})$  and  $K(\mathcal{H})$  respectively for the space of bounded operators on  $\mathcal{H}$  and the space of compact operators on  $\mathcal{H}$ .

### SCHATTEN-VON NEUMANN CLASSES $S_p(\mathcal{H})$

For positive  $x \in B(\mathcal{H})$  we denote

$$\mathrm{Tr}(x) = \sum_{i \in I} \langle x e_i, e_i \rangle \quad (2.1)$$

where  $(e_i)_{i \in I}$  is any orthonormal basis for  $\mathcal{H}$ . For  $p \in (0, \infty)$  we define the Schatten-von Neumann class  $S_p(\mathcal{H})$  as the space of all  $x \in B(\mathcal{H})$  for which

$$\|x\|_p := \mathrm{Tr}(|x|^p)^{\frac{1}{p}} \quad (2.2)$$

is finite. If  $p \in [1, \infty)$  then (2.2) defines a norm turning  $S_p(\mathcal{H})$  into a Banach space that is moreover a 2-sided ideal in  $B(\mathcal{H})$ . We additionally define  $\|\cdot\|_\infty$  equal to the operator norm  $\|\cdot\|$  and put  $S_\infty(\mathcal{H}) = K(\mathcal{H})$ . Moreover, we extend the trace (2.1) linearly to define a bounded map  $\mathrm{Tr} : S_1(\mathcal{H}) \rightarrow \mathbb{C}$ . The Schatten class  $S_1(\mathcal{H})$  can be identified as the predual of  $B(\mathcal{H})$  through the identification  $S_1(\mathcal{H}) \ni x \mapsto \mathrm{Tr}(\cdot x)$ .

### TOPOLOGIES ON $B(\mathcal{H})$

We recall the three most important topologies on  $B(\mathcal{H})$ .

1. The strong operator topology (SOT). A net  $(x_i)_{i \in I}$  converges strongly to  $x$  if  $\|x_i \xi\| \rightarrow \|x \xi\|$  for all  $\xi \in \mathcal{H}$ .
2. The weak operator topology (WOT). A net  $(x_i)_{i \in I}$  converges weakly to  $x$  if  $\langle x_i \xi, \eta \rangle \rightarrow \langle x \xi, \eta \rangle$  for all  $\xi, \eta \in \mathcal{H}$ .
3. The  $\sigma$ -weak topology. A net  $(x_i)_{i \in I}$  converges  $\sigma$ -weakly to  $x$  if  $\sum_{n \geq 1} \langle x_i \xi_n, \eta_n \rangle \rightarrow \sum_{n \geq 1} \langle x \xi_n, \eta_n \rangle$  for all sequences  $(\xi_n)_{n \geq 1}$ ,  $(\eta_n)_{n \geq 1}$  in  $\mathcal{H}$  for which  $\sum_{n \geq 1} \|\xi_n\|^2$  and  $\sum_{n \geq 1} \|\eta_n\|^2$  are finite.

We note that convergence in norm implies strong convergence implies weak convergence. Furthermore, we note also that  $\sigma$ -weak convergence implies weak convergence. In fact, on the unit ball the weak operator topology and the  $\sigma$ -weak topology coincide.

### $C^*$ -ALGEBRAS AND TENSOR PRODUCTS

An algebra  $A$  is a vector space equipped with a multiplication, i.e. a map  $A \times A \rightarrow A$  that is bilinear and associative. A  $*$ -algebra is an algebra  $A$  equipped with an involution  $*$ , i.e. an antilinear map  $A \rightarrow A$  satisfying  $(a^*)^* = a$  and  $(ab)^* = b^* a^*$  for  $a, b \in A$ . When  $A$  is unital, we denote by  $1_A$  the unit of  $A$ . A  $C^*$ -norm on a  $*$ -algebra  $A$  is a norm  $\|\cdot\|$  satisfying  $\|xy\| \leq \|x\| \|y\|$ ,  $\|x^*\| = \|x\|$  and  $\|x^* x\| = \|x\|^2$  for  $x, y \in A$ . A  $C^*$ -algebra is a  $*$ -algebra  $A$  equipped with a  $C^*$ -norm that makes  $A$  into a Banach space. We recall the following notions of tensor products.

1. For vector spaces  $V, W$  we let  $V \otimes_{\mathrm{alg}} W$  be the algebraical tensor product.
2. For Hilbert spaces  $\mathcal{H}, \mathcal{K}$  we let  $\mathcal{H} \otimes \mathcal{K}$  be the Hilbert space tensor product, which is the Hilbert space completion of  $\mathcal{H} \otimes_{\mathrm{alg}} \mathcal{K}$  with respect to the inner product given by  $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle_{\mathcal{H}} \langle \eta_1, \eta_2 \rangle_{\mathcal{K}}$

3. For  $C^*$ -algebras  $A$  and  $B$  we let  $A \otimes_{\min} B$  be the minimal tensor product and let  $A \otimes_{\max} B$  be the maximal tensor product. These are the completions of  $A \otimes_{\text{alg}} B$  w.r.t the  $C^*$ -norms  $\|\cdot\|_{\min}$  respectively  $\|\cdot\|_{\max}$  that are defined by the property that  $\|\cdot\|_{\min} \leq \|\cdot\| \leq \|\cdot\|_{\max}$  for any  $C^*$ -norm  $\|\cdot\|$  on  $A \otimes_{\text{alg}} B$ .
4. For (concrete)  $C^*$ -algebras  $A \subseteq B(\mathcal{H})$  and  $B \subseteq B(\mathcal{K})$  we can construct an embedding  $\pi : A \otimes_{\text{alg}} B \rightarrow B(\mathcal{H} \otimes \mathcal{K})$  as  $\pi(a \otimes b)(\xi \otimes \eta) = (a\xi) \otimes (b\eta)$ . The norm closure of  $\pi(A \otimes_{\text{alg}} B)$  is called the *spatial tensor product*. By [Tak02, Theorem 4.19] the spatial tensor product agrees with the minimal tensor product  $A \otimes_{\min} B$ .
5. For linear maps  $T_i : V_i \rightarrow W_i$  between vector spaces  $V_i$  and  $W_i$ , we denote by  $T_1 \otimes T_2 : V_1 \otimes_{\text{alg}} V_2 \rightarrow W_1 \otimes_{\text{alg}} W_2$  the map given by  $(T_1 \otimes T_2)(x \otimes y) = T_1(x) \otimes T_2(y)$ . When  $T_i$  is a bounded map between Hilbert spaces  $\mathcal{H}_i$  and  $\mathcal{K}_i$  then we extend  $T_1 \otimes T_2$  to a map from  $\mathcal{H}_1 \otimes \mathcal{H}_2$  to  $\mathcal{K}_1 \otimes \mathcal{K}_2$ .

We remark for  $n \geq 1$  that the algebraic tensor product  $A \otimes_{\text{alg}} \text{Mat}_n(\mathbb{C})$  can be equipped with a unique norm making it into a  $C^*$ -algebra [BO08, Proposition 3.3.2].

### OPERATOR SPACES

An operator space is a (norm-)closed subspace of  $B(\mathcal{H})$  for some complex Hilbert space  $\mathcal{H}$ . Given an operator space  $B \subseteq B(\mathcal{H})$  we identify  $B \otimes \text{Mat}_n(\mathbb{C})$  with the corresponding (closed) subspace of  $B(\mathcal{H} \otimes \mathbb{C}^n)$ . A bounded map  $\theta : B_1 \rightarrow B_2$  between operator spaces  $B_1, B_2$  is said to be a *completely isometry* if for all  $n \geq 1$  the map  $\theta \otimes \text{Id}_{\text{Mat}_n(\mathbb{C})}$  between  $B_1 \otimes \text{Mat}_n(\mathbb{C})$  and  $B_2 \otimes \text{Mat}_n(\mathbb{C})$  is an isometry. Furthermore we denote,

$$\|\theta\|_{\text{cb}} := \sup_{n \geq 1} \|\theta \otimes \text{Id}_{\text{Mat}_n(\mathbb{C})}\|$$

and call  $\theta$  *completely bounded* whenever  $\|\theta\|_{\text{cb}}$  is finite. We call  $\theta$  *completely contractive* whenever  $\|\theta\|_{\text{cb}} \leq 1$ . A bounded map  $\theta : A_1 \rightarrow A_2$  between  $C^*$ -algebras is called a *homomorphism* if  $\theta(ab) = \theta(a)\theta(b)$  for  $a, b \in A_1$ . We call  $\theta$  a  *$*$ -homomorphism* (or *representation*) if moreover  $\theta(a^*) = \theta(a)^*$  for  $a \in A$ . We recall the following dilation theorem.

**Theorem 2.1.1** (Theorem 8.4 in [Pau02]). *Let  $A$  be a  $C^*$ -algebra and  $\theta : A \rightarrow B(\mathcal{H})$  be a completely bounded map. Then there is a Hilbert space  $\mathcal{K}$ , a representation  $\pi : A \rightarrow B(\mathcal{K})$  and bounded maps  $V_1, V_2 : \mathcal{H} \rightarrow \mathcal{K}$  such that*

$$\theta(a) = V_1^* \pi(a) V_2 \quad a \in A \tag{2.3}$$

and  $\|\theta\|_{\text{cb}} = \|V_1\| \|V_2\|$ . Moreover, if  $\|\theta\|_{\text{cb}} = 1$  then  $V_1$  and  $V_2$  may be taken to be isometries.

We also remark that if  $A$  is a  $C^*$ -algebra and  $\theta : A \rightarrow B(\mathcal{H})$  is any map that can be written as (2.3), then it is completely bounded. Furthermore, when for  $i = 1, 2$  we are given completely bounded maps  $T_i$  between  $C^*$ -algebras  $A_i$  and  $B_i$  then we may extend  $T_1 \otimes T_2$  to a map from  $A_1 \otimes_{\min} A_2$  to  $B_1 \otimes_{\min} B_2$ .

For operator spaces,  $V, W$  we denote  $V \otimes_{\mathfrak{h}} W$  for their Haagerup tensor product, see [ER00, Chapter 9].

### 2.1.2. VON NEUMANN ALGEBRAS

Given a complex Hilbert space  $\mathcal{H}$  and a subset  $S \subseteq B(\mathcal{H})$  we denote by

$$S' := \{x \in B(\mathcal{H}) : xy - yx = 0 \text{ for } y \in S\}$$

the commutant of  $S$ . Moreover, we denote by  $S'' := (S')'$  the double commutant of  $S$ . A *von Neumann algebra*  $M$  is a subset  $M \subseteq B(\mathcal{H})$  that satisfies  $M'' = M$ . Equivalently, a von Neumann algebra  $M$  is a SOT-closed unital  $*$ -subalgebra of  $B(\mathcal{H})$  for some complex Hilbert space  $\mathcal{H}$ . For a von Neumann algebra  $M \subseteq B(\mathcal{H})$  we denote  $M^+ := \{a^*a : a \in M\}$  for the cone of positive elements,  $U(M)$  for the group of unitaries,  $P(M)$  for the lattice of projections and  $Z(M) := M' \cap M$  for the center. We denote the unit of  $M$  by  $1_M (= \text{Id}_{\mathcal{H}})$ . We call  $M$  a *factor* if  $Z(M) = \mathbb{C}1_M$ . We call  $N \subseteq M$  a *von Neumann subalgebra* when  $N$  is a von Neumann algebra and  $1_N = 1_M$ . For a projection  $p \in P(M')$  we call the von Neumann algebra  $Mp|_{p\mathcal{H}} \subseteq B(p\mathcal{H})$  the *reduction* of  $M$ , and for a projection  $p \in P(M)$  we call the von Neumann algebra  $pMp|_{p\mathcal{H}} \subseteq B(p\mathcal{H})$  a *corner* of  $M$ . If  $M \subseteq B(\mathcal{H})$  and  $N \subseteq B(\mathcal{K})$  are von Neumann algebras, we denote  $M\bar{\otimes}N \subseteq B(\mathcal{H} \otimes \mathcal{K})$  for their *von Neumann tensor product*, which is the SOT-closure of their spatial tensor product. We say that von Neumann algebras  $M$  and  $N$  are isomorphic if there is a  $*$ -isomorphism from  $M$  to  $N$ , i.e. a bijective linear map  $\theta : M \rightarrow N$  that satisfies  $\theta(xy) = \theta(x)\theta(y)$  and  $\theta(x^*) = \theta(x)^*$  for  $x, y \in M$ . In this case we write  $M \simeq N$ , or sometimes just  $M = N$ . We say that  $M$  and  $N$  are stably isomorphic if  $M\bar{\otimes}B(\mathcal{H}) \simeq N\bar{\otimes}B(\mathcal{H})$  where  $\mathcal{H}$  is the separable infinite-dimensional Hilbert space. For  $u \in U(M)$  we denote  $\text{Ad}_u : M \rightarrow M$  for the  $*$ -isomorphism given by  $\text{Ad}_u(x) = uxu^*$ . For von Neumann subalgebra  $N \subseteq 1_M M 1_M$  we denote by

$$\text{Nor}_M(N) = \{u \in U(M) : uNu^* = N\}$$

the group of *normalizers* of  $N$  inside  $M$ . Furthermore, we define the  $*$ -algebra of *quasi-normalizers* of  $N$  inside  $M$  as

$$\text{qNor}_M(N) = \{x \in M : \exists n, m \in \mathbb{N}, x_1, \dots, x_n, y_1, \dots, y_m : xN \subseteq \sum_{i=1}^n Nx_i, Nx \subseteq \sum_{j=1}^m y_j N\}.$$

#### COMPLETELY POSITIVE MAPS, COMPLETELY BOUNDED MAPS, STATES AND TRACES

Let  $M$  and  $N$  be von Neumann algebras (or unital  $C^*$ -algebras). A linear map  $\theta : M \rightarrow N$  is called *unital* if it maps the unit of  $M$  to the unit of  $N$ , i.e.  $\theta(1_M) = 1_N$ . A linear map  $\theta$  is called *positive* if it maps positive elements to positive elements, i.e.  $\theta(M^+) \subseteq N^+$ . A positive map is called *faithful* if  $\theta(a^*a) > 0$  whenever  $a$  is non-zero. We call a linear map  $\theta : M \rightarrow N$  *completely positive* if  $\theta \otimes \text{Id}_{\text{Mat}_n(\mathbb{C})}$  is a positive map for all  $n \geq 1$ . We call a linear map  $\theta$  *unital completely positive* (u.c.p) if the map is both unital and completely positive. We note that every u.c.p map is completely contractive [Pau02, Proposition 3.2].

A linear map  $\theta : M \rightarrow N$  is called *normal* if it is continuous for the  $\sigma$ -weak topology. We note that a positive map  $\theta$  between von Neumann algebras  $M$  and  $N$  is normal if and only if for any increasing net  $(a_i)_i$  in  $M$  we have  $\theta(\sup_i a_i) = \sup \theta(a_i)$ , see [Sak12, Theorem 1.13.2].

For a von Neumann algebra  $M$  (or  $C^*$ -algebra) its (Banach space) dual  $M^*$  is the space of all bounded linear functionals  $\varphi : M \rightarrow \mathbb{C}$ . A *state* on  $M$  is a positive linear functional

$\varphi \in M^*$  of norm  $\|\varphi\| = 1$ . Given a von Neumann subalgebra  $N \subseteq M$ , a state on  $M$  is called *N-central* if  $\tau(ab) = \tau(ba)$  for all  $a \in N$ ,  $b \in M$ . A state on  $M$  is called *tracial* if it is *M-central*, i.e. if  $\varphi(ab) = \varphi(ba)$  for all  $a, b \in M$ . A weight on  $M$  is a map  $\varphi : M^+ \rightarrow [0, \infty]$  satisfying  $\varphi(a + b) = \varphi(a) + \varphi(b)$  and  $\varphi(\lambda a) = \lambda \varphi(a)$  for  $a, b \in M$  and  $\lambda \in [0, \infty)$ . For a weight  $\varphi$  we denote

$$\mathfrak{M}_\varphi = \text{Span}\{a \in M^+ : \varphi(a) < \infty\} \quad (2.4)$$

We may extend a weight linearly to a map  $\varphi : \mathfrak{M}_\varphi \rightarrow \mathbb{C}$ . We call a weight semifinite if  $\mathfrak{M}_\varphi$  is  $\sigma$ -weakly dense in  $M$ . We call a weight *finite* if  $\varphi(x) < \infty$  for all  $x \in M^+$ . We call a weight *tracial* if  $\varphi(x^*x) = \varphi(xx^*)$  for  $x \in M$ . We call a weight *faithful* if  $\varphi(x^*x) > 0$  for  $x \in M$ . We call a weight *normal* if  $\varphi(\sup_i a_i) = \sup_i \varphi(a_i)$  for every increasing net  $(a_i)_{i \in I}$  in  $M^+$ . We note that the map  $\text{Tr}$  from (2.1) defines a normal semifinite tracial weight on  $B(\mathcal{H})$ . Furthermore, we observe that all states are weights (by restriction) and that all finite weights extend linearly to positive functionals on  $M$ .

For a normal state  $\varphi$  on a von Neumann algebra  $M$  (or a state  $\varphi$  on a  $C^*$ -algebra) put

$$\mathcal{N}_\varphi = \{a \in M : \varphi(a^*a) = 0\} \quad (2.5)$$

and denote  $L^2(M, \varphi)$  for the GNS-Hilbert space, which is the completion of  $M/\mathcal{N}_\varphi$  with respect to the inner product  $\langle x + \mathcal{N}_\varphi, y + \mathcal{N}_\varphi \rangle := \varphi(y^*x)$ . We let  $\pi : M \rightarrow B(L^2(M, \varphi))$  be the GNS-representation, which is the  $*$ -homomorphism given by  $\pi(a)(b + \mathcal{N}_\varphi) = ab + \mathcal{N}_\varphi$ .

When  $\varphi$  is normal and faithful we call  $(M, \varphi)$  a *statial von Neumann algebra*. For  $\tau$  is a normal faithful tracial state, we call  $(M, \tau)$  a *tracial von Neumann algebra*. We denote  $L^1(M, \tau)$  for the Banach space completion of  $M$  w.r.t. the norm  $\|x\|_1 = \tau(|x|)$ . For a von Neumann algebra  $M$  we denote by  $M_*$  the predual of  $M$ , which is a Banach space such that  $(M_*)^* \simeq M$ . We can identify  $M_*$  with the space of all  $\sigma$ -weakly continuous linear functionals. When  $(M, \tau)$  is a tracial von Neumann algebra, the predual  $M_*$  is isomorphic to  $L^1(M, \tau)$  under the identification  $L^1(M, \tau) \ni x \mapsto \tau(\cdot x) \in M_*$ . We will sometimes require a von Neumann algebra  $M$  to have a separable predual. This is equivalent with saying that  $M$  can be faithfully represented as  $M \rightarrow B(\mathcal{H})$  on a separable Hilbert space  $\mathcal{H}$ .

#### SUPPORT, PROJECTIONS, TYPE CLASSIFICATION AND AMPLIFICATIONS

For  $x \in M$  we denote  $|x| := \sqrt{x^*x}$  for its absolute value,  $\Re(x) := \frac{x+x^*}{2}$  for its real part and  $\Im(x) := \frac{x-x^*}{2i}$  for its imaginary part. For self-adjoint  $x \in M$  we denote  $x_+ := \frac{|x|+x}{2}$  for its positive part and  $x_- := \frac{|x|-x}{2}$  for its negative part, and we note that  $x_-$  and  $x_+$  satisfy  $x_-x_+ = 0$ . For  $x \in M$  we denote the *left support* (resp. *right support*) of  $x$  by  $\mathbf{l}(x)$  (resp.  $\mathbf{r}(x)$ ) which is the smallest projection  $p \in P(M)$  such that  $px = x$  (resp.  $xp = x$ ). We recall that any  $x \in M$  can be written as a polar decomposition  $x = u|x|$  where  $u \in M$  is a partial isometry with  $uu^* = \mathbf{l}(x)$  and  $u^*u = \mathbf{r}(x)$ . For projections  $p, q \in P(M)$  we say that

We denote the *support* of  $x$  by  $\mathbf{s}(x) := \mathbf{l}(x) \vee \mathbf{r}(x) \in M$  (i.e.  $\mathbf{s}(x)$  is the smallest projection larger than  $\mathbf{l}(x)$  and  $\mathbf{r}(x)$ ). Furthermore, we denote the *central support* of  $x$  by  $\mathbf{z}(x)$ , which is the smallest projection in  $M \cap M'$  such that  $\mathbf{s}(x) \leq \mathbf{z}(x)$ .

Projections  $p, q \in P(M)$  are said to be (*Murray-von Neumann*) *equivalent* (in  $M$ ), denoted  $p \sim q$ , whenever there is a  $v \in M$  such that  $p = v^*v$  and  $q = vv^*$ . We write  $p \leq q$

whenever  $r \leq q$  for some projection  $r$  with  $r \sim p$ . Moreover, we write  $p < q$  when  $p \leq q$  and  $p \neq q$ . A projection  $p \in P(M)$  is called *central* (in  $M$ ) if  $p \in M \cap M'$ . A projection  $p \in P(M)$  is called *abelian* (in  $M$ ) if the corner  $pMp$  is commutative. A non-zero projection  $p \in P(M)$  is called *minimal* (in  $M$ ) if there is no non-zero projection  $q \in P(M)$  satisfying  $q < p$ . A projection  $p \in P(M)$  is called *finite* (in  $M$ ) if  $q \leq p$  with  $q \sim p$  implies  $q = p$  for any projection  $q \in P(M)$ . There are the following types of von Neumann algebras  $M$ .

- Type I if every projection  $0 \neq p \in M$  majorizes an abelian projection  $0 \neq e \in M$ .
- Type II if  $M$  does not contain any non-zero abelian projection and if every non-zero central projection in  $M$  majorizes a non-zero finite projection.
- Type III if  $M$  does not contain any non-zero finite projection.

A von Neumann algebra is called *atomic* if every non-zero projection majorizes a non-zero abelian projection. A von Neumann algebra  $M$  is called *diffuse* if there are no minimal projections. A von Neumann algebra  $M$  is called *finite* if the projection  $1_M$  is finite in  $M$ . Equivalently,  $M$  is finite if and only if there exists a normal faithful tracial state on  $M$ . When  $M$  is moreover a factor, then this trace is in fact unique. A von Neumann algebra that is not finite is called *infinite*. A von Neumann algebra is called *semifinite* if any non-zero central projection  $p \in M$  majorizes a non-zero finite projection.

Every factor is precisely of one of the three types: I, II or III. Factors of type I are always atomic and of the form  $M = B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . When  $M$  is moreover finite then  $n := \dim \mathcal{H} < \infty$  and we call  $M$  a factor of type  $I_n$ . A von Neumann algebra of type I that is not finite, is said to be of type  $I_\infty$ . Von Neumann algebras of type II and III are always diffuse. A type II von Neumann algebra is said to be of type  $II_1$  if it is finite, and otherwise it is said to be of type  $II_\infty$ . Factors  $M$  of type  $II_\infty$  are always semifinite and of the form  $M = N \otimes B(\mathcal{H})$  for some  $II_1$ -factor  $N$  and some infinite-dimensional Hilbert space  $\mathcal{H}$ . Whenever  $M$  is a factor of type  $II_1$ , the unique normal faithful tracial state  $\tau$  satisfies  $\tau(P(M)) = [0, 1]$ . Given  $t \in (0, \infty)$  we then denote by  $M^t$  the amplification of  $M$  by  $t$ . Writing  $t = ns$  for some  $n \geq 1$  and  $s \in (0, 1)$  this amplification is defined as

$$M^t := pMp \bar{\otimes} \text{Mat}_n(\mathbb{C})$$

where  $p \in P(M)$  has trace  $\tau(p) = s$ . We note that this definition is independent of the choice of  $n$ ,  $s$  and  $p$ . Furthermore, we note for  $t \in (0, \infty)$  that the amplification  $M^t$  is stably isomorphic to  $M$ .

#### OPPOSITE ALGEBRA AND THE STANDARD FORM

For an algebra  $A$  its *opposite algebra*  $A^{\text{op}}$  is defined as a vector space as  $A$  and for  $a \in A$  the corresponding element in  $A^{\text{op}}$  is denoted by  $a^{\text{op}}$ . We equip  $A^{\text{op}}$  with the multiplication  $A^{\text{op}} \times A^{\text{op}} \ni (a^{\text{op}}, b^{\text{op}}) \mapsto (ba)^{\text{op}} \in A^{\text{op}}$  making it into an algebra. When  $A$  has an involution, then we equip  $A^{\text{op}}$  with the involution  $(a^{\text{op}})^* := (a^*)^{\text{op}}$ , making it into a  $*$ -algebra. When  $A$  is a  $C^*$ -algebra, then so is  $A^{\text{op}}$  when equipped with the same norm.

Let  $M \subseteq B(\mathcal{H})$  be a von Neumann algebra. Recall that a vector  $\xi \in \mathcal{H}$  is called *cyclic* if  $M\xi$  is dense in  $\mathcal{H}$  and that it is called *separating* if  $x\xi = 0$  implies  $x = 0$  for  $x \in M$ .

The inclusion  $M \subseteq B(\mathcal{H})$  is said to be in *standard form* if there exists a conjugation  $J : \mathcal{H} \rightarrow \mathcal{H}$  (i.e. a conjugate-linear isometric map satisfying  $J^2 = \text{Id}_{\mathcal{H}}$ ) such that the map  $j : M^{\text{op}} \rightarrow M'$  given by  $x^{\text{op}} \mapsto Jx^*J$  defines a  $*$ -isomorphism that acts as the identity on  $Z(M)$ . In this case  $\mathcal{H}$  is called the *standard Hilbert space* for  $M$  and denoted by  $L^2(M)$ . The map  $J$  is called the *modular conjugate operator*. If there exists a vector  $\xi \in \mathcal{H}$  that is both cyclic and separating, then  $M$  is in standard form [SZ19, Introduction (8°) of Chapter 10]. In such case we shall identify  $M^{\text{op}}$  with  $JMJ$ .

### CONDITIONAL EXPECTATIONS

Let  $M$  be a von Neumann algebra and  $N \subseteq M$  a von Neumann subalgebra. A map  $E : M \rightarrow N$  is called a *conditional expectation from  $M$  to  $N$*  if it satisfies the following conditions

1.  $E$  is positive (i.e.  $E(M^+) \subseteq N^+$ )
2.  $E(a) = a$  for  $a \in N$
3.  $E(axb) = aE(x)b$  for  $a, b \in N, x \in M$ .

An equivalent definition is that  $E$  is a projection on  $N$  with norm  $\|E\| = 1$ . We note that conditional expectations are u.c.p maps and therefore satisfy the Schwarz inequality  $E(x)^*E(x) \leq E(x^*x)$  [Pau02, proposition 3.3].

If  $(M, \tau)$  is a tracial von Neumann algebra and  $N \subseteq M$  is a von Neumann subalgebra, then there is a unique conditional expectation  $\mathbb{E}_N$  on  $N$  that is trace-preserving (i.e. satisfies  $\tau(\mathbb{E}_N(x)) = \tau(x)$  for  $x \in M$ ), see [AP17, Theorem 9.1.2]. The map  $\mathbb{E}_N$  is automatically normal and faithful and moreover extends to a contraction on  $L^2(M, \tau)$  as

$$\|\mathbb{E}_N(x)\|_2^2 = \tau(\mathbb{E}_N(x)^*\mathbb{E}_N(x)) \leq \tau(\mathbb{E}_N(x^*x)) = \tau(x^*x) = \|x\|_2^2$$

This  $L^2$ -extension is denoted by  $e_N$  and called the *Jones projection*. We denote by  $\langle M, e_N \rangle$  the *Jones extension* of  $M$ , which is the von Neumann algebra  $(M \cup \{e_N\})''$ .

### LOCALLY MEASURABLE OPERATORS

Let  $M$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$ . Given a linear subspace  $\mathcal{H}_0 \subseteq \mathcal{H}$ , we call a linear operator  $x : \mathcal{H}_0 \rightarrow \mathcal{H}$  *densely defined* if  $\mathcal{H}_0 \subseteq \mathcal{H}$  is dense. We denote  $\text{Dom}(x) := \mathcal{H}_0$  for the domain of  $x$ . We say that a linear operator  $y$  *extends*  $x$ , denoted  $x \subseteq y$ , if  $\text{Dom}(x) \subseteq \text{Dom}(y)$  and  $x\xi = y\xi$  for  $\xi \in \text{Dom}(x)$ . We call a densely defined linear operator  $x$  *closed* if its graph  $\mathcal{G}(x) := \{(\xi, x\xi) : \xi \in \mathcal{H}_0\}$  is a closed subspace of  $\mathcal{H} \oplus \mathcal{H}$ . We call  $x$  *preclosed* if it is densely defined and if the closure of  $\mathcal{G}(x)$  in  $\mathcal{H} \oplus \mathcal{H}$  is the graph of a linear operator called the *closure* of  $x$ . For linear operators  $x : \text{Dom}(x) \rightarrow \mathcal{H}$ ,  $y : \text{Dom}(y) \rightarrow \mathcal{H}$  we let  $x + y$  and  $xy$  be the linear operators with domain  $\text{Dom}(x + y) := \text{Dom}(x) \cap \text{Dom}(y)$  and  $\text{Dom}(xy) := \{\xi \in \text{Dom}(y) : y\xi \in \text{Dom}(x)\}$  respectively and defined in the obvious way.

A densely defined, closed linear operator  $x : \text{Dom}(x) \rightarrow \mathcal{H}$  is said to be *affiliated* with  $M$  if  $yx \subset xy$  for all  $y$  from the commutant  $M'$  of the algebra  $M$ . A linear operator  $x$  affiliated with  $M$  is called *measurable* with respect to  $M$  if  $\chi_{(\lambda, \infty)}(|x|)$  is a finite projection for some  $\lambda > 0$ . Here  $\chi_{(\lambda, \infty)}(|x|)$  is the spectral projection of  $|x|$  corresponding to the interval  $(\lambda, +\infty)$ . We denote the set of all measurable operators by  $S(M)$ . Clearly,  $M$  is a subset of  $S(M)$ . It is clear that if  $M$  is a factor of type I or III then  $S(M) = M$ .

Let  $x, y \in S(M)$ . It is well known that  $x + y$  and  $xy$  are densely-defined and preclosed operators [DPS22]. We define the *strong sum* respectively the *strong product* of  $x$  and  $y$  as the closures of these operators, which we simply also denote by  $x + y$  and  $xy$  respectively. When  $S(M)$  is equipped with the operation of strong sum, operation of strong product, and the  $*$ -operation, it becomes a unital  $*$ -algebra over  $\mathbb{C}$ . It is clear that  $M$  is a  $*$ -subalgebra of  $S(M)$ . Moreover, in the case that  $M$  is finite, every operator affiliated with  $M$  becomes measurable. In particular, the set of all affiliated operators then forms a  $*$ -algebra, which coincides with  $S(M)$ . Following [KL14; KLT20], in the case when the von Neumann algebra  $M$  is finite, we refer to the algebra  $S(M)$  as the Murray-von Neumann algebra associated with  $M$ .

Let  $M$  be semifinite and let  $\tau$  be a faithful normal semifinite trace on  $M$ . A linear operator  $x$  affiliated with  $M$  is called  $\tau$ -*measurable* with respect to  $M$  if  $\tau(\chi_{(\lambda, \infty)}(|x|)) < \infty$  for some  $\lambda > 0$ . We denote the set of all  $\tau$ -measurable operators by  $S(M, \tau)$ . The set  $S(M, \tau)$  is a  $*$ -subalgebra of  $S(M)$  that contains  $M$ . Consider the topology  $t_\tau$  of convergence in measure or *measure topology* on  $S(M, \tau)$ , which is defined by the following neighborhoods of zero:

$$N(\varepsilon, \delta) = \{x \in S(M, \tau) : \exists e \in P(M), \tau(1_M - e) \leq \delta, xe \in M, \|xe\| \leq \varepsilon\},$$

where  $\varepsilon, \delta$  are positive numbers. The algebra  $S(M, \tau)$  equipped with the measure topology is a topological  $*$ -algebra and  $F$ -space [DPS22].

A linear operator  $x$  affiliated with  $M$  is called *locally measurable* with respect to  $M$  if there exist increasing central projections  $(p_n)$  in  $P(Z(M))$  converging strongly to  $1_M$ , and such that  $xp_n \in S(M)$ . The set  $LS(M)$  of locally measurable operators forms a  $*$ -algebra with respect to the operations of a strong sum and a strong product. It is clear that if  $M$  is a factor then  $LS(M) = S(M)$ .

### 2.1.3. BIMODULES

Let  $A$  be a  $*$ -algebra. A *left Hilbert  $A$ -module*, or simply a *left  $A$ -module*, is a Hilbert space  $\mathcal{H}$  together with a left  $A$  action, i.e. a  $*$ -homomorphism  $\pi_l : A \rightarrow B(\mathcal{H})$ . For  $a \in A, \xi \in \mathcal{H}$  we simply write  $a\xi$  for  $\pi_l(a)\xi$ . A *right  $A$ -module* is a Hilbert space with a right  $A$  action, i.e. a  $*$ -homomorphism  $\pi_r : A^{\text{op}} \rightarrow B(\mathcal{H})$ . For  $a \in A, \xi \in \mathcal{H}$  we write  $\xi a$  for  $\pi_r(a^{\text{op}})\xi$ . Let  $A, B$  be  $*$ -algebras. An  $A - B$ -bimodule is a Hilbert space  $\mathcal{H}$  that is both a left  $A$ -module and a right  $B$ -module and such that  $\pi_l(A)$  and  $\pi_r(B^{\text{op}})$  commute. For  $\eta \in \mathcal{H}, a \in A, b \in B$  we can write  $\pi_l(a)\pi_r(b)\eta$  by  $a\eta b$  without ambiguity. To emphasize that  $\mathcal{H}$  is a  $A - B$  bimodule we sometimes write  ${}_A\mathcal{H}_B$  for  $\mathcal{H}$ . When  $A = B$  we simply call  $\mathcal{H}$  an  $A$  bimodule. In case  $A, B$  are also  $C^*$ -algebras we require that both actions are continuous as maps  $A \rightarrow B(\mathcal{H})$  (and therefore contractive). In case  $A, B$  are von Neumann algebra we require both actions to be normal. We refer to these bimodules as  $A - B$  bimodules and it should be clear from the context whether this is a bimodule over a  $*$ -algebra,  $C^*$ -algebra or von Neumann algebra.

Given a von Neumann algebra  $M$ , we call a Banach space  $X$  a *Banach  $M$ -bimodule* if we are given homomorphisms  $\pi_l : M \rightarrow B(X)$  and  $\pi_r : M^{\text{op}} \rightarrow B(X)$  for which  $\pi_l(M)$  and  $\pi_r(M^{\text{op}})$  commute. A linear map  $\delta$  from a von Neumann algebra  $M$  to a Banach  $M$ -bimodule  $X$  is called a *derivation* if  $\delta(ab) = \delta(a)b + a\delta(b)$  for  $a, b \in M$ . For a tracial

von Neumann algebra  $(M, \tau)$  we remark that  $L^1(M, \tau)$  is a Banach  $M$ -Bimodule with the obvious actions.

### THE TRIVIAL BIMODULE AND THE COARSE BIMODULE

Let  $M$  be a von Neumann algebra. The  $M$ - $M$ -bimodule  $L^2(M)$  with the actions  $\pi_l(x)\eta = x\eta$  and  $\pi_r(x^{\text{op}}) = Jx^*J\eta$  is called the *trivial bimodule*. Let  $M, N$  be von Neumann algebras with standard Hilbert spaces  $L^2(M)$  and  $L^2(N)$ . Then the  $M$ - $N$ -bimodule  $L^2(M) \otimes L^2(N)$  with actions  $\pi_l(x)\pi_r(y)(\eta_1 \otimes \eta_2) = (x\eta_1) \otimes (Jy^*J\eta_2)$  is called the *coarse bimodule*.

### CONTAINEMENT, WEAK-CONTAINMENT AND QUASI-CONTAINEMENT

We say that an  $A$ - $B$  bimodule  $\mathcal{H}$  is *contained* in an  $A$ - $B$  bimodule  $\mathcal{K}$  if  $\mathcal{H}$  is a Hilbert subspace of  $\mathcal{K}$  that is invariant under the actions of  $A$  and  $B$ . We say that  $\mathcal{H}$  is *quasi-contained* in  $\mathcal{K}$  if  $\mathcal{H}$  is contained in  $\oplus_{i \in I} \mathcal{K}$  for some index set  $I$  (if  $\mathcal{H}$  is separable we may choose  $I = \mathbb{N}$ ). We say that  $\mathcal{H}$  is *weakly contained* in  $\mathcal{K}$  if for every  $\epsilon > 0$ , every finite sets  $\mathcal{F} \subseteq A$ ,  $\mathcal{G} \subseteq B$  and every  $\xi \in \mathcal{H}$  there exist finitely many  $\eta_j \in \mathcal{K}$  indexed by  $j \in G$  such that for  $x \in \mathcal{F}$ ,  $y \in \mathcal{G}$ ,

$$|\langle x\xi y, \xi \rangle - \sum_{j \in G} \langle x\eta_j y, \eta_j \rangle| < \epsilon.$$

Containment implies quasi-containment which implies weak containment. Note that if  $A$  is a  $*$ -subalgebra of a von Neumann algebra  $M$  and  $K$  is an  $A$  bimodule that is quasi-contained in a  $M$ -bimodule. Then the left and right  $A$  actions on  $K$  are normal and can be extended to  $M$  so that  $K$  is a  $M$ -bimodule.

### POPA'S INTERTWINING-BY-BIMODULE TECHNIQUE

We recall the following definition from the fundamental work of [Pop06c; Pop06d]. In this section we let  $M$  be a finite von Neumann algebra.

**Definition 2.1.2** (Embedding  $A <_M B$ ). *For von Neumann subalgebras  $A \subseteq 1_A M 1_A, B \subseteq 1_B M 1_B$  we will say that  $A$  embeds in  $B$  inside  $M$  (denoted by  $A <_M B$ ) if one of the following equivalent statements holds:*

1. *There exist projections  $p \in A, q \in B$ , a normal  $*$ -homomorphism  $\theta : pAp \rightarrow qBq$  and a non-zero partial isometry  $v \in qMp$  such that  $\theta(x)v = vx$  for all  $x \in pAp$ ;*
2. *There exists no net of unitaries  $(u_i)_i$  in  $A$  such that for any  $x, y \in 1_A M 1_B$  we have that  $\|\mathbb{E}_B(x^* u_i y)\|_2 \rightarrow 0$ ;*
3. *There exists a Hilbert  $A$ - $B$  bimodule  $\mathcal{H} \subseteq L^2(M, \tau)$  such that  $\dim_B \mathcal{H} < \infty$  (see [JS97, Definition 2.2.3] for the definition of  $\dim_B \mathcal{H}$ ).*

We say that  $A$  embeds stably in  $B$  inside  $M$  (denoted by  $A <_M^s B$ ) if for any projection  $r \in A' \cap M$  we have  $Ar <_M B$ .

## 2.2. GRAPHS

We establish notation related to graphs, which will be excessively used throughout the thesis. The graphs we will consider are *simple* (i.e. undirected, no double edges, no self-loops). Formally, this means that a graph  $\Gamma$  consists of a pair  $(V, E)$  where  $V$  is a set and  $E$  is a subset of  $\{\{v, w\} : v, w \in V, v \neq w\}$ . The set  $V$  is called the *vertex set* and the set  $E$  is called the *edge set*. In practice we will identify  $\Gamma$  with the vertex set  $V$  and write  $v \in \Gamma$  to mean that  $v$  is a vertex of  $\Gamma$ . We also write  $|\Gamma|$  for the cardinality of  $V$ , and call it the *size* of the graph. We say  $\Gamma$  is *finite* when  $|\Gamma| < \infty$ . Two vertices  $v, w \in \Gamma$  are said to *share an edge* in  $\Gamma$  if  $\{v, w\} \in E$ . A graph is said to be *complete* if any two distinct vertices  $v, w \in \Gamma$  share an edge. An *induced subgraph* or simply *subgraph* of  $\Gamma$  is a graph  $\Gamma_0$  whose vertex set is a subset of the vertex set of  $\Gamma$  and that is such that  $v, w \in \Gamma_0$  share an edge in  $\Gamma_0$  if and only if  $v, w$  share an edge in  $\Gamma$ . This is denoted as  $\Gamma_0 \subseteq \Gamma$ . We call a subgraph  $\Gamma_0$  of  $\Gamma$  *strict* when  $\Gamma_0 \neq \Gamma$ , and we denote this by  $\Gamma_0 \subsetneq \Gamma$ . We will always identify subsets of the vertex set of  $\Gamma$  with their induced subgraphs. For example, if  $\Gamma_1, \Gamma_2$  are subgraphs of  $\Gamma$ , then so are  $\Gamma_1 \cup \Gamma_2$ ,  $\Gamma_1 \cap \Gamma_2$  and  $\Gamma_1 \setminus \Gamma_2$ . A complete subgraph of  $\Gamma$  is called a *clique* and the set of all cliques is denoted by  $\text{Cliq}(\Gamma)$  (this includes the empty graph). For a vertex  $v \in \Gamma$  we define the *link* of  $v$ , respectively the *star* of  $v$  as

$$\text{Link}_\Gamma(v) := \{w \in \Gamma : v \text{ and } w \text{ share an edge in } \Gamma\} \quad (2.6)$$

$$\text{Star}_\Gamma(v) := \{v\} \cup \text{Link}_\Gamma(v) \quad (2.7)$$

and consider them as a subgraph of  $\Gamma$ . More generally, the link of a subgraph  $\Lambda \subseteq \Gamma$  is defined as

$$\text{Link}_\Gamma(\Lambda) := \bigcap_{v \in \Lambda} \text{Link}_\Gamma(v)$$

with the convention  $\text{Link}_\Gamma(\emptyset) = \Gamma$ . When the graph  $\Gamma$  is fixed, and we only consider subgraphs of  $\Gamma$ , we will omit the subscript  $\Gamma$  in the notation and simply write  $\text{Link}(v)$ ,  $\text{Link}(\Lambda)$  and  $\text{Star}(v)$ . We observe for  $v \in \Gamma$  that it always holds true that  $v \in \text{Link}(\text{Link}(v))$ .

A graph  $\Gamma$  will be called *reducible* if there are disjoint non-empty subgraphs  $\Gamma_1, \Gamma_2 \subseteq \Gamma$  such that  $\Gamma = \Gamma_1 \cup \Gamma_2$  and  $\text{Link}_\Gamma(\Gamma_1) = \Gamma_2$ . The graph  $\Gamma$  will be called *irreducible* if it is not reducible. An *irreducible component* of a graph  $\Gamma$  is a non-empty subgraph  $\Lambda \subseteq \Gamma$  that is irreducible and satisfies  $\text{Link}_\Gamma(\Lambda) = \Gamma \setminus \Lambda$ . For a graph  $\Gamma$  and vertices  $u, v \in \Gamma$  a *path from  $u$  to  $w$*  is a tuple  $P = (v_0, \dots, v_n)$  of vertices  $v_0, \dots, v_n \in \Gamma$  such that  $v_{i-1}$  shares an edge with  $v_i$  for  $i = 1, \dots, n$  and such that  $v_0 = u$  and  $v_n = w$ . The number  $n$  is called the *length* of  $P$  and is denoted by  $|P|$ . A path from  $u$  to  $v$  is called a *geodesic* if it is the shortest path from  $u$  to  $v$ . If a path from  $u$  to  $w$  exists then we say that  $u$  and  $w$  are *connected by a path* and we write  $\text{Dist}_\Gamma(u, w)$  for the minimal length of a path from  $u$  to  $w$ . If such path does not exist we put  $\text{Dist}_\Gamma(u, w) = \infty$ . We define the radius of a non-empty graph  $\Gamma$  as

$$\text{Radius}(\Gamma) = \inf_{u \in \Gamma} \sup_{w \in \Gamma} \text{Dist}_\Gamma(u, w)$$

and put  $\text{Radius}(\Gamma) = 0$  when  $\Gamma$  is empty. We say that a graph  $\Gamma$  is *connected* if any two vertices  $v, w \in \Gamma$  are connected by a path. A *connected component* of a graph  $\Gamma$  is a non-empty subgraph  $\Lambda \subseteq \Gamma$  that is connected and satisfies for  $v \in \Lambda$  that  $\text{Link}_\Gamma(v) \subseteq \Lambda$ .

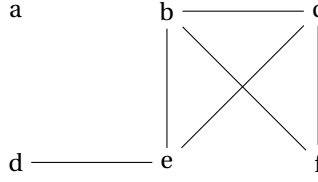


Figure 2.1: A finite graph  $\Gamma$  with vertices  $\{a, b, c, d, e, f\}$  is depicted. As an example, we have

$$\text{Link}(e) = \{b, c, d\} \quad \text{Star}(e) = \{b, c, d, e\} \quad \text{Link}(\{b, e\}) = \{c\}$$

The graph  $\Gamma$  is irreducible, not connected and has radius  $\text{Radius}(\Gamma) = \infty$ . The only irreducible component of  $\Gamma$  is  $\Gamma$  itself. The connected components of  $\Gamma$  are  $\{a\}$  and  $\{b, c, d, e, f\}$ . The cliques of  $\Gamma$  are the empty graph  $\emptyset$  (size 0), the singletons  $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}$  (size 1), and the subgraphs  $\{b, c\}, \{b, e\}, \{b, f\}, \{c, e\}, \{c, f\}, \{d, e\}$  (size 2) and  $\{b, c, e\}, \{b, c, f\}$  (size 3). A path from  $b$  to  $d$  is given by  $P = (b, c, e, d)$ . This is not a geodesic since  $Q = (b, e, d)$  is a path from  $b$  to  $d$  of length  $|Q| = 2 < 3 = |P|$ .

We say that a graph  $\Gamma$  is a *tree* if for every two vertices  $u, w \in \Gamma$  there is a unique path from  $u$  to  $w$ . We say that a graph  $\Gamma$  is a *forest* if its connected components are trees. Two graphs  $\Gamma$  and  $\Lambda$  are said to be isomorphic if there is a bijection  $\iota : \Gamma \rightarrow \Lambda$  between their vertex sets, such that  $v, w \in \Gamma$  share an edge if and only if  $\iota(v), \iota(w)$  share an edge. These definitions are illustrated in Fig. 2.1.

## 2.3. DISCRETE GROUPS

In Section 2.3.1 we discuss for discrete groups  $G$  the Cayley graph  $\text{Cayley}_S(G)$ , hyperbolicity, word lengths and other function  $\psi : G \rightarrow \mathbb{C}$ . In Section 2.3.2 we discuss the group algebra  $\mathbb{C}[G]$ , the reduced group  $C^*$ -algebra  $C_r^*(G)$ , the group von Neumann algebra  $\mathcal{L}(G)$  and the coarse bimodule  $\ell^2(G) \otimes \ell^2(G)$ . In Section 2.3.3 we introduce notation for Coxeter groups and Hecke algebras and furthermore state the definition of graph products of groups. For more background on Coxeter groups we refer to [Hum90][Dav08][Tit09].

### 2.3.1. CAYLEY GRAPHS, WORD LENGTHS AND HYPERBOLICITY

Recall that a *topological group* is a group  $G$  equipped with a topology for which the inversion map  $G \ni g \mapsto g^{-1} \in G$  and the multiplication map  $G \times G \ni (g, h) \mapsto gh \in G$  are continuous. We only consider *discrete groups*, i.e. groups equipped with the discrete topology. For a group  $G$  we always denote by  $e$  its unit element.

#### CAYLEY GRAPH AND WORD LENGTH

A group  $G$  is said to be generated by a subset  $S \subseteq G$  if  $G$  is the only subgroup of  $G$  that contains  $S$ . We say that  $G$  is finitely generated if there exists a finite set  $S$  that generates  $G$ . For a subset  $S \subseteq G$  we put  $S^{-1} = \{s^{-1} : s \in S\}$  and we define the Cayley graph as follows.

**Definition 2.3.1** (Cayley graph). *Let  $G$  be a group that is generated by a set  $S$ . Then the Cayley graph  $\text{Cayley}_S(G)$  is the simple graph with vertex set  $G$  and where distinct  $g, h \in G$  share an edge if and only if  $gh^{-1} \in S \cup S^{-1}$ .*

The Cayley graph  $\text{Cayley}_S(G)$  is connected and for a group element  $g \in G$  we define its word length as

$$|g|_S = \text{Dist}_{\text{Cayley}_S(G)}(e, g).$$

We observe that  $|g|_S$  equals the minimal integer  $n$  such that we can write  $g = g_1 \cdots g_n$  where  $g_i \in S \cup S^{-1}$  for  $1 \leq i \leq n$ .

### HYPERBOLICITY

We state the definition of hyperbolicity for connected graphs.

**Definition 2.3.2** (Hyperbolicity). *A connected graph  $\Gamma$  is called hyperbolic if there exists a  $R > 0$  satisfying the following condition: for every  $u, v, w \in \Gamma$  and every geodesic  $P_1$  from  $u$  to  $v$ , geodesic  $P_2$  from  $v$  to  $w$  and geodesic  $P_3$  from  $w$  to  $u$  we have that  $P_3 \subseteq B_R(P_1 \cup P_2)$  (here  $B_R(P_1 \cup P_2)$  denotes the open ball of radius  $R$  around the set of vertices in  $P_1 \cup P_2$ ).*

A group  $G$  generated by a finite set  $S$  is called *hyperbolic* or *word hyperbolic* if the Cayley graph  $\text{Cayley}_S(G)$  is hyperbolic. This definition is independent of the choice of the generating set  $S$ , see [BO08, Section 5.3]. We emphasize that in this thesis ‘hyperbolic’ and ‘word hyperbolic’ mean the same thing. The terminology ‘word hyperbolic’ is more common in the theory of Coxeter groups.

### FUNCTIONS ON GROUPS

A *length function*  $\psi$  on a discrete group  $G$  is a function  $\psi : G \rightarrow \mathbb{R}_{\geq 0}$  satisfying  $\psi(uv) \leq \psi(u) + \psi(v)$  for all  $u, v \in G$ . If  $G$  is generated by a finite set  $S$  then a typical length function is defined by  $\psi(w) = |w|_S$ . A function  $\psi : G \rightarrow \mathbb{R}$  is called *conditionally of negative type* if  $\psi(e) = 0$ ,  $\psi(g) = \psi(g^{-1})$ ,  $g \in G$  and for all  $n \in \mathbb{N}$  and  $g_1, \dots, g_n \in G$  and  $c_1, \dots, c_n \in \mathbb{C}$  with  $\sum_{i=1}^n c_i = 0$  we have

$$\sum_{i=1}^n \sum_{j=1}^n \overline{c_i} c_j \psi(g_j^{-1} g_i) \leq 0.$$

A function  $\psi : G \rightarrow \mathbb{R}$  is called *positive definite* if for  $n \in \mathbb{N}$  and  $g_1, \dots, g_n \in G$  and  $c_1, \dots, c_n \in \mathbb{C}$  we have

$$\sum_{i=1}^n \sum_{j=1}^n \overline{c_i} c_j \psi(g_j^{-1} g_i) \geq 0.$$

A function  $\psi : G \rightarrow \mathbb{R}$  is called *proper* if the inverse image of a compact set is compact (hence finite as  $G$  is discrete). The function  $\psi$  is called *symmetric* if  $\psi(g^{-1}) = \psi(g)$  for  $g \in G$ .

### 2.3.2. THE GROUP VON NEUMANN ALGEBRA

For a discrete group  $G$  we show the construction of the group algebra  $\mathbb{C}[G]$ , reduced group  $C^*$ -algebra  $C_r^*(G)$  and of the group von Neumann algebra  $\mathcal{L}(G)$ . We shall denote  $\ell^2(G)$  for the space of all square summable functions  $f : G \rightarrow \mathbb{C}$ . This is a Hilbert space with an orthonormal basis given by  $(\delta_t)_{t \in G}$  where  $\delta_t$  is the delta function at  $t \in G$  (i.e.  $\delta_t(s)$  equals 1 if  $s = t$  and equals 0 otherwise). For  $s \in G$  define bounded operators  $\lambda_s, \rho_s : \ell^2(G) \rightarrow \ell^2(G)$  as

$$(\lambda_s f)(t) = f(s^{-1}t)(\rho_s f)(t) = f(ts) \quad (2.8)$$

and observe that  $\lambda_s \delta_t = \delta_{st}$  and  $\rho_s \delta_t = \delta_{ts^{-1}}$  for  $s, t \in G$ . We define the left (resp. right) regular representation

$$G \rightarrow \mathcal{B}(\ell^2(G)) : s \mapsto \lambda_s, \quad (2.9)$$

$$G \rightarrow \mathcal{B}(\ell^2(G)) : s \mapsto \rho_s. \quad (2.10)$$

The *group algebra*  $\mathbb{C}[G]$  is the  $*$ -algebra generated by  $\lambda_s, s \in G$ . The *reduced group  $C^*$ -algebra*  $C_r^*(G)$  is the norm closure of  $\mathbb{C}[G]$ . The group von Neumann algebra  $\mathcal{L}(G)$  is the strong operator topology closure of  $\mathbb{C}[G]$ . The von Neumann algebra  $\mathcal{L}(G)$  is finite with normal faithful tracial state

$$\tau(x) = \langle x \delta_e, \delta_e \rangle, \quad x \in \mathcal{L}(G). \quad (2.11)$$

Furthermore, the commutant  $\mathcal{L}(G)'$  equals the von Neumann algebra generated by the set  $\{\rho_s : s \in G\}$ . We note that we have an identification as Hilbert spaces  $L^2(\mathcal{L}(G)) \simeq \ell^2(G)$  by  $x \mapsto x \delta_e$  with  $x \in \mathbb{C}[G]$ . Under this identification  $\ell^2(G)$  is the trivial bimodule with actions given by the left and right regular representations  $\lambda$  and  $\rho$ . The coarse bimodule is then given by  $\ell^2(G) \otimes \ell^2(G)$  with left and right actions given by

$$x \cdot (\xi \otimes \eta) \cdot y = (x\xi) \otimes (\eta y), \quad \xi, \eta \in \ell^2(G).$$

We simply call  $\ell^2(G) \otimes \ell^2(G)$  with these bimodule actions the coarse bimodule of  $G$ . We also summarize that

$$G \subseteq \mathbb{C}[G] \subseteq C_r^*(G) \subseteq \mathcal{L}(G) \subseteq \ell^2(G),$$

where the first inclusion is given by  $s \mapsto \lambda_s$  and the others were discussed above.

### 2.3.3. COXETER GROUPS, HECKE-ALGEBRAS AND GRAPH PRODUCTS

Let  $S$  be a (possibly infinite) set. A *Coxeter matrix* on  $S$  is a symmetric matrix  $M = (m_{s,t})_{s,t \in S}$  (indexed by  $S$ ) with  $m_{s,s} = 1$  for  $s \in S$  and  $m_{s,t} = m_{t,s} \in \{2, 3, \dots\} \cup \{\infty\}$  for  $s \neq t$ . We write  $\mathcal{W} := \langle S | M \rangle$  to denote the corresponding *Coxeter group*, which is defined by

$$\mathcal{W} = \langle S | (st)^{m_{s,t}} = e \text{ for } s, t \in S \rangle \quad (2.12)$$

that is,  $\mathcal{W}$  is the group generated by  $S$  subject to the relations  $(st)^{m_{s,t}} = e$  for  $s, t \in S$ . When  $m_{s,t} = \infty$ , we mean that no relation of the form  $(st)^k = e$  exists for  $k \geq 1$ . We call  $(\mathcal{W}, S)$  a *Coxeter system*. When such a system is fixed, we write  $|\cdot|$  for the length function  $|\cdot|_S$ . We call a Coxeter system *finite rank* if  $S$  is finite. An element  $v \in S$  is referred to as a *letter*, and an element  $\mathbf{v} \in \mathcal{W}$  is referred to as a *word*. We will say that an expression  $\mathbf{w} = \mathbf{w}_1 \mathbf{w}_2 \cdots \mathbf{w}_n$  is *reduced* if  $|\mathbf{w}| = |\mathbf{w}_1| + |\mathbf{w}_2| + \dots + |\mathbf{w}_n|$ . We will say that a word  $\mathbf{w}$  *starts with*  $\mathbf{v}$  if  $|\mathbf{w}| = |\mathbf{v}| + |\mathbf{v}^{-1}\mathbf{w}|$  and we will say that  $\mathbf{w}$  *ends with*  $\mathbf{v}$  if  $|\mathbf{w}| = |\mathbf{w}\mathbf{v}^{-1}| + |\mathbf{v}|$ .

#### HECKE ALGEBRAS

Fix a Coxeter system  $\mathcal{W} = \langle S | M \rangle$ . Let  $\mathbf{q} = (q_s)_{s \in S}$  with  $q_s > 0$  for  $s \in S$  and such that  $q_s = q_t$  whenever  $s, t \in S$  are conjugate in  $\mathcal{W}$ . In this thesis we shall call such tuples *Hecke tuples*.

Moreover, we will denote  $p_s(\mathbf{q}) = \frac{q_s - 1}{\sqrt{q_s}}$ . We can as in [Dav08, Theorem 19.1.1] define for  $s \in S$  the operators  $T_s^{(\mathbf{q})} : \ell^2(\mathcal{W}) \rightarrow \ell^2(\mathcal{W})$  given by

$$T_s^{(\mathbf{q})}(\delta_{\mathbf{w}}) = \begin{cases} \delta_{s\mathbf{w}} & |s\mathbf{w}| > |\mathbf{w}| \\ \delta_{s\mathbf{w}} + p_s(\mathbf{q})\delta_{\mathbf{w}} & |s\mathbf{w}| < |\mathbf{w}| \end{cases}.$$

For these operators we have

$$\begin{aligned} \langle T_s^{(\mathbf{q})}(\delta_{\mathbf{w}}), \delta_{\mathbf{z}} \rangle &= \langle \delta_{s\mathbf{w}}, \delta_{\mathbf{z}} \rangle + \langle p_s(\mathbf{q})\delta_{\mathbf{w}}, \delta_{\mathbf{z}} \rangle \mathbb{1}(|s\mathbf{w}| < |\mathbf{w}|) \\ &= \langle \delta_{\mathbf{w}}, \delta_{s\mathbf{z}} \rangle + \langle \delta_{\mathbf{w}}, p_s(\mathbf{q})\delta_{\mathbf{z}} \rangle \mathbb{1}(|s\mathbf{z}| < |\mathbf{z}|) \\ &= \langle \delta_{\mathbf{w}}, T_s^{(\mathbf{q})}(\delta_{\mathbf{z}}) \rangle \end{aligned}$$

that is  $(T_s^{(\mathbf{q})})^* = T_s^{(\mathbf{q})}$ . For a word  $\mathbf{w} \in \mathcal{W}$  with a reduced expression  $\mathbf{w} = w_1 \dots w_k$  we can set

$$T_{\mathbf{w}}^{(\mathbf{q})} = T_{w_1}^{(\mathbf{q})} \dots T_{w_k}^{(\mathbf{q})},$$

which is well-defined by [Dav08, Theorem 19.1.1]. We note that we have  $(T_{\mathbf{w}}^{(\mathbf{q})})^* = T_{\mathbf{w}^{-1}}^{(\mathbf{q})}$  and  $T_{\mathbf{w}}^{(\mathbf{q})}(\delta_e) = \delta_{\mathbf{w}}$ . Furthermore for  $s \in S$  and  $\mathbf{w} \in \mathcal{W}$  they satisfy the equations

$$\begin{aligned} T_s^{(\mathbf{q})} T_{\mathbf{w}}^{(\mathbf{q})} &= T_{s\mathbf{w}}^{(\mathbf{q})} + p_s(\mathbf{q}) T_{\mathbf{w}}^{(\mathbf{q})} \mathbb{1}(|s\mathbf{w}| < |\mathbf{w}|), \\ T_{\mathbf{w}}^{(\mathbf{q})} T_s^{(\mathbf{q})} &= T_{\mathbf{w}s}^{(\mathbf{q})} + p_s(\mathbf{q}) T_{\mathbf{w}}^{(\mathbf{q})} \mathbb{1}(|\mathbf{w}s| < |\mathbf{w}|). \end{aligned}$$

Note that the first equation holds by the proof of [Dav08, Theorem 19.1.1], and the second equation follows by taking the adjoint on both sides.

We will denote  $\mathbb{C}_{\mathbf{q}}[\mathcal{W}]$  for the  $*$ -algebra spanned by the linear basis  $\{\delta_{\mathbf{w}} : \mathbf{w} \in \mathcal{W}\}$ . We furthermore denote  $C_{r,\mathbf{q}}^*(\mathcal{W}) \subseteq B(\ell^2(\mathcal{W}))$  for the reduced  $C^*$ -algebra obtained by taking the norm closure of  $\mathbb{C}_{\mathbf{q}}[\mathcal{W}]$ . Finally, we define the Hecke von Neumann algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  as the strong closure of  $C_{r,\mathbf{q}}^*(\mathcal{W})$ . We equip the von Neumann algebra with the faithful finite trace  $\tau(x) = \langle x\delta_e, \delta_e \rangle$ . For  $q > 0$  we write  $\mathcal{N}_q(\mathcal{W})$  for the Hecke algebra corresponding to the tuple  $\mathbf{q} = (q_s)_{s \in S}$  with  $q_s = q$  for  $s \in S$ . We note here that when  $q$  is taken equal to 1, then  $(\mathcal{N}_q(\mathcal{W}), \tau)$  is simply the group von Neumann algebra  $\mathcal{L}(\mathcal{W})$  with canonical trace  $\tau$ .

### RIGHT-ANGLED COXETER GROUPS

A Coxeter group  $\mathcal{W}$  is called *right-angled* if it can be represented in the form  $\mathcal{W} = \langle S|M \rangle$  where  $m_{s,t} \in \{2, \infty\}$  for  $s \neq t$ . Let  $\Gamma$  be a simple graph. We will write  $\mathcal{W}_{\Gamma}$  for the right-angled Coxeter group  $\mathcal{W}_{\Gamma} := \langle S|M \rangle$  where  $S = \Gamma$  and  $M = (m_{s,t})_{s,t \in S}$  satisfies  $m_{s,t} = 1$  if  $s = t$ ,  $m_{s,t} = 2$  when  $s, t$  share an edge and  $m_{s,t} = \infty$  otherwise. By [Gre90, Theorem 4.12] every right-angled Coxeter group corresponds to a unique simple graph.

For  $\mathbf{v}, \mathbf{w} \in \mathcal{W}_{\Gamma}$ , we say that  $\mathbf{v}$  is a *subword* of  $\mathbf{w}$  if we can write  $\mathbf{w}$  in reduced form  $\mathbf{w} = \mathbf{v}_1 \mathbf{v} \mathbf{v}_2$  for some  $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{W}_{\Gamma}$ . A Coxeter word  $\mathbf{w} \in \Gamma$  with reduced expression  $\mathbf{w} = w_1 \dots w_n$  is called a *clique word* if  $w_i$  commutes with  $w_j$  for all  $1 \leq i, j \leq n$ . We can define a partial order on the set of clique words by writing  $\mathbf{v} \subseteq \mathbf{w}$  if  $\mathbf{v}$  is a subword of  $\mathbf{w}$ . We observe that a clique word  $\mathbf{w}$  in  $\mathcal{W}_{\Gamma}$  uniquely corresponds to a subgraphs of  $\Gamma$  (the graph of all letters in

**w).** For clique words  $\mathbf{w} = w_1 \cdots w_n$ ,  $\mathbf{v} = v_1 \cdots v_m$  we write  $\mathbf{w} \cap \mathbf{v}$  for the clique word with letters  $\{w_1, \dots, w_n\} \cap \{v_1, \dots, v_m\}$ . If  $\mathbf{w}$  and  $\mathbf{v}$  moreover commute, we write  $\mathbf{w} \cup \mathbf{v}$  for the clique word with letters  $\{w_1, \dots, w_n\} \cup \{v_1, \dots, v_m\}$ . For a word  $\mathbf{u} \in \mathcal{W}_\Gamma$  we write  $\mathbf{s}_l(\mathbf{u})$  for the maximal clique word that  $\mathbf{u}$  start with and write  $\mathbf{s}_r(\mathbf{u})$  for the maximal clique word that  $\mathbf{u}$  ends with.

For a subset  $S \subseteq \mathcal{W}_\Gamma$  we will write

$$\mathcal{W}_\Gamma(S) := \{\mathbf{w} \in \mathcal{W}_\Gamma : \mathbf{u}\mathbf{w} \text{ is reduced for all } \mathbf{u} \in S\} \quad (2.13)$$

$$\mathcal{W}'_\Gamma(S) := \{\mathbf{w} \in \mathcal{W}_\Gamma : \mathbf{w}\mathbf{u} \text{ is reduced for all } \mathbf{u} \in S\} \quad (2.14)$$

This notation will in particular be used when  $S \subseteq \Gamma \subseteq \mathcal{W}_\Gamma$  is a subgraph of  $\Gamma$  or when  $S = \{\mathbf{u}\}$  is a singleton. In the latter case we simply write  $\mathcal{W}_\Gamma(\mathbf{u})$  respectively  $\mathcal{W}'_\Gamma(\mathbf{u})$  for  $\mathcal{W}_\Gamma(\{\mathbf{u}\})$  respectively  $\mathcal{W}'_\Gamma(\{\mathbf{u}\})$ . Furthermore, when the graph  $\Gamma$  is fixed, we will omit the subscript  $\Gamma$  in the notation.

### GRAPH PRODUCTS OF GROUPS

Given a simple graph  $\Gamma$  and groups  $G_v$  for  $v \in \Gamma$ . Let  $G$  be the free product of the groups  $(G_v)_{v \in \Gamma}$ . Let  $H \subseteq G$  be the normal subgroup generated by

$$\{ghg^{-1}h^{-1} : g \in G_v, h \in G_w \text{ for } v, w \in \Gamma \text{ that share an edge}\}$$

The graph product  $G_\Gamma = *_v G_v$  is defined by the quotient  $G_\Gamma := G/H$ .

## 2.4. GRAPH PRODUCTS OF OPERATOR ALGEBRAS

At the end of previous section we defined graph products of groups. In this section we show the construction of the reduced graph product and the von Neumann algebraic graph products as in [CF17]. First, in Section 2.4.1 we construct the graph product of pointed Hilbert spaces, which will be needed in Section 2.4.2 where we construct the reduced graph product of unital  $C^*$ -algebras. In Section 2.4.3 we define the von Neumann algebraic graph product.

### 2.4.1. THE GRAPH PRODUCT OF POINTED HILBERT SPACES

Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $(\mathcal{H}_v, \xi_v)$  be a pair of a Hilbert space  $\mathcal{H}_v$  and a unit vector  $\xi_v \in \mathcal{H}_v$ . For  $v \in \Gamma$  we denote  $\mathring{\mathcal{H}}_v := \xi_v^\perp$  for the orthogonal complement. Furthermore, for a vector  $\eta \in \mathcal{H}_v$  we write  $\mathring{\eta} := \eta - \langle \eta, \xi \rangle \xi \in \mathring{\mathcal{H}}_v$  for the projection in  $\mathring{\mathcal{H}}_v$ . For every word  $\mathbf{w} \in \mathcal{W}_\Gamma$  with  $\mathbf{w} \neq e$  we fix a reduced representative  $(w_1, \dots, w_n)$  and define the Hilbert space

$$\mathring{\mathcal{H}}_{\mathbf{w}} := \mathring{\mathcal{H}}_{w_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{w_n}. \quad (2.15)$$

We also set

$$\mathring{\mathcal{H}}_e := \mathbb{C}\Omega \quad (2.16)$$

where the vector  $\Omega$  is called the *vacuum* vector. For  $d \geq 0$  define the Hilbert space

$$\mathcal{H}_{\Gamma, d} := \bigoplus_{\mathbf{w} \in \mathcal{W}_\Gamma, |\mathbf{w}|=d} \mathring{\mathcal{H}}_{\mathbf{w}}. \quad (2.17)$$

The *graph product* of the pointed Hilbert spaces  $(\mathcal{H}_v, \xi_v)$  is defined by

$$\mathcal{H}_\Gamma := \bigoplus_{\mathbf{w} \in \mathcal{W}_\Gamma} \mathring{\mathcal{H}}_{\mathbf{w}}. \quad (2.18)$$

and this will also be denoted by  $(\mathcal{H}_\Gamma, \Omega) = *_{v \in \Gamma} (\mathcal{H}_v, \xi_v)$ . We observe for a subgraph  $\Lambda \subseteq \Gamma$  that the Hilbert space  $\mathcal{H}_\Lambda$  is a subspace of  $\mathcal{H}_\Gamma$ .

For a subset  $S \subseteq \mathcal{W}_\Gamma$  we denote

$$\mathcal{H}_\Gamma(S) = \bigoplus_{\mathbf{w} \in \mathcal{W}_\Gamma(S)} \mathring{\mathcal{H}}_{\mathbf{w}} \quad \mathcal{H}_\Gamma^l(S) = \bigoplus_{\mathbf{w} \in \mathcal{W}_\Gamma^l(S)} \mathring{\mathcal{H}}_{\mathbf{w}} \quad (2.19)$$

This notation will in particular be used when  $S \subseteq \Gamma \subseteq \mathcal{W}_\Gamma$  is a subgraph of  $\Gamma$  or when  $S = \{\mathbf{u}\}$  is a singleton and in the latter case we simply write  $\mathcal{H}_\Gamma(\mathbf{u})$  and  $\mathcal{H}_\Gamma^l(\mathbf{u})$  respectively. Furthermore, often we will omit the subscript  $\Gamma$ , and just write  $\mathcal{H}(S)$  and  $\mathcal{H}^l(S)$ .

#### 2.4.2. THE REDUCED GRAPH PRODUCT

Let  $\Gamma$  be a simple graph with to each vertex  $v \in \Gamma$  associated a unital  $C^*$ -algebra  $A_v$  together with a state  $\varphi_v$  on  $A_v$  that is GNS-faithful (meaning the GNS-representations is faithful). For  $v \in \Gamma$  let  $\mathcal{H}_v := L^2(A_v, \varphi)$  denote the GNS-Hilbert space. As by assumption the GNS-representations  $\pi_v : A_v \rightarrow B(\mathcal{H}_v)$  are faithful we may consider  $A_v \subseteq B(\mathcal{H}_v)$  as a subalgebra. Let  $\xi_v \in \mathcal{H}_v$  be a unit vector for which  $\varphi_v(x) = \langle x\xi_v, \xi_v \rangle$  for  $x \in A$ . We will let  $(\mathcal{H}_\Gamma, \Omega) := *_{v \in \Gamma} (\mathcal{H}_v, \xi_v)$  be the graph product of the Hilbert spaces. We put  $\mathring{A}_v := \ker \varphi_v$  and for  $a \in A_v$  write  $\mathring{a} := a - \varphi_v(a)1_{A_v} \in \mathring{A}_v$  and  $\hat{a} := a\xi_v \in \mathcal{H}_v$ . We observe for  $a \in A_v$  that  $\mathring{\hat{a}} = \hat{\mathring{a}}$  and particularly that  $a \in \mathring{A}_v$  implies  $\hat{a} \in \mathring{\mathcal{H}}_v$ .

For an element  $\mathbf{w} \in \mathcal{W}_\Gamma$ ,  $\mathbf{w} \neq e$  with representative  $(w_1 \dots w_l)$  define the algebraic tensor product

$$\mathring{A}_{\mathbf{w}} := \mathring{A}_{w_1} \otimes \dots \otimes \mathring{A}_{w_l}. \quad (2.20)$$

Furthermore, define

$$\mathring{A}_e := B(\mathring{\mathcal{H}}_e). \quad (2.21)$$

Moreover, for  $d \geq 0$  we define the algebraic direct sums

$$\mathbf{A}_{\Gamma, d} := \bigoplus_{\substack{\mathbf{w} \in \mathcal{W}_\Gamma \\ |\mathbf{w}|=d}} \mathring{A}_{\mathbf{w}} \quad (2.22)$$

$$\mathbf{A}_\Gamma := \bigoplus_{\mathbf{w} \in \mathcal{W}_\Gamma} \mathring{A}_{\mathbf{w}} \quad (2.23)$$

In order to define the reduced graph product of the algebras  $(A_v, \varphi_v)$  we will define a linear map  $\lambda : \mathbf{A}_\Gamma \rightarrow B(\mathcal{H}_\Gamma)$ . To define  $\lambda$  we first define maps  $\mathcal{Q}_{(\mathbf{v}_1, \dots, \mathbf{v}_n)}$  for certain words  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathcal{W}_\Gamma$ .

## IDENTIFYING HILBERT SPACES AND OPERATOR ALGEBRAS

Let  $n \geq 1$ , and  $(\mathbf{v}_1, \dots, \mathbf{v}_n) \in \mathcal{W}_\Gamma^n$  be s.t.  $|\mathbf{v}_1 \cdots \mathbf{v}_n| = |\mathbf{v}_1| + \dots + |\mathbf{v}_n|$ . We will define linear maps  $\mathcal{Q}_{(\mathbf{v}_1, \dots, \mathbf{v}_n)}$  which will be used in defining the graph product for operator algebras. Write  $\mathcal{J}$  for the set of all indices  $1 \leq i \leq n$  s.t.  $\mathbf{v}_i \neq e$ . For  $i \in \mathcal{J}$  write  $(v_{(i,1)}, \dots, v_{(i,l_i)})$  for the representative of  $\mathbf{v}_i$ . Also, write  $(\tilde{v}_1, \dots, \tilde{v}_l)$  for the representative of  $\mathbf{v} := \mathbf{v}_1 \cdots \mathbf{v}_n$ . By the assumption it holds that  $l = \sum_{i \in \mathcal{J}} l_i$ . For convenience, we define a bijection  $\sigma$  from  $\{1, \dots, l\}$  to  $\{(i, j) | i \in \mathcal{J}, 1 \leq j \leq l_i\}$  as  $\sigma(m) = (i, j)$  where  $(i, j)$  is uniquely chosen with the property that  $m = j + \sum_{k \in I, k < i} l_k$ . Now, we have by the definitions that  $(v_{\sigma(1)}, \dots, v_{\sigma(l)}) \sim (\tilde{v}_1, \dots, \tilde{v}_l)$ . Therefore, by [CF17, Lemma 2.3] we obtain that there is a unique permutation  $\pi$  of  $\{1, \dots, l\}$  with the property that

$$(v_{\sigma(\pi(1))}, \dots, v_{\sigma(\pi(l))}) = (\tilde{v}_1, \dots, \tilde{v}_l) \quad (2.24)$$

and satisfying that if  $1 \leq i < j \leq l$  are s.t.  $v_{\sigma(i)} = v_{\sigma(j)}$ , then  $\pi(i) < \pi(j)$ .

We will now define a unitary  $\mathcal{Q}_{(\mathbf{v}_1, \dots, \mathbf{v}_n)} : \mathcal{H}_{\mathbf{v}_1} \otimes \dots \otimes \mathcal{H}_{\mathbf{v}_n} \rightarrow \mathcal{H}_{\mathbf{v}_1 \cdots \mathbf{v}_n}$  as follows. For  $i \in \mathcal{J}$  choose pure tensors  $\eta_i = \eta_{i,1} \otimes \dots \otimes \eta_{i,l_i} \in \mathcal{H}_{\mathbf{v}_i}$  and for  $1 \leq i \leq n$  with  $i \notin \mathcal{J}$  denote  $\eta_i = \Omega$ . We define

$$\mathcal{Q}_{(\mathbf{v}_1, \dots, \mathbf{v}_n)}(\eta_1 \otimes \dots \otimes \eta_n) = \begin{cases} \eta_{\sigma(\pi(1))} \otimes \dots \otimes \eta_{\sigma(\pi(l))} & \text{when } \mathcal{J} \neq \emptyset \\ \Omega & \text{when } \mathcal{J} = \emptyset \end{cases} \quad (2.25)$$

and we extend this definition linearly to a bounded map.

Similarly, we define another map  $\mathcal{Q}_{(\mathbf{v}_1, \dots, \mathbf{v}_n)} : \mathring{\mathcal{A}}_{\mathbf{v}_1} \otimes \dots \otimes \mathring{\mathcal{A}}_{\mathbf{v}_n} \rightarrow \mathring{\mathcal{A}}_{\mathbf{v}_1 \cdots \mathbf{v}_n}$ , denoted by the same symbol, as follows. For  $i \in \mathcal{J}$  choose pure tensors  $a_i = a_{i,1} \otimes \dots \otimes a_{i,l_i} \in \mathring{\mathcal{A}}_{\mathbf{v}_i}$  and for  $1 \leq i \leq n$  with  $i \notin \mathcal{J}$  denote  $a_i = \text{Id}_{\mathring{\mathcal{H}}_e}$ . We define

$$\mathcal{Q}_{(\mathbf{v}_1, \dots, \mathbf{v}_n)}(a_1 \otimes \dots \otimes a_n) = \begin{cases} a_{\sigma(\pi(1))} \otimes \dots \otimes a_{\sigma(\pi(l))} & \text{when } \mathcal{J} \neq \emptyset \\ \text{Id}_{\mathring{\mathcal{H}}_\Gamma} & \text{when } \mathcal{J} = \emptyset \end{cases} \quad (2.26)$$

and we extend this definition to a linear map.

## DEFINING THE GRAPH PRODUCT

We will for a subgraph  $\Lambda \subseteq \Gamma$  define unitaries

$$U_\Lambda : \mathcal{H}_\Lambda \otimes \mathcal{H}_\Gamma(\Lambda) \rightarrow \mathcal{H}_\Gamma \quad \text{as} \quad U_\Lambda|_{\mathring{\mathcal{H}}_u \otimes \mathring{\mathcal{H}}_w} = \mathcal{Q}_{(\mathbf{u}, \mathbf{w})} \quad \text{for } \mathbf{u} \in \mathcal{W}_\Lambda, \mathbf{w} \in \mathcal{W}_\Gamma(\Lambda) \quad (2.27)$$

$$U'_\Lambda : \mathcal{H}'_\Gamma(\Lambda) \otimes \mathcal{H}_\Lambda \rightarrow \mathcal{H}_\Gamma \quad \text{as} \quad U'_\Lambda|_{\mathring{\mathcal{H}}_u \otimes \mathring{\mathcal{H}}_w} = \mathcal{Q}_{(\mathbf{u}, \mathbf{w})} \quad \text{for } \mathbf{u} \in \mathcal{W}'_\Gamma(\Lambda), \mathbf{w} \in \mathcal{W}_\Lambda \quad (2.28)$$

and define operators  $\lambda_\Lambda : \mathcal{B}(\mathcal{H}_\Lambda) \rightarrow \mathcal{B}(\mathcal{H}_\Gamma)$  and  $\rho_\Lambda : \mathcal{B}(\mathcal{H}_\Lambda) \rightarrow \mathcal{B}(\mathcal{H}_\Gamma)$  as

$$\lambda_\Lambda(a) = U_\Lambda(a \otimes \text{Id})U_\Lambda^* \quad (2.29)$$

$$\rho_\Lambda(a) = U'_\Lambda(\text{Id} \otimes a)(U'_\Lambda)^*. \quad (2.30)$$

For  $u \in \Gamma$  we simply write  $U_u, U'_u, \lambda_u, \rho_u$  instead of  $U_{\{u\}}, U'_{\{u\}}, \lambda_{\{u\}}, \rho_{\{u\}}$  respectively. The definitions of  $U_u, U'_u$  and  $\lambda_u, \rho_u$  are the same as in [CF17] and the intuition behind these maps is as follows. The unitary  $U'_u$  represents a pure tensor  $\eta = \eta_{v_1} \otimes \dots \otimes \eta_{v_n} \in \mathring{\mathcal{H}}_{\mathbf{v}} \subseteq \mathcal{H}_\Gamma$  by an element in  $\mathcal{H}_u \otimes \mathcal{H}_\Gamma(u)$  by either shuffling the indices (when  $\mathbf{v}$  starts with  $u$ ), or

tensoring with the vector  $\xi_u$  (when  $\mathbf{v}$  does not start with  $u$ ). The operator  $\lambda_u(a)$  acts on  $\eta \in \mathcal{H}_\Gamma$  by rearranging the tensor  $\eta$  using  $U_u^*$ , acting with  $a$  on the part in  $\mathcal{H}_u$ , and subsequently using  $U_u$  to map the vector back to an element from  $\mathcal{H}_\Gamma$ .

This construction also coincides with [CKL21, Section 1.5] where the shuffling is done implicit by using an equivalence relation (called shuffle equivalence) to identify Hilbert spaces  $\mathcal{H}_{w_1} \otimes \cdots \otimes \mathcal{H}_{w_n}$  and  $\mathcal{H}_{w'_1} \otimes \cdots \otimes \mathcal{H}_{w'_n}$  whenever  $w_1 \cdots w_n = w'_1 \cdots w'_n$  are two reduced expressions for the same word. The action is then defined by  $a \cdot \eta = \hat{a} \otimes \eta + \varphi(a)\eta$  when  $\mathbf{v}$  does not start with  $u$ , and  $a \cdot \eta = (a\eta_0) \otimes \eta' + \langle a\eta_0, \xi_u \rangle \eta'$  when  $\mathbf{v}$  starts with  $u$  and  $\eta$  is shuffle equivalent to  $\eta_0 \otimes \eta' \in \mathcal{H}_u \otimes \mathcal{H}_{u^c}$ .

We define a linear map  $\lambda : \mathbf{A}_\Gamma \rightarrow \mathbf{B}(\mathcal{H}_\Gamma)$  for  $\mathbf{w} \in \mathcal{W}_\Gamma$  with representative  $(w_1, \dots, w_t)$  and for a pure tensor  $a = a_1 \otimes \cdots \otimes a_t \in \mathring{\mathbf{A}}_{\mathbf{w}}$  as

$$\lambda(a_1 \otimes \cdots \otimes a_t) = \lambda_{w_1}(a_1) \lambda_{w_2}(a_2) \dots \lambda_{w_t}(a_t) \quad (2.31)$$

and we moreover define  $\lambda(\text{Id}_{\mathcal{H}_e}) = \text{Id}_{\mathcal{H}_\Gamma}$ . We note that  $\lambda$  is injective as  $\hat{a} := \lambda(a)\Omega = \hat{a}_1 \otimes \cdots \otimes \hat{a}_n$  for  $a = a_1 \otimes \cdots \otimes a_n \in \mathring{\mathbf{A}}_{\mathbf{w}}$ . We moreover note that for words  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathcal{W}_\Gamma$  with  $|\mathbf{v}_1| + \dots + |\mathbf{v}_n| = |\mathbf{v}_1 \cdots \mathbf{v}_n|$  and elements  $a_i \in \mathring{\mathbf{A}}_{\mathbf{v}_i}$  we have for  $a = \mathcal{Q}_{(\mathbf{v}_1, \dots, \mathbf{v}_n)}(a_1 \otimes \cdots \otimes a_n)$  that  $\lambda(a) = \lambda(a_1) \dots \lambda(a_n)$ . We call an operator  $a = \lambda_{w_1}(a_1) \cdots \lambda_{w_n}(a_n)$  with  $\mathbf{w} = w_1 \cdots w_n$  reduced and  $w_i \in \mathring{A}_{w_i}$  for  $1 \leq i \leq n$  a *reduced operator*. Sometimes we leave out the embedding  $\lambda_{v_i}$  and simply write  $a = a_1 \cdots a_n$ .

We now define the *reduced graph product* as

$$A_\Gamma := \min_{v, \Gamma}^* (A_v, \varphi_v) := \overline{\lambda(\mathbf{A}_\Gamma)}^{\|\cdot\|} \quad (2.32)$$

Also, for  $d \geq 0$  we define the *homogeneous subspace of degree  $d$*  as

$$A_{\Gamma, d} := \overline{\lambda(\mathbf{A}_{\Gamma, d})}^{\|\cdot\|}. \quad (2.33)$$

Also, for  $\mathbf{v} \in \mathcal{W}_\Gamma$  we define

$$\mathring{A}_{\mathbf{v}} := \overline{\lambda(\mathring{\mathbf{A}}_{\mathbf{v}})}^{\|\cdot\|}. \quad (2.34)$$

We moreover define the graph product state  $\varphi_\Gamma$  on  $A_\Gamma$  (or simply denoted as  $\varphi$ ) by  $\varphi_\Gamma(a) = \langle a\Omega, \Omega \rangle$ . This is a faithful state on  $A_\Gamma$  which restricts to  $\varphi_v \circ \lambda_v^{-1}$  on  $\lambda_v(A_v)$ . The vertex  $C^*$ -algebras  $A_v$  are included in  $A_\Gamma$  through  $\lambda_v$  and we simply identify  $A_v$  as subalgebras of  $A_\Gamma$ . By the universal property [CF17, Proposition 3.12, Proposition 3.22] these inclusions extend to an inclusion of  $A_\Lambda \subseteq A_\Gamma$ , for  $\Lambda \subseteq \Gamma$ . This inclusion admits a unique  $\varphi_\Gamma$ -preserving conditional expectation  $\mathbb{E}_{A_\Lambda} : A_\Gamma \rightarrow A_\Lambda$  that is determined by the following formula, where  $a_1 \dots a_n$  is a reduced operator with  $a_i \in \mathring{A}_{v_i}$ ,

$$\mathbb{E}_{A_\Lambda}(a_1 \dots a_n) = \begin{cases} a_1 \dots a_n, & \forall i, v_i \in \Lambda; \\ 0, & \text{otherwise.} \end{cases} \quad (2.35)$$

We state the following result which we will often use in this thesis (for reduced amalgamated free products we refer to [VDN02, Section 3.8])

**Theorem 2.4.1** (Theorem 3.15 in [CF17]). *Let  $\Gamma$  be a graph and for  $v \in \Gamma$  let  $(A_v, \varphi_v)$  be a  $C^*$ -algebra with a GNS-faithful state  $\varphi_v$ . Let  $A_\Gamma = \ast_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  be their reduced graph product. Fix  $u \in \Gamma$ . There is a  $*$ -isomorphism from the reduced amalgamated free product*

$$\pi : A_{\text{Star}(u)} *_{A_{\text{Link}(u)}} A_{\Gamma \setminus \{u\}} \rightarrow A_\Gamma \quad (2.36)$$

*that is state-preserving and so that  $\pi|_{A_{\text{Star}(u)}}$  and  $\pi|_{A_{\Gamma \setminus \{u\}}}$  are the canonical inclusions.*

**Remark 2.4.2.** While this may be obvious, we remark that the graph product notation depends on the initial notation. For example, If we are given a simple graph  $\Lambda$  and  $C^*$ -algebras  $(B_v, \psi_v)$  we use notation like  $\mathring{B}_v, \mathring{\mathbf{B}}_v, \mathbf{B}_\Lambda, B_\Lambda$  and  $\psi$ .

### 2.4.3. THE VON NEUMANN ALGEBRAIC GRAPH PRODUCT

Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $(M_v, \varphi_v)$  be von Neumann algebras with normal faithful states  $\varphi_v$ . We define the *von Neumann algebraic graph product* as the closure in the strong operator topology of the reduced  $C^*$ -algebraic graph product, i.e.

$$M_\Gamma := \ast_{v \in \Gamma} (M_v, \varphi_v) := \overline{\lambda(M_\Gamma)}^{SOT}. \quad (2.37)$$

Again we define a faithful state  $\varphi$  on  $M_\Gamma$  by  $\varphi(a) = \langle a\Omega, \Omega \rangle$  which is normal in this case. We also define the *homogeneous subspace of degree  $d$*  as

$$M_{\Gamma, d} := \overline{\lambda(M_{\Gamma, d})}^{SOT} \quad (2.38)$$

Also, for  $\mathbf{v} \in \mathcal{W}_\Gamma$  we define

$$\mathring{M}_{\mathbf{v}} := \overline{\lambda(\mathring{\mathbf{M}}_{\mathbf{v}})}^{SOT}. \quad (2.39)$$

We note that the conditional expectations from (2.35) extend to normal conditional expectations  $\mathbb{E}_{M_\Lambda} : M_\Gamma \rightarrow M_\Lambda$ . We also have the following amalgamated free product decomposition.

**Theorem 2.4.3** (Theorem 3.26 in [CF17]). *Let  $\Gamma$  be a graph and for  $v \in \Gamma$  let  $(M_v, \varphi_v)$  be a von Neumann algebra with a normal faithful state  $\varphi_v$ . Let  $M_\Gamma = \ast_{v \in \Gamma} (M_v, \varphi_v)$  be their von Neumann algebraic graph product. Fix  $u \in \Gamma$ . There is a  $*$ -isomorphism from the von Neumann algebraic amalgamated free product*

$$\pi : M_{\text{Star}(u)} *_{M_{\text{Link}(u)}} M_{\Gamma \setminus \{u\}} \rightarrow M_\Gamma \quad (2.40)$$

*that is state-preserving and so that  $\pi|_{M_{\text{Star}(u)}}$  and  $\pi|_{M_{\Gamma \setminus \{u\}}}$  are the canonical inclusions.*

## 2.5. PROPERTIES FOR GROUPS AND OPERATOR ALGEBRAS

We recall some approximation properties, rigidity properties and indecomposability properties and collect some results that explain how these properties are related. For a more detailed exposition we refer to [BO08] and [AP17].

### 2.5.1. APPROXIMATION PROPERTIES

We recap several approximation properties.

### NUCLEAR MAPS AND EXACTNESS

We state the following definitions.

**Definition 2.5.1** (Nuclear maps). *A map  $\theta : A \rightarrow B$  between  $C^*$ -algebras is called nuclear if there are contractive, completely positive maps  $\varphi_n : A \rightarrow \text{Mat}_{k_n}(\mathbb{C})$  and  $\psi_n : \text{Mat}_{k_n}(\mathbb{C}) \rightarrow B$  such that  $\varphi_n \circ \psi_n \rightarrow \theta$  pointwise in the norm topology, i.e.*

$$\|\varphi_n \circ \psi_n(a) - \theta(a)\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

for all  $a \in A$ .

**Definition 2.5.2** (Weakly nuclear maps). *A map  $\theta : M \rightarrow N$  between von Neumann algebras is called weakly nuclear if there are contractive, completely positive maps  $\varphi_n : M \rightarrow \text{Mat}_{k_n}(\mathbb{C})$  and  $\psi_n : \text{Mat}_{k_n}(\mathbb{C}) \rightarrow N$  such that  $\varphi_n \circ \psi_n \rightarrow \theta$  pointwise in the  $\sigma$ -weak topology, i.e.*

$$\varphi(\varphi_n \circ \psi_n(a) - \theta(a)) \rightarrow 0 \text{ as } n \rightarrow \infty$$

for all  $a \in A$  and all normal functionals  $\varphi \in N_*$

**Definition 2.5.3** (Exactness). *A  $C^*$ -algebra  $A$  is exact if there exists a faithful representation  $\pi : A \rightarrow B(\mathcal{H})$  such that  $\pi$  is nuclear. A discrete group  $G$  is exact if  $C_r^*(G)$  is exact.*

### AMENABILITY AND RELATIVE AMENABILITY

We state the notion of amenability for discrete groups, the notion of nuclearity for  $C^*$ -algebras, and the notion of semidiscreteness for von Neumann algebras.

**Definition 2.5.4** (Amenability). *A discrete group  $G$  is called amenable if there exists a net  $(m_k)_k$  of finitely supported, positive definite functions  $m_k : G \rightarrow \mathbb{C}$  that converge pointwise to the constant function  $1_G$ .*

**Definition 2.5.5** (Nuclearity). *A  $C^*$ -algebra  $A$  is called nuclear if the map  $\text{Id}_A : A \rightarrow A$  is nuclear.*

**Definition 2.5.6** (Semidiscreteness). *A von Neumann algebra  $A$  is called semidiscrete if the map  $\text{Id}_A : A \rightarrow A$  is weakly nuclear.*

We state the following result which relates these notions.

**Proposition 2.5.7** (Theorem 2.6.8 in [BO08]). *Let  $G$  be a discrete group. Then  $G$  is amenable if and only if  $C_r^*(G)$  is nuclear if and only if  $\mathcal{L}(G)$  is semidiscrete.*

For von Neumann algebras the notion of semidiscreteness agrees with another property called *injectivity*, see [BO08, Theorem 9.3.3]. Injectivity was introduced in [Loe74] where it was moreover shown to be equivalent to the *extension property* from [HT67].

**Definition 2.5.8** (Extension property). *A von Neumann algebra  $M \subseteq B(\mathcal{H})$  satisfies the extension property if there exists a Banach space projection  $P : B(\mathcal{H}) \rightarrow M$  of norm 1.*

We state a definition and a result of due to Murray and von Neumann.

**Definition 2.5.9** (Hyperfiniteness). *A von Neumann algebra  $M$  is called hyperfinite if there exists a chain  $A_1 \subseteq A_2 \subseteq \dots \subseteq M$  of finite-dimensional  $*$ -algebras  $A_n$  such that  $M$  is the von Neumann algebra generated by  $\bigcup_{n \geq 1} A_n$ , i.e.  $M = (\bigcup_{n \geq 1} A_n)''$ .*

**Proposition 2.5.10** ([MN43]). *All hyperfinite  $II_1$ -factors are isomorphic.*

In the fundamental work of [Con76] Connes showed for von Neumann algebras  $M$  acting on separable von Neumann algebras that the notion of injectivity coincides with hyperfiniteness. Furthermore, in [JKR72, Corollary 6.4] it was shown that all hyperfinite von Neumann algebras satisfy a certain cohomological property, called *amenability*. Connes moreover showed in [Con78] that amenability implies injectivity. Thus all these notions agree when  $M$  has separable predual. Nowadays, the notions of semidiscreteness, injectivity and the extension property are referred to as amenability. Furthermore, in [OP10a] Ozawa and Popa introduced the following notion of *relative amenability*.

**Definition 2.5.11.** *Let  $(M, \tau)$  be a tracial von Neumann algebra and let  $P \subseteq {}_{1_P}M1_P, Q \subseteq M$  be von Neumann subalgebras. Say that  $P$  is amenable relative to  $Q$  inside  $M$  if there exists a  $P$ -central positive functional on  ${}_{1_P}\langle M, e_Q \rangle 1_P$  that restricts to the trace  $\tau$  on  ${}_{1_P}M1_P$ .*

We also remark by [OP10a, Proposition 2.4] that if  $Q$  is hyperfinite and  $P$  is amenable relative to  $Q$  inside  $M$ , then also  $P$  is hyperfinite.

#### WEAK AMENABILITY AND THE CBAP/CCAP AND THE WEAK-\* CBAP/WEAK-\* CCAP

In the following definition we will, for a function  $m : G \rightarrow \mathbb{C}$  on a discrete group  $G$ , denote by  $T_m : C_r^*(G) \rightarrow C_r^*(G)$  the Fourier multiplier given by  $T_m \lambda_g = m(g) \lambda_g$  (whenever  $T_m$  is well-defined).

**Definition 2.5.12** (Weak amenability). *A discrete group  $G$  is said to be weakly amenable with constant  $\Lambda < \infty$  if there is a net  $(m_k)_k$  of finitely supported functions  $m_k : G \rightarrow \mathbb{C}$  that converge pointwise to the constant function  $1_G$  and satisfy  $\sup_k \|T_{m_k}\|_{cb} \leq \Lambda$ . The Cowling-Haagerup constant  $\Lambda_{cb}(G)$  is defined as the infimum of all constants  $\Lambda$  for which such net exists.*

We state the definitions of the CBAP (completely bounded approximation property) and the CCAP (completely contractive approximation property) for  $C^*$ -algebras as well as their analogues for von Neumann algebras

**Definition 2.5.13** (CBAP and CCAP). *A  $C^*$ -algebra  $A$  has the CBAP with constant  $\Lambda < \infty$  if there exists a net  $(\theta_i)_i$  of finite rank maps  $\theta_i : A \rightarrow A$  that converge pointwise to the identity operator  $\text{Id}_A$  in norm (i.e.  $\|\theta_i(a) - a\| \rightarrow 0$  for  $a \in A$ ) and such that  $\sup_i \|\theta_i\|_{cb} \leq \Lambda$ . The Cowling-Haagerup constant  $\Lambda_{cb}(A)$  is defined as the infimum of all constants  $\Lambda$  for which there exists such net. If  $\Lambda_{cb}(A) = 1$  we say that  $A$  has the CCAP.*

**Definition 2.5.14** (weak-\* CBAP and weak-\* CCAP). *A von Neumann algebra  $M$  has the weak-\* CBAP with constant  $\Lambda < \infty$  if there exists a net  $(\theta_i)_i$  of normal, finite rank maps  $\theta_i : M \rightarrow M$  that converge pointwise to the identity operator  $\text{Id}_M$  in the  $\sigma$ -weak topology (i.e.  $\varphi(\theta_i(a) - a) \rightarrow 0$  for  $a \in M$  and normal  $\varphi \in M_*$ ) and such that  $\sup_i \|\theta_i\|_{cb} \leq \Lambda$ . The Cowling-Haagerup constant  $\Lambda_{cb}(M)$  is defined as the infimum of all constants  $\Lambda$  for which there exists such net. If  $\Lambda_{cb}(M) = 1$  we say that  $M$  has the weak-\* CCAP.*

We state the following result which relates these notions.

**Proposition 2.5.15** (Theorem 12.3.8 in [BO08]). *Let  $G$  be a discrete group. Then*

$$\Lambda_{\text{cb}}(G) = \Lambda_{\text{cb}}(C_r^*(G)) = \Lambda_{\text{cb}}(\mathcal{L}(G))$$

### THE HAAGERUP PROPERTY AND QUANTUM MARKOV SEMI-GROUPS

We state the definition of the Haagerup property for groups.

**Definition 2.5.16.** *A discrete group  $G$  has the Haagerup property if there exists a net  $(\varphi_i)_i$  of positive definite functions  $\varphi_i : G \rightarrow \mathbb{C}$  that vanish at infinity and converge pointwise to  $1_G$ .*

We state the definition of the Haagerup property for a finite von Neumann algebra  $(M, \tau)$  and remark by [Jol02, Proposition 2.4] that this is independent of the trace  $\tau$ .

**Definition 2.5.17.** *A finite von Neumann algebra  $(M, \tau)$  has the Haagerup property if there exists a net  $(\Phi_i)_i$  of normal, completely positive maps  $\Phi_i : M \rightarrow M$  such that for  $x \in M$  we have  $\|\Phi_i(x) - x\|_2 \rightarrow 0$  as  $i \rightarrow \infty$  and so that  $\tau \circ \Psi \leq \tau$ .*

We state the definition of a quantum Markov semigroup and state a result which relates it to the Haagerup property.

**Definition 2.5.18.** *A quantum Markov semigroup (QMS) on a finite von Neumann algebra  $(M, \tau)$  is a family  $(\Phi_t)_{t \geq 0}$  of normal, trace preserving u.c.p maps  $\Phi_t : M \rightarrow M$  such that*

1. *The family  $(\Phi_t)_{t \geq 0}$  forms a semigroup, i.e.  $\Phi_{t+s} = \Phi_t \circ \Phi_s$  for  $t, s \geq 0$  and  $\Phi_0 = \text{Id}_M$ .*
2. *For  $x \in M$  the map  $t \mapsto \Phi_t(x)$  is strongly continuous.*
3. *For  $t \geq 0$  and  $x, y \in M$  we have  $\tau(\Phi_t(x)y) = \tau(x\Phi_t(y))$  (symmetric).*

We stress that we assume the QMS to be symmetric, so QMS always means symmetric QMS.

**Proposition 2.5.19** (Theorem 1 in [JM04]). *A finite von Neumann algebra  $(M, \tau)$  with separable predual has the Haagerup property if and only if there exists a QMS  $(\Phi_t)_{t \geq 0}$  on  $M$  such that for  $t > 0$  the maps  $\Phi_t : M \rightarrow M$  extend to compact operators on  $L^2(M, \tau)$ .*

### 2.5.2. RIGIDY PROPERTIES OF VON NEUMANN ALGEBRAS

We recall versions of the Akemann-Ostrand property and recall the notions of Cartan subalgebras, primeness, solidity and strong solidity and how they are related.

#### AKEMANN-OSTRAND PROPERTIES (AO), $(\text{AO})^+$ AND STRONG (AO)

We state the definition of versions of the Akemann-Ostrand property.

**Definition 2.5.20** (Property (AO) [AO75] and  $(\text{AO})^+$  [Iso15a]). *A finite von Neumann algebra  $M$  possesses the Akemann Ostrand property (AO) if there are  $\sigma$ -weakly dense unital  $C^*$ -subalgebra  $A, B \subseteq M$  such that*

1.  $A$  is locally reflexive [BO08, Definition 9.1.2]

2. There exists a u.c.p map

$$\theta : A \otimes B^{\text{op}} \rightarrow B(L^2(M))$$

such that  $\theta(a \otimes b^{\text{op}}) - ab^{\text{op}}$  is compact for all  $a \in A, b \in A^{\text{op}}$ .

Furthermore, we say that  $M$  possesses the property  $(AO)^+$  if the  $C^*$ -algebras  $A$  and  $B$  can be chosen equal.

Local reflexivity is a certain approximation property satisfied by many  $C^*$ -algebras, including all exact  $C^*$ -algebras [BO08, Corollary 9.4.1]. As we will not deal with local reflexivity directly, we leave out its definition. We note for an exact group  $G$  that  $\mathcal{L}(G)$  has  $(AO)$  if and only if it has  $(AO)^+$ , see [DP23, Corollary 7.18].

We state the definition of strong  $(AO)$ .

**Definition 2.5.21** (Strong property  $(AO)$ , see [HI17]). *Let  $M$  be a von Neumann algebra with standard form  $(M, L^2(M), J, L^2(M)^+)$ . We say that  $M$  has strong property  $(AO)$  if there exist unital  $C^*$ -subalgebras  $A \subseteq M$  and  $C \subseteq B(L^2(M))$  such that:*

- $A$  is  $\sigma$ -weakly dense in  $M$ ,
- $C$  is nuclear and contains  $A$ ,
- The commutators  $[C, JAJ] = \{[c, JaJ] \mid c \in C, a \in A\}$  are contained in the space of compact operators  $K(L^2(M))$ .

We remark by [HI17, Remark 2.7] for general von Neumann algebras that strong  $(AO)$  implies  $(AO)$ . Moreover, under some extra conditions strong  $(AO)$  also implies  $(AO)^+$ . Furthermore, we note by [Iso15b, Lemma 3.1.4] that the von Neumann algebra  $\mathcal{L}(G)$  has strong  $(AO)$  for any hyperbolic discrete group  $G$ .

#### CARTAN SUBALGEBRAS, PRIMENESS, SOLIDITY AND STRONG SOLIDITY

We recall the following definition.

**Definition 2.5.22** (MASA). *A maximal abelian subalgebra (MASA) in a von Neumann algebra  $M$  is von Neumann subalgebra  $A \subseteq M$  that satisfies  $A' \cap M = A$ .*

We remark that every von Neumann algebra  $M$  possesses a MASA (this can be shown using Zorn's Lemma).

**Definition 2.5.23** (Cartan subalgebra). *A Cartan subalgebra  $A$  of a von Neumann algebra  $M$  is a MASA in  $M$  for which the set of normalizers*

$$\text{Nor}_M(A) = \{u \in U(M) : uAu^* = A\}$$

*generate  $M$  as a von Neumann algebra, i.e.  $\text{Nor}_M(A)'' = M$ .*

Without providing the definition of the group measure space  $M := L^\infty(0, 1) \rtimes_\alpha G$ , we remark that for a countable discrete group  $G$  and a free, ergodic action  $\alpha$  the von Neumann algebra  $M$  possesses  $L^\infty(0, 1)$  as a Cartan subalgebra.

**Definition 2.5.24** (Primeness). *A  $II_1$ -factor  $M$  is called prime if it does not decompose as a tensor product  $M = M_1 \bar{\otimes} M_2$  with both  $M_1$  and  $M_2$  diffuse.*

We now state the definitions of solidity and strong solidity that were introduced by Ozawa [Oza04] and Ozawa and Popa [OP10a].

**Definition 2.5.25** (Solidity). *A von Neumann algebra  $M$  is called solid if for every diffuse von Neumann subalgebra  $A \subseteq M$  its relative commutant  $A' \cap M$  is amenable.*

**Definition 2.5.26** (Strong solidity). *A von Neumann algebra  $M$  is called strongly solid if for every diffuse amenable subalgebra  $A \subseteq M$  the set of normalizers  $\text{Nor}_M(A)$  generates a von Neumann algebra that is amenable again.*

The following result provides sufficient conditions to be strongly solid.

**Proposition 2.5.27** (Theorem A in [Iso15a]). *Let  $M$  be a  $II_1$ -factor with separable predual. If  $M$  has condition  $(AO)^+$  and has the weak-\* CBAP, then  $M$  is strongly solid.*

In Proposition 2.5.29 we state some direct implications of solidity/strong solidity. For the interested reader we included a short proof, that uses the following lemma.

**Lemma 2.5.28.** *If  $A$  is a MASA in a diffuse von Neumann algebra  $M$ , then  $A$  is also diffuse.*

*Proof.* Let  $A$  be a MASA in a diffuse von Neumann algebra  $M$ . Suppose  $A$  is not diffuse and let  $p \in A$  be a minimal projection (in  $A$ ). Since  $M$  is diffuse there is a projection  $q \in M$  satisfying  $0 < q < p$ . Now, for  $a \in A$  we have  $qa = qpa = qpap$  since  $q \leq p$  and  $p \in A = A' \cap M$ . Now since  $p$  is minimal in  $A$  we obtain  $pAp = \mathbb{C}p$ . Hence,  $qa = aq$ , i.e.  $q \in A' \cap M = A$ . This contradicts that  $p$  is minimal in  $A$ . We conclude that  $A$  is diffuse.  $\square$

**Proposition 2.5.29.** *A solid factor is amenable or prime. A diffuse, strongly solid von Neumann algebra is amenable or does not possess a Cartan subalgebra. Furthermore, every strongly solid von Neumann algebra is solid.*

*Proof.* Suppose  $M$  is a solid factor. Suppose  $M$  is not prime, we show it is amenable. Indeed, we can write  $M = M_1 \bar{\otimes} M_2$  for some diffuse  $M_1, M_2$ . Then since  $M$  is solid, the relative commutants  $M_2 = M'_1 \cap M$  and  $M_1 = M'_2 \cap M$  are amenable. Hence,  $M = M_1 \bar{\otimes} M_2$  is amenable.

Now suppose  $M$  is diffuse, strongly solid and non-amenable. We show it does not possess a Cartan subalgebra. Indeed, let  $A \subseteq M$  be a MASA in  $M$ . Then  $A$  is diffuse by Lemma 2.5.28. Furthermore,  $A$  is commutative, hence amenable. Thus,  $\text{Nor}_M(A)''$  is again amenable, since  $M$  is strongly solid. Therefore, since  $M$  is non-amenable we have  $\text{Nor}_M(A)'' \neq M$ . Thus  $A$  can not be a Cartan subalgebra. This shows  $M$  does not possess any Cartan subalgebra.

Let  $M$  be a strongly solid von Neumann algebra. We show  $M$  is solid. Indeed, let  $A \subseteq M$  be a diffuse von Neumann subalgebra. We let  $B \subseteq A$  be a MASA in  $A$ . Note that this MASA always exists and is amenable (since it is commutative). Furthermore, note that  $B$  is diffuse by Lemma 2.5.28 since  $A$  is diffuse. Then by strong solidity of  $M$  we have that  $\text{Nor}_M(B)''$  is amenable as well. Observe that  $A' \cap M \subseteq B' \cap M \subseteq \text{Nor}_M(B)''$ . Thus  $A' \cap M$  is amenable. Thus  $M$  is solid.  $\square$

# 3

## CALCULATIONS IN GRAPH PRODUCTS

In this chapter we will not prove any main results, but instead preform some calculations in graph products which will be used in Chapter 6 and also in a few parts of Chapter 5. These calculations involve the annihilation, diagonal and creation operators which were considered in [CKL21]. We introduce new notation and prove some additional results.

This chapter is based on (a small part of) the paper:

- **Matthijs Borst**, *The CCAP for graph products of operator algebras*, [Journal of Functional Analysis](#) 286.8 (2024) 110350.

### 3.1. CREATION, ANNIHILATION AND DIAGONAL OPERATORS

Let  $\Gamma$  be a finite graph and for  $v \in \Gamma$  let  $(A_v, \varphi_v)$  be a  $C^*$ -algebra equipped with a GNS-faithful state. Let  $\mathcal{H}_v := L^2(A_v, \varphi_v)$  and consider  $A_v \subseteq B(\mathcal{H}_v)$ . Let  $A_\Gamma = \ast_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  be the reduced graph product. For  $v \in \Gamma$  denote  $P_v \in B(\mathcal{H}_\Gamma)$  for the projection on the complement of  $\mathcal{H}(v)$ . We make the following definitions.

**Definition 3.1.1.** *We define the annihilation operator  $\lambda_{ann} : A_\Gamma \rightarrow B(\mathcal{H}_\Gamma)$ , the diagonal operator  $\lambda_{dia} : A_\Gamma \rightarrow B(\mathcal{H}_\Gamma)$  and the creation operator  $\lambda_{cre} : A_\Gamma \rightarrow B(\mathcal{H}_\Gamma)$  for a pure tensor  $a = a_1 \otimes \cdots \otimes a_n \in \mathring{A}_{\mathbf{w}} = \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_n}$  with  $\mathbf{w} \in \mathcal{W}_\Gamma$ ,  $\mathbf{w} \neq e$  as*

$$\lambda_{ann}(a_1 \otimes \cdots \otimes a_n) = (P_{w_1}^\perp \lambda(a_1) P_{w_1}) (P_{w_2}^\perp \lambda(a_2) P_{w_2}) \cdots (P_{w_n}^\perp \lambda(a_n) P_{w_n}) \quad (3.1)$$

$$\lambda_{dia}(a_1 \otimes \cdots \otimes a_n) = (P_{w_1} \lambda(a_1) P_{w_1}) (P_{w_2} \lambda(a_2) P_{w_2}) \cdots (P_{w_n} \lambda(a_n) P_{w_n}) \quad (3.2)$$

$$\lambda_{cre}(a_1 \otimes \cdots \otimes a_n) = (P_{w_1} \lambda(a_1) P_{w_1}^\perp) (P_{w_2} \lambda(a_2) P_{w_2}^\perp) \cdots (P_{w_n} \lambda(a_n) P_{w_n}^\perp). \quad (3.3)$$

Furthermore, we put  $\lambda_{ann}(\text{Id}_{\mathcal{H}_e}) = \lambda_{dia}(\text{Id}_{\mathcal{H}_e}) = \lambda_{cre}(\text{Id}_{\mathcal{H}_e}) = \text{Id}_{\mathcal{H}_\Gamma}$  and we extend these maps linearly.

**Lemma 3.1.2.** *Let  $\Gamma$  be a simple graph. Let  $w \in \Gamma$  and let  $\mathbf{v} \in \mathcal{W}_\Gamma$ . Let  $b \in \mathring{\mathbf{A}}_w$  and let  $\eta \in \mathring{\mathcal{H}}_{\mathbf{v}}$ . Then  $\lambda_{ann}(b)\eta \in \mathring{\mathcal{H}}_{w\mathbf{v}}$ ,  $\lambda_{dia}(b)\eta \in \mathring{\mathcal{H}}_{\mathbf{v}}$  and  $\lambda_{cre}(b)\eta \in \mathring{\mathcal{H}}_{w\mathbf{v}}$ . Furthermore,*

1. *If  $\lambda_{ann}(b)\eta$  is non-zero then  $\mathbf{v}$  starts with  $w$ .*
2. *If  $\lambda_{dia}(b)\eta$  is non-zero then  $\mathbf{v}$  starts with  $w$ .*
3. *If  $\lambda_{cre}(b)\eta$  is non-zero then  $\mathbf{v}$  does not start with  $w$ .*

Moreover, if  $\eta$  is a pure tensor, then so are  $\lambda_{ann}(b)\eta$ ,  $\lambda_{dia}(b)\eta$  and  $\lambda_{cre}(b)\eta$ .

*Proof.* Let  $v, \mathbf{w}, b$  and  $\eta$  be given. Note that  $P_w\eta = 0$  when  $\mathbf{v}$  does not start with  $w$  and that  $P_w^\perp\eta = 0$  when  $\mathbf{v}$  starts with  $w$ . This shows  $\lambda_{ann}(b)\eta = \lambda_{dia}(b)\eta = 0$  when  $\mathbf{v}$  does not start with  $w$  and that  $\lambda_{cre}(b)\eta = 0$  when  $\mathbf{v}$  starts with  $v$ . This already shows (1), (2) and (3). To prove the other statements we may assume that  $\eta$  is a pure tensor. We show that  $\lambda_{ann}(b)\eta \in \mathring{\mathcal{H}}_{w\mathbf{v}}$ . We may assume that  $\mathbf{v}$  starts with  $w$  (since  $0 \in \mathring{\mathcal{H}}_{w\mathbf{v}}$ ). Thus write  $\eta = \mathcal{Q}_{(w,w\mathbf{v})}(\eta_1 \otimes \eta_2)$  for some  $\eta_1 \in \mathring{\mathcal{H}}_w$  and  $\eta_2 \in \mathring{\mathcal{H}}_{w\mathbf{v}}$ . Then

$$\lambda_{ann}(b)\eta = P_w\lambda(b)\eta = P_w(\mathcal{Q}_{(w,w\mathbf{v})}((b\mathring{\eta}_1) \otimes \eta_2) + \langle b\Omega, \eta_1 \rangle \eta_2) = \langle b\Omega, \eta_1 \rangle \eta_2 \in \mathring{\mathcal{H}}_{w\mathbf{v}}.$$

We now show that  $\lambda_{dia}(b)\eta \in \mathring{\mathcal{H}}_{\mathbf{v}}$ . Again we may assume that  $\mathbf{v}$  starts with  $w$  (since  $0 \in \mathring{\mathcal{H}}_{\mathbf{v}}$ ). Thus write  $\eta = \mathcal{Q}_{(w,w\mathbf{v})}(\eta_1 \otimes \eta_2)$  for some  $\eta_1 \in \mathring{\mathcal{H}}_w$  and  $\eta_2 \in \mathring{\mathcal{H}}_{w\mathbf{v}}$ . Then

$$\lambda_{dia}(b)\eta = P_w^\perp\lambda(b)\eta = P_w^\perp(\mathcal{Q}_{(w,w\mathbf{v})}((b\mathring{\eta}_1) \otimes \eta_2) + \langle b\Omega, \eta_1 \rangle \eta_2) = \mathcal{Q}_{(w,w\mathbf{v})}((b\mathring{\eta}_1) \otimes \eta_2) \in \mathring{\mathcal{H}}_{\mathbf{v}}.$$

We now show that  $\lambda_{cre}(b)\eta \in \mathring{\mathcal{H}}_{w\mathbf{v}}$ . This time we may assume that  $\mathbf{v}$  does not start with  $w$  (since  $0 \in \mathring{\mathcal{H}}_{w\mathbf{v}}$ ). Then

$$\lambda_{dia}(b)\eta = P_w\lambda(b)\eta = P_w^\perp\mathcal{Q}_{(w,\mathbf{v})}(b \otimes \eta_2) = \mathcal{Q}_{(w,\mathbf{v})}(b \otimes \eta_2) \in \mathring{\mathcal{H}}_{w\mathbf{v}}.$$

The final remark now also follows directly from these calculations.  $\square$

**Definition 3.1.3.** *Let  $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{W}_\Gamma^3$  be s.t.  $\mathbf{w} := \mathbf{w}_1\mathbf{w}_2\mathbf{w}_3$  is a reduced expression. We define a linear map  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)} : \mathbf{A}_\Gamma \rightarrow \mathbf{B}(\mathring{\mathcal{H}}_\Gamma)$  as follows. For a pure tensor  $a \in \mathring{\mathbf{A}}_{\mathbf{w}}$ , there is a unique tensor  $a_1 \otimes a_2 \otimes a_3 \in \mathring{\mathbf{A}}_{\mathbf{w}_1} \otimes \mathring{\mathbf{A}}_{\mathbf{w}_2} \otimes \mathring{\mathbf{A}}_{\mathbf{w}_3}$  s.t.  $a = \mathcal{Q}_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a_1 \otimes a_2 \otimes a_3)$ . We then define*

$$\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a) = \lambda_{cre}(a_1)\lambda_{dia}(a_2)\lambda_{ann}(a_3) \quad (3.4)$$

Furthermore, we define  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a) = 0$  for  $a \in \mathring{\mathbf{A}}_{\mathbf{w}'}$  with  $\mathbf{w}' \neq \mathbf{w}_1\mathbf{w}_2\mathbf{w}_3$ .

The operator  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a)$  must be seen as the part of  $\lambda(a)$  that acts on a pure tensor  $\eta \in \mathring{\mathcal{H}}_{\mathbf{v}}$  precisely by annihilating the  $\mathbf{w}_3$ -part, diagonally acting on a  $\mathbf{w}_2$ -part, and creating a  $\mathbf{w}_1$ -part. We prove the following lemma.

**Lemma 3.1.4.** *Let  $\mathbf{w}, \mathbf{v} \in \mathcal{W}_\Gamma$ , let  $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{W}_\Gamma^3$  with  $\mathbf{w}_1\mathbf{w}_2\mathbf{w}_3$  reduced, let  $a \in \mathring{\mathbf{A}}_{\mathbf{w}}$  and  $\eta \in \mathring{\mathcal{H}}_{\mathbf{v}}$ . Then  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a)\eta \in \mathring{\mathcal{H}}_{\mathbf{u}}$  where  $\mathbf{u} = \mathbf{w}_1\mathbf{w}_3\mathbf{v}$ . Moreover, if  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a)\eta$  is non-zero, we have that  $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2\mathbf{w}_3$ , that  $\mathbf{w}_2$  is a clique word and that  $\mathbf{v}$  and  $\mathbf{u}$  start with  $\mathbf{w}_3^{-1}\mathbf{w}_2$  and  $\mathbf{w}_1\mathbf{w}_2$  respectively. Moreover, if  $a$  and  $\eta$  are pure tensors, then so is  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a)\eta$ .*

*Proof.* Let  $\mathbf{w}, \mathbf{v}, (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3), a$  and  $\eta$  be a stated. It follows directly from Lemma 3.1.2 that  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a)\eta \in \dot{\mathcal{H}}_{\mathbf{u}}$  where  $\mathbf{u} := \mathbf{w}_1\mathbf{w}_2\mathbf{v}$ . Suppose  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a)\eta$  is non-zero. By Definition 3.1.3 we have that  $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2\mathbf{w}_3$ . We may assume that  $a$  is of the form  $a = a_1a_2a_3$  with  $a_i \in \dot{\mathbf{A}}_{\mathbf{w}_i}$ . Then Lemma 3.1.2(1) asserts that  $\eta_1 := \lambda_{(e, e, \mathbf{w}_3)}(a_3)\eta \in \dot{\mathcal{H}}_{\mathbf{w}_3\mathbf{v}}$  and that  $\mathbf{v}$  starts with  $\mathbf{w}_3^{-1}$ . Moreover, Lemma 3.1.2(2) then implies that  $\lambda_{(e, \mathbf{w}_2, \mathbf{w}_3)}(a_2a_3)\eta = \lambda_{(e, \mathbf{w}_2, e)}(a_2)\eta_1 \in \dot{\mathcal{H}}_{\mathbf{w}_3\mathbf{v}}$  and that  $\mathbf{w}_3\mathbf{v}$  starts every letter from  $\mathbf{w}_2$ . This already shows  $\mathbf{w}_2$  is a clique word and that  $\mathbf{v}$  starts with  $\mathbf{w}_3^{-1}\mathbf{w}_2$ . Last, Lemma 3.1.2(3) implies that  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a) = \lambda_{(\mathbf{w}_1, e, e)}(a_1)\lambda_{(e, \mathbf{w}_2, \mathbf{w}_3)}(a_2a_3)\eta \in \dot{\mathcal{H}}_{\mathbf{w}_1\mathbf{w}_3\mathbf{v}}$  and that  $\mathbf{w}_1\mathbf{w}_3\mathbf{v}$  starts with  $\mathbf{w}_1$ . Hence  $\mathbf{u} := \mathbf{w}_1\mathbf{w}_3\mathbf{v}$  starts with  $\mathbf{w}_1\mathbf{w}_2$ . The statements on pure tensors follows directly from Lemma 3.1.2.  $\square$

For convenience we make the following definition.

**Definition 3.1.5.** For an element  $\mathbf{w} \in \mathcal{W}_{\Gamma}$  we define the set of triple splittings

$$\mathcal{S}_{\mathbf{w}} = \left\{ (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{W}_{\Gamma}^3 \left| \begin{array}{l} \mathbf{w} = \mathbf{w}_1\mathbf{w}_2\mathbf{w}_3 \\ \mathbf{w}_2 \text{ is a clique word} \\ |\mathbf{w}| = |\mathbf{w}_1| + |\mathbf{w}_2| + |\mathbf{w}_3| \end{array} \right. \right\} \quad (3.5)$$

and also define  $\mathcal{S}_{\Gamma} = \bigcup_{\mathbf{w} \in \mathcal{W}_{\Gamma}} \mathcal{S}_{\mathbf{w}}$ .

*Remark 3.1.6.* We explain how the definitions of the sets  $\mathcal{S}_{\mathbf{v}}$  relates to permutations defined in [CKL21, Definition 2.3]. Let  $\mathbf{v} = v_1 \cdots v_d \in \mathcal{W}_{\Gamma}$  be a reduced expression, let  $0 \leq l \leq d$ ,  $0 \leq k \leq d - l$  and let  $\mathbf{t}, \mathbf{u}_l, \mathbf{u}_r \in \mathcal{W}_{\Gamma}$  be clique words such that  $\mathbf{u}_l\mathbf{t}, \mathbf{t}\mathbf{u}_r$  are clique words,  $\mathbf{u}_l\mathbf{t}\mathbf{u}_r$  is reduced, and  $|\mathbf{t}| = l$  (in the notation of [CKL21, Definition 2.3]  $\mathbf{t}, \mathbf{u}_l, \mathbf{u}_r$  correspond to the cliques  $\Gamma_0, \Gamma_1, \Gamma_2$ , and the conditions we put on  $\mathbf{t}, \mathbf{u}_l, \mathbf{u}_r$  are equivalent to  $\Gamma_0 \in \text{Cliq}(\Gamma, l)$  and  $(\Gamma_1, \Gamma_2) \in \text{Comm}(\Gamma_0)$ ). Then a permutation  $\sigma (= \sigma_{l, k, \mathbf{t}, \mathbf{u}_l, \mathbf{u}_r}^{\mathbf{v}})$  is defined (if existent) as the permutation such that (1)  $\mathbf{v} = v_{\sigma(1)} \cdots v_{\sigma(d)}$ , (2)  $v_{\sigma(k+1)} \cdots v_{\sigma(k+l)} = \mathbf{t}$ , (3)  $|v_{\sigma(1)} \cdots v_{\sigma(k)} s| = k - 1$  for any letter  $s$  of  $\mathbf{u}_l$ , (4)  $|v_{\sigma(1)} \cdots v_{\sigma(k)} s| = k + 1$  for any letter  $s$  such that  $\mathbf{s}\mathbf{u}_l\mathbf{t}$  is a reduced clique word, (5)  $|s v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}| = d - k - l - 1$  for any letter  $s$  of  $\mathbf{u}_r$ , (6)  $|s v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}| = d - k - l + 1$  for any letter  $s$  such that  $\mathbf{s}\mathbf{u}_r\mathbf{t}$  is a reduced clique word. Furthermore  $\sigma$  is chosen such that the expressions  $\mathbf{v}_1 := v_{\sigma(1)} \cdots v_{\sigma(k)}$ ,  $\mathbf{v}_2 := v_{\sigma(k+1)} \cdots v_{\sigma(k+l)}$  and  $\mathbf{v}_3 := v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}$  are the representatives of their equivalence classes and such that  $v_i = v_j$  for  $i < j$  implies  $\sigma(i) < \sigma(j)$ . Such permutation, if existent, is unique. We make a few remarks on the definition of  $\sigma$ . First of all we note that conditions (3)+(4) are equivalent to

$$\mathbf{s}_r(v_{\sigma(1)} \cdots v_{\sigma(k)} \mathbf{t}) = \mathbf{u}_l \mathbf{t}$$

and similarly that conditions (5)+(6) are equivalent to

$$\mathbf{s}_l(\mathbf{t} v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}) = \mathbf{u}_r \mathbf{t}.$$

Secondly, we note that, when  $\sigma$  exists, the obtained triple  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  lies in  $\mathcal{S}_{\mathbf{v}}$ . In fact, for  $\mathbf{v} = v_1 \cdots v_d \in \mathcal{W}_{\Gamma}$ , this correspondence

$$(l, k, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \leftrightarrow (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

between tuples  $(l, k, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$  for which  $\sigma_{l, k, \mathbf{t}, \mathbf{u}_l, \mathbf{u}_r}^{\mathbf{v}}$  exists, and tuples  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  in  $\mathcal{S}_{\mathbf{v}}$ , is bijective. Indeed, for  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in \mathcal{S}_{\mathbf{v}}$  the tuple  $(l, k, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$  such that the corresponding permutation  $\sigma$  satisfies  $\mathbf{v}_1 = v_{\sigma(1)} \cdots v_{\sigma(k)}$ ,  $\mathbf{v}_2 = v_{\sigma(k+1)} \cdots v_{\sigma(k+l)}$ ,  $\mathbf{v}_3 = v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}$  is given by  $k = |\mathbf{v}_1|$ ,  $l = |\mathbf{v}_2|$ ,  $\mathbf{t} = \mathbf{v}_2$ ,  $\mathbf{u}_l = \mathbf{s}_r(\mathbf{v}_1\mathbf{t})\mathbf{t}$ ,  $\mathbf{u}_r = \mathbf{s}_l(\mathbf{t}\mathbf{v}_3)\mathbf{t}$ .

The following lemma was essentially proven in [CKL21, Lemma 2.5, Proposition 2.6], and tells in what ways an element  $a \in \lambda(\mathbf{A}_\Gamma)$  can act on a vector.

**Lemma 3.1.7.** *We have for  $\mathbf{w} \in \mathcal{W}_\Gamma$  and  $a \in \mathbf{A}_{\mathbf{w}}$  that*

$$\lambda = \sum_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_\Gamma} \lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)} \quad (3.6)$$

$$\lambda(a) = \sum_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}}} \lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a). \quad (3.7)$$

Moreover, for  $\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3$  a reduced expression in  $\mathcal{W}_\Gamma$  we have that  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)} = 0$  whenever  $\mathbf{w}_2$  is not a clique word.

*Proof.* Let  $\mathbf{w} = w_1 \cdots w_d \in \mathcal{W}_\Gamma$  and  $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}}$  and let  $\sigma$  be the corresponding permutation with  $\mathbf{w}_1 = w_{\sigma(1)} \cdots w_{\sigma(k)}$ ,  $\mathbf{w}_2 = w_{\sigma(k+1)} \cdots w_{\sigma(k+l)}$  and  $\mathbf{w}_3 = w_{\sigma(k+l+1)} \cdots w_d$ . Then, for  $a = a_1 \otimes \cdots \otimes a_d \in \mathbf{A}_{\mathbf{w}}$  we have

$$\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a) = \quad (3.8)$$

$$= \lambda_{cre}(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}) \quad (3.9)$$

$$\cdot \lambda_{dia}(a_{\sigma(k+1)} \otimes \cdots \otimes a_{\sigma(k+l)}) \quad (3.10)$$

$$\cdot \lambda_{ann}(a_{\sigma(k+l+1)} \otimes \cdots \otimes a_{\sigma(d)}) \quad (3.11)$$

$$= (P_{w_{\sigma(1)}} \lambda_{w_{\sigma(1)}}(a_{\sigma(1)}) P_{w_{\sigma(1)}}^\perp) \cdots (P_{w_{\sigma(k)}} \lambda_{w_{\sigma(k)}}(a_{\sigma(k)}) P_{w_{\sigma(k)}}^\perp) \quad (3.12)$$

$$\cdot (P_{w_{\sigma(k+1)}} \lambda_{w_{\sigma(k+1)}}(a_{\sigma(k+1)}) P_{w_{\sigma(k+1)}}) \cdots (P_{w_{\sigma(k+l)}} \lambda_{w_{\sigma(k+l)}}(a_{\sigma(k+l)}) P_{w_{\sigma(k+l)}}) \quad (3.13)$$

$$\cdot (P_{w_{\sigma(k+l+1)}}^\perp \lambda_{w_{\sigma(k+l+1)}}(a_{\sigma(k+l+1)}) P_{w_{\sigma(k+l+1)}}) \cdots (P_{w_{\sigma(d)}}^\perp \lambda_{w_{\sigma(d)}}(a_{\sigma(d)}) P_{w_{\sigma(d)}}). \quad (3.14)$$

Equation (3.7) now follows from [CKL21, Proposition 2.6] and from the bijective correspondence between the tuples  $(l, k, \mathbf{u}, \mathbf{u}', \mathbf{t})$  and the elements in  $\mathcal{S}_{\mathbf{w}}$  as described in Remark 3.1.6. Equation (3.6) then follows from linearity and the fact that  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(b) = 0$  whenever  $b \in \mathbf{A}_{\mathbf{w}'}$  with  $\mathbf{w}' \neq \mathbf{w}$ . Last, we note that by [CKL21, Lemma 2.5] we have  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a) = 0$  whenever  $\mathbf{w}_2$  is not a clique word, which completes the proof.  $\square$

We now prove the following lemma.

**Lemma 3.1.8.** *Let  $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{W}_\Gamma$  with  $|\mathbf{v}_1 \mathbf{v}_2| = |\mathbf{v}_1| + |\mathbf{v}_2|$ . Let  $\eta \in \mathcal{H}_{\mathbf{v}_1 \mathbf{v}_2}$  be a pure tensor, and write  $\eta = \mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\eta_1 \otimes \eta_2)$  for some  $\eta_1 \otimes \eta_2 \in \mathcal{H}_{\mathbf{v}_1} \otimes \mathcal{H}_{\mathbf{v}_2}$ . Let  $\mathbf{w} \in \mathcal{W}_\Gamma$  and let  $a \in \mathbf{A}_{\mathbf{w}}$ . The following holds*

1. *If  $|\mathbf{v}_1| = |\mathbf{w}| + |\mathbf{w} \mathbf{v}_1|$  then also  $|\mathbf{w} \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w} \mathbf{v}_1| + |\mathbf{v}_2|$  and*

$$\lambda_{ann}(a)\eta = \mathcal{Q}_{(\mathbf{w} \mathbf{v}_1, \mathbf{v}_2)}(\lambda_{ann}(a)\eta_1 \otimes \eta_2) \quad (3.15)$$

$$\lambda_{dia}(a)\eta = \mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{dia}(a)\eta_1 \otimes \eta_2). \quad (3.16)$$

2. *If  $|\mathbf{w} \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w}| + |\mathbf{v}_1 \mathbf{v}_2|$  then also  $|\mathbf{w} \mathbf{v}_1| = |\mathbf{w}| + |\mathbf{v}_1|$  and*

$$\lambda_{cre}(a)\eta = \mathcal{Q}_{(\mathbf{w} \mathbf{v}_1, \mathbf{v}_2)}(\lambda_{cre}(a)\eta_1 \otimes \eta_2). \quad (3.17)$$

3. If  $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}}$  and if  $|\mathbf{v}_1| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_1|$  and  $|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2|$ , then also  $|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1| + |\mathbf{v}_2|$  and

$$\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a)\eta = \mathcal{Q}_{(\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1, \mathbf{v}_2)}(\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a)\eta_1 \otimes \eta_2). \quad (3.18)$$

*Proof.* (1) Assume that  $|\mathbf{v}_1| = |\mathbf{w}| + |\mathbf{w} \mathbf{v}_1|$ . Then

$$|\mathbf{v}_1 \mathbf{v}_2| - |\mathbf{w}| \leq |\mathbf{w} \mathbf{v}_1 \mathbf{v}_2| \leq |\mathbf{w} \mathbf{v}_1| + |\mathbf{v}_2| = |\mathbf{v}_1| + |\mathbf{v}_2| - |\mathbf{w}| = |\mathbf{v}_1 \mathbf{v}_2| - |\mathbf{w}|. \quad (3.19)$$

Hence,  $|\mathbf{w} \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w} \mathbf{v}_1| + |\mathbf{v}_2|$ , which proves the remark. We now prove that the equations by induction to the length  $|\mathbf{w}|$ . First of all, it is clear that the statement holds when  $\mathbf{w} = e$ , as then  $\lambda_{ann}(a) = \lambda_{dia}(a) = a \in \mathbb{C} \text{Id}_{\mathcal{H}_e}$ .

Thus assume that  $|\mathbf{w}| \geq 1$  and that the statement holds for  $\tilde{\mathbf{w}}$  with  $|\tilde{\mathbf{w}}| \leq |\mathbf{w}| - 1$ . Write  $\mathbf{w} = \tilde{\mathbf{w}} w$  with  $\tilde{\mathbf{w}} \in \mathcal{W}_{\Gamma}$  and  $w \in \Gamma$  and s.t.  $|\tilde{\mathbf{w}}| = |\mathbf{w}| - 1$ . Then we also have  $|\mathbf{v}_1| = |w| + |\mathbf{w} \mathbf{v}_1|$ . Let us write  $a = \mathcal{Q}_{(\tilde{\mathbf{w}}, w)}(a_1 \otimes a_2)$  with  $a_1 \otimes a_2 \in \mathring{\mathbf{A}}_{\tilde{\mathbf{w}}} \otimes \mathring{\mathbf{A}}_w$ . Then  $\lambda_{ann}(a) = \lambda_{ann}(a_1) \lambda_{ann}(a_2)$ .

Now, write  $\eta = \mathcal{Q}_{(w, w \mathbf{v}_1, \mathbf{v}_2)}(\eta_w \otimes \eta'_1 \otimes \eta_2)$  for some  $\eta_w \otimes \eta'_1 \otimes \eta_2 \in \mathcal{H}_w \otimes \mathcal{H}_{w \mathbf{v}_1} \otimes \mathcal{H}_{\mathbf{v}_2}$  and define

$$\eta' = \mathcal{Q}_{(w \mathbf{v}_1, \mathbf{v}_2)}(\eta'_1 \otimes \eta_2) \quad (3.20)$$

$$\eta_1 = \mathcal{Q}_{(w, w \mathbf{v}_1)}(\eta_w \otimes \eta'_1) \quad (3.21)$$

so that also  $\eta = \mathcal{Q}_{(w, w \mathbf{v}_1, \mathbf{v}_2)}(\eta_w \otimes \eta') = \mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\eta_1 \otimes \eta_2)$ .

We now have, using the definitions, that

$$\lambda_{ann}(a_2)\eta = P_w^\perp \lambda_w(a_2) P_w \eta = P_w^\perp U_w((a_2 \eta_w) \otimes \eta') = \langle a_2 \eta_w, \xi_w \rangle \eta' \quad (3.22)$$

$$\lambda_{ann}(a_2)\eta_1 = P_w^\perp \lambda_w(a_2) P_w \eta_1 = P_w^\perp U_w((a_2 \eta_w) \otimes \eta'_1) = \langle a_2 \eta_w, \xi_w \rangle \eta'_1 \quad (3.23)$$

and

$$\lambda_{dia}(a_2)\eta = P_w U_w((a_2 \eta_w) \otimes \eta') = \mathcal{Q}_{(w, w \mathbf{v}_1, \mathbf{v}_2)}((a_2 \eta_w) \otimes \eta') \quad (3.24)$$

$$\lambda_{dia}(a_2)\eta_1 = P_w U_w((a_2 \eta_w) \otimes \eta'_1) = \mathcal{Q}_{(w, w \mathbf{v}_1)}((a_2 \eta_w) \otimes \eta'_1). \quad (3.25)$$

Now this means that

$$\lambda_{ann}(a_2)\eta = \varphi_w(a_2 \eta_w) \eta' \quad (3.26)$$

$$= \mathcal{Q}_{(w \mathbf{v}_1, \mathbf{v}_2)}(\langle a_2 \eta_w, \xi_w \rangle \eta'_1 \otimes \eta_2) \quad (3.27)$$

$$= \mathcal{Q}_{(w \mathbf{v}_1, \mathbf{v}_2)}(\lambda_{ann}(a_2)\eta_1 \otimes \eta_2) \quad (3.28)$$

and

$$\lambda_{dia}(a_2)\eta = \mathcal{Q}_{(w, w \mathbf{v}_1, \mathbf{v}_2)}((a_2 \eta_w) \otimes \eta') \quad (3.29)$$

$$= \mathcal{Q}_{(w, w \mathbf{v}_1, \mathbf{v}_2)}((a_2 \eta_w) \otimes \eta'_1 \otimes \eta_2) \quad (3.30)$$

$$= \mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\mathcal{Q}_{(w, w \mathbf{v}_1)}((a_2 \eta_w) \otimes \eta'_1) \otimes \eta_2) \quad (3.31)$$

$$= \mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{dia}(a_2)\eta_1 \otimes \eta_2). \quad (3.32)$$

Now, we note that  $|w\mathbf{v}_1| = |\tilde{\mathbf{w}}| + |\tilde{\mathbf{w}}w\mathbf{v}_1|$  so that using the induction hypothesis and the fact that  $|\tilde{\mathbf{w}}| = |\mathbf{w}| - 1$  we find

$$\lambda_{ann}(a)\eta = \lambda_{ann}(a_1)\lambda_{ann}(a_2)\eta \quad (3.33)$$

$$= \lambda_{ann}(a_1)\mathcal{Q}_{(w\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{ann}(a_2)\eta_1 \otimes \eta_2) \quad (3.34)$$

$$= \mathcal{Q}_{(\tilde{\mathbf{w}}w\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{ann}(a_1)\lambda_{ann}(a_2)\eta_1 \otimes \eta_2) \quad (3.35)$$

$$= \mathcal{Q}_{(\mathbf{w}\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{ann}(a)\eta_1 \otimes \eta_2). \quad (3.36)$$

Similarly

$$\lambda_{dia}(a)\eta = \lambda_{dia}(a_1)\lambda_{dia}(a_2)\eta \quad (3.37)$$

$$= \lambda_{dia}(a_1)\mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{dia}(a_2)\eta_1 \otimes \eta_2) \quad (3.38)$$

$$= \mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{dia}(a_1)\lambda_{dia}(a_2)\eta_1 \otimes \eta_2) \quad (3.39)$$

$$= \mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{dia}(a)\eta_1 \otimes \eta_2). \quad (3.40)$$

This finishes the induction, and proves the statement.

(2) Assume that  $|\mathbf{w}\mathbf{v}_1\mathbf{v}_2| = |\mathbf{w}| + |\mathbf{v}_1\mathbf{v}_2|$ . Then

$$|\mathbf{w}\mathbf{v}_1\mathbf{v}_2| \leq |\mathbf{w}\mathbf{v}_1| + |\mathbf{v}_2| \leq |\mathbf{w}| + |\mathbf{v}_1| + |\mathbf{v}_2| = |\mathbf{w}| + |\mathbf{v}_1\mathbf{v}_2| = |\mathbf{w}\mathbf{v}_1\mathbf{v}_2|. \quad (3.41)$$

Hence  $|\mathbf{w}\mathbf{v}_1| = |\mathbf{w}| + |\mathbf{v}_1|$ , which shows the first remark. Again we prove the equation by induction to the length  $|\mathbf{w}|$ . Again, it is clear that the statement holds when  $\mathbf{w} = e$ . Thus assume that  $|\mathbf{w}| \geq 1$  and that the statement holds for  $\tilde{\mathbf{w}}$  with  $|\tilde{\mathbf{w}}| \leq |\mathbf{w}| - 1$ . Write  $\mathbf{w} = \tilde{\mathbf{w}}w$  with  $\tilde{\mathbf{w}} \in \mathcal{W}_\Gamma$  and  $w \in \Gamma$  and s.t.  $|\tilde{\mathbf{w}}| = |\mathbf{w}| - 1$ . Then we also have  $|w\mathbf{v}_1\mathbf{v}_2| = |w| + |\mathbf{v}_1\mathbf{v}_2|$ . Let us write  $a = \mathcal{Q}_{(\tilde{\mathbf{w}}, w)}(a_1 \otimes a_2)$  with  $a_1 \otimes a_2 \in \mathring{\mathbf{A}}_{\tilde{\mathbf{w}}} \otimes \mathring{\mathbf{A}}_w$ . Then  $\lambda_{cre}(a) = \lambda_{cre}(a_1)\lambda_{cre}(a_2)$ .

We now have by definition

$$\lambda_{cre}(a_2)\eta = P_w\lambda_w(a_2)P_w^\perp\eta = (P_wU_w)((a_2\xi_w) \otimes \eta) = \mathcal{Q}_{(w, \mathbf{v}_1\mathbf{v}_2)}(\hat{a}_2 \otimes \eta) \quad (3.42)$$

$$\lambda_{cre}(a_2)\eta_1 = P_w\lambda_w(a_2)P_w^\perp\eta_1 = (P_wU_w)((a_2\xi_w) \otimes \eta_1) = \mathcal{Q}_{(w, \mathbf{v}_1)}(\hat{a}_2 \otimes \eta_1). \quad (3.43)$$

Now this means that

$$\lambda_{cre}(a_2)\eta = \mathcal{Q}_{(w, \mathbf{v}_1\mathbf{v}_2)}(\hat{a}_2 \otimes \eta) \quad (3.44)$$

$$= \mathcal{Q}_{(w, \mathbf{v}_1, \mathbf{v}_2)}(\hat{a}_2 \otimes \eta_1 \otimes \eta_2) \quad (3.45)$$

$$= \mathcal{Q}_{(w\mathbf{v}_1, \mathbf{v}_2)}(\mathcal{Q}_{(w, \mathbf{v}_1)}(\hat{a}_2 \otimes \eta_1) \otimes \eta_2) \quad (3.46)$$

$$= \mathcal{Q}_{(w\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{cre}(a_2)\eta_1 \otimes \eta_2). \quad (3.47)$$

Now, we note that  $|\tilde{\mathbf{w}}w\mathbf{v}_1\mathbf{v}_2| = |\tilde{\mathbf{w}}| + |w\mathbf{v}_1\mathbf{v}_2|$  so that using the induction hypothesis and the fact that  $|\tilde{\mathbf{w}}| = |\mathbf{w}| - 1$  we find

$$\lambda_{cre}(a)\eta = \lambda_{cre}(a_1)\lambda_{cre}(a_2)\eta \quad (3.48)$$

$$= \lambda_{cre}(a_1)\mathcal{Q}_{(w\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{cre}(a_2)\eta_1 \otimes \eta_2) \quad (3.49)$$

$$= \mathcal{Q}_{(\tilde{\mathbf{w}}w\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{cre}(a_1)\lambda_{cre}(a_2)\eta_1 \otimes \eta_2) \quad (3.50)$$

$$= \mathcal{Q}_{(\mathbf{w}\mathbf{v}_1, \mathbf{v}_2)}(\lambda_{cre}(a)\eta_1 \otimes \eta_2). \quad (3.51)$$

This finishes the induction, and proves the statement.

(3) Let  $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}}$  be s.t  $|\mathbf{v}_1| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_1|$  and  $|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2|$ . We will write  $\lambda_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}(a) = \lambda_{cre}(a_1) \lambda_{dia}(a_2) \lambda_{ann}(a_3)$  for some  $a_1 \otimes a_2 \otimes a_3 \in \mathring{\mathbf{A}}_{\mathbf{w}_1} \otimes \mathring{\mathbf{A}}_{\mathbf{w}_2} \otimes \mathring{\mathbf{A}}_{\mathbf{w}_3}$ . Now, first, as  $|\mathbf{v}_1| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_1|$ , we also have

$$|\mathbf{v}_1| \leq |\mathbf{w}_3| + |\mathbf{w}_3 \mathbf{v}_1| \quad (3.52)$$

$$\leq |\mathbf{w}_2| + |\mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_1| \quad (3.53)$$

$$= |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_1| \quad (3.54)$$

$$= |\mathbf{v}_1| \quad (3.55)$$

and therefore  $|\mathbf{v}_1| = |\mathbf{w}_3| + |\mathbf{w}_3 \mathbf{v}_1|$ . By (1) this gives us

$$\lambda_{ann}(a_3)\eta = \mathcal{Q}_{(\mathbf{w}_3 \mathbf{v}_1, \mathbf{v}_2)}(\lambda_{ann}(a_3)\eta_1 \otimes \eta_2) \quad (3.56)$$

and also  $|\mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w}_3 \mathbf{v}_1| + |\mathbf{v}_2|$ . Now, we also find

$$|\mathbf{w}_3 \mathbf{v}_1| = |\mathbf{v}_1| - |\mathbf{w}_3| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_1| - |\mathbf{w}_3| = |\mathbf{w}_2| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_1|. \quad (3.57)$$

Let us set  $\mathbf{v}'_1 = \mathbf{w}_3 \mathbf{v}_1$  and  $\mathbf{v}'_2 = \mathbf{v}_2$ , so that  $|\mathbf{v}'_1 \mathbf{v}'_2| = |\mathbf{v}'_1| + |\mathbf{v}'_2|$  and  $|\mathbf{v}'_1| = |\mathbf{w}_2| + |\mathbf{w}_2 \mathbf{v}'_1|$ . Moreover set  $\eta' = \lambda_{ann}(a_3)\eta$  and  $\eta'_1 = \lambda_{ann}(a_3)\eta_1$  and  $\eta'_2 = \eta_2$ . Now  $\eta' = \mathcal{Q}_{(\mathbf{v}'_1, \mathbf{v}'_2)}(\eta'_1 \otimes \eta'_2)$  and we see that the conditions for applying (1) are satisfied. This thus gives us that

$$\lambda_{dia}(a_2)\lambda_{ann}(a_3)\eta = \mathcal{Q}_{(\mathbf{w}_3 \mathbf{v}_1, \mathbf{v}_2)}(\lambda_{dia}(a_2)\lambda_{ann}(a_3)\eta_1 \otimes \eta_2). \quad (3.58)$$

Now, set  $\tilde{\mathbf{v}}_1 = \mathbf{v}'_1 = \mathbf{w}_3 \mathbf{v}_1$  and  $\tilde{\mathbf{v}}_2 = \mathbf{v}'_2 = \mathbf{v}_2$  so that again  $|\tilde{\mathbf{v}}_1 \tilde{\mathbf{v}}_2| = |\tilde{\mathbf{v}}_1| + |\tilde{\mathbf{v}}_2|$ . Also we get  $|\mathbf{w}_1 \tilde{\mathbf{v}}_1 \tilde{\mathbf{v}}_2| = |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1| + |\mathbf{v}_2| = |\mathbf{w}_1 \tilde{\mathbf{v}}_1| + |\tilde{\mathbf{v}}_2|$ . Also set  $\tilde{\eta} = \lambda_{dia}(a_2)\lambda_{ann}(a_3)\eta$  and  $\tilde{\eta}_1 = \lambda_{dia}(a_2)\lambda_{ann}(a_3)\eta_1$  and  $\tilde{\eta}_2 = \eta_2$ . Then  $\tilde{\eta} = \mathcal{Q}_{(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2)}(\tilde{\eta}_1 \otimes \tilde{\eta}_2)$  and all conditions for applying (2) are satisfied. By (2) we thus get

$$\lambda_{cre}(a_1)\lambda_{dia}(a_2)\lambda_{ann}(a_3)\eta = \mathcal{Q}_{(\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1, \mathbf{v}_2)}(\lambda_{cre}(a_1)\lambda_{dia}(a_2)\lambda_{ann}(a_3)\eta_1 \otimes \eta_2) \quad (3.59)$$

and moreover  $|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_1|$ . The previous equation is precisely what we needed to show, and we moreover obtain  $|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_1| + |\mathbf{v}_2| = |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1| + |\mathbf{v}_2|$ , which proves the statement.  $\square$



# 4

## BIMODULE COEFFICIENTS, RIESZ TRANSFORMS AND STRONG SOLIDITY

In deformation-rigidity theory, it is often important to know whether certain bimodules are weakly contained in the coarse bimodule. Consider a bimodule  $\mathcal{H}$  over the group algebra  $\mathbb{C}[G]$  with  $G$  a discrete group. The starting point of this chapter is that if a dense set of the so-called coefficients of  $\mathcal{H}$  is contained in the Schatten  $S_p$  class  $p \in [2, \infty)$ , then the  $n$ -fold tensor power  $\mathcal{H}_G^{\otimes n}$  for  $n \geq \frac{p}{2}$  is quasi-contained in the coarse bimodule. We apply this to gradient bimodules associated with the carré du champ of a symmetric quantum Markov semigroup.

For Coxeter groups, we give a number of characterizations of having coefficients in  $S_p$  for the gradient bimodule constructed from the word length function. We get equivalence of: (1) the gradient- $S_p$  property, (2) smallness at infinity of a natural compactification of the Coxeter group, and for a large class of Coxeter groups, (3) walks in the Coxeter diagram called parity paths. We derive several strong solidity results. In particular, we obtain current strong solidity results for right-angled Hecke von Neumann algebras beyond right-angled Coxeter groups that are small at infinity.

This chapter is based on the paper:

- **Matthijs Borst, Martijn Caspers and Mateusz Wasilewski**, *Bimodule coefficients, Riesz transforms on Coxeter groups and strong solidity*, [Groups, Geometry, and Dynamics](#) 18.2 (2023) pp. 501–549.

### 4.1. INTRODUCTION

This chapter establishes bridges between the Riesz transform in modern harmonic analysis and von Neumann algebra theory. The original Riesz transform can be defined as

follows. Consider the positive unbounded Laplace operator  $\Delta$  and the directional gradient  $\nabla_j$  on  $L^2(\mathbb{R}^n)$  given by

$$\Delta = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}, \quad \nabla_j = \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq n.$$

Then the Riesz transform  $R_j = \nabla_j \circ \Delta^{-\frac{1}{2}}$  for  $1 \leq j \leq n$  is an isometry on  $L^2(\mathbb{R}^n)$  that has been studied extensively in classical harmonic analysis in the context of Fourier multipliers, singular integral operators and Calderón-Zygmund theory.

Riesz transforms can be defined abstractly for any  $C_0$ -semigroup of positive measure preserving unital contractions on  $L^\infty(X, \mu)$ , with  $(X, \mu)$  a finite Borel measure space. Such semigroups admit a generator  $\Delta$  and a natural replacement of the gradient  $\nabla$  known as the *carré du champ*. The Riesz transform is then defined as  $\nabla \circ \Delta^{-\frac{1}{2}}$ . These Riesz transforms were studied by Meyer [Mey84] for (commutative) Gaussian algebras and their study was continued by Bakry [Bak85], [Bak87], Gundy [Gun86], Pisier [Pis88], amongst others. This in particular involves an analysis of diffusion semigroups on compact Riemannian manifolds with lower bounds on the Ricci curvature [Bak87]. Furthermore, in the non-commutative situation, Clifford algebras were considered by Lust-Piquard [Lus99], [Lus98]. Also the Riesz transform was studied on general groups [JMP18] using certain multipliers associated with cocycles.

In this chapter, we study Riesz transforms associated with non-commutative generalizations of diffusion semigroups: (symmetric) quantum Markov semigroups. Let  $M$  be a finite von Neumann algebra and  $\Phi = (\Phi_t)_{t \geq 0}$  a point-strongly continuous semigroup of trace preserving unital completely positive maps. Such a semigroup comes with a generator  $\Delta$ . The proper replacement of the gradient is played by a bilinear form that is a non-commutative version of the carré du champ. For simplicity, we consider mostly quantum Markov semigroups of Fourier multipliers associated with a discrete group  $G$ , acting on the group algebra  $\mathbb{C}[G]$ . Then the carré du champ allows the construction of a  $\mathbb{C}[G]$  bimodule  $\mathcal{H}_\nabla$  and a derivation, i.e. a map satisfying the Leibniz rule,  $\nabla : \mathbb{C}[G] \rightarrow \mathcal{H}_\nabla$  such that (here formally)  $\Delta = \nabla^* \nabla$ . So  $\nabla$  is a root of  $\Delta$  just as in the case of the Laplace operator and the gradient. We refer to Cipriani and Sauvageot [CS03] where also the analytical framework is established. Then there is an isometry  $\nabla \circ \Delta^{-\frac{1}{2}} : \ell^2(G) \rightarrow \mathcal{H}_\nabla$  called the *Riesz transform*. This Riesz transform was studied in the context of  $q$ -Gaussian algebras [CIW21], [Lus99], [Lus98] and compact quantum groups [Cas22], [Cas21].

In the current chapter we are interested in applications of the Riesz transform to group von Neumann algebras of discrete groups; we focus on Coxeter groups but we also obtain results for other groups.

Recall that to a discrete group  $G$  we may associate the group von Neumann algebra  $\mathcal{L}(G)$  which is the von Neumann algebra generated by the left regular representation. Let  $\mathbb{F}_2$  be the free group with two generators. In his fundamental papers on free probability Voiculescu [Voi96] showed that  $\mathcal{L}(\mathbb{F}_2)$  does not possess a Cartan subalgebra, meaning that there does not exist a maximal abelian subalgebra (MASA) of  $\mathcal{L}(\mathbb{F}_2)$  whose normalizer generates  $\mathcal{L}(\mathbb{F}_2)$ . An important consequence is that  $\mathcal{L}(\mathbb{F}_2)$  does not non-trivially decompose as a crossed product and cannot be constructed from an equivalence relation with a cocycle as was shown by Feldman and Moore [FM77a], [FM77b]. In [OP10a]

Ozawa and Popa gave an alternative proof of the Voiculescu's result. They showed that  $\mathcal{L}(\mathbb{F}_2)$  is strongly solid. This means that the normalizer of any diffuse amenable von Neumann subalgebra of  $\mathcal{L}(\mathbb{F}_2)$  generates a von Neumann algebra that is amenable again. Since  $\mathcal{L}(\mathbb{F}_2)$  is nonamenable and since MASAs are diffuse it automatically follows that  $\mathcal{L}(\mathbb{F}_2)$  does not possess a Cartan subalgebra. After [OP10a] many von Neumann algebras were proven to be strongly solid, see e.g. [Iso15a], [OP10b], [PV14b] and references given there. As a consequence of the methods in this chapter we are able to prove such strong solidity results as well.

To motivate the first part of this chapter we recall the following theorem from [CIW21]. We do not explain for now the technical terms that occur in this theorem but in the subsequent paragraph we explain what the crucial part is. Theorem 4.1.1 itself is actually not that hard to prove; however its consequences (see [PV14b], [Iso15a]) and proving that its assumptions hold in examples is rather intricate.

**Theorem 4.1.1** (Proposition 5.2 in [CIW21]). *Let  $\mathcal{H}$  be a  $\mathbb{C}[G]$  bimodule and let  $V : \ell^2(G) \rightarrow \mathcal{H}$  be bounded. Assume that  $\mathcal{H}$  is quasi-contained in the coarse bimodule of  $G$ , that  $V$  is almost bimodular and that  $V^*V$  is Fredholm. Assume that  $C_r^*(G)$  is locally reflexive. Then  $\mathcal{L}(G)$  satisfies  $AO^+$ .*

The Akemann-Ostrand property  $AO^+$  (as in [Iso15a]) will be used frequently in this chapter for which we refer to Definition 4.2.11. If  $G$  is weakly amenable then  $AO^+$  implies strong solidity [PV14b], [Iso15a]. The Coxeter groups in this thesis are weakly amenable [Fen02], [Jan02] as are all hyperbolic discrete groups [Oza08].

In view of Theorem 4.1.1 we are mostly still interested in two things: (1) constructing almost bimodular maps  $V : \ell^2(G) \rightarrow \mathcal{H}$  with  $\mathcal{H}$  a  $\mathbb{C}[G]$  bimodule; (2) showing that the  $\mathbb{C}[G]$  bimodule  $\mathcal{H}$  is quasi-contained in the coarse bimodule  $\ell^2(G) \otimes \ell^2(G)$  of  $G$ . It turns out that very often the Riesz transform is an almost bimodular map. Further, we provide comprehensible conditions that show that the gradient bimodule is quasi-contained in the coarse bimodule. We will develop general theory for this as follows.

In the first part of this chapter we study bimodules over  $\mathbb{C}[G]$  and their *coefficients*. We define coefficients of a  $\mathbb{C}[G]$  bimodule as a certain map  $\mathbb{C}[G] \rightarrow \mathbb{C}[G]$ . This notion occurs for instance in [AP17, Section 13] for von Neumann algebras; the more algebraic notion we present here is more convenient for our purposes. Since  $\mathbb{C}[G] \subseteq \ell^2(G)$  a coefficient determines a densely defined map  $\ell^2(G) \rightarrow \ell^2(G)$ . We study when these maps are contained in the Schatten von Neumann non-commutative  $L_p$ -space  $S_p$ .

For two  $\mathbb{C}[G]$  bimodules  $\mathcal{H}_1$  and  $\mathcal{H}_2$  we shall also show that  $\mathcal{H}_1 \otimes \mathcal{H}_2$  has a natural  $\mathbb{C}[G]$  bimodule structure and we denote this bimodule by  $\mathcal{H}_1 \otimes_G \mathcal{H}_2$ . As a Hilbert space  $\mathcal{H}_1 \otimes_G \mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}_2$ . Recall that the coarse bimodule of  $G$  is given by  $\ell^2(G) \otimes \ell^2(G)$  where the left action of  $\mathbb{C}[G]$  is on the first tensor leg and the right action on the second tensor leg. In Section 4.2 we prove the following, amongst other results (except for part (4), which is proved in Section 4.3, see Corollary 4.3.13).

**Theorem 4.1.2.** *Let  $\mathcal{H}, \mathcal{H}_1$  and  $\mathcal{H}_2$  be  $\mathbb{C}[\Gamma]$  bimodules.*

1. *If a dense set of coefficients of  $\mathcal{H}$  are in  $S_2$  then  $\mathcal{H}$  is a  $\mathcal{L}(G)$  bimodule that is quasi-contained in the coarse bimodule of  $G$ .*

2. If a dense set of coefficients of  $\mathcal{H}_i, i = 1, 2$  is contained in  $S_{p_i}, p_i \in [1, \infty)$  then a dense set of coefficients of  $\mathcal{H}_1 \otimes_G \mathcal{H}_2$  is contained in  $S_p$  where  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ .
3. If  $V_i : \ell^2(G) \rightarrow \mathcal{H}_i, i = 1, 2$  is almost  $\mathbb{C}[G]$  bimodular then so is  $V_1 * V_2 := (V_1 \otimes V_2) \circ \Delta_G : \ell^2(G) \rightarrow \mathcal{H}_1 \otimes_G \mathcal{H}_2$  where  $\Delta_G : \ell^2(G) \rightarrow \ell^2(G) \otimes \ell^2(G)$  is the comultiplication.
4. Consider a proper length function  $\psi : G \rightarrow \mathbb{Z}_{\geq 0}$  that is conditionally of negative type, defined on a finitely generated group  $G$ . Then the associated Riesz transform  $R : \ell^2(G) \rightarrow \ell^2(G)_\nabla$  is almost bimodular.

Theorem 4.1.2 provides a clear strategy towards obtaining the input of Theorem 4.1.1. Namely we start with a proper length function  $\psi : G \rightarrow \mathbb{R}$  that is conditionally of negative type. We construct the associated gradient bimodule  $\mathcal{H}_\nabla$  and show that its coefficients are in  $S_p$  for some  $p \in [1, \infty)$ . By tensoring we obtain a bimodule  $(\mathcal{H}_\nabla)_G^{\otimes n}, n \geq \lceil \frac{p}{2} \rceil$  and a map

$$V^{*n} : \ell^2(G) \rightarrow (\mathcal{H}_\nabla)_G^{\otimes n},$$

with the desired properties of Theorem 4.1.1. This is the rough idea of our strategy. We say ‘rough’ since in all applications we need some suitably adapted variation of this idea.

In the second part of this chapter we analyse when coefficients of a gradient bimodule  $\mathcal{H}_\nabla$  are in  $S_p, p \in [1, \infty)$ . In order to do so we recall the property *gradient- $S_p$*  for quantum Markov semigroups from [Cas21], [CIW21]. If a quantum Markov semigroup has gradient- $S_p$  then a dense set of coefficients of  $\mathcal{H}_\nabla$  are in  $S_p$ ; consequently  $\mathcal{H}_\nabla$  is quasi-contained in the coarse bimodule of  $G$ .

We first show (Lemma 4.3.11) that if  $\psi : \Gamma \rightarrow \mathbb{Z}$  is a proper length function that is conditionally of negative type then gradient  $S_p, p \in [1, \infty)$  for the associated quantum Markov semigroup is independent of  $p$ . Then we analyse when the word length function of a general (finite rank) Coxeter group is gradient- $S_p$ . We find the following characterization.

**Theorem A** (Theorem 4.4.15). *Let  $\mathcal{W} = \langle S|M \rangle$  be a finite rank Coxeter system. Fix  $p \in [1, \infty]$ . The following are equivalent:*

1. *The quantum Markov semigroup associated with the word length is gradient- $S_p$ .*
2. *For all  $s \in S$  the set  $\{\mathbf{w} \in \mathcal{W} : \mathbf{w}s = s\mathbf{w}\}$  is finite.*
3. *The Coxeter system  $\langle S|M \rangle$  is small at infinity (as in [Kli23b]).*

In particular for right-angled Coxeter groups these statements are equivalent to the Coxeter group being a free product of finite abelian Coxeter groups, see [Kli23b]. This shows that gradient- $S_p$  is rather rare. However with the right tensor techniques it can still be turned into a very useful property. We also provide an almost characterization of when the equivalent statements of Theorem A hold in the following theorem. For the definition of the graph  $\text{Graph}_S(\mathcal{W})$  we refer to Definition 4.4.5. The definition of a parity path is given in Definition 4.4.6.

**Theorem B** (Theorem 4.4.8 and Theorem 4.4.9). *Let  $\mathcal{W} = \langle S|M \rangle$  be a Coxeter group. If there does not exist a cyclic parity path in  $\text{Graph}_S(\mathcal{W})$  then the semigroup  $(\Phi_t)_{t \geq 0}$  associated to the word length  $|\cdot|_S$  is gradient- $S_p$  for all  $p \in [1, \infty]$ . The converse holds true if  $m_{i,j} \neq 2$  for all  $i, j$ .*

Section 4.4 shows that it is usually easy to determine whether  $\text{Graph}_S(\mathcal{W})$  has a parity path, see Corollaries 4.4.11 and 4.4.12.

Next we obtain strong solidity results for Hecke von Neumann algebras, i.e. for  $\mathbf{q}$ -deformations of group von Neumann algebras of Coxeter groups. The following theorem extends [Kli23b, Theorem 0.7] in the case of a right-angled Coxeter system. What is of particular interest is that our methods really improve on the approach based on compactifications and boundaries in [Kli23b]. More precisely, [Kli23b] shows that if the action of a right-angled Coxeter group on a natural boundary associated with it is small at infinity, then actually the Coxeter group is a free product of finite (commutative) Coxeter groups. So the approach in [Kli23b, Theorem 0.7] cannot be extended to the current generality.

**Theorem C** (Theorem 4.7.5). *Let  $\Gamma$  be a finite simple graph and let  $\mathbf{q} = (q_v)_{v \in \Gamma}$  with  $q_v > 0$ . Assume*

$$\Lambda := \{r \in \Gamma : \exists s, t \in \Gamma \text{ such that } r \in \text{Link}_\Gamma(s) \cap \text{Link}_\Gamma(t), s \notin \text{Star}_\Gamma(t)\}$$

*is a clique in  $\Gamma$ . Then the Hecke von Neumann algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_\Gamma)$  satisfies the Akemann-Ostrand property  $AO^+$  and is strongly solid.*

We note that a large part of the analysis in proving Theorem C applies to general Hecke algebras. However, the strong solidity properties are still pending on whether certain semigroups extend to quantum Markov semigroups. In the final Section 4.8 of this chapter we summarize some open problems.

**Structure of the chapter.** Section 4.2 contains results on bimodules and their coefficients and we prove Theorem 4.1.2. Section 4.3 introduces quantum Markov semigroups, the gradient bimodule and the Riesz transform. We also derive many of the basic properties. Section 4.4 proves Theorem A and Theorem B. Note that here we also establish the Corollaries 4.4.11 and 4.4.12 which make it easy to see if a Coxeter group is small at infinity. Section 4.5 contains an analysis of quantum Markov semigroups with weights on the generators. This applies mostly to right-angled Coxeter groups and it is crucial in the later sections. Section 4.6 proves strong solidity results for Coxeter groups using tensor methods. In Section 4.7 we prove Theorem C. We have included Section 4.8 to list some problems that are left open.

## 4.2. COEFFICIENTS OF BIMODULES

In this section we study bimodules over the group algebra of a discrete group and provide sufficient criteria for when such a bimodule is quasi-contained in the coarse bimodule. We also consider tensor products of such bimodules.

### 4.2.1. COEFFICIENTS AND QUASI-CONTAINMENT

Let  $G$  be a discrete group with group algebra  $\mathbb{C}[G]$ , reduced group  $C^*$ -algebra  $C_r^*(G)$  and group von Neumann algebra  $\mathcal{L}(G)$ . They include naturally

$$\mathbb{C}[G] \subseteq C_r^*(G) \subseteq \mathcal{L}(G).$$

In turn  $\mathcal{L}(G) \subseteq \ell^2(G)$  by  $x \mapsto x\delta_e$ . Hence we may and will view  $\mathbb{C}[G]$  as the subspace of  $\ell^2(G)$  of functions with finite support. Now a  $\mathbb{C}[G]$  bimodule will be a Hilbert space  $\mathcal{H}$  with commuting left and right actions of  $G$  and thus of  $\mathbb{C}[G]$  by extending the actions linearly.

**Definition 4.2.1** (Coefficients). *Let  $\mathcal{H}$  be a  $\mathbb{C}[G]$  bimodule. Let  $\xi, \eta \in \mathcal{H}$  be such that there exists a map  $T_{\xi, \eta} : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$  such that*

$$\tau(T_{\xi, \eta}(x)y) = \langle x\xi y, \eta \rangle, \quad x, y \in \mathbb{C}[G]. \quad (4.1)$$

*Then  $T_{\xi, \eta}$  is called the coefficient of  $\mathcal{H}$  at  $\xi, \eta$ . Set  $T_\xi := T_{\xi, \xi}$ . We say that the coefficient  $T_{\xi, \eta}$  is in  $S_p$  with  $p \in [1, \infty]$  if  $T_{\xi, \eta}$  exists and extends to a bounded operator  $T_{\xi, \eta} : \ell^2(G) \rightarrow \ell^2(G)$  that is moreover in the Schatten class  $S_p := S_p(\ell^2(G))$ .*

Note that if the map  $T_{\xi, \eta}$  is existent then it is uniquely determined by (4.1). Indeed, if  $T'_{\xi, \eta}$  is another map with this property then  $\tau((T_{\xi, \eta} - T'_{\xi, \eta})(x)y) = 0$  for all  $x, y \in \mathbb{C}[G]$  so that  $T'_{\xi, \eta} = T_{\xi, \eta}$ .

**Remark 4.2.2.** In [AP17, Definition 13.1.6] the notion of a coefficient of a von Neumann bimodule is defined. Definition 4.2.1 is an algebraic analogue which is more convenient for our purposes. The reason that we work in this algebraic setting is that the bimodules we consider in this chapter are *a priori* not necessarily von Neumann bimodules. In fact for the gradient bimodules we consider in Section 4.3 this is not even true in general. However, under the conditions of Proposition 4.2.3 the normal extensions of the left and right actions automatically exist.

**Proposition 4.2.3** (Quasi-containment). *Let  $\mathcal{H}$  be a  $\mathbb{C}[G]$  bimodule. Suppose that there exists a dense subset  $\mathcal{H}_0 \subset \mathcal{H}$  such that for any  $\xi \in \mathcal{H}_0$  the coefficient  $T_\xi : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$  is in  $S_2$ . Then the left and right  $\mathbb{C}[G]$  actions on  $\mathcal{H}$  extend to (bounded) normal  $\mathcal{L}(G)$  actions and the  $\mathcal{L}(G)$  bimodule  $\mathcal{H}$  is quasi-contained in the coarse bimodule  $\ell^2(G) \otimes \ell^2(G)$ .*

*Proof.* Take  $\xi \in \mathcal{H}_0$ . Define the functional

$$\rho : \mathbb{C}[G] \otimes_{\text{alg}} \mathbb{C}[G]^{\text{op}} \rightarrow \mathbb{C} : x \otimes y^{\text{op}} \mapsto \langle x \cdot \xi \cdot y, \xi \rangle.$$

For  $x, y \in \mathbb{C}[G]$  by definition of  $T_\xi$ ,

$$\rho(x \otimes y^{\text{op}}) = \langle x \cdot \xi \cdot y, \xi \rangle = \tau(T_\xi(x)y) = \tau(yT_\xi(x)) = \langle T_\xi(x), y^* \rangle_\tau.$$

Now as  $T_\xi$  is Hilbert-Schmidt there exists a vector  $\zeta_\xi \in \ell^2(G) \otimes \ell^2(G)$  such that

$$\rho(x \otimes y^{\text{op}}) = \langle x \otimes y^{\text{op}}, \zeta_\xi \rangle = \langle (x \otimes y^{\text{op}}) \cdot (1 \otimes 1), \zeta_\xi \rangle.$$

This shows that  $\rho$  extends contractively to  $C_r^*(G) \otimes_{\min} C_r^*(G)$ . Moreover, this shows that  $\rho$  extends to a normal contractive map on the von Neumann algebraic tensor product  $\mathcal{L}(G) \overline{\otimes} \mathcal{L}(G) \rightarrow \mathbb{C}$ . By Kaplansky's density theorem this extension of  $\rho$  is moreover positive. Since  $\ell^2(G) \otimes \ell^2(G)$  is the standard form of  $\mathcal{L}(G) \overline{\otimes} \mathcal{L}(G)^{\text{op}}$  there exists  $\eta \in \ell^2(G) \otimes \ell^2(G)$  such that

$$\rho(x \otimes y^{\text{op}}) = \langle x \cdot \eta \cdot y, \eta \rangle, \quad x, y \in \mathcal{L}(G).$$

This proves that the conditions of [CIW21, Lemma 2.2] are fulfilled and hence  $\mathcal{H}$  is quasi-contained in the coarse bimodule. We already observed in the preliminaries that this quasi-containment implies that the left and right actions extend to normal actions of  $\mathcal{L}(G)$ .  $\square$

A subset  $\mathcal{H}_{00} \subseteq \mathcal{H}$  of a  $\mathbb{C}[G]$  bimodule  $\mathcal{H}$  is called *cyclic* if  $\mathcal{H}_0 := \text{span} \mathbb{C}[\Gamma] \mathcal{H}_{00} \mathbb{C}[\Gamma]$  is dense in  $\mathcal{H}$ . The following lemma tells us that we can reduce Proposition 4.2.3 to checking the property only for the coefficient in a cyclic subset.

**Lemma 4.2.4** (Reduction to cyclic subset). *Suppose that  $\mathcal{H}_{00} \subseteq \mathcal{H}$  is a subset whose coefficients  $T_{\xi, \eta}$  for  $\xi, \eta \in \mathcal{H}_{00}$  are in  $S_2$ . Then for  $\xi, \eta \in \mathcal{H}_0 := \text{span} \mathbb{C}[G] \mathcal{H}_{00} \mathbb{C}[G]$  the coefficients  $T_{\xi, \eta}$  are in  $S_2$ . Consequently, if  $\mathcal{H}_{00}$  is cyclic then  $\mathcal{H}$  is a  $\mathcal{L}(G)$  bimodule that is quasi-contained in the coarse bimodule  $\ell^2(G) \otimes \ell^2(G)$ .*

*Proof.* Let  $\xi' = \lambda_g \xi \lambda_h$  and  $\eta' = \lambda_s \eta \lambda_t$  for some  $g, h, s, t \in G$  and  $\xi, \eta \in \mathcal{H}_{00}$ . We have that

$$\begin{aligned} \tau(T_{\xi', \eta'}(x)y) &= \langle x \xi' y, \eta' \rangle = \langle x \lambda_g \xi \lambda_h y, \lambda_s \eta \lambda_t \rangle = \langle \lambda_{s^{-1}} x \lambda_g \xi \lambda_h y \lambda_{t^{-1}}, \eta \rangle \\ &= \tau(T_{\xi, \eta}(\lambda_{s^{-1}} x \lambda_g) \lambda_h y \lambda_{t^{-1}}) = \tau(\lambda_{t^{-1}} T_{\xi, \eta}(\lambda_{s^{-1}} x \lambda_g) \lambda_h y). \end{aligned}$$

This shows that  $T_{\xi', \eta'}(x) = \lambda_{t^{-1}} T_{\xi, \eta}(\lambda_{s^{-1}} x \lambda_g) \lambda_h$  and so  $T_{\xi', \eta'}$  is in  $S_2$ . The first statement then follows by linearity. By Proposition 4.2.3 we find that  $\mathcal{H}$  is quasi-contained in the coarse bimodule  $\ell^2(G) \otimes \ell^2(G)$  in case  $\mathcal{H}_{00}$  is cyclic.  $\square$

#### 4.2.2. TENSORING BIMODULES

If  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are two  $\mathbb{C}[G]$  bimodules then we can construct a bimodule  $\mathcal{H}_1 \otimes_G \mathcal{H}_2$ , which, as a Hilbert space, is the same as  $\mathcal{H}_1 \otimes \mathcal{H}_2$  and the actions are given by

$$s \cdot (\xi \otimes \eta) := s\xi \otimes s\eta \quad \text{and} \quad (\xi \otimes \eta)s := \xi s \otimes \eta s, \quad \xi \in \mathcal{H}_1, \eta \in \mathcal{H}_2, s \in G.$$

The actions extend by linearity to actions of  $\mathbb{C}[G]$ . If we take an  $n$ -fold tensor power of a given bimodule  $\mathcal{H}$ , it will be denoted by  $\mathcal{H}_G^{\otimes n}$ . For later use we also recall that the comultiplication

$$\Delta_G: \mathbb{C}[G] \rightarrow \mathbb{C}[G] \otimes \mathbb{C}[G]$$

is given by the linear extension of the assignment  $g \mapsto g \otimes g, g \in G$ . Then  $\Delta_G$  extends to an isometry  $\ell^2(G) \rightarrow \ell^2(G) \otimes \ell^2(G)$  which we still denote by  $\Delta_G$ .

**Lemma 4.2.5.** *Let  $1 \leq p, q, r \leq \infty$  with  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ . There exists a bounded bilinear map*

$$S_p \times S_q \rightarrow S_r : (x, y) \mapsto \Delta_G^*(x \otimes y) \Delta_G.$$

*Proof.* For  $r = 1$  take  $(x, y) \in S_p \times S_q$  both positive so that  $\Delta_G^*(x \otimes y)\Delta_G \in S_1$  is positive. Then

$$\begin{aligned} \|\Delta_G^*(x \otimes y)\Delta_G\|_r &= \tau(\Delta_G^*(x \otimes y)\Delta_G) = \sum_{g \in G} \langle xg, g \rangle \langle yg, g \rangle \\ &\leq \left( \sum_{g \in G} \langle xg, g \rangle^p \right)^{\frac{1}{p}} \left( \sum_{g \in G} \langle yg, g \rangle^q \right)^{\frac{1}{q}} = \|x\|_p \|y\|_q. \end{aligned}$$

As every element in  $S_p$  and  $S_q$  can be written as a linear combination of 4 positive elements with smaller or equal norm the lemma follows for  $r = 1$ . Now, take  $r = \infty$ . Then also  $p = q = \infty$  and for  $(x, y) \in S_p \times S_q$  we see that  $\Delta_G^*(x \otimes y)\Delta_G \in S_r$ . Furthermore, we have the norm estimate

$$\|\Delta_G^*(x \otimes y)\Delta_G\| \leq \|\Delta_G^*\| \cdot \|x \otimes y\| \cdot \|\Delta_G\| \leq \|x\| \cdot \|y\|.$$

The lemma then follows from bilinear complex interpolation [BL12, Theorem 4.4.1]  $\square$

**Lemma 4.2.6.** *Let  $1 \leq p, q, r \leq \infty$  with  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ . Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be  $\mathbb{C}[G]$  bimodules and let  $\xi \in \mathcal{H}_1$  and  $\eta \in \mathcal{H}_2$ . Suppose that the coefficient  $T_\xi$  is in  $S_p$  and the coefficient  $T_\eta$  is in  $S_q$ . Then the coefficient  $T_{\xi \otimes \eta}$  of  $\mathcal{H}_1 \otimes_\Gamma \mathcal{H}_2$  is in  $S_r$ .*

*Proof.* We have for  $s, t \in G$ ,

$$\begin{aligned} \tau(T_{\xi \otimes \eta}(s)t) &= \langle s\xi t \otimes s\eta t, \xi \otimes \eta \rangle \\ &= \langle s\xi t, \xi \rangle \langle s\eta t, \eta \rangle \\ &= \tau(T_\xi(s)t) \tau(T_\eta(s)t). \end{aligned}$$

It follows that  $T_{\xi \otimes \eta} = \Delta_G^*(T_\xi \otimes T_\eta)\Delta_G$ . We conclude the proof by Lemma 4.2.5.  $\square$

**Proposition 4.2.7.** *Let  $\mathcal{H}$  be a  $\mathbb{C}[G]$  bimodule such that for a dense subset of  $\mathcal{H}$  the coefficients are in  $S_p$ . Then the bimodule  $\mathcal{H}_G^{\otimes n}$  is quasi-contained in the coarse bimodule for any  $n \geq \frac{p}{2}$ .*

*Proof.* By Lemma 4.2.6 (and induction) we get that a dense subset of coefficients of  $\mathcal{H}_G^{\otimes n}$  is in  $S_{\frac{p}{n}} \subset S_2$ , so by Proposition 4.2.3 we get the quasi-containment.  $\square$

**Definition 4.2.8.** *Let  $\mathcal{H}$  and  $\mathcal{K}$  be  $\mathbb{C}[G]$  bimodules. A linear map  $V : \mathcal{H} \rightarrow \mathcal{K}$  is called almost bimodular if for every  $x, y \in \mathbb{C}[G]$  the map*

$$\mathcal{H} \rightarrow \mathcal{K} : \xi \mapsto xV(\xi)y - V(x\xi)y,$$

*is compact.*

**Lemma 4.2.9.** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be bimodules over  $\mathbb{C}[G]$ . Suppose  $V_1 : \ell^2(G) \rightarrow \mathcal{H}_1$  and  $V_2 : \ell^2(G) \rightarrow \mathcal{H}_2$  are almost bimodular bounded linear maps. Then*

$$V_1 * V_2 := (V_1 \otimes V_2) \circ \Delta_G : \ell^2(G) \rightarrow \mathcal{H}_1 \otimes_G \mathcal{H}_2$$

*is almost bimodular.*

*Proof.* It suffices to check the almost bimodularity for  $x = s$  and  $y = t$ , as the general case will follow by taking linear combinations. For a map  $V : \ell^2(G) \rightarrow \mathcal{H}$  with  $\mathcal{H}$  a  $\mathbb{C}[G]$  bimodule we will write  $(sVt)(\xi) = sV(\xi)t$  and  $V^{s,t}(\xi) := V(s\xi t)$  where  $\xi \in \ell^2(G)$ . It follows from the definitions that

$$s(V_1 * V_2)t = (s \otimes s) \cdot ((V_1 \otimes V_2) \circ \Delta_G) \cdot (t \otimes t) = (sV_1 t * sV_2 t).$$

Further, for  $\xi \in \ell^2(G)$ ,

$$\begin{aligned} (V_1 * V_2)^{s,t}(\xi) &= (V_1 \otimes V_2)\Delta_G(s\xi t) \\ &= (V_1 \otimes V_2)((s \otimes s)\Delta_G(\xi)(t \otimes t)) = (V_1^{s,t} * V_2^{s,t})(\xi). \end{aligned}$$

Hence

$$(V_1 * V_2)^{s,t} = V_1^{s,t} * V_2^{s,t}.$$

Therefore we have

$$s(V_1 * V_2)t - (V_1 * V_2)^{s,t} = ((sV_1 t - V_1^{s,t}) * sV_2 t) + (V_1^{s,t} * (sV_2 t - V_2^{s,t})). \quad (4.2)$$

By our assumption the operators  $sV_1 t - V_1^{s,t}$  and  $sV_2 t - V_2^{s,t}$  are compact. So it suffices to check that if  $K$  is compact and  $T$  is bounded then both  $K * T$  and  $T * K$  are compact. To check that, for every finite subset  $F \subset G$  consider the corresponding finite rank orthogonal projection  $P_F$  onto the linear span of  $\delta_s \in \ell^2(G)$ ,  $s \in F$ . We can easily check that  $\Delta_G \circ P_F = (P_F \otimes \text{Id}) \circ \Delta_G = (\text{Id} \otimes P_F) \circ \Delta_G$ . It follows that  $(K * T)P_F = (KP_F * T)$ , so  $(K * T)P_F - K * T = (KP_F - K) * T$ . Further,

$$\|(K * T)P_F - K * T\| \leq \|KP_F - K\| \|T\|.$$

By compactness of  $K$  we see that  $\|KP_F - K\|$  goes to 0 as  $F$  increases. So  $K * T$  can be approximated in norm by finite rank operators and thus is compact. The proof for  $T * K$  is the same. Hence the operator in (4.2) is compact, i.e.  $V_1 * V_2$  is almost bimodular.  $\square$

**Lemma 4.2.10.** *For  $i = 1, 2$  suppose that  $V_i : \ell^2(G) \rightarrow \mathcal{H}_i$  is a partial isometry to a  $\mathbb{C}[G]$  bimodule  $\mathcal{H}_i$  such that  $\ker(V_i)$  is spanned linearly by a subset  $F_i \subseteq G$ . Then  $V_1 * V_2$  is a partial isometry whose kernel is the linear span of  $F_1 \cup F_2$ .*

*Proof.* The comultiplication  $\Delta_G$  is an isometry  $\ell^2(G) \rightarrow \ell^2(G) \otimes \ell^2(G)$ . Clearly  $\Delta_G(s) = s \otimes s$  is contained in  $\ker(V_1 \otimes V_2)$  if  $s$  is in  $F_1 \cup F_2$ . Further,  $V_1 \otimes V_2$  is isometric on  $\ker(V_1)^\perp \otimes \ker(V_2)^\perp$  and so it is certainly isometric on the linear span of  $\Delta_G(s) = s \otimes s$ ,  $s \in G \setminus (F_1 \cup F_2)$ . These observations conclude the lemma.  $\square$

### 4.2.3. THE AKEMANN-OSTRAND PROPERTY $\text{AO}^+$ AND STRONG SOLIDITY

This section serves as a blackbox that connects the theory that we develop in this chapter to a central concept in deformation-rigidity theory: strong solidity. Firstly we recall a version of the Akemann-Ostrand property that was introduced in [Iso15a].

**Definition 4.2.11.** *A finite von Neumann algebra  $M$  has property  $\text{AO}^+$  if there exists a  $\sigma$ -weakly dense unital  $C^*$ -subalgebra  $A \subseteq M$  such that:*

1.  $A$  is locally reflexive [BO08, Section 9];
2. There exists a unital completely positive map  $\theta : A \otimes_{\min} A^{\text{op}} \rightarrow B(L^2(M))$  such that  $\theta(a \otimes b^{\text{op}}) - ab^{\text{op}}$  is compact for all  $a, b \in A$ .

The following theorem will be the main tool to prove that certain von Neumann algebras have  $\text{AO}^+$  using the Riesz transforms in this chapter.

**Theorem 4.2.12** (Proposition 5.2 in [CIW21]). *Let  $\mathcal{H}$  be a  $\mathbb{C}[G]$  bimodule and let  $V : \ell^2(G) \rightarrow \mathcal{H}$  be bounded. Assume that  $\mathcal{H}$  is quasi-contained in the coarse bimodule of  $G$ , that  $V$  is almost bimodular and that  $V^*V$  is Fredholm. Assume that  $C_r^*(G)$  is locally reflexive. Then  $\mathcal{L}(G)$  satisfies  $\text{AO}^+$ .*

The following theorem in turn yields the strong solidity results from  $\text{AO}^+$ . For the notion of weak amenability we refer to [BO08, Section 12.3]. If  $G$  is a weakly amenable discrete group then  $C_r^*(G)$  is automatically locally reflexive. All Coxeter groups are weakly amenable [Fen02], [Jan02] as well as simple Lie groups of real rank one [CH85], [CH89]. We recall that amenability of a von Neumann algebra was defined in the introduction and preliminaries. We note that, in this chapter, amenability and weak amenability shall not appear explicitly in the proofs. We recall that a von Neumann algebra is called *diffuse* if it does not contain minimal projections.

**Definition 4.2.13.** *A finite von Neumann algebra  $M$  is called strongly solid if for every diffuse amenable von Neumann subalgebra  $B \subseteq M$  we have that the normalizer*

$$\text{Nor}_M(B) := \{u \in M : u \text{ unitary and } uBu^* = B\},$$

*generates a von Neumann algebra that is amenable again.*

**Theorem 4.2.14** (See [PV14b] and [Iso15a]). *Let  $G$  be a discrete weakly amenable group such that  $\mathcal{L}(G)$  satisfies  $\text{AO}^+$ . Then  $\mathcal{L}(G)$  is strongly solid.*

### 4.3. QUANTUM MARKOV SEMIGROUPS, GRADIENTS AND THE RIESZ TRANSFORMS

In this section we study quantum Markov semigroups of Fourier multipliers on the group von Neumann algebra of a discrete group. We introduce the associated Riesz transform which takes values in a certain bimodule that we call the ‘gradient bimodule’ or the bimodule associated with the ‘carré du champ’. Our main goal is to analyze when the coefficients of this bimodule are in the Schatten  $S_p$  space and consequently when this bimodule is quasi-contained in the coarse bimodule. We also show that under very natural conditions the Riesz transform is an almost bimodular map in the sense of Section 4.2.

#### 4.3.1. QUANTUM MARKOV SEMIGROUPS, THE GRADIENT BIMODULE AND THE RIESZ TRANSFORM

We start defining the Riesz transform of a quantum Markov semigroup (QMS). Recall the definition of a QMS.

**Definition 4.3.1.** A quantum Markov semigroup (QMS) on a finite von Neumann algebra  $(M, \tau)$  is a semigroup  $\Phi = (\Phi_t)_{t \geq 0}$  of normal unital completely positive maps  $\Phi_t : M \rightarrow M$  that are trace preserving ( $\tau \circ \Phi_t = \tau$ ,  $t \geq 0$ ) and such that for every  $x \in M$  the map  $t \mapsto \Phi_t(x)$  is strongly continuous. We shall moreover assume that a quantum Markov semigroup is symmetric meaning that for every  $x, y \in M$  and  $t \geq 0$  we have  $\tau(\Phi_t(x)y) = \tau(x\Phi_t(y))$ . So QMS always means symmetric QMS.

Fix a QMS  $\Phi = (\Phi_t)_{t \geq 0}$  on a finite von Neumann algebra  $M$  with a normal faithful tracial state  $\tau$ . By the Kadison-Schwarz inequality there exists a semigroup of contractions  $(\Phi_t^{(2)})_{t \geq 0}$  on  $L^2(M) = L^2(M, \tau)$  by

$$\Phi_t^{(2)}(x\Omega_\tau) = \Phi_t(x)\Omega_\tau, \quad x \in M.$$

Here  $\Omega_\tau = 1_M$  is the cyclic vector in  $L^2(M)$ . The semigroup  $(\Phi_t^{(2)})_{t \geq 0}$  is moreover point-norm continuous, i.e. it is continuous for the strong topology on  $B(L^2(M))$ . By a special case of the Hille-Yosida theorem there exists an unbounded positive self-adjoint operator  $\Delta$  on  $L^2(M)$  such that  $\Phi_t^{(2)} = \exp(-t\Delta)$ . We will assume the existence of a  $\sigma$ -weakly dense  $*$ -subalgebra  $A \subseteq M$  such that  $A\Omega_\tau \subseteq \text{Dom}(\Delta)$  and  $\Delta(A\Omega_\tau) \subseteq A\Omega_\tau$ . By identifying  $a \in A$  with  $a\Omega_\tau \in L^2(M)$  we may and will view  $\Delta$  as a map  $A \rightarrow A$ . We now introduce the carré du champ or gradient as

$$\Gamma : A \times A \rightarrow A : (a, b) \mapsto \frac{1}{2}(\Delta(b^*)a + b^*\Delta(a) - \Delta(b^*a)).$$

Let  $\mathcal{H}$  be any  $A$  bimodule, i.e. we recall  $\mathcal{H}$  is a Hilbert space with commuting left and right actions of  $A$ . For  $a, b \in A, \xi, \eta \in \mathcal{H}$  we set the possibly degenerate inner product on  $A \otimes_{\text{alg}} \mathcal{H}$  by

$$\langle a \otimes \xi, b \otimes \eta \rangle = \langle \Gamma(a, b)\xi, \eta \rangle.$$

The Hilbert space obtained by quotienting out the degenerate part of this inner product and taking the completion shall be denoted by  $\mathcal{H}_\nabla$ . We denote by  $a \otimes_\nabla \xi$  the element  $a \otimes \xi$  identified in  $\mathcal{H}_\nabla$ . For  $x, y, a \in A$  and  $\xi \in \mathcal{H}$  we define commuting left and right actions by

$$x \cdot (a \otimes_\nabla \xi) = xa \otimes_\nabla \xi - x \otimes_\nabla a\xi, \quad (a \otimes_\nabla \xi) \cdot y = a \otimes_\nabla \xi y. \quad (4.3)$$

In this chapter we shall only deal with the case  $\mathcal{H} = L^2(M)$  with actions by left and right multiplication of  $M$ . In this case the actions (4.3) extend to contractive actions on the norm closure of  $A$ . We do not say anything about whether the actions are normal at this point, but rather use Proposition 4.2.3 to show that they are normal in the cases that are relevant. We define a derivation

$$\nabla : A \rightarrow L^2(M)_\nabla : a \mapsto a \otimes_\nabla \Omega_\tau.$$

More precisely,  $\nabla$  satisfies the Leibniz rule

$$\nabla(xy) = x\nabla(y) + \nabla(x)y, \quad x, y \in A$$

with respect to the module actions (4.3). This fact uses that  $\tau$  is tracial. Since  $\Phi_t$  is  $\tau$ -preserving it follows that for  $x \in A$  we have  $\langle \Delta(x)\Omega_\tau, \Omega_\tau \rangle = \frac{d}{dt}|_{t=0} \langle \Phi_t(x)\Omega_\tau, \Omega_\tau \rangle = 0$  (upper derivative). Therefore, as  $\Delta \geq 0$ ,

$$\begin{aligned} \|\nabla(a)\|^2 &= \langle \Gamma(a, a)\Omega_\tau, \Omega_\tau \rangle \\ &= \frac{1}{2} (\langle \Delta(a)\Omega_\tau, a\Omega_\tau \rangle + \langle a\Omega_\tau, \Delta(a)\Omega_\tau \rangle - \langle \Delta(a^*a)\Omega_\tau, \Omega_\tau \rangle) \\ &= \frac{1}{2} (\langle \Delta^{\frac{1}{2}}(a)\Omega_\tau, \Delta^{\frac{1}{2}}(a)\Omega_\tau \rangle + \langle \Delta^{\frac{1}{2}}(a)\Omega_\tau, \Delta^{\frac{1}{2}}(a)\Omega_\tau \rangle - 0) \\ &= \|\Delta^{\frac{1}{2}}(a)\Omega_\tau\|^2. \end{aligned}$$

It follows that we have an isometric map

$$\nabla \Delta^{-\frac{1}{2}} : \ker(\Delta)^\perp \rightarrow L^2(M)_\nabla.$$

We extend this map to a partial isometry

$$R_\Phi : L^2(M) \rightarrow L^2(M)_\nabla$$

by defining it to have  $\ker(\Delta)$  as its kernel. We call  $R_\Phi$  the *Riesz transform*.

**Remark 4.3.2.** This Riesz transform was also used in [CIW21, Section 5]. Note that mapping that was introduced in [CIW21, Section 5, Eqn. (5.1)] differs from  $R_\Phi$  only on  $\ker(\Delta)$ . If the kernel of  $\Delta$  is finite-dimensional then  $R_\Phi$  agrees with [CIW21, Eqn. (5.1)] up to a finite rank perturbation. In particular this is the case if  $\Delta \geq 0$  has a compact resolvent. The results of [CIW21, Section 5] stay intact under this finite rank perturbation.

#### 4.3.2. COEFFICIENTS OF THE GRADIENT BIMODULE

We now start our analysis of coefficients of the gradient bimodule. The following definition of ‘gradient- $S_p$ ’ that first occurred in [Cas21] (for  $p = 2$ ) and [CIW21] (for general  $p$ ) plays a central role in this chapter. The definition may depend on the choice of the  $\sigma$ -weakly dense subalgebra  $A$  of  $M$  which we fixed before in our notation. This thesis contains the first results for the gradient- $S_p$  property in the context of group algebras.

**Definition 4.3.3.** Let  $p \in [1, \infty]$ . Consider a QMS  $\Phi$  on a finite von Neumann algebra  $(M, \tau)$  with generator  $\Delta$  and a dense  $*$ -subalgebra  $A \subseteq M$  as in Section 4.3.1. Then  $\Phi$  is called gradient- $S_p$  if for every  $a, b \in A$  the map

$$\Psi^{a,b} : A \rightarrow A : x \mapsto \Delta(axb) + a\Delta(x)b - \Delta(ax)b - a\Delta(xb),$$

extends as  $x\Omega_\tau \mapsto \Psi^{a,b}(x)\Omega_\tau$  to a bounded map on  $L^2(M)$  that is moreover in the Schatten  $p$ -class  $S_p = S_p(L^2(M))$ .

**Remark 4.3.4.** Since  $\Delta$  is self-adjoint we have for  $a, b, x, y \in A$ ,

$$\begin{aligned} \langle \Psi^{a,b}(x)\Omega_\tau, y\Omega_\tau \rangle &= \langle (\Delta(axb) + a\Delta(x)b - \Delta(ax)b - a\Delta(xb))\Omega_\tau, y\Omega_\tau \rangle \\ &= \langle x\Omega_\tau, (\Delta(a^*yb^*) + a^*\Delta(y)b^* - \Delta(a^*x)b^* - a^*\Delta(yb^*))\Omega_\tau \rangle \\ &= \langle x\Omega_\tau, \Psi^{a^*,b^*}(y)\Omega_\tau \rangle. \end{aligned}$$

So it follows that

$$(\Psi^{a,b})^* = \Psi^{a^*,b^*}, \quad a, b \in A. \quad (4.4)$$

The following lemma simplifies verifying whether a QMS has the gradient- $S_p$  property.

**Lemma 4.3.5** (Condition that implies Gradient- $S_p$  property). *Let  $p \in [1, \infty]$ . Let  $A_0 \subseteq A$  be a self-adjoint subset that generates  $A$  as a  $*$ -algebra. Then  $(\Phi_t)_{t \geq 0}$  is gradient- $S_p$  if and only if for all  $a, b \in A_0$  we have that  $\Psi^{a,b}$  is in  $S_p$ .*

*Proof.* The only if statement follows directly from the definition of gradient- $S_p$ . We will prove the other direction. We must prove that  $\Psi^{a,b}$  is in  $S_p$  for every  $a, b \in A$ . Since  $A_0$  is self-adjoint,  $A$  is generated by  $A_0$  as an algebra. So  $A$  is spanned linearly by  $(A_0)^n$ ,  $n \in \mathbb{N}$ . Note that the map  $\Psi^{a,b}$  depends linearly on both  $a$  and  $b$ . So in order to prove that  $\Psi^{a,b}$  is in  $S_p$  for all  $a, b \in A$  it suffices to prove that  $\Psi^{a,b}$  is in  $S_p$  for all  $a, b \in (A_0)^n$  for every  $n \in \mathbb{N}_{\geq 1}$ . We shall prove this latter statement by induction on  $n$ . The case  $n = 1$  holds by assumption of the lemma. We now assume that we have proved the statement for  $n$  and shall prove it for  $n + 1$ .

First note that for  $u_1, u_2, v, w \in A$  we have

$$\begin{aligned} \Psi^{u_1 u_2, w}(v) &= \Delta(u_1 u_2 v w) + u_1 u_2 \Delta(v) w - \Delta(u_1 u_2 v) w - u_1 u_2 \Delta(v w) \\ &= (\Delta(u_1 u_2 v w) + u_1 \Delta(u_2 v) w - \Delta(u_1 u_2 v) w - u_1 \Delta(u_2 v w)) \\ &\quad + u_1 (\Delta(u_2 v w) + u_2 \Delta(v) w - \Delta(u_2 v) w - u_2 \Delta(v w)) \\ &= \Psi^{u_1, w}(u_2 v) + u_1 \Psi^{u_2, w}(v), \end{aligned} \tag{4.5}$$

and likewise for  $u, v, w_1, w_2 \in A$  we have

$$\Psi^{u, w_2 w_1}(v) = \Psi^{u, w_1}(v w_2) + \Psi^{u, w_2}(v) w_1. \tag{4.6}$$

Combining these expressions we see that for  $u = u_1 u_2$  and  $w = w_2 w_1$  we have

$$\begin{aligned} \Psi^{u, w}(v) &= \Psi^{u_1 u_2, w}(v) \\ &= \Psi^{u_1, w}(u_2 v) + u_1 \Psi^{u_2, w}(v) \\ &= \Psi^{u_1, w_2 w_1}(u_2 v) + u_1 \Psi^{u_2, w_2 w_1}(v) \\ &= (\Psi^{u_1, w_1}(u_2 v w_2) + \Psi^{u_1, w_2}(u_2 v) w_1) + u_1 (\Psi^{u_2, w_1}(v w_2) + \Psi^{u_2, w_2}(v) w_1). \end{aligned} \tag{4.7}$$

By the induction hypothesis we have that  $\Psi^{u_1, w_1}, \Psi^{u_1, w_2}, \Psi^{u_2, w_1}, \Psi^{u_2, w_2}$  are all in  $S_p$ . Since the  $S_p$  class forms an ideal in  $B(L^2(M, \tau))$  we have that the four operators in (4.7) are all in  $S_p$ . This finishes the induction and thus shows that the associated semigroup is gradient- $S_p$ .  $\square$

### 4.3.3. ALMOST BIMODULARITY OF THE RIESZ TRANSFORM

Next we analyze when the Riesz transform is almost bimodular. Therefore we introduce the following notions. We say that a QMS  $\Phi$  on a finite von Neumann algebra is *filtered* if the generator  $\Delta$  has a compact resolvent (i.e.  $(\Delta - z)^{-1}$  is compact for some  $z \in \mathbb{C}$ ) and for every eigenvalue  $\lambda$  of  $\Delta$  there exists a (necessarily finite dimensional) subspace  $A(\lambda) \subseteq A$  such that  $A(\lambda)\Omega_\tau$  equals the eigenspace of  $\Delta$  at eigenvalue  $\lambda$ . Moreover, we assume that for an increasing enumeration  $(\lambda_n)_{n \geq 0}$  of the eigenvalues of  $\Delta$  we have for all  $k, l \geq 0$  that

$$A = \bigoplus_{n=0}^{\infty} A(\lambda_n), \quad A(\lambda_l) A(\lambda_k) \subseteq \bigoplus_{n=0}^{l+k} A(\lambda_n).$$

We will further say that  $\Delta$  has *subexponential growth* if

$$\lim_{k \rightarrow \infty} \frac{\lambda_{k+1}}{\lambda_k} = 1.$$

*Remark 4.3.6.* In [Cas22] a more general notion of filtering and subexponential growth was considered for central Fourier multipliers on compact quantum groups. The current ‘linear’ type of definition suffices however for our purposes.

**Theorem 4.3.7** (Theorem 5.12 of [CIW21]). *Suppose that a QMS  $\Phi$  on a finite von Neumann algebra  $M$  is filtered with subexponential growth. Then the Riesz transform  $R_\Phi : L^2(M) \rightarrow L^2(M)_\nabla$  is almost bimodular.*

#### 4.3.4. SEMIGROUPS OF FOURIER MULTIPLIERS ON GROUP VON NEUMANN ALGEBRAS

Now consider the case that  $M$  is a group von Neumann algebra  $\mathcal{L}(G)$  of a discrete group  $G$  and  $A = \mathbb{C}[G]$ . The following theorem is a version of Schönberg’s theorem.

**Theorem 4.3.8** (See Appendix C of [BHV08]). *Let  $\psi : G \rightarrow \mathbb{R}$ . The following are equivalent:*

1.  $\psi$  is conditionally of negative type.
2. There exists a (recall: symmetric) QMS  $\Phi = (\Phi_t)_{t \geq 0}$  on  $M$  determined by

$$\Phi_t(\lambda_g) = \exp(-t\psi(g))\lambda_g, \quad g \in G.$$

We will call a QMS  $\Phi$  as in Theorem 4.3.8 a *QMS of Fourier multipliers* or a QMS associated with  $\psi : G \rightarrow \mathbb{R}$ . Note that we assume such QMS’s to be symmetric. We view the generator of this semigroup as a map on  $\mathbb{C}[G]$  which is given by

$$\Delta_\psi : \mathbb{C}[G] \rightarrow \mathbb{C}[G] : \lambda_g \mapsto \psi(g)\lambda_g.$$

The following Theorem 4.3.9 connects Definition 4.3.3 to Section 4.2.

**Theorem 4.3.9.** *Consider a QMS  $\Phi = (\Phi_t)_{t \geq 0}$  of Fourier multipliers on  $\mathcal{L}(G)$ . Let*

$$\mathcal{H}_{00} = \{a \otimes_\nabla c \in \ell^2(G)_\nabla : a, c \in \mathbb{C}[G]\} \subseteq \ell^2(G)_\nabla.$$

*If  $\Phi$  is gradient- $S_p$  with  $p \in [1, \infty]$  then for every  $\xi, \eta \in \text{span} \mathbb{C}[G] \mathcal{H}_{00} \mathbb{C}[G]$  the coefficient  $T_{\xi, \eta}$  is in  $S_p$ .*

*Proof.* Let  $a, b, c, d, x, y \in \mathbb{C}[G]$  and let  $\xi = a \otimes_\nabla c, \eta = b \otimes_\nabla d$  be elements of  $\mathcal{H}_{00}$ . We have

$$\begin{aligned} 2\langle x \cdot (a \otimes_\nabla c) \cdot y, b \otimes_\nabla d \rangle &= 2\langle xa \otimes_\nabla cy - x \otimes_\nabla acy, b \otimes_\nabla d \rangle \\ &= 2\langle \Gamma(xa, b)cy - \Gamma(x, b)acy, d \rangle_\tau \\ &= \langle (b^* \Delta(xa) + \Delta(b^*)xa - \Delta(b^*xa) - b^* \Delta(x)a - \Delta(b^*)xa + \Delta(b^*x)a)cy, d \rangle_\tau \\ &= \langle (\Delta(b^*x)a + b^* \Delta(xa) - \Delta(b^*xa) - b^* \Delta(x)a)cy, d \rangle_\tau \\ &= -\langle \Psi^{b^*, a}(x)cy, d \rangle_\tau \\ &= -\tau(d^* \Psi^{b^*, a}(x)cy). \end{aligned}$$

We conclude that

$$-2T_{\xi,\eta}(x) = d^* \Psi^{b^*,a}(x)c.$$

In particular if  $\Psi^{b^*,a}$  is in  $S_p$  then so is  $T_{\xi,\eta}$ . The result now follows from Lemma 4.2.4.  $\square$

Let us now show that in the case of semigroups of Fourier multipliers, the case gradient- $S_p$  is conceptually much easier to understand. Consider again a QMS  $\Phi = (\Phi_t)_{t \geq 0}$  of Fourier multipliers associated with a function  $\psi : G \rightarrow \mathbb{R}$  that is conditionally of negative type. Let  $\Delta_\psi : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$  be as before. For  $u, w \in G$  we define a function  $\gamma_{u,w}^\psi : G \rightarrow \mathbb{R}$  as

$$\gamma_{u,w}^\psi(v) = \psi(uvw) + \psi(v) - \psi(uv) - \psi(vw). \quad (4.8)$$

We have that the function  $\gamma_{u,w}^\psi$  is related to the operator  $\Psi^{\lambda_u, \lambda_w}$  associated with  $\Delta_\psi$  as follows

$$\Psi_{\Delta_\psi}^{\lambda_u, \lambda_v}(\lambda_v) = \Delta_\psi(\lambda_{uvw}) + \lambda_u \Delta_\psi(\lambda_v) \lambda_w - \Delta_\psi(\lambda_{uv}) \lambda_w - \lambda_u \Delta_\psi(\lambda_{vw}) = \gamma_{u,w}^\psi(v) \lambda_{uv}.$$

Now as by (4.4) we have  $(\Psi^{\lambda_u, \lambda_w})^* = \Psi^{\lambda_u^*, \lambda_w^*} = \Psi^{\lambda_{u^{-1}}, \lambda_{w^{-1}}}$  we obtain that

$$|\Psi^{\lambda_u, \lambda_w}|^2(\lambda_v) = \Psi^{\lambda_{u^{-1}}, \lambda_{w^{-1}}} \Psi^{\lambda_u, \lambda_w}(\lambda_v) = \gamma_{u^{-1}, w^{-1}}^\psi(uvw) \gamma_{u,w}^\psi(v) \lambda_v = |\gamma_{u,w}^\psi(v)|^2 \lambda_v. \quad (4.9)$$

Now for  $p \in [1, \infty)$ , this then means that  $|\Psi^{\lambda_u, \lambda_w}|^p(\lambda_v) = |\gamma_{u,w}^\psi(v)|^p \lambda_v$  and therefore, as  $\{\lambda_v\}_{v \in \Gamma}$  forms an orthonormal basis, we have that

$$\|\Psi^{\lambda_u, \lambda_w}\|_{S_p} = \left( \sum_{v \in \Gamma} |\Psi^{\lambda_u, \lambda_w}|^p(\lambda_v) \right)^{\frac{1}{p}} = \|\gamma_{u,w}^\psi\|_{\ell^p(G)}. \quad (4.10)$$

In order to check whether  $\Psi^{\lambda_u, \lambda_w}$  is in  $S_p$  we thus need to check whether  $\gamma_{u,w}^\psi \in \ell^p(G)$ . Moreover, for  $p = \infty$ , the condition that  $\Psi^{\lambda_u, \lambda_w} \in S_p$  means that  $\Psi^{\lambda_u, \lambda_w}$  is a compact operator, which is precisely the case when  $\gamma_{u,w}^\psi \in c_0(G)$ , i.e. when  $\gamma_{u,w}^\psi$  vanishes at infinity.

The above calculations, together with Lemma 4.3.5, give us a simple condition to check for  $p \in [1, \infty]$  whether the semigroup  $(\Phi_t)_{t \geq 0}$  is gradient- $S_p$ .

**Lemma 4.3.10.** *Let  $p \in [1, \infty)$ . Let  $G_0 \subseteq G$  be a subset that generates a discrete group  $G$  with  $G_0^{-1} = G_0$ . Let  $\Phi = (\Phi_t)_{t \geq 0}$  be a QMS associated with a proper function  $\psi : G \rightarrow \mathbb{R}$  that is conditionally of negative type. If  $\gamma_{u,w}^\psi \in \ell^p(G)$  for all  $u, w \in G_0$  then the QMS  $\Phi$  is gradient- $S_p$ . The same holds true for  $p = \infty$  when  $\ell^p(G)$  is replaced with  $c_0(G)$ .*

*Proof.* We denote  $A_0 := \{\lambda_g : g \in G_0\} \subseteq \mathbb{C}[G]$ . Since  $G_0^{-1} = G_0$  and  $G_0$  generates  $G$  we have that  $A_0$  is self-adjoint and generates  $\mathbb{C}[G]$  as an algebra. Now, if for  $u, w \in G_0$  we have that  $\gamma_{u,w}^\psi \in \ell^p(G)$  then by (4.10) we have that  $\Psi^{\lambda_u, \lambda_w} \in S_p$ . Then Lemma 4.3.5 shows that  $\Phi$  is gradient- $S_p$ . The proof is similar for  $p = \infty$ .  $\square$

**Lemma 4.3.11.** *Let  $\Phi = (\Phi_t)_{t \geq 0}$  be a QMS associated to a proper symmetric function  $\psi : G \rightarrow \mathbb{Z}$  that is conditionally of negative type. If  $\Phi$  is gradient- $S_p$  for some  $p \in [1, \infty]$  then for every  $u, v \in G$  the function  $\gamma_{u,v}^\psi : G \rightarrow \mathbb{Z}$  has compact support. In particular by (4.9) we find that  $\Psi^{\lambda_u, \lambda_v}$  is of finite rank and  $\Phi$  is gradient- $S_p$  for all  $p \in [1, \infty]$ .*

*Proof.* If  $\psi$  takes integer values then so does  $\gamma_{u,v}^\psi$  for all  $u, v \in G$ . Therefore  $\gamma_{u,v}^\psi$  is contained in  $\ell^p(G)$ ,  $p \in [1, \infty)$  or  $c_0(G)$  if and only if  $\gamma_{u,v}^\psi$  has compact support. The remainder of the lemma is directly clear.  $\square$

### 4.3.5. ALMOST BIMODULARITY OF THE RIESZ TRANSFORM FOR LENGTH FUNCTIONS

We show that a QMS of Fourier multipliers associated with a  $\mathbb{Z}_{\geq 0}$ -valued length function automatically satisfies the conditions of Theorem 4.3.7. Recall that  $\psi : G \rightarrow \mathbb{Z}_{\geq 0}$  is a length function if

$$\psi(uw) \leq \psi(u) + \psi(w) \quad \text{for all } u, w \in G. \quad (4.11)$$

**Theorem 4.3.12.** *Let  $\psi : G \rightarrow \mathbb{Z}_{\geq 0}$  be a proper length function that is conditionally of negative type. Then  $\Delta_\psi$  is moreover filtered. If  $\psi(G) = \mathbb{Z}_{\geq 0}$  or if  $G$  is finitely generated then  $\Delta_\psi$  has subexponential growth.*

*Proof.* First of all we have that  $(1 + \Delta_\psi)^{-1}(\lambda_\nu) = (1 + \psi(\nu))^{-1}\lambda_\nu$  for all  $\nu \in \Gamma$ . As  $\psi$  is proper this shows that  $(1 + \Delta_\psi)^{-1}$  is a compact operator on  $\ell^2(G)$ . Consider the finite dimensional spaces

$$\mathbb{C}[G](l) := \text{Span}\{\lambda_\nu \in \mathbb{C}[G] : \psi(\nu) = l\} \quad \text{for } l \in \mathbb{Z}_{\geq 0}. \quad (4.12)$$

Then  $\mathbb{C}[G](l)\Omega_\tau$  equals the eigenspace of  $\Delta_\psi$  at the eigenvalue  $l$ . We have

$$\mathbb{C}[G] = \bigoplus_{l \geq 0} \mathbb{C}[G](l) \quad \mathbb{C}[G](l)\mathbb{C}[G](k) \subseteq \bigoplus_{j=0}^{l+k} \mathbb{C}[G](j) \quad \text{for } l, k \geq 0 \quad (4.13)$$

where  $\bigoplus$  denotes the algebraic direct sum. The first equality holds because  $\psi$  only takes positive integer values and the second equality holds because  $\psi$  is a length function, i.e. (4.11). This shows that  $\Delta_\psi$  is filtered.

That  $\Delta_\psi$  has subexponential growth follows in the first case from the fact that  $\mathbb{Z}_{\geq 0}$  is the set of eigenvalues and we have  $(l+1)/l \rightarrow 1$  as  $l \rightarrow \infty$ . In case  $G$  is generated by a finite set  $G_0$  we set  $K := \{\max \psi(u) : u \in G_0\}$ . Then (4.11) implies that  $\mathbb{Z}_{\geq 0} \setminus \psi(G)$  cannot contain an interval of length  $K+1$ . Hence if  $\lambda_0 \leq \lambda_1 \leq \dots$  is an increasing enumeration of  $\psi(G)$  then  $\lambda_{k+1} \leq \lambda_k + K$ . Hence  $\lambda_{k+1}/\lambda_k \rightarrow 1$  as  $k \rightarrow \infty$ .  $\square$

**Corollary 4.3.13.** *Assume that  $G$  is finitely generated. Let  $\psi : G \rightarrow \mathbb{Z}_{\geq 0}$  be a proper length function that is conditionally of negative type. Let  $\Phi$  be the associated QMS of Fourier multipliers. Then the Riesz transform  $R_\Phi : \ell^2(G) \rightarrow \ell^2(G)_\nabla$  is almost bimodular.*

*Proof.* This follows from Theorem 4.3.7 and Theorem 4.3.12.  $\square$

**Theorem 4.3.14.** *Assume that  $G$  is finitely generated and that  $C_r^*(G)$  is locally reflexive. If there exists a proper length function  $\psi : G \rightarrow \mathbb{Z}_{\geq 0}$  that is conditionally of negative type such that the associated QMS is gradient- $S_p$  for some  $p \in [1, \infty)$ . Then  $\mathcal{L}(G)$  has  $AO^+$ .*

*Proof.* Let  $\mathcal{H}_\nabla := \ell^2(G)_\nabla$  be the gradient bimodule. Let  $n \geq \frac{p}{2}$ . Then by Proposition 4.2.7 the bimodule  $(\mathcal{H}_\nabla)_G^{\otimes n}$  is quasi-contained in the coarse bimodule. Let  $R_\Phi : \ell^2(G) \rightarrow \mathcal{H}_\nabla$  be the Riesz transform. The kernel of  $R_\Phi$  is spanned by all  $\delta_g$  with  $\psi(g) = 0$ . Since  $\psi$  is proper  $\ker(R_\Phi)$  is finite dimensional. By Corollary 4.3.13 we see that  $R_\Phi$  is almost bimodular. By Lemma 4.2.9 and Lemma 4.2.10 the convolved Riesz transform  $R_\Phi^{*n} : \ell^2(G) \rightarrow (\mathcal{H}_\nabla)_G^{\otimes n}$  is an almost bimodular partial isometry. Therefore we obtain  $AO^+$  from Theorem 4.2.12.

Note that in fact we could have avoided the tensor products in this proof by using Lemma 4.3.11 instead.  $\square$

#### 4.4. CHARACTERIZING GRADIENT- $S_p$ FOR COXETER GROUPS

In this section we will consider the case of Coxeter groups. For any Coxeter group the word length defines a proper length function that is conditionally of negative type [BJS88] (see also [Tit09, p. 2.22]). Therefore it determines a QMS of Fourier multipliers. The aim of this section is to find characterizations of when this specific QMS is gradient- $S_p$ .

Throughout Sections 4.4.1 – 4.4.4 we give an almost characterization of gradient- $S_p$  in terms of the Coxeter diagram. In particular we give sufficient conditions for gradient- $S_p$  that are easy to verify in Corollary 4.4.11 and Corollary 4.4.12. We also argue that these conditions are necessary for a large class of Coxeter groups. In Section 4.4.5 we show that gradient- $S_p$  is equivalent to smallness at infinity of the Coxeter group. More precisely, a certain natural compactification of the Coxeter group that was considered in [CL11], [Kli23b] (see also [Kli23a]), [LT15] is small at infinity. This result can be understood directly after Section 4.4.1.

Consider a finite set  $S = \{s_1, \dots, s_n\}$  and a symmetric matrix  $M = (m_{ij})_{1 \leq i, j \leq n}$  with  $m_{i,j} \in \mathbb{N} \cup \{\infty\}$  satisfying  $m_{i,i} = 1$  and  $m_{i,j} \geq 2$  whenever  $i \neq j$ . Occasionally we write  $m_{s_i, s_j}$  for  $m_{i,j}$ ; this notation is convenient when considering  $m_{s,t}$  without referring to the indices of the generators  $s, t \in S$ . We let  $\mathcal{W} = \langle S | M \rangle$  be a Coxeter system. In this chapter, all Coxeter systems considered are finite rank. For convenience, we will by  $\psi_S$  denote the word length function

$$\psi_S : \mathcal{W} \rightarrow \mathbb{Z}_{\geq 0} : \mathbf{w} \mapsto |\mathbf{w}|.$$

We state the following result.

**Theorem 4.4.1** (See [BJS88]). *For any Coxeter group the map  $\psi_S : \mathcal{W} \rightarrow \mathbb{Z}_{\geq 0}$  is conditionally of negative type.*

Therefore by Theorem 4.3.8 there exists a QMS of Fourier multipliers on  $\mathcal{L}(\mathcal{W})$  associated with the word length function  $\psi_S$ . The aim of the current Section 4.4 is to describe when this QMS has gradient- $S_p$ . Recall that by Lemmas 4.3.10 and 4.3.11 we must thus investigate for generators  $u, w \in S$  when precisely  $\gamma_{u,w}^{\psi_S}$  is finite rank where  $\gamma_{u,w}^{\psi_S}$  was defined in (4.8).

##### 4.4.1. DESCRIBING SUPPORT OF THE FUNCTION $\gamma_{u,w}^{\psi_S}$

The aim of this subsection is to describe the support of  $\gamma_{u,w}^{\psi_S}$  explicitly. In fact, in anticipation of Section 4.5 we will give this description for more general length functions  $\psi$ . Let  $\mathbb{1}(\cdot)$  be the indicator function which equals 1 if the statement within brackets is true.

**Lemma 4.4.2.** *Let  $\mathcal{W} = \langle S | M \rangle$  be a Coxeter group. Suppose  $\psi : \mathcal{W} \rightarrow \mathbb{R}$  is conditionally of negative type satisfying  $\psi(\mathbf{w}) = \psi(w_1) + \dots + \psi(w_k)$  whenever  $\mathbf{w} = w_1 \dots w_k$  is a reduced expression. Then for  $u, w \in S$  and  $\mathbf{v} \in \mathcal{W}$  we have that*

$$|\gamma_{u,w}^{\psi}(\mathbf{v})| = 2\psi(u)\mathbb{1}(u\mathbf{v} = \mathbf{v}w) = 2\psi(w)\mathbb{1}(u\mathbf{v} = \mathbf{v}w).$$

*Proof.* We first note that, since we have  $u^2 = w^2 = e$  as they are generators, we have that

$$\gamma_{u,w}^{\psi}(\mathbf{v}) = \gamma_{u,w}^{\psi}(u\mathbf{v}w) = -\gamma_{u,w}^{\psi}(u\mathbf{v}) = -\gamma_{u,w}^{\psi}(\mathbf{v}w).$$

When  $\mathbf{v}$  is fixed, we can let  $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$  be such that  $|\mathbf{z}| = \min\{|\mathbf{v}|, |u\mathbf{v}|, |\mathbf{v}w|, |u\mathbf{v}w|\}$ . Then we have  $|\gamma_{u,w}^\psi(\mathbf{z})| = |\gamma_{u,w}^\psi(\mathbf{v})|$ . Furthermore, because  $|\mathbf{z}|$  is minimal we have  $|u\mathbf{z}| = |\mathbf{z}w| = |\mathbf{z}| + 1$ . Thus, if  $\mathbf{z} = z_1 \dots z_k$  is a reduced expression for  $\mathbf{z}$  we have that  $uz_1 \dots z_k$  and  $z_1 \dots z_k w$  are reduced expressions for  $u\mathbf{z}$  respectively  $\mathbf{z}w$ . Therefore,  $\psi(u\mathbf{z}) = \psi(u) + \psi(\mathbf{z})$  and  $\psi(\mathbf{z}w) = \psi(\mathbf{z}) + \psi(w)$ . Hence

$$\begin{aligned} \gamma_{u,w}^\psi(\mathbf{z}) &= \psi(u\mathbf{z}w) + \psi(\mathbf{z}) - \psi(u\mathbf{z}) - \psi(\mathbf{z}w) \\ &= \psi(u\mathbf{z}w) - \psi(\mathbf{z}) - \psi(u) - \psi(w). \end{aligned}$$

Now, since  $|u\mathbf{z}| = |\mathbf{z}| + 1$  we either have that  $|u\mathbf{z}w| = |\mathbf{z}| + 2$  or  $|u\mathbf{z}w| = |\mathbf{z}|$ . We shall consider these two separate cases, from which the result will follow.

In the first case we have that  $uz_1 \dots z_k w$  is reduced so that  $\psi(u\mathbf{z}w) = \psi(u) + \psi(\mathbf{z}) + \psi(w)$  and therefore  $|\gamma_{u,w}^\psi(\mathbf{v})| = |\gamma_{u,w}^\psi(\mathbf{z})| = 0$ . We note that in this case also  $u\mathbf{v} \neq \mathbf{v}w$ . Namely,  $u\mathbf{v} = \mathbf{v}w$  would imply  $u\mathbf{z} = \mathbf{z}w$  and hence  $u\mathbf{z}w = \mathbf{z}$ , which contradicts that  $|u\mathbf{z}w| = |\mathbf{z}| + 2$ .

In the second case we have that  $uz_1 \dots z_k w$  is not reduced. Therefore, by the exchange condition (see [Dav08, Theorem 3.3.4.]) and the fact that  $|u\mathbf{z}w| = |\mathbf{z}| < |\mathbf{z}w|$  we have that  $uz_1 \dots z_k w$  is equal to  $z_1 \dots z_{i-1} z_{i+1} \dots z_k w$  for some index  $1 \leq i \leq k$ , or that  $uz_1 \dots z_k w = z_1 \dots z_k$ . Now in the former case we also have that  $u\mathbf{z} = z_1 \dots z_{i-1} z_{i+1} \dots z_k$  so that  $|u\mathbf{z}| < |\mathbf{z}|$  which is a contradiction. In this case we must thus have that  $u\mathbf{z}w = \mathbf{z}$  and hence  $u\mathbf{z} = \mathbf{z}w$ . This then implies that  $\psi(u\mathbf{z}w) = \psi(\mathbf{z})$  and  $\psi(u) = \psi(u\mathbf{z}) - \psi(\mathbf{z}) = \psi(\mathbf{z}w) - \psi(\mathbf{z}) = \psi(w)$ . In this case we thus obtain that

$$\gamma_{u,w}^\psi(\mathbf{z}) = \psi(u\mathbf{z}w) - \psi(\mathbf{z}) - \psi(u) - \psi(w) = -2\psi(u) = -2\psi(w)$$

which shows that  $|\gamma_{u,w}^\psi(\mathbf{v})| = |\gamma_{u,w}^\psi(\mathbf{z})| = 2\psi(u) = 2\psi(w)$  in this case.

The result now follows from these cases. Namely, either we have that  $|\gamma_{u,w}^\psi(\mathbf{v})| = 0$  and that  $\mathbf{v}$  does not satisfy  $u\mathbf{v} = \mathbf{v}w$ , or we have that  $|\gamma_{u,w}^\psi(\mathbf{v})| = 2\psi(u) = 2\psi(w)$  and that  $\mathbf{v}$  does satisfy  $u\mathbf{v} = \mathbf{v}w$ . This thus shows us that  $|\gamma_{u,w}^\psi(\mathbf{v})| = 2\psi(u)\mathbb{1}(u\mathbf{v} = \mathbf{v}w) = 2\psi(w)\mathbb{1}(u\mathbf{v} = \mathbf{v}w)$ . □

#### 4.4.2. A CHARACTERIZATION IN TERMS OF COXETER DIAGRAMS

We note that for the word length  $\psi_S$  we have  $\psi_S(s) > 0$  for all generators  $s \in S$ . Now by Lemma 4.4.2, in order to see when  $\gamma_{u,w}^{\psi_S}$  is finite-rank, we have to know what kind of words  $\mathbf{v} \in \mathcal{W}$  have the property that  $u\mathbf{v} = \mathbf{v}w$ . For this we introduce some notation.

For distinct  $i, j \in \{1, \dots, |S|\}$  we will, whenever the label  $m_{i,j}$  is finite, denote  $k_{i,j} = \lfloor \frac{m_{i,j}}{2} \rfloor \geq 1$ . Now if  $m_{i,j}$  is even, then  $m_{i,j} = 2k_{i,j}$  and we set  $\mathbf{r}_{i,j} = s_i(s_j s_i)^{k_{i,j}-1}$ . If  $m_{i,j}$  is odd, then  $m_{i,j} = 2k_{i,j} + 1$  and we set  $\mathbf{r}_{i,j} = (s_i s_j)^{k_{i,j}}$ . Furthermore we set

$$a_{i,j} = s_i \quad b_{i,j} = \begin{cases} s_i & m_{i,j} \text{ even} \\ s_j & m_{i,j} \text{ odd} \end{cases} \quad c_{i,j} = s_j \quad d_{i,j} = \begin{cases} s_j & m_{i,j} \text{ even} \\ s_i & m_{i,j} \text{ odd} \end{cases}. \quad (4.14)$$

Then  $a_{i,j}$  and  $b_{i,j}$  are respectively the first and last letter of the word  $\mathbf{r}_{i,j}$ . Furthermore when  $m_{i,j}$  is even we have

$$c_{i,j} \mathbf{r}_{i,j} = s_j s_i (s_j s_i)^{k_{i,j}-1} = (s_j s_i)^{k_{i,j}} = (s_i s_j)^{k_{i,j}} = \mathbf{r}_{i,j} s_j = \mathbf{r}_{i,j} d_{i,j},$$

and when  $m_{i,j}$  is odd we have

$$c_{i,j} \mathbf{r}_{i,j} = s_j (s_i s_j)^{k_{i,j}} = s_i (s_j s_i)^{k_{i,j}} = \mathbf{r}_{i,j} s_i = \mathbf{r}_{i,j} d_{i,j}.$$

Thus in either case  $c_{i,j} \mathbf{r}_{i,j} = \mathbf{r}_{i,j} d_{i,j}$ .

For given generators  $u, w \in S$  we will now check for what kind of words  $\mathbf{v} \in \mathcal{W}$  with  $|\mathbf{v}| \leq |u\mathbf{v}|, |\mathbf{v}w|$  we have that  $u\mathbf{v} = \mathbf{v}w$ . In Proposition 4.4.4 we then give a precise description of the support of  $\gamma_{u,w}^{\psi_S}$ .

**Lemma 4.4.3.** *For generators  $u, w \in S$  and a word  $\mathbf{v} \in \mathcal{W}$  with  $|\mathbf{v}| \leq |u\mathbf{v}|, |\mathbf{v}w|$  we have  $u\mathbf{v} = \mathbf{v}w$  if and only if  $\mathbf{v}$  can be written in the reduced form  $\mathbf{v} = \mathbf{r}_{i_1, j_1} \dots \mathbf{r}_{i_k, j_k}$  so that  $u = c_{i_1, j_1}$  and  $w = d_{i_k, j_k}$  and so that for  $l = 1, \dots, k-1$  we have that  $c_{i_{l+1}, j_{l+1}} = d_{i_l, j_l}$  and  $a_{i_{l+1}, j_{l+1}} \notin \{s_{i_l}, s_{j_l}\}$  and  $b_{i_l, j_l} \notin \{s_{i_{l+1}}, s_{j_{l+1}}\}$ .*

*Proof.* First, suppose that  $\mathbf{v}$  can be written in the given form  $\mathbf{v} = \mathbf{r}_{i_1, j_1} \dots \mathbf{r}_{i_k, j_k}$  with the given conditions on  $c_{i_l, j_l}$  and  $d_{i_l, j_l}$ . Then since we have  $c_{i_l, j_l} \mathbf{r}_{i_l, j_l} = \mathbf{r}_{i_l, j_l} d_{i_l, j_l} = \mathbf{r}_{i_l, j_l} c_{i_{l+1}, j_{l+1}}$  for  $l = 1, \dots, k-1$ , and since  $u = c_{i_1, j_1}$  and  $w = d_{i_k, j_k}$  we have  $u\mathbf{v} = \mathbf{v}w$ , which shows the ‘if’ direction.

We now prove the opposite direction. First note that the statement holds for  $\mathbf{v} = e$  as this can be written as the empty word. We now prove by induction on  $n$  that for  $\mathbf{v} \in \mathcal{W}$  with  $|\mathbf{v}| \geq 1$  and  $|\mathbf{v}| \leq n$  and  $|\mathbf{v}| \leq |u\mathbf{v}|, |\mathbf{v}w|$  and  $u\mathbf{v} = \mathbf{v}w$  for some  $u, w \in S$ , we can write  $\mathbf{v}$  in the given form. Note first that the statement holds for  $n = 0$ , since then no such  $\mathbf{v} \in \mathcal{W}$  exists. Thus, assume that the statement holds for  $n-1$ , we prove the statement for  $n$ . Let  $u, w \in S$  and  $\mathbf{v} \in \mathcal{W}$  be with  $|\mathbf{v}| = n$  and  $|u\mathbf{v}| = |\mathbf{v}w| = |\mathbf{v}| + 1$  and  $u\mathbf{v} = \mathbf{v}w$ . Let  $v_1 \dots v_n$  be a reduced expression for  $\mathbf{v}$ . Then the expression  $u v_1 \dots v_n$  and  $v_1 \dots v_n w$  are reduced expressions for  $u\mathbf{v} = \mathbf{v}w$ . In particular we have  $u \neq v_1$ . Set  $m := m_{u, v_1}$ . Now, since  $u\mathbf{v}$  and  $\mathbf{v}w$  are equal and  $u \neq v_1$ , we can as in the proof of [Dav08, theorem 3.4.2(ii)] find a reduced expression  $y_1 \dots y_{n+1}$  for  $u\mathbf{v}$  with  $n \geq m-1$  so that  $y_1 \dots y_m = u v_1 u v_1 \dots u$  whenever  $m$  is odd, and  $y_1 \dots y_m = u v_1 \dots u v_1$  whenever  $m$  is even. This is to say that if we let  $i_0, j_0 \in \{1, \dots, |S|\}$  be such that  $v_1 = s_{i_0}$  and  $u = s_{j_0}$ , then we have that  $\mathbf{r}_{i_0, j_0} = y_2 \dots y_m$  and  $c_{i_0, j_0} = s_{j_0} = u$ . Note that by the proof of [Dav08, theorem 3.4.2(ii)] we have in particular that  $m < \infty$ . Now moreover, since  $y_1 = u$  we have that  $y_2 \dots y_{n+1} w$  is an expression for  $\mathbf{v}w$ , and this expression is reduced since  $|\mathbf{v}w| = n+1$ .

Now suppose that  $m = n+1$ , then  $\mathbf{v} = \mathbf{r}_{i_0, j_0}$  and  $i_0 \neq j_0$  since  $u \neq v_1$ . Now, we have  $u = s_{j_0} = c_{i_0, j_0}$  and furthermore, since  $\mathbf{r}_{i_0, j_0} d_{i_0, j_0} = c_{i_0, j_0} \mathbf{r}_{i_0, j_0} = u\mathbf{v} = \mathbf{v}w = \mathbf{r}_{i_0, j_0} w$ , also  $w = d_{i_0, j_0}$ . Thus in this case we can write  $\mathbf{v}$  in the given form.

Now suppose  $m < n+1$  and define  $\mathbf{v}' = y_{m+1} \dots y_{n+1}$  and  $u' = d_{i_0, j_0}$  and  $w' = w$ . Note that since  $u = s_{j_0} = c_{i_0, j_0}$  and  $u' = d_{i_0, j_0}$  we have

$$\mathbf{r}_{i_0, j_0} u' \mathbf{v}' = u \mathbf{r}_{i_0, j_0} \mathbf{v}' = u\mathbf{v} = \mathbf{v}w = \mathbf{r}_{i_0, j_0} \mathbf{v}' w'.$$

Therefore  $u' \mathbf{v}' = \mathbf{v}' w'$ . Moreover  $|u' \mathbf{v}'| = |\mathbf{v}' w'| = |\mathbf{v}'| + 1$  since  $y_{m+1} \dots y_{n+1} w$  is a reduced expression for  $\mathbf{v}' w$ . Now, since also  $|\mathbf{v}'| \geq 1$  and  $|\mathbf{v}'| \leq n-1$  we have by the induction

hypothesis that there is a reduced expression  $\mathbf{v}' = \mathbf{r}_{i_1, j_1} \dots \mathbf{r}_{i_k, j_k}$  for some indices  $i_l, j_l \in \{1, \dots, |S|\}$  with  $i_l \neq j_l$  so that  $u' = c_{i_1, j_1}$  and  $w' = d_{i_k, j_k}$  and so that for  $l = 1, \dots, k-1$  we have that  $c_{i_{l+1}, j_{l+1}} = d_{i_l, j_l}$  and  $a_{i_{l+1}, j_{l+1}} \notin \{s_{i_l}, s_{j_l}\}$  and  $b_{i_l, j_l} \notin \{s_{i_{l+1}}, s_{j_{l+1}}\}$ . Hence we can write  $\mathbf{v} = \mathbf{r}_{i_0, j_0} \mathbf{v}' = \mathbf{r}_{i_0, j_0} \dots \mathbf{r}_{i_k, j_k}$ . We also have  $u = s_{j_0} = c_{i_0, j_0}$  and  $w = w' = d_{i_k, j_k}$  and  $d_{i_0, j_0} = u' = c_{i_1, j_1}$ . Furthermore, since  $|\mathbf{v}| = n = (m-1) + (n-m+1) = |\mathbf{r}_{i_0, j_0}| + |\mathbf{v}'|$ , and since the expression for  $\mathbf{v}'$  is reduced we thus have that the expression for  $\mathbf{v}$  is also reduced. Now suppose that  $b_{i_0, j_0} \in \{s_{i_1}, s_{j_1}\}$ . We note that  $b_{i_0, j_0} \neq d_{i_0, j_0} = c_{i_1, j_1} \neq a_{i_1, j_1}$ . Now as also  $c_{i_1, j_1}, a_{i_1, j_1} \in \{s_{i_1}, s_{j_1}\}$  we obtain that  $a_{i_1, j_1} = b_{i_0, j_0}$ . However as  $\mathbf{r}_{i_0, j_0}$  ends with  $b_{i_0, j_0}$  and as  $\mathbf{r}_{i_1, j_1}$  starts with  $a_{i_1, j_1}$  we then obtain that  $\mathbf{r}_{i_0, j_0} \mathbf{r}_{i_1, j_1}$  is not a reduced expression. This contradicts the fact that the expression for  $\mathbf{v}$  is reduced. Likewise, if  $a_{i_1, j_1} \in \{s_{i_0}, s_{j_0}\}$  we have because of the fact that  $a_{i_1, j_1} \neq c_{i_1, j_1} = d_{i_0, j_0} \neq b_{i_0, j_0}$  and  $d_{i_0, j_0}, b_{i_0, j_0} \in \{s_{i_0}, s_{j_0}\}$  that  $a_{i_1, j_1} = b_{i_0, j_0}$ . This then shows that  $\mathbf{r}_{i_0, j_0} \mathbf{r}_{i_1, j_1}$  is not a reduced expression, which contradicts the fact that the expression for  $\mathbf{v}$  is reduced. This proves the lemma.  $\square$

**Proposition 4.4.4.** *Let  $u, w \in S$ . Then  $\mathbf{z} \in \text{supp}(\gamma_{u,w}^{\psi_S})$  if and only if  $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$ , where  $\mathbf{v}$  is a word as in Lemma 4.4.3.*

*Proof.* It is clear that if  $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$  where  $\mathbf{v}$  is of the form of Lemma 4.4.3, that we then have that  $u\mathbf{z} = \mathbf{z}w$ , and hence by Lemma 4.4.2 that  $\psi_{u,w}^{\psi_S}(\mathbf{z}) \neq 0$ . For the other direction we suppose that  $\mathbf{z} \in \text{supp}(\gamma_{u,w}^{\psi_S})$ . Then we have that  $u\mathbf{z} = \mathbf{z}w$  holds by Lemma 4.4.2. Now there is a  $\mathbf{v} \in \{\mathbf{z}, u\mathbf{z}, \mathbf{z}w, u\mathbf{z}w\}$  such that  $|\mathbf{v}| \leq |u\mathbf{v}|, |\mathbf{v}w|$ . This word  $\mathbf{v}$  moreover satisfies  $u\mathbf{v} = \mathbf{v}w$  as we had  $u\mathbf{z} = \mathbf{z}w$ . Now, this means that  $\mathbf{v}$  can be written in an expression as in Lemma 4.4.3. Last, we note that  $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$ , which finishes the proof.  $\square$

#### 4.4.3. PARITY PATHS IN COXETER DIAGRAM

In Proposition 4.4.4 we showed precisely for what kind of words  $\mathbf{v} \in \mathcal{W}$  we have  $\mathbf{v} \in \text{supp}(\gamma_{u,w}^{\psi_S})$ . The question is now whether this support is finite or infinite. It follows from the proposition that the support is finite if and only if there exist only finitely many words  $\mathbf{v} \in \mathcal{W}$  that can be written in the form  $\mathbf{v} = \mathbf{r}_{i_1, j_1} \dots \mathbf{r}_{i_k, j_k}$  with the condition from Lemma 4.4.3. To answer the question on whether this is the case, we shall identify these expressions with certain walks in a graph. The following defines essentially the Coxeter diagram with the difference that in a Coxeter diagram the edges that are labeled with  $m_{i,j} = 2$  are deleted and those labeled with  $m_{i,j} = \infty$  are added. Recall that a graph is simplicial if it contains no double edges and no edges from a point to itself.

**Definition 4.4.5.** *We will let  $\text{Graph}_S(\mathcal{W}) = (V, E)$  be the complete simplicial graph with vertex set  $V = S$  and labels  $m_{i,j}$  on the edges  $\{s_i, s_j\} \in E$ .*

**Definition 4.4.6.** *Let  $k \geq 1$  and  $i_l, j_l \in \{1, \dots, |S|\}$  for  $l = 1, \dots, k$ . Let*

$$P = (s_{j_1}, s_{i_1}, s_{j_2}, \dots, s_{j_k}, s_{i_k})$$

*be a walk in the  $\text{Graph}_S(\mathcal{W})$ , which has even length. We will say that  $P$  is a parity path if the edges of  $P$  have finite labels, and if (1)  $i_l \neq j_l$  for all  $l$ ; (2) for  $l = 1, \dots, k-1$  we have  $s_{j_{l+1}} = d_{i_l, j_l}$  and (3)  $i_{l+1} \notin \{i_l, j_l\}$ . We will moreover call the parity path  $P$  a cyclic parity path if the path  $\bar{P} := (s_{j_1}, s_{i_1}, \dots, s_{j_k}, s_{i_k}, s_{j_1}, s_{i_1})$  is a parity path.*

The intuition for a parity path is that if you walk an edge with odd label, you have to stay there for one turn and then continue your walk over a different edge than you came from. Furthermore, when you walk an edge with an even label you have to return directly over the same edge, and then continue your walk using another edge. Note that in both cases you may still use same edges as before at a later point in your walk. A cyclic parity path is defined so that walking the same path any number of times in a row gives you a parity path.

We state the following definition.

**Definition 4.4.7.** *An elementary M-operation on a word  $v_1 \dots v_k$  is one of the following operations*

1. *Delete a subword of the form  $s_i s_i$ .*
2. *Replace an alternating subword of the form  $s_i s_j s_i s_j \dots$  of length  $m_{i,j}$  by the alternating word  $s_j s_i s_j s_i \dots$  of the same length.*

*A word is called M-reduced if it cannot be shortened by elementary M-operations.*

We shall now show in the following two Theorems that the gradient- $S_p$  property of the semigroup  $(\Phi_t)_{t \geq 0}$  on  $\mathcal{L}(\mathcal{W})$  associated to the word length  $\psi_S$ , is almost equivalent with the non-existence of cyclic parity paths in  $\text{Graph}_S(\mathcal{W})$ .

**Theorem 4.4.8.** *Let  $\mathcal{W} = \langle S|M \rangle$  be a Coxeter system. Suppose there is a cyclic parity path*

$$P = (s_{j_1}, s_{i_1}, s_{j_2}, \dots, s_{j_k}, s_{i_k})$$

*in  $\text{Graph}_S(\mathcal{W})$  in which the labels  $m_{i_l, j_l}, m_{i_l, i_{l+1}}, m_{j_l, i_{l+1}}$  are all unequal to 2. Then the semigroup  $(\Phi_t)_{t \geq 0}$  associated to the word length  $\psi_S$  is not gradient- $S_p$  for any  $p \in [1, \infty]$ .*

*Proof.* Suppose the assumptions hold. Then we have that there exists a parity path of the form  $\bar{P} = (s_{j_1}, s_{i_1}, s_{j_2}, \dots, s_{j_k}, s_{i_k}, s_{j_{k+1}}, s_{i_{k+1}})$  where  $s_{i_1} = s_{i_{k+1}}$  and  $s_{j_1} = s_{j_{k+1}}$ . We will denote  $\mathbf{v}_1 = \mathbf{r}_{i_1, j_1} \dots \mathbf{r}_{i_k, j_k}$ . We note that by the definition of a parity path we have  $d_{i_l, j_l} = s_{j_{l+1}} = c_{i_{l+1}, j_{l+1}}$  for  $l = 1, \dots, k-1$  and  $d_{i_k, j_k} = s_{j_{k+1}} = s_{j_1} = c_{i_1, j_1}$ . We now define  $u = c_{i_1, j_1} = d_{i_k, j_k}$ . Now we thus have  $u\mathbf{v}_1 = \mathbf{v}_1 u$ . This means by Lemma 4.4.2 that  $\gamma_{u, u}^{\psi_S}(\mathbf{v}_1) \neq 0$ . We show that  $\psi_S(\mathbf{v}_1) \geq k$ . To see this, note that  $a_{i_{l+1}, j_{l+1}} = s_{i_{l+1}} \notin \{s_{i_l}, s_{j_l}\}$  by the definition of the parity path. Furthermore, since  $b_{i_l, j_l} \neq d_{i_l, j_l} = c_{i_{l+1}, j_{l+1}}$  and  $b_{i_l, j_l} \neq a_{i_{l+1}, j_{l+1}}$  (as  $a_{i_{l+1}, j_{l+1}} \notin \{s_{i_l}, s_{j_l}\} \ni b_{i_l, j_l}$ ) and  $a_{i_{l+1}, j_{l+1}} = s_{i_{l+1}} \neq s_{j_{l+1}} = c_{i_{l+1}, j_{l+1}}$  we have that  $b_{i_l, j_l} \notin \{a_{i_{l+1}, j_{l+1}}, c_{i_{l+1}, j_{l+1}}\} = \{s_{i_{l+1}}, s_{j_{l+1}}\}$ . Now, since there are no labels  $m_{i_l, j_l}$  equal to 2 we have that the sub-words  $\mathbf{r}_{i_l, j_l}$  contain both elements  $s_{i_l}$  and  $s_{j_l}$ . This means, since  $a_{i_{l+1}, j_{l+1}} \notin \{s_{i_l}, s_{j_l}\}$  and  $b_{i_l, j_l} \notin \{s_{i_{l+1}}, s_{j_{l+1}}\}$ , that the only sub-words of  $\mathbf{v}_1$  of the form  $s_i s_j s_i \dots s_i s_j$  or  $s_i s_j s_i \dots s_j s_i$  are the sub-words of  $\mathbf{r}_{i_l, j_l}$  for some  $l = 1, \dots, k$ , and the words  $b_{i_l, j_l} a_{i_{l+1}, j_{l+1}}$  for  $l = 1, \dots, k-1$ . For an alternating subword  $\mathbf{x}$  of  $\mathbf{r}_{i, j}$  for some  $i, j$  we have that  $\mathbf{x}$  is an alternating sequence of  $s_i$ 's and  $s_j$ 's and further

$$|\mathbf{x}| \leq |\mathbf{r}_{i, j}| \leq m_{i, j} - 1.$$

Furthermore, for a word  $s_i s_j$  with  $s_i = b_{i_l, j_l}$  and  $s_j = a_{i_{l+1}, j_{l+1}}$  for some  $l = 1, \dots, k-1$  (in which case we have  $i \in \{i_l, j_l\}$  and  $j = i_{l+1}$ ) we have that

$$|s_i s_j| = 2 \leq \min\{m_{i_l, i_{l+1}}, m_{j_l, i_{l+1}}\} - 1 \leq m_{i, j} - 1.$$

Furthermore, there are no sub-words of  $\mathbf{v}_1$  of the form  $s_i s_i$ . This means that the expression for  $\mathbf{v}_1$  is  $M$ -reduced, and therefore, by [Dav08, Theorem 3.4.2], that the expression is reduced. This means that  $\psi_S(\mathbf{v}_1) \geq k$ . Now, since we can create cyclic parity paths  $P_n$  by walking over  $P$  a  $n$  number of times, we can create  $\mathbf{v}_n \in \mathcal{W}$  with  $\psi_S(\mathbf{v}_n) \geq nk$  and  $\gamma_{u,u}^{\psi_S}(\mathbf{v}_n) \neq 0$ . Therefore  $\gamma_{u,u}^{\psi_S}$  is not finite rank, and hence the semigroup  $(\Phi_t)_{t \geq 0}$  is not gradient- $S_p$  for any  $p \in [1, \infty]$ .  $\square$

**Theorem 4.4.9.** *Let  $\mathcal{W} = \langle S|M \rangle$  be a Coxeter group. If there does not exist a cyclic parity path in  $\text{Graph}_S(\mathcal{W})$  then the semigroup  $(\Phi_t)_{t \geq 0}$  associated to the word length  $\psi_S$  is gradient- $S_p$  for all  $p \in [1, \infty]$ .*

*Proof.* Suppose that  $(\Phi_t)_{t \geq 0}$  is not gradient- $S_p$  for some  $p \in [1, \infty]$ . We will show that a cyclic parity path exists. Namely, since the semigroup is not gradient- $S_p$ , there exist by Lemma 4.3.11 generators  $u, w \in S$  for which  $\gamma_{u,w}^{\psi_S}$  is not finite rank. Set  $m = \max\{m_{i,j} : 1 \leq i, j \leq |S| \setminus \{\infty\}\}$ . We can thus let  $\mathbf{z} \in \text{supp}(\gamma_{u,w}^{\psi_S})$  be with  $\psi_S(\mathbf{z}) > m|S|^2 + 2$ . Then by Proposition 4.4.4 there is a  $\mathbf{v} \in \{\mathbf{z}, u\mathbf{z}, \mathbf{z}w, u\mathbf{z}w\}$  such that we can write  $\mathbf{v}$  in reduced form  $\mathbf{v} = \mathbf{r}_{i_1, j_1} \dots \mathbf{r}_{i_k, j_k}$  with the conditions as in Lemma 4.4.3. Now define the path  $P = (s_{j_1}, s_{i_1}, \dots, s_{j_k}, s_{i_k})$ . We show that this is a parity path. By the properties that we obtained from Lemma 4.4.3, we have that  $i_l \neq j_l$  and that  $m_{i_l, j_l} < \infty$  for all  $l$ . Moreover  $s_{j_{l+1}} = c_{i_{l+1}, j_{l+1}} = d_{i_l, j_l}$  and  $s_{i_l} = a_{i_l, j_l} \notin \{s_{i_{l+1}}, s_{j_{l+1}}\}$ . This shows that  $P$  is a parity path. Note furthermore that since  $\psi_S(\mathbf{v}) \geq \psi_S(\mathbf{z}) - 2 > m|S|^2$ , we have that  $P$  has length  $|P| = 2k \geq 2 \frac{\psi_S(\mathbf{v})}{m} > 2|S|^2$ . Therefore, there must exist indices  $l < l'$  such that  $(s_{j_l}, s_{i_l}) = (s_{j_{l'}}, s_{i_{l'}})$ . The sub-path  $(s_{j_l, s_{i_l}}, \dots, s_{j_{l'-1}, s_{i_{l'-1}}})$  then is a cyclic parity path.  $\square$

#### 4.4.4. CHARACTERIZATION OF GRAPHS THAT CONTAIN CYCLIC PARITY PATHS

In the previous subsection, in Theorem 4.4.8 and Theorem 4.4.9 we have shown that the gradient- $S_p$  property is almost equivalent to the non-existence of a cyclic parity path. We shall now characterize in Proposition 4.4.10 precisely when a graph possesses a cyclic parity path. The content of this proposition is moreover visualized in Figure 4.1. Thereafter we state two corollaries that follow from this proposition and from Theorem 4.4.8 and Theorem 4.4.9. These corollaries give an ‘almost’ complete characterization of the types of Coxeter systems for which the semigroup associated to  $\psi_S$  is gradient- $S_p$ .

The following proposition shows exactly when a cyclic parity path  $P$  in the graph  $\text{Graph}_S(\mathcal{W})$  exists. Recall that a forest is a union of trees. A graph is a tree if it has no loops/cycles.

**Proposition 4.4.10.** *Let us denote  $V = S$  and  $E_0 = \{\{i, j\} : m_{i,j} \in 2\mathbb{N}\}$  and  $E_1 = \{\{i, j\} : m_{i,j} \in 2\mathbb{N} + 1\}$ . Then there does not exist a cyclic parity path  $P$  in  $\text{Graph}_S(\mathcal{W})$  if and only if  $(V, E_1)$  is a forest, and for every connected component  $C$  of  $(V, E_1)$  there is at most one edge*

### Graphs with and without a cyclic parity path

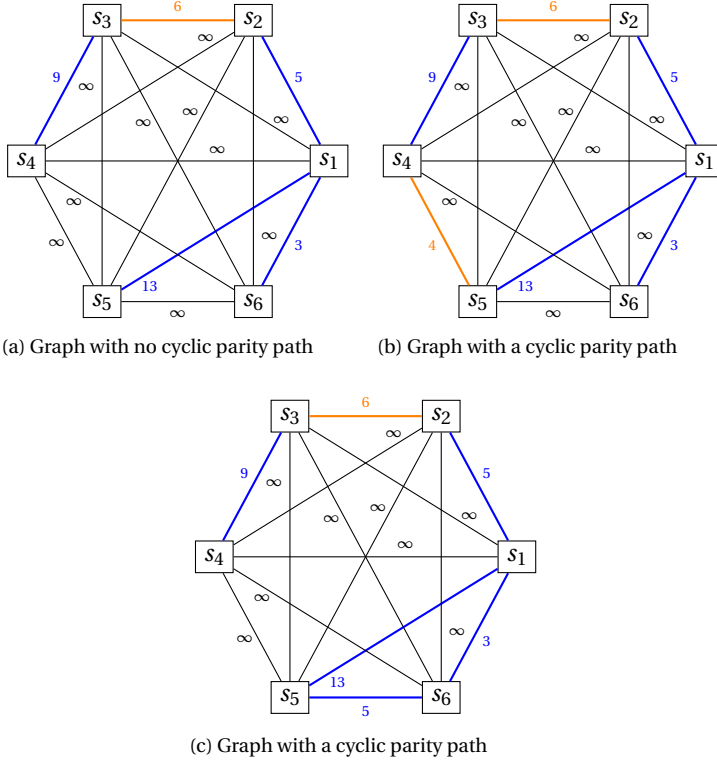


Figure 4.1: The graph  $\text{Graph}_S(\mathcal{W})$  is denoted for three different Coxeter systems  $\mathcal{W} = \langle S|M \rangle$  with  $|S| = 6$ . In each of the graphs the label  $m_{i,j}$  is shown on the edge  $\{s_i, s_j\}$ . We colored the edge orange when the label is even, we colored it blue when the label is odd, and we colored it black when the label is infinity. The relations we imposed on the generators are almost the same in the three cases. They only differ on the edges  $\{s_4, s_5\}$  and  $\{s_5, s_6\}$ . The graph in (A) satisfies the assumptions of Proposition 4.4.10 and hence does not contain a cyclic parity path. The graph in (B) does not satisfy the assumptions of the proposition as for the connected component  $C = \{s_3, s_4\}$  of  $(V, E_1)$  there are two distinct edges  $\{s_2, s_3\}$  and  $\{s_4, s_5\}$  with even label and with (at least) one endpoint in  $C$ . Therefore the graph contains a cyclic parity path. One is given by  $P = (s_3, s_2, s_3, s_4, s_4, s_5, s_4, s_3)$  (another cyclic parity path uses the node  $s_1$ ). The graph in (C) does also not satisfy the assumptions of the proposition as it contains a cycle with odd labels. Here a cyclic parity path is given by  $P = (s_1, s_5, s_5, s_6, s_6, s_1)$  (another cyclic parity path is obtained by walking in reverse order).

$\{t, r\} \in E_0$  with  $t \in C$  and  $r \notin C$ , and for every connected component  $C$  of  $(V, E_1)$  there is no edge  $\{t, t'\} \in E_0$  with  $t, t' \in C$ .

*Proof.* First suppose that  $(V, E_1)$  is not a forest. Then there is a cycle  $Q = (s_{j_1}, s_{j_2}, \dots, s_{j_k}, s_{j_1})$  in  $(V, E_1)$ . Now, since all edges are odd, this means that

$$P = (s_{j_1}, s_{j_2}, s_{j_2}, s_{j_3}, s_{j_3}, \dots, s_{j_k}, s_{j_k}, s_{j_1})$$

is a cyclic parity path. Indeed, if we denote  $j_{k+1} := j_1$  and  $j_{k+2} := j_2$ , then  $j_l \neq j_{l+1}$  for

$l = 1, \dots, k$  and we have  $s_{j_{l+1}} = d_{j_{l+1}, j_l}$  and  $j_{l+2} \notin \{j_{l+1}, j_l\}$ , which shows all conditions hold.

Now suppose that there is a connected component  $C$  of  $(V, E_1)$  for which there are two distinct edges  $\{t_1, r_1\}, \{t_2, r_2\} \in E_0$  with  $t_1, t_2 \in C$  and  $r_1, r_2 \notin C$ . If  $t_1 = t_2$  then  $r_1 \neq r_2$  and a cyclic parity path is given by  $P = (t_1, r_1, t_1, r_2)$ . In the case that  $t_1$  and  $t_2$  are distinct there is a simple path  $Q = (t_1, s_{j_1}, \dots, s_{j_k}, t_2)$  in  $(V, E_1)$  from  $t_1$  to  $t_2$ . The path

$$P = (t_1, s_{j_1}, s_{j_1}, s_{j_2}, s_{j_2}, \dots, s_{j_k}, s_{j_k}, t_2, t_2, r_2, t_2, s_{j_k}, s_{j_k}, s_{j_{k-1}}, s_{j_{k-1}}, \dots, s_{j_1}, s_{j_1}, t_1, t_1, r_1)$$

then is a cyclic parity path. Indeed, just as the previous case we have that the paths

$$P_1 := (t_1, s_{j_1}, s_{j_1}, s_{j_2}, s_{j_2}, \dots, s_{j_k}, s_{j_k}, t_2)$$

and

$$P_2 := (t_2, s_{j_k}, s_{j_k}, s_{j_{k-1}}, s_{j_{k-1}}, \dots, s_{j_1}, s_{j_1}, t_1)$$

are parity paths, since they are obtained from a simple path in  $(V, E_1)$ . We then only have to check that in the middle and at the start/end of the path  $P$  the conditions are satisfied. For the middle, we see that indeed  $r_2 \notin \{s_{j_k}, t_2\}$  as the label of the edge between  $t_2$  and  $r_2$  is even. Furthermore, since  $P_1$  is a parity path we have that  $s_{j_k} \neq t_2$ . Thus also  $s_{j_k} \notin \{t_2, r_2\}$ . Furthermore, if we let  $i, j$  be such that  $t_2 = s_j$ ,  $r_2 = s_i$ , then since  $m_{j_k, j}$  is odd, we have that  $t_2 = d_{j, j_k}$  and since  $m_{i, j}$  is even we have  $t_2 = d_{i, j}$ . This shows all conditions in the middle. The conditions at the start/end hold by symmetry. Thus  $P$  is indeed a cyclic parity path.

Now, suppose that there is a connected component  $C$  of  $(V, E_1)$  for which there exists an edge  $\{t, t'\} \in E_0$  with  $t, t' \in C$ . Then we can, similar to what we just did, obtain a cyclic parity path by taking  $t_1 = t$  and  $t_2 = t'$  and  $r_1 = t'$  and  $r_2 = t$ .

We now prove the other direction. Thus, suppose that  $(V, E_1)$  is a forest and that for every connected component  $C$  there is at most edge  $\{t, r\} \in E_0$  with  $t \in C$  and  $r \in V$ , and that for every connected component there is no edge  $\{t, t'\} \in E_0$  with  $t, t' \in C$ . Suppose there exists a cyclic parity path  $P = (s_{j_1}, s_{i_1}, \dots, s_{j_k}, s_{i_k})$  in  $(V, E_0 \cup E_1)$ , we show that this gives a contradiction. First suppose that  $P$  only has odd edges. Then  $s_{j_{l+1}} = d_{i_l, j_l} = s_{i_l}$  for  $l = 1, \dots, k-1$  and  $s_{j_1} = d_{i_k, j_k} = s_{i_k}$ , and thus  $P = (s_{i_k}, s_{i_1}, s_{i_1}, s_{i_2}, s_{i_2}, \dots, s_{i_{k-1}}, s_{i_k})$ . However, since also  $i_{l+1} \notin \{i_l, j_l\} = \{i_l, i_{l-1}\}$ , this means that  $Q = (s_{i_1}, s_{i_2}, \dots, s_{i_k}, s_{i_1})$  is a cycle in  $(V, E_1)$ . But this is not possible since  $(V, E_1)$  is a forest, which gives the contradiction. We thus assume that there is an index  $l$  such that the label  $m_{i_l, j_l}$  is even. By choosing the starting point of  $P$  as  $j_l$  instead of  $j_1$ , we can assume that  $m_{i_1, j_1}$  is even. Now in that case we have  $s_{j_2} = d_{i_1, j_1} = s_{j_1}$ . We must moreover have  $i_2 \notin \{i_1, j_1\}$  as  $P$  is a parity path. Now as the edges  $\{i_1, j_1\}$  and  $\{i_2, j_2\}$  are thus distinct, and share an endpoint, we obtain that  $m_{i_2, j_2}$  is odd. This means that  $j_3 = d_{i_2, j_2} = i_2 \neq j_2$ . Now the sub-path  $(s_{j_2}, s_{i_2}, \dots, s_{j_k}, s_{i_k}, s_{j_1}, s_{i_1})$  is also a parity path. Denote  $j_{k+1} = j_1$  and  $i_{k+1} = i_1$  and let  $3 < k' \leq k+1$  be the smallest index such that  $s_{j_{k'}} = s_{j_2}$ . Note that such  $k'$  exists since  $s_{j_{k+1}} = s_{j_1} = s_{j_2}$ . Then the sub-path  $P' := (s_{j_2}, s_{i_2}, \dots, s_{j_{k'}}, s_{i_{k'}})$  is a parity path, and the labels  $m_{i_l, j_l}$  for  $l = 2, \dots, k' - 1$  are odd since  $s_{j_2}$  is the only vertex in its connected component in  $(V, E_1)$  that is connected by an edge in  $E_0$ . Thus, just like in the previous case we have that  $P' := (s_{i_{k'}}, s_{i_2}, s_{i_2}, s_{i_3}, \dots, s_{i_{k'-1}}, s_{i_{k'}})$ . Now this means that the

path  $Q = (s_{i_{k'}}, s_{i_2}, s_{i_3}, \dots, s_{i_{k'}})$  contains a cycle, which is a contradiction with the fact that  $(V, E_1)$  is a forest. This proves the lemma.  $\square$

We now state two corollaries that directly follow from Theorem 4.4.8, Theorem 4.4.9 and Theorem 4.4.10.

**Corollary 4.4.11.** *Let  $\mathcal{W} = \langle S|M \rangle$  be a Coxeter system and fix  $p \in [1, \infty]$ . Let us denote  $E_0 = \{(i, j) : m_{i,j} \in 2\mathbb{N}\}$  and  $E_1 = \{(i, j) : m_{i,j} \in 2\mathbb{N} + 1\}$ . Then the semigroup  $(\Phi_t)_{t \geq 0}$  on  $\mathcal{L}(\mathcal{W})$  associated to the word length  $\psi_S$  is gradient- $S_p$  if  $(S, E_1)$  is a forest, and if for every connected component  $C$  of  $(S, E_1)$  there is at most one edge  $\{t, r\} \in E_0$  with  $t \in C$  and  $r \notin C$  and no edge  $\{t, t'\} \in E_0$  with  $t, t' \in C$ .*

**Corollary 4.4.12.** *Let  $\mathcal{W} = \langle S|M \rangle$  be a Coxeter system satisfying  $m_{i,j} \neq 2$  for all  $i, j$ . Fix  $p \in [1, \infty]$ . Let us denote  $E_0 = \{(i, j) : m_{i,j} \in 2\mathbb{N}\}$  and  $E_1 = \{(i, j) : m_{i,j} \in 2\mathbb{N} + 1\}$ . Then the semigroup  $(\Phi_t)_{t \geq 0}$  on  $\mathcal{L}(\mathcal{W})$  associated to the word length  $\psi_S$  is gradient- $S_p$  if and only if  $(S, E_1)$  is a forest, and for every connected component  $C$  of  $(S, E_1)$  there is at most one edge  $\{t, r\} \in E_0$  with  $t \in C$  and  $r \notin C$  and no edge  $\{t, t'\} \in E_0$  with  $t, t' \in C$ .*

We would also like to point out the following result from [Bra+02, Example 5.1]. It follows that the Coxeter groups are in some cases actually equal. In such cases we have obtained the gradient- $S_p$  property for multiple quantum Markov semigroups.

**Proposition 4.4.13.** *Let  $\mathcal{W}_i = \langle S_i|M_i \rangle$  be Coxeter systems for  $i = 1, 2$  such that  $\text{Graph}_{S_1}(\mathcal{W}_1)$  has no edges of even label, and such that the edges of odd label form a tree. Then if  $\text{Graph}_{S_1}(\mathcal{W}_2)$  has the same set of labels as  $\text{Graph}_{S_2}(\mathcal{W}_2)$  (counting multiplicities), then the Coxeter groups are equal, that is  $\mathcal{W}_1 = \mathcal{W}_2$ .*

#### 4.4.5. SMALLNESS AT INFINITY

We recall the construction of a natural compactification and boundary associated with a finite rank Coxeter group. We base ourselves mostly on the very general construction from [Kli23b] but in the case of Coxeter groups this boundary was also considered in [CL11], [LT15]. In [Kli23b] then smallness at infinity was studied as well as its connection to the Gromov boundary, which generally is different from the construction below.

Let  $\mathcal{W} = \langle S|M \rangle$  be a finite rank Coxeter system and let  $\text{Cayley}_S(\mathcal{W})$  be its Cayley graph which has vertex set  $\mathcal{W}$  and  $\mathbf{w}, \mathbf{v} \in \mathcal{W}$  are connected by an edge if and only if  $\mathbf{w} = \mathbf{v}s$  for some  $s \in S$ . We see  $\text{Cayley}_S(\mathcal{W})$  as a rooted graph with  $e \in \mathcal{W}$  the root. We say that  $\mathbf{w} \leq \mathbf{v}$  if there exists a geodesic (=shortest path) from  $e$  to  $\mathbf{v}$  passing through  $\mathbf{w}$ . An infinite geodesic path is a sequence  $\alpha = (\alpha_i)_{i \in \mathbb{N}}$  such that: (1)  $\alpha_i \in \mathcal{W}$ , (2)  $\alpha_i$  and  $\alpha_{i+1}$  have distance 1 in the Cayley graph, (3)  $(\alpha_i)_{i=0, \dots, n}$  is a shortest path (geodesic) from  $\alpha_0$  to  $\alpha_n$  for every  $n$ . For every  $\mathbf{w} \in \mathcal{W}$  we have either  $\mathbf{w} \leq \alpha_i$  for all large enough  $i$  or  $\mathbf{w} \not\leq \alpha_i$  for all large enough  $i$ . We write  $\mathbf{w} \leq \alpha$  in the former case and  $\mathbf{w} \not\leq \alpha$  in the latter case. We define an equivalence relation  $\sim$  by saying that for two infinite geodesics  $\alpha$  and  $\beta$  we have  $\alpha \sim \beta$  if for all  $\mathbf{w} \in \mathcal{W}$  both implications  $\mathbf{w} \leq \alpha \Leftrightarrow \mathbf{w} \leq \beta$  hold. Let  $\partial(\mathcal{W}, S)$  be the set of infinite geodesics modulo  $\sim$ . Define  $\overline{(\mathcal{W}, S)} = \mathcal{W} \cup \partial(\mathcal{W}, S)$ . We equip  $\overline{(\mathcal{W}, S)}$  with the topology generated by the subbase consisting of

$$\mathcal{U}_{\mathbf{w}} := \left\{ \alpha \in \overline{(\mathcal{W}, S)} : \mathbf{w} \leq \alpha \right\}, \quad \mathcal{U}_{\mathbf{w}}^c := \left\{ \alpha \in \overline{(\mathcal{W}, S)} : \mathbf{w} \not\leq \alpha \right\},$$

with  $\mathbf{w} \in \mathcal{W}$ . Then  $\overline{(\mathcal{W}, S)}$  contains  $\mathcal{W}$  as an open dense subset and the left translation action of  $\mathcal{W}$  on  $\mathcal{W}$  extends to a continuous action on  $\overline{(\mathcal{W}, S)}$  (see [Kli23b]). This means that  $\overline{(\mathcal{W}, S)}$  is a compactification of  $\mathcal{W}$  in the sense of [BO08, Definition 5.3.15] and  $\partial(\mathcal{W}, S)$  is the boundary. We now recall the following definition from [BO08, Definition 5.3.15].

**Definition 4.4.14.** *We will say that a finite rank Coxeter system  $(\mathcal{W}, S)$  is small at infinity if the compactification  $\overline{(\mathcal{W}, S)}$  is small at infinity. This means that for every sequence  $(x_i)_{i \in \mathbb{N}} \in \mathcal{W}$  converging to a boundary point  $z \in \partial(\mathcal{W}, S)$  and for every  $\mathbf{w} \in \mathcal{W}$  we have that  $x_i \mathbf{w} \rightarrow z$ .*

The following is the main theorem of this subsection. The authors are indebted to Mario Klisse for noting the connections in this theorem as well as its proof.

**Theorem 4.4.15.** *Let  $\mathcal{W} = \langle S | M \rangle$  be a Coxeter system. Fix  $p \in [1, \infty]$ . The following are equivalent:*

1. *The QMS  $(\Phi_t)_{t \geq 0}$  associated with the word length  $\psi_S$  is gradient- $S_p$  on  $\mathcal{L}(\mathcal{W})$ .*
2. *For all  $u, w \in S$  the set  $\{\mathbf{v} \in \mathcal{W} : u\mathbf{v} = \mathbf{v}w\}$  is finite.*
3. *For all  $s \in S$  the set  $\{\mathbf{v} \in \mathcal{W} : s\mathbf{v} = \mathbf{v}s\}$  is finite.*
4. *The Coxeter system  $\mathcal{W} = \langle S | M \rangle$  is small at infinity.*

*Proof.* (1) is equivalent to saying that for all  $u, v \in S$  we have that  $\gamma_{u,v}^{\psi_S}$  has compact support by Lemma 4.3.11. By Lemma 4.4.2 this is equivalent to (1). The equivalence between (3) and (4) was proven in [Kli23b, Theorem 0.3]. The implication (2)  $\implies$  (3) is immediate.

Now assume (4). We shall prove that (2) holds by contradiction. So suppose that  $|\{\mathbf{v} : u\mathbf{v} = \mathbf{v}w\}| = \infty$  for some  $u, w \in S$ . Choose a sequence  $(\mathbf{v}_i)_i$  in  $\{\mathbf{v} : u\mathbf{v} = \mathbf{v}w\}$  which has increasing word length. By the compactness of the compactification  $\overline{(\mathcal{W}, S)}$  [Kli23b, Proposition 2.8] this implies that (by possibly going over to a subsequence) the sequence  $(\mathbf{v}_i)_i$  converges to a boundary point  $z$ . Now, by the smallness at infinity and the assumption that  $u\mathbf{v}_i = \mathbf{v}_i w$  we have that  $z = \lim_i \mathbf{v}_i w = \lim_i u\mathbf{v}_i = u \cdot z$ . We have either  $u \leq z$  or  $u \not\leq z$  but not both in the partial order from [Kli23b, Lemma 2.2]. Further,  $u \not\leq z$  iff  $u \leq u \cdot z = z$  which yields a contradiction.  $\square$

**Remark 4.4.16.** We refer to [Kli23b, Theorem 0.3] for yet another statement that is equivalent to the statements in Theorem 4.4.15. A consequence of [Kli23b, Theorem 0.3] is that Coxeter groups that are small at infinity are word hyperbolic. Conversely, not every word hyperbolic Coxeter group is small at infinity. The simplest example is probably the Coxeter group generated by  $S = \{s_1, s_2, s_3, s_4\}$  where  $m_{i,j} = 2$  if  $|i - j| = 1$  and  $m_{i,j} = \infty$  otherwise. We thus see that not for every hyperbolic Coxeter group we have the gradient- $S_p$  property for the QMS associated with the word length. However, in Section 4.6 we show that using tensoring we may still use our methods for such Coxeter groups.

**Remark 4.4.17.** It is known that every discrete hyperbolic group is strongly solid by combining results in [HG04] (to get  $\text{AO}^+$  using amenable actions on the Gromov boundary), [Oza08] (for weak amenability, see [Fen02], [Jan02] for general Coxeter groups) and

[PV14b] (for Theorem 4.2.14). Condition  $AO^+$  may also be obtained by Theorem 4.3.14 for the Coxeter groups that admit a QMS with gradient- $S_p$ . However, Remark 4.4.16 shows that this covers a smaller class than [HG04] and so our methods – for now at least – do not improve on existing methods concerning strong solidity questions.

There are still two large benefits of the results in this section. Firstly, given a Coxeter system  $\mathcal{W} = \langle S|M \rangle$  it is not directly clear whether it is small at infinity. A combination of Theorem 4.4.15 and Corollaries 4.4.11 and 4.4.12 gives in many cases an easy way to see whether a Coxeter group is small at infinity. Secondly, for now we may not improve on current strong solidity results but in Section 4.6 we show that using the tensor methods of Section 4.2 we may prove strong solidity for all hyperbolic right-angled Coxeter groups. This gives an alternative path to the method of [HG04] (still not outweighing known results). In Section 4.7 this alternative path also gives strong solidity results for Hecke von Neumann algebras. Here we really improve on existing results as the methods of [HG04] can only be applied in a limited way, see [Kli23b, Theorem 3.15 and Corollary 3.17].

*Remark 4.4.18.* By Theorem 4.4.15 (see [Kli23b, Theorem 0.3]) smallness at infinity or gradient- $S_p$  can be characterized in terms of the finiteness of the centralizers of the generators. Such centralizers can be analyzed using the methods from [All13], [Bri96].

## 4.5. GRADIENT- $S_p$ SEMIGROUPS ASSOCIATED TO WEIGHTED WORD LENGTHS ON COXETER GROUPS

In this section we will consider proper length functions on Coxeter groups that are conditionally of negative type and are different from the standard word length. We can then consider the quantum Markov semigroups associated to these other functions, and study the gradient- $S_p$  property of these semigroups. We show that these other semigroups may have the gradient- $S_p$  properties in cases where the semigroup associated to the word length  $\psi_S$  fails to be gradient- $S_p$ . For  $p \in [1, \infty]$  this gives us new examples of Coxeter groups  $\mathcal{W}$  for which there exist a gradient- $S_p$  quantum Markov semigroup on  $\mathcal{L}(\mathcal{W})$ . These results will turn out to be crucial in Section 4.6.

### 4.5.1. WEIGHTED WORD LENGTHS

For non-negative weights  $\mathbf{x} = (x_1, \dots, x_{|S|})$  we consider, if existent, the function  $\psi_{\mathbf{x}} : \mathcal{W} \rightarrow \mathbb{R}$  by taking the word length with respect to the weights  $\mathbf{x}$  on the generators (see below). These functions are conditionally of negative definite type as follows for instance as a special case of [BS94, Theorem 1.1]. Here we give another purely group theoretical proof.

Fix again a (finite rank) Coxeter group  $\mathcal{W} = \langle S|M \rangle$ . Recall that the graph  $Graph_S(\mathcal{W})$  was defined in Definition 4.4.5. Let  $Graph'_S(\mathcal{W})$  be the subgraph of  $Graph_S(\mathcal{W})$  that has vertex set  $S$  and edge set  $E = \{(s_i, s_j) : 3 \leq m_{i,j} := m_{s_i, s_j} < \infty\}$ . Then let  $\mathcal{C}_i$  be the connected component in  $Graph'_S(\mathcal{W})$  that contains  $s_i$ .

**Lemma 4.5.1.** *Let  $\mathcal{W} = \langle S|M \rangle$  be a Coxeter group. Then if  $\mathbf{x} \in [0, \infty)^{|S|}$  is such that  $x_i = x_j$  whenever  $\mathcal{C}_i = \mathcal{C}_j$ , then the function*

$$\psi_{\mathbf{x}} : \mathcal{W} \rightarrow [0, \infty),$$

given for a word  $\mathbf{w} = w_1 \dots w_k$  in reduced expression by  $\psi_{\mathbf{x}}(\mathbf{w}) = \sum_{i=1}^{|\mathbf{S}|} x_i |\{l : w_l = s_i\}|$  is well-defined and is conditionally of negative type.

*Proof.* Let  $\mathbf{n} = (n_1, \dots, n_{|\mathbf{S}|}) \in \mathbb{N}^{|\mathbf{S}|}$  be such that  $n_i = n_j$  whenever  $\mathcal{C}_i = \mathcal{C}_j$ . We will construct a new Coxeter group  $\widetilde{\mathcal{W}}_{\mathbf{n}} = \langle S_{\mathbf{n}} | M_{\mathbf{n}} \rangle$  as follows. We denote  $S_{\mathbf{n}} = \{s_{i,k} : 1 \leq i \leq |\mathbf{S}|, 1 \leq k \leq n_i\}$  for the set of letters. We then define  $M_{\mathbf{n}} : S_{\mathbf{n}} \rightarrow \mathbb{N} \cup \{\infty\}$  as:

$$m_{\mathbf{n}, s_{i,k}, s_{j,l}} = \begin{cases} m_{s_i, s_j} & \mathcal{C}_i = \mathcal{C}_j \text{ and } k = l \\ 2 & \mathcal{C}_i = \mathcal{C}_j \text{ and } k \neq l \\ m_{s_i, s_j} & \mathcal{C}_i \neq \mathcal{C}_j \end{cases}.$$

We put  $\widetilde{\mathcal{W}}_{\mathbf{n}} := \langle S_{\mathbf{n}} | M_{\mathbf{n}} \rangle$ . We now define a homomorphism  $\varphi_{\mathbf{n}} : \mathcal{W} \rightarrow \widetilde{\mathcal{W}}_{\mathbf{n}}$  given for generators by  $\varphi_{\mathbf{n}}(s_i) = s_{i,1} s_{i,2} \dots s_{i,n_i}$ . We note that  $\varphi_{\mathbf{n}}(s_i)^2 = s_{i,1} \dots s_{i,n_i} s_{i,n_i} \dots s_{i,1} = s_{i,1}^2 \dots s_{i,n_i}^2 = e$ . Furthermore, when  $\mathcal{C}_i = \mathcal{C}_j$  we have that  $n_i = n_j$  and

$$(\varphi_{\mathbf{n}}(s_i) \varphi_{\mathbf{n}}(s_j))^m = (s_{i,1} \dots s_{i,n_i} s_{j,1} \dots s_{j,n_j})^m = (s_{i,1} s_{j,1})^m (s_{i,2} s_{j,2})^m \dots (s_{i,n_i} s_{j,n_j})^m.$$

This means that in this case  $(\varphi_{\mathbf{n}}(s_i) \varphi_{\mathbf{n}}(s_j))^{m_{s_i, s_j}} = e$ . If  $\mathcal{C}_i \neq \mathcal{C}_j$  then either  $m_{s_i, s_j} = 2$  or  $m_{s_i, s_j} = \infty$ . If  $m_{s_i, s_j} = 2$  then also  $\varphi_{\mathbf{n}}(s_i) \varphi_{\mathbf{n}}(s_j) = s_{i,1} \dots s_{i,n_i} s_{j,1} \dots s_{j,n_j} = s_{j,1} \dots s_{j,n_j} s_{i,1} \dots s_{i,n_i} = \varphi_{\mathbf{n}}(s_j) \varphi_{\mathbf{n}}(s_i)$  holds. Therefore, we can extend  $\varphi_{\mathbf{n}}$  to words  $\mathbf{w} = w_1 \dots w_k \in \mathcal{W}$  by defining  $\varphi_{\mathbf{n}}(\mathbf{w}) = \varphi_{\mathbf{n}}(w_1) \dots \varphi_{\mathbf{n}}(w_k)$ . By what we just showed, this map is well-defined. Furthermore, from the definition it follows that this map is a homomorphism. Moreover, we note that if  $\mathbf{w} = w_1 \dots w_k \in \mathcal{W}$  is a reduced expression, then  $\varphi_{\mathbf{n}}(\mathbf{w}) = \varphi_{\mathbf{n}}(w_1) \dots \varphi_{\mathbf{n}}(w_k)$  is also a reduced expression. This means in particular that  $\varphi_{\mathbf{n}}$  is injective. Furthermore, if we denote  $\tilde{\psi}_{\mathbf{n}}$  for the word length on  $\widetilde{\mathcal{W}}_{\mathbf{n}}$ , then we have that for a word  $\mathbf{w} = w_1 \dots w_k \in \mathcal{W}$  written in a reduced expression that

$$\tilde{\psi}_{\mathbf{n}} \circ \varphi_{\mathbf{n}}(\mathbf{w}) = \sum_{i=1}^k \tilde{\psi}_{\mathbf{n}}(\varphi_{\mathbf{n}}(w_i)) = \sum_{i=1}^{|\mathbf{S}|} \tilde{\psi}_{\mathbf{n}}(\varphi_{\mathbf{n}}(s_i)) |\{l : w_l = s_i\}| = \sum_{i=1}^{|\mathbf{S}|} n_i |\{l : w_l = s_i\}|.$$

Now fix  $\mathbf{x} \in [0, \infty)^{|\mathbf{S}|}$  with  $x_i = x_j$  whenever  $\mathcal{C}_i = \mathcal{C}_j$ . For  $m \in \mathbb{N}$  define  $\mathbf{n}_m \in \mathbb{N}^{|\mathbf{S}|}$  by  $(\mathbf{n}_m)_i = \lceil mx_i \rceil + 1 \in \mathbb{N}$ . Now, for  $\mathbf{w} \in \mathcal{W}$  with reduced expression  $\mathbf{w} = w_1 \dots w_k$  we have

$$\begin{aligned} & \left| \frac{1}{m} \tilde{\psi}_{\mathbf{n}_m} \circ \varphi_{\mathbf{n}_m}(\mathbf{w}) - \sum_{i=1}^{|\mathbf{S}|} x_i |\{l : w_l = s_i\}| \right| \leq \sum_{i=1}^{|\mathbf{S}|} \left| \frac{(\mathbf{n}_m)_i}{m} - x_i \right| |\{l : w_l = s_i\}| \\ &= \sum_{i=1}^{|\mathbf{S}|} \frac{|\lceil mx_i \rceil + 1 - mx_i|}{m} |\{l : w_l = s_i\}| \leq \sum_{i=1}^{|\mathbf{S}|} \frac{2}{m} |\{l : w_l = s_i\}| \leq \frac{2|\mathbf{w}|}{m}, \end{aligned}$$

and hence  $\frac{1}{m} \tilde{\psi}_{\mathbf{n}_m} \circ \varphi_{\mathbf{n}_m}(\mathbf{w}) \rightarrow \sum_{i=1}^{|\mathbf{S}|} x_i |\{l : w_l = s_i\}|$  as  $m \rightarrow \infty$ . This shows in particular that  $\psi_{\mathbf{x}}$  is well defined. Now, since  $\frac{1}{m} \tilde{\psi}_{\mathbf{n}_m} \circ \varphi_{\mathbf{n}_m} \rightarrow \psi_{\mathbf{x}}$  point-wise. Since  $\frac{1}{m} \tilde{\psi}_{\mathbf{n}_m} \circ \varphi_{\mathbf{n}_m}$  is conditionally of negative type we have by [BHV08, Proposition C.2.4(ii)] that  $\psi_{\mathbf{x}}$  is conditionally of negative type.  $\square$

*Remark 4.5.2.* By Lemma 4.5.1 in the case of a right-angled Coxeter group  $\mathcal{W} = \langle S | M \rangle$  we have that every weight  $\mathbf{x} \in [0, \infty)^{|\mathbf{S}|}$  defines a function that is conditionally of negative type.

*Remark 4.5.3.* For a general Coxeter group  $\mathcal{W} = \langle S|M \rangle$  and arbitrary non-negative weights  $\mathbf{x} \in [0, \infty)^{|S|}$  the weighted word length is not well-defined. Indeed, if  $s_i, s_j \in S$  are such that  $m_{s_i, s_j}$  is odd, then for  $k_{i,j} := \lfloor \frac{1}{2} m_{s_i, s_j} \rfloor$  we have that  $(s_i s_j)^{k_{i,j}} s_i$  and  $s_j (s_i s_j)^{k_{i,j}}$  are two reduced expressions for the same word, but the values of  $|\{l : w_l = s_i\}|$  and  $|\{l : w_l = s_j\}|$  depend on the choice of the reduced expressions.

We shall now turn to examine when a weighted word length is proper. Fix again a Coxeter system  $\mathcal{W} = \langle S|M \rangle$ . Let  $T \subseteq S$  be a subset of the generators such that for  $i = 1, \dots, |S|$  either  $\mathcal{C}_i \subseteq T$  or  $\mathcal{C}_i \cap T = \emptyset$ . We set

$$\psi_T = \psi_{\mathbf{x}}, \quad \text{with } \mathbf{x} \in [0, \infty)^{|S|} \text{ defined by } \mathbf{x}(i) = \chi_T(i),$$

where  $\chi_T$  is the indicator function on  $T$ . Then by Lemma 4.5.1 we have that  $\psi_T : \mathcal{W} \rightarrow \mathbb{R}$  is a well-defined function that is conditionally of negative type. We give the following characterization on when the function  $\psi_T$  is proper.

**Proposition 4.5.4.** *The function  $\psi_T$  is proper if and only if the elements  $S \setminus T$  generate a finite subgroup.*

*Proof.* Indeed, if the generated group  $H$  is infinite, then  $\psi_T$  is not proper as  $\psi_T|_H = 0$ . On the other hand, if the generated group  $H$  contains  $N < \infty$  elements, then for a reduced expression  $\mathbf{w} = w_1 \dots w_k \in \mathcal{W}$  we can not have that  $w_l, w_{l+1}, \dots, w_{l+N} \in S \setminus T$  for some  $1 \leq l \leq k - N$  as the expressions  $w_l, w_l w_{l+1}, w_l w_{l+1} w_{l+2}, \dots$  would all be distinct elements in  $H$ . This thus implies that  $\psi_T(\mathbf{w}) > \frac{|\mathbf{w}|}{N+1} - 1$  which shows that  $\psi_T$  is proper in this case.  $\square$

#### 4.5.2. GRADIENT- $S_p$ PROPERTY WITH RESPECT TO WEIGHTED WORD LENGTHS ON RIGHT-ANGLED COXETER GROUPS

In this subsection we shall consider a finite rank Coxeter group  $\mathcal{W} = \langle S|M \rangle$ . By Remark 4.5.2 it follows that for any  $\mathbf{x} \in [0, \infty)^{|S|}$  we have that  $\psi_{\mathbf{x}} : \mathcal{W} \rightarrow \mathbb{R}$  is well-defined and conditionally of negative definite type. We note also that  $\psi_{\mathbf{x}}(\mathbf{w}) = \psi_{\mathbf{x}}(w_1) + \dots + \psi_{\mathbf{x}}(w_k)$  when  $\mathbf{w} = w_1 \dots w_k$  is a reduced expression. Therefore by Lemma 4.4.2 we have that  $\gamma_{u,w}^{\psi_{\mathbf{x}}}(\mathbf{v}) \neq 0$  for  $u, w \in S$  and  $\mathbf{v} \in \mathcal{W}$  if and only if  $u\mathbf{v} = \mathbf{v}w$  and  $\psi_{\mathbf{x}}(u) > 0$ .

**Theorem 4.5.5.** *Let  $\mathcal{W} = \langle S|M \rangle$  be a finite rank, right-angled Coxeter group. Let  $\mathbf{x} \in [0, \infty)^{|S|}$  and  $p \in [1, \infty]$ . Suppose the function  $\psi_{\mathbf{x}} : \mathcal{W} \rightarrow \mathbb{R}$  is proper. Then, the semigroup  $(\Phi_t)_{t \geq 0}$  induced by  $\psi_{\mathbf{x}}$  is gradient- $S_p$  if and only if there do not exist (distinct) generators  $r, s, t \in \Gamma$  with  $m_{r,s} = m_{r,t} = 2$ ,  $m_{s,t} = \infty$  and  $\psi_{\mathbf{x}}(r) > 0$ .*

*Proof.* Suppose that  $(\Phi_t)_{t \geq 0}$  is not gradient- $S_p$  for some  $p \in [1, \infty]$ . We will show the generators with the given properties exist. Namely, there are generators  $u, w \in S$  for which  $\gamma_{u,w}^{\psi_{\mathbf{x}}}$  is not finite rank. We can thus let  $\mathbf{v} \in \mathcal{W}$  with  $|\mathbf{v}| > |S| + 1$  be such that  $\gamma_{u,w}^{\psi_{\mathbf{x}}}(\mathbf{v}) \neq 0$ . Then  $u\mathbf{v} = \mathbf{v}w$  and  $\psi_{\mathbf{x}}(u), \psi_{\mathbf{x}}(w) > 0$  by Lemma 4.4.2. We note moreover that by [Dav08, Lemma 3.3.3] we have that  $u = w$  because these elements are conjugate and the Coxeter group is right-angled. We can now let  $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$  be such that  $|\mathbf{z}| \leq |u\mathbf{z}|, |\mathbf{z}w|$ . Then the equality  $u\mathbf{z} = \mathbf{z}w$  also holds. Therefore, we can write  $\mathbf{z}$  in reduced form  $\mathbf{z} = \mathbf{r}_{i_1, j_1} \dots \mathbf{r}_{i_k, j_k}$  with the conditions as in Lemma 4.4.3. Now, as  $m_{i_l, j_l} := m_{s_{i_l}, s_{j_l}} < \infty$  we must have  $m_{s_{i_l}, s_{j_l}} = 2$  for  $l = 1, \dots, k$ . Hence  $\mathbf{z} = s_{i_1} s_{i_2} \dots s_{i_k}$ . Furthermore  $s_{j_{l+1}} = s_{j_l}$  for  $l = 1, \dots, k-1$  since  $m_{s_{i_l}, s_{j_l}}$  is even. We define  $r = s_{j_1}$ . Then  $r = c_{i_1, j_1} = u$  so that  $\psi_{\mathbf{x}}(r) > 0$ .

Furthermore, since  $k = |\mathbf{z}| \geq |\mathbf{v}| - 1 > |S|$  there exist indices  $l < l'$  such that  $m_{s_{i_l}, s_{i_{l'}}} = \infty$ . We then set  $s = s_{i_l}$  and  $t = s_{i_{l'}}$ . Then  $m_{s,r} = m_{s_{i_l}, s_{j_l}} = 2$  and likewise  $m_{t,r} = 2$ . This shows that all stated properties hold for  $r, s, t$ .

For the other direction, suppose that there exist  $r, s, t \in S$  with  $m_{r,s} = m_{r,t} = 2$  and  $m_{s,t} = \infty$  and  $\psi_{\mathbf{x}}(r) > 0$ . Define the words  $\mathbf{v}_n = (st)^n$ . Then we have  $|\mathbf{v}_n| = 2n$  and hence  $\{\mathbf{v}_n\}_{n \geq 1}$  are all distinct. Moreover, we have  $r\mathbf{v}_n = \mathbf{v}_nr$  and  $\psi_{\mathbf{x}}(r) > 0$ . This means by Lemma 4.4.2 that  $\gamma_{r,r}^{\psi_{\mathbf{x}}}(\mathbf{v}_n) = \psi_{\mathbf{x}}(r) > 0$  for  $n \geq 1$ . Thus the semigroup  $(\Phi_t)_{t \geq 0}$  is not gradient- $S_p$ .  $\square$

## 4.6. STRONG SOLIDITY FOR HYPERBOLIC RIGHT-ANGLED COXETER GROUPS

We conclude this chapter with two applications that combines all the techniques that we have developed so far. This section contains the first application. We prove that any right-angled hyperbolic Coxeter group has a strongly solid group von Neumann algebra. This result was surely known before; it follows for instance from [PV14b]. Nevertheless we present our alternative proof to demonstrate the techniques that we have established in this chapter. For the rest of this section fix a *right-angled* Coxeter group  $\mathcal{W}_{\Gamma}$  associated to a finite graph  $\Gamma$ . We shall use the following characterisation of word hyperbolicity.

**Theorem 4.6.1** (See [Dav08]). *Let  $\mathcal{W}_{\Gamma}$  be a right-angled Coxeter group associated to a finite graph  $\Gamma$ . The following are equivalent:*

1. *The Coxeter group  $\mathcal{W}_{\Gamma}$  is word hyperbolic.*
2. *The graph  $\Gamma$  does not contain the cyclic graph  $\mathbb{Z}_4$  of size  $|\mathbb{Z}_4| = 4$  as a subgraph.*

Our aim is to prove the following. The proof is based on Proposition 4.6.3 and Lemma 4.6.4 which we prove at the end. Recall that a clique of  $\Gamma$  is a complete subgraph  $\Lambda \subseteq \Gamma$ . We denote by  $\text{Cliq}(\Gamma)$  the set of all cliques. Note that this precisely correspond to the set of those subgraphs  $\Lambda \subseteq \Gamma$  that generate a finite Coxeter subgroup  $\mathcal{W}_{\Lambda} \subseteq \mathcal{W}_{\Gamma}$ .

**Theorem 4.6.2.** *Let  $\mathcal{W}_{\Gamma}$  be a word hyperbolic right-angled Coxeter group associated to a finite graph  $\Gamma$ . Then  $\mathcal{L}(\mathcal{W}_{\Gamma})$  satisfies  $\text{AO}^+$  and is strongly solid.*

*Proof.* For  $\Lambda \in \text{Cliq}(\Gamma)$  the function  $\psi_{\Gamma \setminus \Lambda}$  is proper (see Proposition 4.5.4) and conditionally of negative type (see Lemma 4.5.1). We may therefore consider the QMS  $\Phi_{\Lambda}$  associated with  $\psi_{\Gamma \setminus \Lambda}$ , the associated gradient  $\mathbb{C}[\mathcal{W}_{\Gamma}]$  bimodule  $\mathcal{H}_{\mathcal{W}_{\Gamma}, \Lambda} := \ell^2(\mathcal{W}_{\Gamma})_{\nabla \Phi_{\Lambda}}$  and the Riesz transform  $R_{\mathcal{W}_{\Gamma}, \Lambda} : \ell^2(\mathcal{W}_{\Gamma}) \rightarrow \mathcal{H}_{\mathcal{W}_{\Gamma}, \Lambda}$ . The Riesz transform  $R_{\mathcal{W}_{\Gamma}, \Lambda}$  is then a partial isometry with a finite dimensional kernel spanned by  $\delta_u, u \in \mathcal{W}_{\Lambda}$ . Furthermore,  $R_{\mathcal{W}_{\Gamma}, \Lambda}$  is almost bimodular by Corollary 4.3.13. We now consider the  $\otimes_G$  tensor product of bimodules with  $G = \mathcal{W}_{\Gamma}$  over all  $\Lambda \in \text{Cliq}(\Gamma)$  as was defined in Section 4.2.2,

$$\mathcal{H}_{\mathcal{W}_{\Gamma}} = \bigotimes_{\Lambda \in \text{Cliq}(\Gamma)} \mathcal{H}_{\mathcal{W}_{\Gamma}, \Lambda}. \quad (4.15)$$

We note that the order in which the tensor products are taken is not relevant for our analysis. Consider the convolution product of Riesz transforms

$$R_{\mathcal{W}_{\Gamma}} = *_{\Lambda \in \text{Cliq}(\Gamma)} R_{\mathcal{W}_{\Gamma}, \Lambda} : \ell^2(\mathcal{W}_{\Gamma}) \rightarrow \mathcal{H}_{\mathcal{W}_{\Gamma}}.$$

By Lemma 4.2.9 and Lemma 4.2.10 we see that  $R_{\mathcal{W}_\Gamma}$  is an almost bimodular partial isometry whose kernel is spanned by all vectors  $\delta_u$  where  $u \in \mathcal{W}_\Lambda$  for some  $\Lambda \in \text{Cliq}(\Gamma)$ . In particular the kernel of  $R_{\mathcal{W}_\Gamma}$  is finite dimensional. Let  $\mathcal{K} \subseteq \mathcal{H}_{\mathcal{W}_\Gamma}$  be the smallest  $\mathbb{C}[\mathcal{W}_\Gamma]$  subbimodule containing the image of  $R_{\mathcal{W}_\Gamma}$ . Then  $R_{\mathcal{W}_\Gamma} : \ell^2(\mathcal{W}_\Gamma) \rightarrow \mathcal{K}$  is still an almost bimodular partial isometry with finite dimensional kernel.

Recall that  $C_r^*(\mathcal{W}_\Gamma)$  is locally reflexive and  $\mathcal{L}(\mathcal{W}_\Gamma)$  has the weak- $*$  completely bounded approximation property as  $\mathcal{W}_\Gamma$  is weakly amenable (see [Fen02], [Jan02]). It then follows from Theorem 4.2.12 that if  $\mathcal{K}$  is quasi-contained in the coarse bimodule of  $\mathcal{W}_\Gamma$  then  $\mathcal{L}(\mathcal{W}_\Gamma)$  satisfies  $\text{AO}^+$ . Consequently,  $\mathcal{L}(\mathcal{W}_\Gamma)$  is strongly solid by Theorem 4.2.14. The proof that  $\mathcal{K}$  is quasi-contained in the coarse bimodule of  $\mathcal{W}_\Gamma$  is given in Proposition 4.6.3 below.  $\square$

**Proposition 4.6.3.** *The  $\mathbb{C}[\mathcal{W}_\Gamma]$  bimodule  $\mathcal{K}$  defined in the proof of Theorem 4.6.2 is quasi-contained in the coarse bimodule of the word hyperbolic right-angled Coxeter group  $\mathcal{W}_\Gamma$ .*

*Proof.* We shall prove that a cyclic set of coefficients is in  $S_2$  so that the proposition follows from Lemma 4.2.4. Let us denote  $\mathcal{H}_{00} \subseteq \mathcal{K}$  for the sets of all the vectors

$$\xi_{\mathbf{v}} := (*_{\Lambda \in \text{Cliq}(\Gamma)} R_{\mathcal{W}_\Gamma, \Lambda})(\delta_{\mathbf{v}}) = \bigotimes_{\Lambda \in \text{Cliq}(\Gamma)} \lambda_{\mathbf{v}} \otimes_{\nabla_\Lambda} \delta_e, \quad \mathbf{v} \in \mathcal{W}_\Gamma.$$

Here we used the symbol  $\otimes_{\nabla_\Lambda}$  to denote elements in the gradient bimodule constructed from  $\Lambda$ . By construction of  $\mathcal{K}$  we have that  $\mathcal{H}_{00}$  is cyclic for  $\mathcal{K}$ . For  $\xi_{\mathbf{u}}, \xi_{\mathbf{w}} \in \mathcal{H}_{00}$  we now inspect the coefficient  $T_{\xi_{\mathbf{u}}, \xi_{\mathbf{w}}}$ . We have for  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{W}_\Gamma, y \in \mathbb{C}[\mathcal{W}_\Gamma]$ ,

$$\begin{aligned} \tau(T_{\xi_{\mathbf{w}}, \xi_{\mathbf{u}}}(\lambda_{\mathbf{v}})y) &= \langle \lambda_{\mathbf{v}} \cdot \xi_{\mathbf{w}} \cdot y, \xi_{\mathbf{u}} \rangle_{\mathcal{H}_{\mathcal{W}_\Gamma}} \\ &= \prod_{\Lambda \in \text{Cliq}(\Gamma)} \langle \lambda_{\mathbf{v}} \cdot (\lambda_{\mathbf{w}} \otimes_{\nabla_\Lambda} \delta_e) \cdot y, \lambda_{\mathbf{u}} \otimes_{\nabla_\Lambda} \delta_e \rangle_{\mathcal{H}_{\mathcal{W}_\Gamma, \Lambda}} \\ &= \prod_{\Lambda \in \text{Cliq}(\Gamma)} \langle \Psi_{\Lambda}^{\lambda_{\mathbf{u}^{-1}}, \lambda_{\mathbf{w}}}(\lambda_{\mathbf{v}}) \delta_e y, \delta_e \rangle \\ &= \prod_{\Lambda \in \text{Cliq}(\Gamma)} \gamma_{\mathbf{u}^{-1}, \mathbf{w}}^{\Psi_{\Gamma \setminus \Lambda}}(\mathbf{v}) \langle \lambda_{\mathbf{u}^{-1} \mathbf{v} \mathbf{w}} \delta_e y, \delta_e \rangle. \end{aligned}$$

Define the function

$$\tilde{\gamma}_{\mathbf{u}, \mathbf{w}}(\mathbf{v}) = \prod_{\Lambda \in \text{Cliq}(\Gamma)} \gamma_{\mathbf{u}, \mathbf{w}}^{\Psi_{\Gamma \setminus \Lambda}}(\mathbf{v}). \quad (4.16)$$

Then, if  $\tilde{\gamma}_{\mathbf{u}^{-1}, \mathbf{w}}(\mathbf{v}) = 0$  we have that  $\tau(T_{\xi_{\mathbf{w}}, \xi_{\mathbf{u}}}(\lambda_{\mathbf{v}})y) = 0$  for all  $y \in \mathbb{C}[\mathcal{W}_\Gamma]$  and consequently  $T_{\xi_{\mathbf{w}}, \xi_{\mathbf{u}}}(\lambda_{\mathbf{v}}) = 0$ . We thus have that  $T_{\xi_{\mathbf{w}}, \xi_{\mathbf{u}}}$  is finite rank whenever  $\tilde{\gamma}_{\mathbf{u}^{-1}, \mathbf{w}}$  has finite support. In Lemma 4.6.4 we shall show that the function  $\tilde{\gamma}_{\mathbf{u}, \mathbf{w}}$  has finite rank for all  $\mathbf{u}, \mathbf{w} \in \mathcal{W}_\Gamma$  so that we conclude the proof.  $\square$

In order to prove Lemma 4.6.4 rigorously we shall introduce some notation here. A tuple  $(w_1, \dots, w_k)$  with  $w_i \in \Gamma$  will be call *reduced* if the expression  $w_1 \dots w_k$  is reduced. Furthermore, we will call the tuple *semi-reduced* whenever  $w_i \in \Gamma \cup \{e\}$  for  $1 \leq i \leq k$  and  $|w_1 \dots w_k| + |\{l : w_l = e\}| = k$ . We will say that a pair  $(i, j)$  with  $i < j$  *collapses* for a tuple  $(w_1, \dots, w_k)$  whenever  $w_i = w_j \neq e$  and the elements  $\{w_l : i \leq l \leq j\}$  pair-wise commute.

In that case we will call the tuple  $(w_1, \dots, w_{i-1}, e, w_{i+1}, \dots, w_{j-1}, e, w_{j+1}, \dots, w_k)$  the *tuple obtained from  $(w_1, \dots, w_k)$  by collapsing on the pair  $(i, j)$* . We note that the word  $w_1 \dots w_k$  corresponding to  $(w_1, \dots, w_k)$  equals the word  $w_1 \dots w_{i-1} e w_{i+1} \dots w_{j-1} e w_{j+1} \dots w_k$  corresponding to the collapsed tuple. The notation that we introduced here is convenient because it keeps indices aligned correctly. We also note that a tuple  $(w_1, \dots, w_k)$  is semi-reduced if and only if we cannot collapse on any pair  $(i, j)$ . Hence, for a general tuple we can obtain a semi-reduced tuple by subsequently collapsing on pairs  $(i_1, j_1), \dots, (i_q, j_q)$ .

**Lemma 4.6.4.** *For a right-angled word hyperbolic Coxeter group  $\mathcal{W}_\Gamma$  associated to a finite graph  $\Gamma$ , we have that for  $\mathbf{u}, \mathbf{w} \in \mathcal{W}_\Gamma$  the function  $\tilde{\gamma}_{\mathbf{u}, \mathbf{w}} : \mathcal{W}_\Gamma \rightarrow \mathbb{R}$  defined in (4.16) has finite support.*

*Proof.* Let  $\mathbf{u} = u_1 \dots u_{n_1}, \mathbf{v} = v_1 \dots v_{n_2}, \mathbf{w} = w_1 \dots w_{n_3} \in \mathcal{W}_\Gamma$  written in reduced expression. We will moreover assume that  $|\mathbf{v}| > |\mathbf{u}| + |\mathbf{w}| + |\Gamma| + 2$ . We will show that for such words we have  $\tilde{\gamma}_{\mathbf{u}, \mathbf{w}}(\mathbf{v}) = 0$ . This then shows that  $\tilde{\gamma}_{\mathbf{u}, \mathbf{w}}$  has finite support.

Let  $(u'_1, \dots, u'_{n_1}, v'_1, \dots, v'_{n_2})$  be the semi-reduced tuple obtained by subsequently collapsing the tuple  $(u_1, \dots, u_{n_1}, v_1, \dots, v_{n_2})$  on pairs  $(i'_1, j'_1), \dots, (i'_{q_1}, j'_{q_1})$ . Then we must have  $i'_l \leq n_1$  and  $j'_l > n_1$  since the expressions for  $\mathbf{u}$  and  $\mathbf{v}$  were reduced. Also  $|\mathbf{uv}| = |\mathbf{u}| + |\mathbf{v}| - 2q_1$  and more generally for a weight  $\mathbf{x} \in [0, \infty)^{|\Gamma|}$  we have

$$\psi_{\mathbf{x}}(\mathbf{uv}) = \psi_{\mathbf{x}}(\mathbf{u}) + \psi_{\mathbf{x}}(\mathbf{v}) - 2 \sum_{l=1}^{q_1} \psi_{\mathbf{x}}(u'_{i'_l}).$$

Likewise let  $(v''_1, \dots, v''_{n_2}, w''_1, \dots, w''_{n_3})$  be the semi-reduced tuple obtained by subsequently collapsing the tuple  $(v_1, \dots, v_{n_2}, w_1, \dots, w_{n_3})$  on pairs  $(i''_1, j''_1), \dots, (i''_{q_2}, j''_{q_2})$ . Then we must have  $i''_l \leq n_2$  and  $j''_l > n_2$  since the expressions for  $\mathbf{v}$  and  $\mathbf{w}$  were reduced. Also  $|\mathbf{vw}| = |\mathbf{v}| + |\mathbf{w}| - 2q_2$  and more generally for a weight  $\mathbf{x} \in [0, \infty)^{|\Gamma|}$  we have

$$\psi_{\mathbf{x}}(\mathbf{vw}) = \psi_{\mathbf{x}}(\mathbf{v}) + \psi_{\mathbf{x}}(\mathbf{w}) - 2 \sum_{l=1}^{q_2} \psi_{\mathbf{x}}(w''_{j''_l - n_2}).$$

Let us denote

$$\Lambda = \{v_j : j \in \{1, \dots, n_2\} \setminus \{j'_1 - n_1, \dots, j'_{q_1} - n_1\} \cup \{i''_1, \dots, i''_{q_2}\}\}.$$

Now since  $n_2 = |\mathbf{v}| > |\mathbf{u}| + |\mathbf{w}| + |\Gamma| + 2 \geq q_1 + q_2 + |\Gamma| + 2$  we have that  $|\Lambda| \geq |\Gamma| + 2$ . Hence, there are two elements  $g_1, g_2 \in \Lambda$  that do not mutually commute. Now, if  $s_1, s_2 \in \Gamma$  commute with all elements in  $\Lambda$ , then  $s_1, s_2$  commute with both  $g_1$  and  $g_2$  so that by the word hyperbolicity of  $\mathcal{W}_\Gamma$  (see (2) of Theorem 4.6.1) we must have that also  $s_1$  commutes with  $s_2$ . We now let  $\Lambda_0 \subseteq \Gamma$  be the set of all generators that commute with all elements in  $\Lambda$ , i.e.  $\Lambda_0 = \bigcap_{v \in \Lambda} \text{Star}_\Gamma(v)$ . Then by what we just mentioned we have that the elements in  $\Lambda_0$  pair-wise commute, i.e.  $\Lambda_0 \in \text{Cliq}(\Gamma)$ .

Now, for  $i = 1, \dots, n_1$  let us set  $\widetilde{u}_i = u'_i$  and for  $i = 1, \dots, n_3$  set  $\widetilde{w}_i = w''_i$ . Furthermore, for  $i = 1, \dots, n_2$  set  $\widetilde{v}_i = e$  whenever either  $v'_i = e$  or  $v''_i = e$  but not both, and set  $\widetilde{v}_i = v_i$  otherwise. Let us also denote  $\widetilde{\mathbf{u}} = \widetilde{u}_1 \dots \widetilde{u}_{n_2}, \widetilde{\mathbf{v}} = \widetilde{v}_1 \dots \widetilde{v}_{n_2}$  and  $\widetilde{\mathbf{w}} = \widetilde{w}_1 \dots \widetilde{w}_{n_3}$ .

We claim that  $\widetilde{\mathbf{u}}\widetilde{\mathbf{v}}\widetilde{\mathbf{w}} = \mathbf{u}\mathbf{v}\mathbf{w}$ . Indeed, we have that  $\mathbf{u}\mathbf{v}\mathbf{w} = \mathbf{u}v''_1 \dots v''_{n_2} w''_1 \dots w''_{n_3}$ . Now we can collapse  $(u_1, \dots, u_{n_1}, v''_1, \dots, v''_{n_2}, w''_1, \dots, w''_{n_3})$  subsequently on the pairs  $(i'_l, j'_l)$  for  $l = 1, \dots, q_1$  except when  $v''_{j'_l-n_1} \neq v_{j'_l-n_1}$  for some  $1 \leq l \leq q_1$ , in which case  $v''_{j'_l-n_1} = e$ . If this is the case then  $j'_l - n_1 = i''_{k_l}$  for some  $k_l \in \{1, \dots, q_2\}$ . In particular it follows that in this case  $u_{i'_l} = v_{j'_l-n_1} = v_{i''_{k_l}} = w_{j''_{k_l}-n_2}$  and that this element commutes with all elements in  $\Lambda$ . Therefore  $u_{i'_l} \in \Lambda_0$ . We can then simply interchange the elements at index  $i'_l$  (which is  $u_{i'_l}$ ) and the element at index  $j'_l$  (which is  $v''_{j'_l-n_1} = e$ ). This manipulation does not change the word, and allows us to continue collapsing on the remaining pairs. Once we are done collapsing on all pairs we have obtained the tuple  $(\widetilde{u}_1, \dots, \widetilde{u}_{n_1}, \widetilde{v}_1, \dots, \widetilde{v}_{n_2}, \widetilde{w}_1, \dots, \widetilde{w}_{n_3})$ . This thus shows us that  $\mathbf{u}\mathbf{v}\mathbf{w} = \widetilde{\mathbf{u}}\widetilde{\mathbf{v}}\widetilde{\mathbf{w}}$ . It also shows us that  $\widetilde{v}_{j'_l-n_1} \in \{e\} \cup \Lambda_0$  for  $l = 1, \dots, q_2$ . Note that also by definition  $\widetilde{u}_{i'_l} = e$  for  $l = 1, \dots, q_1$  and  $\widetilde{w}_{j''_{k_l}-n_2} = e$  for  $l = 1, \dots, q_2$ . Therefore we also have that  $\psi_{\Gamma \setminus \Lambda_0}(\widetilde{u}_{i'_l}) = \psi_{\Gamma \setminus \Lambda_0}(e) = 0$  for  $l = 1, \dots, q_1$  and likewise  $\psi_{\Gamma \setminus \Lambda_0}(\widetilde{w}_{j''_{k_l}-n_2}) = 0$  for  $l = 1, \dots, q_2$ . Furthermore  $\psi_{\Gamma \setminus \Lambda_0}(\widetilde{v}_{j'_l-n_1}) = 0$  for  $l = 1, \dots, q_1$  and  $\psi_{\Gamma \setminus \Lambda_0}(\widetilde{v}_{i''_{k_l}}) = 0$  for  $l = 1, \dots, q_2$ .

If we can collapse  $(\widetilde{u}_1, \dots, \widetilde{u}_{n_1}, \widetilde{v}_1, \dots, \widetilde{v}_{n_2}, \widetilde{w}_1, \dots, \widetilde{w}_{n_3})$  on some pair  $(i, j)$  then we must have  $i \leq n_1$  and  $j > n_1 + n_2$ . Indeed otherwise we have that either  $(u'_1, \dots, u'_{n_1}, v'_1, \dots, v'_{n_2})$  or  $(v''_1, \dots, v''_{n_2}, w''_1, \dots, w''_{n_3})$  is not semi-reduced, which is a contradiction. We will let  $q \geq 0$  and let  $(i_1, j_1), \dots, (i_q, j_q)$  be pairs on which we can subsequently collapse the tuple  $(\widetilde{u}_1, \dots, \widetilde{u}_{n_1}, \widetilde{v}_1, \dots, \widetilde{v}_{n_2}, \widetilde{w}_1, \dots, \widetilde{w}_{n_3})$  to obtain a semi-reduced tuple. Then we thus must have  $i_l \leq n_1$  and  $j_l > n_1 + n_2$ . This thus implies that for  $l = 1, \dots, q$  we have that  $\widetilde{u}_{i_l} = \widetilde{w}_{j_l-n_1-n_2}$  commutes with the elements from  $\Lambda$ . Therefore we have  $\{\widetilde{u}_{i_l} : l = 1, \dots, q\} = \{\widetilde{w}_{j_l-n_1-n_2} : l = 1, \dots, q\} \subseteq \Lambda_0$ . Now, we have that

$$\begin{aligned} \psi_{\Gamma \setminus \Lambda_0}(\mathbf{u}\mathbf{v}\mathbf{w}) &= \psi_{\Gamma \setminus \Lambda_0}(\mathbf{u}) + \psi_{\Gamma \setminus \Lambda_0}(\mathbf{v}) + \psi_{\Gamma \setminus \Lambda_0}(\mathbf{w}) \\ &\quad - 2 \left[ \sum_{l=1}^{q_1} \psi_{\Gamma \setminus \Lambda_0}(u_{i'_l}) + \sum_{l=1}^{q_2} \psi_{\Gamma \setminus \Lambda_0}(w_{i''_{k_l}-n_2}) + \sum_{l=1}^q \psi_{\Gamma \setminus \Lambda_0}(\widetilde{u}_{i_l}) \right] \\ &= \psi_{\Gamma \setminus \Lambda_0}(\mathbf{u}\mathbf{v}) + \psi_{\Gamma \setminus \Lambda_0}(\mathbf{v}\mathbf{w}) - \psi_{\Gamma \setminus \Lambda_0}(\mathbf{v}) + 2 \sum_{l=1}^q \psi_{\Gamma \setminus \Lambda_0}(\widetilde{u}_{i_l}) \\ &= \psi_{\Gamma \setminus \Lambda_0}(\mathbf{u}\mathbf{v}) + \psi_{\Gamma \setminus \Lambda_0}(\mathbf{v}\mathbf{w}) - \psi_{\Gamma \setminus \Lambda_0}(\mathbf{v}). \end{aligned}$$

This shows that  $\gamma_{\mathbf{u}, \mathbf{w}}^{\psi_{\Gamma \setminus \Lambda_0}}(\mathbf{v}) = 0$ . Therefore, as  $\Lambda_0 \in \text{Cliq}(\Gamma)$  we obtain that  $\widetilde{\gamma}_{\mathbf{u}, \mathbf{w}}(\mathbf{v}) = 0$ . Now as this holds for every  $\mathbf{v} \in \mathcal{W}_\Gamma$  with  $|\mathbf{v}| > |\mathbf{u}| + |\mathbf{w}| + |\Gamma| + 2$ , we obtain that  $\widetilde{\gamma}_{\mathbf{u}, \mathbf{w}}$  has finite support.  $\square$

## 4.7. STRONG SOLIDITY OF HECKE VON NEUMANN ALGEBRAS

In this final section we obtain strong solidity results for Hecke von Neumann algebras. These are  $q$ -deformations of the group (von Neumann) algebra of a Coxeter group. If  $q = 1$  we retrieve the classical case of a group (von Neumann) algebra of a Coxeter group.

For the Hecke deformations our methods turn out to improve on existing strong solidity results. In [Kli23b, Theorem 0.7] it was shown that for Coxeter groups that are small at infinity, their Hecke von Neumann algebras satisfy the condition  $\text{AO}^+$ . If such Hecke von Neumann algebras have the weak- $*$  completely bounded approximation property

(weak-\* CBAP) then they are strongly solid by [Iso15a, Theorem A] (this is a generalisation of Theorem 4.2.14 from [PV14b]). The weak-\* CBAP was proved in [Cas20] for Hecke von Neumann algebras of right-angled Coxeter groups (see also Example 6.3.7); For general Coxeter groups this is an open problem. Therefore right-angled Coxeter groups that are small at infinity have strongly solid Hecke von Neumann algebras. It was proved in [Kli23b] that such Coxeter groups are in fact free products of abelian Coxeter groups; hence this result is somewhat more limited than one would hope for.

It is natural to ask whether these strong solidity results for Hecke von Neumann algebras apply to more general word hyperbolic Coxeter groups. In the group case ( $q = 1$ ) this is exactly Theorem 4.6.2. However, the results from [Kli23b] and in particular [Kli23b, Corollary 3.17] show that it is hard to extend current methods beyond free products of abelian Coxeter groups. A typical right-angled word hyperbolic Coxeter group that was not covered before this chapter is given by

$$\langle \{s_1, s_2, s_3, s_4\} | M = (m_{i,j})_{i,j} \rangle \quad \text{with } m_{i,j} = 2 \text{ if } |i - j| = 1 \text{ and } m_{i,j} = \infty \text{ otherwise.} \quad (4.17)$$

In this section we prove that also the Hecke deformations of this Coxeter group satisfy  $\text{AO}^+$  and are strongly solid. The precise statement is contained in Theorem 4.7.5.

#### 4.7.1. COEFFICIENTS FOR GRADIENT BIMODULES OF HECKE ALGEBRAS

Let  $\mathcal{W} = \langle S | M \rangle$  be a finite rank Coxeter group. We use the notation of Hecke-algebras from the preliminaries. Fix a Hecke-tuple  $\mathbf{q} = (q_s)_{s \in S}$ . We will simply write  $T_{\mathbf{w}}$  instead of  $T_{\mathbf{w}}^{(\mathbf{q})}$  and  $p_s$  instead of  $p_s(\mathbf{q})$ . We let  $\psi : \mathcal{W} \rightarrow \mathbb{R}$  be proper and conditionally of negative type. Define

$$\Delta_\psi := \Delta_\psi^{(\mathbf{q})} : \mathbb{C}_{\mathbf{q}}[\mathcal{W}] \rightarrow \mathbb{C}_{\mathbf{q}}[\mathcal{W}] : T_{\mathbf{w}} \mapsto \psi(\mathbf{w}) T_{\mathbf{w}},$$

and for  $t \geq 0$ ,

$$\Phi_t := \Phi_t^{(\mathbf{q})} : \mathbb{C}_{\mathbf{q}}[\mathcal{W}] \rightarrow \mathbb{C}_{\mathbf{q}}[\mathcal{W}] : T_{\mathbf{w}} \mapsto \exp(-t\psi(\mathbf{w})) T_{\mathbf{w}}. \quad (4.18)$$

We will now work under the following assumption.

**Assumption 4.7.1.** *For  $t \geq 0$  the map  $\Phi_t$  extends to a normal unital completely positive map  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}) \rightarrow \mathcal{N}_{\mathbf{q}}(\mathcal{W})$ .*

The main point of the assumption is the complete positivity of  $\Phi_t$ ; the unitality is automatic since  $\psi(e) = e$  and also the existence of a normal extension can usually be proved using a standard argument once one knows that  $\Phi_t$  is bounded (see the final paragraph of the proof of [Cas20, Theorem 4.13]).

The assumption holds in case  $\mathbf{q} = 1$  by Schönberg's theorem and in case  $\mathcal{W}$  is right-angled by combining [Cas20, Corollary 3.4, Proposition 3.7] and [CF17, Proposition 2.30]. Note that if the assumption holds then  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  satisfies the Haagerup property since  $\psi$  is proper. In general we do not know whether Assumption 4.7.1 holds. In fact, it is not even known whether  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  has the Haagerup property unless  $\mathcal{W}$  is right-angled (see [Cas20, Section 3]) or  $\mathbf{q} = 1$  (see [BJS88]).

It is standard to check that if Assumption 4.7.1 holds then  $\Phi = (\Phi_t)_{t \geq 0}$  is a symmetric quantum Markov semigroup. For the continuity property note that  $\Phi_t$  is a contractive semigroup on  $L^2(\mathcal{N}_{\mathbf{q}}(\mathcal{W}), \tau)$  and then use that on the unit ball of  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  the strong topology equals the  $L^2(\mathcal{N}_{\mathbf{q}}(\mathcal{W}), \tau)$ -topology.

We shall now investigate the gradient- $S_p$ ,  $p \in [1, \infty]$  property for  $\Phi$  with respect to the  $\sigma$ -weak dense subalgebra  $A := \mathbb{C}_{\mathbf{q}}[\mathcal{W}]$  of  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$ . The set  $A_0 := \{T_s : s \in S\}$  forms a self-adjoint set that generates the  $*$ -algebra  $A$ . Therefore by Lemma 4.3.5 in order to check the gradient- $S_p$  property for  $\Phi$  we only have to check that  $\Psi^{T_u, T_w}$  given in Definition 4.3.3 is in  $S_p = S_p(L^2(\mathcal{N}_{\mathbf{q}}(\mathcal{W}), \tau))$  for generators  $u, w \in S$ . To check this we shall make some calculations to obtain a simplified expression for  $\Psi^{T_u, T_w}$ .

Fix  $u, w \in S$  and let  $\mathbf{v} \in \mathcal{W}$ . We have by the multiplication rules that

$$\begin{aligned} T_u(T_{\mathbf{v}}T_w) &= T_uT_{\mathbf{v}w} + T_uT_{\mathbf{v}}p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) \\ &= T_{u\mathbf{v}w} + p_uT_{\mathbf{v}w}\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|) \\ &\quad + (T_{u\mathbf{v}} + p_uT_{\mathbf{v}}\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|))p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|). \end{aligned}$$

We can now make the following calculations

$$\begin{aligned} \Delta_{\psi}(T_uT_{\mathbf{v}}T_w) &= \psi(u\mathbf{v}w)T_{u\mathbf{v}w} + \psi(\mathbf{v}w)p_uT_{\mathbf{v}w}\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|) \\ &\quad + \psi(u\mathbf{v})T_{u\mathbf{v}}p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) + \psi(\mathbf{v})p_uT_{\mathbf{v}}p_w\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|)\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|), \\ T_u\Delta_{\psi}(T_{\mathbf{v}})T_w &= \psi(\mathbf{v})(T_{u\mathbf{v}w} + p_uT_{\mathbf{v}w}\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|)) \\ &\quad + \psi(\mathbf{v})(T_{u\mathbf{v}} + p_uT_{\mathbf{v}}\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|))p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|), \\ T_u\Delta_{\psi}(T_{\mathbf{v}}T_w) &= \psi(\mathbf{v}w)T_uT_{\mathbf{v}w} + \psi(\mathbf{v})T_uT_{\mathbf{v}}p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|), \\ &= \psi(\mathbf{v}w)(T_{u\mathbf{v}w} + p_uT_{\mathbf{v}w}\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|)) \\ &\quad + \psi(\mathbf{v})(T_{u\mathbf{v}} + p_uT_{\mathbf{v}}\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|))p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|), \\ \Delta_{\psi}(T_uT_{\mathbf{v}})T_w &= \psi(u\mathbf{v})T_{u\mathbf{v}}T_w + \psi(\mathbf{v})p_uT_{\mathbf{v}}T_w\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|) \\ &= \psi(u\mathbf{v})(T_{u\mathbf{v}w} + T_{u\mathbf{v}}p_w\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|)) \\ &\quad + \psi(\mathbf{v})p_u(T_{\mathbf{v}w} + T_{\mathbf{v}}p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|))\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|). \end{aligned}$$

Let  $\psi_S$  be again the word length function on  $\mathcal{W}$ . Now by collecting all previous terms we

get

$$\begin{aligned}
\Psi^{T_u, T_w}(T_{\mathbf{v}}) &= \Delta_{\psi}(T_u T_{\mathbf{v}} T_w) + T_u \Delta_{\psi}(T_{\mathbf{v}}) T_w - T_u \Delta_{\psi}(T_{\mathbf{v}} T_w) - \Delta_{\psi}(T_u T_{\mathbf{v}}) T_w \\
&= (\psi(u\mathbf{v}w) + \psi(\mathbf{v}) - \psi(\mathbf{v}w) - \psi(u\mathbf{v})) T_{u\mathbf{v}w} \\
&\quad + [(\psi(u\mathbf{v}) + \psi(\mathbf{v}) - \psi(\mathbf{v})) \mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) - \psi(u\mathbf{v}) \mathbb{1}(|u\mathbf{v}w| < |u\mathbf{v}|)] T_{u\mathbf{v}} p_w \\
&\quad + [(\psi(\mathbf{v}w) + \psi(\mathbf{v}) - \psi(\mathbf{v}w)) \mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|) - \psi(\mathbf{v}) \mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|)] p_u T_{\mathbf{v}w} \\
&\quad + (\psi(\mathbf{v}) + \psi(\mathbf{v}) - \psi(\mathbf{v}) - \psi(\mathbf{v})) \mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|) \mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) p_u T_{\mathbf{v}} p_w \\
&= \gamma_{u,w}^{\psi}(\mathbf{v}) T_{u\mathbf{v}w} \\
&\quad + \psi(u\mathbf{v}) (\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) - \mathbb{1}(|u\mathbf{v}w| < |u\mathbf{v}|)) T_{u\mathbf{v}} p_w \\
&\quad + \psi(\mathbf{v}) (\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|) - \mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|)) p_u T_{\mathbf{v}w} \\
&= \gamma_{u,w}^{\psi}(\mathbf{v}) T_{u\mathbf{v}w} \\
&\quad + \psi(u\mathbf{v}) \left( \frac{|\mathbf{v}| - |\mathbf{v}w| + 1}{2} - \frac{|u\mathbf{v}| - |u\mathbf{v}w| + 1}{2} \right) T_{u\mathbf{v}} p_w \\
&\quad + \psi(\mathbf{v}) \left( \frac{|\mathbf{v}w| - |u\mathbf{v}w| + 1}{2} - \frac{|\mathbf{v}| - |u\mathbf{v}| + 1}{2} \right) p_u T_{\mathbf{v}w} \\
&= \gamma_{u,w}^{\psi}(\mathbf{v}) T_{u\mathbf{v}w} + \frac{1}{2} (|u\mathbf{v}w| + |\mathbf{v}| - |\mathbf{v}w| - |u\mathbf{v}|) (\psi(u\mathbf{v}) T_{u\mathbf{v}} p_w - \psi(\mathbf{v}) p_u T_{\mathbf{v}w}) \\
&= \gamma_{u,w}^{\psi}(\mathbf{v}) T_{u\mathbf{v}w} + \frac{1}{2} \gamma_{u,w}^{\psi_S}(\mathbf{v}) (\psi(u\mathbf{v}) T_{u\mathbf{v}} p_w - \psi(\mathbf{v}) p_u T_{\mathbf{v}w}).
\end{aligned}$$

Now when  $u\mathbf{v} \neq \mathbf{v}w$  we have by Lemma 4.4.2 that  $\gamma_{u,w}^{\psi_S}(\mathbf{v}) = 0$ . When  $u\mathbf{v} = \mathbf{v}w$  we have  $|\frac{1}{2}\gamma_{u,w}^{\psi_S}(\mathbf{v})| = \psi_S(u) = 1$ . In this case the elements  $u$  and  $w$  are also conjugate and therefore  $p_u = p_w$ . Combining these facts we obtain the simplified formula

$$\Psi^{T_u, T_w}(T_{\mathbf{v}}) = \gamma_{u,w}^{\psi}(\mathbf{v}) T_{u\mathbf{v}w} + \frac{1}{2} \gamma_{u,w}^{\psi_S}(\mathbf{v}) (\psi(u\mathbf{v}) - \psi(\mathbf{v})) T_{u\mathbf{v}} p_w. \quad (4.19)$$

We will proceed under the further assumption that  $\psi$  is a length function.

**Assumption 4.7.2.** *We shall assume from this point that the proper, conditionally of negative type function  $\psi : \mathcal{W} \rightarrow \mathbb{R}$  is also a length function.*

Using the fact that  $\{T_{\mathbf{v}}\}_{\mathbf{v} \in \mathcal{W}}$  is an orthonormal basis for  $L^2(\mathcal{N}_{\mathbf{q}}(\mathcal{W}), \tau)$  we obtain that for the  $S_2$ -norm of  $\Psi^{T_u, T_w}$  we have the following bound

$$\begin{aligned}
\|\Psi^{T_u, T_w}\|_{S_2}^2 &= \sum_{\mathbf{v} \in \mathcal{W}} \langle \Psi^{T_u, T_w}(T_{\mathbf{v}}), \Psi^{T_u, T_w}(T_{\mathbf{v}}) \rangle \\
&= \sum_{\mathbf{v} \in \mathcal{W}} \left[ |\gamma_{u,w}^{\psi}(\mathbf{v})|^2 + \frac{1}{4} |\gamma_{u,w}^{\psi_S}(\mathbf{v})|^2 |\psi(u\mathbf{v}) - \psi(\mathbf{v})|^2 |p_u|^2 \right] \\
&\leq \|\gamma_{u,w}^{\psi}\|_{\ell^2(\mathcal{W})}^2 + \frac{1}{4} |\psi(u)|^2 p_u^2 \|\gamma_{u,w}^{\psi_S}\|_{\ell^2(\mathcal{W})}^2.
\end{aligned} \quad (4.20)$$

We are then thus interested in functions  $\psi$  for which this bound is finite for all  $u, w \in S$ .

**Theorem 4.7.3.** *Let  $\mathcal{W} = \langle S|M \rangle$  be a finite-rank Coxeter group. Let  $\mathbf{q} = (q_s)_{s \in S}$  with  $q_s > 0$ . Let  $T \subseteq S$  be a subset that generates a finite subgroup  $\mathcal{W}_T \subseteq \mathcal{W}_S$ , i.e.  $T \in \text{Cliq}(S|M)$ . Suppose that  $\psi := \psi_{S \setminus T}$  satisfies Assumption 4.7.1. Then the QMS on  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  determined by (4.18) associated with  $\psi_{S \setminus T}$  is gradient- $S_2$ .*

*Proof.* lemma:gradient-Sp-weighted-word-length-right-angled-Coxeter-group □

**Theorem 4.7.4.** *Let  $\Gamma$  be a finite simple graph and let  $\mathbf{q} = (q_v)_{v \in \Gamma}$  with  $q_v > 0$ . Assume*

$$\Lambda := \{r \in \Gamma : \exists s, t \in \Gamma \text{ such that } r \in \text{Link}_{\Gamma}(s) \cap \text{Link}_{\Gamma}(t), s \notin \text{Star}_{\Gamma}(t)\} \quad (4.21)$$

*is a clique in  $\Gamma$ . Then the QMS on  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma})$  determined by (4.18) associated with  $\psi_{\Gamma \setminus \Lambda}$  is gradient- $S_2$ .*

*Proof.* It follows from Theorem 4.5.5 that for  $u, w \in \Gamma$  we have that  $\gamma_{u,w}^{\psi_{\Gamma \setminus \Lambda}} \in \ell^2(\mathcal{W}_{\Gamma})$ . Now if  $u \in \Lambda$  then  $\psi_{\Gamma \setminus \Lambda}(u) = 0$  and hence by (4.20),

$$\|\Psi^{T_u, T_w}\|_{S_2}^2 \leq \|\gamma_{u,w}^{\psi_{\Gamma \setminus \Lambda}}\|_{\ell^2(\mathcal{W}_{\Gamma})}^2 < \infty.$$

If  $u \in \Gamma \setminus \Lambda$  then  $\psi_{S \setminus \Lambda}(u) = 1$  and therefore by Lemma 4.4.2 we have

$$|\gamma_{u,w}^{\psi_{\Gamma \setminus \Lambda}}(\mathbf{v})| = 2\psi_{\Gamma \setminus \Lambda}(u)\mathbb{1}(u\mathbf{v} = \mathbf{v}w) = 2\psi_{\Gamma}(u)\mathbb{1}(u\mathbf{v} = \mathbf{v}w) = |\gamma_{u,w}^{\psi_{\Gamma}}(\mathbf{v})|.$$

This means that in this case  $\gamma_{u,w}^{\psi_{\Gamma \setminus \Lambda}} = \gamma_{u,w}^{\psi_{\Gamma}} \in \ell^2(\mathcal{W}_{\Gamma})$ . We conclude from (4.20) that

$$\|\Psi^{T_u, T_w}\|_{S_2}^2 \leq \|\gamma_{u,w}^{\psi_{\Gamma \setminus \Lambda}}\|_{\ell^2(\mathcal{W}_{\Gamma})}^2 + \frac{1}{4}p_u^2 \cdot \|\gamma_{u,w}^{\psi_{\Gamma}}\|_{\ell^2(\mathcal{W}_{\Gamma})}^2 < \infty.$$

□

**Theorem 4.7.5.** *Let  $\Gamma$  be a finite graph and let  $\mathcal{W}_{\Gamma}$  be the corresponding right-angled Coxeter group. Let  $\mathbf{q} = (q_v)_{v \in \Gamma}$  with  $q_v > 0$ . Assume that (4.21) is contained in  $\text{Cliq}(\Gamma)$ . Then  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma})$  satisfies  $\text{AO}^+$  and is strongly solid.*

*Proof.* Theorem 4.7.4 shows that the QMS  $\Phi$  on  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma})$  associated with the length function  $\psi_{\Gamma \setminus \Lambda}$  is gradient- $S_2$ . Therefore by Theorem 4.3.9 we see that a dense set of coefficients of the associated gradient bimodule  $L^2(\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma}), \tau)_{\nabla}$  is in  $S_2$ . Note that Theorem 4.3.9 is stated only for groups, but a straightforward adaptation of the computations in the proof yields the same result for Hecke algebras. Hence the gradient bimodule is quasi-contained in the coarse bimodule of  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma})$  by [CIW21, Theorem 3.9] (see also Proposition 4.2.3). The Riesz transform is then an isometry  $R_{\Phi} : L^2(\mathcal{N}_{\mathbf{q}}(\mathcal{W}), \tau) \rightarrow L^2(\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma}), \tau)_{\nabla}$ . The kernel of  $R_{\Phi}$  is given by the space spanned by the vectors  $T_{\mathbf{w}}$  with  $\mathbf{w}$  in the (finite) group  $\mathcal{W}_{\Lambda}$ . Essentially in the same way as in the group case ( $\mathbf{q} = 1$ ) one checks that  $\Phi$  is filtered with subexponential growth. Therefore by Theorem 4.3.7 we see that  $R_{\Phi}$  is almost bimodular. By [CKL21, Theorem 6.1]  $C_{r,\mathbf{q}}^*(\mathcal{W}_{\Gamma})$  is exact and hence locally reflexive [BO08]. We may now invoke Theorem [CIW21, Proposition 5.2] (see also Theorem 4.2.12) to conclude that  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma})$  satisfies  $\text{AO}^+$ . By [Cas20, Theorem A]  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma})$  satisfies the weak- $*$  completely bounded approximation property. Hence [Iso15a, Theorem A] (see also Theorem 4.2.14) shows that  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  is strongly solid. □

*Remark 4.7.6.* The strong solidity result of Theorem 4.7.5 can also be proved by combining the results in this chapter with the methods of [Cas21], [OP10b], [Pet08] without using condition  $\text{AO}^+$ .

*Remark 4.7.7.* The set (4.21) can be understood as all elements in  $S$  that belong to exactly one maximal clique.

## 4.8. DISCUSSION

We list two natural problems.

*Problem 4.8.1.* Consider a Coxeter system  $\mathcal{W} = \langle S|M \rangle$  and  $\mathbf{q} = (q_s)_{s \in S}$  with  $q_s > 0, s \in S$  such that  $q_s = q_t$  whenever  $s, t \in S$  are conjugate in  $\mathcal{W}$ . Does the Hecke von Neumann algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  have the Haagerup property and/or the weak-\* completely bounded approximation property? An affirmative answer for both properties is known in case  $q_s = 1$  for all  $s \in S$  [BJS88], [Fen02], [Jan02] or in case  $\mathcal{W} = \langle S|M \rangle$  is right-angled [Cas20]. For general cases these properties are open. In particular we do not know in which generality Assumption 4.7.1 holds for  $\psi = \psi_S$  the (unweighted) word length function.

*Problem 4.8.2.* For a right-angled word hyperbolic Coxeter system  $\mathcal{W} = \langle S|M \rangle$  and  $\mathbf{q} = (q_s)_{s \in S}$  a tuple with  $q_s > 0, s \in S$  we ask if the Hecke-algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  is strongly solid? The cases obtained in Theorem 4.7.5 are word hyperbolic but do not exhaust all word hyperbolic right-angled Coxeter groups. In case  $q_s = 1, s \in S$  the tensor product techniques from Section 4.6 allows one to improve the results of Section 4.7 to all word hyperbolic right-angled Coxeter groups. However, such tensor products of bimodules are unavailable unless  $q_s = 1, s \in S$  by the absence of a suitable comultiplication for Hecke algebras.

We will resolve parts of these problems in the coming chapters. Indeed, in Chapter 5 (Theorem 5.6.13) we precisely characterize for finite rank, right-angled Coxeter groups when  $\mathcal{L}(\mathcal{W}_{\Gamma})$  is strong solidity in terms of the graph  $\Gamma$ . More generally we even characterize strong solidity for arbitrary graph products of von Neumann algebras (Theorem 5.6.7); in particular right-angled Hecke-von Neumann algebras  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}_{\Gamma})$ . Furthermore, in Chapter 6 we study the weak-\* completely contractive approximation property (weak-\* CCAP) for graph products. We show that  $M_{\Gamma} = *_{v, \Gamma} (M_v, \varphi_v)$  possesses the weak-\* CCAP whenever  $\dim M_v < \infty$  for  $v \in \Gamma$ . This in particular shows that  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  possesses the weak-\* CCAP whenever  $\mathcal{W}$  can be written as graph product  $\mathcal{W} = *_{v, \Gamma} \mathcal{W}_v$  of finite Coxeter groups  $\mathcal{W}_v$ .

# 5

## RIGID GRAPH PRODUCTS

We prove rigidity properties for von Neumann algebraic graph products. We introduce the notion of rigid graphs and define a class of  $\text{II}_1$ -factors named  $\mathcal{C}_{\text{Rigid}}$ . For von Neumann algebras in this class we show a unique rigid graph product decomposition. In particular, we obtain unique prime factorization results and unique free product decomposition results for new classes of von Neumann algebras. Furthermore, we show that for many graph products of  $\text{II}_1$ -factors we can, up to some constant, retrieve the radius of the graph from the graph product. We also prove several technical results concerning relative amenability and embeddings of (quasi)-normalizers in graph products. Furthermore, we give sufficient conditions for a graph product to be nuclear and characterize strong solidity, primeness and free-indecomposability for graph products.

This chapter is based on the papers:

- **Matthijs Borst and Martijn Caspers**, *Classification of right-angled Coxeter groups with a strongly solid von Neumann algebra*, [Journal de Mathématiques Pures et Appliquées](#) 189 (2024) 103591.
- **Matthijs Borst, Martijn Caspers and Enli Chen**, *Rigid graph products*, Preprint submitted to journal: [Arxiv:2408.06171v2](#).

### 5.1. INTRODUCTION

The advent of Popa's deformation-rigidity theory has led to major applications to the structure of von Neumann algebras and their decomposability properties for crossed products, tensor products and free products. For instance, in [OP10a] Ozawa and Popa studied the notion of strongly solid von Neumann algebras (see Definition 5.6.1) and proved that the free group factors possess this property. Consequently, these von Neumann algebras do not admit certain crossed product decompositions, and they are prime factors (see Definition 5.7.1), meaning that they can not decompose as tensor products in non-trivial way (see also [Oza04], [Pop83]). More general prime factorization results

were then obtained in e.g. [CKP16; CSU13; HI17; Iso17; OP04; Pet08; Sak09; SW13]. In the same spirit, decompositions of von Neumann algebras in terms of free products and Kurosh type results were studied in e.g. [HU16; IPP08; Oza06; Pet08].

This chapter contributes to decomposability and rigidity results for von Neumann algebras that appear as graph products. We will prove rigidity results for graph products of von Neumann algebras. We first discuss our main result Theorem F which establishes unique rigid graph product decompositions. Thereafter, we give new unique prime factorization results and unique free product decomposition results. Furthermore, we state results that characterize primeness, free indecomposability and strong solidity for graph products. Hereafter, we present other main results that are needed in the proofs. Last, we give an overview of the structure of the chapter.

### 5.1.1. UNIQUE RIGID GRAPH PRODUCT DECOMPOSITION

Our main result, Theorem F, concerns the question whether from the graph product  $*_{v,\Gamma}(M_v, \varphi_v)$  we can, under some conditions, retrieve the graph  $\Gamma$  and the vertex von Neumann algebras  $M_v$ . Such questions have already been studied for graph products of groups. In [Gre90, Theorem 4.12] Green showed the following rigidity result, which for graph products  $*_{v,\Gamma}G_v$  of prime cycles  $G_v$  asserts that the graph  $\Gamma$  and the vertex groups  $G_v$  can be retrieved from the graph product group.

**Theorem (Green).** *Let  $\Gamma, \Lambda$  be finite graphs,  $G_\Gamma := *_{v,\Gamma}G_v$  and  $H_\Lambda := *_{w,\Lambda}H_w$  be graph products of groups  $G_v := \mathbb{Z}/p_v\mathbb{Z}$  and  $H_w := \mathbb{Z}/q_w\mathbb{Z}$  with some prime numbers  $p_v, q_w$ . If  $G_\Gamma$  and  $H_\Lambda$  are isomorphic, then there is a graph isomorphism  $\alpha : \Gamma \rightarrow \Lambda$  such that  $H_{\alpha(v)} \simeq G_v$ .*

In the current chapter we prove an analogy of this result for graph products  $M_\Gamma = *_{v,\Gamma}(M_v, \tau_v)$  of tracial von Neumann algebras  $(M_v, \tau_v)$ . Earlier rigidity results for von Neumann algebraic graph products have already been proven in [CDD22, Theorem A and C] for group von Neumann algebras  $M_v := \mathcal{L}(G_v)$  for certain discrete property (T) groups  $G_v$  and for graphs  $\Gamma$  from a class called  $\text{CC}_1$ . In our main result, Theorem F, we also prove rigidity results for graph products of von Neumann algebras  $M_\Gamma = *_{v,\Gamma}(M_v, \tau_v)$ . Our result compares to [CDD22; CDD23a] as follows. On the one hand we cover a much richer class of graphs than  $\text{CC}_1$  and our vertex von Neumann algebras  $M_v$  come from a different class than [CDD22; CDD23a]. In this chapter  $M_v$  are not even necessarily group von Neumann algebras. On the other hand the type of rigidity obtained in [CDD22; CDD23a] is stronger as it recovers the groups up to isomorphism, and not just the von Neumann algebras. Furthermore, [CDD22; CDD23a] obtains a so-called superrigidity result, meaning that the group can be recovered from an isomorphism of  $\mathcal{L}(G)$  with any other group von Neumann algebra, whereas our rigidity results are usually for an isomorphism of two von Neumann algebras in the class  $\mathcal{C}_{\text{Rigid}}$  introduced below. Such a superrigidity result is simply not true in the context of the current chapter as we argue in Remark 5.5.6.

The condition we impose on the vertex von Neumann algebras  $M_v$  is that they lie in the class  $\mathcal{C}_{\text{Vertex}}$  of all non-amenable  $\text{II}_1$ -factors that satisfy property strong (AO) (see

Definition 5.5.4) and have separable preduals. This is a natural class of von Neumann algebras including the (interpolated) free group factors  $\mathcal{L}(\mathbb{F}_t)$  for  $1 < t < \infty$ , the group von Neumann algebras  $\mathcal{L}(G)$  of non-amenable hyperbolic icc groups  $G$  [HG04],  $q$ -Gaussian von Neumann algebras  $M_q(H_{\mathbb{R}})$  associated with real Hilbert spaces  $H_{\mathbb{R}}$  with  $2 \leq \dim(H_{\mathbb{R}}) < \infty$  [Bor+23, Remark 4.5], [Kuz23], free orthogonal quantum groups [VV07] as well as several common series of easy quantum groups and free wreath products of quantum groups [Cas22, Theorem 0.5].

The condition we impose on the graph  $\Gamma$  is that each vertex  $v$  satisfies  $\text{Link}(\text{Link}(v)) = \{v\}$ . Such graphs, which we call *rigid*, form a large natural class of graphs containing for example complete graphs and cyclic graphs with at least 5 vertices. We also observe that all graphs in  $\text{CC}_1$  are rigid (see Remark 5.2.10). We stress that some restrictions on the graphs need to be imposed. Indeed, for general graphs  $\Gamma$ , and graph products  $M_{\Gamma} = *_{v \in \Gamma} (M_v, \tau_v)$  with  $M_v \in \mathcal{C}_{\text{Vertex}}$ , it is not possible to retrieve the graphs  $\Gamma$  from  $M_{\Gamma}$  (see Remark 5.5.6). This is due to the fact that the free product  $(M_v, \tau_v) * (M_w, \tau_w)$  of factors  $M_v, M_w \in \mathcal{C}_{\text{Vertex}}$  again lies again in the class  $\mathcal{C}_{\text{Vertex}}$  (see Remark 5.5.5).

We now state our main result which shows rigidity for the class  $\mathcal{C}_{\text{Rigid}}$  of all graph products  $M_{\Gamma} = *_{v \in \Gamma} (M_v, \tau_v)$  with  $\Gamma$  non-empty, finite, rigid graphs and with  $M_v \in \mathcal{C}_{\text{Vertex}}$ .

**Theorem F** (Theorem 5.5.19 and Theorem 5.7.5). *Let  $\Gamma$  be finite rigid graphs and for  $v \in \Gamma$  let  $M_v$  be von Neumann algebras in the class  $\mathcal{C}_{\text{Vertex}}$  with faithful normal state  $\tau_v$ . Let  $M_{\Gamma} = *_{v \in \Gamma} (M_v, \tau_v)$  be their graph product. Suppose there is another graph product decomposition of  $M_{\Gamma}$  over another rigid graph  $\Lambda$  and other von Neumann algebras  $N_w \in \mathcal{C}_{\text{Vertex}}$ ,  $w \in \Lambda$ , i.e.  $M_{\Gamma} = *_{w \in \Lambda} (N_w, \tau_w)$ . Then there is a graph isomorphism  $\alpha : \Gamma \rightarrow \Lambda$ , and for each  $v \in \Gamma$  there is a unitary  $u_v \in M_{\Gamma}$  and a real number  $0 < t_v < \infty$  such that:*

$$M_{\text{Star}(v)} = u_v^* N_{\text{Star}(\alpha(v))} u_v \quad \text{and} \quad M_v \simeq N_{\alpha(v)}^{t_v}. \quad (5.1)$$

Furthermore, for the connected component  $\Gamma_v \subseteq \Gamma$  of any vertex  $v \in \Gamma$ , we have  $M_{\Gamma_v} = u_v^* N_{\alpha(\Gamma_v)} u_v$ ; and for any irreducible component  $\Gamma_0 \subseteq \Gamma$ ,  $\exists t_0 \in (0, \infty)$  such that  $M_{\Gamma_0} \simeq N_{\alpha(\Gamma_0)}^{t_0}$ .

We remark that in the setting of [CDD22, Theorem 7.9], it is possible to obtain unitary conjugacy between the vertex von Neumann algebras  $M_v = \mathcal{L}(G_v)$ . In our setting it is generally only possible to obtain isomorphisms up to amplification between the vertex von Neumann algebras. The reason is that the tensor product  $M_v \overline{\otimes} M_w$  of  $\text{II}_1$ -factors is isomorphic to the tensor product  $M_v^t \overline{\otimes} M_w^{1/t}$  for any  $0 < t < \infty$ . For certain subgraphs  $\Gamma_0 \subseteq \Gamma$  we do however obtain unitary conjugacy of the graph products  $M_{\Gamma_0}$  to  $N_{\alpha(\Gamma_0)}$  inside  $M_{\Gamma}$ . Indeed, this is the case when  $\Gamma_0$  is a connected component of  $\Gamma$  or is of the form  $\Gamma_0 = \text{Star}(v)$  for some vertex  $v$  of  $\Gamma$ . Moreover, for  $\Gamma_0$  an irreducible component of  $\Gamma$  we are able to show that  $M_{\Gamma_0}$  is isomorphic to a amplification of  $N_{\alpha(\Gamma_0)}$ .

### 5.1.2. UNIQUE PRIME FACTORIZATION

For classes of von Neumann algebras we are interested in unique prime factorization results. Recall that a  $\text{II}_1$ -factor  $M$  is prime if it can not decompose as a tensor product

$M = M_1 \overline{\otimes} M_2$  of diffuse factors  $M_1, M_2$ . The first example of a prime factor was given by Popa in [Pop83]. Thereafter, Ge showed in [Ge96] that  $\mathcal{L}(\mathbb{F}_n)$  is a prime factor for  $n \geq 2$  by computing Voiculescu's free entropy. Later, in [Oza04] Ozawa introduced a new property, called solidity, which for non-amenable factors implies primeness. He showed that all finite von Neumann algebras satisfying the Akemann-Ostrand property are solid. We note that in particular all von Neumann algebras in  $\mathcal{C}_{\text{Vertex}}$  are prime. There are many more examples of prime factors, see e.g. [BHV18; CSS18; CKP16; CSU13; DHI19; Pet08; Sak09; SW13].

Given a class  $\mathcal{C}$  of von Neumann algebras, a natural question is whether any von Neumann algebra  $M \in \mathcal{C}$  has a tensor product decomposition  $M = M_1 \overline{\otimes} \cdots \overline{\otimes} M_m$  for some  $m \geq 1$  and prime factors  $M_1, \dots, M_m \in \mathcal{C}$ , which is called prime factorization inside  $\mathcal{C}$ , and whether the prime factorization is unique. This is to say, given another prime factorization  $M = N_1 \overline{\otimes} \cdots \overline{\otimes} N_n$ , with  $n \geq 1$  and prime factors  $N_1, \dots, N_n \in \mathcal{C}$ , do we have  $n = m$  and, up to permutation of the indices, any  $M_i$  is isomorphic to an amplification of  $N_i$ . The first unique prime factorization (UPF) results were established by Ozawa and Popa in [OP04] for tensor products of group von Neumann algebras  $\mathcal{L}(G_v)$  for certain groups  $G_v$ . The groups they considered included non-amenable, icc groups that are hyperbolic or are discrete subgroups of connected simple Lie groups of rank one. Later, in [Iso17] Isono studied UPF results for free quantum group factors. Thereafter, by combining results from [OP04] and [Iso17], Houdayer and Isono showed in [HI17] more general UPF results for tensor products of factors from a class called  $\mathcal{C}_{(\text{AO})}$ . We note that our class  $\mathcal{C}_{\text{Vertex}}$  is very similar to  $\mathcal{C}_{(\text{AO})}$  and that  $\mathcal{C}_{\text{Vertex}} \subseteq \mathcal{C}_{(\text{AO})}$ . In the setting of graph products, UPF results have been obtained in [CSS18, Theorem 6.16] under the condition that the vertex von Neumann algebras are group von Neumann algebras.

We observe that we can use Theorem F to obtain UPF results. Indeed, let  $\mathcal{C}_{\text{Complete}}$  be the class of all tensor products of von Neumann algebras in  $\mathcal{C}_{\text{Vertex}}$ . If in Theorem F we restrict our attention to complete graphs (which are rigid) then we precisely obtain UPF results for the class  $\mathcal{C}_{\text{Complete}}$  (see Corollary 5.5.21). This partially retrieves the UPF results from [HI17]. To obtain more general UPF results we prove the following result which characterizes primeness for graph products of  $\text{II}_1$ -factors (see also Theorems 5.7.11 and 5.7.12 in the case the vertex von Neumann algebras are not  $\text{II}_1$ -factors).

**Theorem H** (Theorem 5.7.4). *Let  $\Gamma$  be a finite graph of size  $|\Gamma| \geq 2$ . For any  $v \in \Gamma$ , let  $M_v$  be a  $\text{II}_1$ -factor. The graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  is prime if and only if  $\Gamma$  is irreducible.*

We then use Theorem F and Theorem H to prove the following theorem which covers UPF results for a new class of von Neumann algebras (see Remark 5.7.7).

**Theorem I** (Theorem 5.7.6). *Any von Neumann algebra  $M \in \mathcal{C}_{\text{Rigid}}$  has a prime factorization inside  $\mathcal{C}_{\text{Rigid}}$ , i.e.*

$$M = M_1 \overline{\otimes} \cdots \overline{\otimes} M_m, \quad (5.2)$$

for some  $m \geq 1$  and prime factors  $M_1, \dots, M_m \in \mathcal{C}_{\text{Rigid}}$ .

Suppose  $M$  has another prime factorization inside  $\mathcal{C}_{\text{Rigid}}$ , i.e.

$$M = N_1 \bar{\otimes} \cdots \bar{\otimes} N_n, \quad (5.3)$$

for some  $n \geq 1$ , and prime factors  $N_1, \dots, N_n \in \mathcal{C}_{\text{Rigid}}$ . Then  $m = n$  and there is a permutation  $\sigma$  of  $\{1, \dots, m\}$  such that  $M_i$  is stably isomorphic to  $N_{\sigma(i)}$  for  $1 \leq i \leq m$ .

### 5.1.3. UNIQUE FREE PRODUCT DECOMPOSITION

In [Oza06] Ozawa extended the results [OP04] for tensor products to the setting of free products. In particular, he showed for  $M = M_1 * \cdots * M_m$  a von Neumann algebraic free products of non-prime, non-amenable, semiexact finite factors  $M_1, \dots, M_m$  that if  $M = N_1 * \cdots * N_n$  is another free product decomposition into non-prime, non-amenable, semiexact finite factors  $N_1, \dots, N_n$ , then  $m = n$  and, up to permutation of the indices,  $M_i$  unitarily conjugates to  $N_i$  inside  $M$  for each  $1 < i < m$ . This can be seen as a von Neumann algebraic version of the Kurosh isomorphism theorem [Kur34], which states that any discrete group uniquely decomposes as a free product of freely-indecomposable groups. Versions of Ozawa's result were later shown for other classes of von Neumann algebras, see [Ash09], [IPP08], [Pet08]. In [HU16] these results were then extended by Houdayer and Ueda to a single, large class of von Neumann algebras. Other Kurosh type theorems have recently been obtained in [Dri23, Corollary 8.1], [DE24b, Corollary 1.8].

In the current chapter we obtain unique free product decomposition results for a new class of von Neumann algebras. First, we prove the following result which characterizes precisely when a graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  can decompose as tracial free product of  $\text{II}_1$ -factors.

**Theorem J** (Theorem 5.8.1). *Let  $\Gamma$  be a finite graph of size  $|\Gamma| \geq 2$ , and for each  $v \in \Gamma$  let  $M_v$  be  $\text{II}_1$ -factor with separable predual. Then the graph product  $M_\Gamma := *_{v \in \Gamma} (M_v, \tau_v)$  can decompose as a tracial free product  $M_\Gamma = (M_1, \tau_1) * (M_2, \tau_2)$  of  $\text{II}_1$ -factors  $M_1, M_2$  if and only if  $\Gamma$  is not connected.*

Using Theorem F and Theorem J we obtain unique free product decomposition for the class  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ .

**Theorem K** (Theorem 5.8.2). *Any von Neumann algebra  $M \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  can decompose as a tracial free product inside  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ , i.e.*

$$M = M_1 * \cdots * M_m, \quad (5.4)$$

for some  $m \geq 1$  and factors  $M_1, \dots, M_m \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  that can not decompose as any tracial free product of  $\text{II}_1$ -factors.

Suppose  $M$  can decompose as another tracial free product inside  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ , i.e.

$$M = N_1 * \cdots * N_n,$$

for some  $n \geq 1$  and factors  $N_1, \dots, N_n \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  that can not decompose as tracial free product of  $\text{II}_1$ -factors. Then  $m = n$  and there is a permutation  $\sigma$  of  $\{1, \dots, m\}$  such that  $N_i$  unitarily conjugate to  $M_{\sigma(i)}$  in  $M$ .

Let us remark that von Neumann algebras in the class  $\mathcal{C}_{\text{Complete}} \setminus \mathcal{C}_{\text{Vertex}}$  are examples of non-prime, non-amenable, semiexact, finite factors. Thus Ozawa's result in particular asserts a unique free product decomposition for free products of factors in  $\mathcal{C}_{\text{Complete}} \setminus \mathcal{C}_{\text{Vertex}}$ . The same result is also covered by Theorem K since any free product of factors in  $\mathcal{C}_{\text{Complete}} \setminus \mathcal{C}_{\text{Vertex}}$  lies in the class  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ . We observe that, in contrast to Ozawa's result, in Theorem K it is possible for the factors  $M_1, \dots, M_m$  to be prime. More generally, we remark that the result of Theorem K is not covered by the result from [HU16] (see Remark 5.8.4). Thus our examples of unique free product decompositions are again new.

#### 5.1.4. GRAPH RADIUS RIGIDITY

We are interested in the question whether from the graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  of  $\text{II}_1$ -factors  $M_v$  we can retrieve the radius of the graph  $\Gamma$ . To study this question we introduce the notion of the radius of a von Neumann algebra  $M$  (see Definition 5.9.3). As we show in the following theorem, we are in many cases able to estimate the radius of the von Neumann algebra  $M_\Gamma$  with the radius of the graph  $\Gamma$ .

**Theorem G** (Theorem 5.9.6 and Theorem 5.9.11). *Let  $\Gamma$  be a finite, non-complete graph. For  $v \in \Gamma$  let  $M_v$  be a  $\text{II}_1$ -factor and let  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  be the tracial graph product. Suppose one of the following holds true.*

1. *For all  $v \in \Gamma$  the vertex algebra  $M_v$  possesses strong (AO) and has separable predual.*
2. *For all  $v \in \Gamma$  we have  $M_v = \mathcal{L}(G_v)$  for some countable icc group  $G_v$ .*

*Then*

$$\text{Radius}(\Gamma) - 2 \leq \text{Radius}(M_\Gamma) \leq \max\{2, \text{Radius}(\Gamma)\}$$

The above result allows us to distinguish certain von Neumann algebras coming from graph products. In particular, for graph products  $R_{\Gamma_i} = *_{v \in \Gamma_i} (R_v, \tau_v)$  of hyperfinite  $\text{II}_1$ -factors  $R_v$ , we are able to show that  $R_{\Gamma_1} \neq R_{\Gamma_2}$  whenever  $2 \leq \text{Radius}(\Gamma_1) < \text{Radius}(\Gamma_2) - 2$  (see Remark 5.9.7).

We remark that when  $\Lambda_i$  for  $i = 1, 2$  are graph of size  $2 \leq |\Lambda_1| =: n < |\Lambda_2| =: m$  and with no edges, then  $R_{\Lambda_1} = \mathcal{L}(\mathbb{F}_n)$  and  $R_{\Lambda_2} = \mathcal{L}(\mathbb{F}_m)$  by [Dyk94]. In this case, it is very hard to distinguish  $R_{\Lambda_1}$  from  $R_{\Lambda_2}$  as this is precisely the free factor problem. Of course, Theorem G is of no use here since  $\text{Radius}(\Lambda_1) = \infty = \text{Radius}(\Lambda_2)$ .

#### 5.1.5. STRONG SOLIDITY

For a finite von Neumann algebra  $M$  the notion of strong solidity was introduced by Ozawa and Popa in [OP10a]. This property, which in particular implies solidity, asserts that for any diffuse amenable von Neumann subalgebra  $A \subseteq M$ , its normalizers  $\text{Nor}_M(A)$  generates a von Neumann algebra that is amenable. This property implies that for a non-amenable von Neumann algebra it does not have a Cartan subalgebra, and hence can not decompose as a crossed product in a natural way. In [OP10a], it was shown in that the free group factors  $\mathcal{L}(\mathbb{F}_r)$  are strong solidity. Nowadays, many examples of strongly solid von Neumann algebras are known, see e.g. [Cas22; CS13; DP23; Iso15a; PV14b]. Moreover, we remark that using the resolution of the Peterson-Thom conjecture

(see [Hay22], [BC24b], [BC24a]), it has been shown in [HJE24] that the free group factors even satisfy a strengthened version of strong solidity.

In this chapter we study strong solidity for graph products of von Neumann algebras. We fully characterize strong solidity for arbitrary graph products.

**Theorem D** (Theorem 5.6.7). *Let  $\Gamma$  be a finite graph, and for each  $v \in \Gamma$  let  $M_v (\neq \mathbb{C})$  be a von Neumann algebra with normal faithful trace  $\tau_v$ . Then  $M_\Gamma$  is strongly solid if and only if the following conditions are satisfied:*

1. *For each vertex  $v \in \Gamma$  the von Neumann algebra  $M_v$  is strongly solid;*
2. *For each subgraph  $\Lambda \subseteq \Gamma$  with  $M_\Lambda$  non-amenable, we have that  $M_{\text{Link}(\Lambda)}$  is not diffuse;*
3. *For each subgraph  $\Lambda \subseteq \Gamma$  with  $M_\Lambda$  non-amenable and diffuse, we have moreover that  $M_{\text{Link}(\Lambda)}$  is atomic.*

We remark that for a large class of vertex von Neumann algebras  $M_v$  it can be verified whether the conditions (1), (2) and (3) hold true for the graph products  $M_\Lambda$  and  $M_{\text{Link}(\Lambda)}$ . For group von Neumann algebras of right-angled Coxeter groups we obtain a simple characterization of strong solidity, see Theorem 5.6.13. More generally, Theorem D completes the characterization of strong solidity for right-angled Hecke von Neumann algebras (using Theorem 5.6.12 from [CKL21], [RS23]). Partial results in this direction had already been obtained in [Cas20] and in Chapter 4

### 5.1.6. OTHER RESULTS

The proofs of the stated theorems require several main results that are of independent interest, which we present here. Firstly, we give sufficient conditions for a graph product of unital  $C^*$ -algebras to be nuclear. This is a generalization of Ozawa's result for free products [Oza02] and is needed in the proof of Theorem F.

**Theorem L** (Theorem 5.3.4). *Let  $A_\Gamma = *_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  be the reduced  $C^*$ -algebraic graph product of nuclear, unital  $C^*$ -algebras  $A_v$  with GNS-faithful state  $\varphi_v$ . Let  $\mathcal{H}_v := L^2(A_v, \varphi_v)$  and let  $\pi_v : A_v \rightarrow \mathcal{B}(\mathcal{H}_v)$  be the GNS-representation. If for any  $v \in \Gamma$ ,  $\pi_v(A_v)$  contains the space of compact operators  $K(\mathcal{H}_v)$ , then  $A_\Gamma$  is nuclear.*

The following result is the graph product analogue of [HI17, Theorem 5.1] and [Oza06, Theorem 3.3], and is crucial in the proof of Theorem F for establishing the graph isomorphism.

**Theorem M** (Theorem 5.5.15). *Let  $(M_\Gamma, \tau) = *_{v \in \Gamma} (M_v, \tau_v)$  be the graph product of finite von Neumann algebras  $M_v$  that satisfy condition strong (AO) and have separable preduals. Let  $Q \subseteq M_\Gamma$  be a diffuse von Neumann subalgebra. At least one of the following holds:*

1. *The relative commutant  $Q' \cap M_\Gamma$  is amenable;*
2. *There exists  $\Gamma_0 \subseteq \Gamma$  such that  $Q \prec_{M_\Gamma} M_{\Gamma_0}$  and  $\text{Link}(\Gamma_0) \neq \emptyset$ .*

The following result concerning relative amenability is needed in the proof of the characterizations given in Theorem D, Theorem G and Theorem H.

**Theorem N** (Theorem 5.4.8). *Let  $\Gamma$  be a graph with subgraphs  $\Gamma_1, \Gamma_2 \subseteq \Gamma$ . For each  $v \in \Gamma$  let  $(M_v, \tau_v)$  be a von Neumann algebra with a normal faithful trace. Let  $P \subset M_\Gamma$  be a von Neumann subalgebra that is amenable relative to  $M_{\Gamma_i}$  inside  $M_\Gamma$  for  $i = 1, 2$ . Then  $P$  is amenable relative to  $M_{\Gamma_1 \cap \Gamma_2}$  inside  $M_\Gamma$ .*

### 5.1.7. CHAPTER OVERVIEW

In Section 5.2 we introduce the notion of rigid graphs and study some basic properties. Here, we also define graph products of graphs and precisely characterize when a graph product of graphs is rigid. In Section 5.3 we prove Theorem L which establishes sufficient conditions for a graph product to be nuclear. In Section 5.4 we prove some technical results concerning graph products. In particular, using calculation for iterated conditional expectations in graph products we prove Theorem N regarding relative-amenability, and prove some embedding results for graph products. In Section 5.5 we prove Theorem M which we then use to prove the major part of Theorem F. In Section 5.6 we prove Theorem D which characterizes strong solidity for graph products. In Section 5.7 we prove Theorem H which characterizes primeness in graph products. Moreover, we also complete the proof of Theorem F and we prove Theorem I which establishes UPF results for the class  $\mathcal{C}_{\text{Rigid}}$ . In Section 5.8 we prove Theorem J which characterizes free-indecomposability for graph products and we prove Theorem K which establishes unique free product decomposition results for the class  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ . In Section 5.9 we define the radius of a von Neumann algebra and prove Theorem G which for graph products provides good estimates on the radius of the graph. Last, in Section 5.10 we discuss some open questions and state a conjecture.

## 5.2. RIGID GRAPHS

In this section we introduce the notion of rigid graphs.

**Definition 5.2.1** (Rigid graphs). *We say that a simple graph  $\Gamma$  is rigid if for every  $v \in \Gamma$  we have  $\text{Link}_\Gamma(\text{Link}_\Gamma(v)) = \{v\}$ . When  $|\Gamma| \geq 2$  this means in particular for each  $v \in \Gamma$  that  $\text{Link}_\Gamma(v)$  is not empty.*

*Example 5.2.2.* We give some examples of rigid graphs which are easy to check:

1. By the convention  $\text{Link}_\Gamma(\emptyset) = \Gamma$  it follows that if  $|\Gamma| = 1$  then  $\Gamma$  is rigid.
2. Any complete graph is rigid.
3. For  $n \geq 2$  let  $\mathbb{Z}_n = \{1, \dots, n\}$  be the cyclic graph of length  $n$ , i.e.  $i, j$  share an edge if and only if  $|i - j| = 1$  or  $\{i, j\} = \{1, n\}$ . Then for  $n \geq 5$  the graph  $\mathbb{Z}_n$  is rigid. Note also that  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  are rigid, but  $\mathbb{Z}_4$  is not.
4. Consider  $\mathbb{Z}$  as the infinite cyclic graph, i.e.  $i, j$  share an edge in  $\mathbb{Z}$  if and only if  $|i - j| = 1$ . Then  $\mathbb{Z}$  is rigid.

We will now define the notion of graph products of graphs, and construct a large variety of rigid graphs.

**Definition 5.2.3.** Let  $\Gamma$  be a simple graph and for each  $v \in \Gamma$  let  $\Lambda_v$  be a simple graph. We denote  $\Lambda_\Gamma := *_{v \in \Gamma} \Lambda_v$  for the graph product of the graphs  $\{\Lambda_v\}_{v \in \Gamma}$ . This is defined as the graph with vertices set

$$\{(v, s) : v \in \Gamma, s \in \Lambda_v\}, \quad (5.5)$$

where vertices  $(v, s)$  and  $(w, t)$  share an edge in  $\Lambda_\Gamma$  if either  $v = w$  and  $t, s$  share an edge in  $\Lambda_v$  or  $v \neq w$  and  $v, w$  share an edge in  $\Gamma$ .

We observe that  $\Lambda_\Gamma$  contains the graphs  $\Lambda_v$  for  $v \in \Gamma$  as (mutually disjoint) subgraphs. Furthermore, we observe that if we take  $|\Lambda_v| = 1$  for each  $v \in \Gamma$  then  $\Lambda_\Gamma = \Gamma$ .

**Remark 5.2.4.** For a simple graph  $\Gamma$  and graphs  $\{\Lambda_v\}_{v \in \Gamma}$ , and groups  $G_w$  and von Neumann algebras  $(N_w, \varphi_w)$  with normal GNS-faithful state, with  $w \in \Lambda_v$ , we have

$$*_{w \in \Lambda_\Gamma} G_w = *_{v \in \Gamma} (*_{w \in \Lambda_v} G_w) \quad (5.6)$$

$$*_{w \in \Lambda_\Gamma} (N_w, \varphi_w) = *_{v \in \Gamma} (*_{w \in \Lambda_v} (N_w, \varphi_w)). \quad (5.7)$$

Indeed, this follows by the defining universal property of graph products of groups as well as its analogue for operator algebras that can be found in [CF17, Proposition 3.22].

**Lemma 5.2.5.** Let  $\Gamma$  be a simple graph and for each  $v \in \Gamma$  let  $\Lambda_v$  be a non-empty graph. Then the graph product graph  $\Lambda_\Gamma$  is rigid if and only if for each vertex  $v \in \Gamma$  the graph  $\Lambda_v$  is rigid and the vertex  $v$  satisfies at least one of the following conditions:

1.  $\text{Link}_\Gamma(\text{Link}_\Gamma(v)) = \{v\}$ ;
2.  $|\Lambda_v| \geq 2$ .

*Proof.* We may assume  $\Gamma$  is non-empty. First, suppose the conditions in the lemma are satisfied. We show  $\Lambda_\Gamma$  is rigid. Let  $(v, j) \in \Lambda_\Gamma$  for some  $v \in \Gamma$ ,  $j \in \Lambda_v$ . Let  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(\text{Link}_{\Lambda_\Gamma}(v, j))$ . We need to show that  $(z, k) = (v, j)$ .

Suppose first that  $|\Lambda_v| \geq 2$ . Then, as  $\Lambda_v$  is rigid, we have that  $\text{Link}_{\Lambda_v}(j)$  is non-empty. Let  $l \in \text{Link}_{\Lambda_v}(j)$ . Then  $(v, l) \in \text{Link}_{\Lambda_\Gamma}(v, j)$  and similarly  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(v, l)$ . If  $z \neq v$  then by the definition of the graph product graph this implies  $z \in \text{Link}_\Gamma(v)$ . But then, again by the definition of the graph product graph, we obtain  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(v, j)$ . However, as  $(z, k) \notin \text{Link}_{\Lambda_\Gamma}(z, k)$ , this contradicts that  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(\text{Link}_{\Lambda_\Gamma}(v, j))$ . We conclude that  $z = v$ . Hence, since  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(v, l)$  we obtain that  $k \in \text{Link}_{\Lambda_v}(l)$ . Since this holds true for all  $l \in \text{Link}_{\Lambda_v}(j)$ , we obtain that  $k \in \text{Link}_{\Lambda_v}(\text{Link}_{\Lambda_v}(j))$ , so that  $k = j$  by rigidity of  $\Lambda_v$ . Thus  $(z, k) = (v, j)$ .

Now suppose that  $|\Lambda_v| < 2$ , i.e.  $\Lambda_v = \{j\}$ , and just assume that  $\text{Link}_\Gamma(\text{Link}_\Gamma(v)) = \{v\}$ . If  $|\Gamma| = 1$  then  $\Lambda_\Gamma = \Lambda_v$  is rigid. Thus we can assume  $|\Gamma| \geq 2$ . Then  $\text{Link}_\Gamma(v)$  must be non-empty since  $\text{Link}(\emptyset) = \Gamma \neq \{v\}$ . Take  $w \in \text{Link}_\Gamma(v)$ . Then, as by assumption  $\Lambda_w$  is non-empty, we can pick  $i \in \Lambda_w$ . Now  $(w, i) \in \text{Link}_{\Lambda_\Gamma}(v, j)$ , by the definition of the graph  $\Lambda_\Gamma$ . Thus  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(w, i)$ . If  $w = z$  then  $z \in \text{Link}_\Gamma(v)$  and so also  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(v, j)$ . But as  $(z, k) \notin \text{Link}_{\Lambda_\Gamma}(z, k)$ , this contradicts that  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(\text{Link}_{\Lambda_\Gamma}(v, j))$ . Thus  $w \neq z$ , and therefore, as  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(w, i)$ , we obtain that  $z \in \text{Link}_\Gamma(w)$ . Therefore, since this holds for any  $w \in \text{Link}_\Gamma(v)$ , we obtain that  $z \in \text{Link}_\Gamma(\text{Link}_\Gamma(v)) = \{v\}$  and thus  $z = v$ . Thus as  $k \in \Lambda_z = \Lambda_v = \{j\}$ , we obtain  $(z, k) = (v, j)$ .

We now prove the reverse direction. First, suppose there is a vertex  $v \in \Gamma$  such that  $\Lambda_v$  is not rigid. Take  $j \in \Lambda_v$  such that  $\text{Link}_{\Lambda_v}(\text{Link}_{\Lambda_v}(j)) \neq \{j\}$  so that we can choose  $k \in \text{Link}_{\Lambda_v}(\text{Link}_{\Lambda_v}(j))$  with  $k \neq j$ . Now, one can check that  $(v, k) \in \text{Link}_{\Lambda_\Gamma}(\text{Link}_{\Lambda_\Gamma}(v, j))$ , hence  $\Lambda_\Gamma$  is not rigid.

Now suppose there is vertex  $v \in \Gamma$  such that  $\text{Link}_\Gamma(\text{Link}_\Gamma(v)) \neq \{v\}$  and  $|\Lambda_v| = 1$ , i.e.  $\Lambda_v = \{j\}$  for some  $j$ . Then  $\text{Link}_{\Lambda_\Gamma}(v, j) = \bigcup_{w \in \text{Link}_\Gamma(v)} \{(w, i) : i \in \Lambda_w\}$ . We can choose a  $z \in \text{Link}_\Gamma(\text{Link}_\Gamma(v))$  with  $z \neq v$  and let  $k \in \Lambda_z$ . Then we see that  $(z, k) \in \text{Link}_{\Lambda_\Gamma}(\text{Link}_{\Lambda_\Gamma}(v, j))$ , which shows  $\Lambda_\Gamma$  is not rigid.  $\square$

By the result of Lemma 5.2.5, it is possible to construct many different rigid graphs using the rigid graphs from Example 5.2.2.

*Remark 5.2.6.* Let  $\Gamma$  be a rigid graph. Then any connected component of  $\Gamma$  is rigid and any irreducible component of  $\Gamma$  is rigid. Indeed, if  $\Lambda_1, \dots, \Lambda_n$  are the irreducible components of  $\Gamma$  and we let  $\Pi = \{1, \dots, n\}$  be a complete graph, then  $\Gamma = *_{v, \Pi} \Lambda_v = \Lambda_\Pi$ . Hence, by Lemma 5.2.5 and rigidity of  $\Gamma$  we obtain that the graphs  $\Lambda_1, \dots, \Lambda_n$  are rigid. Similarly, if we let  $\Lambda'_1, \dots, \Lambda'_m$  be connected components of  $\Gamma$  and we let  $\Pi' = \{1, \dots, m\}$  be a graph with no edges, then  $\Gamma = *_{v, \Pi'} \Lambda'_v = \Lambda'_{\Pi'}$  so that by Lemma 5.2.5 and rigidity of  $\Gamma$  we obtain that  $\Lambda'_1, \dots, \Lambda'_m$  are rigid.

We now define the core of a graph.

**Definition 5.2.7** (Core of a graph). *Let  $\Gamma$  be a simple graph. We say that two vertices  $v, w \in \Gamma$  are core equivalent, with notation  $v \sim w$ , if  $\text{Star}(v) = \text{Star}(w)$ . Let  $\bar{v}$  be the core equivalence class of  $v \in \Gamma$ . We define the core of  $\Gamma$ , with notation  $\mathcal{C}\Gamma$ , as the graph whose vertices set is the set of all core equivalence classes of  $\Gamma$ . The edges set of  $\mathcal{C}\Gamma$  is defined by declaring that  $\bar{v}, \bar{w} \in \mathcal{C}\Gamma$  with  $\bar{v} \neq \bar{w}$  share an edge in  $\mathcal{C}\Gamma$  if and only if  $v, w$  share an edge in  $\Gamma$ .*

We remark that  $\mathcal{C}\mathcal{C}\Gamma = \mathcal{C}\Gamma$ , that is, the core of the core of a graph is equal to the core of the graph. In the following lemma we show that any graph can be written as a graph product over its core.

**Lemma 5.2.8.** *Let  $\Gamma$  be a simple graph. For  $\bar{v} \in \mathcal{C}\Gamma$  let  $\Lambda_{\bar{v}}$  be the complete graph of size  $|\Lambda_{\bar{v}}| = |\bar{v}|$ . Then  $\Gamma \simeq \Lambda_{\mathcal{C}\Gamma}$ . Furthermore, if  $\mathcal{C}\Gamma$  is rigid, then so is  $\Gamma$ .*

*Proof.* Indeed, as for  $\bar{v} \in \mathcal{C}\Gamma$  we have  $|\bar{v}| = |\Lambda_{\bar{v}}|$ , we can build a bijection  $\iota_{\bar{v}} : \bar{v} \rightarrow \Lambda_{\bar{v}}$ . We then define the bijection  $\iota : \Gamma \rightarrow \Lambda_{\mathcal{C}\Gamma}$  as  $\iota(v) = (\bar{v}, \iota_{\bar{v}}(v))$ . We show this is a graph isomorphism. Let  $v \neq w \in \Gamma$ . If  $v, w$  do not share an edge in  $\Gamma$  then  $\bar{v} \neq \bar{w}$  and  $\bar{v}, \bar{w}$  do not share an edge in  $\mathcal{C}\Gamma$ . Hence  $(\bar{v}, \iota_{\bar{v}}(v))$  and  $(\bar{w}, \iota_{\bar{w}}(w))$  do not share an edge in  $\Lambda_{\mathcal{C}\Gamma}$ . Now suppose  $v, w$  do share an edge in  $\Gamma$ . If  $\bar{v} = \bar{w}$  then since  $\Lambda_{\bar{v}} = \Lambda_{\bar{w}}$  is complete we obtain that  $(\bar{v}, \iota_{\bar{v}}(v))$  and  $(\bar{w}, \iota_{\bar{w}}(w))$  share an edge in  $\Lambda_{\mathcal{C}\Gamma}$ . On the other hand, if  $\bar{v} \neq \bar{w}$ , then  $\bar{v}, \bar{w}$  share an edge in  $\mathcal{C}\Gamma$  so that also  $(\bar{v}, \iota_{\bar{v}}(v))$  and  $(\bar{w}, \iota_{\bar{w}}(w))$  share an edge in  $\Lambda_{\mathcal{C}\Gamma}$ . This shows that  $\iota$  is an isomorphism and hence  $\Gamma \simeq \Lambda_{\mathcal{C}\Gamma}$ .

We prove the last statement. Suppose  $\mathcal{C}\Gamma$  is rigid. Since for each  $\bar{v} \in \mathcal{C}\Gamma$  the graph  $\Lambda_{\bar{v}}$  is rigid (since it is complete) and since by rigidity of  $\mathcal{C}\Gamma$  we have  $\text{Link}_{\mathcal{C}\Gamma}(\text{Link}_{\mathcal{C}\Gamma}(\bar{v})) = \bar{v}$ , we obtain by Lemma 5.2.5 that  $\Lambda_{\mathcal{C}\Gamma}$  is rigid. Thus  $\Gamma \simeq \Lambda_{\mathcal{C}\Gamma}$  is rigid.  $\square$

We make two remarks on Lemma 5.2.8

*Remark 5.2.9.* We remark that if a simple graph  $\Gamma$  is rigid, then its core is, in general, not rigid. Indeed, let  $\Pi = \{v, w\}$  denote the simple graph of size 2 with no edges and let  $\Lambda_v, \Lambda_w$  denote complete graphs of size  $|\Lambda_v|, |\Lambda_w| \geq 2$ . Then the graph  $\Gamma := \Lambda_\Pi$  is rigid by Lemma 5.2.5 but  $\mathcal{C}\Gamma = \Pi$  is not rigid.

*Remark 5.2.10.* If a graph  $\Gamma$  is in the class  $CC_1$  as described in [CDD22] then  $\Gamma$  is rigid. Indeed if  $\Gamma$  is  $CC_1$  then its core  $\mathcal{C}\Gamma$ , which is in fact also  $CC_1$ , is given by the graph of [CDD22, Eqn. (1.1)]. This graph is rigid as can be checked directly from the very definition of rigidity. We can then apply Lemma 5.2.8 to obtain that  $\Gamma$  is rigid. It thus follows that the graphs considered in the current chapter form a much richer class than [CDD22].

### 5.3. GRAPH PRODUCTS OF NUCLEAR $C^*$ -ALGEBRAS

The aim of this section is to give a sufficient condition for when the reduced graph product of nuclear  $C^*$ -algebras is nuclear again. Such a result cannot hold in full generality as it is clear from the fact that the free product of amenable discrete groups is non-amenable as soon as one group has at least 2 elements and the other group has at least 3 elements. Hence the stability result in this section requires particular conditions on the states with respect to which we take the graph product. Such a result was obtained by Ozawa in [Oza02] for amalgamated free products and we use the amalgamated free product decomposition of graph products (Theorem 2.4.1) to show that the same holds for graph products.

Let  $\Gamma$  be a finite simple graph. Let  $(A_v, \varphi_v)$  with  $v \in \Gamma$  be unital  $C^*$ -algebras  $A_v$ , GNS-faithful states  $\varphi_v$  and GNS-representation  $\pi_v$  of  $A_v$  on the Hilbert space  $\mathcal{H}_v = L^2(A_v, \varphi_v)$ .

For Hilbert  $C^*$ -modules we refer to [Lan95]. Consider the reduced graph product  $C^*$ -algebras  $(A_\Lambda, \varphi_\Lambda)$  for any  $\Lambda \subseteq \Gamma$  which is a subalgebra of  $(A_\Gamma, \varphi_\Gamma)$  with conditional expectation  $\mathbb{E}_\Lambda$ .

**Definition 5.3.1.** We construct a Hilbert  $C^*$ -module  $\mathcal{H}_{\mathbb{E}_\Lambda}$  as the completion of  $A_\Gamma$  with respect to the  $A_\Lambda$ -valued inner product

$$\langle a, b \rangle_{\mathbb{E}_\Lambda} = \mathbb{E}_\Lambda(b^* a)$$

and the corresponding Hilbert  $A_\Lambda$ -module norm  $\|a\| = \|\langle a, a \rangle_{\mathbb{E}_\Lambda}\|^{1/2}$ . Let  $\pi_{\mathbb{E}_\Lambda} : A_\Gamma \rightarrow B(\mathcal{H}_{\mathbb{E}_\Lambda})$  be the GNS-representation of  $A_\Gamma$  on the Hilbert  $C^*$ -module  $\mathcal{H}_{\mathbb{E}_\Lambda}$  by adjointable operators. Then  $\pi_{\mathbb{E}_\Lambda}$  is given by extending left multiplication

$$\pi_{\mathbb{E}_\Lambda}(x)a = xa, x \in A_\Gamma, a \in A_\Gamma \subseteq \mathcal{H}_{\mathbb{E}_\Lambda}$$

and we shall omit  $\pi_{\mathbb{E}_\Lambda}$  in the notation if the module action is clear.

**Definition 5.3.2.** An operator on the Hilbert  $A_\Lambda$ -module  $\mathcal{H}_{\mathbb{E}_\Lambda}$  is called finite rank if it is in the linear span of operators of the form

$$\theta_{\eta_2, \eta_1} : \xi \mapsto \eta_2 \langle \xi, \eta_1 \rangle_{\mathbb{E}_\Lambda}, \quad \eta_i \in \mathcal{H}_{\mathbb{E}_\Lambda}.$$

The closure of the space of all finite rank operators are defined as the space of compact operators  $K(\mathcal{H}_{\mathbb{E}_\Lambda})$ .

**Lemma 5.3.3.** *Suppose there exists  $v \in \Gamma$  such that  $\Gamma = \text{Star}(v)$ . If  $\pi_v(A_v)$  contains  $\mathcal{K}(\mathcal{H}_v)$  then  $\pi_{\mathbb{E}_{\text{Link}(v)}}(A_{\text{Star}(v)})$  contains  $\mathcal{K}(\mathcal{H}_{\mathbb{E}_{\text{Link}(v)}})$ .*

*Proof.* We have that  $A_{\text{Star}(v)} = A_v \otimes A_{\text{Link}(v)}$  where the tensor product is the minimal tensor product and under this correspondence we have

$$\langle a \otimes b, c \otimes d \rangle_{\mathbb{E}_{\text{Link}(v)}} = \varphi_v(c^* a) d^* b, \quad a, c \in A_v, b, d \in A_{\text{Link}(v)}.$$

We thus may identify  $\mathcal{H}_{\mathbb{E}_{\text{Link}(v)}}$  as the closure of the algebraic tensor product  $\mathcal{H}_v \otimes A_{\text{Link}(v)}$  with respect to the inner product  $\langle \xi \otimes b, \eta \otimes d \rangle = \langle \xi, \eta \rangle d^* b$ . Further, under this correspondence  $\pi_{\mathbb{E}_{\text{Link}(v)}} = \pi_v \otimes \pi_l$  where  $\pi_l(x)a = xa$ ,  $x, a \in A_{\text{Link}(v)}$  is the left multiplication. Let  $p_v$  be the projection of  $\mathcal{H}_v$  onto  $\mathbb{C}\xi_v$ . Then  $p_v \otimes 1$  equals the extension of  $\mathbb{E}_{\text{Link}(v)}$  as a bounded map on  $\mathcal{H}_{\mathbb{E}_{\text{Link}(v)}}$  identified with the closure of  $\mathcal{H}_v \otimes A_{\text{Link}(v)}$ . As by assumption  $p_v$  lies in  $\pi_v(A_v)$  it thus follows that  $p_v \otimes 1$  lies in  $\pi_{\mathbb{E}_{\text{Link}(v)}}(A_{\text{Star}(v)})$ . It thus follows that for  $a, c, x \in A_v, b, d, y \in A_{\text{Link}(v)}$  we have

$$\theta_{a \otimes b, c \otimes d}(x \otimes y) = \varphi_v(c^* x) a \otimes b d^* y = \pi_{\mathbb{E}_{\text{Link}(v)}}(a \otimes b)(p_v \otimes 1) \pi_{\mathbb{E}_{\text{Link}(v)}}(c^* \otimes d^*)(x \otimes y).$$

The right hand side is contained in  $\pi_{\mathbb{E}_{\text{Link}(v)}}(A_{\text{Star}(v)})$ . Hence  $\pi_{\mathbb{E}_{\text{Link}(v)}}(A_{\text{Star}(v)})$  contains a dense set of finite rank operators and hence must contain all compact operators.  $\square$

**Theorem 5.3.4.** *Let  $\Gamma$  be a simple graph. If for each  $v \in \Gamma$ ,  $A_v$  is nuclear and  $\pi_v(A_v)$  contains the compact operators  $\mathcal{K}(\mathcal{H}_v)$ , then  $A_\Gamma$  is nuclear.*

*Proof.* It suffices to prove the theorem for  $\Gamma$  a finite graph as inductive limits of inclusions of nuclear  $C^*$ -algebras are nuclear.

Our proof proceeds by induction to the number of vertices in  $\Gamma$ . So we assume that for any  $\Lambda \subsetneq \Gamma$  we have proved that  $A_\Lambda$  is nuclear. We shall prove that  $A_\Gamma$  is nuclear.

If  $\Gamma$  is complete then  $A_\Gamma$  is the minimal tensor product of  $A_v$ ,  $v \in \Gamma$  which is nuclear as each  $A_v$  is nuclear.

Assume  $\Gamma$  is not complete. Then we may take  $v \in \Gamma$  such that  $\text{Star}(v) \neq \Gamma$ . By Theorem 2.4.1 we obtain

$$A_\Gamma = A_{\text{Star}(v)} *_{A_{\text{Link}(v)}} A_{\Gamma \setminus \{v\}},$$

where all graph products and amalgamated free products are reduced. By induction  $A_{\text{Star}(v)}$  and  $A_{\Gamma \setminus \{v\}}$  are nuclear. Further the GNS-representation of  $A_{\text{Star}(v)}$  with respect to its conditional expectation onto  $A_{\text{Link}(v)}$  contains all compact operators by Lemma 5.3.3. Hence [Oza02, Theorem 1.1] concludes that  $A_\Gamma$  is nuclear.  $\square$

## 5.4. RELATIVE AMENABILITY, QUASI-NORMALIZERS AND EMBEDDINGS IN GRAPH PRODUCTS

In this section we establish the required machinery we need throughout the chapter. First in Section 5.4.1 we discuss how to calculate conditional expectations in graph products. This will be used in Section 5.4.2 to prove a result concerning relative amenability in graph products. The calculations from Section 5.4.1 will furthermore be used in Section 5.4.3 to keep control of certain quasi-normalizers in graph products. Last, in Section 5.4.4 we apply results from Section 5.4.3 to establish a unitary embedding of certain subalgebras in graph products.

### 5.4.1. CALCULATING CONDITIONAL EXPECTATIONS IN GRAPH PRODUCTS

For a simple graph  $\Gamma$ , a graph product  $(M_\Gamma, \varphi_\Gamma) = *_{v \in \Gamma} (M_v, \varphi_v)$  and subgraphs  $\Gamma_1, \Gamma_2 \subseteq \Gamma$ , we will in Proposition 5.4.3 calculate iterated conditional expectations of the form  $\mathbb{E}_{M_{\Gamma_2}}(a \mathbb{E}_{M_{\Gamma_1}}(x)b)$  for  $a, b, x \in M_\Gamma$  (here  $\mathbb{E}_{M_\Lambda}$  is the condition expectation that preserves the state  $\varphi_\Gamma$ ). In [BC23] we have done these calculations in the setting of Coxeter groups, i.e. the setting  $M_v = \mathcal{L}(\mathbb{Z}/2\mathbb{Z})$  for all  $v \in \Gamma$ . We will present here a generalisation of our calculations to the setting of general graph products. However, we note that such general calculations for graph products were already done in [Cha+24].

For our calculations we need the following combinatorial result concerning words in Coxeter groups. Here, for  $\mathbf{u} \in \mathcal{W}_\Gamma$  we denote  $\text{Link}(\mathbf{u}) = \text{Link}(\Lambda_{\mathbf{u}})$  where  $\Lambda_{\mathbf{u}}$  is the set of all letters that occur in  $\mathbf{u}$ . Alternatively,  $\text{Link}(\mathbf{u})$  can be described as the set of all  $w \in \Gamma \setminus \Lambda_{\mathbf{u}}$  such that  $w\mathbf{u} = \mathbf{u}w$ .

**Lemma 5.4.1.** *Let  $\Gamma$  be a graph and let  $\Gamma_1, \Gamma_2 \subseteq \Gamma$  be subgraphs. Let  $\mathbf{w} \in \mathcal{W}_{\Gamma_1}, \mathbf{u}, \mathbf{u}' \in \mathcal{W}_\Gamma$  be such that  $\mathbf{u}$  and  $\mathbf{u}'$  do not have a letter in  $\Gamma_1$  at the start and do not have a letter in  $\Gamma_2$  at the end. Then the following are equivalent:*

1.  $\mathbf{u}^{-1}\mathbf{w}\mathbf{u}' \in \mathcal{W}_{\Gamma_2}$ ;
2.  $\mathbf{u} = \mathbf{u}'$  and  $\mathbf{w} \in \mathcal{W}_{\Gamma_1 \cap \Gamma_2 \cap \text{Link}(\mathbf{u})}$ .

*Proof.* We show that (1)  $\implies$  (2); the other direction is trivial. Suppose that  $\mathbf{w}$  contains a letter  $b$  in  $\Gamma_1$  which is not contained in  $\Gamma_2$ , say that we write  $\mathbf{w} = \mathbf{w}_1 b \mathbf{w}_2$  as a reduced expression. We may assume that  $\mathbf{w}_1$  does not end on any letters commuting with  $b$  by moving those letters into  $\mathbf{w}_2$ . Then as  $\mathbf{u}'$  does not have letters from  $\Gamma_1$  at the start we see that  $\mathbf{w}\mathbf{u}'$  contains the letter  $b$ ; more precisely we may write a reduced expression  $\mathbf{w}\mathbf{u}' = \mathbf{w}_1 b \mathbf{w}_3 \mathbf{u}''$  where  $\mathbf{w}_3$  is a start of  $\mathbf{w}_2$  and  $\mathbf{u}''$  is a tail of  $\mathbf{u}'$ . Since  $\mathbf{u}^{-1}\mathbf{w}\mathbf{u}'$  is contained in  $\mathcal{W}_{\Gamma_2}$  the letter  $b$  cannot occur anymore in its reduced expression. We have  $\mathbf{u}^{-1}\mathbf{w}\mathbf{u}' = \mathbf{u}^{-1}\mathbf{w}_1 b \mathbf{w}_3 \mathbf{u}''$  (possibly non-reduced). Now if a letter at the end of  $\mathbf{u}^{-1}$  deletes the letter  $b$  then this would mean that  $\mathbf{u}$  has a letter in  $\Gamma_1$  up front (either  $b$  itself or a letter from  $\mathbf{w}_1$ ) which is not possible. We conclude that  $\mathbf{w} \in \mathcal{W}_{\Gamma_1 \cap \Gamma_2}$ .

Write  $\mathbf{u} = \mathbf{v}\mathbf{u}_1$  and  $\mathbf{u}' = \mathbf{v}\mathbf{u}'_1$  (both reduced) where  $\mathbf{u}_1, \mathbf{u}'_1 \in \mathcal{W}_\Gamma$  and where  $\mathbf{v} \in \mathcal{W}_\Gamma$  such that  $\mathbf{v}$  commutes with  $\mathbf{w}$ . Moreover we can assume that  $\mathbf{u}_1, \mathbf{u}'_1, \mathbf{v}$  are chosen such that  $|\mathbf{v}|$  is maximal over all possible choices. Now, suppose that  $\mathbf{u}'_1 \neq e$ . Let  $d$  be a letter at the end of  $\mathbf{u}'_1$ . Then  $d \notin \Gamma_2$  by assumption on  $\mathbf{u}'$  (as  $d$  is also at the end of  $\mathbf{u}'$ ). Now  $\mathbf{u}_1^{-1}\mathbf{w}\mathbf{u}'_1 = \mathbf{u}^{-1}\mathbf{w}\mathbf{u}' \in \mathcal{W}_{\Gamma_2}$ , which implies that  $d$  is deleted, i.e.  $\mathbf{u}_1^{-1}\mathbf{w}\mathbf{u}'_1$  is not reduced. Thus a letter  $c$  at the start of  $\mathbf{u}_1$  must delete a letter at the end of  $\mathbf{u}_1^{-1}\mathbf{w}$ . If  $c$  deletes a letter from  $\mathbf{w}$  then in particular  $c \in \Gamma_1 \cap \Gamma_2$  (as  $\mathbf{w} \in \mathcal{W}_{\Gamma_1 \cap \Gamma_2}$ ). However, as  $\mathbf{u}'$  does not start with letters from  $\Gamma_1$  this implies that  $|\mathbf{v}| \geq 1$ . Now, every letter of  $\mathbf{v}$  commutes with the letters from  $\mathbf{w}$  (by assumption on  $\mathbf{v}$ ). However, not every letter of  $\mathbf{v}$  commutes with  $c$ , since  $c$  is not at the start of  $\mathbf{u}'$ . From this we conclude that  $c$  is not a letter of  $\mathbf{w}$ , a contradiction. We conclude that  $c$  is not deleted by a letter from  $\mathbf{w}$ , and thus that  $c$  must commute with  $\mathbf{w}$ , and that  $c$  deletes a letter at the end of  $\mathbf{u}_1^{-1}$  i.e. a letter at the start of  $\mathbf{u}_1$ . Hence, we can write  $\mathbf{u}_1 = c\mathbf{u}_2$  and  $\mathbf{u}'_1 = c\mathbf{u}'_2$  (both reduced) for some  $\mathbf{u}_2, \mathbf{u}'_2 \in \mathcal{W}$ . But then  $\mathbf{u} = \mathbf{v}c\mathbf{u}_2$  and  $\mathbf{u}' = \mathbf{v}c\mathbf{u}'_2$  and we have that  $\mathbf{v}c$  commutes with  $\mathbf{w}$ . This contradicts the maximality of  $|\mathbf{v}|$ . We conclude that  $\mathbf{u}'_1 = e$ . Now as  $\mathbf{u}_1^{-1}\mathbf{w} = \mathbf{u}_1^{-1}\mathbf{w}\mathbf{u}'_1 = \mathbf{u}^{-1}\mathbf{w}\mathbf{u}'$  lies in  $\mathcal{W}_{\Gamma_2}$  by assumption

and as  $\mathbf{w} \in \mathcal{W}_{\Gamma_1 \cap \Gamma_2}$  we obtain that  $\mathbf{u}_1^{-1} \in \mathcal{W}_{\Gamma_2}$ . But  $\mathbf{u}_1$  does not end with a letter from  $\Gamma_2$  by assumption on  $\mathbf{u}$  (since letters at the end of  $\mathbf{u}_1$  are also at the end of  $\mathbf{u}$ ). This implies that  $\mathbf{u}_1 = e$ . This shows  $\mathbf{u} = \mathbf{v} = \mathbf{u}'$  and that  $\mathbf{u} (= \mathbf{v})$  commutes with  $\mathbf{w}$ .  $\square$

In the following lemma we calculate conditional expectations in graph products. We use the explicit graph product notation from the preliminaries. Furthermore, as in Chapter 3 we consider for  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \in \mathcal{S}_\Gamma$  the annihilation/diagonal/creation operator  $\lambda_{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)} : \mathbf{M}_\Gamma \rightarrow \mathbf{B}(\mathcal{H}_\Gamma)$ . Furthermore, we recall that for a von Neumann subalgebra  $Q \subseteq M$  that  $e_Q : L^2(M, \tau) \rightarrow L^2(Q, \tau)$  denotes the Jones projection.

**Lemma 5.4.2.** *Let  $\Gamma$  be a graph and let  $\Gamma_1, \Gamma_2 \subseteq \Gamma$  be subgraphs. For  $v \in \Gamma$  let  $(M_v, \varphi_v)_{v \in \Gamma}$  be von Neumann algebras with normal faithful states and let  $(M_\Gamma, \varphi) = *_{v \in \Gamma} (M_v, \varphi_v)$  be the von Neumann algebraic graph product. Let  $a_1 \in \mathring{M}_{\mathbf{u}^{-1}}$  and  $a_2 \in \mathring{M}_{\mathbf{w}}$  and  $a_3 \in \mathring{M}_{\mathbf{u}'}$  where  $\mathbf{w} \in \mathcal{W}_{\Gamma_1}$  and  $\mathbf{u}, \mathbf{u}' \in \mathcal{W}_\Gamma$  are such that  $\mathbf{u}, \mathbf{u}'$  do not start with letters from  $\Gamma_1$  and do not end with letters from  $\Gamma_2$ . Then*

$$\mathbb{E}_{M_{\Gamma_2}}(a_1 a_2 a_3) = \begin{cases} \varphi(a_1 a_3) a_2, & \mathbf{u}^{-1} \mathbf{w} \mathbf{u}' \in \mathcal{W}_{\Gamma_2}; \\ 0, & \text{else.} \end{cases} \quad (5.8)$$

Moreover, for  $x \in M_{\Gamma_1}$  we have  $\mathbb{E}_{M_{\Gamma_2}}(a_1 x a_3) = \varphi(a_1 a_3) \mathbb{E}_{M_{\Gamma_1 \cap \Gamma_2 \cap \text{Link}(\mathbf{u})}}(x)$ .

*Proof.* First assume that  $a_i = \lambda(c_i)$  with  $c_1 \in \mathring{M}_{\mathbf{u}^{-1}}$ ,  $c_2 \in \mathring{M}_{\mathbf{w}}$ ,  $c_3 \in \mathring{M}_{\mathbf{u}'}$ . We put  $\eta := a_2 a_3 \Omega \in \mathring{\mathcal{H}}_{\mathbf{w} \mathbf{u}'}$  (observe that  $\mathbf{w} \mathbf{u}'$  is reduced). Now

$$\mathbb{E}_{M_{\Gamma_2}}(a_1 a_2 a_3) \Omega = e_{M_{\Gamma_2}} a_1 a_2 a_3 e_{M_{\Gamma_2}} \Omega = e_{M_{\Gamma_2}} a_1 \eta.$$

Furthermore, by Lemma 3.1.7 we have that

$$a_1 = \lambda(c_1) = \sum_{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \in \mathcal{S}_{\mathbf{u}^{-1}}} \lambda_{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}(c_1).$$

Let  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \in \mathcal{S}_{\mathbf{u}^{-1}}$  and suppose  $e_{M_{\Gamma_2}} \lambda_{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}(c_1) \eta$  is non-zero. Then by Lemma 3.1.4 we have  $\lambda_{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}(c_1) \eta \in \mathring{\mathcal{H}}_{\mathbf{v}}$  where  $\mathbf{v} = \mathbf{u}_1 \mathbf{u}_3 \mathbf{w} \mathbf{u}'$  (possibly non-reduced) and moreover that  $\mathbf{v}$  starts with  $\mathbf{u}_1 \mathbf{u}_2$ . Moreover, since  $e_{M_{\Gamma_2}} \lambda_{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}(c_1) \eta$  is non-zero we have that  $\mathbf{v} \in \mathcal{W}_{\Gamma_2}$ . Then as  $\mathbf{v}$  starts with  $\mathbf{u}_1 \mathbf{u}_2$  and as  $\mathbf{u}_1 \mathbf{u}_2$  does not start with letters from  $\Gamma_2$  as this is true for  $\mathbf{u}^{-1}$ , we have that  $\mathbf{u}_1 \mathbf{u}_2 = e$ . As  $\mathbf{u}_1 \mathbf{u}_2$  is reduced by definition of  $\mathcal{S}_{\mathbf{u}^{-1}}$ , we obtain  $\mathbf{u}_1 = \mathbf{u}_2 = e$ . Hence, as  $\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3 = \mathbf{u}^{-1}$  we obtain  $\mathbf{u}_3 = \mathbf{u}^{-1}$ . We conclude that  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = (e, e, \mathbf{u}^{-1})$  and moreover that  $\mathbf{u}^{-1} \mathbf{w} \mathbf{u}' = \mathbf{u}_1 \mathbf{u}_3 \mathbf{w} \mathbf{u}' = \mathbf{v} \in \mathcal{W}_{\Gamma_2}$ .

Now suppose that  $\mathbf{u}^{-1} \mathbf{w} \mathbf{u}' \notin \mathcal{W}_{\Gamma_2}$ . Then by the above we obtain for all  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \in \mathcal{S}_{\mathbf{u}^{-1}}$  that  $e_{M_{\Gamma_2}} \lambda_{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}(c_1) \eta = 0$  and hence  $e_{M_{\Gamma_2}} a_1 \eta = 0$ . But then  $\mathbb{E}_{M_{\Gamma_2}}(a_1 a_2 a_3) \Omega = e_{M_{\Gamma_2}} a_1 \eta = 0$ . Hence  $\mathbb{E}_{M_{\Gamma_2}}(a_1 a_2 a_3) = 0$ .

Now, suppose  $\mathbf{u}^{-1} \mathbf{w} \mathbf{u}' \in \mathcal{W}_{\Gamma_2}$ . By Lemma 5.4.1 we obtain  $\mathbf{u} = \mathbf{u}'$  and  $\mathbf{w} \in \mathcal{W}_{\Gamma_1 \cap \Gamma_2 \cap \text{Link}(\mathbf{u})}$ . Therefore, since  $\mathbf{w}$  and  $\mathbf{u}$  commute and have no letters in common, we obtain that the

operator  $\lambda_{(e,e,\mathbf{u}^{-1})}(c_1)$  commutes with  $\lambda(c_2)$ . Now

$$\begin{aligned}\lambda_{(e,e,\mathbf{u}^{-1})}(c_1)\eta &= \lambda_{(e,e,\mathbf{u}^{-1})}(c_1)\lambda(c_2)\lambda_{(\mathbf{u},e,e)}(c_3)\Omega \\ &= \lambda(c_2)\lambda_{(e,e,\mathbf{u}^{-1})}(c_1)\lambda_{(\mathbf{u},e,e)}(c_3)\Omega \\ &= \lambda(c_2)\varphi(\lambda(c_1)\lambda(c_3))\Omega \\ &= \varphi(a_1 a_3) a_2 \Omega\end{aligned}$$

and hence

$$\begin{aligned}e_{M_{\Gamma_2}} a_1 \eta &= \sum_{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \in \mathcal{S}_{\mathbf{u}^{-1}}} e_{M_{\Gamma_2}} \lambda_{(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}(c_1) \eta \\ &= e_{M_{\Gamma_2}} \lambda_{(e,e,\mathbf{u}^{-1})}(c_1) \eta \\ &= e_{M_{\Gamma_2}} \varphi(a_1 a_3) a_2 \Omega \\ &= \varphi(a_1 a_3) a_2 \Omega.\end{aligned}$$

Thus  $\mathbb{E}_{M_{\Gamma_2}}(a_1 a_2 a_3) \Omega = \varphi(a_1 a_3) a_2 \Omega$  and thus  $\mathbb{E}_{M_{\Gamma_2}}(a_1 a_2 a_3) = \varphi(a_1 a_3) a_2$ . This shows (5.8) by density of  $\lambda(\dot{M}_{\mathbf{v}}) \subseteq \dot{M}_{\mathbf{v}}$  for  $\mathbf{v} \in \mathcal{W}_{\Gamma}$ .

To prove the second statement, let  $x \in M_{\Gamma_1}$  and write  $x = \sum_{\mathbf{w} \in \mathcal{W}_{\Gamma_1}} x_{\mathbf{w}}$  with  $x_{\mathbf{w}} \in \dot{M}_{\mathbf{w}}$ . Then by what we just proved, we get, where  $\chi$  is the indicator function,

$$\mathbb{E}_{M_{\Gamma_2}}(a_1 x a_3) = \sum_{\mathbf{w} \in \mathcal{W}_{\Gamma_1}} \mathbb{E}_{M_{\Gamma_2}}(a_1 x_{\mathbf{w}} a_3) = \varphi(a_1 a_3) \sum_{\mathbf{w} \in \mathcal{W}_{\Gamma_1}} x_{\mathbf{w}} \chi_{\mathcal{W}_{\Gamma_2}}(\mathbf{u}^{-1} \mathbf{w} \mathbf{u}'). \quad (5.9)$$

We now claim that

$$\varphi(a_1 a_3) \chi_{\mathcal{W}_{\Gamma_2}}(\mathbf{u}^{-1} \mathbf{w} \mathbf{u}') = \varphi(a_1 a_3) \chi_{\mathcal{W}_{\Gamma_1} \cap \Gamma_2 \cap \text{Link}(\mathbf{u})}(\mathbf{w}). \quad (5.10)$$

Indeed, when  $\mathbf{u} \neq \mathbf{u}'$  then  $\varphi(a_1 a_3) = 0$  so that both sides of (5.10) equal 0. Furthermore, in case  $\mathbf{u} = \mathbf{u}'$  we have by Lemma 5.4.1 that the conditions  $\mathbf{u}^{-1} \mathbf{w} \mathbf{u}' \in \mathcal{W}_{\Gamma_2}$  and  $\mathbf{w} \in \mathcal{W}_{\Gamma_1 \cap \Gamma_2 \cap \text{Link}(\mathbf{u})}$  are equivalent, which establishes (5.10). Now, combining (5.9) and (5.10) we obtain

$$\mathbb{E}_{M_{\Gamma_2}}(a_1 x a_3) = \varphi(a_1 a_3) \sum_{\mathbf{w} \in \mathcal{W}_{\Gamma_1 \cap \Gamma_2 \cap \text{Link}(\mathbf{u})}} x_{\mathbf{w}} = \varphi(a_1 a_3) \mathbb{E}_{M_{\Gamma_1 \cap \Gamma_2 \cap \text{Link}(\mathbf{u})}}(x).$$

This concludes the proof.  $\square$

**Proposition 5.4.3.** *Let  $\Gamma$  be a graph and let  $\Gamma_1, \Gamma_2$  be subgraphs. Let  $\mathbf{u}, \mathbf{v} \in \mathcal{W}_{\Gamma}$  and write  $\mathbf{u} = \mathbf{u}_l \mathbf{u}_c \mathbf{u}_r$  and  $\mathbf{v} = \mathbf{v}_l \mathbf{v}_c \mathbf{v}_r$  (both reduced) with  $\mathbf{u}_l, \mathbf{v}_l \in \mathcal{W}_{\Gamma_1}$ ,  $\mathbf{u}_r, \mathbf{v}_r \in \mathcal{W}_{\Gamma_2}$  and such that  $\mathbf{u}_c, \mathbf{v}_c$  do not start with letters from  $\Gamma_1$  and do not end with letters from  $\Gamma_2$ .*

*For  $v \in \Gamma$  let  $(M_v, \varphi_v)$  be a von Neumann algebra with a normal faithful state. Let  $a = a_l a_c a_r$  and  $b = b_l b_c b_r$  where  $a_l \in \dot{M}_{\mathbf{u}_l}$ ,  $a_c \in \dot{M}_{\mathbf{u}_c}$ ,  $a_r \in \dot{M}_{\mathbf{u}_r}$  and  $b_l \in \dot{M}_{\mathbf{v}_l}$ ,  $b_c \in \dot{M}_{\mathbf{v}_c}$ ,  $b_r \in \dot{M}_{\mathbf{v}_r}$ . Then for  $x \in M_{\Gamma}$  we have*

$$\mathbb{E}_{M_{\Gamma_2}}(a^* \mathbb{E}_{M_{\Gamma_1}}(x) b) = \varphi(a_c^* b_c) a_r^* \mathbb{E}_{M_{\Gamma_1 \cap \Gamma_2 \cap \text{Link}(\mathbf{u}_c)}}(a_l^* x b_l) b_r.$$

*Proof.* As  $a_l^*, b_l \in M_{\Gamma_1}$  and  $a_r^*, b_r \in M_{\Gamma_2}$  we have

$$\mathbb{E}_{M_{\Gamma_2}}(a^* \mathbb{E}_{M_{\Gamma_1}}(x)b) = a_r^* \mathbb{E}_{M_{\Gamma_2}}(a_c^* \mathbb{E}_{M_{\Gamma_1}}(a_l^* x b_l) b_c) b_r. \quad (5.11)$$

Now as  $\mathbb{E}_{M_{\Gamma_1}}(a_l^* x b_l) \in M_{\Gamma_1}$  and  $a_c^* \in \dot{M}_{u_c^{-1}}$  and  $b_c \in \dot{M}_{v_c}$  we have by Lemma 5.4.2 that

$$\begin{aligned} \mathbb{E}_{M_{\Gamma_2}}(a_c^* \mathbb{E}_{M_{\Gamma_1}}(a_l^* x b_l) b_c) &= \varphi(a_c^* b_c) \mathbb{E}_{M_{\Gamma_1 \cap \Gamma_2 \cap \text{Link}(u_c)}}(\mathbb{E}_{M_{\Gamma_1}}(a_l^* x b_l)) \\ &= \varphi(a_c^* b_c) \mathbb{E}_{M_{\Gamma_1 \cap \Gamma_2 \cap \text{Link}(u_c)}}(a_l^* x b_l). \end{aligned} \quad (5.12)$$

This proves the statement by combining (5.11) and (5.12).  $\square$

#### 5.4.2. RELATIVE AMENABILITY IN GRAPH PRODUCTS

Given a finite von Neumann algebra  $M$  with normal faithful tracial state  $\tau$  and let  $Q \subseteq M$  be a von Neumann subalgebra. We recall that  $\langle M, e_Q \rangle$  is the Jones extension which is the von Neumann subalgebra of  $B(L^2(M, \tau))$  generated by  $M \cup \{e_Q\}$  equipped with the tracial weight  $\text{Tr} : \langle M, e_Q \rangle^+ \rightarrow [0, \infty]$  whose linear extension satisfies  $\text{Tr}(x e_Q y) = \tau(xy)$ . Let

$$T_Q : L^1(\langle M, e_Q \rangle, \text{Tr}) \rightarrow L^1(M, \tau),$$

be the unique map defined through  $\tau(T_Q(y)x) = \text{Tr}(yx)$  for all  $y \in L^1(\langle M, e_Q \rangle, \text{Tr})$ ,  $x \in M$ . Then  $T_Q$  is the predual of the inclusion map  $M \subset \langle M, e_Q \rangle$  and thus is contractive and preserves positivity. For the following definition of relative amenability we refer to [PV14a, Definition 2.2, Proposition 2.4].

**Definition 5.4.4.** Let  $(M, \tau)$  be a tracial von Neumann algebra and let  $P \subseteq {}_{1_P} M {}_{1_P}$ ,  $Q \subseteq M$  be von Neumann subalgebras. We say that  $P$  is amenable relative to  $Q$  inside  $M$  if there exists a  $P$ -central positive functional on  ${}_{1_P} \langle M, e_Q \rangle {}_{1_P}$  that restricts to the trace  $\tau$  on  ${}_{1_P} M {}_{1_P}$ .

*Remark 5.4.5.* Assume the inclusion  $P \subseteq M$  is not unital. Let  $p = 1_M - 1_P$ . Set  $\tilde{P} = P \oplus \mathbb{C}p$  which is a unital subalgebra of  $M$ . We claim:  $P$  is amenable relative to  $Q$  inside  $M$  if and only if  $\tilde{P}$  is amenable relative to  $Q$  inside  $M$ . Indeed, for the if part, choose a  $\tilde{P}$ -central positive functional  $\tilde{\Omega}$  on  $\langle M, e_Q \rangle$  that restricts to  $\tau$  on  $M$ . Set  $\Omega$  to be the restriction of  $\tilde{\Omega}$  to  ${}_{1_P} \langle M, e_Q \rangle {}_{1_P}$  which then clearly witnesses relative amenability of  $P$ . For the only if part, let  $\Omega$  be a  $P$ -central positive functional on  ${}_{1_P} \langle M, e_Q \rangle {}_{1_P}$  that restricts to  $\tau$  on  ${}_{1_P} M {}_{1_P}$  then we set  $\tilde{\Omega}(x) = \Omega(pxp) + \tilde{\tau}((1_M - p)x(1_M - p))$  for any positive functional  $\tilde{\tau}$  extending  $\tau$  from  $(1_M - p)M(1_M - p)$  to  $(1_M - p)\langle M, e_Q \rangle(1_M - p)$ . Clearly  $\tilde{\Omega}$  witnesses the relative amenability of  $\tilde{P}$ .

Using the calculations of conditional expectations we will prove Theorem 5.4.8 which asserts that when a von Neumann algebra  $P \subseteq M_\Gamma$  is amenable relative to  $M_{\Gamma_i}$  inside  $M_\Gamma$  for some subgraphs  $\Gamma_i \subseteq \Gamma$  for  $i = 1, 2$ , then  $P$  is also amenable relative to  $M_{\Gamma_1 \cap \Gamma_2}$  inside  $M_\Gamma$ . We need the following proposition.

**Proposition 5.4.6** (Proposition 2.4 of [PV14a]). Assume  $P, Q \subseteq M$  are von Neumann subalgebras. Then  $P$  is amenable relative to  $Q$  inside  $M$  if and only if there exists a net  $(\xi_j)_j \in L^2(\langle M, e_Q \rangle, \text{Tr})^+$  such that:

1.  $0 \leq T_Q(\xi_j^2) \leq 1_M$  for all  $j$  and  $\lim_j \|T_Q(\xi_j^2) - 1_M\|_1 = 0$ ;
2. For all  $y \in P$  we have  $\lim_j \|y\xi_j - \xi_j y\|_2 = 0$ .

Before we prove Theorem 5.4.8, we will in Remark 5.4.7 do some bimodule computations for the Connes tensor product. Let  $M, Q, N$  be tracial von Neumann algebras and let  ${}_M\mathcal{H}_Q$  and  ${}_Q\mathcal{K}_N$  be bimodules. Recall that a vector  $\xi \in \mathcal{H}$  is called *right  $Q$ -bounded* if there exists  $C > 0$  s.t.  $\|\xi y\| \leq C\|y\|$  for all  $y \in Q$ . For a right  $Q$ -bounded vector  $\xi \in \mathcal{H}$  we define  $L(\xi) \in B(L^2(Q, \tau), \mathcal{H})$  as  $L(\xi)x = \xi x$  where  $x \in Q$ . Then, for right  $Q$ -bounded vectors  $\xi, \eta \in \mathcal{H}$  we have that  $L(\eta)^*L(\xi) \in Q$ . We denote by  $\mathcal{H}_0 \subseteq \mathcal{H}$  the subspace of all right  $Q$ -bounded vectors. We equip the algebraic tensor product  $\mathcal{H}_0 \otimes \mathcal{K}$  with the (possibly degenerate) inner product

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{\mathcal{H}_0 \otimes_Q \mathcal{K}} := \langle L(\xi_2)^*L(\xi_1)\eta_1, \eta_2 \rangle_{\mathcal{K}}. \quad (5.13)$$

The Connes tensor product  $\mathcal{H} \otimes_Q \mathcal{K}$  is the Hilbert space obtained from  $\mathcal{H}_0 \otimes_{\text{alg}} \mathcal{K}$  by quotienting out the degenerate part and taking a completion. The Hilbert space  $\mathcal{H} \otimes_Q \mathcal{K}$  is a  $M$ - $N$  bimodule with the action

$$x \cdot (\xi \otimes_Q \eta) \cdot y = (x\xi) \otimes_Q (\eta y).$$

*Remark 5.4.7.* We calculate the operator  $L(\xi_2)^*L(\xi_1)$  for certain bimodules and vectors  $\xi_1, \xi_2 \in \mathcal{H}_0$ . Let  $(M, \tau)$  be a tracial von Neumann algebra and let  $P, Q \subseteq M$  be von Neumann subalgebras with  $Q$  unital. Consider the bimodule  ${}_P L^2(M, \tau)_Q$ . Let  $x, y \in M$ . Then  $x, y$  are right  $Q$ -bounded and thus  $L(x), L(y) : L^2(Q, \tau) \rightarrow L^2(M, \tau)$  are well-defined. We calculate  $L(x)^*L(y)$ . For  $q_1, q_2 \in L^2(Q, \tau)$  we have

$$\langle L(x)^*L(y)q_1, q_2 \rangle = \langle yq_1, xq_2 \rangle = \tau(q_2^*x^*yq_1) = \tau(q_2^*\mathbb{E}_Q(x^*y)q_1) = \langle \mathbb{E}_Q(x^*y)q_1, q_2 \rangle. \quad (5.14)$$

Thus  $L(x)^*L(y) = \mathbb{E}_Q(x^*y)$ .

Let  $R \subseteq M$  be a unital von Neumann subalgebra and let  $N = \langle M, e_R \rangle$ , where  $e_R$  denotes the Jones projection of the inclusion  $R \subseteq M$ . We consider the bimodule  ${}_P L^2(N, \text{Tr})_Q$ . For  $x, x', y, y' \in M$  we have that  $xe_R y$  and  $x'e_R y'$  are right  $Q$ -bounded vectors as they are elements in  $N$ . We calculate  $L(xe_R y)^*L(x'e_R y')$ . For  $q_1, q_2 \in Q$  we have,

$$\begin{aligned} \langle L(xe_R y)^*L(x'e_R y')q_1, q_2 \rangle &= \langle x'e_R y'q_1, xe_R yq_2 \rangle \\ &= \text{Tr}(q_2^*y^*e_R x^*x'e_R y'q_1) \\ &= \text{Tr}(q_2^*y^*\mathbb{E}_R(x^*x')e_R y'q_1) \\ &= \tau(q_2^*y^*\mathbb{E}_R(x^*x')y'q_1) \\ &= \tau(\mathbb{E}_Q(q_2^*y^*\mathbb{E}_R(x^*x')y'q_1)) \\ &= \tau(q_2^*\mathbb{E}_Q(y^*\mathbb{E}_R(x^*x')y')q_1) \\ &= \langle \mathbb{E}_Q(y^*\mathbb{E}_R(x^*x')y')q_1, q_2 \rangle. \end{aligned}$$

Thus we obtain  $L(xe_R y)^*L(x'e_R y') = \mathbb{E}_Q(y^*\mathbb{E}_R(x^*x')y')$ .

We present Theorem 5.4.8 which we proved in [BC23, Theorem 3.7] in the setting of right-angled Coxeter groups (i.e.  $M_v = \mathcal{L}(\mathbb{Z}/2\mathbb{Z})$  for  $v \in \Gamma$ ) and later in [BCC24, Theorem 5.3] in the general setting. The proof of the theorem follows [PV14a, Proposition 2.7] but in our case the subalgebras are not regular. We furthermore remark that the bimodule computations we do in the proof of Theorem 5.4.8 are also related to those done in [Cha+24, Section 5].

**Theorem 5.4.8.** *Let  $\Gamma$  be a graph and let  $\Gamma_1, \Gamma_2 \subseteq \Gamma$  be subgraphs. For  $v \in \Gamma$  let  $(M_v, \tau_v)$  be a von Neumann algebra with a normal faithful trace. Let  $P \subset 1_P M_\Gamma 1_P$  be a von Neumann subalgebra that is amenable relative to  $M_{\Gamma_i}$  inside  $M_\Gamma$  for  $i = 1, 2$ . Then  $P$  is amenable relative to  $M_{\Gamma_1 \cap \Gamma_2}$  inside  $M_\Gamma$ .*

*Proof.* By Remark 5.4.5 we may assume without loss of generality that the inclusion  $P \subseteq M_\Gamma$  is unital and use the characterisation of relative amenability given by Proposition 5.4.6. Put  $Q_i := M_{\Gamma_i}$  for  $i = 1, 2$ . As before, let  $T_i = T_{Q_i} : L^1(\langle M_\Gamma, e_{Q_i} \rangle) \rightarrow L^1(M_\Gamma)$  be the contraction determined by  $\tau(T_i(S)x) = \text{Tr}_i(Sx)$  for  $S \in L^1(\langle M_\Gamma, e_{Q_i} \rangle)$  and  $x \in M_\Gamma$ . Since  $P$  is amenable relative to  $Q_i$ , Proposition 5.4.6 implies the existence of nets  $(\mu_j^i)_j$  in  $L^2(\langle M_\Gamma, e_{Q_i} \rangle)^+$  satisfying

$$0 \leq T_i((\mu_j^i)^2) \leq 1, \quad \|T_i((\mu_j^i)^2) - 1\|_1 \rightarrow 0, \quad \|y\mu_j^i - \mu_j^i y\|_2 \rightarrow 0, \text{ for all } y \in P, \quad (5.15)$$

where the limits are taken over  $j$ . Consider the  $M_\Gamma$ - $M_\Gamma$  bimodule

$$\mathcal{H} = L^2(\langle M_\Gamma, e_{Q_1} \rangle) \otimes_{M_\Gamma} L^2(\langle M_\Gamma, e_{Q_2} \rangle).$$

*Claim:* As in [PV14a] we claim that tensor products  $\mu_j := \mu_{j_1}^1 \otimes \mu_{j_2}^2 \in \mathcal{H}$  for certain  $j = (j_1, j_2)$  can be combined into a net such that

$$\|y\mu_j - \mu_j y\| \rightarrow 0, \quad |\langle x\mu_j, \mu_j \rangle - \tau(x)| \rightarrow 0,$$

for all  $y \in P$ ,  $x \in M_\Gamma$ , where the limit is taken over  $j$ . Let us now prove this claim in the next paragraphs which repeats the argument used in [PV14a, Proposition 2.4].

*Proof of the claim.* Take  $\mathcal{F} \subseteq P$ ,  $\mathcal{G} \subseteq M_\Gamma$  finite and let  $\varepsilon > 0$ . Set  $\mathcal{G}^1 := \mathcal{G}$  and fix  $j_1$  such that  $\|y\mu_{j_1}^1 - \mu_{j_1}^1 y\|_2 \leq \varepsilon$  for all  $y \in \mathcal{F}$  and  $|\langle x\mu_{j_1}^1, \mu_{j_1}^1 \rangle - \tau(x)| \leq \varepsilon$  for all  $x \in \mathcal{G}^1$ . As  $0 \leq T_1((\mu_{j_1}^1)^2) \leq 1$  and as  $T_1$  preserves positivity, it follows that for  $x \in M_\Gamma$  the element  $T_1(\mu_{j_1}^1 x \mu_{j_1}^1) \in L^1(M_\Gamma, \tau)$  is bounded in the uniform norm and thus belongs to  $M$ . Set  $\mathcal{G}^2 := T_1(\mu_{j_1}^1 \mathcal{G}^1 \mu_{j_1}^1) \subseteq M_\Gamma$ , which is finite. We may proceed from  $\mathcal{F}$  and  $\mathcal{G}^2$  to find  $j_2$  such that  $\|y\mu_{j_2}^2 - \mu_{j_2}^2 y\|_2 \leq \varepsilon$  for all  $y \in \mathcal{F}$  and  $|\langle x\mu_{j_2}^2, \mu_{j_2}^2 \rangle - \tau(x)| \leq \varepsilon$  for all  $x \in \mathcal{G}^2$ . Put  $j = (j_1, j_2)$  and set  $\mu_j = \mu_{j_1}^1 \otimes \mu_{j_2}^2$ . For  $y \in \mathcal{F}$  it follows by the triangle inequality that

$$\|y\mu - \mu y\| \leq \|(y\mu_{j_1}^1 - \mu_{j_1}^1 y) \otimes_{M_\Gamma} \mu_{j_2}^2\| + \|\mu_{j_1}^1 \otimes_{M_\Gamma} (y\mu_{j_2}^2 - \mu_{j_2}^2 y)\| \leq 2\varepsilon.$$

Now, by construction of the sets  $\mathcal{G}^i$  and the vectors  $\mu_{j_i}^i$  we see that for  $x \in \mathcal{G}$  that

$$\begin{aligned} |\langle x\mu, \mu \rangle - \tau(x)| &\leq |\langle x\mu, \mu \rangle - \langle x\mu_{j_2}^2, \mu_{j_2}^2 \rangle| + |\langle x\mu_{j_2}^2, \mu_{j_2}^2 \rangle - \tau(x)| \\ &\leq |\langle T_1(\mu_{j_1}^1 x \mu_{j_1}^1) \mu_{j_2}^2, \mu_{j_2}^2 \rangle - \langle x\mu_{j_2}^2, \mu_{j_2}^2 \rangle| + |\langle x\mu_{j_2}^2, \mu_{j_2}^2 \rangle - \tau(x)| \\ &\leq 2\varepsilon \end{aligned}$$

Taking  $j = j(\mathcal{F}, \mathcal{G})$  with increasing sets  $\mathcal{F}$  and  $\mathcal{G}$  as before gives a net of vectors  $\mu_j \in H$  with the property that

$$\|y\mu_j - \mu_j y\| \rightarrow 0, \quad |\langle x\mu_j, \mu_j \rangle - \tau(x)| \rightarrow 0$$

for all  $y \in P$  and  $x \in M_\Gamma$ . This proves the claim.

*Remainder of the proof.* The net  $(\mu_j)_j$  in particular shows that the bimodule  ${}_M L^2(M)_P$  is weakly contained in  ${}_M \mathcal{H}_P$ . Denote

$\mathcal{V} = \{\mathbf{v} \in \mathcal{W}_\Gamma : \mathbf{v} \text{ does not start with letters from } \Gamma_1 \text{ and does not end with letters from } \Gamma_2\}$ ,

and define the subspace  $\mathcal{H}_0 \subseteq \mathcal{H}$  as

$$\mathcal{H}_0 = \text{Span}\{xe_{Q_1}y \otimes_{M_\Gamma} e_{Q_2}z : \mathbf{v} \in \mathcal{V}, x, z \in M_\Gamma, y \in \dot{M}_{\mathbf{v}}\},$$

which is dense in  $\mathcal{H}$ . Indeed, clearly the span of all operators of the form  $xe_{Q_1}y_1 \otimes_{M_\Gamma} y_2 e_{Q_2}z$  for some reduced operators  $x \in M_{\mathbf{u}}$ ,  $z \in M_{\mathbf{w}}$  and  $y_i \in M_{\mathbf{v}_i}$  for some  $\mathbf{u}, \mathbf{w}, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{W}_\Gamma$  is dense in  $\mathcal{H}$ . However, it can be seen that all these operators are also contained in  $\mathcal{H}_0$ , which shows that  $\mathcal{H}_0 \subseteq \mathcal{H}$  is dense.

We define a bimodule  $\mathcal{K} := L^2(M_\Gamma) \otimes_{Q_0} L^2(M_\Gamma)$  and a map  $U : \mathcal{H}_0 \rightarrow L^2(M_\Gamma) \otimes_{Q_0} \mathcal{K}$  as

$$xe_{Q_1}y \otimes_{M_\Gamma} e_{Q_2}z \mapsto x \otimes_{Q_0} y \otimes_{Q_0} z \quad x, z \in M_\Gamma, \mathbf{v} \in \mathcal{V}, y \in \dot{M}_{\mathbf{v}}.$$

Fix  $x, x', z, z' \in M_\Gamma$ . For  $\mathbf{v}, \mathbf{v}' \in \mathcal{V}$  and  $y \in \dot{M}_{\mathbf{v}}, y' \in \dot{M}_{\mathbf{v}'}$  we have by Proposition 5.4.3 that

$$\mathbb{E}_{Q_0}(y^* \mathbb{E}_{Q_0}(x^* x') y') = \tau(y^* y') \mathbb{E}_{Q_0 \cap \text{Link}(\mathbf{v})}(x^* x') = \mathbb{E}_{Q_1}(y^* \mathbb{E}_{Q_2}(x^* x') y'). \quad (5.16)$$

Hence for general  $y, y' \in M_\Gamma$  we have  $\mathbb{E}_{Q_0}(y^* \mathbb{E}_{Q_0}(x^* x') y') = \mathbb{E}_{Q_1}(y^* \mathbb{E}_{Q_2}(x^* x') y')$  by linearity. Combining this with the bimodule computations from Remark 5.4.7 we obtain

$$\begin{aligned} \langle x' \otimes_{Q_0} y' \otimes_{Q_0} z', x \otimes_{Q_0} y \otimes_{Q_0} z \rangle_{L^2(M) \otimes_{Q_0} \mathcal{K}} &= \langle \mathbb{E}_{Q_0}(x^* x') y' \otimes_{Q_0} z', y \otimes_{Q_0} z \rangle_{\mathcal{K}} \\ &= \langle \mathbb{E}_{Q_0}(y^* \mathbb{E}_{Q_0}(x^* x') y') z', z \rangle \\ &= \langle \mathbb{E}_{Q_2}(y^* \mathbb{E}_{Q_1}(x^* x') y') z', z \rangle \\ &= \langle T_{Q_2}(\mathbb{E}_{Q_2}(y^* \mathbb{E}_{Q_1}(x^* x') y') e_{Q_2} z'), z \rangle \\ &= \langle T_{Q_2}(e_{Q_2} y^* \mathbb{E}_{Q_1}(x^* x') y' e_{Q_2} z'), z \rangle \\ &= \langle y^* \mathbb{E}_{Q_1}(x^* x') y' e_{Q_2} z', e_{Q_2} T_{Q_2}^*(z) \rangle \\ &= \langle \mathbb{E}_{M_\Gamma}(y^* \mathbb{E}_{Q_1}(x^* x') y') e_{Q_2} z', e_{Q_2} z \rangle \\ &= \langle x' e_{Q_1} y' \otimes_{M_\Gamma} e_{Q_2} z', x e_{Q_1} y \otimes_{M_\Gamma} e_{Q_2} z \rangle_{\mathcal{H}} \end{aligned}$$

Thus  $U$  extends to an isometry  $\mathcal{H} \rightarrow L^2(M_\Gamma) \otimes_{Q_0} \mathcal{K}$ , which clearly is  $M_\Gamma$ - $M_\Gamma$ -bimodular. This shows that  ${}_M L^2(M_\Gamma)_P$  is weakly contained in  ${}_M L^2(M_\Gamma) \otimes_{Q_0} \mathcal{K}_P$ , which by [PV14a, Proposition 2.4 (3)] means that  $P$  is amenable relative to  $Q_0 = Q_1 \cap Q_2$  inside  $M_\Gamma$ .  $\square$

### 5.4.3. EMBEDDINGS OF QUASI-NORMALIZERS IN GRAPH PRODUCTS

We prove Proposition 5.4.13 and Proposition 5.4.14 concerning embeddings in graph products. To prove Proposition 5.4.13 we need some auxiliary lemmas. First, we state Lemma 5.4.9 which was essentially proven in [Vae08, Remark 3.8]. The result is surely known but for completeness we give the proof.

**Lemma 5.4.9.** *Let  $A, B_1, \dots, B_n, Q \subseteq M$  be von Neumann subalgebras with  $B_i \subseteq Q$ . Assume that  $A <_M Q$  but  $A \not\prec_M B_i$  for any  $i = 1, \dots, n$ . Then there exist projections  $p \in A, q \in Q$ , a non-zero partial isometry  $v \in qMp$  and a normal  $*$ -homomorphism  $\theta : pAp \rightarrow qQq$  such that  $\theta(x)v = vx, x \in pAp$  and such that  $\theta(pAp) \not\prec_Q B_i$  for any  $i = 1, \dots, n$ . Moreover, it may be assumed that  $p$  is majorized by the support of  $\mathbb{E}_A(v^*v)$ .*

*Proof.* Let  $p \in A, q \in Q$  and  $\theta : pAp \rightarrow qQq$  be a normal  $*$ -homomorphism such that there is a partial isometry  $v \in qMp$  such that  $\theta(x)v = vx$  for all  $x \in pAp$ . We first prove that without loss of generality we can assume that  $p$  is majorized by the support of  $\mathbb{E}_A(v^*v)$ .

Let  $z$  be the support of  $\mathbb{E}_A(v^*v)$ . As  $p\mathbb{E}_A(v^*v)p = \mathbb{E}_A(pv^*vp) = \mathbb{E}_A(v^*v)$  it follows that  $z \in pAp$ . Further for  $x \in pAp$  we have  $x\mathbb{E}_A(v^*v) = \mathbb{E}_A(xv^*v) = \mathbb{E}_A(v^*\theta(x)v) = \mathbb{E}_A(v^*vx) = \mathbb{E}_A(v^*v)x$  so that  $z \in (pAp)'$ . We conclude  $z \in (pAp)' \cap pAp$ . Now let  $p' := pz \in A$ , let  $\theta' : p'Ap' \rightarrow qQq$  be the restriction of  $\theta$  to  $p'Ap'$  and let  $v' := vz \in qMp'$ . Then for  $x \in p'Ap'$  we have  $\theta'(x)v' = \theta(x)vz = vxz = vzx$ . We claim further that  $v'$  is non-zero. Indeed,  $v' = vz = 0$  iff  $zv^*vz = 0$  iff  $0 = \mathbb{E}_A(zv^*vz) = z\mathbb{E}_A(v^*v)z$ . But as  $v$  is non-zero  $\mathbb{E}_A(v^*v)$  is non-zero and hence  $z\mathbb{E}_A(v^*v)z \neq 0$  by construction of  $z$ . We conclude that  $v' \neq 0$ . In all the tuple  $(\theta, p', q, v')$  witnesses that  $A <_M Q$  and the support of  $\mathbb{E}_A((v')^*v')$  majorizes  $p'$ .

For the remainder of the proof one just follows [BC23, Lemma 2.1] which does not affect the assumption that  $p$  is majorized by the support of  $\mathbb{E}_A(v^*v)$ .  $\square$

The following lemma is similar to [DHI19, Remark 2.3].

**Lemma 5.4.10.** *Let  $(M, \tau)$  be a tracial von Neumann algebra and let  $A, B_1, \dots, B_n$  be (possibly non-unital) von Neumann subalgebras of  $M$ . Assume  $A \not\prec_M B_k$  for  $k = 1, \dots, n$ . Then there is a single net  $(u_i)_i$  of unitaries in  $A$  such that for  $1 \leq k \leq n$  and  $a, b \in 1_A M 1_{B_k}$  we have  $\|\mathbb{E}_{B_k}(a^* u_i b)\|_2 \rightarrow 0$  as  $i \rightarrow \infty$*

*Proof.* Put

$$\tilde{B} = \bigoplus_{1 \leq k \leq n} B_k, \quad \tilde{M} = \bigoplus_{1 \leq k \leq n} M. \quad (5.17)$$

Let  $\pi : M \rightarrow \tilde{M}$  be the (normal) diagonal embedding  $\pi(x) = \bigoplus_{k=1}^n x$ . Suppose  $\pi(A) <_{\tilde{M}} \tilde{B}$ . Then there are projections  $p \in (A), q \in \tilde{B}$ , a normal  $*$ -homomorphism  $\theta : p\pi(A)p \rightarrow q\tilde{B}q$  and a non-zero partial isometry  $v \in q\tilde{M}p$  s.t.  $\theta(x)v = vx$  for  $x \in p\pi(A)p$ . For  $k = 1, \dots, n$  let  $\pi_k : \tilde{M} \rightarrow M$  be the coordinate projections. Denote  $p_k := \pi_k(p) \in A, q_k := \pi_k(q) \in B_k$  and  $v_k := \pi_k(v) \in \pi_k(q\tilde{M}p) = q_k M p_k$ . Define a normal  $*$ -homomorphism  $\theta_k : p_k A p_k \rightarrow q_k B_k q_k$  as  $\theta_k(x) = \pi_k(\theta(\pi(x)))$ . Then  $\theta_k(x)v_k = \pi_k(\theta(\pi(x))v) = \pi_k(v\pi(x)) = v_k \pi_k(\pi(x)) = v_k x$ . Since  $0 \neq v = \bigoplus_{k=1}^n v_k$  there is  $1 \leq k_0 \leq n$  s.t.  $v_{k_0} \neq 0$ . This then shows that  $A <_{M_{k_0}} B_{k_0}$  which is a contradiction. We conclude that  $\pi(A) \not\prec_{\tilde{M}} \tilde{B}$ .

Thus, there is a net of unitaries  $(\tilde{u}_i)_i$  in  $\pi(A)$  such that for  $a', b' \in 1_{\pi(A)}\widetilde{M}1_{\widetilde{B}}$  we have  $\|\mathbb{E}_{\widetilde{B}}(a'^* \tilde{u}_i b')\|_2 \rightarrow 0$  as  $i \rightarrow \infty$ . Let  $u_i \in A$  be the unitary s.t.  $\pi(u_i) = \tilde{u}_i$ . Fix  $1 \leq k \leq n$  and let  $a, b \in 1_A M 1_{B_k}$ . We can choose  $\tilde{a}, \tilde{b} \in 1_{\pi(A)}\widetilde{M}1_{\widetilde{B}}$  s.t.  $\pi_k(\tilde{a}) = a$  and  $\pi_k(\tilde{b}) = b$ . We have

$$\|\mathbb{E}_{B_k}(a^* u_i b)\|_2 = \|\mathbb{E}_{\pi_k(\widetilde{B})}(\pi_k(\tilde{a}^* \tilde{u}_i \tilde{b}))\|_2 \leq \|\mathbb{E}_{\widetilde{B}}(\tilde{a}^* \tilde{u}_i \tilde{b})\|_2 \rightarrow 0 \quad \text{as } i \rightarrow \infty. \quad (5.18)$$

This shows the net  $(u_i)_i$  satisfies the stated property.  $\square$

In order to have control over quasi-normalizers we need the following lemma. The lemma is stated in [Vae07, Lemma D.3] for sequences, but holds equally well for nets.

*Remark 5.4.11.* Consider an inclusion  $B \subseteq 1_B M 1_B$  of finite von Neumann algebras with conditional expectation  $\mathbb{E}_B : 1_B M 1_B \rightarrow B$ . We extend it to  $\mathbb{E}_B : M \rightarrow B$  by setting  $\mathbb{E}_B(x) = \mathbb{E}_B(1_B x 1_B)$ . Fix a normal faithful tracial state  $\tau$  on  $M$ . If  $p \in B$  is a non-zero projection then  $p$  is in the multiplicative domain of  $\mathbb{E}_B$  and so  $\mathbb{E}_B : pMp \rightarrow pBp$  is a conditional expectation. If  $\mathbb{E}_B$  preserves  $\tau$  then it also preserves the normal faithful tracial state  $\tau(p)^{-1}\tau$  on  $pMp$ .

**Lemma 5.4.12** (Lemma D.3 in [Vae07]). *Let  $(M, \tau)$  be a finite von Neumann algebra with normal faithful trace  $\tau$  and let  $B \subseteq 1_B M 1_B$  and  $A \subseteq 1_A B 1_A$  be von Neumann subalgebras. Suppose there is a net of unitaries  $(u_i)_i$  in  $A$  such that for all  $a, b \in M$  with  $\mathbb{E}_B(a) = \mathbb{E}_B(b) = 0$  we have*

$$\|\mathbb{E}_B(au_i b)\|_2 \rightarrow 0 \text{ as } i \rightarrow \infty. \quad (5.19)$$

*Then if  $n \geq 1$ ,  $x_0, x_1, \dots, x_n \in M$  satisfy  $Ax_0 \subseteq \sum_{k=1}^n x_k B$  then we have that  $1_A x_0 1_A \in B$ .*

*Proof.* We put  $B_0 = 1_A B 1_A$  and  $M_0 = 1_A M 1_A$  so that  $A \subseteq B_0 \subseteq M_0$  are unital inclusions. We observe  $B_0 = B \cap M_0$ . Now let  $a, b \in M_0$  be such that  $\mathbb{E}_{B_0}(a) = \mathbb{E}_{B_0}(b) = 0$ . Then by Remark 5.4.11 with  $p = 1_A$  we find  $\mathbb{E}_B(a) = \mathbb{E}_{B_0}(a) = 0$  and similarly  $\mathbb{E}_B(b) = 0$ . Thus by assumption  $\|\mathbb{E}_B(au_i b)\|_2 \rightarrow 0$  as  $i \rightarrow \infty$ . Hence, since  $B_0 = 1_A B 1_A$  we obtain  $\|\mathbb{E}_{B_0}(au_i b)\|_2 \rightarrow 0$  as  $i \rightarrow \infty$ . Choose a central projection  $z \in B \cap B'$  such that there exists  $m \geq 1$  and partial isometries  $v_i \in B$  for  $1 \leq i \leq m$  with  $v_i v_i^* \leq 1_A$  and  $\sum_{i=1}^m v_i^* v_i = z$ . Now let  $n \geq 1$ ,  $x_0, x_1, \dots, x_n \in M$  be such that  $Ax_0 \subseteq \sum_{k=1}^n x_k B$ . Then

$$A(1_A x_0 z 1_A) = (Ax_0 z) 1_A \subseteq \sum_{k=1}^n x_k B z 1_A = \sum_{k=1}^n \sum_{i=1}^m x_k (v_i^* 1_A v_i) B 1_A \subseteq \sum_{k=1}^n \sum_{i=1}^m x_k v_i^* B_0.$$

Multiplying both sides from the left with  $1_A$  gives  $A(1_A x_0 z 1_A) \subseteq \sum_{k=1}^n \sum_{i=1}^m 1_A x_k v_i^* B_0$  where  $1_A x_k v_i^* \in 1_A M 1_A$ . By the existence of the net  $(u_i)_i$  this implies, by applying [Vae07, Lemma D.3] to the inclusions  $A \subseteq B_0 \subseteq M_0$ , that  $1_A x_0 z 1_A \in B$ . As we may let  $z$  approximate  $1_B$  in the strong topology we find that  $1_A x_0 1_A \in B$ .  $\square$

We are now able to show the following result. The second statement in the proposition should be compared to [Ioa15, Lemma 9.4]. While the inclusion  $M_\Lambda \subseteq M_\Gamma$  is generally not mixing, we still have enough control over the (quasi)-normalizer of subalgebras. The proof of Proposition 5.4.13(1) uses Lemma 5.4.10, Lemma 5.4.12 and the results from Section 5.4.1 for calculating conditional expectations in graph products. The proof of Proposition 5.4.13(2) uses (1) and Lemma 5.4.9.

**Proposition 5.4.13.** *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $(M_v, \tau_v)$  be a finite von Neumann algebra with normal faithful trace  $\tau_v$ . Let  $\Lambda \subseteq \Gamma$  be a subgraph, and  $\{\Lambda_j\}_{j \in \mathcal{J}}$  be a non-empty, finite collection of subgraphs of  $\Gamma$ . Define*

$$\Lambda_{\text{emb}} := \Lambda \cup \bigcap_{j \in \mathcal{J}} \bigcup_{v \in \Lambda \setminus \Lambda_j} \text{Link}_{\Gamma}(v). \quad (5.20)$$

*Let  $A \subseteq 1_A M_{\Gamma} 1_A$  be a von Neumann subalgebra.*

*1. If  $A \subseteq 1_A M_{\Lambda} 1_A$  and  $A \not\prec_{M_{\Gamma}} M_{\Lambda_j}$  for all  $j \in \mathcal{J}$  then the following properties hold true:*

- (a) There is a net  $(u_i)_i$  of unitaries in  $A$  such that for all  $a, b \in 1_A M_{\Gamma} 1_A$  with  $\mathbb{E}_{M_{\Lambda_{\text{emb}}}}(a) = \mathbb{E}_{M_{\Lambda_{\text{emb}}}}(b) = 0$  we have  $\|\mathbb{E}_{M_{\Lambda_{\text{emb}}}}(au_i b)\|_2 \rightarrow 0$ ;*
- (b)  $1_A \text{qNor}_{M_{\Gamma}}(A)'' 1_A \subseteq M_{\Lambda_{\text{emb}}}$ ;*
- (c) For any unitary  $u \in M_{\Gamma}$  satisfying  $u^* A u \subseteq M_{\Lambda_{\text{emb}}}$  we have  $1_A u 1_A \in M_{\Lambda_{\text{emb}}}$ .*

*2. Denote  $P = \text{Nor}_{M_{\Gamma}}(A)''$  and let  $r \in P \cap P'$  be a projection. If  $r A \prec_{M_{\Gamma}} M_{\Lambda}$  and  $r A \not\prec_{M_{\Gamma}} M_{\Lambda_j}$  for  $j \in \mathcal{J}$  then  $r P \prec_{M_{\Gamma}} M_{\Lambda_{\text{emb}}}$ .*

*We remark that if  $\{\Lambda_j\}_{j \in \mathcal{J}}$  enumerates all strict subgraphs of  $\Lambda$  then  $\Lambda_{\text{emb}} = \Lambda \cup \text{Link}_{\Gamma}(\Lambda)$ .*

*Proof. (1)*

By Lemma 5.4.10 we can build a net of unitaries  $(u_i)_i$  in  $A$  such that for any  $a, b \in M_{\Gamma}$  and any  $j \in \mathcal{J}$  we have  $\|\mathbb{E}_{M_{\Lambda_j}}(au_i b)\|_2 \rightarrow 0$  when  $i \rightarrow \infty$ . We show the net  $(u_i)_i$  satisfies the properties of (1a). Let  $b \in \dot{M}_{\mathbf{v}}$  and  $c \in \dot{M}_{\mathbf{w}}$  for some  $\mathbf{v}, \mathbf{w} \in \mathcal{W}_{\Gamma} \setminus \mathcal{W}_{\Lambda_{\text{emb}}}$ . Write  $\mathbf{v} = \mathbf{v}_l \mathbf{v}_c \mathbf{v}_r$  and  $\mathbf{w} = \mathbf{w}_l \mathbf{w}_c \mathbf{w}_r$  with  $\mathbf{v}_l, \mathbf{w}_l \in \mathcal{W}_{\Lambda_{\text{emb}}}$ ,  $\mathbf{v}_r, \mathbf{w}_r \in \mathcal{W}_{\Lambda}$  and such that  $\mathbf{v}_c$  and  $\mathbf{w}_c$  do not start with letter from  $\Lambda_{\text{emb}}$  nor do they end with letters from  $\Lambda$ . Now write  $b = b_l b_c b_r$  and  $c = c_l c_c c_r$  with  $b_l \in \dot{M}_{\mathbf{v}_l}, c_l \in \dot{M}_{\mathbf{w}_l}, b_c \in \dot{M}_{\mathbf{v}_c}, c_c \in \dot{M}_{\mathbf{w}_c}$  and  $b_r \in \dot{M}_{\mathbf{v}_r}, c_r \in \dot{M}_{\mathbf{w}_r}$ . Then as  $\mathbf{v} \notin \mathcal{W}_{\Lambda_{\text{emb}}}$  and  $\mathbf{v}_l \in \mathcal{W}_{\Lambda_{\text{emb}}}$  and  $\mathbf{v}_r \in \mathcal{W}_{\Lambda} \subseteq \mathcal{W}_{\Lambda_{\text{emb}}}$ , we have  $\mathbf{v}_c \notin \mathcal{W}_{\Lambda_{\text{emb}}}$  and hence there is a letter  $v$  of  $\mathbf{v}_c$  such that  $v \notin \Lambda_{\text{emb}}$ . Thus, there is an index  $j \in \mathcal{J}$  such that  $v \notin \bigcup_{w \in \Lambda \setminus \Lambda_j} \text{Link}_{\Gamma}(w)$ . Hence  $\text{Link}(v) \subseteq \Gamma \setminus (\Lambda \setminus \Lambda_j) = \Lambda_j \cup (\Gamma \setminus \Lambda)$  and thus  $\Lambda \cap \text{Link}(\mathbf{v}_c) \subseteq \Lambda_j$ . Using Proposition 5.4.3 we get,

$$\begin{aligned} \|\mathbb{E}_{M_{\Lambda_{\text{emb}}}}(b^* u_i c)\|_2 &= \|\mathbb{E}_{M_{\Lambda_{\text{emb}}}}(b^* \mathbb{E}_{M_{\Lambda}}(u_i) c)\|_2 \\ &= \|\tau(b_c^* c_c) b_r^* \mathbb{E}_{M_{\Lambda \cap \text{Link}(\mathbf{v}_c)}}(b_l^* u_i c_l) c_r\|_2 \\ &= \|\tau(b_c^* c_c) b_r^* \mathbb{E}_{M_{\Lambda \cap \text{Link}(\mathbf{v}_c)}}(\mathbb{E}_{M_{\Lambda_j}}(b_l^* u_i c_l)) c_r\|_2 \\ &\leq \|b_c\|_2 \|c_c\|_2 \|b_r\| \|c_r\| \|\mathbb{E}_{M_{\Lambda_j}}(b_l^* u_i c_l)\|_2. \end{aligned}$$

We see that this expression converges to 0 when  $i \rightarrow \infty$ . Thus, more generally, for  $b, c \in M_{\Gamma}$  with  $\mathbb{E}_{M_{\Lambda_{\text{emb}}}}(b) = \mathbb{E}_{M_{\Lambda_{\text{emb}}}}(c) = 0$ , we obtain  $\|\mathbb{E}_{M_{\Lambda_{\text{emb}}}}(b^* u_i c)\|_2 \rightarrow 0$  when  $i \rightarrow \infty$ , which shows (1a).

(1b) Observe that if  $x \in \text{qNor}_{M_{\Gamma}}(A)$  then for some  $n \geq 1$  and  $x_1, \dots, x_n \in M_{\Gamma}$  we have  $Ax \subseteq \sum_{k=1}^n x_k A \subseteq \sum_{k=1}^n x_k M_{\Lambda_{\text{emb}}}$ . Therefore by the existence of the net  $(u_i)_i$  shown by (1a) and by Lemma 5.4.12, we have that  $1_A x 1_A \in M_{\Lambda_{\text{emb}}}$ . This shows  $1_A \text{qNor}_{M_{\Gamma}}(A) 1_A \subseteq M_{\Lambda_{\text{emb}}}$  and thus proves (1b).

(1c) Let  $u \in M_\Gamma$  be a unitary for which  $u^*Au \subseteq M_{\Lambda_{\text{emb}}}$ . Then  $Au \subseteq uM_{\Lambda_{\text{emb}}}$  so again by the existence of the net  $(u_i)_i$  shown by (1a) and by Lemma 5.4.12, we obtain that  $1_A u 1_A \in M_{\Lambda_{\text{emb}}}$ .

(2) By replacing  $\{\Lambda_j\}_{j \in \mathcal{J}}$  with  $\{\Lambda_j \cap \Lambda\}_{j \in \mathcal{J}}$  we may assume that  $\Lambda_j \subseteq \Lambda$  for  $j \in \mathcal{J}$ . We observe that  $r$  is central in  $A$ , which we will use a number of times in the proof. By Lemma 5.4.9 the assumptions imply that there exist projections  $p \in rA, q \in M_\Lambda$  a non-zero partial isometry  $v \in qM_\Gamma p$  and a normal  $*$ -homomorphism  $\theta : pAp \rightarrow qM_\Lambda q$  such that  $\theta(x)v = vx$  for all  $x \in pAp$  and such that moreover  $\theta(pAp) \not\prec_{M_\Lambda} M_{\Lambda_j}$  for  $j \in \mathcal{J}$ . From (1) we see that  $\theta(p)q\text{Nor}_{M_\Gamma}(\theta(pAp))\theta(p) \subseteq M_{\Lambda_{\text{emb}}}$ .

Now take  $u \in \text{Nor}_{M_\Gamma}(A)$ . We follow the proof of [Pop06c, Lemma 3.5] or [Ioa15, Lemma 9.4]. Take  $z \in A$  a central projection such that  $z = \sum_{j=1}^n v_j v_j^*$  with  $v_j \in A$  partial isometries such that  $v_j^* v_j \leq p$ . Then

$$pzupz(pAp) \subseteq pzuA = pzuA = pAzu \subseteq \sum_{j=1}^n (pAv_j)v_j^*u \subseteq \sum_{j=1}^n (pAp)v_j^*u,$$

and similarly  $(pAp)pzupz \subseteq \sum_{j=1}^n uv_j(pAp)$ . We conclude that  $pzupz \in q\text{Nor}_{pMp}(pAp)$ .

Now if  $x \in q\text{Nor}_{pMp}(pAp)$  then by direct verification we see that we have that  $vxv^* \in \theta(p)q\text{Nor}_{qM_\Gamma q}(\theta(pAp))\theta(p)$ . It follows that  $vpzupzv^*$ , with  $u \in \text{Nor}_{M_\Gamma}(A)$  as before, is contained in  $\theta(p)q\text{Nor}_{qM_\Gamma q}(\theta(pAp))\theta(p)$  which was contained in  $M_{\Lambda_{\text{emb}}}$ . We may take the projections  $z$  to approximate the central support of  $p$  and therefore  $vu v^* = vpv p v^* \in M_{\Lambda_{\text{emb}}}$ . Hence  $v\text{Nor}_{M_\Gamma}(A)''v^* \subseteq M_{\Lambda_{\text{emb}}}$ . Set  $p_1 = v^*v \in pA'p$ . Note that  $p_1 \leq p \leq r$ . As both  $A$  and  $A'$  are contained in  $\text{Nor}_{M_\Gamma}(A)''$  we find that  $p_1 \in \text{Nor}_{M_\Gamma}(A)''$  (as  $p \in A$ ). So we have the  $*$ -homomorphism  $\rho : p_1\text{Nor}_{M_\Gamma}(A)''p_1 = p_1r\text{Nor}_{M_\Gamma}(A)''p_1 \rightarrow M_{\Lambda_{\text{emb}}} : x \mapsto vxv^*$  with  $v \in qM_\Gamma p_1$  and clearly  $\rho(x)v = vx$ . We conclude that  $r\text{Nor}_{M_\Gamma}(A)'' \prec_{M_\Gamma} M_{\Lambda_{\text{emb}}}$ .  $\square$

We prove the following result concerning embeddings in graph products.

**Proposition 5.4.14.** *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$ , and let  $(M_v, \tau_v)$  be a tracial von Neumann algebra. Fix  $v \in \Gamma$  and let  $N \subseteq M_v$  be diffuse. If  $N \prec_{M_\Gamma} M_\Lambda$  for some subgraph  $\Lambda \subseteq \Gamma$ , then  $v \in \Lambda$ . In particular if  $\Lambda = \{w\}$ , a singleton set, then  $v = w$ .*

*Proof.* Let  $\Lambda \subseteq \Gamma$  be a subgraph with  $v \notin \Lambda$ . We show that  $N \not\prec_{M_\Gamma} M_\Lambda$ . Since  $N$  is diffuse, we can choose a net  $(u_k)_k$  of unitaries in  $N$  such that  $\tau(u_k) = 0$  and  $u_k \rightarrow 0$   $\sigma$ -weakly. Since  $\lambda(M_\Gamma)$  is a dense subspace of  $M_\Gamma$ , it is sufficient to show for any reduced operators  $x = x_1 x_2 \dots x_m, y = y_1 y_2 \dots y_n$ , s.t.  $x_i \in \dot{M}_{v_i}, y_i \in \dot{M}_{w_i}$ , we have  $\|\mathbb{E}_{M_\Lambda}(xu_k y)\|_2 \rightarrow 0$ . Indeed, writing  $x = x'a, y = by'$ , where  $a, b \in M_v$  and where  $x'$  respectively  $y'$  is a reduced operator without letter  $v$  at the end respectively start. Then

$$xu_k y = x'au_k by' = x'\tau(au_k b)y' + x'(au_k b - \tau(au_k b))y'.$$

On the one hand,  $\mathbb{E}_{M_\Lambda}(x'\tau(au_k b)y') = \tau(au_k b)\mathbb{E}_{M_\Lambda}(x'y') = \langle u_k b, a^* \rangle \mathbb{E}_{M_\Lambda}(x'y') \rightarrow 0$ . On the other hand, we write  $x' = x''d, y' = ey''$ , where  $d, e \in M_{\text{Link}(v)}$  and where  $x''$  respec-

tively  $y''$  has no letter from  $\text{Star}(v)$  at the end respectively at the start. Then we have

$$\begin{aligned} x'(au_k b - \tau(au_k b))y' &= x''d(au_k b - \tau(au_k b))ey'' \\ &= x''de(au_k b - \tau(au_k b))y'' \\ &= \sum_i x''f_i(au_k b - \tau(au_k b))y'', \end{aligned}$$

where we write  $de = \sum_i f_i$  and  $f_i \in M_{\text{Link}(v)}$  reduced. Since  $x''f_i(au_k b - \tau(au_k b))y''$  is reduced and  $v \notin \Lambda$  we obtain that  $\mathbb{E}_{M_\Lambda}(x''f_i(au_k b - \tau(au_k b))y'') = 0$ . Thus  $\|\mathbb{E}_{M_\Lambda}(xu_k y)\|_2 \rightarrow 0$ , which completes the proof.  $\square$

*Remark 5.4.15.* We remark in particular for any graph  $\Gamma$ ,  $\text{II}_1$ -factors  $\{M_v\}_{v \in \Gamma}$  and a finite subgraph  $\Lambda \subseteq \Gamma$  that  $\text{qNor}_{M_\Gamma}(M_\Lambda)'' = \text{Nor}_{M_\Gamma}(M_\Lambda)'' = M_{\Lambda \cup \text{Link}(\Lambda)}$ . Indeed, clearly  $M_\Lambda, M_{\text{Link}(\Lambda)} \subseteq \text{Nor}_{M_\Gamma}(M_\Lambda)''$  (as  $M_{\text{Link}(\Lambda)} = M'_\Lambda \cap M_\Gamma$ ) so that  $M_{\Lambda \cup \text{Link}(\Lambda)} \subseteq \text{Nor}_{M_\Gamma}(M_\Lambda)'' \subseteq \text{qNor}_{M_\Gamma}(M_\Lambda)''$ . Furthermore, by Proposition 5.4.14 we have  $M_\Lambda \not\prec_{M_\Gamma} M_{\tilde{\Lambda}}$  for any strict subgraph  $\tilde{\Lambda} \subsetneq \Lambda$  so that by Proposition 5.4.13 we obtain  $\text{qNor}_{M_\Gamma}(M_\Lambda)'' \subseteq M_{\Lambda \cup \text{Link}(\Lambda)}$ .

#### 5.4.4. UNITARY CONJUGACY IN GRAPH PRODUCTS

We prove Theorem 5.4.16 which gives sufficient conditions for a subalgebra  $Q \subseteq M_\Gamma$  to unitarily embed in a subalgebra  $M_{\Lambda_{\text{emb}}}$ . This can be seen as a generalization of [Oza06, Theorem 3.3] where a unitary embedding is proven for free products. The proof of Theorem 5.4.16 combines (the second half of) the proof of [Oza06, Theorem 3.3] with results of Section 5.4.3 concerning embeddings in graph products.

**Theorem 5.4.16.** *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $(M_v, \tau_v)$  be a  $\text{II}_1$ -factor with normal faithful trace  $\tau_v$ . Let  $Q \subseteq M_\Gamma$  be a subfactor whose relative commutant  $Q' \cap M_\Gamma$  is also a factor. Let  $\Lambda \subseteq \Gamma$  be a subgraph and let  $\{\Lambda_j\}_{j \in \mathcal{J}}$  be a non-empty, finite collection of subgraphs of  $\Lambda$ . Suppose  $Q \prec_{M_\Gamma} M_\Lambda$  and  $Q \not\prec_{M_\Gamma} M_{\Lambda_j}$  for  $j \in \mathcal{J}$ . Then there is a unitary  $u \in M_\Gamma$  such that  $u^*Qu \subseteq M_{\Lambda_{\text{emb}}}$ , where  $\Lambda_{\text{emb}}$  is defined as in (5.20).*

*Proof.* Since  $Q \prec_{M_\Gamma} M_\Lambda$  and  $Q \not\prec_{M_\Gamma} M_{\Lambda_j}$  for  $j \in \mathcal{J}$  we have by Lemma 5.4.9 that there are projections  $q \in Q$ ,  $e \in M_\Lambda$ , a normal  $*$ -homomorphism  $\theta : qQq \rightarrow eM_\Lambda e$  and a non-zero partial isometry  $v \in eM_\Gamma q$  such that  $\theta(x)v = vx$  for  $x \in qQq$  and such that moreover  $\theta(qQq) \not\prec_{M_\Gamma} M_{\Lambda_j}$  for  $j \in \mathcal{J}$ . We may moreover assume that  $q$  is majorized by the support of  $\mathbb{E}_Q(v^*v)$ . Let  $q_0 \in Q$  be a non-zero projection with  $q_0 \leq q$  and trace  $\tau(q_0) = \frac{1}{m}$  for some  $m \geq 1$ . Put  $v_0 := vq_0$ . Note that  $v^*v \in (qQq)' \cap qM_\Gamma q$ . Then  $\mathbb{E}_Q(v_0^*v_0) = \mathbb{E}_Q(q_0v^*vq_0) = \mathbb{E}_Q(q_0v^*v) = q_0\mathbb{E}_Q(v^*v)$  and the latter expression is non-zero by the assumption that the support of  $\mathbb{E}_Q(v^*v)$  majorizes  $q$ . As  $\mathbb{E}_Q$  is faithful  $v_0 \neq 0$ . Define  $\theta_0 : q_0Qq_0 \rightarrow eM_\Lambda e$  as  $\theta_0 := \theta|_{q_0Qq_0}$ . Then for  $x \in q_0Qq_0$  we have  $\theta_0(x)v_0 = \theta(x)vq_0 = vxq_0 = v_0x$ . Automatically this implies  $v_0^*v_0 \in (q_0Qq_0)' \cap q_0M_\Gamma q_0$ . Furthermore for  $j \in \mathcal{J}$ , the corner  $\theta_0(q_0Qq_0) = \theta_0(q_0)\theta(qQq)\theta(q_0)$  does not embed in  $M_{\Lambda_j}$  inside  $M_\Gamma$  since  $\theta(qQq)$  does not embed in  $M_{\Lambda_j}$  inside  $M_\Gamma$ . Hence, by Proposition 5.4.13(1b) we obtain  $\theta(q_0)\text{Nor}_{M_\Gamma}(\theta_0(q_0Qq_0))''\theta(q_0) \subseteq M_{\Lambda_{\text{emb}}}$ .

Since  $Q$  is a factor and  $\tau(q_0) = \frac{1}{m}$  we can for  $j = 1, \dots, m$  choose a partial isometry  $u_j$  in  $Q$  such that  $u_j^*u_j = q_0$  and  $\sum_{j=1}^m u_j u_j^* = 1_{M_\Gamma}$ . We may moreover assume that  $u_1 = q_0$ .

We define a projection  $q' := \sum_{j=1}^m u_j v_0^* v_0 u_j^* \in M_\Gamma$ . We show that  $q' \in Q' \cap M_\Gamma$ . Indeed, let  $y \in Q$ . Then using that  $v_0^* v_0 \in (q_0 Q q_0)'$  and  $u_j^* y u_j \in q_0 Q q_0$  for  $j = 1, \dots, n$  we get

$$\begin{aligned} q' y &= \sum_{j=1}^m u_j (v_0^* v_0) u_j^* y = \sum_{j=1}^m \sum_{i=1}^m u_j (v_0^* v_0) (u_j^* y u_i) u_i^* \\ &= \sum_{j=1}^m \sum_{i=1}^m u_j (u_j^* y u_i) (v_0^* v_0) u_i^* = \sum_{i=1}^m y u_i (v_0^* v_0) u_i^* = y q'. \end{aligned}$$

and thus  $q' \in Q' \cap M_\Gamma$ . We observe that  $v_0^* v_0 = q_0 q' q_0 = q_0 q'$  which shows in particular that  $q'$  is non-zero (since  $v_0 \neq 0$ ). Since  $Q' \cap M$  is a (finite) factor and  $q'$  is a non-zero projection, we can choose a projection  $q'_0 \in Q' \cap M$  with  $q'_0 \leq q'$  and  $\tau(q'_0) = \frac{1}{n}$  for some  $n \geq 1$ . Since  $Q' \cap M_\Gamma$  is a factor and since  $\tau(q'_0) = \frac{1}{n}$  we can find partial isometries  $u'_1, \dots, u'_n \in Q' \cap M_\Gamma$  with  $(u'_k)^* u'_k = q'_0$  for  $k = 1, \dots, n$  and such that  $\sum_{k=1}^n u'_k (u'_k)^* = 1_{M_\Gamma}$ .

We put  $v_{00} := v_0 q'_0 = v_0 q_0 q'_0 \in e M_\Gamma q_0$ . Observe that  $v_{00}^* v_{00} = q'_0 v_0^* v_0 q'_0 = q'_0 q_0$  has trace  $\tau(v_{00}^* v_{00}) = \tau(q'_0) \tau(q_0) = \frac{1}{nm}$  so in particular  $v_{00}$  is non-zero. For  $x \in q_0 Q q_0$  we have  $\theta_0(x) v_{00} = \theta_0(x) v_0 q'_0 = v_0 x q'_0 = v_{00} x$ . Therefore,  $v_{00} v_{00}^* \in \theta_0(q_0 Q q_0)' \cap M_\Gamma$ . Since  $v_{00} v_{00}^* \leq \theta(q_0)$  we obtain  $v_{00} v_{00}^* \in \theta(q_0) \text{Nor}_{M_\Gamma}(\theta_0(q_0 Q q_0))' \theta(q_0) \subseteq M_{\Lambda_{\text{emb}}}$  using the first paragraph.

Since  $M_{\Lambda_{\text{emb}}}$  is a factor (as it is a graph product of  $\text{II}_1$ -factors), and since  $\tau(v_{00} v_{00}^*) = \frac{1}{nm}$  there exist for  $j = 1, \dots, m, k = 1, \dots, n$  partial isometries  $w_{j,k} \in M_{\Lambda_{\text{emb}}}$  with  $w_{j,k} w_{j,k}^* = v_{00} v_{00}^*$  and  $\sum_{j=1}^m \sum_{k=1}^n w_{j,k}^* w_{j,k} = 1_{M_\Gamma}$ . Put  $u := \sum_{j=1}^m \sum_{k=1}^n u_j u_k^* v_{00}^* w_{j,k} \in M_\Gamma$  and observe that  $u$  is a unitary. Now for  $x \in Q$  we have

$$\begin{aligned} u^* x u &= \sum_{j_1=1}^m \sum_{k_1=1}^n \sum_{j_2=1}^m \sum_{k_2=1}^n w_{j_1,k_1}^* v_{00} (u_{k_1}')^* (u_{j_1}^* x u_{j_2}) u_{k_2}' v_{00}^* w_{j_1,k_2} \\ &= \sum_{j_1=1}^m \sum_{k=1}^n \sum_{j_2=1}^m w_{j_1,k}^* v_{00} (u_{j_1}^* x u_{j_2}) q'_0 v_{00}^* w_{j_1,k} \\ &= \sum_{j_1=1}^m \sum_{k=1}^n \sum_{j_2=1}^m w_{j_1,k}^* \theta_0(u_{j_1}^* x u_{j_2}) v_{00} v_{00}^* w_{j_1,k} \\ &= \sum_{j_1=1}^m \sum_{k=1}^n \sum_{j_2=1}^m w_{j_1,k}^* \theta_0(u_{j_1}^* x u_{j_2}) w_{j_1,k} \in M_{\Lambda_{\text{emb}}}. \end{aligned}$$

Hence  $u^* Q u \subseteq M_{\Lambda_{\text{emb}}}$ . □

## 5.5. GRAPH PRODUCT RIGIDITY

The aim of this section is to prove Theorem 5.5.19. This provides a rather general class of graphs and von Neumann algebras such that the graph product completely remembers the graph and the vertex von Neumann algebra up to stable isomorphism. Note that we cannot expect to cover all graphs as this would imply the free factor problem and which is beyond reach of our methods. The class of rigid graphs as presented in Section 5.2 is therefore natural.

### 5.5.1. VERTEX VON NEUMANN ALGEBRAS

We define classes of von Neumann algebras for which we first recall a version of the Akemann-Ostrand property [HI17].

**Definition 5.5.1.** A von Neumann algebra  $M$  with standard form  $(M, L^2(M), J, L^2(M)^+)$  is said to possess strong property (AO) if there exist unital  $C^*$ -subalgebras  $A \subseteq M$  and  $C \subseteq B(L^2(M))$  such that:

- $A$  is  $\sigma$ -weakly dense in  $M$ ,
- $C$  is nuclear and contains  $A$ ,
- The commutators  $[C, JAJ] = \{[c, JaJ] \mid c \in C, a \in A\}$  are contained in the space of compact operators  $K(L^2(M))$ .

We recall that a wide class of examples of von Neumann algebras with property strong (AO) comes from hyperbolic groups.

**Theorem 5.5.2** (See Lemma 3.1.4 of [Iso15b] and remarks before). *Let  $G$  be a discrete hyperbolic group. Consider the anti-linear isometry  $J$  determined by*

$$J: \ell^2(G) \rightarrow \ell^2(G) : \delta_s \mapsto \delta_{s^{-1}}, \quad s \in G.$$

*Then there is a nuclear  $C^*$ -algebra  $C$  such that:*

1.  $C_r^*(G) \subseteq C \subseteq B(\ell^2(G))$ .
2.  $C$  contains all compact operators.
3. The commutator  $[C, J C_r^*(G) J]$  is contained in the space of compact operators.

**Remark 5.5.3.** In view of Section 5.3 it is worth to note that we may always assume without loss of generality that  $C$  contains the space of compact operators by replacing  $C$  by  $C + K(L^2(M))$  if necessary, see [HI17, Remark 2.7]. This fact also underlies Theorem 5.5.2.

**Definition 5.5.4.** We define the following classes of von Neumann algebras:

- Let  $\mathcal{C}_{\text{Vertex}}$  denote the class of  $\text{II}_1$ -factors  $M$  with separable predual  $M_*$  that satisfy condition strong (AO) and which are non-amenable;
- Let  $\mathcal{C}_{\text{Complete}}$  denote the class of all von Neumann algebraic graph products  $(M_\Gamma, \tau) = *_{v \in \Gamma} (M_v, \tau_v)$  of tracial von Neumann algebras  $(M_v, \tau_v)$  in  $\mathcal{C}_{\text{Vertex}}$  taken over non-empty, finite, complete graphs  $\Gamma$ ;
- Let  $\mathcal{C}_{\text{Rigid}}$  denote the class of all von Neumann algebraic graph products  $(M_\Gamma, \tau) = *_{v \in \Gamma} (M_v, \tau_v)$  of tracial von Neumann algebras  $(M_v, \tau_v)$  in  $\mathcal{C}_{\text{Vertex}}$  taken over non-empty, finite, rigid graphs  $\Gamma$ .

**Remark 5.5.5.** We remark that  $\mathcal{C}_{\text{Vertex}} \subseteq \mathcal{C}_{\text{Complete}} \subseteq \mathcal{C}_{\text{Rigid}}$ . Furthermore,

1. The class  $\mathcal{C}_{\text{Vertex}}$  is closed under taking free products (see [HI17, Example 2.8(5)]). Moreover, all von Neumann algebras  $M \in \mathcal{C}_{\text{Vertex}}$  are solid and prime, see [Oza04];

2. The class  $\mathcal{C}_{\text{Complete}}$  is closed under taking tensor products. Moreover, we observe that  $\mathcal{C}_{\text{Complete}}$  coincides with the class of tensor products of factors from  $\mathcal{C}_{\text{Vertex}}$ ;
3. The class  $\mathcal{C}_{\text{Rigid}}$  is closed under taking graph products over non-empty, finite, *rigid* graphs by Remark 5.2.4 and Lemma 5.2.5. In particular, it is closed under tensor products;
4. The class  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  is closed under taking graph products over *arbitrary* non-empty, finite graphs by Remark 5.2.4 and Lemma 5.2.5. In particular, it is closed under tensor products and under free products.

*Remark 5.5.6.* We show that it may happen that a graph product over a rigid graph is isomorphic to a graph product over a non-rigid graph; even if all vertex von Neumann algebras come from the class  $\mathcal{C}_{\text{Vertex}}$ . Consider the graph  $\mathbb{Z}_4$  defined in Example 5.2.2(3). The graph  $\mathbb{Z}_4$  is rigid. For  $v \in \mathbb{Z}_4$  let  $G_v$  be a countable discrete group. Let  $H_v = G_v * G_{v+2}$ . We have for the graph products of groups that

$$*_{v, \mathbb{Z}_4} G_v = (G_0 * G_2) \times (G_1 * G_3) = *_{v, \mathbb{Z}_2} H_v.$$

We now set  $G_v = \mathbb{F}_2$  and  $H_v = \mathbb{F}_4$  to be free groups with 2 and 4 generators respectively. Set  $M_v = \mathcal{L}(\mathbb{F}_2)$ ,  $v \in \mathbb{Z}_4$  and  $N_v = \mathcal{L}(\mathbb{F}_4)$ ,  $v \in \mathbb{Z}_2$  equipped with their tracial Plancherel states  $\tau_v$ . Then  $M_v$  and  $N_v$  are in class  $\mathcal{C}_{\text{Vertex}}$  and  $*_{v, \mathbb{Z}_4}(M_v, \tau_v) = *_{v, \mathbb{Z}_2}(N_v, \tau_v)$ . We have thus given an example of a rigid and non-rigid graph that give isomorphic graph products.

### 5.5.2. KEY RESULT FOR EMBEDDINGS OF DIFFUSE SUBALGEBRAS IN GRAPH PRODUCTS

In this section we fix the following notation. Let  $\Gamma$  be a simple graph. For  $v \in \Gamma$  let  $(M_v, \tau_v)$  be a tracial von Neumann algebra ( $M_v \neq \mathbb{C}$ ) that satisfies strong (AO) and has a separable predual. Let  $(M_\Gamma, \tau_\Gamma) = *_{v \in \Gamma} (M_v, \tau_v)$  be the von Neumann algebraic graph product. For  $v \in \Gamma$  let  $\mathcal{H}_v = L^2(M_v, \tau_v)$  and let  $\mathcal{H}_\Gamma$  be the graph product of these Hilbert spaces, which is the standard Hilbert space of  $M_\Gamma$  [CF17]. We denote by  $J : \mathcal{H}_\Gamma \rightarrow \mathcal{H}_\Gamma$  the modular conjugation. Let  $B_v = B(\mathcal{H}_v)$ . Let  $\Omega_v = 1_{M_v}$  as a vector in  $\mathcal{H}_v$  and let  $\omega_v(x) = \langle x\Omega_v, \Omega_v \rangle$ ,  $x \in B_v$ . Then  $\omega_v$  is a GNS-faithful, but not faithful, state on  $B_v$  and the GNS-space of  $\omega_v$  can canonically be identified with  $\mathcal{H}_v$ . The reduced  $C^*$ -algebraic graph product  $(B_\Gamma, \omega_\Gamma) = *_{v \in \Gamma}^{\min} (B_v, \omega_v)$  gives then by construction a  $C^*$ -subalgebra  $B$  of  $B(\mathcal{H}_\Gamma)$ . We let  $\lambda_v : B_v \rightarrow B$  be the canonical embedding. Furthermore we let  $\rho_v : B_v^{\text{op}} \rightarrow B^{\text{op}}$  be the map  $\rho_v(x^{\text{op}}) = J\lambda_v(x)^*J$ . As for  $v \in \Gamma$  the von Neumann algebra  $M_v$  has strong property (AO) by assumption (as  $M_v \in \mathcal{C}_{\text{Vertex}}$ ), there are unital  $C^*$ -subalgebras  $C_v \subseteq B_v$  and  $A_v \subseteq M_v \cap C_v$  such that

1. The  $C^*$ -algebra  $A_v$  are  $\sigma$ -weakly dense in  $M_v$ ,
2. The  $C^*$ -algebra  $C_v$  are nuclear,
3. The commutators  $[C_v, J_v A_v J_v]$  are contained in  $K(\mathcal{H}_v)$ .

As in [HI17, Remarks 2.7 (1)] we may and will moreover assume that  $K(\mathcal{H}_v) \subseteq C_v$ . We let  $(C_\Gamma, \omega_\Gamma) = *_{v \in \Gamma}^{\min}(C_v, \omega_v)$  and  $(A_\Gamma, \omega_\Gamma) = *_{v \in \Gamma}^{\min}(A_v, \omega_v)$  be the reduced graph products of the  $C^*$ -algebras. Observe that we now have

$$A_\Gamma \subseteq M_\Gamma \subseteq B_\Gamma \text{ and } A_\Gamma \subseteq C_\Gamma \subseteq B_\Gamma,$$

and the states  $\omega_\Gamma$  defined through the different graph products coincide.

**Lemma 5.5.7.**  $C_\Gamma$  is nuclear.

*Proof.* The vector  $\Omega_v$  is cyclic for  $M_v$ . Furthermore,  $A_v$  is  $\sigma$ -weakly dense in  $M_v$  by assumption and so  $\Omega_v$  is also cyclic for  $A_v$ . It follows that the GNS-representation  $\pi_v$  of  $C_v$  with respect to  $\omega_v$  is unitarily equivalent with the canonical representation given by the inclusion  $C_v \subseteq B(\mathcal{H}_v)$ , see [Con97, Theorem VIII.5.14 (b)]. We assumed that  $K(\mathcal{H}_v) \subseteq C_v$  and that  $C_v$  is nuclear and so we may apply Theorem 5.3.4 to conclude that  $C_\Gamma$  is nuclear.  $\square$

**Definition 5.5.8.** For  $\Lambda \subseteq \Gamma$  we define the  $C^*$ -algebra

$$D_\Lambda = U'_\Lambda(K(\mathcal{H}'(\Lambda)) \otimes B(\mathcal{H}_\Lambda))(U'_\Lambda)^*.$$

The tensor product in the definition of  $D_\Lambda$  is understood as the spatial (minimal) tensor product, which is the norm closure of the algebraic tensors acting on the tensor product Hilbert space. In particular  $D_\emptyset = K(\mathcal{H}_\Gamma)$ .

**Lemma 5.5.9.** Let  $v \in \Gamma$ . We have  $B_\Gamma D_{\text{Link}(v)} B_\Gamma \subseteq D_{\text{Link}(v)}$ .

*Proof.* We note that the proof we give here in particular also works if  $\text{Link}(v)$  is empty; though in that case the statement trivially follows from the fact that  $D_\emptyset = K(\mathcal{H}_\Gamma)$  is an ideal in  $B(\mathcal{H}_\Gamma)$ . Take  $x \in B(\mathcal{H}_w)$ . Then if  $w \notin \text{Link}(v)$  we have that  $\mathcal{H}'(\text{Link}(v))$  is an invariant subspace of  $x$  and

$$x = U'_{\text{Link}(v)}(x \otimes 1)U'^*_{\text{Link}(v)}. \quad (5.21)$$

Now suppose that  $w \in \text{Link}(v)$ . Let  $P$  be the orthogonal projection of  $\mathcal{H}'(\text{Link}(v))$  onto  $\mathcal{H}'(\text{Link}(v)) \cap \mathcal{H}_{\text{Link}(w)}$ . Then

$$x = U'_{\text{Link}(v)}(xP^\perp \otimes 1)U'^*_{\text{Link}(v)} + U'_{\text{Link}(v)}(P \otimes x)U'^*_{\text{Link}(v)}. \quad (5.22)$$

From the decompositions (5.21), (5.22) we see that  $x D_{\text{Link}(v)}, D_{\text{Link}(v)} x \subseteq D_{\text{Link}(v)}$ . As  $B_\Gamma$  is the closed linear span of products of elements in  $B(\mathcal{H}_w), w \in \Gamma$  the proof follows.  $\square$

Denote  $P_\Omega$  for the orthogonal projection onto  $\mathbb{C}\Omega$ .

**Lemma 5.5.10.** Let  $v, w \in \Gamma$ . Let  $a \in B_v, b \in B_w$ . Then

$$[a, Jb] = \begin{cases} U'_{\text{Star}(v)}(P_\Omega \otimes [a, JbJ])(U'_{\text{Star}(v)})^*, & v = w; \\ 0, & v \neq w. \end{cases} \quad (5.23)$$

*Proof.* If  $v \neq w$  then the result follows from [CF17, Proposition 3.3]. Suppose  $v = w$ . Let  $\mathbf{v}_1 \in \mathcal{W}'_1(\text{Star}(v))$  and  $\mathbf{v}_2 \in \mathcal{W}_{\text{Star}(v)}$ , and put  $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2$ . Let  $\eta_1 \in \mathcal{H}_{\mathbf{v}_1}$ ,  $\eta_2 \in \mathcal{H}_{\mathbf{v}_2}$  be pure tensors and denote  $\eta := U'_{\text{Star}(v)}(\eta_1 \otimes \eta_2) \in \mathcal{H}_{\mathbf{v}}$ . We claim

$$a\eta = \begin{cases} U'_{\text{Star}(v)}(\eta_1 \otimes (a\eta_2)), & \text{if } \mathbf{v}_1 = e; \\ U'_{\text{Star}(v)}((a\eta_1) \otimes \eta_2), & \text{if } \mathbf{v}_1 \neq e. \end{cases} \quad (5.24)$$

Indeed, if  $\mathbf{v}_1 = e$  then  $\eta_1 = \Omega$  and  $\eta = \eta_2$  up to scalar multiplication, so that  $U'_{\text{Star}(v)}(\eta_1 \otimes (a\eta_2)) = a\eta_2 = a\eta$ . Thus suppose  $\mathbf{v}_1 \neq e$ . Then it follows that  $\nu \mathbf{v}_1 \in \mathcal{W}'(\text{Star}(v))$  since  $\mathbf{v}_1 \in \mathcal{W}'(\text{Star}(v))$ .

Suppose  $\nu \mathbf{v}$  is reduced. Then also  $\nu \mathbf{v}_1$  is reduced, and we have  $a\eta = \lambda_{(\nu, e, e)}(a)\eta$  and  $a\eta_1 = \lambda_{(\nu, e, e)}(a)\eta_1$ . It follows from Lemma 3.1.8(3) that

$$\begin{aligned} a\eta &= \lambda_{(\nu, e, e)}(a)\eta \\ &= \mathcal{Q}_{(\nu \mathbf{v}_1, \mathbf{v}_2)}((\lambda_{(\nu, e, e)}(a)\eta_1) \otimes \eta_2) \\ &= \mathcal{Q}_{(\nu \mathbf{v}_1, \mathbf{v}_2)}((a\eta_1) \otimes \eta_2), \end{aligned}$$

and therefore, as  $\nu \mathbf{v}_1 \in \mathcal{W}'(\text{Star}(v))$  and  $\mathbf{v}_2 \in \mathcal{W}_{\text{Star}(v)}$ , we obtain  $a\eta = U'_{\text{Star}(v)}((a\eta_1) \otimes \eta_2)$ .

Now, suppose  $\nu \mathbf{v}$  is not reduced. Then also  $\nu \mathbf{v}_1$  is not reduced as  $\mathbf{v}_1 \in \mathcal{W}'(\text{Star}(v))$  and  $\mathbf{v}_1 \neq e$ . We have  $a\eta = \lambda_{(e, \nu, e)}(a)\eta + \lambda_{(e, e, \nu)}(a)\eta$  and  $a\eta_1 = \lambda_{(e, \nu, e)}(a)\eta_1 + \lambda_{(e, e, \nu)}(a)\eta_1$ . Again, using Lemma 3.1.8(3) we obtain

$$\begin{aligned} a\eta &= \lambda_{(e, \nu, e)}(a)\eta + \lambda_{(e, e, \nu)}(a)\eta \\ &= \mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}((\lambda_{(e, \nu, e)}(a)\eta_1) \otimes \eta_2) + \mathcal{Q}_{(\nu \mathbf{v}_1, \mathbf{v}_2)}((\lambda_{(e, e, \nu)}(a)\eta_1) \otimes \eta_2). \end{aligned}$$

And thus

$$a\eta = U'_{\text{Star}(v)}(\lambda_{(e, \nu, e)}(a)\eta_1 \otimes \eta_2) + U'_{\text{Star}(v)}(\lambda_{(e, e, \nu)}(a)\eta_1 \otimes \eta_2) = U'_{\text{Star}(v)}((a\eta_1) \otimes \eta_2).$$

This shows (5.24).

We now claim that

$$JbJ\eta = U'_{\text{Star}(v)}(\eta_1 \otimes JbJ\eta_2). \quad (5.25)$$

First, by [CF17, Proposition 2.20] we observe that  $J\eta_1 \in \mathcal{H}_{\mathbf{v}_1^{-1}}$ ,  $J\eta_2 \in \mathcal{H}_{\mathbf{v}_2^{-1}}$  and  $J\eta = J\mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\eta_1 \otimes \eta_2) = \mathcal{Q}_{(\mathbf{v}_2^{-1}, \mathbf{v}_1^{-1})}(J\eta_2 \otimes J\eta_1) \in \mathcal{H}_{\mathbf{v}^{-1}}$ . Furthermore, note that  $\nu \mathbf{v}_2^{-1} = \mathbf{v}_2^{-1} \nu$  and  $\nu \mathbf{v}_2 \in \mathcal{W}_{\text{Star}(v)}$ .

Suppose that  $\nu \mathbf{v}^{-1}$  is reduced. Then  $\nu \mathbf{v}_2^{-1}$  is also reduced. Hence, similar as before we obtain  $bJ\eta = \mathcal{Q}_{(\nu \mathbf{v}_2^{-1}, \mathbf{v}_1^{-1})}((bJ\eta_2) \otimes J\eta_1)$ . Hence

$$JbJ\eta = \mathcal{Q}_{(\mathbf{v}_1, \nu \mathbf{v}_2)}(\eta_1 \otimes (JbJ\eta_2)) = U'_{\text{Star}(v)}(\eta_1 \otimes (JbJ\eta_2)).$$

Now, suppose that  $\nu \mathbf{v}^{-1}$  is not reduced. Then  $\nu \mathbf{v}_2^{-1}$  is not reduced. Similar as before we obtain

$$bJ\eta = \mathcal{Q}_{(\mathbf{v}_2^{-1}, \mathbf{v}_1^{-1})}((\lambda_{(e, \nu, e)}(b)J\eta_2) \otimes J\eta_1) + \mathcal{Q}_{(\nu \mathbf{v}_2^{-1}, \mathbf{v}_1^{-1})}((\lambda_{(e, e, \nu)}(b)J\eta_2) \otimes J\eta_1).$$

Hence

$$\begin{aligned} JbJ\eta &= \mathcal{Q}_{(\mathbf{v}_1, \mathbf{v}_2)}(\eta_1 \otimes (\lambda_{(e, v, e)}(b)J\eta_2)) + \mathcal{Q}_{(\mathbf{v}_1, v\mathbf{v}_2)}(\eta_1 \otimes (\lambda_{(e, e, v)}(b)J\eta_2)) \\ &= U'_{\text{Star}(v)}(\eta_1 \otimes (J\lambda_{(e, v, e)}(b)J\eta_2)) + U'_{\text{Star}(v)}(\eta_1 \otimes (J\lambda_{(e, e, v)}(b)J\eta_2)) \\ &= U'_{\text{Star}(v)}(\eta_1 \otimes (JbJ\eta_2)), \end{aligned}$$

which shows (5.25). Now, combining (5.24) with (5.25) we obtain

$$[a, JbJ]\eta = \begin{cases} U'_{\text{Star}(v)}(\eta_1 \otimes ([a, JbJ]\eta_2)), & \text{if } \mathbf{v}_1 = e; \\ 0, & \text{if } \mathbf{v}_1 \neq e \end{cases}$$

and the statement follows.  $\square$

**Lemma 5.5.11.** *For  $v, w \in \Gamma, c \in C_v, a \in A_w$  we have  $[c, JaJ] \in D_{\text{Link}(v)}$ .*

*Proof.* If  $v \neq w$  it actually holds since by [CF17, Proposition 2.3]  $[c, JaJ] = 0$ . So assume  $v = w$ . Lemma 5.5.10 gives that

$$[c, JaJ] = U'_{\text{Star}(v)}(P_\Omega \otimes [c, JaJ])(U'_{\text{Star}(v)})^*. \quad (5.26)$$

In what follows we will use the decomposition of Section 2.4 applied to  $\text{Link}(v)$  as a subgraph of  $\text{Star}(v)$ , opposed to  $\text{Link}(v)$  as a subgraph of  $\Gamma$ , and correspondingly define the Hilbert space  $\mathcal{H}'(\text{Link}(v))$  with respect to this inclusion. We thus have a natural unitary

$$U''_{\text{Link}(v)} : \mathcal{H}'_{\text{Star}(v)}(\text{Link}(v)) \otimes \mathcal{H}_{\text{Link}(v)} \rightarrow \mathcal{H}_{\text{Star}(v)}.$$

Further as  $v$  commutes with all vertices in  $\text{Link}(v)$  it follows that with respect to this decomposition we have  $\mathcal{H}'_{\text{Star}(v)}(\text{Link}(v)) = \mathcal{H}_v$ . So

$$U''_{\text{Link}(v)} : \mathcal{H}_v \otimes \mathcal{H}_{\text{Link}(v)} \rightarrow \mathcal{H}_{\text{Star}(v)}.$$

For  $x \in B(\mathcal{H}_v)$  we get that

$$x = U''_{\text{Link}(v)}(x \otimes 1)(U''_{\text{Link}(v)})^*. \quad (5.27)$$

Set the unitary

$$U''_v := U'_{\text{Star}(v)}(1 \otimes U''_{\text{Link}(v)}) : \mathcal{H}'_\Gamma(\text{Star}(v)) \otimes \mathcal{H}_v \otimes \mathcal{H}_{\text{Link}(v)} \rightarrow \mathcal{H}_\Gamma.$$

Combining (5.26) and (5.27) we have

$$[c, JaJ] = U''_v(P_\Omega \otimes [c, JaJ] \otimes 1)U''_v^*,$$

where  $[c, JaJ]$  on the left hand side acts on  $\mathcal{H}_\Gamma$  and on the right hand side on  $\mathcal{H}_v$ . As we assumed  $[c, JaJ] \in K(\mathcal{H}_v)$  it follows that  $[c, JaJ]$  is contained in

$$U''_v(K(\mathcal{H}'(\text{Star}(v))) \otimes K(\mathcal{H}_v) \otimes 1)U''_v^* = U'_{\text{Link}(v)}(K(\mathcal{H}'(\text{Link}(v))) \otimes 1)U'^*_{\text{Link}(v)} \subseteq D_{\text{Link}(v)},$$

and thus the lemma is proved.  $\square$

Let  $Q \subseteq M_\Gamma$  be an amenable von Neumann subalgebra. As explained in [OP04, p. 228] there exists a conditional expectation  $\Psi_Q : B(\mathcal{H}_\Gamma) \rightarrow Q'$  that is *proper* in the sense that for any  $a \in B(\mathcal{H}_\Gamma)$  we have that  $\Psi_Q(a)$  is in the  $\sigma$ -weak closure of

$$\text{Conv}\{uau^* \mid u \in \mathcal{U}(Q)\},$$

where  $\text{Conv}$  denotes the convex hull.

**Lemma 5.5.12.** *Let  $Q \subseteq M_\Gamma$  be an amenable von Neumann subalgebra. If there is  $\Lambda \in \Gamma$  such that  $Q \not\prec_{M_\Gamma} M_\Lambda$ , then  $D_\Lambda$  is contained in  $\ker \Psi_Q$ .*

*Proof.* Let  $p \in K(\mathcal{H}'(\Lambda))$  be a finite rank projection. We first claim that

$$U'_\Lambda(p \otimes 1)U'^*_\Lambda \in \ker \Psi_Q.$$

We prove this claim by contradiction so suppose that  $d := \Psi_Q(U'_\Lambda(p \otimes 1)U'^*_\Lambda) \neq 0$ . First observe that for  $a \in M_\Lambda$  we have

$$J a J = U'_\Lambda(1 \otimes J_\Lambda a J_\Lambda)U'^*_\Lambda,$$

where  $J_\Lambda$  is the modular conjugation operator of  $M_\Lambda$  acting on  $\mathcal{H}_\Lambda$ . It follows in particular that

$$(J M_\Lambda J)' = U'_\Lambda(B(\mathcal{H}'(\Lambda)) \bar{\otimes} M_\Lambda)U'^*_\Lambda.$$

Any  $u \in U(Q)$  commutes with  $M'_\Gamma = J M_\Gamma J$  and so certainly it commutes with  $J M_\Lambda J$ . As  $\Psi_Q$  is proper we find that  $d$  as defined above thus commutes with  $J M_\Lambda J$ . Thus  $d \in U'_\Lambda(B(\mathcal{H}'(\Lambda)) \bar{\otimes} M_\Lambda)U'^*_\Lambda$ . Let  $\text{Tr}$  the trace on  $B(\mathcal{H}'(\Lambda))$  and let  $\Phi_\Lambda$  be the center valued trace of  $M_\Lambda$  onto  $Z(M_\Lambda) = M_\Lambda \cap M'_\Lambda$ . Using again that  $\Psi_Q$  is proper we find by lower semi-continuity [Tak02, Theorem VII.11.1] that for any normal (necessarily tracial) state  $\tau$  on the center  $Z(M_\Lambda)$  we have

$$(\text{Tr} \otimes (\tau \circ \Phi_\Lambda))(U'^*_\Lambda d U'_\Lambda) \leq (\text{Tr} \otimes (\tau \circ \Phi_\Lambda))(p \otimes 1) < \infty.$$

Let  $e$  be a spectral projection of  $d$  corresponding to the interval  $[\|d\|/2, \|d\|]$ . Then

$$(\text{Tr} \otimes (\tau \circ \Phi_\Lambda))(U'^*_\Lambda e U'_\Lambda) \leq 2(\text{Tr} \otimes (\tau \circ \Phi_\Lambda))(U'^*_\Lambda d U'_\Lambda) < \infty.$$

Thus it follows that  $(\text{Tr} \otimes \Phi_\Lambda)(U'^*_\Lambda e U'_\Lambda) < \infty$ . Then  $\mathcal{K} := e \mathcal{H}_\Gamma$  is a  $Q$ - $M_\Lambda$  sub-bimodule of  $\mathcal{H}_\Gamma$  with  $\dim_{M_\Lambda}(\mathcal{K}) < \infty$  and  $\mathcal{H}_\Gamma$  is the standard representation Hilbert space of  $M_\Gamma$ . It thus follows from Definition 2.1.2(3) that  $Q \prec_{M_\Gamma} M_\Lambda$ . This contradicts the assumptions and the claim is proved.

Taking linear spans and closures it thus follows from the previous paragraph that

$$U'_\Lambda(K(\mathcal{H}'(\Lambda)) \otimes 1)U'^*_\Lambda \subseteq \ker \Psi_Q.$$

Using the multiplicative domain of  $\Psi_Q$  it follows then that

$$U'_\Lambda(K(\mathcal{H}'(\Lambda)) \otimes B(\mathcal{H}_\Lambda))U'^*_\Lambda \subseteq \ker \Psi_Q.$$

This concludes the proof. □

**Lemma 5.5.13.** *Let  $Q \subseteq M_\Gamma$  be an amenable von Neumann subalgebra. Assume that for every  $v \in \Gamma$  we have  $Q \not\prec_{M_\Gamma} M_{\text{Link}(v)}$ . Then we have  $[C_\Gamma, JA_\Gamma J] \subseteq \ker(\Psi_Q)$ .*

*Proof.* The commutator  $[C_\Gamma, JA_\Gamma J]$  is contained in the closed linear span of the sets

$$C_\Gamma[C_w, JA_v J]C_\Gamma, \quad v, w \in \Gamma.$$

We have, as  $C_\Gamma \subseteq B_\Gamma$ , by Lemma 5.5.11 and Lemma 5.5.9 that

$$C_\Gamma[C_w, JA_v J]C_\Gamma \subseteq B_\Gamma D_{\text{Link}(v)} B_\Gamma \subseteq D_{\text{Link}(v)}.$$

By Lemma 5.5.12 we see that  $D_{\text{Link}(v)}, v \in \Gamma$  is contained in the kernel of  $\Psi_Q$ . We thus conclude that  $[C_\Gamma, JA_\Gamma J]$  is contained in  $\ker \Psi_Q$ .

Now  $[C_\Gamma, JA_\Gamma J]$  is contained in the closed linear span of the sets

$$JA_\Gamma J[C_\Gamma, JA_v J]JA_\Gamma J, \quad v \in \Gamma.$$

Note  $JA_\Gamma J$  is contained in  $M'_\Gamma$  so certainly in  $Q'$ . As  $\Psi_Q$  is a  $Q'$ -bimodule map it follows that  $JA_\Gamma J[C_\Gamma, JA_v J]JA_\Gamma J$  is contained in  $\ker \Psi_Q$ . This finishes the proof.  $\square$

**Lemma 5.5.14.** *Let  $Q \subseteq M_\Gamma$  be an amenable von Neumann subalgebra. Assume that for every  $v \in \Gamma$  we have  $Q \not\prec_{M_\Gamma} M_{\text{Link}(v)}$ . The map*

$$\begin{aligned} \Theta : A_\Gamma \otimes JA_\Gamma J &\rightarrow B(\mathcal{H}_\Gamma) \\ a \otimes JbJ &\mapsto \Psi_Q(aJbJ). \end{aligned} \tag{5.28}$$

*is continuous with respect to the minimal tensor norm.*

*Proof.* Observe that  $\Psi_Q$  is a  $Q'$ -bimodule map and we have  $JA_\Gamma J \subseteq M'_\Gamma \subseteq Q'$ . It thus follows from Lemma 5.5.13 that for  $x \in C_\Gamma$  and  $y \in JA_\Gamma J$  we have

$$\Psi_Q(x)y = \Psi_Q(xy) = \Psi_Q(yx + [x, y]) = \Psi_Q(yx) = y\Psi_Q(x).$$

So  $\Psi_Q(C_\Gamma) \subseteq (JA_\Gamma J)' = M_\Gamma$ . Now consider the composition of maps, see [BO08, Theorem 3.3.7 and 3.5.3],

$$\tilde{\Theta} : C_\Gamma \otimes_{\max} JA_\Gamma J \xrightarrow{\Psi_Q \otimes \text{Id}} M_\Gamma \otimes_{\max} JA_\Gamma J \xrightarrow{m} B(\mathcal{H}),$$

where  $m$  is the multiplication map. Note that  $C_\Gamma$  is nuclear by Lemma 5.5.7. Thus  $C_\Gamma \otimes_{\max} JA_\Gamma J = C_\Gamma \otimes_{\min} JA_\Gamma J$ . Then the restriction of  $\tilde{\Theta}$  to  $A_\Gamma \otimes_{\min} JA_\Gamma J$  gives the map  $\Theta$ .  $\square$

**Theorem 5.5.15.** *Let  $\Gamma$  be a finite simple graph. Let  $(M_\Gamma, \tau) = \ast_{v \in \Gamma} (M_v, \tau_v)$  be a graph product of finite von Neumann algebras  $M_v$  ( $\neq \mathbb{C}$ ) that satisfy condition strong (AO) and have separable preduals. Let  $Q \subseteq M_\Gamma$  be a diffuse von Neumann subalgebra. At least one of the following holds:*

1. *The relative commutant  $Q' \cap M_\Gamma$  is amenable;*
2. *There exists a non-empty  $\Gamma_0 \subseteq \Gamma$  such that  $\text{Link}(\Gamma_0) \neq \emptyset$  and  $Q \prec_{M_\Gamma} M_{\Gamma_0}$ .*

*Proof.* We first show we can reduce it to the case that  $Q$  is amenable. Indeed, suppose we have proven that every amenable diffuse subalgebra  $Q_0 \subseteq M_\Gamma$  satisfies (1) or (2). Let  $Q \subseteq M_\Gamma$  be an arbitrary diffuse subalgebra. Then by [HI17, Corollary 4.7] there is an amenable diffuse von Neumann subalgebra  $Q_0 \subseteq Q$  such that for subgraphs  $\Lambda \subseteq \Gamma$  we have  $Q_0 \not\prec_{M_\Gamma} M_\Lambda$  whenever  $Q \not\prec_{M_\Gamma} M_\Lambda$ . If  $Q$  does not satisfy (2), then neither does  $Q_0$ . Hence  $Q_0$  satisfies (1), so  $Q'_0 \cap M_\Gamma$  is amenable. Hence also the subalgebra  $Q' \cap M_\Gamma \subseteq Q'_0 \cap M_\Gamma$  is amenable, i.e.  $Q$  satisfies (1), which shows the reduction.

We now prove the statement with the notation introduced in this section. Assume (2) does not hold and we shall prove (1). By assumption for  $\Lambda \subseteq \Gamma$  with  $\text{Link}(\Lambda) \neq \emptyset$  we have  $Q \not\prec_{M_\Gamma} M_\Lambda$ . In particular we have for all  $v \in \Gamma$  with  $\text{Link}(v)$  non-empty that  $v$  is contained in  $\text{Link}(\text{Link}(v))$  and so  $Q \not\prec_{M_\Gamma} M_{\text{Link}(v)}$ . If  $\text{Link}(v)$  is empty then  $M_{\text{Link}(v)} = \mathbb{C}$  and so  $Q \not\prec_{M_\Gamma} M_{\text{Link}(v)}$  as  $Q$  is diffuse. It follows now from Lemma 5.5.14 that  $\Theta$  defined in (5.28) is bounded for the minimal tensor norm.

Each  $A_v$  is exact being included in the nuclear  $C^*$ -algebra  $C_v$ . Therefore the  $C^*$ -algebra  $A_\Gamma$  is exact by [CF17, Corollary 3.17]. Furthermore, the inclusions  $A_\Gamma \subseteq M_\Gamma$  and  $JA_\Gamma J \subseteq M'_\Gamma$  are  $\sigma$ -weakly dense.

The conclusions of the previous two paragraphs show that the assumptions of [Oza06, Lemma 2.1] are satisfied and this lemma concludes that  $Q' \cap M_\Gamma$  is amenable.  $\square$

### 5.5.3. UNIQUE RIGID GRAPH PRODUCT DECOMPOSITION

We will prove our main result Theorem 5.5.19 which asserts for a graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v) \in \mathcal{C}_{\text{Rigid}}$  with  $M_v \in \mathcal{C}_{\text{Vertex}}$  that we can retrieve the rigid graph  $\Gamma$  and retrieve the vertex von Neumann algebras  $M_v$  up to stable isomorphism. To prove the result we need the following lemmas.

**Lemma 5.5.16** (Lemma 3.5 of [Vae08]). *If  $A \subseteq 1_A M 1_A, B \subseteq 1_B M 1_B$  are von Neumann subalgebras and  $A <_M B$ , then  $B' \cap 1_B M 1_B <_M A' \cap 1_A M 1_A$ .*

**Lemma 5.5.17** (Lemma 2.4 in [DHI19], see also [Vae08]). *Let  $(M, \tau)$  be a tracial von Neumann algebra and let  $P \subseteq 1_P M 1_P, Q \subseteq 1_Q M 1_Q$  and  $R \subseteq 1_R M 1_R$  be von Neumann subalgebras. Then the following hold:*

1. *Assume that  $P <_M Q$  and  $Q <_M^s R$ . Then  $P <_M R$ ;*
2. *Assume that, for any non-zero projection  $z \in \text{Nor}_{1_P M 1_P}(P)' \cap 1_P M 1_P \subseteq Z(P' \cap 1_P M 1_P)$ , we have  $Pz <_M Q$ . Then  $P <_M^s Q$ .*

*In particular, we note that if  $Q' \cap 1_Q M 1_Q$  is a factor and  $P <_M Q$  and  $Q <_M R$  then  $P <_M R$ .*

**Lemma 5.5.18.** *Let  $\Gamma$  be a finite graph. For  $v \in \Gamma$ , let  $M_v, N_v$  be  $\text{II}_1$ -factors and put  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  and  $N_\Gamma = *_{v \in \Gamma} (N_v, \tilde{\tau}_v)$ . Suppose  $\iota: N_\Gamma \rightarrow M_\Gamma$  is a  $*$ -isomorphism and for  $v \in \Gamma$  we have*

$$\iota(N_v) <_{M_\Gamma} M_v \quad \text{and} \quad M_v <_{M_\Gamma} \iota(N_v).$$

*Then the following holds true:*

1. *For  $v \in \Gamma$  there is a unitary  $u_v \in M_\Gamma$  such that  $u_v^* \iota(N_{\text{Star}(v)}) u_v = M_{\text{Star}(v)}$ .*

2. Let  $\Lambda_0 \subseteq \Lambda \subseteq \Gamma$  be subgraphs with  $\iota(N_\Lambda) = M_\Lambda$ . Then  $\iota(N_{\Lambda \cup \text{Link}_\Gamma(\Lambda_0)}) = M_{\Lambda \cup \text{Link}_\Gamma(\Lambda_0)}$ .
3. Let  $P = (v_1, \dots, v_n)$  be a path in  $\Gamma$  and denote  $\Gamma_0 := \bigcup_{i=1}^n \text{Star}(v_i)$ . If there exist  $1 \leq j \leq n$  and a subgraph  $\Lambda \subseteq \Gamma_0$  such that  $v_j \in \Lambda$  and  $\iota(N_\Lambda) = M_\Lambda$ , then  $\iota(N_{\Gamma_0}) = M_{\Gamma_0}$ .
4. Let  $\Gamma_0$  be a connected component of  $\Gamma$ . If there is a non-empty subgraph  $\Lambda \subseteq \Gamma_0$  with  $\iota(N_\Lambda) = M_\Lambda$  then  $\iota(N_{\Gamma_0}) = M_{\Gamma_0}$ .

*Proof.* (1) As  $\iota(N_v) <_{M_\Gamma} M_v$  and  $\iota(N_v) \not\prec_{M_\Gamma} M_\emptyset$  (since  $N_v$  diffuse), and since  $\iota(N_v)$  and  $\iota(N_v)' \cap M_\Gamma (= \iota(N_{\text{Link}(v)}))$  are factors, we obtain by Theorem 5.4.16 a unitary  $u_v \in M_\Gamma$  such that  $u_v^* \iota(N_v) u_v \subseteq M_{\text{Star}(v)}$ . By assumption  $M_v <_{M_\Gamma} \iota(N_v)$  so that  $M_v <_{M_\Gamma} u_v^* \iota(N_v) u_v$ . If  $u_v^* \iota(N_v) u_v <_{M_\Gamma} M_{\text{Link}(v)}$  then  $u_v^* \iota(N_v) u_v <_{M_\Gamma}^s M_{\text{Link}(v)}$  by Lemma 5.5.17 (2), since  $M_v' \cap M_\Gamma$  is a factor. Consequently, by Lemma 5.5.17 (1) we obtain  $M_v <_{M_\Gamma} M_{\text{Link}(v)}$ , which gives a contradiction by Proposition 5.4.14. We conclude that  $u_v^* \iota(N_v) u_v \not\prec_{M_\Gamma} M_{\text{Link}(v)}$ . Now, since  $u_v^* \iota(N_v) u_v \subseteq M_{\text{Star}(v)}$  and  $u_v^* \iota(N_v) u_v \not\prec_{M_\Gamma} M_{\text{Link}(v)}$  we have by Proposition 5.4.13(1b) that  $\text{Nor}_{M_\Gamma}(u_v^* \iota(N_v) u_v) \subseteq M_{\text{Star}(v)}$ , hence  $u_v^* \iota(N_{\text{Star}(v)}) u_v \subseteq M_{\text{Star}(v)}$ .

By symmetry there is also a unitary  $\tilde{u}_v \in M_\Gamma$  such that  $\tilde{u}_v^* M_{\text{Star}(v)} \tilde{u}_v \subseteq \iota(N_{\text{Star}(v)})$ . Hence

$$u_v^* \tilde{u}_v^* M_{\text{Star}(v)} \tilde{u}_v u_v \subseteq u_v^* \iota(N_{\text{Star}(v)}) u_v \subseteq M_{\text{Star}(v)}. \quad (5.29)$$

Hence, since  $M_{\text{Star}(v)} \not\prec_{M_\Gamma} M_{\tilde{\Lambda}}$  for any strict subgraph  $\tilde{\Lambda} \subsetneq \text{Star}(v)$  we obtain by Proposition 5.4.13(1c) that  $\tilde{u}_v u_v \in M_{\text{Star}(v)}$ . From this we conclude that the inclusions in (5.29) are in fact equalities so  $u_v^* \iota(N_{\text{Star}(v)}) u_v = M_{\text{Star}(v)}$ .

(2) Let  $\Lambda_0 \subseteq \Lambda$  be a subgraph. Then  $\iota(N_{\Lambda_0}) \subseteq \iota(N_\Lambda) = M_\Lambda$  and by the assumptions  $\iota(N_{\Lambda_0}) \not\prec_{M_\Gamma} M_{\tilde{\Lambda}}$  for any strict subgraph  $\tilde{\Lambda} \subsetneq \Lambda_0$ . Hence, by Proposition 5.4.13(1b) we obtain that  $\iota(N_{\text{Link}(\Lambda_0)}) \subseteq \text{Nor}_{M_\Gamma}(\iota(N_{\Lambda_0}))'' \subseteq M_{\Lambda \cup \text{Link}(\Lambda_0)}$ . Thus  $\iota(N_{\Lambda \cup \text{Link}(\Lambda_0)}) \subseteq M_{\Lambda \cup \text{Link}(\Lambda_0)}$ . By symmetry we also obtain that  $M_{\Lambda \cup \text{Link}(\Lambda_0)} \subseteq \iota(N_{\Lambda \cup \text{Link}(\Lambda_0)})$  so we get the equality.

(3) As  $v_j \in \Lambda$  and  $\iota(N_\Lambda) = M_\Lambda$ , using (2) we obtain that  $\iota(N_{\Lambda \cup \text{Star}(v_j)}) = \iota(N_{\Lambda \cup \text{Link}(v_j)}) = M_{\Lambda \cup \text{Link}(v_j)} = M_{\Lambda \cup \text{Star}(v_j)}$ . Now for  $1 \leq i \leq n$  with  $|i - j| = 1$  we have  $v_i \in \Lambda \cup \text{Star}(v_j)$ . Hence, applying (2) again we obtain  $\iota(N_{\Lambda \cup \text{Star}(v_j) \cup \text{Star}(v_i)}) = M_{\Lambda \cup \text{Star}(v_j) \cup \text{Star}(v_i)}$ . Repeating the same argument at most  $n$  times we obtain  $\iota(N_{\Gamma_0}) = M_{\Gamma_0}$ .

(4) Let  $P = (v_1, \dots, v_n)$  be a path in  $\Gamma$  traversing all vertices in  $\Gamma_0$ . Then  $\Gamma_0$  is equal to  $\bigcup_{i=1}^n \text{Star}(v_i)$ . Now since  $\Lambda \subseteq \Gamma_0$  is non-empty, we can choose  $1 \leq j \leq n$  s.t.  $v_j \in \Lambda$ . Now by (3) we obtain  $\iota(N_{\Gamma_0}) = M_{\Gamma_0}$ .  $\square$

**Theorem 5.5.19.** *Let  $\Gamma$  be a finite rigid graph. For  $v \in \Gamma$ , let  $M_v$  be von Neumann algebras in the class  $\mathcal{C}_{\text{Vertex}}$ . Let  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$ . Suppose  $M_\Gamma = *_{w \in \Lambda} (N_w, \tau_w)$  for another finite rigid graph  $\Lambda$  and other von Neumann algebras  $N_w \in \mathcal{C}_{\text{Vertex}}$  for  $w \in \Lambda$ . Then there is a graph isomorphism  $\alpha : \Gamma \rightarrow \Lambda$ , and for each  $v \in \Gamma$  there is a unitary  $u_v \in M_\Gamma$  and a real number  $0 < t_v < \infty$  such that*

$$M_{\text{Star}(v)} = u_v^* N_{\text{Star}(\alpha(v))} u_v \quad \text{and} \quad M_v \simeq N_{\alpha(v)}^{t_v}. \quad (5.30)$$

Furthermore, for each  $v \in \Gamma$  its connected component  $\Gamma_v \subseteq \Gamma$  satisfies  $M_{\Gamma_v} = u_v^* N_{\alpha(\Gamma_v)} u_v$ .

*Proof.* First we construct the graph isomorphism  $\alpha$ . Take  $v \in \Gamma$ . As the vertex von Neumann algebras are factors we have

$$M'_{\text{Link}(v)} \cap M = M_{\text{Link}(\text{Link}(v))} = M_v.$$

In particular  $M'_{\text{Link}(v)} \cap M$  is non-amenable. Therefore Theorem 5.5.15 implies that there exists  $\Lambda_0 \subseteq \Lambda$  such that  $M_{\text{Link}(v)} \prec_{N_\Gamma} N_{\Lambda_0}$  and  $\text{Link}(\Lambda_0) \neq \emptyset$ . Thus taking relative commutants (Lemma 5.5.16) we find that  $N_{\text{Link}(\Lambda_0)} \prec_{M_\Gamma} M_v$ .

So we have shown that for every  $v \in \Gamma$  there exists a subgraph  $\alpha(v) \subseteq \Lambda$  that occurs as the link of a set such that  $N_{\alpha(v)} \prec_{M_\Gamma} M_v$ . Conversely, by symmetry, for every  $w \in \Lambda$  there exists  $\beta(w) \subseteq \Gamma$  that occurs as the link of a set such that  $M_{\beta(w)} \prec_{M_\Gamma} N_w$ .

Let again  $v \in \Gamma$ . Then for any  $w \in \alpha(v)$  we have  $N_w \prec_{M_\Gamma} M_v$  and consequently as  $N'_w \cap M_\Gamma$  is a factor  $N_w \prec_{M_\Gamma}^s M_v$ , see Lemma 5.5.17(2). Therefore, by transitivity of stable embeddings, i.e. Lemma 5.5.17(1), we find  $M_{\beta(w)} \prec_{M_\Gamma} M_v$ . Hence for any  $v' \in \beta(w)$  we have  $M_{v'} \prec_{M_\Gamma} M_v$ . But then by Proposition 5.4.14 we see that  $v' = v$ . Hence  $\beta(w) = v$  for any  $w \in \alpha(v)$  and in particular is a singleton set. So we have proved that for  $v \in \Gamma$  we have  $\beta(\alpha(v)) := \bigcup_{w \in \alpha(v)} \beta(w) = v$  and by symmetry for  $w \in \Lambda$  we have  $\alpha(\beta(w)) = w$ . But this can only happen if the values of  $\alpha$  and  $\beta$  are singletons and  $\alpha$  and  $\beta$  are inverses of each other.

If  $v \in \Gamma$  then we know that  $N_{\alpha(v)} \prec_{M_\Gamma} M_v$  and  $M_v \prec_{M_\Gamma} N_{\alpha(v)}$ . Taking relative commutants, using again factoriality of the vertex von Neumann algebras, we find

$$M_{\text{Link}(v)} \prec_{M_\Gamma} N_{\text{Link}(\alpha(v))}, \quad N_{\text{Link}(\alpha(v))} \prec_{M_\Gamma} M_{\text{Link}(v)}.$$

Now take  $v' \in \text{Link}(v)$  so that the first of these embeddings gives  $M_{v'} \prec_{M_\Gamma} N_{\text{Link}(\alpha(v))}$ , hence  $M_{v'} \prec_{M_\Gamma}^s N_{\text{Link}(\alpha(v))}$  by Lemma 5.5.17(2). Then again by Lemma 5.5.17(1) we obtain  $N_{\alpha(v')} \prec_{N_\Gamma} N_{\text{Link}(\alpha(v))}$ . This then implies by Proposition 5.4.14 that  $\alpha(v') \in \text{Link}(\alpha(v))$ . So we conclude that  $\alpha$  preserves edges. Similarly  $\beta$  preserves edges, and it follows that  $\alpha : \Gamma \rightarrow \Lambda$  is a graph isomorphism.

Since  $\Gamma \simeq \Lambda$  we obtain by Lemma 5.5.18(1) that for each  $v \in \Gamma$  there is a unitary  $u_v \in M_\Gamma$  such that  $u_v^* N_{\text{Star}(\alpha(v))} u_v = M_{\text{Star}(v)}$ . Consider the  $*$ -isomorphism  $\iota_v := \text{Ad}_{u_v^*} : N_\Gamma \rightarrow M_\Gamma$  which satisfies  $\iota_v(N_{\text{Star}(\alpha(v))}) = M_{\text{Star}(v)}$ . Then by Lemma 5.5.18(4) we obtain for the connected component  $\Gamma_v \subseteq \Gamma$  of  $v$  that  $u_v^* N_{\Gamma_v} u_v = \iota_v(N_{\Gamma_v}) = M_{\Gamma_v}$ .

We show the isomorphism of vertex von Neumann algebras up to amplification. Let  $w \in \Gamma$ . Since  $\iota_w(N_{\text{Star}(\alpha(w))}) = M_{\text{Star}(w)}$  and since  $\iota_w(N_{\text{Link}(\alpha(w))})' \cap M_{\text{Star}(w)} = \iota_w(N_{\alpha(w)})$  is non-amenable, we obtain by Theorem 5.5.15 that  $\iota_w(N_{\text{Link}(\alpha(w))}) \prec_{M_{\text{Star}(w)}} M_{\Gamma_1}$  for some subgraph  $\Gamma_1 \subseteq \text{Star}(w)$  with  $\text{Link}_{\text{Star}(w)}(\Gamma_1) \neq \emptyset$ . Thus, by Lemma 5.5.16 we obtain that  $M_{\text{Link}(\Gamma_1)} \prec_{M_{\text{Star}(w)}} \iota_w(N_{\alpha(w)})$ . Let  $v \in \text{Link}(\Gamma_1)$  (which is non-empty). Then  $M_v \prec_{M_{\text{Star}(w)}} \iota_w(N_{\alpha(w)})$  and, as before,  $\iota_w(N_{\alpha(w)}) \prec_{M_\Gamma}^s M_w$ . Hence  $M_v \prec_{M_\Gamma} M_w$  and so  $v = w$  by Proposition 5.4.14. Therefore  $M_w \prec_{M_{\text{Star}(w)}} \iota_w(N_{\alpha(w)})$ . Analogously,  $\iota_w(N_{\alpha(w)}) \prec_{M_{\text{Star}(w)}} M_w$ .

Since we are dealing with  $\text{II}_1$ -factors these embeddings are also with expectation, i.e.  $\iota_w(N_{\alpha(w)}) \leq_{M_{\text{Star}(w)}} M_w$  as in [HI17, Definition 4.1]. Thus, since  $M_{\text{Star}(w)} = M_w \bar{\otimes} M_{\text{Link}(w)}$  we obtain by [HI17, Lemma 4.13] non-zero projections  $p_w, q_w \in M_{\text{Star}(w)}$ , a partial isometry  $v_w \in M_{\text{Star}(w)}$  with  $v_w^* v_w = p_w$  and  $v_w v_w^* = q_w$  and a subfactor  $P_w \subseteq q_w \iota_w(N_{\alpha(w)}) q_w$  so that

$$q_w \iota_w(N_{\alpha(w)}) q_w = v_w M_w v_w^* \bar{\otimes} P_w, \quad v_w M_{\text{Link}(w)} v_w^* = P_w \bar{\otimes} q_w \iota_w(N_{\text{Link}(\alpha(w))}) q_w.$$

Since  $N_w$  is prime, so is  $q_w \iota_w(N_{\alpha(w)}) q_w$ . Hence, as  $v_w M_w v_w^*$  is a  $\text{II}_1$ -factor, we obtain that  $P_w$  is a factor of type  $\text{I}_n$  for some  $n \in \mathbb{N}$ . We conclude that  $N_{\alpha(w)}$  is isomorphic to

some amplification of  $M_w$ .

□

We state two corollaries that follow from Theorem 5.5.19. The following result tells us when a rigid graph product  $M_\Gamma$  can decompose as graph product over another rigid graph  $\Lambda$ .

**Corollary 5.5.20.** *Let  $\Gamma, \Lambda$  be finite rigid graphs. Let  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  be the graph product of factors  $M_v \in \mathcal{C}_{\text{Vertex}}$ . The following are equivalent:*

1. *We can write  $\Gamma = *_{w \in \Lambda} \Gamma_w$  for some non-empty graphs  $\Gamma_w, w \in \Lambda$ ;*
2. *We can write  $M_\Gamma = *_{w \in \Lambda} (M_w, \tau_w)$  for some factors  $M_w \in \mathcal{C}_{\text{Rigid}}, w \in \Lambda$ .*

*Proof.* Suppose we can write  $\Gamma = *_{w \in \Lambda} \Gamma_w$  for non-empty graphs  $\Gamma_w$  for  $w \in \Lambda$ . Note that  $\Gamma_w$  is rigid by Lemma 5.2.5. Now by Remark 5.2.4 we have  $M_\Gamma = *_{w \in \Lambda} (M_w, \tau_w)$  where  $M_w := M_{\Gamma_w} \in \mathcal{C}_{\text{Rigid}}$ .

For the other direction, suppose  $M_\Gamma = *_{w \in \Lambda} (M_w, \tau_w)$  for some  $M_w \in \mathcal{C}_{\text{Rigid}}$  for  $w \in \Lambda$ . Then there are non-empty, finite rigid graphs  $\Gamma_w$  and factors  $N_v \in \mathcal{C}_{\text{Vertex}}$  for  $v \in \Gamma_w$  such that  $M_w = *_{v \in \Gamma_w} (N_v, \tau_v)$  for  $w \in \Lambda$ . Hence, by Remark 5.2.4 we obtain  $M_\Gamma = *_{v \in \Gamma} (N_v, \tau_v)$ . Since  $\Gamma_\Lambda$  is rigid by Lemma 5.2.5, we obtain by Theorem 5.5.19 that  $\Gamma \simeq \Gamma_\Lambda = *_{w \in \Lambda} \Gamma_w$ . □

The following corollary states a unique prime factorization for the class  $\mathcal{C}_{\text{Complete}}$ . This result recovers the result of [HI17] for a slightly smaller class.

**Corollary 5.5.21.** *Any von Neumann algebra  $M \in \mathcal{C}_{\text{Complete}}$  can decompose as tensor product*

$$M = M_1 \bar{\otimes} \cdots \bar{\otimes} M_m \quad (5.31)$$

*for some  $m \geq 1$  and prime factors  $M_1, \dots, M_m \in \mathcal{C}_{\text{Vertex}}$ .*

*Furthermore, suppose  $M \simeq N$  for*

$$N = N_1 \bar{\otimes} \cdots \bar{\otimes} N_n, \quad (5.32)$$

*where  $n \geq 1$ , and  $N_1, \dots, N_n \in \mathcal{C}_{\text{Vertex}}$  are other prime factors. Then  $n = m$  and there is a permutation  $\alpha$  of  $\{1, \dots, m\}$  such that  $M_i$  is isomorphic to an amplification of  $N_{\alpha(i)}$ .*

*Proof.* Since  $M \in \mathcal{C}_{\text{Complete}}$ , there is a non-empty finite complete graph  $\Gamma$  and factors  $M_v \in \mathcal{C}_{\text{Vertex}}$  for  $v \in \Gamma$  such that  $M = *_{v \in \Gamma} (M_v, \tau_v)$ . Hence  $M = \bar{\otimes}_{v \in \Gamma} M_v$  since  $\Gamma$  is complete. Moreover, for each  $v \in \Gamma$  the factor  $M_v$  is prime by Remark 5.5.5(1). This shows (5.31) with  $m = |\Gamma|$ . For each  $1 \leq i \leq n$  we have  $N_i \in \mathcal{C}_{\text{Vertex}}$  since it is prime. Let  $\Lambda$  be a complete graph with  $n$  vertices. Then  $N = \bar{\otimes}_{1 \leq i \leq n} N_i = *_{v \in \Lambda} (N_i, \tau_i)$ . Since  $\Gamma$  and  $\Lambda$  are rigid we obtain by Theorem 5.5.19 a graph isomorphism  $\alpha : \Gamma \rightarrow \Lambda$  such that  $M_i$  is isomorphic to an amplification of  $N_{\alpha(i)}$ . In particular,  $n = |\Lambda| = |\Gamma| = m$ . □

## 5.6. CLASSIFICATION OF STRONG SOLIDITY FOR GRAPH PRODUCTS

We state the definition of strong solidity. We recall the assumption that inclusions of von Neumann algebras are understood as unital inclusions.

**Definition 5.6.1.** *A von Neumann algebra  $M$  is called strongly solid if for any diffuse, amenable, von Neumann subalgebra  $A \subseteq M$ ,  $\text{Nor}_M(A)''$  is also amenable.*

*Remark 5.6.2.* Note that a tracial von Neumann algebra that is not diffuse must be strongly solid as it contains no diffuse unital subalgebras at all.

In Section 5.6.1 we characterize strong solidity for graph products  $M_\Gamma$  of tracial von Neumann algebras  $(M_v, \tau_v)$ . In Section 5.6.2 we then show that for many concrete cases this makes it possible to verify whether the graph product is strongly solid.

### 5.6.1. STRONG SOLIDITY MAIN RESULT

We prove the main result Theorem 5.6.7. which characterizes strong solidity for graph products. We use the following result concerning amalgamated free products.

**Theorem 5.6.3** (Theorem A of [Vae14]). *Let  $(N_1, \tau_1), (N_2, \tau_2)$  be tracial von Neumann algebras with a common von Neumann subalgebra  $B \subseteq N_i$  satisfying  $\tau_1|_B = \tau_2|_B$ , and denote  $N := N_1 *_B N_2$  for their amalgamated free product. Let  $A \subseteq 1_A N 1_A$  be a von Neumann algebra that is amenable relative to  $N_1$  or  $N_2$  inside  $N$ . Put  $P = \text{Nor}_{1_A N 1_A}(A)''$ . Then at least one of the following is true:*

1.  $A <_N B$ ;
2.  $P <_N N_i$  for some  $i = 1, 2$ ;
3.  $P$  is amenable relative to  $B$  inside  $N$ .

Furthermore, we use the following results that are rather standard.

**Proposition 5.6.4.** *Let  $N \subseteq M$  be a von Neumann subalgebra and assume  $N$  is strongly solid. Let  $A \subseteq M$  be a diffuse amenable von Neumann subalgebra and  $P = \text{Nor}_M(A)''$  and  $z \in P \cap P'$  be a non-zero projection. Assume that  $zP <_M N$ . Then  $zP$  has an amenable direct summand.*

*Proof.* We follow [Vae14, Proof of Corollary C]. As  $zP <_M N$ , using the characterization [Vae08, Theorem 3.2.2], (following [Pop06c]), there exists a non-zero projection  $p \in M_n(\mathbb{C}) \otimes N$  and a normal unital  $*$ -homomorphism  $\varphi : zP \rightarrow p(M_n(\mathbb{C}) \otimes N)p$ . So  $\varphi(Az)$  is a diffuse amenable von Neumann subalgebra of  $M_n(\mathbb{C}) \otimes N$  and  $\tilde{P} = \text{Nor}_{p(M_n(\mathbb{C}) \otimes N)p}(\varphi(Az))''$  contains  $\varphi(Pz)$ . As  $N$  is strongly solid, so is its amplification  $p(M_n(\mathbb{C}) \otimes N)p$  [Hou10, Proposition 5.2] and hence  $\tilde{P}$  is amenable. So  $\varphi(Pz)$  is amenable and therefore  $Pz$  contains an amenable direct summand.  $\square$

Recall that a von Neumann algebra  $M$  is atomic if any projection in  $M$  majorizes a minimal projection. If  $M$  is atomic it is a direct sum of type I factors. We state the following proposition.

**Proposition 5.6.5.** *Let  $N = N_1 \overline{\otimes} N_2$  be a tensor product of finite von Neumann algebras  $N_1, N_2$ . The following statements hold:*

1. *Suppose  $N_1$  is non-amenable and diffuse and  $N$  is strongly solid. Then  $N_2$  is atomic;*
2. *Suppose  $N_1$  is non-amenable and  $N_2$  is diffuse. Then  $N$  is not strongly solid;*
3. *Suppose  $N_1$  is strongly solid and diffuse and  $N_2$  is atomic. Then  $N$  is strongly solid.*

*Proof.* (1) Write  $N_2 = N_c \oplus N_d$  with  $N_c$  either 0 or a diffuse von Neumann algebra and  $N_d$  an atomic von Neumann algebra. Assume  $N_c \neq 0$ . Let  $A \subseteq N_c, B \subseteq N_1$  be diffuse amenable von Neumann subalgebras. Then  $C := \mathbb{C}1_{N_1} \overline{\otimes} A \oplus B \overline{\otimes} \mathbb{C}1_{N_d} \subseteq N$  is diffuse and amenable. Furthermore,  $\text{Nor}_N(C)''$  contains  $N_1 \overline{\otimes} A \oplus B \overline{\otimes} \mathbb{C}1_{N_d}$  which is non-amenable. This contradicts that  $N$  is strongly solid and we conclude that  $N_c = 0$ .

(2) Take any diffuse amenable subalgebra  $A \subseteq N_2$ , for instance we may take  $A$  to be a maximal abelian subalgebra. Then  $\mathbb{C}1_{N_1} \overline{\otimes} A$  is a diffuse amenable subalgebra of  $N$  and  $\text{Nor}_N(\mathbb{C}1_{N_1} \overline{\otimes} A)''$  contains  $N_1 \overline{\otimes} A$  which is non-amenable. Hence  $N$  is not strongly solid.

(3) As  $N_2$  is atomic we may identify  $N_2$  with  $\bigoplus_{\alpha \in I} \text{Mat}_{n_\alpha}(\mathbb{C})$  where  $I$  is some index set and  $n_\alpha \in \mathbb{N}_{\geq 1}$ . Let  $1_\alpha$  be the unit of  $\text{Mat}_{n_\alpha}(\mathbb{C})$ . Let  $A \subseteq N_1 \overline{\otimes} N_2$  be a diffuse amenable von Neumann subalgebra. Then  $1_\alpha A \subseteq N_1 \otimes \text{Mat}_{n_\alpha}(\mathbb{C})$ . So  $\text{Nor}_{N_1 \otimes \text{Mat}_{n_\alpha}(\mathbb{C})}(1_\alpha A)''$  is amenable by [Hou10, Proposition 5.2] since  $N_1$  is strong solid and diffuse. Since

$$\text{Nor}_N(A)'' = \bigoplus_{\alpha \in I} \text{Nor}_{N_1 \otimes \text{Mat}_{n_\alpha}(\mathbb{C})}(1_\alpha A)''$$

and direct sums of amenable von Neumann algebras are amenable we conclude that  $\text{Nor}_N(A)''$  is amenable. It follows that  $N$  is strongly solid.  $\square$

We classify atomicity for graph products.

**Proposition 5.6.6.** *Let  $(M_\Gamma, \tau_\Gamma) = *_{v \in \Gamma} (M_v, \tau_v)$  be a graph product of tracial von Neumann algebras over a finite graph  $\Gamma$ . Then  $M_\Gamma$  is atomic if and only if  $\Gamma$  is complete and each  $M_v$  is atomic.*

*Proof.* Any subalgebra of an atomic von Neumann algebra is atomic again. It follows that each  $M_v$  is atomic. If  $\Gamma$  would not be complete then we may pick  $v, w \in \Gamma$  not sharing an edge and  $(M_v, \tau_v) * (M_w, \tau_w) \subseteq M_\Gamma$ . However,  $(M_v, \tau_v) * (M_w, \tau_w)$  is not atomic by [Ued11]. So  $\Gamma$  is complete. Conversely if  $\Gamma$  is complete and each  $M_v$  is atomic then  $M = \overline{\bigotimes}_{v \in \Gamma} M_v$  is atomic.  $\square$

We now classify strong solidity for graph products in terms of conditions on subgraphs. These conditions can be verified in most cases (see Proposition 5.6.6, Proposition 5.6.8, Proposition 5.6.9 and Theorem 5.6.12).

**Theorem 5.6.7.** *Let  $\Gamma$  be a finite graph and for each  $v \in \Gamma$  let  $M_v (\neq \mathbb{C})$  be a von Neumann algebra with normal faithful trace  $\tau_v$ . Then  $M_\Gamma$  is strongly solid if and only if the following conditions are satisfied:*

1. *For every vertex  $v \in \Gamma$  the von Neumann algebra  $M_v$  is strongly solid;*

2. For every subgraph  $\Lambda \subseteq \Gamma$  with  $M_\Lambda$  non-amenable, we have that  $M_{\text{Link}(\Lambda)}$  is not diffuse;
3. For every subgraph  $\Lambda \subseteq \Gamma$  with  $M_\Lambda$  non-amenable and diffuse, we have moreover that  $M_{\text{Link}(\Lambda)}$  is atomic.

*Proof.* Suppose  $M_\Gamma$  is strongly solid, we show that conditions (1), (2) and (3) are satisfied. Since strong solidity passes to subalgebras, as follows from its very definition, we obtain that (1) is satisfied. Furthermore, suppose  $\Gamma_0 \subseteq \Gamma$  is a subgraph for which  $M_{\Gamma_0}$  is non-amenable. We have  $M_{\Gamma_0 \cup \text{Link}(\Gamma_0)} = M_{\Gamma_0} \overline{\otimes} M_{\text{Link}(\Gamma_0)}$  which is strongly solid being a von Neumann subalgebra of  $M_\Gamma$ . Hence, Proposition 5.6.5(2) shows that  $M_{\text{Link}(\Gamma_0)}$  cannot be diffuse. This concludes (2). If  $M_{\Gamma_0}$  is diffuse then Proposition 5.6.5(1) shows that  $M_{\text{Link}(\Gamma_0)}$  is atomic. This concludes (3).

We now show the reverse direction. The proof is based on induction to the number of vertices of the graph. The statement clearly holds when  $\Gamma = \emptyset$  since in that case  $M_\Gamma = \mathbb{C}$  is strongly solid.

*Induction.* Let  $\Gamma$  be a non-empty graph, and assume by induction that Theorem 5.6.7 is proved for any strictly smaller subgraph of  $\Gamma$ , i.e. with less vertices. Assume conditions (1), (2) and (3) are satisfied for  $\Gamma$ . Observe that condition (1), (2) and (3) are then satisfied for all subgraphs of  $\Gamma$  as well. Hence by the induction hypothesis we obtain that  $M_{\Gamma_0}$  is strongly solid for all strict subgraphs  $\Gamma_0 \subsetneq \Gamma$ . We shall show that  $M_\Gamma$  is strongly solid. Let  $A \subseteq M$  be diffuse and amenable and denote  $P = \text{Nor}_M(A)''$ . We will show that  $P$  is amenable.

Suppose there is  $v \in \Gamma$  with  $\text{Star}(v) = \Gamma$ . Then we can decompose the graph product as  $M_\Gamma = M_v \overline{\otimes} M_{\Gamma \setminus \{v\}}$ . Now  $M_v$  is strongly solid by condition (1), and  $M_{\Gamma \setminus \{v\}}$  is strongly solid by the induction hypothesis as  $\Gamma \setminus \{v\} \subsetneq \Gamma$  is a strict subgraph. When both  $M_v$  and  $M_{\Gamma \setminus \{v\}}$  are amenable then  $M_\Gamma = M_v \overline{\otimes} M_{\Gamma \setminus \{v\}}$  is also amenable, and hence  $M_\Gamma$  is strongly solid. We can thus assume that  $M_v$  or  $M_{\Gamma \setminus \{v\}}$  is non-amenable. If  $M_v$  is non-amenable we need to separate two cases.

- If  $M_v$  is non-amenable and not diffuse then by condition (2) neither  $M_{\Gamma \setminus \{v\}}$  is diffuse and hence neither is  $M_\Gamma = M_v \overline{\otimes} M_{\Gamma \setminus \{v\}}$ . Then certainly  $M_\Gamma$  is strongly solid by the absence of (unital) diffuse subalgebras.
- If  $M_v$  is non-amenable and diffuse then by condition (3) we obtain that  $M_{\text{Link}(v)}$  ( $= M_{\Gamma \setminus \{v\}}$ ) is atomic, so that by Proposition 5.6.5(3) we have  $M_\Gamma = M_{\text{Link}(v)} \overline{\otimes} M_v$  is strongly solid.

The case when  $M_{\Gamma \setminus \{v\}}$  is non-amenable can be treated in the same way by swapping the roles of  $M_v$  and  $M_{\Gamma \setminus \{v\}}$  in the previous argument. We summarize that our proof is complete in case there is  $v \in \Gamma$  with  $\text{Star}(v) = \Gamma$ .

Now we assume that for all  $v \in \Gamma$  we have  $\text{Star}(v) \neq \Gamma$ . Pick  $v \in \Gamma$  and set  $\Gamma_1 := \text{Star}(v)$  and  $\Gamma_2 := \Gamma \setminus \{v\}$ . By Theorem 2.4.3 we can decompose  $M_\Gamma = M_{\Gamma_1} *_{M_{\Gamma_1 \cap \Gamma_2}} M_{\Gamma_2}$ . Moreover, as  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_1 \cap \Gamma_2$  are strict subgraphs of  $\Gamma$  we obtain by our induction hypothesis that  $M_{\Gamma_1}$ ,  $M_{\Gamma_2}$  and  $M_{\Gamma_1 \cap \Gamma_2}$  are strongly solid.

Let  $z \in P \cap P'$  be a central projection such that  $zP$  has no amenable direct summand. Note that  $zP \subseteq \text{Nor}_{zM_\Gamma z}(zA)''$ . As  $zA$  is amenable, it is amenable relative to  $M_{\Gamma_1}$  in  $M_\Gamma$ . Therefore by Theorem 5.6.3 at least one of the following three holds.

1.  $zA \prec_{M_\Gamma} M_{\Gamma_1 \cap \Gamma_2}$ ;
2. There is  $i \in \{1, 2\}$  such that  $zP \prec_{M_\Gamma} M_{\Gamma_i}$ ;
3.  $zP$  is amenable relative to  $M_{\Gamma_1 \cap \Gamma_2}$  inside  $M_\Gamma$ .

We now analyse each of the cases.

*Case (2).* In Case (2) we have that Proposition 5.6.4 together with the induction hypothesis shows that  $zP$  has an amenable direct summand in case  $z \neq 0$ . This is a contradiction so we conclude  $z = 0$  and hence  $P$  is amenable.

*Case (1).* In Case (1), since  $zA \prec_{M_\Gamma} M_{\Gamma_1 \cap \Gamma_2}$  but  $zA \not\prec_{M_\Gamma} \mathbb{C} = M_\emptyset$ , there is a subgraph  $\Lambda \subseteq \Gamma_1 \cap \Gamma_2$  such that  $zA \prec_{M_\Gamma} M_\Lambda$  but  $zA \not\prec_{M_\Gamma} M_{\tilde{\Lambda}}$  for any strict subgraph  $\tilde{\Lambda} \subseteq \Lambda$ . Put  $\Lambda_{\text{emb}} := \Lambda \cup \text{Link}(\Lambda)$ . Observe that  $\Lambda_{\text{emb}}$  contains at least  $v$  and  $\Lambda$ . Furthermore, by Proposition 5.4.13(2) we obtain that  $zP \prec_{M_\Gamma} M_{\Lambda_{\text{emb}}}$ . If  $\Lambda_{\text{emb}} \neq \Gamma$  then  $M_{\Lambda_{\text{emb}}}$  is strongly solid by the induction hypothesis. Therefore, in case  $z \neq 0$  we obtain by Proposition 5.6.4 that  $zP$  has an amenable direct summand, which is a contradiction. Thus  $z = 0$ , and  $P$  is amenable. Hence  $M_\Gamma$  is strongly solid.

We can thus assume that  $\Lambda_{\text{emb}} = \Gamma$ . Suppose  $M_\Lambda$  is non-amenable. Again we separate two cases:

- Assume that  $M_\Lambda$  is non-amenable and diffuse. Then by condition (3) we have that  $M_{\text{Link}(\Lambda)}$  is atomic and by Proposition 5.6.6 we see that  $\text{Link}(\Lambda)$  must be complete. But as  $v \in \text{Link}(\Lambda)$  this implies that  $\text{Link}(\Lambda) \subseteq \text{Star}(v) = \Gamma_1$  and thus  $\Lambda_{\text{emb}} \subseteq \Gamma_1$ . Therefore  $\Lambda_{\text{emb}}$  is a strict subgraph of  $\Gamma$ , a contradiction. So this case does not occur;
- Assume that  $M_\Lambda$  is non-amenable and not diffuse. Then by (2)  $M_{\text{Link}(\Lambda)}$  is not diffuse either. As  $M_\Gamma = M_\Lambda \overline{\otimes} M_{\text{Link}(\Lambda)}$  we find that  $M_\Gamma$  is not diffuse and thus strongly solid by absence of diffuse (unital) subalgebras.

Next suppose  $M_{\text{Link}(\Lambda)}$  is non-amenable. Again we separate two cases:

- Assume that  $M_{\text{Link}(\Lambda)}$  is non-amenable and diffuse. Then  $M_{\text{Link}(\text{Link}(\Lambda))} = M_\Lambda$  is atomic by (3). But then  $zA \prec_{M_\Gamma} M_\Lambda$  with  $zA$  diffuse leads to a contradiction;
- Assume that  $M_{\text{Link}(\Lambda)}$  is non-amenable and not diffuse. Then by (2) also  $M_\Lambda$  is not diffuse and so  $M_\Gamma = M_\Lambda \overline{\otimes} M_{\text{Link}(\Lambda)}$  is not diffuse and thus strongly solid.

So we are left with the case that  $M_\Lambda$  and  $M_{\text{Link}(\Lambda)}$  are amenable. In this case,  $M_\Gamma = M_{\Lambda_{\text{emb}}} = M_\Lambda \overline{\otimes} M_{\text{Link}(\Lambda)}$  is amenable and hence  $M_\Gamma$  is strongly solid.

*Remainder of the proof of the main theorem in the situation that Case (1) and Case (2) never occur.* We first recall that if we can find a single vertex  $v$  as above such that we are in case (1) or (2) then the proof is finished. Otherwise for any vertex  $v \in \Gamma$  we are in case

(3). So  $zP$  is amenable relative to  $M_{\text{Link}(v)}$  inside  $M_\Gamma$ . As  $\bigcap_{v \in \Gamma} \text{Link}(v) \subseteq \bigcap_{v \in \Gamma} \Gamma \setminus \{v\} = \emptyset$  we obtain by iteratively using Theorem 5.4.8 that  $zP$  is amenable relative to  $\bigcap_{v \in V} M_{\text{Link}(v)} = \mathbb{C}$ , that is  $zP$  is amenable. So  $z = 0$  and we conclude again that  $P$  is amenable.  $\square$

### 5.6.2. CLASSIFYING STRONG SOLIDITY IN SPECIFIC CASES

We show that in many concrete cases that one can verify whether or not a graph product  $M_\Gamma$  is strongly solid. Theorem 5.6.7 tells us how to decide whether  $M_\Gamma$  is strongly solid. For this we need to know for each vertex  $v$  whether or not  $M_v$  is strongly solid. Furthermore, we need to know for each subgraph  $\Lambda \subseteq \Gamma$  whether or not  $M_\Lambda$  is atomic, diffuse, or non-amenable. We observe that in concrete cases we can verify whether  $M_\Lambda$  is diffuse, atomic or non-amenable. Indeed, atomicity is classified in Proposition 5.6.6. Furthermore, amenability was classified in [Cha+24]. Moreover, in [Cha+24] diffuseness was classified under the condition that each vertex algebra  $M_v$  contains a unitary element of trace 0, i.e. a Haar unitary. This in particular applies to the case where  $M_v$  is either diffuse or a group von Neumann algebra. We state these results here.

**Proposition 5.6.8** (Proposition 6.3 of [Cha+24]). *Let  $\Gamma$  be a simple graph. For  $v \in \Gamma$  let  $M_v$  ( $\neq \mathbb{C}$ ) be a von Neumann algebra with normal faithful state  $\varphi_v$ . Then the graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \varphi_v)$  is amenable if and only if the following conditions hold:*

1. *Each vertex von Neumann algebra  $M_v, v \in \Gamma$  is amenable;*
2. *If  $v \neq w \in \Gamma$  share no edge, then  $\dim M_v = \dim M_w = 2$  and  $\text{Link}(\{v, w\}) = \Gamma \setminus \{v, w\}$ .*

**Proposition 5.6.9** (Theorem E of [Cha+24]). *Let  $(M_\Gamma, \tau_\Gamma) = *_{v \in \Gamma} (M_v, \tau_v)$  be a graph product of tracial von Neumann algebras over a finite graph  $\Gamma$ . Assume that each  $M_v, v \in \Gamma$  contains a unitary  $u_v$  with  $\tau_v(u_v) = 0$ . Then  $M_\Gamma$  is diffuse if either (a) there is  $v \in \Gamma$  with  $M_v$  diffuse; (b)  $\Gamma$  is not a complete graph.*

In case not every vertex von Neumann algebra contain a unitary of trace 0 the situation becomes more subtle and the analysis becomes significantly more intricate. However, if the vertex von Neumann algebras are 2-dimensional then the results in [Gar16], [RS23], [CKL21] again yield a classification of diffuseness (and amenability) of graph products.

**Definition 5.6.10.** *Suppose that  $M_{v,q_v}, q_v \in (0, 1]$  is the 2-dimensional Hecke algebra which is the  $*$ -algebra spanned by the unit  $1_v$  and an element  $T_{v,q_v}$  satisfying the Hecke relation*

$$(T_{v,q_v} - q_v^{\frac{1}{2}})(T_{v,q_v} + q_v^{-\frac{1}{2}}) = 0, \quad T_{v,q_v}^* = T_{v,q_v}.$$

*Define the tracial state  $\tau_v$  by setting  $\tau_v(T_{v,q_v}) = 0$  and  $\tau_v(1_v) = 1$ . For a finite graph  $\Gamma$  and  $\mathbf{q} := (q_v)_{v \in \Gamma} \in (0, 1]^\Gamma$  we let  $M_{\Gamma,\mathbf{q}} = *_{v \in \Gamma} (M_v, \tau_{v,q_v})$  be the graph product von Neumann algebra which is called the right-angled Hecke von Neumann algebra.*

**Remark 5.6.11.** Note that  $(M_{v,q_v}, \tau_v)$  is isomorphic to  $\mathbb{C}^2$  with tracial state  $\tau_\alpha(x \oplus y) := \alpha x + (1 - \alpha)y$  with  $\alpha = \frac{1}{2} \left( 1 + \sqrt{1 - \frac{4}{p_v(q)^2 + 4}} \right)$  where  $p_v(q) := \frac{q_v - 1}{\sqrt{q_v}} \in (-\infty, 0]$ . Hence a general 2-dimensional von Neumann algebra with a (necessarily tracial) faithful state is of the form  $(M_{v,q_v}, \tau_q)$  and Hecke algebras correspond to a general graph product of 2-dimensional von Neumann algebras.

Let  $L$  be the graph with 3 points and no edges and  $L^+$  be the graph with 3 points and 1 edge between two of the points.

**Theorem 5.6.12** (Theorem A of [RS23], Theorem 6.2 of [CKL21]). *Let  $\Gamma$  be a finite graph and  $\mathbf{q} := (q_v)_{v \in \Gamma} \in (0, 1]^\Gamma$ . Then*

1. *The Hecke von Neumann algebra  $M_{\Gamma, \mathbf{q}}$  is not diffuse if and only if the sum  $\sum_{\mathbf{w} \in \mathcal{W}_\Gamma} q_{\mathbf{w}}$ , converges where  $q_{\mathbf{w}} = q_{w_1} \dots q_{w_n}$  and  $\mathbf{w} = w_1 \dots w_n$  reduced;*
2.  *$M_{\Gamma, \mathbf{q}}$  is non-amenable if and only if  $\mathcal{W}_\Gamma$  is non-amenable if and only if  $L$  or  $L^+$  is a subgraph of  $\Gamma$ .*

Hence, by Theorem 5.6.7 and Proposition 5.6.6 and Theorem 5.6.12 the classification of strongly solid right-angled Hecke von Neumann algebras is complete. Partial results toward this classification had been obtained earlier in [Cas20] and in Chapter 4. We state the following result for the specific case of group von Neumann algebras.

### Graphs $K_{2,3}$ and $K_{2,3}^+$



Figure 5.1: We depict the graph  $K_{2,3}$  and the graph  $K_{2,3}^+$ .

**Theorem 5.6.13.** *Let  $\mathcal{W}_\Gamma$  be a right-angled Coxeter group. The following are equivalent:*

1. *The von Neumann algebra  $\mathcal{L}(\mathcal{W}_\Gamma)$  is strongly solid.*
2. *The Coxeter group  $\mathcal{W}_\Gamma$  does not contain  $\mathbb{Z} \times \mathbb{F}_2$  as a subgroup.*
3. *The graph  $\Gamma$  does not contain  $K_{2,3}$  nor  $K_{2,3}^+$  as a subgraph (see Figure 5.1).*

*Proof.* (1)  $\implies$  (2) Suppose  $\mathcal{W}_\Gamma$  contains  $\mathbb{Z} \times \mathbb{F}_2$  as a subgroup, we show that  $\mathcal{L}(\mathcal{W}_\Gamma)$  is not strongly solid. Note that  $\mathcal{L}(\mathbb{Z}) \bar{\otimes} \mathcal{L}(\mathbb{F}_2) \subseteq \mathcal{L}(\mathcal{W}_\Gamma)$  is a von Neumann subalgebra. We see that the subalgebra  $\mathcal{L}(\mathbb{Z}) \subseteq \mathcal{L}(\mathcal{W}_\Gamma)$  is amenable (since it is commutative) and is diffuse, while  $\text{Nor}_{\mathcal{L}(\mathcal{W}_\Gamma)}(\mathcal{L}(\mathbb{Z}))''$  is non-amenable as it contains  $\mathcal{L}(\mathbb{F}_2)$ . This shows  $\mathcal{L}(\mathcal{W}_\Gamma)$  is not strongly solid.

(2)  $\implies$  (3) Suppose  $\Gamma$  contains  $K_{2,3}$  or  $K_{2,3}^+$  as a subgraph. Then there are distinct vertices  $a, b, c, d, e \in \Gamma$  such that:  $a, b \notin \text{Link}(c)$ ,  $d \notin \text{Link}(e)$  and  $d, e \in \text{Link}(\{a, b, c\})$ . Write  $\mathbb{F}_2 = \langle g_1, g_2 \rangle$ . Put  $k = 10$ . It can be seen that the map  $\phi: \mathbb{Z} \times \mathbb{F}_2 \rightarrow \mathcal{W}_\Gamma$  given by  $\phi((0, g_1)) = (ac)^k$ ,  $\phi((0, g_2)) = (bc)^k$  and  $\phi((1, 0)) = de$  extends to an injective group homomorphism. Thus  $\mathcal{W}_\Gamma$  contains  $\mathbb{Z} \times \mathbb{F}_2$  as a subgroup.

(3)  $\implies$  (1) Suppose  $\Gamma$  does not contain  $K_{2,3}$  or  $K_{2,3}^+$  as a subgraph. Put  $M_v = \mathcal{L}(\mathbb{Z}/2\mathbb{Z})$  with the canonical trace  $\tau_v$  so that  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v) = \mathcal{L}(\mathcal{W}_\Gamma)$ . We show  $M_\Gamma$  is strongly solid by showing that all conditions in Theorem 5.6.7 are satisfied. Clearly  $M_v$  is strongly solid for  $v \in \Gamma$ . We show the other conditions are also satisfied. Let  $\Lambda \subseteq \Gamma$  for which  $M_\Lambda (= \mathcal{L}(\mathcal{W}_\Lambda))$  is non-amenable. From Proposition 5.6.8 it follows that  $\Lambda$  contains three distinct elements  $a, b, c$  such that  $a, b \notin \text{Link}(c)$ . We claim that  $\text{Link}_\Gamma(\Lambda)$  is complete. Indeed, if  $\Lambda$  contains two distinct vertices  $d, e$  that share no edge, then the graph  $\{a, b, c, d, e\}$  is either isomorphic to  $K_{2,3}$  or to  $K_{2,3}^+$  which is a contradiction. Thus  $\text{Link}_\Gamma(\Lambda)$  is complete and so  $\text{Link}_\Gamma(\Lambda)$  is finite. Hence  $M_{\text{Link}(\Lambda)}$  is atomic (and not diffuse). This shows the conditions of Theorem 5.6.7 are satisfied. Thus  $\mathcal{L}(\mathcal{W}_\Gamma)$  is strongly solid.  $\square$

## 5.7. CLASSIFICATION OF PRIMENESS FOR GRAPH PRODUCTS

We start by recalling the definition of primeness.

**Definition 5.7.1.** A  $\text{II}_1$ -factor  $M$  is called *prime* if it can not factorize as a tensor product  $M = M_1 \bar{\otimes} M_2$  with  $M_1, M_2$  diffuse.

We study primeness for graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  of tracial von Neumann algebras  $M_v$ . In Section 5.7.1 we prove Theorem 5.7.4 which characterizes primeness for graph products of  $\text{II}_1$ -factors. In Section 5.7.2 we use this to prove Theorem 5.7.5 concerning irreducible components in rigid graph products. Moreover, we prove Theorem 5.7.6 which establishes UPF results for the class  $\mathcal{C}_{\text{Rigid}}$ . Last, in Section 5.7.3 we extend the primeness characterization from Theorem 5.7.4 to a larger class of graph products.

### 5.7.1. PRIMENESS RESULTS FOR GRAPH PRODUCTS OF $\text{II}_1$ -FACTORS

We prove Lemma 5.7.2 which we use in Lemma 5.7.3 to give sufficient conditions for a graph product to be either prime or amenable. For graph products of  $\text{II}_1$ -factors we then characterize primeness in Theorem 5.7.4

**Lemma 5.7.2.** *Let  $\Gamma$  be a finite graph that is irreducible. For  $v \in \Gamma$  let  $M_v (\neq \mathbb{C})$  be a von Neumann algebra with a normal faithful trace  $\tau_v$ . Suppose  $N \subseteq M_\Gamma$  is a diffuse von Neumann subalgebra. The following are equivalent:*

1.  $N \not\prec_{M_\Gamma} M_{\Gamma_0}$  for any strict subgraph  $\Gamma_0 \subsetneq \Gamma$ ;
2.  $\text{Nor}_{M_\Gamma}(N)'' \not\prec_{M_\Gamma} M_{\Gamma_0}$  for any strict subgraph  $\Gamma_0 \subsetneq \Gamma$ .

*Proof.* As  $N \subseteq \text{Nor}_{M_\Gamma}(N)''$ , it is clear that (1)  $\implies$  (2). We will show that (2)  $\implies$  (1).

As  $N \subseteq M_\Gamma$  is a subalgebra, we have that  $N \prec_{M_\Gamma} M_\Gamma$ . Therefore, there is a (minimal) subgraph  $\Lambda \subseteq \Gamma$  such that  $N \prec_{M_\Gamma} M_\Lambda$  and  $N \not\prec_{M_\Gamma} M_{\tilde{\Lambda}}$  for all strict subgraphs  $\tilde{\Lambda} \subsetneq \Lambda$ . By Proposition 5.4.13 (2) we obtain that  $\text{Nor}_{M_\Gamma}(N)'' \prec_{M_\Gamma} M_{\Lambda_{\text{emb}}}$  where  $\Lambda_{\text{emb}} = \Lambda \cup \text{Link}(\Lambda)$ . Now by our assumption this implies that  $\Lambda_{\text{emb}} = \Gamma$ . Now, as  $\Gamma$  is irreducible and  $\Gamma = \Lambda \cup \text{Link}(\Lambda)$  we have that  $\Lambda$  or  $\text{Link}(\Lambda)$  is empty. As  $N \not\prec_{M_\Gamma} \mathbb{C}1_{M_\Gamma}$  (since  $N$  is diffuse) and  $N \prec_{M_\Gamma} M_\Lambda$  we must have that  $\Lambda$  is non-empty, and thus that  $\text{Link}(\Lambda)$  is empty. Thus  $\Lambda = \Gamma$ , and this proves the statement.  $\square$

**Lemma 5.7.3.** *Let  $\Gamma$  be a finite, irreducible graph with  $|\Gamma| \geq 2$ . For  $v \in \Gamma$  let  $M_v$  ( $\neq \mathbb{C}$ ) be a von Neumann algebra with a normal faithful trace  $\tau_v$ . Suppose the graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  is a  $\text{II}_1$ -factor and  $M_\Gamma \not\prec_{M_\Gamma} M_{\Gamma_0}$  for any strict subgraph  $\Gamma_0 \subsetneq \Gamma$ . Then  $M_\Gamma$  is prime or amenable.*

*Proof.* Suppose that  $M_\Gamma$  is not prime, we show it is amenable. As  $M_\Gamma$  is not prime, we can write  $M_\Gamma = N_1 \bar{\otimes} N_2$  with  $N_1, N_2$  both diffuse. We observe that  $\text{Nor}_{M_\Gamma}(N_1)'' = M_\Gamma$ . Therefore, using our assumption on  $M_\Gamma$  and applying Lemma 5.7.2 we obtain that  $N_1 \not\prec_{M_\Gamma} M_{\Gamma_0}$  for any strict subgraph  $\Gamma_0 \subsetneq \Gamma$ .

As  $N_2$  is diffuse it contains a diffuse amenable von Neumann subalgebra  $A \subseteq N_2$ . Now observe that  $\text{Nor}_{M_\Gamma}(A)''$  contains  $N_1$  and hence  $\text{Nor}_{M_\Gamma}(A)'' \not\prec_{M_\Gamma} M_{\Gamma_0}$  for any strict subgraph  $\Gamma_0 \subsetneq \Gamma$ . Thus, again by Lemma 5.7.2 we obtain that  $A \not\prec_{M_\Gamma} M_{\Gamma_0}$  for any strict subgraph  $\Gamma_0 \subsetneq \Gamma$ .

Let  $v \in \Gamma$  and put  $\Gamma_1 := \text{Star}(v)$  and  $\Gamma_2 := \Gamma \setminus \{v\}$ . We can write

$$M_\Gamma = M_{\Gamma_1} *_{M_{\text{Link}(v)}} M_{\Gamma_2}. \quad (5.33)$$

As  $A$  is amenable relative to  $M_{\Gamma_1}$  inside  $M_\Gamma$  (as  $A$  is amenable), we obtain using Theorem 5.6.3 that at least one of the following holds:

1.  $A \prec_{M_\Gamma} M_{\text{Link}(v)}$ ;
2.  $\text{Nor}_{M_\Gamma}(A)'' \prec_{M_\Gamma} M_{\Gamma_i}$  for some  $i \in \{1, 2\}$ ;
3.  $\text{Nor}_{M_\Gamma}(A)''$  is amenable relative to  $M_{\text{Link}(v)}$  inside  $M_\Gamma$ .

Now as  $\Gamma_1, \Gamma_2$  and  $\text{Link}(v)$  are strict subgraphs of  $\Gamma$  (as  $\Gamma$  is irreducible and  $|\Gamma| \geq 2$ ), we obtain that only option (3) is possible. Thus  $\text{Nor}_{M_\Gamma}(A)''$  is amenable relative to  $M_{\text{Link}(v)}$  inside  $M_\Gamma$ . Note that  $v \in \Gamma$  was chosen arbitrarily. Thus, applying Theorem 5.4.8 repeatedly, and using that  $\bigcap_{v \in \Gamma} \text{Link}(v) = \emptyset$ , we obtain that  $\text{Nor}_{M_\Gamma}(A)''$  is amenable relative to  $M_\emptyset (= \mathbb{C}1_{M_\Gamma})$  inside  $M_\Gamma$ , i.e.  $\text{Nor}_{M_\Gamma}(A)''$  is amenable. Hence the subalgebra  $N_1 \subseteq \text{Nor}_{M_\Gamma}(A)''$  is amenable as well. Interchanging the roles of  $N_1$  and  $N_2$  we obtain that  $N_2$  is also amenable, and hence  $M_\Gamma = N_1 \bar{\otimes} N_2$  is amenable.  $\square$

We characterize primeness for graph products of  $\text{II}_1$ -factors.

**Theorem 5.7.4.** *Let  $\Gamma$  be a finite graph of size  $|\Gamma| \geq 2$ . For each  $v \in \Gamma$  let  $M_v$  be a  $\text{II}_1$ -factor. Then the graph product  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  is prime if and only if  $\Gamma$  is irreducible.*

*Proof.* Take the simple graph  $\Gamma$  with  $|\Gamma| \geq 2$  and the  $\text{II}_1$ -factors  $(M_v, \tau_v)$  for  $v \in \Gamma$ . By [CF17, Theorem 1.2] the von Neumann algebra  $M_\Gamma$  is a factor. Furthermore, by Proposition 5.4.14 we have that  $M_\Gamma \not\prec_{M_\Gamma} M_{\Gamma_0}$  for any strict subgraph  $\Gamma_0 \subsetneq \Gamma$ . Suppose that  $\Gamma$  is irreducible. Then by applying Lemma 5.7.3 we obtain that  $M_\Gamma$  is prime or amenable. Since  $\Gamma$  is irreducible and has size  $|\Gamma| \geq 2$  we obtain that  $\Gamma$  is not complete. We then see by Proposition 5.6.8 that  $M_\Gamma$  is non-amenable. Thus  $M_\Gamma$  is prime, which shows one direction. Now suppose  $\Gamma$  is reducible, so that we can decompose  $\Gamma = \Gamma_1 \cup \Gamma_2$  with  $\Gamma_1, \Gamma_2 \subseteq \Gamma$  non-empty and such that  $\text{Link}(\Gamma_1) = \Gamma_2$ . But then we can decompose  $M_\Gamma = M_{\Gamma_1} \bar{\otimes} M_{\Gamma_2}$  as a tensor product and again by [CF17, Theorem 1.2]  $M_{\Gamma_1}$  and  $M_{\Gamma_2}$  are  $\text{II}_1$ -factors. Thus  $M_\Gamma$  is not prime.  $\square$

### 5.7.2. UNIQUE PRIME FACTORIZATION RESULTS

We proof Theorem 5.7.5 which strengthens the statement of Theorem 5.5.19 by showing for irreducible components  $\Gamma_0$  that  $M_{\Gamma_0}$  is isomorphic to an amplification of  $N_{\alpha(\Gamma_0)}$ . We then use this result to proof Theorem 5.7.6 which establishes UPF results for the class  $\mathcal{C}_{\text{Rigid}}$ .

**Theorem 5.7.5.** *Given a finite rigid graph  $\Gamma$ . For each  $v \in \Gamma$  let  $M_v \in \mathcal{C}_{\text{Vertex}}$ . Let  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  be the graph product. Suppose  $M_\Gamma = *_{w \in \Lambda} (N_w, \tau_w)$ , with another finite rigid graph  $\Lambda$  and other von Neumann algebras  $N_w \in \mathcal{C}_{\text{Vertex}}$ . Let  $\alpha : \Gamma \rightarrow \Lambda$  be the graph isomorphism from Theorem 5.5.19. Then for any irreducible component  $\Gamma_0 \subseteq \Gamma$ ,  $M_{\Gamma_0}$  is isomorphic to an amplification of  $N_{\alpha(\Gamma_0)}$ .*

*Proof.* We observe that  $M'_{\Gamma \setminus \Gamma_0} \cap M_\Gamma = M_{\Gamma_0}$  is non-amenable. Hence, by Theorem 5.5.15 we obtain a subgraph  $\Lambda_0 \subseteq \Lambda$  such that  $M_{\Gamma \setminus \Gamma_0} <_{M_\Gamma} N_{\Lambda_0}$  and  $\text{Link}_\Lambda(\Lambda_0) \neq \emptyset$ . Choose  $\tilde{\Lambda}_0 \subseteq \Lambda_0$  minimal with the property that  $M_{\Gamma \setminus \Gamma_0} <_{M_\Gamma} N_{\tilde{\Lambda}_0}$ . We show  $\tilde{\Lambda}_0 = \alpha(\Gamma \setminus \Gamma_0)$ . By Proposition 5.4.13(2) we have  $N_\Lambda = M_\Gamma = \text{Nor}_{M_\Gamma}(M_{\Gamma \setminus \Gamma_0})'' <_{M_\Gamma} N_{\Lambda_{\text{emb}}}$  where  $\Lambda_{\text{emb}} = \tilde{\Lambda}_0 \cup \text{Link}_\Lambda(\tilde{\Lambda}_0)$ . By Proposition 5.4.14 we conclude  $\Lambda_{\text{emb}} = \Lambda$ . We note for  $v \in \Gamma \setminus \Gamma_0$  that  $N_{\alpha(v)} <_{M_\Gamma} M_{\Gamma \setminus \Gamma_0}$  and  $M_{\Gamma \setminus \Gamma_0} <_{M_\Gamma}^s N_{\tilde{\Lambda}_0}$  by Lemma 5.5.17(2). Hence by Lemma 5.5.17(1) we obtain  $N_{\alpha(v)} <_{M_\Gamma} N_{\tilde{\Lambda}_0}$ . Thus  $\alpha(\Gamma \setminus \Gamma_0) \subseteq \tilde{\Lambda}_0$  by Proposition 5.4.14. Put  $S = \tilde{\Lambda}_0 \cap \alpha(\Gamma_0)$ . Then

$$S \cup \text{Link}_{\alpha(\Gamma_0)}(S) = (\tilde{\Lambda}_0 \cup \text{Link}_\Lambda(S)) \cap \alpha(\Gamma_0) \supseteq (\tilde{\Lambda}_0 \cup \text{Link}_\Lambda(\tilde{\Lambda}_0)) \cap \alpha(\Gamma_0) = \alpha(\Gamma_0).$$

Since the graph  $\alpha(\Gamma_0)$  is irreducible, we conclude that  $S = \emptyset$  or  $S = \alpha(\Gamma_0)$ . Now, if  $S = \alpha(\Gamma_0)$  then  $\alpha(\Gamma_0) \subseteq \tilde{\Lambda}_0$ , so that  $\Lambda = \alpha(\Gamma_0) \cup \alpha(\Gamma \setminus \Gamma_0) \subseteq \tilde{\Lambda}_0$ . But since  $\tilde{\Lambda}_0 \subseteq \Lambda_0 \subseteq \Lambda$  this implies  $\Lambda_0 = \Lambda$ , which contradicts the fact that  $\text{Link}_\Lambda(\Lambda_0) \neq \emptyset$ . We conclude that  $S = \emptyset$  and thus  $\tilde{\Lambda}_0 = \alpha(\Gamma \setminus \Gamma_0)$ .

We have obtained  $M_{\Gamma \setminus \Gamma_0} <_{M_{\Gamma_0}} N_{\alpha(\Gamma \setminus \Gamma_0)}$ . Taking relative commutants, by Lemma 5.5.16, we get  $N_{\alpha(\Gamma_0)} <_{M_\Gamma} M_{\Gamma_0}$ . Since we are dealing with  $\text{II}_1$ -factors, these embeddings are also with expectation, i.e.  $N_{\alpha(\Gamma_0)} \leq_{M_\Gamma} M_{\Gamma_0}$  as in [HI17, Definition 4.1]. Thus, since  $M_\Gamma = M_{\Gamma_0} \bar{\otimes} M_{\Gamma \setminus \Gamma_0}$  we obtain by [HI17, Lemma 4.13] non-zero projections  $p, q \in M_\Gamma$  and a partial isometry  $v \in M_\Gamma$  with  $v^* v = p$  and  $v v^* = q$  and a subfactor  $P \subseteq q N_{\alpha(\Gamma_0)} q$  so that

$$q N_{\alpha(\Gamma_0)} q = v M_{\Gamma_0} v^* \bar{\otimes} P \quad \text{and} \quad v M_{\Gamma \setminus \Gamma_0} v^* = P \bar{\otimes} q N_{\alpha(\Gamma \setminus \Gamma_0)} q.$$

By Theorem 5.7.4 we have that  $N_{\alpha(\Gamma_0)}$  is prime. Hence  $q N_{\alpha(\Gamma_0)} q$  is prime. Thus, since  $v M_{\Gamma_0} v^*$  is a  $\text{II}_1$ -factor, we obtain that  $P$  is a type  $\text{I}_n$  factor for some  $n \in \mathbb{N}$ . We conclude that  $N_{\alpha(\Gamma_0)}$  is isomorphic to some amplification of  $M_{\Gamma_0}$ .  $\square$

**Theorem 5.7.6.** *Any von Neumann algebra  $M \in \mathcal{C}_{\text{Rigid}}$  have a prime factorization inside  $\mathcal{C}_{\text{Rigid}}$ , i.e.*

$$M = M_1 \bar{\otimes} \cdots \bar{\otimes} M_m, \tag{5.34}$$

for some  $m \geq 1$  and prime factors  $M_1, \dots, M_m \in \mathcal{C}_{\text{Rigid}}$ .

Suppose there is another prime factorization of  $M$  inside  $\mathcal{C}_{\text{Rigid}}$ , i.e.

$$M = N_1 \bar{\otimes} \cdots \bar{\otimes} N_n, \tag{5.35}$$

for another  $n \geq 1$  and other prime factors  $N_1, \dots, N_n \in \mathcal{C}_{\text{Rigid}}$ . Then  $m = n$  and there is a permutation  $\sigma$  of  $\{1, \dots, m\}$  such that  $M_i$  is isomorphic to some amplification of  $N_{\sigma(i)}$ .

*Proof.* Since  $M \in \mathcal{C}_{\text{Rigid}}$ , we can write  $M = *_{v, \Gamma}(M_v, \tau_v)$  for some finite rigid graph  $\Gamma$  and some  $M_v \in \mathcal{C}_{\text{Vertex}}$  for  $v \in \Gamma$ . Let  $\Gamma_1, \dots, \Gamma_m$  be the irreducible components of  $\Gamma$ . Let  $\Pi = \{1, \dots, m\}$  be the complete graph with  $m$  vertices and put  $M_i = M_{\Gamma_i}$  for  $i \in \Pi$ . Then since  $\Gamma = \Gamma_\Pi$  we have by Remark 5.2.4 that  $M = *_{v, \Gamma}(M_v, \tau_v) = *_{i, \Pi}(*_{v, \Gamma_i}(M_v, \tau_v)) = M_1 \bar{\otimes} \dots \bar{\otimes} M_m$ . Now, for  $i \in \Pi$  we have by Theorem 5.7.4 that  $M_i$  is prime since  $\Gamma_i$  is irreducible. Note furthermore that  $\Gamma_i$  is rigid by Remark 5.2.6 and hence  $M_i \in \mathcal{C}_{\text{Rigid}}$ .

Now since  $N_i \in \mathcal{C}_{\text{Rigid}}$  for  $i \in \{1, 2, \dots, n\}$ , we can write  $N_i = *_{v, \Lambda_i}(N_v, \tau_v)$  for some non-empty, finite, rigid graph  $\Lambda_i$ . We note that  $\Lambda_i$  is irreducible by Theorem 5.7.4 since  $N_i$  is prime. Let  $\Pi' = \{1, \dots, n\}$  be a complete graph with  $n$  vertices and put  $\Lambda := \Lambda_{\Pi'}$  which is rigid by Lemma 5.2.5. Then by Remark 5.2.4 we have  $M = N_1 \bar{\otimes} \dots \bar{\otimes} N_n = *_{i, \Pi'}(*_{v, \Lambda_i}(N_v, \tau_v)) = *_{v, \Lambda}(N_v, \tau_v)$ . Hence, we can apply Theorem 5.5.19 to obtain a graph isomorphism  $\alpha : \Gamma \rightarrow \Lambda$ . We note that  $\Lambda_1, \dots, \Lambda_n$  are the irreducible components of  $\Lambda$  and that  $\Gamma_1, \dots, \Gamma_m$  are the irreducible components of  $\Gamma$ . Since  $\alpha$  is a graph isomorphism, this implies that  $m = n$  and that there is a permutation  $\sigma$  of  $\{1, \dots, m\}$  such that  $\alpha(\Gamma_i) = \Lambda_{\sigma(i)}$ . Now, for  $1 \leq i \leq m$  we obtain by Theorem 5.5.19 a real number  $0 < t_i < \infty$  such that  $M_i = M_{\Gamma_i} \simeq N_{\alpha(\Gamma_i)}^{t_i} = N_{\Lambda_{\sigma(i)}}^{t_i} = N_{\sigma(i)}^{t_i}$ .  $\square$

*Remark 5.7.7.* In Fig. 5.2 we give an example of a von Neumann algebra for which we obtain a unique prime factorization. This example was not yet covered by [HI17, Theorem A] since the graph  $\Gamma$  is not complete. The example is also not covered by [CSS18, Theorem 6.16] in case the vertex von Neumann algebras  $M_v \in \mathcal{C}_{\text{Vertex}}$  are not known to be group von Neumann algebras. Examples of such  $M_v$  can be found as von Neumann algebras of free orthogonal quantum groups [VV07] or  $q$ -Gaussian algebras of finite dimensional Hilbert spaces and  $q \in (-1, 1)$  sufficiently far away from 0, see [Bor+23, Remark 4.5] which is essentially proved in [Kuz23]. We emphasize that it is not known whether such von Neumann algebras are group von Neumann algebras; we do not make the more definite claim that they cannot be isomorphic to group von Neumann algebras.

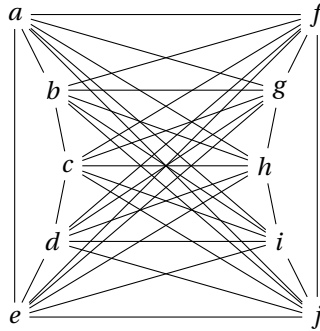


Figure 5.2: An example of a rigid graph  $\Gamma$  is depicted. Let  $M_v \in \mathcal{C}_{\text{Vertex}}$  for  $v \in \Gamma$ . Then Theorem 5.7.6 obtains for  $M_\Gamma = *_{v, \Gamma}(M_v, \tau_v)$  the unique prime factorization  $M_\Gamma = M_{\Gamma_1} \bar{\otimes} M_{\Gamma_2}$ , where  $\Gamma_1 = \{a, b, c, d, e\}$  and  $\Gamma_2 = \{f, g, h, i, j\}$  are the irreducible components of  $\Gamma$ .

### 5.7.3. PRIMENESS RESULTS FOR OTHER GRAPH PRODUCTS

In case the von Neumann algebras  $M_v$  are not (all) type  $\text{II}_1$ -factors, it is interesting to know whether the condition  $M_\Gamma \not\prec_{M_\Gamma} M_{\Gamma_0}$  for any strict subgraph  $\Gamma_0 \subsetneq \Gamma$ , is satisfied. In Lemma 5.7.10 we will give sufficient conditions for the property to hold. To prove this, we need the following lemma.

**Lemma 5.7.8.** *Let  $\Gamma$  be a graph and for  $v \in \Gamma$  let  $(M_v, \tau_v)$  be a tracial von Neumann algebra. Let  $\Lambda \subseteq \Gamma$  be a subgraph and let  $\mathbf{u} \in \mathcal{W}_\Gamma \setminus \mathcal{W}_\Lambda$ . Let  $\mathbf{v}, \mathbf{v}' \in \mathcal{W}_\Gamma$  be such that every letters at the start of  $\mathbf{v}$  respectively  $\mathbf{v}'$  does not commute with any letters at the end of  $\mathbf{u}^{-1}$  respectively  $\mathbf{u}$ . Let  $\mathbf{w}, \mathbf{w}' \in \mathcal{W}_\Gamma$  with  $|\mathbf{w}| \leq |\mathbf{v}|$  and  $|\mathbf{w}'| \leq |\mathbf{v}'|$ . Then*

$$\mathbb{E}_{M_\Lambda}(axb) = 0 \quad \text{for } a \in \dot{M}_{\mathbf{w}}, x \in \dot{M}_{\mathbf{v}^{-1}\mathbf{u}\mathbf{v}'}, b \in \dot{M}_{\mathbf{w}'}. \quad (5.36)$$

*Proof.* Let  $\mathbf{u}, \mathbf{v}, \mathbf{v}', \mathbf{w}, \mathbf{w}'$  be given as stated. Observe by the assumptions on  $\mathbf{v}$  and  $\mathbf{v}'$  that in particular  $\mathbf{v}^{-1}\mathbf{u}\mathbf{v}'$  is reduced. Denote

$$\mathcal{H}(\mathbf{u}) := \bigoplus_{\mathbf{w}_0 \in \mathcal{W}(\mathbf{u})} \dot{\mathcal{H}}_{\mathbf{w}_0}, \quad \mathbf{M}(\mathbf{u}) := \bigoplus_{\mathbf{w}_0 \in \mathcal{W}(\mathbf{u})} \dot{M}_{\mathbf{w}_0}. \quad (5.37)$$

Observe for  $y_1 \in \lambda(\mathbf{M}(\mathbf{u}^{-1}))$ ,  $y_2 \in \dot{M}_{\mathbf{u}}$  and  $y_3 \in \lambda(\mathbf{M}(\mathbf{u}))$  that if we denote  $y := y_1^* y_2 y_3$  and write  $y = \sum_{\mathbf{w} \in \mathcal{W}_\Gamma} y_{\mathbf{w}}$  where  $y_{\mathbf{w}} \in \dot{M}_{\mathbf{w}}$ , then we have that  $y_{\mathbf{w}} = 0$  whenever  $\mathbf{w}$  does not contain  $\mathbf{u}$  as a subword. Thus, in particular  $\mathbb{E}_{M_\Lambda}(y_1^* y_2 y_3) = 0$ . We will apply this to obtain the result.

Let  $x \in \dot{M}_{\mathbf{v}^{-1}\mathbf{u}\mathbf{v}'}$  be a pure tensor, and let  $x_1 \in \dot{M}_{\mathbf{v}}$ ,  $x_2 \in \dot{M}_{\mathbf{u}}$  and  $x_3 \in \dot{M}_{\mathbf{v}'}$  be s.t.  $\lambda(x) = \lambda(x_1)^* \lambda(x_2) \lambda(x_3)$ . Let  $a \in \dot{M}_{\mathbf{w}}$  and  $b \in \dot{M}_{\mathbf{w}'}$ . Let  $\omega \in \mathcal{S}_{\mathbf{v}'}$ , then we can write  $\omega = (\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3)$  for some  $\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3 \in \mathcal{W}_\Gamma$  with  $\mathbf{v}' = \mathbf{v}'_1 \mathbf{v}'_2 \mathbf{v}'_3$ .

By Lemma 3.1.4 we have  $\eta_\omega := \lambda_{(\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3)}(x_3) b \Omega \in \dot{\mathcal{H}}_{\mathbf{v}'_0}$  where  $\mathbf{v}'_0 = \mathbf{v}'_1 \mathbf{v}'_3 \mathbf{w}'$ . We show that  $\eta_\omega \in \mathcal{H}(\mathbf{u})$ . In particular, we can assume that  $\eta_\omega$  is non-zero, so that  $\mathbf{w}'$  starts with  $(\mathbf{v}'_3)^{-1} \mathbf{v}'_2$  and  $\mathbf{v}'_0$  starts with  $\mathbf{v}'_1 \mathbf{v}'_2$ . If  $\mathbf{v}'_1 \mathbf{v}'_2 = e$  then  $\mathbf{v}'_3 = \mathbf{v}'$ , so that  $|\mathbf{v}'_3| + |\mathbf{v}'_3 \mathbf{w}'| = |\mathbf{w}'| \leq |\mathbf{v}'| = |\mathbf{v}'_3|$  and therefore  $\mathbf{v}'_3 \mathbf{w}' = e$ . We then conclude that  $\eta_\omega \in \dot{\mathcal{H}}_e \subseteq \mathcal{H}(\mathbf{u})$ . Thus, suppose  $\mathbf{v}'_1 \mathbf{v}'_2 \neq e$ . Then  $\mathbf{v}'_1 \mathbf{v}'_2 \mathbf{w}'_0 (= \mathbf{v}'_0)$  starts with a letter  $v'_0$  at the start of  $\mathbf{v}'$ . Now, by the assumption on  $\mathbf{v}'$  we obtain that  $v'_0$  does not commute with elements at the end of  $\mathbf{u}$ . This implies that  $\mathbf{u}\mathbf{v}'_0$  is reduced and so  $\eta_\omega \in \mathcal{H}(\mathbf{u})$ . Now, as  $\lambda(x_3) \lambda(b) \Omega = \sum_{\omega \in \mathcal{S}_{\mathbf{v}'}} \lambda_\omega(x_3) \lambda(b) \Omega \in \mathcal{H}(\mathbf{u})$  we obtain that  $y_3 := \lambda(x_3) \lambda(b) \in \mathbf{M}(\mathbf{u})$ . In a similar way we obtain  $y_1 := \lambda(x_1) \lambda(a)^* \in \mathbf{M}(\mathbf{u}^{-1})$ . Hence, putting  $y_2 := \lambda(x_2)$  we obtain that  $\mathbb{E}_{M_\Lambda}(\lambda(a) \lambda(x) \lambda(b)) = \mathbb{E}_{M_\Lambda}(y_1^* y_2 y_3) = 0$ . The result now follows by density of  $\lambda(\dot{M}_{\mathbf{z}}) \subseteq \dot{M}_{\mathbf{z}}$  for  $\mathbf{z} \in \mathcal{W}_\Gamma$ .  $\square$

**Corollary 5.7.9.** *Let  $\Gamma$  be a graph,  $\Lambda \subseteq \Gamma$  be a subgraph and let  $\mathbf{u} \in \mathcal{W}_\Gamma \setminus \mathcal{W}_\Lambda$ . Let  $\mathbf{v}, \mathbf{v}' \in \mathcal{W}_\Gamma$  be such that every letters at the start of  $\mathbf{v}$  respectively  $\mathbf{v}'$  does not commute with any letters at the end of  $\mathbf{u}^{-1}$  respectively  $\mathbf{u}$ . Let  $\mathbf{w}, \mathbf{w}' \in \mathcal{W}_\Gamma$  with  $|\mathbf{w}| \leq |\mathbf{v}|$  and  $|\mathbf{w}'| \leq |\mathbf{v}'|$ . Then  $\mathbf{w}\mathbf{v}^{-1}\mathbf{u}\mathbf{v}'\mathbf{w}' \notin \mathcal{W}_\Lambda$ .*

*Proof.* For  $v \in \Gamma$  let  $M_v := \mathcal{L}(\mathbb{Z}/2\mathbb{Z})$ , so that  $M_\Gamma = \mathcal{L}(\mathcal{W}_\Gamma)$ . Take  $a = \lambda_{\mathbf{w}}$ ,  $x = \lambda_{\mathbf{v}^{-1}\mathbf{u}\mathbf{v}'}$  and  $b = \lambda_{\mathbf{w}'}$ . Then Lemma 5.7.8 shows that  $\mathbb{E}_{M_{\Gamma \setminus \Lambda}}(\lambda_{\mathbf{w}\mathbf{v}^{-1}\mathbf{u}\mathbf{v}'\mathbf{w}}) = \mathbb{E}_{M_\Lambda}(axb) = 0$ . This means that  $\mathbf{w}_1 \mathbf{v}^{-1} \mathbf{u} \mathbf{v}' \mathbf{w}_2 \notin \mathcal{W}_\Lambda$ .  $\square$

**Lemma 5.7.10.** *Let  $\Gamma$  be a finite graph of size  $|\Gamma| \geq 3$  such that for any  $v \in \Gamma$ ,  $\text{Star}(v) \neq \Gamma$ . For  $v \in \Gamma$  let  $(M_v, \tau_v)$  be a von Neumann algebra with a normal faithful trace. Suppose for any  $v \in \Gamma$  there is a unitary  $u_v \in M_\Gamma$  with  $\tau_v(u_v) = 0$ . Then  $M_\Gamma \not\prec_{M_\Gamma} M_\Lambda$  for any strict subgraph  $\Lambda \subsetneq \Gamma$ .*

*Proof.* First observe that the Coxeter group  $\mathcal{W}_\Gamma$  is icc since  $|\Gamma| \geq 3$  and  $\text{Star}(v) \neq \Gamma$  for all  $v \in \Gamma$ . Now let  $\Lambda \subsetneq \Gamma$  be a strict subgraph and fix  $v \in \Gamma \setminus \Lambda$ . As the conjugacy class  $\{v^{-1}\nu v : \nu \in \mathcal{W}_\Gamma\}$  is infinite, we can for  $n \in \mathbb{N}$  choose  $\mathbf{v}_n \in \mathcal{W}_\Gamma$  such that  $|\mathbf{v}_n^{-1}\nu\mathbf{v}_n| \geq 2n+1$ . If a letter  $s$  commuting with  $\nu$  is at the start of  $\mathbf{v}_n$  then we can replace  $\mathbf{v}_n$  with  $\tilde{\mathbf{v}}_n := s\mathbf{v}_n \in \mathcal{W}_\Gamma$  which does not start with  $s$  and is such that  $\tilde{\mathbf{v}}_n^{-1}\nu\tilde{\mathbf{v}}_n = \mathbf{v}_n^{-1}\nu\mathbf{v}_n$ . Repeating the argument, we may thus assume that every letter at the start of  $\mathbf{v}_n$  does not commute with  $\nu$ . Then in particular  $\mathbf{v}_n^{-1}\nu\mathbf{v}_n$  is reduced and  $|\mathbf{v}_n| \geq n$ . Let  $(v_{n,1}, \dots, v_{n,l_n})$  be a reduced expression for  $\mathbf{v}_n^{-1}\nu\mathbf{v}_n$  and define  $u_n := u_{v_{n,1}} \dots u_{v_{n,l_n}} \in \dot{M}_{\mathbf{v}_n^{-1}\nu\mathbf{v}_n}$ . Then  $u_n$  is a unitary and for any  $\mathbf{w}, \mathbf{w}' \in \mathcal{W}_\Gamma$  with  $|\mathbf{w}|, |\mathbf{w}'| \leq n$  and  $a \in \dot{M}_{\mathbf{w}}, b \in \dot{M}_{\mathbf{w}'}$ , we have by Lemma 5.7.8 that

$$\mathbb{E}_{M_\Lambda}(au_nb) = 0. \quad (5.38)$$

We take  $x, y \in M_\Gamma$  and  $\epsilon > 0$ . We can choose  $x_0 \in M_\Gamma$  of the form  $x_0 = \sum_{i=1}^{K_1} x_i$  for some  $K_1 \geq 1$ ,  $x_i \in \dot{M}_{\mathbf{w}_i}$  with some  $\mathbf{w}_i \in \mathcal{W}_\Gamma$ , and with  $\|y\| \cdot \|x_0 - x\|_2 \leq \epsilon$ . We can now also choose  $y_0 \in M_\Gamma$  of the form  $y_0 = \sum_{i=1}^{K_2} y_i$  for some  $K_2 \geq 1$ ,  $y_i \in \dot{M}_{\mathbf{w}'_i}$ , with some  $\mathbf{w}'_i \in \mathcal{W}_\Gamma$  and  $\|x_0\| \cdot \|y_0 - y\|_2 \leq \epsilon$ . Put  $l_1 := \max_{1 \leq i \leq K_1} |\mathbf{w}_i|$ ,  $l_2 := \max_{1 \leq i \leq K_2} |\mathbf{w}'_i|$  and  $l = \max\{l_1, l_2\}$ . Let  $n \geq l$  so that by (5.38) and linearity we have  $\mathbb{E}_{M_\Lambda}(x_0 u_n y_0) = 0$  and hence

$$\mathbb{E}_{M_\Lambda}(x u_n y) = \mathbb{E}_{M_\Lambda}((x - x_0) u_n y) + \mathbb{E}_{M_\Lambda}(x_0 u_n (y - y_0)). \quad (5.39)$$

Furthermore,

$$\|(x - x_0) u_n y\|_2 \leq \|x - x_0\|_2 \cdot \|u_n y\| \leq \epsilon, \quad (5.40)$$

$$\|x_0 u_n (y - y_0)\|_2 \leq \|x_0 u_n\| \cdot \|y - y_0\|_2 \leq \epsilon. \quad (5.41)$$

Thus, as the conditional expectation  $\mathbb{E}_{M_\Lambda}$  is  $\|\cdot\|_2$ -decreasing (this follows from the Schwarz inequality [Pau02, Proposition 3.3] as  $\mathbb{E}_{M_\Lambda}$  is trace-preserving and u.c.p.), we obtain for  $n \geq l$  that

$$\|\mathbb{E}_{M_\Lambda}(x u_n y)\|_2 \leq 2\epsilon. \quad (5.42)$$

This shows for any  $x, y \in M_\Gamma$  that  $\|\mathbb{E}_{M_\Lambda}(x u_n y)\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ . By Definition 2.1.2(2) this means that  $M_\Gamma \not\prec_{M_\Gamma} M_\Lambda$ .  $\square$

**Theorem 5.7.11.** *Let  $\Gamma$  be a irreducible finite graph of size  $|\Gamma| \geq 3$  and for  $v \in \Gamma$ , let  $M_v$  ( $\neq \mathbb{C}$ ) be a von Neumann algebra with a normal faithful trace  $\tau_v$  such that there exists a unitary  $u_v \in M_v$  with  $\tau_v(u_v) = 0$ . Then  $M_\Gamma$  is a prime factor.*

*Proof.* It follows from [Cha+24, Theorem E] and our assumptions that  $M_\Gamma$  is a  $\text{II}_1$ -factor. Furthermore, by Lemma 5.7.10 we have that  $M_\Gamma \not\prec_{M_\Gamma} M_\Lambda$  for any strict subgraph  $\Lambda \subsetneq \Gamma$ . Hence, by Lemma 5.7.3 we obtain that  $M_\Gamma$  is either prime or amenable. Since  $\Gamma$  is irreducible and  $|\Gamma| \geq 3$  it follows from Proposition 5.6.8 that  $M_\Gamma$  is non-amenable. Hence,  $M_\Gamma$  is prime.  $\square$

**Theorem 5.7.12.** *Let  $\Gamma$  be a finite graph. For  $v \in \Gamma$ , let  $M_v (\neq \mathbb{C})$  be a von Neumann algebra with a normal faithful trace  $\tau_v$  and assume that  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  is a  $\text{II}_1$ -factor. Then  $M_\Gamma$  is prime if and only if there is an irreducible component  $\Lambda \subseteq \Gamma$  for which  $M_\Lambda$  is prime and  $M_{\Gamma \setminus \Lambda}$  is finite-dimensional.*

*Proof.* Suppose there is an irreducible component  $\Lambda \subseteq \Gamma$  for which  $M_\Lambda$  is prime and  $\dim M_{\Gamma \setminus \Lambda} < \infty$ . Then  $M_\Gamma = M_\Lambda \bar{\otimes} M_{\Gamma \setminus \Lambda}$  is prime as it is a matrix amplification of  $M_\Lambda$ .

For the other direction, suppose that  $M_\Gamma$  is a prime factor. Denote

$$\Lambda := \{v \in \Gamma : \text{Star}_\Gamma(v) \neq \Gamma \text{ or } \dim M_v = \infty\}.$$

If  $w \in \Gamma \setminus \Lambda$  then  $\text{Star}_\Gamma(w) = \Gamma$  and  $\dim M_w < \infty$ , so  $w \in \text{Link}_\Gamma(\Lambda)$ . Hence  $\text{Link}_\Gamma(\Lambda) = \Gamma \setminus \Lambda$  and so  $M_\Gamma = M_\Lambda \bar{\otimes} M_{\Gamma \setminus \Lambda}$ . Now, since  $\Gamma \setminus \Lambda$  is complete, and since  $\dim M_v < \infty$  for  $v \in \Gamma \setminus \Lambda$  we have that  $M_{\Gamma \setminus \Lambda}$  is finite-dimensional. Hence, since  $M_\Gamma$  is a prime factor also  $M_\Lambda$  is a prime factor.

We now show that the graph  $\Lambda$  is irreducible so that from  $\text{Link}_\Gamma(\Lambda) = \Gamma \setminus \Lambda$  it follows that  $\Lambda$  is an irreducible component of  $\Gamma$ . Suppose there is a non-empty subgraph  $\Lambda_1 \subseteq \Lambda$  s.t.  $\Lambda_2 := \Lambda \setminus \Lambda_1$  is non-empty and  $\text{Link}_\Lambda(\Lambda_1) = \Lambda_2$ . We show a contradiction. We can write  $M_\Lambda = M_{\Lambda_1} \bar{\otimes} M_{\Lambda_2}$ . Hence, by primeness of the factor  $M_\Lambda$  there is  $i \in \{1, 2\}$  s.t.  $\dim M_{\Lambda_i} < \infty$ . Let  $v \in \Lambda_i$ . Since  $\dim M_{\Lambda_i} < \infty$  we have  $\dim M_v < \infty$ . Hence, since  $v \in \Lambda$  we have by definition of  $\Lambda$  that  $\text{Star}_\Gamma(v) \neq \Gamma$ . Let  $w \in \Gamma \setminus \text{Star}_\Gamma(v)$ . Then  $\text{Star}_\Gamma(w) \neq \Gamma$  so that  $w \in \Lambda$ . Furthermore,  $w \notin \text{Link}_\Gamma(v)$  so that  $w \notin \text{Link}_\Lambda(\Lambda_i) = \Lambda \setminus \Lambda_i$ , i.e.  $w \in \Lambda_i$ . Hence, since the vertices  $v, w$  in  $\Lambda_i$  share no edge we have  $\dim M_{\Lambda_i} = \infty$ , which is a contradiction. Thus  $\Lambda$  is irreducible.  $\square$

## 5.8. CLASSIFICATION OF FREE INDECOMPOSABILITY FOR GRAPH PRODUCTS

In this section we study free-indecomposability for graph product of  $\text{II}_1$ -factors. In Theorem 5.8.1 we characterize for graph products of  $\text{II}_1$ -factors (with separable predual) when they can decompose as tracial free products of  $\text{II}_1$ -factors. In Theorem 5.8.2 we combine this result with Theorem 5.5.19 to show unique free product decompositions for von Neumann algebras in the class  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ . Hereafter, we show that Theorem 5.8.1 and Theorem 5.8.2 really cover new examples. Indeed, in Proposition 5.8.3 we give sufficient conditions for a graph product to not possess a Cartan-subalgebra, which in Remark 5.8.4 we use to give examples of freely indecomposable von Neumann algebras  $M \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  that are not in the class  $\mathcal{C}_{\text{anti-free}}$  from [HU16]. In Remark 5.8.5 we show that the unique free product decomposition from Theorem 5.8.2 also covers new examples.

**Theorem 5.8.1.** *Let  $\Gamma$  be a finite graph of size  $|\Gamma| \geq 2$ , and for  $v \in \Gamma$  let  $(M_v, \tau_v)$  be tracial  $\text{II}_1$ -factor with separable predual. Then the graph product  $M_\Gamma := *_{v \in \Gamma} (M_v, \tau_v)$  can decompose as a tracial free product  $M_\Gamma = (M_1, \tau_1) * (M_2, \tau_2)$  of  $\text{II}_1$ -factors  $M_1, M_2$  if and only if  $\Gamma$  is not connected.*

*Proof.* Let  $\Gamma$  and  $(M_v, \tau_v)_{v \in \Gamma}$  be given. If  $\Gamma$  is not connected then for any connected component  $\Gamma_0$  of  $\Gamma$  we have  $M_\Gamma = (M_{\Gamma_0}, \tau_1) * (M_{\Gamma \setminus \Gamma_0}, \tau_2)$ , which shows one direction.

For the other direction suppose that  $\Gamma$  is connected. Assume  $M_\Gamma = (M_1, \tau_1) * (M_2, \tau_2)$  for some  $\text{II}_1$ -factors  $M_1, M_2$ . Fix  $v \in \Gamma$  and by [OP04, Proposition 13] let  $N_0 \subseteq M_v$  be an amenable  $\text{II}_1$ -subfactor with  $N'_0 \cap M_\Gamma = M'_v \cap M_\Gamma$ . Then  $N_0$  is amenable relative to  $M_i$  inside  $M$  for  $i = 1, 2$ . Therefore, by Theorem 5.6.3 one of the following holds true:

1.  $N_0 \prec_{M_\Gamma} \mathbb{C}1_{M_\Gamma}$ ;
2.  $\text{Nor}_{M_\Gamma}(N_0)'' \prec_{M_\Gamma} M_i$  for some  $1 \leq i \leq 2$ ;
3.  $\text{Nor}_{M_\Gamma}(N_0)''$  is amenable relative to  $\mathbb{C}1_{M_\Gamma}$  inside  $M_\Gamma$ .

Since  $N_0$  is diffuse, we can not have (1).

We show that (2) is also not satisfied. Suppose  $\text{Nor}_{M_\Gamma}(N_0)'' \prec_{M_\Gamma} M_1$ . Since  $N_0$  is diffuse, it can not embed in any type I factor. It follows that  $\text{Nor}_{M_\Gamma}(N_0)'' \not\prec_{M_\Gamma} M_\emptyset$ . Therefore, since  $N_0$  and  $N'_0 \cap M_\Gamma = M'_v \cap M_\Gamma = M_{\text{Link}(v)}$  are factors we obtain by Theorem 5.4.16 that  $u^* \text{Nor}_{M_\Gamma}(N_0)'' u \subseteq M_1$  for some unitary  $u \in M_\Gamma$ .

Now take  $w \in \Gamma$  arbitrarily. Since  $\Gamma$  is connected there is a path  $P$  from  $v$  to  $w$ , i.e.  $P = (v_0, v_1, \dots, v_n)$  for some  $n \geq 0$  and vertices  $v_0, v_1, \dots, v_n \in \Gamma$  such that  $v_i \in \text{Link}(v_{i-1})$  for  $1 \leq i \leq n$  and such that  $v_0 = v$  and  $v_n = w$ . As  $|\Gamma| \geq 2$  we can moreover choose this path such that it has length  $n \geq 1$ .

For  $i \in \{1, \dots, n\}$  put  $N_i := M_{v_i}$ . Then, as  $v_i \in \text{Link}(v_{i-1})$  we obtain  $N_i \subseteq \text{Nor}_{M_\Gamma}(N_{i-1})''$ . Since  $u^* \text{Nor}_{M_\Gamma}(N_0)'' u \subseteq M_1$  we obtain  $u^* N_1 u \subseteq M_1$ . Then since  $u^* N_1 u \not\prec_{M_\Gamma} M_\emptyset$  (since  $u^* N_1 u$  is diffuse) we obtain by Proposition 5.4.13(1b) that  $\text{Nor}_{M_\Gamma}(u^* N_1 u)'' \subseteq M_1$ . Note that  $\text{Nor}_{M_\Gamma}(u^* N_1 u) = u^* \text{Nor}_{M_\Gamma}(N_1) u$  so that  $\text{Nor}_{M_\Gamma}(u^* N_1 u)'' = u^* \text{Nor}_{M_\Gamma}(N_1)'' u$ . Thus we obtain  $u^* \text{Nor}_{M_\Gamma}(N_1)'' u \subseteq M_1$ . Continuing in this way we obtain  $u^* \text{Nor}_{M_\Gamma}(N_i)'' u \subseteq M_1$  for all  $0 \leq i \leq n$ . Thus, in particular  $u^* M_w u \subseteq u^* \text{Nor}_{M_\Gamma}(N_{n-1})'' u \subseteq M_1$ . Since  $w$  was arbitrary, we obtain that  $M_w \subseteq u M_1 u^*$  for each  $w \in \Gamma$ . But this implies  $M_\Gamma = (\bigcup_{w \in \Gamma} M_w)'' \subseteq u M_1 u^*$ . Hence  $M_\Gamma = M_1$ , which is a contradiction. We conclude that  $\text{Nor}_{M_\Gamma}(N_0) \not\prec_{M_\Gamma} M_1$ . By symmetry also  $\text{Nor}_{M_\Gamma}(N_0) \not\prec_{M_\Gamma} M_2$ . We obtain that (2) is not satisfied.

We conclude that (3) is satisfied, i.e.  $\text{Nor}_{M_\Gamma}(N_0)''$  is amenable. Hence  $M_{\text{Link}(v)} \subseteq \text{Nor}_{M_\Gamma}(N_0)''$  is amenable as well. Therefore, by Proposition 5.6.8 we obtain that  $\text{Link}(v)$  is a clique and that  $M_w$  is amenable for any  $w \in \text{Link}(v)$ . We observe that  $v \in \Gamma$  was arbitrary, thus for each vertex  $z \in \Gamma$  its  $\text{Link}(z)$  is a clique. Since  $\Gamma$  is connected, it follows that  $\Gamma$  is a complete graph. Moreover, for any  $v \in \Gamma$  choose  $z \in \Gamma \setminus \{v\}$  we have  $M_v \subseteq M_{\text{Link}(z)}$ , which shows that  $M_v$  is amenable. Hence  $M_\Gamma$  is a tensor product of amenable  $\text{II}_1$ -factors and so  $M_\Gamma$  is amenable. But the amenable  $\text{II}_1$ -factor can not decompose as a free product of type  $\text{II}_1$ -factors. This gives a contradiction and we conclude that  $M_\Gamma$  can not decompose as free product of  $\text{II}_1$ -factors.  $\square$

**Theorem 5.8.2.** *Any von Neumann algebra  $M \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  can decompose as tracial free product inside  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$ :*

$$M = M_1 * \dots * M_m, \quad (5.43)$$

for some  $m \geq 1$  and  $\text{II}_1$ -factors  $M_1, \dots, M_m \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  that can not decompose as any tracial free product of  $\text{II}_1$ -factors.

Furthermore, suppose  $M$  has another free product decomposition:

$$M = N_1 * \cdots * N_n,$$

for some  $n \geq 1$  and other  $\Pi_1$ -factors  $N_1, \dots, N_n \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  that can not decompose as tracial free product of  $\Pi_1$ -factors. Then  $m = n$  and there is a permutation  $\sigma$  of  $\{1, \dots, m\}$  such that for each  $i$ ,  $N_i$  is unitarily conjugate to  $M_{\sigma(i)}$  in  $M$ .

*Proof.* Since  $M \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  we can write  $M = M_\Gamma = *_{v, \Gamma} (M_v, \tau_v)$  for some rigid graph  $\Gamma$  of size  $|\Gamma| \geq 2$  and some  $\Pi_1$ -factors  $M_v \in \mathcal{C}_{\text{Vertex}}$ . Let  $\Gamma_1, \dots, \Gamma_m$  be the connected components of  $\Gamma$ , which are rigid by Remark 5.2.6. We let  $\Pi = \{1, \dots, m\}$  be the graph with  $m$  vertices and no edges. We claim that  $|\Gamma_i| \geq 2$  for all  $i \in \Pi$ . Indeed, if  $m = 1$  then  $\Pi = \{1\}$  and  $\Gamma_1 = \Gamma$  so that  $|\Gamma_i| = |\Gamma| \geq 2$  for all  $i \in \Pi$ . On the other hand, if  $m \geq 2$  then  $\text{Link}_\Pi(\text{Link}_\Pi(i)) = \Pi \neq \{i\}$  for all  $i \in \Pi$ , so it follows from Lemma 5.2.5 and rigidity of  $\Gamma_\Pi \simeq \Gamma$  that  $|\Gamma_i| \geq 2$  for all  $i \in \Pi$ .

We denote  $M_i := M_{\Gamma_i} \in \mathcal{C}_{\text{Rigid}}$  for  $i \in \Pi$ . By Theorem 5.5.19 and rigidity of  $\Gamma_i$  and the fact that  $|\Gamma_i| \geq 2$  it follows that  $M_i \notin \mathcal{C}_{\text{Vertex}}$ . Furthermore, since  $\Gamma_i$  is connected we obtain by Theorem 5.8.1 that  $M_i$  can not decompose as tracial free product of  $\Pi_1$ -factors. By Remark 5.2.4 we conclude that  $M_\Gamma = *_{v, \Gamma} (M_v, \tau_v) = *_{i, \Pi} (M_{\Gamma_i}, \tau_i) = M_1 * \cdots * M_m$  which shows (5.43).

Now let  $n \geq 1$  and let  $N_1, \dots, N_n \in \mathcal{C}_{\text{Rigid}}$  be  $\Pi_1$ -factors that can not decompose as tracial free product of  $\Pi_1$ -factors. Since  $N_i \in \mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{Vertex}}$  we can write  $N_i = *_{z, \Lambda_i} (N_{(i, z)}, \tau_{(i, z)})$  where  $\Lambda_i$  is a rigid graph and  $(N_{(i, z)})_{z \in \Lambda_i}$  are  $\Pi_1$ -factors in  $\mathcal{C}_{\text{Vertex}}$ . Observe for  $1 \leq i \leq n$  that  $|\Lambda_i| \geq 2$  since  $N_i \notin \mathcal{C}_{\text{Vertex}}$  and that  $\Lambda_i$  is connected by Theorem 5.8.1 since  $N_i$  can not decompose as tracial free product of  $\Pi_1$ -factors. Let  $\Pi' = \{1, \dots, n\}$  be the graph with  $n$  vertices and no edges. Then by Remark 5.2.4 we have:

$$M = N_1 * \cdots * N_n = *_{i, \Pi'} (*_{v, \Lambda_i} (N_{(i, v)}, \tau_{(i, v)})) \simeq *_{w, \Lambda_{\Pi'}} (N_w, \tau_w) = N_{\Lambda_{\Pi'}}.$$

Then since  $\Lambda_{\Pi'}$  is rigid by Lemma 5.2.5, we obtain by Theorem 5.5.19 that  $\Lambda_{\Pi'} \simeq \Gamma$ . The connected components of  $\Lambda_{\Pi'}$  respectively  $\Gamma$  are  $\Lambda_1, \dots, \Lambda_n$  respectively  $\Gamma_1, \dots, \Gamma_m$ . Hence  $n = m$ . Moreover, Theorem 5.5.19 asserts, for some permutation  $\sigma$  of  $\{1, \dots, m\}$ , that  $\tilde{N}_{\Lambda_i} (= N_i)$  is unitarily conjugate to  $M_{\Gamma_{\sigma(i)}} (= M_{\sigma(i)})$  in  $M_\Gamma$ .  $\square$

We give sufficient conditions for absence of Cartan-subalgebras in graph products. We note that in [Cas20] absence of Cartan was studied for right-angled Hecke algebras and that in [CE23] absence of Cartan was fully characterized for von Neumann algebras associated to graph products of groups.

For a non-empty connected graph  $\Gamma$  we define its radius as

$$\text{Radius}(\Gamma) := \inf_{s \in \Gamma} \sup_{t \in \Gamma} \text{Dist}_\Gamma(s, t), \quad (5.44)$$

where  $\text{Dist}_\Gamma(s, t)$  denotes the minimal length of a path in  $\Gamma$  from  $s$  to  $t$ . Furthermore, we set  $\text{Radius}(\Gamma) = 0$  if  $\Gamma$  is empty and set  $\text{Radius}(\Gamma) = \infty$  if  $\Gamma$  is not connected.

**Proposition 5.8.3.** *Let  $\Gamma$  be a graph with  $\text{Radius}(\Gamma) \geq 3$  and for  $v \in \Gamma$  let  $M_v$  be a  $\Pi_1$ -factor with normal faithful trace  $\tau_v$ . Then  $M_\Gamma = *_{v, \Gamma} (M_v, \tau_v)$  does not possess a Cartan-subalgebra.*

*Proof.* Suppose  $M_\Gamma$  has a Cartan subalgebra  $A \subseteq M_\Gamma$ . Fix  $v \in \Gamma$ . Then  $M_\Gamma = M_{\text{Star}(v)} * M_{\text{Link}(v)}$   $M_{\Gamma \setminus \{v\}}$ . Since  $A$  is amenable, one of the statements of Theorem 5.6.3 must hold. Since  $\text{Radius}(\Gamma) \geq 3$ , we have  $\text{Star}(v) \neq \Gamma$ . Hence  $\text{Nor}_{M_\Gamma}(A)'' = M_\Gamma \not\prec_{M_\Gamma} M_{\text{Star}(v)}$  and  $\text{Nor}_{M_\Gamma}(A)'' = M_\Gamma \not\prec_{M_\Gamma} M_{\Gamma \setminus \{v\}}$  by Proposition 5.4.14. Thus we must have that  $A \prec_{M_\Gamma} M_{\text{Link}(v)}$  or that  $\text{Nor}_{M_\Gamma}(A)''$  is amenable relative to  $M_{\text{Link}(v)}$  inside  $M_\Gamma$ . Suppose that  $A \prec_{M_\Gamma} M_{\text{Link}(v)}$  then since  $A \not\prec_{M_\Gamma} M_\emptyset$  we obtain by Proposition 5.4.13(1b) that  $M_\Gamma = \text{Nor}_{M_\Gamma}(A)'' \subseteq M_{\Lambda_{\text{emb}}}$  where  $\Lambda_{\text{emb}} = \text{Link}(v) \cup \bigcup_{w \in \text{Link}(v)} \text{Link}(w)$ . We see that  $\text{Radius}(\Lambda_{\text{emb}}) \leq 2$  (indeed take as center  $v$ ). Hence, since  $\text{Radius}(\Gamma) \geq 3$  we have  $M_\Gamma = \text{Nor}_{M_\Gamma}(A)'' \subseteq M_{\Lambda_{\text{emb}}} \subsetneq M_\Gamma$ , a contradiction. We conclude that  $\text{Nor}_{M_\Gamma}(A)'' (= M_\Gamma)$  is amenable relative to  $M_{\text{Link}(v)}$  in  $M_\Gamma$ . Since  $v$  was arbitrary we obtain using Theorem 5.4.8 that  $M_\Gamma$  is amenable. This is a contradiction since  $M_\Gamma$  is non-amenable by Proposition 5.6.8 (since  $\text{Radius}(\Gamma) \geq 3$ ). Thus  $M_\Gamma$  does not have a Cartan subalgebra.  $\square$

*Remark 5.8.4.* We argue that we find new classes of finite von Neumann algebras that are freely indecomposable. More precisely we argue that Theorem 5.8.1 covers von Neumann algebras that are not in the class  $\mathcal{C}_{\text{anti-free}}$  from [HU16]. Indeed, let  $\Gamma$  be a graph with  $\text{Radius}(\Gamma) \geq 3$  (hence  $\Gamma$  is irreducible) and for  $v \in \Gamma$  let  $M_v$  be a  $\text{II}_1$ -factor with separable predual and possessing the Haagerup property. Then the  $\text{II}_1$ -factor  $M_\Gamma$  does not lie in the class  $\mathcal{C}_{\text{anti-free}}$  from [HU16]. Indeed, (i)  $M_\Gamma$  is prime by Theorem 5.7.4, (ii)  $M_\Gamma$  is full (so no property Gamma) by [Cha+24, Theorem E], (iii)  $M_\Gamma$  does not have a Cartan subalgebra by Proposition 5.8.3, and (iv)  $M_\Gamma$  has the Haagerup property (so no property (T) by [CJ85, Theorem 3]) by [CF17, Theorem 0.2]. If  $\Gamma$  is moreover connected and rigid and if  $M_v$  lies in  $\mathcal{C}_{\text{Vertex}}$  for each  $v \in \Gamma$ , then  $M_\Gamma$  lies in  $\mathcal{C}_{\text{Rigid}}$  and can not decompose as free product of  $\text{II}_1$ -factors. As a concrete example, take the cyclic graph  $\Gamma = \mathbb{Z}_n$  for some  $n \geq 6$  and for each  $v \in \Gamma$  let  $M_v = \mathcal{L}(\mathbb{F}_2) \in \mathcal{C}_{\text{Vertex}}$  which has the Haagerup property by [BO08, Theorem 12.2.5]. Then  $M_\Gamma$  is a  $\text{II}_1$ -factor in  $\mathcal{C}_{\text{Rigid}} \setminus \mathcal{C}_{\text{anti-free}}$  that can not decompose as a (tracial) reduced free product of  $\text{II}_1$ -factors.

*Remark 5.8.5.* We argue that the unique free product decompositions from Theorem 5.8.2 are not covered by [HU16] nor [DE24b]. Indeed, let  $\Gamma$  be a graph whose connected components  $\Gamma_i$  for  $i = 1, \dots, m$  are of the form  $\mathbb{Z}_{n_i}$  for some  $n_i \geq 6$ . Observe that  $\Gamma$  is rigid. For  $v \in \Gamma$  put  $M_v = \mathcal{L}(\mathbb{F}_2) \in \mathcal{C}_{\text{Vertex}}$ . Then Theorem 5.8.2 asserts the unique free product decomposition  $M_\Gamma = M_{\Gamma_1} * \dots * M_{\Gamma_m}$ . Since the factors  $M_{\Gamma_i}$  for  $i = 1, \dots, m$  are not in the class  $\mathcal{C}_{\text{anti-free}}$ , this result is not covered by [HU16]. Furthermore, we note for  $i = 1, \dots, m$  that the group  $*_{v \in \Gamma_i} \mathbb{F}_2$  is properly proximal by [DE24a, Proposition 3.7] since  $\text{Radius}(\Gamma_i) \geq 3$ . Hence, also [DE24b, Corollary 1.8] does not apply.

## 5.9. GRAPH RADIUS RIGIDITY

In this section we generalize the ideas from the proof of Theorem 5.8.1 and show that we can, in certain cases, retrieve the radius of the graph  $\Gamma$  from the graph product  $M_\Gamma$ . In Section 5.9.1 we introduce the notion of the radius of a von Neumann algebra. Furthermore, we establish good estimates on  $\text{Radius}(M_\Gamma)$  in terms of the radius of  $\Gamma$  whenever the vertex algebras  $M_v$  possess the property strong (AO). In Section 5.9.2 we establish similar estimates when the vertex algebras  $M_v$  are group von Neumann algebras  $\mathcal{L}(G_v)$  of countable icc groups  $G_v$ .

### 5.9.1. RADIUS OF VON NEUMANN ALGEBRAS

We introduce the following definition for a simple graph.

**Definition 5.9.1.** Let  $\Gamma$  be a simple graph and let  $\Lambda \subseteq \Gamma$  be a subgraph. For  $d \in \mathbb{Z}_{\geq 0}$  put

$$B_{\Gamma}(\Lambda; d) = \{v \in \Gamma : \text{Dist}_{\Gamma}(v, w) \leq d \text{ for some } w \in \Lambda\}.$$

which is the closed ball of size  $d$  around  $\Lambda$ . Furthermore, define  $B_{\Gamma}(\Lambda; \infty) = \bigcup_{d \geq 1} B_{\Gamma}(\Lambda; d)$ .

We will now introduce a similar definition for von Neumann algebras.

**Definition 5.9.2.** Let  $M$  be a diffuse von Neumann algebra and  $A \subseteq M$  a diffuse von Neumann subalgebra. For  $d \geq 0$  we define the von Neumann algebra  $B_M(A; d)$  inductively. Put  $B_M(A; 0) = A$  and for  $d \geq 1$  define

$$B_M(A; d) = \left( \bigcup_{\substack{B \subseteq B_M(A; d-1) \\ \text{diffuse vNa}}} \text{Nor}_M(B) \right)''$$

Moreover, we also define

$$B_M(A; \infty) = \left( \bigcup_{d \geq 0} B_M(A; d) \right)''$$

We remark for  $n, m \in \mathbb{Z}_{\geq 0}$  that  $B_M(A; n + m) = B_M(B_M(A; n); m)$ .

Recall that the radius of a graph  $\Gamma$  was defined in (5.44) and note that it is equal to the infimum of all  $d \in \mathbb{Z}_{\geq 0}$  for which there exists a vertex  $v \in \Gamma$  with  $B_{\Gamma}(v; d) = \Gamma$ . In a similar way we can introduce the notion of the radius of a von Neumann algebras.

**Definition 5.9.3.** Let  $M$  be a diffuse von Neumann algebra. We define  $\text{Radius}(M)$  as the infimum of all  $d \in \mathbb{Z}_{\geq 0}$  such that there exists a diffuse, amenable subfactor  $A \subseteq M$  for which  $A' \cap M$  is a non-amenable factor and such that  $B_M(A; d) = M$ .

We remark that the definition of  $\text{Radius}(M)$  would be more natural with the relaxation that  $A$  can be any diffuse amenable von Neumann subalgebra satisfying  $B_M(A; d) = M$ . However, we need the extra restrictions in order to get appropriate lower bounds on  $\text{Radius}(M)$ .

**Proposition 5.9.4.** Let  $\Gamma$  be a finite simple graph and let  $\Lambda \subseteq \Gamma$  be a subgraph. Let  $M_{\Gamma} = *_{v \in \Gamma} (M_v, \tau_v)$  be a graph product of  $\text{II}_1$ -factors with separable preduals. Then

1. For  $d \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$  we have  $B_{M_{\Gamma}}(M_{\Lambda}; d) = M_{B_{\Gamma}(\Lambda; d)}$
2. If  $\Gamma$  is not complete then  $\text{Radius}(M_{\Gamma}) \leq \max\{2, \text{Radius}(\Gamma)\}$ .

*Proof.* (1) The statement holds true for  $d = 0$  since  $B_{M_{\Gamma}}(M_{\Lambda}, 0) = M_{\Lambda} = M_{B_{\Gamma}(\Lambda, 0)}$ . We show the statement for  $d = 1$ . Let  $A \subseteq M_{\Lambda}$  be amenable and diffuse. Then  $A \not\prec_{M_{\Gamma}} \mathbb{C}$ . Let  $\{\Lambda_j\}_{j \in \mathcal{J}}$  be the family  $\{\emptyset\}$ . Then by Proposition 5.4.13(1b) we obtain  $\text{qNor}_{M_{\Gamma}}(A)'' \subseteq M_{\Lambda_{\text{emb}}}$  where  $\Lambda_{\text{emb}} = \Lambda \cup \bigcup_{v \in \Lambda} \text{Link}_{\Gamma}(v) = B_{\Gamma}(\Lambda; 1)$ . Hence  $B_M(M_{\Lambda}; 1) \subseteq M_{B_{\Gamma}(\Lambda; 1)}$ . To show equality, take  $w \in B_{\Gamma}(\Lambda; 1) \setminus \Lambda$  and let  $v \in \Lambda$  such that  $v$  and  $w$  share an edge. Let  $A \subseteq M_v$  be

an amenable and diffuse. Then  $\text{Nor}_{M_\Gamma}(A)'' \supseteq M_{\text{Link}(\nu)} \supseteq M_w$ . Hence,  $M_w \subseteq B_{M_\Gamma}(M_\Lambda; 1)$ . Hence, we obtain equality. Now let  $d \geq 1$  and suppose the statement holds true for  $d - 1$ . Then

$$B_{M_\Gamma}(M_\Lambda; d) = B_{M_\Gamma}(B_{M_\Gamma}(M_\Lambda; d - 1); 1) = B_{M_\Gamma}(M_{B_\Gamma(\Lambda; d-1)}; 1) = M_{B_\Gamma(B_\Gamma(\Lambda; d-1); 1)} = M_{B_\Gamma(\Lambda; d)}$$

This proves the statement by induction for  $d \in \mathbb{N}$ . The statement for  $d = \infty$  follows automatically.

(2) Put  $r = \text{Radius}(\Gamma)$ . We know  $r \geq 1$  and furthermore we may assume  $r < \infty$  since otherwise the statement is trivial. Let  $\nu \in \Gamma$  such that  $B_\Gamma(\nu; r) = \Gamma$ . Observe, since  $\Gamma$  is not complete, that  $\nu$  can be chosen such that  $\text{Link}_\Gamma(\nu)$  is not a clique in  $\Gamma$ . By [OP04, Proposition 13] we may let  $A \subseteq M_\nu$  be a diffuse amenable subfactor for which  $A' \cap M_\Gamma = M'_\nu \cap M_\Gamma = M_{\text{Link}(\nu)}$ . Thus  $A' \cap M_\Gamma$  is a non-amenable factor. We show that  $B_{M_\Gamma}(A; r) = M_\Gamma$ . We see that

$$M_{\text{Link}(\nu)} \subseteq \text{Nor}_{M_\Gamma}(A)'' \subseteq B_{M_\Gamma}(A; 1) \subseteq B_{M_\Gamma}(M_\nu; 1) \subseteq M_{B_\Gamma(\nu; 1)}$$

Hence,

$$M_{B_\Gamma(\text{Link}_\Gamma(\nu); 1)} = B_{M_\Gamma}(M_{\text{Link}(\nu)}; 1) \subseteq B_{M_\Gamma}(B_{M_\Gamma}(A; 1); 1) \subseteq B_{M_\Gamma}(M_{B_\Gamma(\nu; 1)}; 1) = M_{B_\Gamma(\nu; 2)} \quad (5.45)$$

Now, observe that  $B_{M_\Gamma}(A; 2) = B_{M_\Gamma}(B_{M_\Gamma}(A; 1); 1)$  and  $B_\Gamma(\text{Link}_\Gamma(\nu); 1) = B_\Gamma(\nu; 2)$ . If  $r \leq 2$  then  $B_\Gamma(\nu; 2) = \Gamma$  which shows that  $\text{Radius}(M_\Gamma) \leq 2 = \max\{2, r\}$ . Thus assume  $r \geq 2$ . By (5.45) we obtain  $B_\Gamma(A; 2) = M_{B_\Gamma(\nu; 2)}$ . Thus we obtain

$$B_{M_\Gamma}(A; r) = B_{M_\Gamma}(B_{M_\Gamma}(A; 2); r - 2) = B_{M_\Gamma}(M_{B_\Gamma(\nu; 2)}; r - 2) = M_{B_\Gamma(\nu; r)} = M_\Gamma$$

This shows  $\text{Radius}(M_\Gamma) \leq r = \max\{2, r\}$ . □

**Proposition 5.9.5.** *Let  $\Gamma$  be a finite simple graph. Let  $M_\Gamma = *_{\nu \in \Gamma} (M_\nu, \tau_\nu)$  be a graph product of  $\text{II}_1$ -factors  $M_\nu$ . Let  $K \geq 1$  be a constant. Suppose for any amenable diffuse subfactor  $A \subseteq M_\Gamma$  with  $A' \cap M_\Gamma$  a non-amenable factor there is a subgraph  $\Lambda \subseteq \Gamma$  with  $\text{Radius}(B_\Gamma(\Lambda, 1)) \leq K$  such that  $A <_M M_\Lambda$ . Then*

$$\text{Radius}(\Gamma) - K \leq \text{Radius}(M_\Gamma).$$

*Proof.* Denote  $R = \text{Radius}(M_\Gamma)$ . We may assume  $R < \infty$ . Let  $A \subseteq M_\Gamma$  be an amenable, diffuse subfactor for which  $A' \cap M_\Gamma$  is a non-amenable factor and for which  $B_{M_\Gamma}(A; R) = M_\Gamma$ . By assumption  $A <_{M_\Gamma} M_\Lambda$  for some subgraph  $\Lambda \subseteq \Gamma$  with  $\text{Radius}(B_\Gamma(\Lambda; 1)) \leq K$ . Let  $\{\Lambda_j\}_{j \in \mathcal{J}}$  denote the non-empty family  $\{\emptyset\}$ . Then by Theorem 5.4.16 we obtain a unitary  $u \in M_\Gamma$  so that  $u^* A u \subseteq M_{\Lambda_{\text{emb}}}$  where  $\Lambda_{\text{emb}} = B_\Gamma(\Lambda; 1)$ . Hence, for  $d \geq 0$  we obtain

$$u^* B_{M_\Gamma}(A; d) u = B_{M_\Gamma}(u^* A u; d) \subseteq B_{M_\Gamma}(M_{B_\Gamma(\Lambda; 1)}; d) = M_{B_\Gamma(B_\Gamma(\Lambda; 1); d)}$$

Then

$$M_\Gamma = u^* B_{M_\Gamma}(A; R) u \subseteq M_{B_\Gamma(B_\Gamma(\Lambda; 1); R)}$$

so that  $\Gamma = B_\Gamma(B_\Gamma(\Lambda; 1); R)$  Therefore we obtain

$$\text{Radius}(\Gamma) \leq \text{Radius}(B_\Gamma(\Lambda; 1)) + R \leq K + \text{Radius}(M_\Gamma)$$

which completes the proof.  $\square$

**Theorem 5.9.6.** *Let  $\Gamma$  be a finite simple graph that is not complete. Let  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  be a graph product of  $\text{II}_1$ -factors  $M_v$  that satisfy condition strong (AO) and have separable predual. Then*

$$\text{Radius}(\Gamma) - 2 \leq \text{Radius}(M_\Gamma) \leq \max\{2, \text{Radius}(\Gamma)\}$$

*In particular this holds true when  $M_\Gamma$  is a graph products of hyperfinite  $\text{II}_1$ -factors.*

*Proof.* The upper bound is due to Proposition 5.9.4(2). To obtain the lower bound we show that the condition of Proposition 5.9.5 is satisfied with constant  $K = 2$ . Let  $A \subseteq M_\Gamma$  be amenable and diffuse and such that  $A' \cap M_\Gamma$  is non-amenable. By Theorem 5.5.15 we obtain  $A \prec_{M_\Gamma} M_\Lambda$  for some non-empty subgraph  $\Lambda \subseteq \Gamma$  with  $\text{Link}(\Lambda)$  non-empty. Let  $v \in \text{Link}(\Lambda)$ . Then  $\Lambda \subseteq \text{Link}(v)$ . Hence,  $B_\Gamma(\Lambda; 1)$  equals  $B_\Gamma(v, 2)$  and has radius at most 2. This proves the lower bound.  $\square$

*Remark 5.9.7.* We use Theorem 5.9.6 to distinguish certain von Neumann algebras coming from graph products. Indeed, let  $\Gamma$  and  $\Lambda$  be finite, graphs with  $2 \leq \text{Radius}(\Gamma) < \text{Radius}(\Lambda) - 2$ . Let  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v)$  and  $N_\Lambda = *_{v \in \Lambda} (N_v, \tau_v)$  be graph products of factors  $M_v, N_v$  satisfying the conditions from Theorem 5.9.6. Then we obtain

$$\text{Radius}(M_\Gamma) \leq \text{Radius}(\Gamma) < \text{Radius}(\Lambda) - 2 \leq \text{Radius}(N_\Lambda)$$

Thus, in particular  $M_\Gamma \neq N_\Lambda$ .

### 5.9.2. RADIUS ESTIMATES FOR GRAPH PRODUCTS GROUPS

We now show that the statement of Theorem 5.9.6 also holds true when the vertex von Neumann algebras  $M_v$  are group von Neumann algebras  $\mathcal{L}(G_v)$  of countable icc groups (Theorem 5.9.11). We state the following definitions.

**Definition 5.9.8.** *Let  $G$  be a countable discrete group and let  $\mathcal{S}$  be a family of subgroups of  $G$ . Then a subset  $F \subseteq G$  is called small relative to  $\mathcal{S}$  if*

$$F \subseteq \bigcup_{i=1}^k g_i G_i h_i$$

*for some  $k \geq 1$ , groups  $G_1, \dots, G_k \in \mathcal{S}$  and elements  $g_1, \dots, g_k, h_1, \dots, h_k \in G$ .*

**Definition 5.9.9.** *Let  $G$  be a countable discrete group and let  $\mathcal{S}$  be a family of subgroups of  $G$ . Let  $V \subseteq \mathcal{L}(G)$  be a norm bounded subset. We write*

$$V \subseteq_{\text{approx}} \mathcal{L}(\mathcal{S})$$

*if for every  $\epsilon > 0$  there is a subset  $F \subseteq G$  that is small relative to  $\mathcal{S}$  and satisfies for all  $v \in V$  that  $\|v - P_F(v)\|_2 \leq \epsilon$  (here  $P_F : \ell^2(G) \rightarrow \ell^2(F)$  denotes the orthogonal projection).*

The following proposition is similar to [CSS18, Claim 6.15] and follows from the results in [Vae13]. In the proof we write  $(B)_1$  for the closed unit ball of the von Neumann algebra  $B$ .

**Proposition 5.9.10.** *Let  $\Gamma$  be a finite simple graph and for  $v \in \Gamma$  let  $G_v$  be a countable icc group. Let  $G_\Gamma = *_{v \in \Gamma} G_v$  be the graph product and let  $B \subseteq \mathcal{L}(G_\Gamma)$  be a von Neumann subalgebra for which  $B' \cap \mathcal{L}(G_\Gamma)$  is a factor. Let  $\{\Lambda_i\}_{i \in I}$  be a collection of subgraphs of  $\Gamma$  and let  $\Lambda = \bigcap_i \Lambda_i$  be their intersection. If  $B \prec_{\mathcal{L}(G_\Gamma)} \mathcal{L}(G_{\Lambda_i})$  for all  $i$  then  $B \prec_{\mathcal{L}(G_\Gamma)} \mathcal{L}(G_\Lambda)$*

*Proof.* Assume  $B \prec_{\mathcal{L}(G_\Gamma)} \mathcal{L}(G_{\Lambda_i})$  for  $i \in I$ . We show  $B \prec_{\mathcal{L}(G_\Gamma)} \mathcal{L}(G_\Lambda)$ . For  $i \in I$  we can by [Vae13, Lemma 2.5] obtain a non-zero projection  $q_i \in B' \cap \mathcal{L}(G_\Gamma)$  such that for  $\mathcal{S}_i := \{G_{\Lambda_i}\}$  we have

$$(Bq_i)_1 \subset_{\text{approx}} \mathcal{L}(\mathcal{S}_i)$$

Moreover, by [Vae13, Proposition 2.6] we may assume  $q_i \in Z(\text{Nor}_{\mathcal{L}(G_\Gamma)}(B)'')$ . Note that

$$q_i \in Z(\text{Nor}_{\mathcal{L}(G_\Gamma)}(B)'') \cap (B' \cap \mathcal{L}(G_\Gamma)) \subseteq Z(B' \cap \mathcal{L}(G_\Gamma)) = \mathbb{C}1 \quad (5.46)$$

Thus  $q_i = 1$ . Denote

$$\mathcal{S} = \left\{ \bigcap_{i \in I} h_i G_{\Lambda_i} h_i^{-1} \mid h_i \in G_\Gamma \text{ for } i \in I \right\}.$$

From [Vae13, Lemma 2.7] it follows that  $(B)_1 \subset_{\text{approx}} \mathcal{L}(\mathcal{S})$ . Then from [AM15, Proposition 3.4] for each  $(h_i)_{i \in I}$ ,  $h_i \in G_\Gamma$  there is a subgraph  $\Lambda_0 \subseteq \Lambda$  and  $k \in G_\Gamma$  such that

$$\bigcap_{i \in I} h_i G_{\Lambda_i} h_i^{-1} = k G_{\Lambda_0} k^{-1} \subseteq k G_\Lambda k^{-1}.$$

Thus, putting  $\mathcal{S}_0 = \{G_\Lambda\}$  it follows that  $(B)_1 \subseteq_{\text{approx}} \mathcal{L}(\mathcal{S}_0)$  and hence by [Vae13, Lemma 2.5] we obtain  $B \prec_{\mathcal{L}(G_\Gamma)} \mathcal{L}(G_\Lambda)$ .  $\square$

**Theorem 5.9.11.** *Let  $\Gamma$  be a finite simple graph that is not complete. For  $v \in \Gamma$  let  $G_v$  be a countable icc group. Let  $G_\Gamma = *_{v \in \Gamma} G_v$  be the graph product. Then*

$$\text{Radius}(\Gamma) - 2 \leq \text{Radius}(\mathcal{L}(G_\Gamma)) \leq \max\{2, \text{Radius}(\Gamma)\}$$

*Proof.* The upper bound on  $\text{Radius}(\mathcal{L}(G_\Gamma))$  follows immediately from Proposition 5.9.4 since  $\mathcal{L}(G_v)$  is a  $\text{II}_1$ -factor for  $v \in \Gamma$ . To prove the lower bound we show the condition of Proposition 5.9.5 is satisfied with  $K = 2$ . Put  $M_v = \mathcal{L}(G_v)$  and let  $M_\Gamma = *_{v \in \Gamma} (M_v, \tau_v) = \mathcal{L}(G_\Gamma)$  be the graph product. Let  $R \subseteq M_\Gamma$  be an amenable  $\text{II}_1$ -factor for which  $R' \cap M_\Gamma$  is a non-amenable factor. We need to show that  $R \prec_{M_\Gamma} M_\Lambda$  for some  $\Lambda \subseteq \Gamma$  with  $\text{Radius}(B_\Gamma(\Lambda; 1)) \leq 2$ . Let  $I$  be the set of all vertices  $v$  in  $\Gamma$  for which  $\text{Nor}_{M_\Gamma}(R)''$  is amenable relative to  $M_{\text{Link}_\Gamma(v)}$  inside  $M_\Gamma$ . By Theorem 5.4.8 we obtain that  $\text{Nor}_{M_\Gamma}(R)''$  is amenable relative to  $M_{\text{Link}_\Gamma(I)}$  inside  $M_\Gamma$ . Since  $\text{Nor}_{M_\Gamma}(R)''$  is non-amenable (as it contains  $R' \cap M_\Gamma$ ), we obtain that  $\text{Link}_\Gamma(I)$  is non-empty. Let  $w \in \text{Link}_\Gamma(I)$ . Then  $I \subseteq B_\Gamma(w; 1)$  so that  $B_\Gamma(I; 1) \subseteq B_\Gamma(w, 2)$ . Thus since  $w \in B_\Gamma(I; 1)$  we see that  $B_\Gamma(I; 2)$  has radius at most 2.

Now let  $J \subseteq \Gamma$  be the set of all  $v \in \Gamma$  for which  $R \prec_{M_\Gamma} M_{\Gamma \setminus \{v\}}$ . Then since  $R' \cap \mathcal{L}(G_\Gamma)$  is a factor we obtain by Proposition 5.9.10 that  $R \prec_{M_\Gamma} M_{\Gamma \setminus J}$ . Now, if  $\Gamma \setminus J \subseteq I$  then  $R \prec_{M_\Gamma} M_I$  which shows that we may take  $\Lambda = I$ . Thus assume  $\Gamma \setminus J \not\subseteq I$ . Take  $v \in \Gamma \setminus J$  with  $v \notin I$ . We can decompose

$$M_\Gamma = M_{\text{Star}(v)} *_{M_{\text{Link}(v)}} M_{\text{Link}(v)}.$$

Since  $R$  is amenable we get by Theorem 5.6.3 that at least one of the following holds true

1.  $R \prec_{M_\Gamma} M_{\text{Link}(v)}$
2.  $\text{Nor}_{M_\Gamma}(R)'' \prec_{M_\Gamma} M_{\text{Star}(v)}$  or  $\text{Nor}_{M_\Gamma}(R)'' \prec_{M_\Gamma} M_{\Gamma \setminus \{v\}}$ .
3.  $\text{Nor}_{M_\Gamma}(R)''$  is amenable relative to  $M_{\text{Link}(v)}$  inside  $M_\Gamma$ .

Since  $v$  is not in  $I \cup J$  we must have that  $R \prec_{M_\Gamma} M_{\text{Link}(v)}$  or  $\text{Nor}_{M_\Gamma}(R)'' \prec_{M_\Gamma} M_{\text{Star}(v)}$ . Thus in particular we obtain  $R \prec_{M_\Gamma} M_{\text{Star}(v)}$ . Observe that  $B_\Gamma(\text{Star}(v); 1)$  has radius at most 2. Hence we may take  $\Lambda := \text{Star}(v)$ . This finishes the proof.  $\square$

## 5.10. DISCUSSION

We discuss problems concerning rigidity of graph products and state a conjecture. In Theorem 5.5.19 we obtained rigidity for graph products when the vertex algebras  $M_\nu$  lie in the class  $\mathcal{C}_{\text{Vertex}}$  and the graph  $\Gamma$  is rigid. We can not apply Theorem 5.5.19 to graph products  $R_\Gamma = *_{\nu, \Gamma}(R_\nu, \tau_\nu)$  of hyperfinite  $\text{II}_1$ -factors since  $R_\nu \notin \mathcal{C}_{\text{Vertex}}$ . In fact, the result of Theorem 5.5.19 for hyperfinite  $\text{II}_1$ -factors does not even hold true since  $R_\Gamma = R_\Lambda$  for any complete graphs  $\Gamma, \Lambda$  (which are rigid). We are interested to know for what class  $\mathcal{S}$  of graphs we can distinguish  $R_\Gamma$  from  $R_\Lambda$ .

*Problem 5.10.1.* Describe a class of finite graphs  $\mathcal{S}$  such that:

1. Let  $\Gamma, \Lambda \in \mathcal{S}$  and let  $R_\Gamma = *_{\nu, \Gamma}(R_\nu, \tau_\nu)$  and  $R_\Lambda = *_{\nu, \Lambda}(R_\nu, \tau_\nu)$  be tracial graph products of hyperfinite  $\text{II}_1$ -factors. If  $R_\Gamma \simeq R_\Lambda$  then  $\Gamma \simeq \Lambda$ .
2. Let  $\Gamma$  be any finite graph. Then there is a graph  $\Lambda \in \mathcal{S}$  such that the tracial graph products  $R_\Gamma = *_{\nu, \Gamma}(R_\nu, \tau_\nu)$  and  $R_\Lambda = *_{\nu, \Lambda}(R_\nu, \tau_\nu)$  of hyperfinite  $\text{II}_1$ -factors are isomorphic.

Observe that Problem 5.10.1 is very hard. Indeed, for a finite graph  $\Gamma$  with no edges we have by [Dyk94] that  $R_\Gamma = \mathcal{L}(\mathbb{F}_n)$  whenever  $n := |\Gamma| \geq 2$ . Thus, to solve Problem 5.10.1 one would first have to solve the free factor problem. To simplify Problem 5.10.1 we may remove condition (2) and loosely require the class  $\mathcal{S}$  to be sufficiently large. In Remark 5.9.7 we were already able to distinguish graph products  $R_\Gamma$  and  $R_\Lambda$  based on the radius of the graphs  $\Gamma$  and  $\Lambda$ . Furthermore, as we show in the next remark, Theorem 5.5.19 can be used to distinguish certain graph products of hyperfinite  $\text{II}_1$ -factors.

*Remark 5.10.2.* Let  $\Gamma_i$  for  $i = 1, 2$  be a rigid graph. For  $i = 1, 2$  and  $v \in \Gamma_i$  let  $\Lambda_{v,i}$  be a graph of size  $n_{v,i} := |\Lambda_{v,i}| \geq 2$  and with no edges. Let  $\Lambda_{\Gamma_i} = *_{\nu, \Gamma_i} \Lambda_{\nu,i}$  be the graph product graph. Observe that  $R_{\Lambda_{\Gamma_i}} = *_{\nu, \Gamma_i}(R_{\Lambda_{\nu,i}}, \tau) = *_{\nu, \Gamma_i}(\mathcal{L}(\mathbb{F}_{n_{\nu,i}}), \tau)$ . Therefore, if  $R_{\Lambda_{\Gamma_1}} \simeq R_{\Lambda_{\Gamma_2}}$  then by Theorem 5.5.19 we obtain  $\Gamma_1 \simeq \Gamma_2$  since  $\Gamma_i$  is rigid and  $\mathcal{L}(\mathbb{F}_{n_{\nu,i}}) \in \mathcal{C}_{\text{Vertex}}$  for  $i = 1, 2, v \in \Gamma_i$ . This shows that  $R_{\Lambda_{\Gamma_1}} \not\simeq R_{\Lambda_{\Gamma_2}}$  whenever  $\Gamma_1 \not\simeq \Gamma_2$ .

In the following remark we show a difficulty that can arise.

*Remark 5.10.3.* Let  $\Gamma$  be a graph whose two irreducible components  $\Gamma_1$  and  $\Gamma_2$  are graphs with no edges and of size  $|\Gamma_1| = |\Gamma_2| = 3$ . Similar, let  $\Lambda$  be a graph whose two irreducible components  $\Lambda_1$  and  $\Lambda_2$  are graphs with no edges and of size  $|\Lambda_1| = 2$  and  $|\Lambda_2| = 5$ . While  $\Gamma \not\simeq \Lambda$  we see using the amplification formula (1.8) from [R  d94] that

$$R_\Gamma = \mathcal{L}(\mathbb{F}_3) \otimes \mathcal{L}(\mathbb{F}_3) = \mathcal{L}(\mathbb{F}_3)^{\sqrt{2}} \otimes \mathcal{L}(\mathbb{F}_3)^{1/\sqrt{2}} = \mathcal{L}(\mathbb{F}_2) \otimes \mathcal{L}(\mathbb{F}_5) = R_\Lambda$$

Say that a finite graph  $\Gamma$  is *graph product prime (gpp)* if, whenever  $\Gamma \simeq *_{v \in \Pi} \Lambda_v$  for some graph  $\Pi$  and non-empty graphs  $(\Lambda_v)_{v \in \Pi}$ , then either  $|\Pi| = 1$  or  $\Pi \simeq \Gamma$  (but not both). There are many examples of gpp graphs. For example, for  $n \geq 5$  the cyclic graph  $\mathbb{Z}_n$  of size  $|\mathbb{Z}_n| = n$  is gpp. However, the class of gpp graphs is in some sense also restrictive. Indeed, a gpp graph  $\Gamma$  is always connected and irreducible unless  $|\Gamma| = 2$ . We also note that the graph  $\Gamma$  of size  $|\Gamma| = 1$  is not gpp. We state the following problem.

**Problem 5.10.4.** Let  $\Gamma$  and  $\Lambda$  be finite, gpp graphs. Let  $R_\Gamma = *_{v \in \Gamma} (R_v, \tau_v)$  be the graph product of hyperfinite  $\text{II}_1$ -factors and let  $M_\Lambda = *_{v \in \Lambda} (M_v, \tau_v)$  be the graph product of arbitrary  $\text{II}_1$ -factors. Does  $R_\Gamma \simeq M_\Lambda$  imply that  $\Gamma \simeq \Lambda$  and  $R_v \simeq M_v$  for  $v \in \Gamma$ ?

We note that an affirmative answer to Problem 5.10.4 would imply  $R * R \neq R * \mathcal{L}(\mathbb{F}_2)$  which would already resolve the free factor problem. Therefore, we add a restriction and state the following weaker conjecture which we believe is closer to the horizon.

**Conjecture 5.10.5.** *The class of finite gpp graphs satisfies condition (1) of Problem 5.10.1.*

We state another rigidity problem.

**Problem 5.10.6.** We observe that in Remark 5.9.7 we were able to retrieve the radius of the graph  $\Gamma$  (up to constant) from the graph product  $M_\Gamma$  without imposing any condition on the  $\text{II}_1$ -factors  $M_v$ , except that they are group von Neumann algebras. Such graph products  $M_\Gamma$  can generally decompose as graph products in different ways. For example, this is the case when  $\Gamma$  non-trivially decomposes as a graph product of graphs. Thus while the graph may generally not uniquely be retrieved from the graph product  $M_\Gamma$ , we are able to retrieve the radius (up to a constant). We wonder what other graph properties can be retrieved like this, without imposing strong conditions on the vertex algebras. In particular, we ask whether, under some conditions, we can retrieve the diameter (length of largest geodesic), or the girth (length of smallest cycle) of the graph from the graph product. We note that, of course the diameter of the graph satisfies  $\text{Radius}(\Gamma) \leq \text{Diam}(\Gamma) \leq 2 \text{Radius}(\Gamma)$  hence can, up to a factor 2, be retrieved from the graph product in the setting of Theorem 5.9.11. However we ask for a more precise estimate.

Last, we state a problem concerning strong solidity.

**Problem 5.10.7.** Let  $\mathcal{W} = \langle S|M \rangle$  be a Coxeter group and let  $\mathbf{q} \in \mathbb{R}_+^S$  be a Hecke-tuple. We ask if the Hecke-von Neumann algebra  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  is strongly solid. This question is answered by Theorem 5.6.7 in the case  $\mathcal{W}$  is right-angled (or a graph product of finite Coxeter groups), but it remains open for general Coxeter groups.

# 6

## THE CCAP FOR GRAPH PRODUCTS OF OPERATOR ALGEBRAS

For a simple graph  $\Gamma$  and for unital  $C^*$ -algebras with GNS-faithful states  $(A_v, \varphi_v)$  for  $v \in \Gamma$ , we consider the reduced graph product  $(A_\Gamma, \varphi) = \ast_{v \in \Gamma}^{\min} (A_v, \varphi_v)$ , and show that if every  $C^*$ -algebra  $A_v$  has the completely contractive approximation property (CCAP) and satisfies some additional condition, then the graph product has the CCAP as well. The additional condition imposed is satisfied in natural cases, for example for the reduced group  $C^*$ -algebra of a discrete group  $G$  that possesses the CCAP.

This result is an extension of the result of Ricard and Xu in [RX06, Proposition 4.11] where they prove this result under the same conditions for free products. Moreover, our result also extends the result of Reckwerdt in [Rec17, Theorem 5.5], where he proved for groups that weak amenability with Cowling-Haagerup constant 1 is preserved under graph products. Our result further covers many new cases coming from Hecke-algebras and discrete quantum groups.

The content of this chapter is based on the paper:

- **Matthijs Borst**, *The CCAP for graph products of operator algebras*, [Journal of Functional Analysis](#) 286.8 (2024) 110350.

### 6.1. INTRODUCTION

In this chapter we study the CCAP and weak- $*$  CCAP for operator algebraic graph products. In the setting of groups, graph products were introduced by Green in [Gre90]. They preserve many interesting properties like: soficity [CHR14], residual finiteness [Gre90], rapid decay [CHR11] and other properties, see [AM15; Chi12; HM95; HW99]. In particular, approximation properties like the Haagerup property [AD14] and weak-amenability with constant 1 [Rec17] are also preserved by graph products of groups.

Graph products of operator algebras were introduced in [CF17] by Caspers and Fima. In their paper, they also showed stability of exactness (for  $C^*$ -algebras), Haagerup property,  $\text{II}_1$ -factoriality (for von Neumann algebras) and rapid decay (for certain discrete quantum groups) under graph products. Also, in [Cas16] it was proven that embeddability is preserved under graph products.

The notion of weak amenability for groups originates from the work of Haagerup [Haa78], De Cannière-Haagerup [CH85] and Cowling-Haagerup [CH89]. The corresponding notion for unital  $C^*$ -algebras is given by the completely bounded approximation property (CBAP) in the sense that a discrete group is weakly amenable if and only if its reduced group  $C^*$ -algebra possesses the CBAP. We say that a  $C^*$ -algebra  $A$  has the CBAP if there exists a net of completely bounded maps  $V_n : A \rightarrow A$  that are finite rank, converge to the identity in the point-norm topology and such that  $\sup_n \|V_n\|_{\text{cb}} \leq \Lambda < \infty$  for some constant  $\Lambda$ . The minimal such  $\Lambda$  is called the Cowling-Haagerup constant. If the Cowling-Haagerup constant is 1, then we say that  $A$  has the completely contractive approximation property (CCAP).

Weak amenability and the CBAP/CCAP play a crucial role in functional analysis and operator algebras. Already in case of the group  $G = \mathbb{Z}$  weak amenability allows, in a way, to approximate a Fourier series by its partial sums. In operator space theory the CBAP has led to a deep understanding of several group  $C^*$ - and von Neumann algebras. Already the results by Cowling and Haagerup [CH89] allow for the distinction of group von Neumann algebras of lattices in the Lie groups  $\text{Sp}(1, n)$ ,  $n \geq 2$ . Later, Ozawa and Popa used the (weak-\*) CCAP in deformation/rigidity theory of von Neumann algebras [OP10a]. Much more recently also graph products have appeared in the deformation-rigidity programme, see e.g. [Cas20], [CE23], [CDD22], [DE24a]. This line of investigation, especially beyond the realm of group algebras, motivates the study of the CCAP for general graph products.

In this chapter we are concerned with showing that the CCAP is preserved under graph products. While we are not able to show this in full, we prove this under a mild extra condition on the algebras  $(A_v, \varphi_v)$ , similar to the one imposed by [RX06] for proving the same result for free products. The conditions that we impose are stated in Section 6.5, and we abbreviate them by saying that the algebra *has a u.c.p. extension for the CCAP*. This condition is satisfied by many natural unital  $C^*$ -algebras, under which finite-dimensional ones (with a GNS-faithful state), reduced  $C^*$ -algebras of discrete groups (with the Plancherel state) that possess the CCAP [RX06], and reduced  $C^*$ -algebras of compact quantum groups (with the Haar state) whose discrete dual quantum group is weakly amenable with Cowling-Haagerup constant 1 [Fre12]. Our main result is the following:

**Theorem O** (Theorem 6.5.2). *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $(A_v, \varphi_v)$  be unital  $C^*$ -algebras that have a u.c.p. extension for the CCAP. Then the reduced graph product  $(A_\Gamma, \varphi) = \ast_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  has the CCAP.*

Along the way we also obtain the following result for von Neumann algebras.

**Theorem P** (Corollary 6.3.4). *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $M_v$  be a finite-dimensional von Neumann algebra together with a normal faithful state  $\varphi_v$ . Then the von Neumann algebraic graph product  $(M_\Gamma, \varphi) = *_{v \in \Gamma} (M_v, \varphi_v)$  has the weak-\* CCAP.*

The method for proving above results is, on a large scale, similar to [RX06]. However, at most points, the proofs get more involved in order to work for graph products. This becomes most clear in Section 6.2, where we have to use different methods to show the completely boundedness of the word-length projection maps  $P_{\Gamma,d} : A_\Gamma \rightarrow A_\Gamma$  that project on  $A_{\Gamma,d}$ , the homogeneous subspace of order  $d$ . For these maps we show for  $d \geq 1$  the linear bound  $\|P_{\Gamma,d}\|_{\text{cb}} \leq C_\Gamma d$ , where  $C_\Gamma$  is some constant only depending on the graph  $\Gamma$ . In Section 6.3 we show that the graph product map  $\theta$  of state-preserving u.c.p. maps  $\theta_v$  on unital  $C^*$ -algebras  $A_v$ , is again a state-preserving u.c.p. map on the reduced graph product  $A_\Gamma$ . Together with our bound on  $\|P_{\Gamma,d}\|_{\text{cb}}$  we are then able to show the preliminary result, Corollary 6.3.4, that, when all  $C^*$ -algebras, respectively von Neumann algebras, are finite-dimensional, the reduced graph product has the CCAP, respectively the weak-\* CCAP. In Section 6.4 we consider the same problem as in Section 6.3, but now for state-preserving completely bounded maps. We show that the graph product map  $T$  of state-preserving completely bounded maps  $T_v$  defines a completely bounded map, when restricted to a homogeneous subspace  $A_{\Gamma,d}$  (i.e.  $T_d := T|_{A_{\Gamma,d}}$  is completely bounded). In order to do this we consider the operator spaces  $X_d$  from [CKL21] (analogous to [RX06]) and use the Khintchine type inequality [CKL21, Theorem 2.9] they proved. We moreover construct other operator spaces  $\tilde{X}_d$  and prove the ‘reversed’ Khintchine type inequality (Theorem 6.4.2). Finally, in Section 6.5, using all our previous results, we are then able to show the main result Theorem O (Theorem 6.5.2).

Our results extends [RX06] (as well as [Rec17]) in a natural way, and provides a unified approach to proving the CCAP and weak-\* CCAP for various operator algebras. Specifically, Theorem P can be applied to the von Neumann algebraic graph product  $*_{v \in \Gamma} \mathcal{N}_{q_v}(\mathcal{W}_v)$  of Hecke-algebras of finite Coxeter groups. Such a graph product is itself a Hecke-algebra, and by the result we obtained, possesses the weak-\* CCAP. This result is new, and was previously only known, by [Cas20, Theorem A], for the case that  $\mathcal{W}_v$  is right-angled for all  $v$ . Furthermore, the main theorem, Theorem O, can be applied to give new examples of  $C^*$ -algebras that possess the CCAP, for example the graph product  $*_{v \in \Gamma}^{\min} (A_v, \varphi_v)$ , where some algebras  $A_v$  are finite-dimensional, and others are reduced group  $C^*$ -algebras of discrete groups that possess the CCAP.

## 6.2. POLYNOMIAL GROWTH OF WORD-LENGTH PROJECTIONS

In this section we shall fix a simple finite graph  $\Gamma$ , together with unital  $C^*$ -algebras  $A_v$  for  $v \in \Gamma$  and states  $\varphi_v$  on  $A_v$  for which the GNS representation is faithful. We shall look at the reduced graph product  $(A_\Gamma, \varphi) = *_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  and investigate for  $d \geq 0$  the natural projections  $P_{\Gamma,d} : A_\Gamma \rightarrow A_{\Gamma,d}$ . The main result of this section, Theorem 6.2.10, is that these maps are completely bounded, and that we can obtain a bound on  $\|P_{\Gamma,d}\|_{\text{cb}}$  that depends only linearly on  $d$ . To prove this, we can not use the same method as [RX06], since that relies on the fact that each element either does not act diagonally on a pure tensor  $\eta \in \hat{\mathcal{H}}_\nu \subseteq \mathcal{H}_\Gamma$ , or acts diagonally on  $\eta$  on precisely one letter. This holds true for

elements in the free product, but not generally for elements in the graph product, as they may act diagonally on any clique. Therefore, we will instead introduce completely contractive maps  $H_\tau$  (and completely bounded maps  $\tilde{H}_\rho$ ) and write  $P_{\Gamma,d}$  as linear combination of these. For this we have to do some technical graph product computations.

### 6.2.1. THE MAPS $H_\tau$

We introduce some extra notation. Let  $\mathcal{W}_\Gamma$  be the right-angled Coxeter group associated to the graph  $\Gamma$ . Recall, for a word  $\mathbf{w} \in \mathcal{W}_\Gamma$  we defined  $\mathbf{s}_l(\mathbf{w})$  and  $\mathbf{s}_r(\mathbf{w})$  as the maximal clique words that  $\mathbf{w}$  respectively starts with and ends with. Recall that for a word  $\mathbf{u} \in \mathcal{W}_\Gamma$ , we defined

$$\mathcal{W}(\mathbf{u}) = \{\mathbf{w} \in \mathcal{W}_\Gamma : |\mathbf{uw}| = |\mathbf{u}| + |\mathbf{w}|\} \quad \mathcal{W}'(\mathbf{u}) = \{\mathbf{w} \in \mathcal{W}_\Gamma : |\mathbf{wu}| = |\mathbf{w}| + |\mathbf{u}|\}$$

For  $n \geq 0$ ,  $\mathbf{u} \in \mathcal{W}_\Gamma$  we now define

$$\begin{aligned} \widetilde{\mathcal{W}}(\mathbf{u}) &= \{\mathbf{w} \in \mathcal{W}(\mathbf{u}) : \mathbf{s}_l(\mathbf{uw}) = \mathbf{s}_l(\mathbf{u})\} & \widetilde{\mathcal{W}}'(\mathbf{u}) &= \{\mathbf{w} \in \mathcal{W}'(\mathbf{u}) : \mathbf{s}_r(\mathbf{wu}) = \mathbf{s}_r(\mathbf{u})\} \\ \widetilde{\mathcal{W}}_n(\mathbf{u}) &= \{\mathbf{w} \in \widetilde{\mathcal{W}}(\mathbf{u}) : |\mathbf{w}| = n\} & \widetilde{\mathcal{W}}'_n(\mathbf{u}) &= \{\mathbf{w} \in \widetilde{\mathcal{W}}'(\mathbf{u}) : |\mathbf{w}| = n\}. \end{aligned}$$

Now, let  $\mathbf{u} \in \mathcal{W}_\Gamma$  and let  $\mathbf{u}_L, \mathbf{u}_R \in \mathcal{W}_\Gamma$  be s.t.  $|\mathbf{u}| = |\mathbf{u}\mathbf{u}_L^{-1}| + |\mathbf{u}_L|$  and  $|\mathbf{u}| = |\mathbf{u}_R| + |\mathbf{u}_R^{-1}\mathbf{u}|$ , i.e.  $\mathbf{u}_L$  is some word that  $\mathbf{u}$  ends with and  $\mathbf{u}_R$  is some word that  $\mathbf{u}$  starts with. Then we have for  $\mathbf{w}_L \in \mathcal{W}(\mathbf{u})$  and  $\mathbf{w}_R \in \mathcal{W}'(\mathbf{u})$  that  $\mathbf{u}_L\mathbf{w}_L$  and  $\mathbf{w}_R\mathbf{u}_R$  are reduced expressions. Let  $n \geq 0$ . We define

$$\begin{aligned} \mathcal{H}(\mathbf{u}, \mathbf{u}_L) &= \bigoplus_{\mathbf{w} \in \mathcal{W}(\mathbf{u})} \mathcal{H}_{\mathbf{u}_L\mathbf{w}}^\circ & \mathcal{H}'(\mathbf{u}, \mathbf{u}_R) &= \bigoplus_{\mathbf{w} \in \mathcal{W}'(\mathbf{u})} \mathcal{H}_{\mathbf{w}\mathbf{u}_R}^\circ \\ \widetilde{\mathcal{H}}(\mathbf{u}, \mathbf{u}_L) &= \bigoplus_{\mathbf{w} \in \widetilde{\mathcal{W}}(\mathbf{u})} \mathcal{H}_{\mathbf{u}_L\mathbf{w}}^\circ & \widetilde{\mathcal{H}}'(\mathbf{u}, \mathbf{u}_R) &= \bigoplus_{\mathbf{w} \in \widetilde{\mathcal{W}}'(\mathbf{u})} \mathcal{H}_{\mathbf{w}\mathbf{u}_R}^\circ \\ \widetilde{\mathcal{H}}_n(\mathbf{u}, \mathbf{u}_L) &= \bigoplus_{\mathbf{w} \in \widetilde{\mathcal{W}}_n(\mathbf{u})} \mathcal{H}_{\mathbf{u}_L\mathbf{w}}^\circ & \widetilde{\mathcal{H}}'_n(\mathbf{u}, \mathbf{u}_R) &= \bigoplus_{\mathbf{w} \in \widetilde{\mathcal{W}}'_n(\mathbf{u})} \mathcal{H}_{\mathbf{w}\mathbf{u}_R}^\circ. \end{aligned}$$

For  $\mathbf{u} \in \mathcal{W}_\Gamma$  and  $n \geq 0$  we moreover define

$$\widetilde{\mathcal{H}}_n(\mathbf{u}) = \bigoplus_{\substack{\mathbf{w}_1 \in \widetilde{\mathcal{W}}'_n(\mathbf{u}) \\ \mathbf{w}_2 \in \mathcal{W}'(\mathbf{u})}} \mathcal{H}_{\mathbf{w}_1\mathbf{u}\mathbf{w}_2}^\circ.$$

We note that for  $\mathbf{w}_1 \in \widetilde{\mathcal{W}}'_n(\mathbf{u})$  and  $\mathbf{w}_2 \in \mathcal{W}'(\mathbf{u})$  we have that  $\mathbf{w}_1\mathbf{u}\mathbf{w}_2$  is a reduced expression. Indeed, it is clear that  $\mathbf{w}_1\mathbf{u}$  and  $\mathbf{u}\mathbf{w}_2$  are reduced by definition. Now, since moreover  $\mathbf{s}_r(\mathbf{w}_1\mathbf{u}) = \mathbf{s}_r(\mathbf{u})$ , we have that no letter from  $\mathbf{w}_1$  can cancel out a letter of  $\mathbf{w}_2$ , so that the expression is reduced.

**Definition 6.2.1.** Let  $\mathbf{u} \in \mathcal{W}_\Gamma$  and let  $\mathbf{r} \in \mathcal{W}_\Gamma$  be any clique word that  $\mathbf{u}$  ends with. Then  $\mathbf{ur}$  is a word in  $\mathcal{W}_\Gamma$  that  $\mathbf{u}$  starts with, and  $|\mathbf{ur}| + |\mathbf{r}| = |\mathbf{u}|$ . For  $n \geq 0$  we define a partial isometry  $V_n^{\mathbf{u}, \mathbf{r}} : \mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma \rightarrow \mathcal{H}_\Gamma$  with initial subspace  $\widetilde{\mathcal{H}}'_n(\mathbf{u}, \mathbf{ur}) \otimes \mathcal{H}(\mathbf{u}, \mathbf{r})$  and final subspace  $\mathcal{H}_n(\mathbf{u})$  as

$$V_n^{\mathbf{u}, \mathbf{r}}|_{\mathcal{H}_{\mathbf{v}_r\mathbf{ur}}^\circ \otimes \mathcal{H}_{\mathbf{r}\mathbf{v}_{tail}}^\circ} = \mathcal{Q}_{(\mathbf{v}_r\mathbf{ur}, \mathbf{r}\mathbf{v}_{tail})} \quad \text{for } \mathbf{v}_r \in \widetilde{\mathcal{W}}'_n(\mathbf{u}), \mathbf{v}_{tail} \in \mathcal{W}(\mathbf{u}).$$

We note that this is well-defined. Indeed, as just pointed out, for  $\mathbf{v}_r \in \widetilde{\mathcal{W}}'_n(\mathbf{u})$  and  $\mathbf{v}_{tail} \in \mathcal{W}(\mathbf{u})$  we have that  $\mathbf{v}_r \mathbf{u} \mathbf{v}_{tail}$  is reduced. Therefore, we get  $|\mathbf{v}_r \mathbf{u} \mathbf{v}_{tail}| \leq |\mathbf{v}_r \mathbf{u} \mathbf{r}| + |\mathbf{r} \mathbf{v}_{tail}| \leq |\mathbf{v}_r| + |\mathbf{u} \mathbf{r}| + |\mathbf{r}| + |\mathbf{v}_{tail}| = |\mathbf{v}_r| + |\mathbf{u}| + |\mathbf{v}_{tail}| = |\mathbf{v}_r \mathbf{u} \mathbf{v}_{tail}|$ . This shows that  $|\mathbf{v}_r \mathbf{u} \mathbf{r}| + |\mathbf{r} \mathbf{v}_{tail}| = |\mathbf{v}_r \mathbf{u} \mathbf{v}_{tail}|$ , so that  $\mathcal{Q}_{(\mathbf{v}_r \mathbf{u} \mathbf{r}, \mathbf{r} \mathbf{v}_{tail})}$  is well-defined.

**Definition 6.2.2.** We denote

$$\mathcal{T}_\Gamma = \left\{ (\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{W}_\Gamma^3 \left| \begin{array}{l} \mathbf{u}_l \mathbf{t}, \mathbf{t} \mathbf{u}_r \text{ clique words,} \\ \mathbf{u}_l \mathbf{t} \mathbf{u}_r \text{ reduced} \end{array} \right. \right\}.$$

We remark that it follows from the definition that  $\mathbf{u}_l, \mathbf{u}_r$  and  $\mathbf{t}$  must also be clique words and that  $\mathbf{u}_l \mathbf{u}_r$  must be reduced.

**Definition 6.2.3.** Let  $(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_\Gamma$ . Also let  $\mathbf{r} \in \mathcal{W}_\Gamma$  be a sub-clique word of  $\mathbf{t}$  and let  $n_l, n_r \geq 0$ . For the tuple  $\tau = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t}, \mathbf{r})$  define a map  $H_\tau : \mathcal{B}(\mathcal{H}_\Gamma) \rightarrow \mathcal{B}(\mathcal{H}_\Gamma)$  as

$$H_\tau(a) = V_{n_l}^{(\mathbf{u}_l \mathbf{t}), \mathbf{r}} (a \otimes \text{Id}_{\mathcal{H}_\Gamma}) \left( V_{n_r}^{(\mathbf{u}_r \mathbf{t}), \mathbf{r}} \right)^*.$$

It is clear that  $H_\tau$  is completely contractive.

*Example 6.2.4.* We note that the partial isometry  $V_0^{e,e} : \mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma \rightarrow \mathcal{H}_\Gamma$  has initial subspace  $\widetilde{\mathcal{H}}'_0(e, e) \otimes \mathcal{H}(e, e) = \mathbb{C} \Omega \otimes \mathcal{H}_\Gamma$  and final subspace  $\widetilde{\mathcal{H}}_0(e) = \mathcal{H}_\Gamma$  and that on  $\mathbb{C} \Omega \otimes \mathcal{H}_\Gamma$  it is given by  $V_0^{e,e}(z \Omega \otimes \eta) = z \eta$  for  $z \in \mathbb{C}$ ,  $\eta \in \mathcal{H}_\Gamma$ . Setting  $\tau = (0, 0, e, e, e, e)$  and letting  $a \in \mathbf{A}_\Gamma$  be a pure tensor  $a = a_1 \otimes \cdots \otimes a_t$ , we can for  $\eta \in \mathcal{H}_\Gamma$  calculate  $H_\tau(\lambda(a))\eta = V_0^{e,e}(\lambda(a) \Omega \otimes \eta)$ . Now, if  $\lambda(a) \Omega \notin \mathbb{C} \Omega$ , then we get  $H_\tau(\lambda(a))\eta = 0$ . On the other hand, if  $\hat{a} = \lambda(a) \Omega \in \mathbb{C} \Omega$ , then we must have that  $\lambda(a) \in \mathbb{C} \text{Id}_{\mathcal{H}_\Gamma}$  and we get  $H_\tau(a)\eta = a\eta$ . We conclude that  $P_{\Gamma,0} = H_{(0,0,e,e,e,e)}$  and  $\|P_{\Gamma,0}\|_{\text{cb}} = 1$ .

Similarly to Example 6.2.4, we aim to write  $P_{\Gamma,d}$  for  $d \geq 1$  as a linear combination of  $H_\tau$ 's for different tuples  $\tau$ , in order to give a bound on  $\|P_{\Gamma,d}\|_{\text{cb}}$ . To achieve this, we introduce some convenient notation.

**Definition 6.2.5.** Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be closed subspaces of  $\mathcal{H}_\Gamma$ . For an operator  $b \in \mathcal{B}(\mathcal{H}_\Gamma)$  we define a closed subspace  $\mathcal{J}_b(\mathcal{H}_1, \mathcal{H}_2)$  of  $\mathcal{H}_\Gamma$  as

$$\mathcal{J}_b(\mathcal{H}_1, \mathcal{H}_2) = \{\eta \in \mathcal{H}_1 \mid b\eta \in \mathcal{H}_2\}.$$

Recall that for  $\mathbf{w} \in \mathcal{W}_\Gamma$  we defined in Definition 3.1.5 the set of triple splittings

$$\mathcal{S}_{\mathbf{w}} = \left\{ (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{W}_\Gamma^3 \left| \begin{array}{l} \mathbf{w} = \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3 \\ \mathbf{w}_2 \text{ is a clique word} \\ |\mathbf{w}| = |\mathbf{w}_1| + |\mathbf{w}_2| + |\mathbf{w}_3| \end{array} \right. \right\}$$

and also put  $\mathcal{S}_\Gamma = \bigcup_{\mathbf{w} \in \mathcal{W}_\Gamma} \mathcal{S}_{\mathbf{w}}$ . Recall also for  $\omega \in \mathcal{S}_\Gamma$  that in Definition 3.1.3 we defined the annihilation/diagonal/creation operator  $\lambda_\omega$ . We prove the following proposition.

**Proposition 6.2.6.** Let  $(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_\Gamma$ . Also let  $\mathbf{r} \subseteq \mathbf{t}$  be a sub-clique, and let  $n_l, n_r \geq 0$ . Set  $\tau = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t}, \mathbf{r})$ . For  $\mathbf{w} \in \mathcal{W}_\Gamma$  and  $\omega = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}}$  and for pure tensor  $a = a_1 \otimes \cdots \otimes a_t \in \hat{\mathbf{A}}_{\mathbf{w}}$  we have that

$$H_\tau(\lambda_\omega(a)) = \lambda_\omega(a) P_a(\tau, \omega)$$

where  $P_a(\tau, \omega)$  is the projection in  $B(\mathcal{H}_\Gamma)$  on the closed subspace spanned by

$$\bigcup_{\substack{\mathbf{v}_l \in \widetilde{\mathcal{W}}_{n_l}^l(\mathbf{u}_l \mathbf{t}), \mathbf{v}_r \in \widetilde{\mathcal{W}}_{n_r}^r(\mathbf{u}_r \mathbf{t}) \\ \mathbf{v}_{tail} \in \mathcal{W}(\mathbf{u}_l \mathbf{t}) \cap \mathcal{W}(\mathbf{u}_r \mathbf{t}) \\ |\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| \\ |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}|}} \mathcal{I}_{\lambda_\omega(a)}(\mathring{\mathcal{H}}_{\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}}, \mathring{\mathcal{H}}_{\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{v}_{tail}}).$$

*Proof.* We show that the identity holds on pure tensors. First, let  $\mathbf{v} \in \mathcal{W}_\Gamma$  and let  $\eta \in \mathring{\mathcal{H}}_{\mathbf{v}} \subseteq \mathcal{H}_\Gamma$  be a pure tensor s.t.  $\lambda_\omega(a)P_a(\tau, \omega)\eta = 0$ . If  $\eta \perp \mathring{\mathcal{H}}_{n_r}(\mathbf{u}_r \mathbf{t})$ , then clearly  $(V_{n_r}^{\mathbf{u}_r \mathbf{t} \mathbf{r}})^* \eta = 0$  so that  $H_\tau(\lambda_\omega(a))\eta = 0 = \lambda_\omega(a)P_a(\tau, \omega)\eta$ , and we are done. Thus, assume that  $\eta \in \mathring{\mathcal{H}}_{n_r}(\mathbf{u}_r \mathbf{t})$  and  $\eta \neq 0$ , so that  $\eta \in \mathring{\mathcal{H}}_{\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}}$  for some  $\mathbf{v}_r \in \widetilde{\mathcal{W}}_{n_r}^r(\mathbf{u}_r \mathbf{t})$ ,  $\mathbf{v}_{tail} \in \mathcal{W}(\mathbf{u}_r \mathbf{t})$ . Let us write  $V_{n_r}^{\mathbf{u}_r \mathbf{t} \mathbf{r}*} \eta = \eta_1 \otimes \eta_2$  with  $\eta_1 \in \mathring{\mathcal{H}}_{\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}}$ ,  $\eta_2 \in \mathring{\mathcal{H}}_{\mathbf{r} \mathbf{v}_{tail}}$ . Then  $H_\tau(\lambda_\omega(a))\eta = V_{n_l}^{\mathbf{u}_l \mathbf{t} \mathbf{r}}(\lambda_\omega(a)\eta_1 \otimes \eta_2)$ . We can assume that  $0 \neq \lambda_\omega(a)\eta_1 \in \mathring{\mathcal{H}}_{n_l}^l(\mathbf{u}_l \mathbf{t}, \mathbf{u}_l \mathbf{t} \mathbf{r})$  and  $\eta_2 \in \mathcal{H}(\mathbf{u}_l \mathbf{t}, \mathbf{r})$  since otherwise we find directly  $H_\tau(\lambda_\omega(a))\eta = 0$ . Now we thus have that  $\lambda_\omega(a)\eta_1 \in \mathring{\mathcal{H}}_{\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{r}}$  for some  $\mathbf{v}_l \in \widetilde{\mathcal{W}}_{n_l}^l(\mathbf{u}_l \mathbf{t})$  and that  $\eta_2 \in \mathring{\mathcal{H}}_{\mathbf{r} \mathbf{v}'_{tail}}$  for some  $\mathbf{v}'_{tail} \in \mathcal{W}_{n_r}(\mathbf{u}_l \mathbf{t})$ .

As  $\eta_2$  is non-zero, and as  $\eta_2 \in \mathring{\mathcal{H}}_{\mathbf{r} \mathbf{v}_{tail}} \cap \mathring{\mathcal{H}}_{\mathbf{r} \mathbf{v}'_{tail}}$  we find that  $\mathbf{v}_{tail} = \mathbf{v}'_{tail} \in \mathcal{W}(\mathbf{u}_l \mathbf{t}) \cap \mathcal{W}(\mathbf{u}_r \mathbf{t})$ . Also, since  $\eta_1 \in \mathring{\mathcal{H}}_{\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}}$  we find by Lemma 3.1.4 that  $\lambda_\omega(a)\eta_1 \in \mathring{\mathcal{H}}_{\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}}$ . Now, we already had  $\lambda_\omega(a)\eta_1 \in \mathring{\mathcal{H}}_{\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{r}}$  and by the assumption that  $\lambda_\omega(a)\eta_1$  is non-zero, we thus find  $\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{r} = \mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}$ . Moreover, as  $\lambda_\omega(a)\eta_1$  is non-zero, we must have that  $|\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}|$  and  $|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}|$ .

Set  $\mathbf{v}_1 = \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}$  and  $\mathbf{v}_2 = \mathbf{r} \mathbf{v}_{tail}$ , so that  $|\mathbf{v}_1 \mathbf{v}_2| = |\mathbf{v}_1| + |\mathbf{v}_2|$ , and by the above

$$|\mathbf{v}_1| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_1| \quad (6.1)$$

$$|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_1| \quad (6.2)$$

Moreover, we now find

$$\begin{aligned} |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| &\leq |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| \\ &\leq |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_1| + |\mathbf{v}_2| \\ &= |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1| + |\mathbf{v}_2| \\ &= |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| + |\mathbf{r} \mathbf{v}_{tail}| \\ &= |\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{r}| + |\mathbf{r} \mathbf{v}_{tail}| \\ &= |\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{v}_{tail}| \\ &= |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}| \\ &= |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2|. \end{aligned}$$

This shows that

$$|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_1 \mathbf{v}_2| \quad (6.3)$$

Now as  $\eta \in \mathring{\mathcal{H}}_{\mathbf{v}_1 \mathbf{v}_2}$ , and as all conditions of Lemma 3.1.8(3) are satisfied, this gives us

$$\begin{aligned} H_\tau(\lambda_\omega(a))\eta &= V_{n_l}^{\mathbf{u}_l \mathbf{t} \mathbf{r}}(\lambda_\omega(a)\eta_1 \otimes \eta_2) \\ &= \mathcal{Q}_{(\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1, \mathbf{v}_2)}(\lambda_\omega(a)\eta_1 \otimes \eta_2) \\ &= \lambda_\omega(a)\mathcal{Q}_{(\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_1, \mathbf{v}_2)}(\eta_1 \otimes \eta_2) \\ &= \lambda_\omega(a)\eta. \end{aligned}$$

By Lemma 3.1.4  $\lambda_\omega(a)\eta \in \mathring{\mathcal{H}}_{\mathbf{w}_1\mathbf{w}_3\mathbf{v}_1\mathbf{v}_2} = \mathring{\mathcal{H}}_{\mathbf{v}_l\mathbf{u}_l\mathbf{t}\mathbf{v}_{tail}}$ , thus  $\eta \in \mathcal{I}_{\lambda_\omega(a)}(\mathring{\mathcal{H}}_{\mathbf{v}_l\mathbf{u}_l\mathbf{t}\mathbf{v}_{tail}}, \mathring{\mathcal{H}}_{\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{v}_{tail}})$ . By all the conditions we have shown for  $\mathbf{v}_l, \mathbf{v}_r, \mathbf{v}_{tail}$ , and as we have shown that  $|\mathbf{v}_1| = |\mathbf{w}_2\mathbf{w}_3| + |\mathbf{w}_2\mathbf{w}_3\mathbf{v}_1|$  (Equation (6.1)) and  $|\mathbf{w}_1\mathbf{w}_3\mathbf{v}_1\mathbf{v}_2| = |\mathbf{w}_1| + |\mathbf{w}_3\mathbf{v}_1\mathbf{v}_2|$  (Equation (6.3)) it follows that  $P_a(\tau, \omega)\eta = \eta$ . We conclude that  $H_\tau(\lambda_\omega(a))\eta = \lambda_\omega(a)\eta = \lambda_\omega(a)P_a(\tau, \omega)\eta$ .

Alternatively, let  $\eta \in \mathring{\mathcal{H}}_{\mathbf{v}} \subseteq \mathcal{H}_\Gamma$  be a pure vector s.t.  $\lambda_\omega(a)P_a(\tau, \omega)\eta \neq 0$ . Then we must have that  $P_a(\tau, \omega)\eta = \eta$  and moreover that  $\lambda_\omega(a)\eta$  is non-zero. We thus get that  $\eta \in \mathcal{I}_{\lambda_\omega(a)}(\mathring{\mathcal{H}}_{\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{v}_{tail}}, \mathring{\mathcal{H}}_{\mathbf{v}_l\mathbf{u}_l\mathbf{t}\mathbf{v}_{tail}})$  with  $\mathbf{v}_l \in \widetilde{\mathcal{W}}_{n_l}^1(\mathbf{u}_l\mathbf{t})$ ,  $\mathbf{v}_r \in \widetilde{\mathcal{W}}_{n_r}^1(\mathbf{u}_r\mathbf{t})$ ,  $\mathbf{v}_{tail} \in \mathcal{W}(\mathbf{u}_r\mathbf{t}) \cap \mathcal{W}(\mathbf{u}_l\mathbf{t})$  and so that

$$\begin{aligned} |\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{r}| &= |\mathbf{w}_2\mathbf{w}_3| + |\mathbf{w}_2\mathbf{w}_3\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{r}| \\ |\mathbf{w}_1\mathbf{w}_3\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{v}_{tail}| &= |\mathbf{w}_1| + |\mathbf{w}_3\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{v}_{tail}|. \end{aligned}$$

Set  $\mathbf{v}_1 = \mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{r}$  and  $\mathbf{v}_2 = \mathbf{r}\mathbf{v}_{tail}$ , so that  $|\mathbf{v}_1\mathbf{v}_2| = |\mathbf{v}_1| + |\mathbf{v}_2|$ . Moreover the above equations state that  $|\mathbf{v}_1| = |\mathbf{w}_2\mathbf{w}_3| + |\mathbf{w}_2\mathbf{w}_3\mathbf{v}_1|$  and  $|\mathbf{w}_1\mathbf{w}_3\mathbf{v}_1\mathbf{v}_2| = |\mathbf{w}_1| + |\mathbf{w}_3\mathbf{v}_1\mathbf{v}_2|$ . As  $\eta \in \mathring{\mathcal{H}}_{\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{v}_{tail}} \subseteq \mathring{\mathcal{H}}_{n_r}(\mathbf{u}_r\mathbf{t})$ , we can write  $V_{n_r}^{\mathbf{u}_r\mathbf{t}, \mathbf{r}*}\eta = \eta_1 \otimes \eta_2 \in \mathring{\mathcal{H}}_{\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{r}} \otimes \mathring{\mathcal{H}}_{\mathbf{r}\mathbf{v}_{tail}} = \mathring{\mathcal{H}}_{\mathbf{v}_1} \otimes \mathring{\mathcal{H}}_{\mathbf{v}_2}$ . By the above properties we get from Lemma 3.1.8(3) that

$$\lambda_\omega(a)\eta = \mathcal{Q}_{(\mathbf{w}_1\mathbf{w}_3\mathbf{v}_1, \mathbf{v}_2)}(\lambda_\omega(a)\eta_1 \otimes \eta_2) \in \mathring{\mathcal{H}}_{\mathbf{w}_1\mathbf{w}_3\mathbf{v}_1\mathbf{v}_2}.$$

However, we also know that  $\lambda_\omega(a)\eta \in \mathring{\mathcal{H}}_{\mathbf{v}_l\mathbf{u}_l\mathbf{t}\mathbf{v}_{tail}}$ . Therefore, as  $\lambda_\omega(a)\eta$  is non-zero we find  $\mathbf{v}_l\mathbf{u}_l\mathbf{t}\mathbf{v}_{tail} = \mathbf{w}_1\mathbf{w}_3\mathbf{v}_1\mathbf{v}_2 = \mathbf{w}_1\mathbf{w}_3\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{v}_{tail}$ . We thus find  $\mathbf{v}_l\mathbf{u}_l\mathbf{t}\mathbf{r} = \mathbf{w}_1\mathbf{w}_3\mathbf{v}_r\mathbf{u}_r\mathbf{t}\mathbf{r} = \mathbf{w}_1\mathbf{w}_3\mathbf{v}_1$ , and hence  $\lambda_\omega(a)\eta_1 \in \mathring{\mathcal{H}}_{\mathbf{w}_1\mathbf{w}_3\mathbf{v}_1} = \mathring{\mathcal{H}}_{\mathbf{v}_l\mathbf{u}_l\mathbf{t}\mathbf{r}} \subseteq \mathring{\mathcal{H}}_{n_l}^1(\mathbf{u}_l\mathbf{t}, \mathbf{u}_l\mathbf{t}\mathbf{r})$ . Note that  $\eta_2 \in \mathcal{H}(\mathbf{u}_l\mathbf{t}, \mathbf{r})$  by the assumption on  $\mathbf{v}_{tail}$ . Hence, as  $\lambda_\omega(a)\eta_1 \otimes \eta_2 \in \mathring{\mathcal{H}}_{n_l}^1(\mathbf{u}_l\mathbf{t}, \mathbf{u}_l\mathbf{t}\mathbf{r}) \otimes \mathcal{H}(\mathbf{u}_l\mathbf{t}, \mathbf{r})$  we find that

$$\begin{aligned} H_\tau(\lambda_\omega(a))\eta &= V_{n_l}^{\mathbf{u}_l\mathbf{t}, \mathbf{r}}(\lambda_\omega(a)\eta_1 \otimes \eta_2) \\ &= \mathcal{Q}_{(\mathbf{w}_1\mathbf{w}_3\mathbf{v}_1, \mathbf{v}_2)}(\lambda_\omega(a)\eta_1 \otimes \eta_2) \\ &= \lambda_\omega(a)\eta \\ &= \lambda_\omega(a)P_a(\tau, \omega)\eta \end{aligned}$$

which proves the statement.  $\square$

### 6.2.2. THE MAPS $\tilde{H}_\rho$

We shall now introduce other maps,  $\tilde{H}_\rho$ , that are linear combinations of the maps  $H_\tau$  for different  $\tau$ 's, and that satisfy a nice equation. We use these maps to show that  $P_{\Gamma, d}$  is completely bounded, and give a bound on  $\|P_{\Gamma, d}\|_{cb}$ .

**Definition 6.2.7.** Let  $n_l, n_r \geq 0$  and  $(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_\Gamma$ . For  $\mathbf{w} \in \mathcal{W}_\Gamma$  and for the tuple  $\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$  define the set

$$\mathcal{S}_{\mathbf{w}}(\rho) = \left\{ (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}} \left| \begin{array}{l} \mathbf{w}_1 = \mathbf{v}_l\mathbf{u}_l, \mathbf{w}_2 = \mathbf{t} \text{ and } \mathbf{w}_3 = \mathbf{u}_r^{-1}\mathbf{v}_r^{-1} \\ \text{for some } \mathbf{v}_l \in \widetilde{\mathcal{W}}_{n_l}^1(\mathbf{u}_l\mathbf{t}), \mathbf{v}_r \in \widetilde{\mathcal{W}}_{n_r}^1(\mathbf{u}_r\mathbf{t}) \end{array} \right. \right\}.$$

Also denote  $|\rho| := n_l + |\mathbf{u}_l| + |\mathbf{t}| + |\mathbf{u}_r| + n_r$ .

*Remark 6.2.8.* We note that we can partition  $\mathcal{S}_{\mathbf{w}}$  as  $\{\mathcal{S}_{\mathbf{w}}(\rho)\}_{|\rho|=|\mathbf{w}|}$  where we run over all tuples  $\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$  for  $n_l, n_r \geq 0$ ,  $(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_{\Gamma}$  with  $|\rho| = |\mathbf{w}|$ . Indeed, if  $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}}(\rho)$  then  $\mathbf{w}_1 = \mathbf{v}_l \mathbf{u}_l$ ,  $\mathbf{w}_2 = \mathbf{t}$ ,  $\mathbf{w}_3 = \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$  for some  $\mathbf{v}_l \in \widetilde{\mathcal{W}}_{n_l}'(\mathbf{u}_l \mathbf{t})$  and  $\mathbf{v}_r \in \widetilde{\mathcal{W}}_{n_r}'(\mathbf{u}_r \mathbf{t})$  and we obtain that  $\mathbf{t} = \mathbf{w}_2$ ,  $\mathbf{u}_l = (\mathbf{u}_l \mathbf{t}) \mathbf{t} = \mathbf{s}_r(\mathbf{v}_l \mathbf{u}_l \mathbf{t}) \mathbf{t} = \mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2) \mathbf{w}_2$  and  $\mathbf{u}_r = (\mathbf{u}_r \mathbf{t}) \mathbf{t} = \mathbf{s}_r(\mathbf{v}_r \mathbf{u}_r \mathbf{t}) \mathbf{t} = \mathbf{s}_r(\mathbf{w}_3^{-1} \mathbf{w}_2) \mathbf{w}_2$  and  $n_l = |\mathbf{w}_1| - |\mathbf{u}_l| = |\mathbf{w}_1| - |\mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2) \mathbf{w}_2|$  and  $n_r = |\mathbf{w}_3| - |\mathbf{u}_r| = |\mathbf{w}_3| - |\mathbf{s}_r(\mathbf{w}_3^{-1} \mathbf{w}_2) \mathbf{w}_2|$ . Since we can retrieve  $\rho$  from  $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ , this shows the sets  $\mathcal{S}_{\mathbf{w}}(\rho)$  are disjoint.

Now let  $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}}$  and set  $\mathbf{t} := \mathbf{w}_2$ ,  $\mathbf{u}_l := \mathbf{s}_r(\mathbf{w}_1 \mathbf{t}) \mathbf{t}$ ,  $\mathbf{u}_r := \mathbf{s}_r(\mathbf{w}_3^{-1} \mathbf{t}) \mathbf{t}$ . Then  $\mathbf{u}_l \mathbf{t}$  and  $\mathbf{t} \mathbf{u}_r$  are clique words and

$$\begin{aligned} |\mathbf{w}| &\leq |\mathbf{w}_1 \mathbf{w}_2 \mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2)| + |\mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2) \mathbf{w}_2 \mathbf{s}_l(\mathbf{w}_2 \mathbf{w}_3)| + |\mathbf{s}_l(\mathbf{w}_2 \mathbf{w}_3) \mathbf{w}_2 \mathbf{w}_3| \\ &= (|\mathbf{w}_1 \mathbf{w}_2| - |\mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2)|) + |\mathbf{u}_l \mathbf{t} \mathbf{u}_r| + (|\mathbf{w}_2 \mathbf{w}_3| - |\mathbf{s}_l(\mathbf{w}_2 \mathbf{w}_3)|) \\ &= |\mathbf{w}| + |\mathbf{u}_l \mathbf{t} \mathbf{u}_r| - |\mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2)| + |\mathbf{w}_2| - |\mathbf{s}_l(\mathbf{w}_2 \mathbf{w}_3)| \\ &= |\mathbf{w}| + |\mathbf{u}_l \mathbf{t} \mathbf{u}_r| - |\mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2) \mathbf{w}_2| - |\mathbf{w}_2| - |\mathbf{s}_l(\mathbf{w}_2 \mathbf{w}_3) \mathbf{w}_2| \\ &= |\mathbf{w}| + |\mathbf{u}_l \mathbf{t} \mathbf{u}_r| - |\mathbf{u}_l| - |\mathbf{t}| - |\mathbf{u}_r| \\ &\leq |\mathbf{w}|. \end{aligned}$$

Thus all inequalities must be equalities and we get  $|\mathbf{u}_l \mathbf{t} \mathbf{u}_r| = |\mathbf{u}_l| + |\mathbf{t}| + |\mathbf{u}_r|$  so  $\mathbf{u}_l \mathbf{t} \mathbf{u}_r$  is reduced. This shows  $(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_{\Gamma}$ . Now, set  $n_l := |\mathbf{w}_1| - |\mathbf{u}_l| \geq 0$ ,  $n_r := |\mathbf{w}_3| - |\mathbf{u}_r| \geq 0$ . Then we have  $\mathbf{v}_l := \mathbf{w}_1 \mathbf{u}_l^{-1} \in \widetilde{\mathcal{W}}_{n_l}'(\mathbf{u}_l \mathbf{t})$  and  $\mathbf{v}_r := \mathbf{w}_3^{-1} \mathbf{u}_r^{-1} \in \widetilde{\mathcal{W}}_{n_r}'(\mathbf{u}_r \mathbf{t})$ . Set  $\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$  and observe that  $|\rho| = n_l + |\mathbf{u}_l \mathbf{t} \mathbf{u}_r| + n_r = |\mathbf{w}_1| + |\mathbf{w}_2| + |\mathbf{w}_3| = |\mathbf{w}|$ . Now, as  $\mathbf{w}_1 = \mathbf{v}_l \mathbf{u}_l$ ,  $\mathbf{w}_2 = \mathbf{t}$  and  $\mathbf{w}_3 = \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$  we obtain  $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}}(\rho)$ . This proves the claim.

**Proposition 6.2.9.** *For  $n_l, n_r \geq 0$  and  $(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_{\Gamma}$  define for the tuple  $\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$  an operator  $\tilde{H}_{\rho} : \mathcal{B}(\mathcal{H}_{\Gamma}) \rightarrow \mathcal{B}(\mathcal{H}_{\Gamma})$  as*

$$\tilde{H}_{\rho} = \sum_{\mathbf{r} \subseteq \mathbf{t}} (-1)^{|\mathbf{r}|} H_{(n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t}, \mathbf{r})}.$$

*Then we have for  $\mathbf{w} \in \mathcal{W}_{\Gamma}$ ,  $\omega \in \mathcal{S}_{\mathbf{w}}$  and  $a \in \mathbf{A}_{\Gamma}$  that*

$$\tilde{H}_{\rho}(\lambda_{\omega}(a)) = \begin{cases} \lambda_{\omega}(a) & \text{if } \omega \in \mathcal{S}_{\mathbf{w}}(\rho) \\ 0 & \text{else} \end{cases}. \quad (6.4)$$

*Proof.* Let  $\mathbf{w} \in \mathcal{W}_{\Gamma}$ ,  $\omega \in \mathcal{S}_{\mathbf{w}}$  and let  $a = a_1 \otimes \cdots \otimes a_t \in \mathbf{A}_{\Gamma}$  be a pure tensor. By Proposition 6.2.6 we have

$$\tilde{H}_{\rho}(\lambda_{\omega}(a)) = \sum_{\mathbf{r} \subseteq \mathbf{t}} (-1)^{|\mathbf{r}|} \lambda_{\omega}(a) P_a((\rho, \mathbf{r}), \omega).$$

Let  $\mathbf{v} \in \mathcal{W}_{\Gamma}$  and let  $\eta \in \mathcal{H}_{\Gamma}^{\circ} \subseteq \mathcal{H}_{\Gamma}$  be a pure tensor. If  $\lambda_{\omega}(a)\eta = 0$ , then it is clear that  $\tilde{H}_{\rho}(\lambda_{\omega}(a))\eta = 0$ , so that Equation (6.4) applied to  $\eta$  holds in either case. Thus assume  $\lambda_{\omega}(a)\eta \neq 0$ . Let  $\mathcal{I}_{\eta, \omega}$  be the set of all sub-clique words  $\mathbf{r} \subseteq \mathbf{t}$  s.t.  $P_a((\rho, \mathbf{r}), \omega)\eta \neq 0$ , that is

$$\mathcal{I}_{\eta, \omega} = \{\mathbf{r} \subseteq \mathbf{t} \mid P_a((\rho, \mathbf{r}), \omega)\eta \neq 0\}.$$

We prove the proposition using the following steps.

1) We prove that  $\mathcal{J}_{\eta,\omega}$  is closed under taking sub-cliques. Let  $\mathbf{r}_1 \subseteq \mathbf{r}_2 \subseteq \mathbf{t}$ , and suppose that  $\mathbf{r}_2 \in \mathcal{J}_{\eta,\omega}$ . Then we must have  $\eta \in \mathcal{J}_{\lambda_\omega(a)}(\mathcal{H}_{\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}}, \mathcal{H}_{\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{v}_{tail}})$  with  $\mathbf{v}_l \in \mathcal{W}'_{n_l}(\mathbf{u}_l \mathbf{t})$ ,  $\mathbf{v}_r \in \mathcal{W}'_{n_r}(\mathbf{u}_r \mathbf{t})$  and  $\mathbf{v}_{tail} \in \mathcal{W}(\mathbf{u}_l \mathbf{t}) \cap \mathcal{W}(\mathbf{u}_r \mathbf{t})$ , and  $|\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_2| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_2|$  and  $|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}|$ . This means that also

$$\begin{aligned} |\mathbf{v}_r \mathbf{u}_r \mathbf{t}| &\leq |\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_1| + |\mathbf{r}_1| \\ &\leq |\mathbf{w}_1 \mathbf{w}_2| + |\mathbf{w}_1 \mathbf{w}_2 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_1| + |\mathbf{r}_1| \\ &\leq |\mathbf{w}_1 \mathbf{w}_2| + |\mathbf{w}_1 \mathbf{w}_2 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_2| + |\mathbf{r}_2 \mathbf{r}_1| + |\mathbf{r}_1| \\ &= |\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_2| + |\mathbf{r}_2| \\ &= |\mathbf{v}_r \mathbf{u}_r \mathbf{t}| \end{aligned}$$

and so  $|\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_1| = |\mathbf{w}_1 \mathbf{w}_2| + |\mathbf{w}_1 \mathbf{w}_2 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_1|$ . This shows  $P_a((\rho, \mathbf{r}_1), \omega)\eta = \eta$ , hence  $\mathbf{r}_1 \in \mathcal{J}_{\eta,\omega}$ .

2) We prove that  $\mathcal{J}_{\eta,\omega}$  is closed under taking unions. Let  $\mathbf{r}_1, \mathbf{r}_2 \subseteq \mathbf{t}$  be sub-cliques with  $\mathbf{r}_1, \mathbf{r}_2 \in \mathcal{J}_{\eta,\omega}$ . Then  $P_a((\rho, \mathbf{r}_1), \omega)\eta = P_a((\rho, \mathbf{r}_2), \omega)\eta = \eta$ . Moreover, by previous step we moreover have  $P_a((\rho, e), \omega)\eta = \eta$ . We must now have  $\eta \in \mathcal{J}_{\lambda_\omega(a)}(\mathcal{H}_{\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}}, \mathcal{H}_{\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{v}_{tail}})$  with  $\mathbf{v}_l \in \mathcal{W}'_{n_l}(\mathbf{u}_l \mathbf{t})$ ,  $\mathbf{v}_r \in \mathcal{W}'_{n_r}(\mathbf{u}_r \mathbf{t})$  and  $\mathbf{v}_{tail} \in \mathcal{W}(\mathbf{u}_l \mathbf{t}) \cap \mathcal{W}(\mathbf{u}_r \mathbf{t})$ , and  $|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}|$  and moreover

$$|\mathbf{v}_r \mathbf{u}_r \mathbf{t}| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}| \quad (6.5)$$

$$|\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_1| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_1| \quad (6.6)$$

$$|\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_2| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_2|. \quad (6.7)$$

Now we note that also  $|\mathbf{v}_r \mathbf{u}_r \mathbf{t}| = |\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_1| + |\mathbf{r}_1| = |\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_2| + |\mathbf{r}_2|$ , hence

$$|\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}| = |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_1| + |\mathbf{r}_1| = |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}_2| + |\mathbf{r}_2|.$$

As  $\mathbf{r}_1, \mathbf{r}_2$  are cliques, this implies  $\mathbf{r}_1, \mathbf{r}_2 \subseteq \mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t})$  so that for  $\mathbf{r} = \mathbf{r}_1 \cup \mathbf{r}_2$  it holds that  $\mathbf{r} \subseteq \mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t})$ . But this implies

$$|\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}| = |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| + |\mathbf{r}|.$$

Now, as also  $|\mathbf{v}_r \mathbf{u}_r \mathbf{t}| = |\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| + |\mathbf{r}|$  we find using (6.5) that  $|\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}|$ . It now directly follows that  $P((\rho, \mathbf{r}), \omega)\eta = \eta$ . This shows that  $\mathbf{r} \in \mathcal{J}_{\eta,\omega}$ , and thus that  $\mathcal{J}_{\eta,\omega}$  is closed under taking unions.

3) We prove the equation  $\tilde{H}_\rho(\lambda_\omega(a))\eta = \mathbb{1}(\mathcal{J}_{\eta,\omega} = \{e\})\lambda_\omega(a)\eta$ . Here  $\mathbb{1}(\mathcal{J}_{\eta,\omega} = \{e\})$  denotes 1 whenever  $\mathcal{J}_{\eta,\omega} = \{e\}$  is satisfied, and 0 otherwise. In the case that  $\mathcal{J}_{\eta,\omega}$  is empty we directly find  $\tilde{H}_\rho(\lambda_\omega(a))\eta = 0$ , so that the equation is satisfied. Thus assume that  $\mathcal{J}_{\eta,\omega}$  is non-zero. Then as  $\mathcal{J}_{\eta,\omega}$  is closed under taking unions, there exists a maximal element  $\mathbf{r}_{\eta,\omega} \in \mathcal{J}_{\eta,\omega}$ . However, since  $\mathcal{J}_{\eta,\omega}$  is also closed under taking sub-cliques, we then find

$\mathcal{I}_{\eta,\omega} = \{\mathbf{r} \subseteq \mathbf{r}_{\eta,\omega}\}$ . We conclude that

$$\begin{aligned} \tilde{H}_\rho(\lambda_\omega(a))\eta &= \sum_{\mathbf{r} \subseteq \mathbf{t}} (-1)^{|\mathbf{r}|} \lambda_\omega(a) P_a((\rho, \mathbf{r}), \omega) \eta \\ &= \sum_{\mathbf{r} \subseteq \mathbf{r}_{\eta,\omega}} (-1)^{|\mathbf{r}|} \lambda_\omega(a) \eta \\ &= \mathbb{1}(\mathbf{r}_{\eta,\omega} = e) \lambda_\omega(a) \eta \\ &= \mathbb{1}(\mathcal{I}_{\eta,\omega} = \{e\}) \lambda_\omega(a) \eta. \end{aligned}$$

4) We will now show, for a pure tensor  $\eta \in \mathring{\mathcal{H}}_\nu \subseteq \mathcal{H}_\Gamma$  with  $\lambda_\omega(a)\eta \neq 0$ , that  $\mathcal{I}_{\eta,\omega} = \{e\}$  if and only if  $\omega \in \mathcal{S}_\mathbf{w}(\rho)$ . First, suppose that  $\omega \in \mathcal{S}_\mathbf{w}(\rho)$ . Then we can write  $\omega = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ , where  $\mathbf{w}_1 = \mathbf{v}_l \mathbf{u}_l$  and  $\mathbf{w}_2 = \mathbf{t}$  and  $\mathbf{w}_3 = \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$  for some  $\mathbf{v}_l \in \widetilde{\mathcal{W}}_{n_l}'(\mathbf{u}_l \mathbf{t})$  and  $\mathbf{v}_r \in \widetilde{\mathcal{W}}_{n_r}'(\mathbf{u}_r \mathbf{t})$ . Then as  $\lambda_\omega(a)\eta \neq 0$ , we must have that  $\eta \in \mathcal{I}_{\lambda_\omega(a)}(\mathring{\mathcal{H}}_{\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}}, \mathring{\mathcal{H}}_{\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{v}_{tail}})$  for some  $\mathbf{v}_{tail} \in \mathcal{W}(\mathbf{u}_l \mathbf{t}) \cap \mathcal{W}(\mathbf{u}_r \mathbf{t})$ . It is clear that

$$\begin{aligned} |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}| &= |\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{v}_{tail}| \\ &= |\mathbf{v}_l \mathbf{u}_l| + |\mathbf{t} \mathbf{v}_{tail}| \\ &= |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}|. \end{aligned}$$

Moreover, as  $\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \subseteq \mathbf{t}$  it is also clear that  $|\mathbf{v}_r \mathbf{u}_r \mathbf{t}| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}|$ . This shows that  $P_a((\rho, e), \omega)\eta = \eta$ , hence  $e \in \mathcal{I}_{\eta,\omega}$ .

Now let  $\mathbf{r} \subseteq \mathbf{t}$  be a sub-clique with  $\mathbf{r} \neq e$ . Then we have  $\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r} = \mathbf{r}$ . Hence, we have

$$\begin{aligned} |\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| + |\mathbf{r}| &= |\mathbf{v}_r \mathbf{u}_r \mathbf{t}| \\ &= |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}| \\ &= |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| - |\mathbf{r}|. \end{aligned}$$

Now as  $\mathbf{r} \neq e$  we have  $|\mathbf{r}| \geq 1$ , which shows that  $|\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}| \neq |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{r}|$ . This proves that  $P_a((\rho, \mathbf{r}), \omega)\eta = 0$ . Thus  $\mathbf{r} \notin \mathcal{I}_{\eta,\omega}$ . This shows  $\mathcal{I}_{\eta,\omega} = \{e\}$ .

Now, let  $\omega \in \mathcal{S}_\mathbf{w}$  for some  $\mathbf{w} \in \mathcal{W}_\Gamma$  be s.t.  $\mathcal{I}_{\eta,\omega} = \{e\}$ . Then  $P((\rho, e), \omega)\eta = \eta$ . Hence  $\eta \in \mathcal{I}_{\lambda_\omega(a)}(\mathring{\mathcal{H}}_{\mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}}, \mathring{\mathcal{H}}_{\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{v}_{tail}})$  for some  $\mathbf{v}_l \in \widetilde{\mathcal{W}}_{n_l}'(\mathbf{u}_l \mathbf{t})$ ,  $\mathbf{v}_r \in \widetilde{\mathcal{W}}_{n_r}'(\mathbf{u}_r \mathbf{t})$  and  $\mathbf{v}_{tail} \in \mathcal{W}(\mathbf{u}_l \mathbf{t}) \cap \mathcal{W}(\mathbf{u}_r \mathbf{t})$  and  $|\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}| = |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}|$  and  $|\mathbf{v}_r \mathbf{u}_r \mathbf{t}| = |\mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}|$ . Now as also  $\lambda_\omega(a)\eta \in \mathring{\mathcal{H}}_{\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail}}$ , and as  $\lambda_\omega(a)\eta \neq 0$ , we have that  $\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} \mathbf{v}_{tail} = \mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{v}_{tail}$ . Hence,  $\mathbf{w}_1 \mathbf{w}_3 = \mathbf{v}_l \mathbf{u}_l \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$ . Now, as  $P_a((\rho, \mathbf{r}), \omega)\eta = 0$  for all  $\mathbf{r} \subseteq \mathbf{t}$  with  $\mathbf{r} \neq e$ , we must have that  $\mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}) \cap \mathbf{t} = e$ . However, multiplying  $\mathbf{w}_2 \mathbf{w}_3$  with  $\mathbf{v}_r \mathbf{u}_r \mathbf{t}$  removes all letters from  $\mathbf{w}_2 \mathbf{w}_3$ . This means that  $\mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}) \subseteq \mathbf{s}_r(\mathbf{v}_r \mathbf{u}_r \mathbf{t}) = \mathbf{s}_r(\mathbf{u}_r \mathbf{t})$ . Now we also have

$$\begin{aligned} |\mathbf{v}_l \mathbf{u}_l \mathbf{t}| &\leq |\mathbf{w}_2 \mathbf{w}_1^{-1}| + |\mathbf{w}_2 \mathbf{w}_1^{-1} \mathbf{v}_l \mathbf{u}_l \mathbf{t}| \\ &= |\mathbf{w}_2 \mathbf{w}_1^{-1}| + |\mathbf{v}_r \mathbf{u}_r \mathbf{t}| - |\mathbf{w}_2 \mathbf{w}_3| \\ &\leq |\mathbf{w}_2 \mathbf{w}_1^{-1}| + |\mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}| - |\mathbf{w}_2| \\ &= |\mathbf{w}_1| + |\mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}| \\ &= |\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}| \\ &= |\mathbf{v}_l \mathbf{u}_l \mathbf{t}| \end{aligned}$$

so that  $|\mathbf{v}_l \mathbf{u}_l \mathbf{t}| = |\mathbf{w}_2 \mathbf{w}_1^{-1}| + |\mathbf{w}_2 \mathbf{w}_1^{-1} \mathbf{v}_l \mathbf{u}_l \mathbf{t}|$ . Now this means that  $\mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_1^{-1} \mathbf{v}_l \mathbf{u}_l \mathbf{t}) \subseteq \mathbf{s}_r(\mathbf{v}_l \mathbf{u}_l \mathbf{t}) = \mathbf{s}_r(\mathbf{u}_l \mathbf{t})$ . Hence, as  $\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} = \mathbf{w}_2 \mathbf{w}_1^{-1} \mathbf{v}_l \mathbf{u}_l \mathbf{t}$ , we find  $\mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}) \subseteq \mathbf{u}_l \mathbf{t} \cap \mathbf{u}_r \mathbf{t} = \mathbf{t}$ . However, as also  $\mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}) \cap \mathbf{t} = e$ , we conclude that  $\mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t}) = e$ , so  $\mathbf{w}_2 \mathbf{w}_1^{-1} \mathbf{v}_l \mathbf{u}_l \mathbf{t} = \mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{t} = e$ . But this means that  $\mathbf{w}_3^{-1} \mathbf{w}_2 = \mathbf{v}_r \mathbf{u}_r \mathbf{t}$  and  $\mathbf{w}_1 \mathbf{w}_2 = \mathbf{v}_l \mathbf{u}_l \mathbf{t}$ . From this it follows that  $\mathbf{w}_2 \subseteq \mathbf{s}_r(\mathbf{v}_l \mathbf{u}_l \mathbf{t}) \cap \mathbf{s}_r(\mathbf{v}_r \mathbf{u}_r \mathbf{t}) = \mathbf{t}$ . Now, we can not have that  $\mathbf{w}_2 \subseteq \mathbf{t}$  strictly, as this would mean that  $\mathbf{w}_3$  starts with a part of  $\mathbf{t}$  that  $\mathbf{w}_1$  ends with, which would contradict the fact that  $\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3$  is reduced. Thus we now find  $\mathbf{w}_2 = \mathbf{t}$  and then also  $\mathbf{w}_1 = \mathbf{v}_l \mathbf{u}_l$  and  $\mathbf{w}_3 = \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$ . This means that  $\omega \in \mathcal{S}_{\mathbf{w}}(\rho)$ .

5) We now conclude the proof of the proposition as we have shown for  $\mathbf{w} \in \mathcal{W}_{\Gamma}$ ,  $\omega \in \mathcal{S}_{\mathbf{w}}$ , pure tensor  $a = a_1 \otimes \cdots \otimes a_t \in \mathbf{A}_{\Gamma}$  and pure tensor  $\eta \in \mathcal{H}_{\mathbf{v}} \subseteq \mathcal{H}_{\Gamma}$  with  $\lambda_{\omega}(a)\eta \neq 0$  that

$$\tilde{H}_{\rho}(\lambda_{\omega}(a))\eta = \begin{cases} \lambda_{\omega}(a)\eta & \mathcal{I}_{\eta, \omega} = \{e\} \\ 0 & \text{else} \end{cases} = \begin{cases} \lambda_{\omega}(a)\eta & \omega \in \mathcal{S}_{\mathbf{w}}(\rho) \\ 0 & \text{else} \end{cases}.$$

Now, as noted earlier, the equation is also satisfied when  $\eta$  is a pure tensor with  $\lambda_{\omega}(a)\eta = 0$ . Therefore, by linearity and continuity, the equation in the proposition holds for all  $\eta \in \mathcal{H}_{\Gamma}$ . By linearity of  $\tilde{H}_{\rho}$  and  $\lambda_{\omega}$  the equation also holds for all  $a \in \mathbf{A}_{\Gamma}$ . This proves the statement.  $\square$

We now prove our main theorem of this section, that shows that  $\|P_{\Gamma, d}\|_{\text{cb}}$  is polynomially bounded in  $d$ .

**Theorem 6.2.10.** *For  $d \geq 0$  we have (on  $\mathbf{A}_{\Gamma}$ ) that*

$$P_{\Gamma, d} = \sum_{\substack{(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_{\Gamma} \\ 0 \leq n \leq d - |\mathbf{u}_l \mathbf{t} \mathbf{u}_r|}} \sum_{\mathbf{r} \subseteq \mathbf{t}} (-1)^{|\mathbf{r}|} H_{(n, d - n - |\mathbf{u}_l \mathbf{t} \mathbf{u}_r|, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t}, \mathbf{r})}.$$

Moreover, for  $d \geq 1$  we get the linear bound  $\|P_{\Gamma, d}\|_{\text{cb}} \leq C_{\Gamma} d$ , where  $C_{\Gamma}$  denotes the constant

$$C_{\Gamma} = \sum_{(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_{\Gamma}} 2^{|\mathbf{t}|}.$$

*Proof.* For  $d \geq 0$  define

$$\mathcal{T}_{\Gamma, d} = \{\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathbb{Z}_{\geq 0}^2 \times \mathcal{T}_{\Gamma} : |\rho| = d\}. \quad (6.8)$$

We recall for  $\mathbf{w} \in \mathcal{W}_{\Gamma}$  that  $\{\mathcal{S}_{\mathbf{w}}(\rho)\}_{\rho \in \mathcal{T}_{|\mathbf{w}|}}$  is a partition of  $\mathcal{S}_{\mathbf{w}}$  by Remark 6.2.8. Fix some

$a \in \mathbf{A}_\Gamma$ . For  $d \geq 0$  we find using Lemma 3.1.7 that

$$\begin{aligned}
 P_{\Gamma,d}(\lambda(a)) &= \sum_{\mathbf{w} \in \mathcal{W}_\Gamma, |\mathbf{w}|=d} \sum_{\omega \in \mathcal{S}_\mathbf{w}} \lambda_\omega(a) \\
 &= \sum_{\rho \in \mathcal{T}_{\Gamma,d}} \sum_{\mathbf{w} \in \mathcal{W}_\Gamma, |\mathbf{w}|=d} \sum_{\omega \in \mathcal{S}_\mathbf{w}(\rho)} \lambda_\omega(a) \\
 &= \sum_{\rho \in \mathcal{T}_{\Gamma,d}} \tilde{H}_\rho \left( \sum_{\mathbf{w} \in \mathcal{W}_\Gamma} \sum_{\omega \in \mathcal{S}_\mathbf{w}} \lambda_\omega(a) \right) \\
 &= \sum_{\rho \in \mathcal{T}_{\Gamma,d}} \tilde{H}_\rho(\lambda(a)) \\
 &= \sum_{\substack{(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_\Gamma \\ 0 \leq n \leq d - |\mathbf{u}_l \mathbf{t} \mathbf{u}_r|}} \sum_{\mathbf{r} \leq \mathbf{t}} (-1)^{|\mathbf{r}|} H_{(n, d - |\mathbf{u}_l \mathbf{t} \mathbf{u}_r| - n, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t}, \mathbf{r})}(\lambda(a)).
 \end{aligned}$$

Therefore, the equation holds on  $\lambda(\mathbf{A}_\Gamma)$  and hence, by continuity, on  $A_\Gamma$ .

Now let  $d \geq 1$ , we show that the bound holds. We note first that by definition  $V_n^{e,e} = 0$  for  $n \geq 1$ . This implies directly that  $H_{(n, d - n - |\mathbf{u}_l \mathbf{t} \mathbf{u}_r|, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t}, e)} = 0$  for  $0 \leq n \leq d - |\mathbf{u}_l \mathbf{t} \mathbf{u}_r|$  whenever  $(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) = (e, e, e)$ . Therefore we find

$$\begin{aligned}
 \|P_{\Gamma,d}\|_{\text{cb}} &\leq \sum_{\substack{(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_\Gamma \setminus \{(e,e,e)\} \\ 0 \leq n \leq d - |\mathbf{u}_l \mathbf{t} \mathbf{u}_r|}} \sum_{\mathbf{r} \leq \mathbf{t}} \|H_{(n, d - n - |\mathbf{u}_l \mathbf{t} \mathbf{u}_r|, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t}, \mathbf{r})}\|_{\text{cb}} \\
 &\leq \sum_{\substack{(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_\Gamma \setminus \{(e,e,e)\} \\ 0 \leq n \leq d - |\mathbf{u}_l \mathbf{t} \mathbf{u}_r|}} 2^{|\mathbf{t}|} \\
 &\leq \left( \sum_{(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_\Gamma} 2^{|\mathbf{t}|} \right) d.
 \end{aligned}$$

□

### 6.3. GRAPH PRODUCTS OF STATE-PRESERVING U.C.P MAPS

In Section 6.3.1 we show that the graph product of state-preserving u.c.p maps extends to a state-preserving u.c.p map. Thereafter, in Section 6.3.2, we use this to obtain the result that the graph product of finite-dimensional algebras with GNS-faithful states is weakly amenable with constant 1.

#### 6.3.1. GRAPH PRODUCTS OF STATE-PRESERVING UCP MAPS

Let  $\Gamma$  be a graph, and for  $v \in \Gamma$  let  $\theta_v : A_v \rightarrow B_v$  be state-preserving maps between unital  $C^*$ -algebras (with states s.t. the GNS representation is faithful). Let  $(A_\Gamma, \varphi) = *_{v \in \Gamma} (A_v, \varphi_v)$  and  $(B_\Gamma, \psi) = *_{v \in \Gamma} (B_v, \psi_v)$  be their reduced graph products. As  $\theta_v$  is state preserving it maps  $\dot{A}_v$  to  $\dot{B}_v$ . We can look at the map  $\theta : \lambda(\mathbf{A}_\Gamma) \rightarrow \lambda(\mathbf{B}_\Gamma)$  for  $a_1 \otimes \cdots \otimes a_s \in \dot{A}_{v_1} \otimes \cdots \otimes \dot{A}_{v_s}$  for a reduced word  $v_1 \cdots v_s$  given as

$$\theta(\lambda(a_1 \otimes \cdots \otimes a_s)) = \lambda(\theta_{v_1}(a_1) \otimes \cdots \otimes \theta_{v_s}(a_s)) \quad (6.9)$$

and we set  $\theta(\text{Id}) = \text{Id}$ . We denote this map by  $\theta = *_{v \in \Gamma} \theta_v$  and call it the graph product map. The map is clearly state-preserving. To prove the main theorem, we need the result

that the graph product map  $\theta = *_{v,\Gamma}\theta_v$  of state-preserving u.c.p maps  $\theta_v$  extends to a bounded map on the graph product, and that it is again u.c.p. This result was already proven by Blanchard-Dykema in [BD01] for the case of free products. For graph products the result has been proven by Caspers-Fima in [CF17, Proposition 3.30] in the setting of von Neumann algebras.

**Proposition 6.3.1.** [CF17, Proposition 3.30] *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$ , let  $\theta_v : M_v \rightarrow N_v$  be state-preserving normal u.c.p. maps between von Neumann algebras  $M_v$  and  $N_v$  that have faithful normal states. Let  $(M_\Gamma, \varphi) = *_{v,\Gamma}(M_v, \varphi_v)$  and  $(N_\Gamma, \psi) = *_{v,\Gamma}(N_v, \psi_v)$  be the von Neumann algebraic graph products. Then there exists a unique normal u.c.p. map  $\theta : M_\Gamma \rightarrow N_\Gamma$  s.t. for all pure tensors  $a_1 \otimes \cdots \otimes a_s \in \dot{M}_{v_1} \otimes \cdots \otimes \dot{M}_{v_s}$  we have*

$$\theta(\lambda(a_1 \otimes \cdots \otimes a_s)) = \lambda(\theta_{v_1}(a_1) \otimes \cdots \otimes \theta_{v_s}(a_s)). \quad (6.10)$$

The map  $\theta$  will be denoted as  $\theta = *_{\Gamma}\theta_v$

We give here a proof for the case of  $C^*$ -algebras.

**Proposition 6.3.2.** *For  $v \in \Gamma$  let  $\theta_v : A_v \rightarrow B_v$  be state-preserving, unital completely positive maps between unital  $C^*$ -algebras  $(A_v, \varphi_v)$  and  $(B_v, \psi_v)$ , and assume  $\varphi_v$  and  $\psi_v$  are GNS-faithful. Then the graph product map  $\theta = *_{v,\Gamma}\theta_v$  extends to a state-preserving unital completely positive map between the reduced graph products  $A_\Gamma$  and  $B_\Gamma$ .*

*Proof.* We will use the notation  $\mathcal{H}_v^A, \mathcal{H}_v^{\circ A}, \mathcal{H}_\Gamma^A, \lambda^A, \xi_v^A, \Omega^A$ , et cetera, corresponding to the reduced graph product  $(A_\Gamma, \varphi) := *_{v,\Gamma}^{\min}(A_v, \varphi_v)$ , and use similar notation for the reduced graph product  $(B_\Gamma, \psi) := *_{v,\Gamma}^{\min}(B_v, \psi_v)$ . By the Stinespring's dilation theorem, Theorem 2.1.1, we can write  $\theta_v(a) = V_v^* \pi_v(a) V_v$  for some Hilbert space  $\widehat{\mathcal{H}}_v$  and unital  $*$ -homomorphism  $\pi_v : A_v \rightarrow B(\widehat{\mathcal{H}}_v)$  and some isometry  $V_v \in B(\mathcal{H}_v^B, \widehat{\mathcal{H}}_v)$ . We note that for  $a \in A_v$  we have  $\varphi_v(a) = \psi_v(\theta_v(a)) = \langle \theta_v(a) \xi_v^B, \xi_v^B \rangle = \langle \pi_v(a) \widehat{\xi}_v, \widehat{\xi}_v \rangle$  with  $\widehat{\xi}_v := V_v \xi_v^B$ . Also  $\pi_v$  is faithful, as  $\pi_v(a) = 0$  implies for  $b \in A_v$  that

$$0 = \|\pi_v(a) \pi_v(b) \widehat{\xi}_v\|^2 = \|\pi_v(ab) \widehat{\xi}_v\|^2 = \langle \pi_v(b^* a^* ab) \widehat{\xi}_v, \widehat{\xi}_v \rangle = \varphi_v(b^* a^* ab),$$

which implies  $a = 0$  since  $\varphi_v$  is GNS-faithful. By these properties we conclude that we can construct the graph product of the  $A_v$ 's w.r.t. the representations  $\pi_v$ . To distinguish the notation from the other graph products we use *hat*-notation like  $\widehat{\mathcal{H}}_v, \widehat{\mathcal{H}}_v^{\circ}, \widehat{\mathcal{H}}_\Gamma, \widehat{\lambda}, \widehat{\Omega}$ . Define a contraction  $V : \mathcal{H}_\Gamma^B \rightarrow \widehat{\mathcal{H}}_\Gamma$  for  $\eta = \eta_1 \otimes \cdots \otimes \eta_l \in \mathcal{H}_\Gamma^B$  as

$$V|_{\mathcal{H}_\Gamma^B}(\eta_1 \otimes \cdots \otimes \eta_l) = V_{v_1} \eta_1 \otimes \cdots \otimes V_{v_l} \eta_l \quad (6.11)$$

and  $V(\Omega^B) = \widehat{\Omega}$ . We note that  $\eta_i \in \mathcal{H}_{v_i}^B$  implies  $\langle V\eta_i, \widehat{\xi}_{v_i} \rangle = \langle V\eta_i, V\xi_{v_i}^B \rangle = \langle \eta_i, \xi_{v_i}^B \rangle = 0$  and hence  $V\eta_i \in \widehat{\mathcal{H}}_{v_i}^{\circ}$ . This shows that  $V$  is well-defined.

By [CF17, Proposition 3.12], we know that there exists a state-preserving, unital  $*$ -homomorphism  $\pi : A_\Gamma \rightarrow B(\widehat{\mathcal{H}}_\Gamma)$  that for  $a = a_1 \otimes \cdots \otimes a_l \in \dot{A}_\Gamma$  is given by

$$\pi(\lambda^A(a_1 \otimes \cdots \otimes a_l)) = \widehat{\lambda}(\pi_{v_1}(a_1) \otimes \cdots \otimes \pi_{v_l}(a_l)) \quad (6.12)$$

We will now show that  $\theta(\lambda^A(a)) = V^* \pi(\lambda^A(a)) V$  for  $a \in \mathbf{A}_\Gamma$ , which then shows that  $\theta$  can be extended to a u.c.p. map on  $\mathbf{A}_\Gamma$ .

Let  $\eta = \eta_1 \otimes \cdots \otimes \eta_l \in \mathcal{H}_\mathbf{v}^B$  for some  $\mathbf{v} \in \mathcal{W}_\Gamma$  and let  $a \in \mathring{A}_\nu$  for some  $\nu \in \Gamma$ . We will calculate  $\hat{\lambda}_\nu(\pi_\nu(a)) V$ . First suppose that  $\nu \mathbf{v}$  is reduced. We have  $\langle (\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a) \hat{\xi}_\nu, \hat{\xi}_\nu \rangle = \langle \pi_\nu(a) \hat{\xi}_\nu, 0 \rangle = 0$  so that

$$\begin{aligned} \hat{\lambda}_\nu((\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a)) V \eta &= \hat{U}_\nu((\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a) \otimes \text{Id}_{\mathcal{H}_\Gamma})(\hat{\xi}_\nu \otimes V \eta) \\ &= \hat{U}_\nu((\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a) \hat{\xi}_\nu \otimes V \eta) \\ &= \hat{\mathcal{Q}}_{(\nu, \mathbf{v})}(((\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a) \hat{\xi}_\nu \otimes V \eta)). \end{aligned}$$

Also we have  $\langle V_\nu V_\nu^* \pi_\nu(a) \hat{\xi}_\nu, \hat{\xi}_\nu \rangle = \varphi_\nu(a) = 0$  and so we find

$$\begin{aligned} \hat{\lambda}_\nu(V_\nu V_\nu^* \pi_\nu(a)) V \eta &= \hat{U}_\nu(V_\nu V_\nu^* \pi_\nu(a) \otimes \text{Id}_{\mathcal{H}_\Gamma})(\hat{\xi}_\nu \otimes V \eta) \\ &= \hat{U}_\nu(V_\nu V_\nu^* \pi_\nu(a) \hat{\xi}_\nu \otimes V \eta) \\ &= \hat{\mathcal{Q}}_{(\nu, \mathbf{v})}((V_\nu V_\nu^* \pi_\nu(a) \hat{\xi}_\nu \otimes V \eta) \\ &= \hat{\mathcal{Q}}_{(\nu, \mathbf{v})}((V_\nu \theta_\nu(a) \xi_\nu^B) \otimes V \eta) \\ &= V \mathcal{Q}_{(\nu, \mathbf{v})}^B((\theta_\nu(a) \xi_\nu^B) \otimes \eta) \\ &= V \lambda_\nu^B(\theta_\nu(a)) \eta. \end{aligned}$$

Now, suppose instead that  $\mathbf{v}$  starts with  $\nu$ . Then we can write  $\eta = \mathcal{Q}_{(\nu, \nu \mathbf{v})}^B(\eta_0 \otimes \eta')$  for some  $\eta_0 \in \mathcal{H}_\nu^B$  and  $\eta' \in \mathcal{H}_{\nu \mathbf{v}}^B$  and we have  $V \eta = \hat{\mathcal{Q}}_{(\nu, \nu \mathbf{v})}(V_\nu \eta_0 \otimes V \eta')$ . Again we have that  $\langle (\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a) V_\nu \eta_0, \hat{\xi}_\nu \rangle = 0$  and so

$$\begin{aligned} \hat{\lambda}_\nu((\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a)) V \eta &= \hat{U}_\nu((\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a) \otimes \text{Id}_{\mathcal{H}_\Gamma}) \hat{U}_\nu^* V \eta \\ &= \hat{U}_\nu((\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a) \otimes \text{Id}_{\mathcal{H}_\Gamma})(V \eta_0 \otimes V \eta') \\ &= \hat{U}_\nu(((\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a) V_\nu \eta_0) \otimes V \eta') \\ &= \hat{\mathcal{Q}}_{(\nu, \nu \mathbf{v})}(((\text{Id}_{\hat{H}_\nu} - V_\nu V_\nu^*) \pi_\nu(a) V_\nu \eta_0) \otimes V \eta'). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \hat{\lambda}_\nu(V_\nu V_\nu^* \pi_\nu(a)) V \eta &= \hat{U}_\nu(V_\nu V_\nu^* \pi_\nu(a) \otimes \text{Id}_{\mathcal{H}_\Gamma}) \hat{U}_\nu^* V \eta \\ &= \hat{U}_\nu(V_\nu V_\nu^* \pi_\nu(a) \otimes \text{Id}_{\mathcal{H}_\Gamma})(V \eta_0 \otimes V \eta') \\ &= \hat{U}_\nu(V_\nu V_\nu^* \pi_\nu(a) V_\nu \eta_0 \otimes V \eta') \\ &= \hat{U}_\nu((V_\nu \theta_\nu(a) \eta_0) \otimes V \eta') \\ &= V U_\nu^B((\theta_\nu(a) \eta_0) \otimes \eta') \\ &= V U_\nu^B(\theta_\nu(a) \otimes \text{Id}_{\mathcal{H}_\Gamma})(U_u^B)^* \eta \\ &= V \lambda_\nu^B(\theta_\nu(a)) \eta. \end{aligned}$$

Now, when  $a = a_1 \otimes \cdots \otimes a_k \in \mathring{\mathbf{A}}_{\mathbf{W}}$ , then we have

$$\begin{aligned} V^* \pi(\lambda^A(a)) V \eta &= V^* \hat{\lambda}(\pi_{w_1}(a_1)) \cdots \hat{\lambda}(\pi_{w_{k-1}}(a_{k-1})) \hat{\lambda}(V_{w_k} V_{w_k}^* \pi_{w_k}(a_k)) V \eta \\ &\quad + V^* \hat{\lambda}(\pi_{w_1}(a_1)) \cdots \hat{\lambda}(\pi_{w_{k-1}}(a_{k-1})) \hat{\lambda}((\text{Id}_{\mathcal{H}_{w_k}} - V_{w_k} V_{w_k}^*) \pi_{w_k}(a_k)) V \eta \\ &= V^* \hat{\lambda}(\pi_{w_1}(a_1)) \cdots \hat{\lambda}(\pi_{w_{k-1}}(a_{k-1})) \hat{\lambda}(V_{w_k} V_{w_k}^* \pi_{w_k}(a_k)) V \eta \\ &= V^* \pi(\lambda^A(a_1 \otimes \cdots \otimes a_{k-1})) V \lambda^B(\theta_{w_k}(a_k)) \eta. \end{aligned}$$

Note here that the reason why we can remove the second summand is because one tensor leg of  $\hat{\lambda}((\text{Id}_{\mathcal{H}_{w_k}} - V_{w_k} V_{w_k}^*) \pi_{w_k}(a_k)) V \eta$  is of the form  $(\text{Id}_{\mathcal{H}_{w_k}} - V_{w_k} V_{w_k}^*) \pi_{w_k}(a_k) V_{w_k} \eta_0$  for some  $\eta_0 \in \mathcal{H}_{w_k}^B$ . This tensor leg is not changed by the operator  $\pi(\lambda^A(a_1 \otimes \cdots \otimes a_{k-1}))$  as it may not act on the same letter. Now after the application of  $V^*$  we obtain for this tensor leg that  $V_{w_k}^* (\text{Id}_{\mathcal{H}_{w_k}} - V_{w_k} V_{w_k}^*) \pi_{w_k}(a_k) V_{w_k} \eta_0 = 0$ , so that this term vanishes.

By what we showed, it now follows from induction to the tensor length  $k$  that for all  $a \in \mathbf{A}_{\Gamma}$  we have  $V^* \pi(\lambda^A(a)) V = \theta(\lambda^A(a))$ . This then shows the statement.  $\square$

### 6.3.2. CCAP FOR REDUCED GRAPH PRODUCTS OF FINITE-DIMENSIONAL ALGEBRAS

We now state the following generalization of [RX06, Proposition 3.5.] to graph products. The proof uses Theorem 6.2.10 and Proposition 6.3.1 and Proposition 6.3.2 and goes analogously to [RX06, Proposition 3.5.].

**Proposition 6.3.3.** *Let  $\Gamma$  be a finite simple graph. For  $v \in \Gamma$  let  $A_v$  be a unital  $C^*$ -algebra together with a GNS-faithful state  $\varphi_v$ . Let  $(A_{\Gamma}, \varphi) := *_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  be the reduced graph product. For  $d \geq 0$  let  $P_{\Gamma, d} : A_{\Gamma} \rightarrow A_{\Gamma, d}$  be the natural projection. Let  $0 \leq r \leq 1$ ,  $n \in \mathbb{N}$  and define*

$$\mathcal{T}_r = \sum_{k=0}^{\infty} r^k P_{\Gamma, k} \quad \mathcal{T}_{r, n} = \sum_{k=0}^n r^k P_{\Gamma, k}.$$

Then  $\mathcal{T}_r$  and  $\mathcal{T}_{r, n}$  are completely bounded with

$$\|\mathcal{T}_r\|_{\text{cb}} \leq 1 \quad \text{and} \quad \|\mathcal{T}_r - \mathcal{T}_{r, n}\|_{\text{cb}} \leq \frac{C_{\Gamma} n r^n}{(1-r)^2}. \quad (6.13)$$

The maps  $\mathcal{T}_{e^{-t}}$  for  $t \geq 0$  form a one-parameter semi-group of unital completely positive maps on  $A_{\Gamma}$  preserving the state  $\varphi$ . Moreover, the sequence  $(\mathcal{T}_{1-\frac{1}{\sqrt{n}}, n})_{n \geq 1}$  tends pointwise to the identity of  $A_{\Gamma}$  and  $\lim_{n \rightarrow \infty} \|\mathcal{T}_{1-\frac{1}{\sqrt{n}}, n}\|_{\text{cb}} = 1$ .

*Proof.* For  $v \in \Gamma$  we define a state-preserving u.c.p map  $U_{r, v} : A_v \rightarrow A_v$  as  $U_{r, v}(a) = r a + (1-r) \varphi_v(a) \text{Id}_{\mathcal{H}_v}$ . It can be seen that  $*_{v \in \Gamma} U_{r, v} = \mathcal{T}_r$  on  $\lambda(\mathbf{A}_{\Gamma})$  and by Proposition 6.3.2 this map extends to a state-preserving u.c.p map on  $A_{\Gamma}$ . Thus  $\|\mathcal{T}_r\|_{\text{cb}} = 1$ . Furthermore,

$$\|\mathcal{T}_r - \mathcal{T}_{r, n}\|_{\text{cb}} \leq \sum_{k=n+1}^{\infty} r^k \|P_{\Gamma, k}\|_{\text{cb}} \leq C_{\Gamma} \sum_{k=n}^{\infty} k r^k = C_{\Gamma} r \frac{d}{dr} \left( \frac{r^n}{1-r} \right) \quad (6.14)$$

Therefore, as  $\frac{d}{dr} \left( \frac{r^n}{1-r} \right) = nr^{n-1}(1-r)^{-1} + r^n(1-r)^{-2} \leq nr^{n-1}(1-r)^{-2}$  this proves (6.13). It is furthermore clear that  $(\mathcal{T}_{e^{-t}})_{t \geq 0}$  forms a semi-group since  $P_{\Gamma, m} P_{\Gamma, n} = 0$  when  $n \neq m$ . By (6.13) and by the triangle inequality we have  $\|\mathcal{T}_{1-\frac{1}{\sqrt{n}}, n}\|_{\text{cb}} \leq 1 + C_{\Gamma} n^2 (1 - \frac{1}{\sqrt{n}})^n \rightarrow 1$  as  $n \rightarrow \infty$  which shows  $\lim_{n \rightarrow \infty} \|\mathcal{T}_{1-\frac{1}{\sqrt{n}}, n}\|_{\text{cb}} = 1$  since the maps  $\mathcal{T}_{1-\frac{1}{\sqrt{n}}, n}$  are unital. Moreover, on  $\lambda(A_{\Gamma})$  it is clear that  $(\mathcal{T}_{1-\frac{1}{\sqrt{n}}, n})_{n \geq 1}$  tends pointwise to the identity. Therefore, as  $(\mathcal{T}_{1-\frac{1}{\sqrt{n}}, n})_{n \geq 1}$  is uniformly bounded it follows by density that this holds true on  $A_{\Gamma}$  as well.  $\square$

**Corollary 6.3.4.** *For  $v \in \Gamma$  let  $A_v$  be a finite-dimensional  $C^*$ -algebras together with a GNS-faithful state  $\varphi_v$ . Then the reduced graph product  $A_{\Gamma}$  has the CCAP. Similarly, for finite dimensional von Neumann algebras  $M_v$  together with normal faithful states  $\varphi_v$ , we have that the graph product  $M_{\Gamma}$  has the weak-\* CCAP.*

We give an application of this result to Hecke-algebras (for references on Hecke-algebras see [Dav08, Chapter 19]). Let  $\mathcal{W}$  be a Coxeter group generated by some set  $S$  and let  $\mathbf{q} = (q_s)_{s \in S}$  be a Hecke tuple (i.e.  $q_s > 0$  for all  $s \in S$  and  $q_s = q_t$  whenever  $s$  and  $t$  are conjugate in  $\mathcal{W}$ ). Denote  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  for the Hecke algebra corresponding to  $\mathcal{W}$  and  $\mathbf{q}$ . Our application uses the following proposition which asserts that we can decompose Hecke algebras as graph products. This is somewhat similar to Remark 5.2.4. Furthermore, the result for right-angled Coxeter groups is stated in [Cas20, Corollary 3.4].

**Proposition 6.3.5.** *Let  $\Gamma$  be a graph, and for  $v \in \Gamma$  let  $\mathcal{W}_v$  be a Coxeter group generated by a set  $S_v$  and let  $\mathbf{q}_v = (q_{v,s})_{s \in S_v}$  be a Hecke-tuple. Set  $\mathcal{W} = *_{v \in \Gamma} \mathcal{W}_v$  and  $\mathbf{q} := *_{v \in \Gamma} \mathbf{q}_v = (q_{v,s})_{v \in \Gamma, s \in S_v}$ . Then we get a graph product decomposition  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}) = *_{v \in \Gamma} \mathcal{N}_{\mathbf{q}_v}(\mathcal{W}_v)$ .*

*Proof.* This follows from [CF17, Proposition 3.22] by considering the natural embeddings  $\pi_v : \mathcal{N}_{\mathbf{q}_v}(\mathcal{W}_v) \rightarrow \mathcal{N}_{\mathbf{q}}(\mathcal{W})$  that send generators to generators.  $\square$

The following was already known from [Cas20, Theorem A], but we believe our approach is more conceptual.

**Example 6.3.6.** Let  $\mathcal{W}$  be a right-angled Coxeter group generated by a finite set  $S$ , and  $\mathbf{q} = (q_v)_{v \in S}$  a Hecke-tuple. Then as  $\mathcal{W} = *_{v \in \Gamma} (\mathbb{Z}/2\mathbb{Z})$  for some (finite) graph  $\Gamma$ , we can by Proposition 6.3.5 write  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}) = *_{v \in \Gamma} \mathcal{N}_{q_v}(\mathbb{Z}/2\mathbb{Z})$ . As  $\mathcal{N}_{q_v}(\mathbb{Z}/2\mathbb{Z})$  is finite-dimensional we obtain by Corollary 6.3.4 that  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  has the weak-\* CCAP.

The result for the following example is new.

**Example 6.3.7.** Let  $\Gamma$  be a finite simple graph, and for  $v \in \Gamma$  let  $\mathcal{W}_v$  be a finite Coxeter group generated by some set  $S_v$  and let  $\mathbf{q}_v = (q_{v,s})_{s \in S_v}$  be a Hecke-tuple for  $\mathcal{W}_v$ . Then if we let  $\mathcal{W} = *_{v \in \Gamma} \mathcal{W}_v$  and  $\mathbf{q} = *_{v \in \Gamma} \mathbf{q}_v := (q_{v,s})_{v \in \Gamma, s \in S_v}$ , we have by Proposition 6.3.5 that  $\mathcal{N}_{\mathbf{q}}(\mathcal{W}) = *_{v \in \Gamma} \mathcal{N}_{\mathbf{q}_v}(\mathcal{W}_v)$ . Since  $\mathcal{N}_{\mathbf{q}_v}(\mathcal{W}_v)$  is finite-dimensional we obtain by Corollary 6.3.4 that  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  possesses the weak-\* CCAP.

## 6.4. GRAPH PRODUCT OF COMPLETELY BOUNDED MAPS ON $A_{\Gamma,d}$

The main result of this section is Theorem 6.4.3, which shows that the graph product of completely bounded maps  $T_v$  defines a completely bounded map  $T_d$  on the homogeneous subspace  $A_{\Gamma,d}$  of degree  $d$ . The proof of this results follows the lines of [RX06] (where they use the different convention  $\langle \hat{a}, \hat{b} \rangle = \varphi(a^* b)$ ), and uses the construction of the operator space  $X_d$  as in [CKL21] and another operator space  $\tilde{X}_d$ , to extend it to graph products.

### 6.4.1. FREE PRODUCTS AND OPERATOR SPACES

When given a finite graph  $\Gamma$  and algebras  $(A_v, \varphi_v)$  we will denote the reduced free product of the algebras as  $(A_{\Gamma}^f, \varphi^f) = *_v (A_v, \varphi_v)$ . Let  $\Gamma^f$  be the graph with the same vertex set as  $\Gamma$  and no edges. Note that the free product is simply the reduced graph product corresponding to  $\Gamma^f$ , i.e.  $A_{\Gamma}^f = A_{\Gamma^f}$ . For the graph product corresponding to  $\Gamma^f$  we will use notation using superscript  $f$ , that is we will write  $\mathcal{W}_{\Gamma}^f, \lambda_{\Gamma}^f, P_v^f, \mathcal{H}_{\Gamma}^f, \mathcal{H}_{\mathbf{w}}^f, \mathbf{A}_{\mathbf{w}}^f$ , et cetera. We remark that  $\mathcal{H}_{\Gamma} \subseteq \mathcal{H}_{\Gamma}^f$  and  $\mathbf{A}_{\Gamma} \subseteq \mathbf{A}_{\Gamma}^f$  as linear subspaces and that  $A_v = A_v^f$  for  $v \in \Gamma$ . For  $\mathbf{w} \in \mathcal{W}_{\Gamma} \setminus \{e\}$  with representative  $(w_1, \dots, w_n)$  we will define  $\mathcal{H}_{\mathbf{w}} = \mathcal{H}_{w_1} \otimes \dots \otimes \mathcal{H}_{w_n}$  and  $\mathbf{A}_{\mathbf{w}} = A_{w_1} \otimes \dots \otimes A_{w_n}$ , and we define  $\mathcal{H}_e = \mathbb{C}\Omega$  and  $\mathbf{A}_e = B(\mathcal{H}_e)$ . Define a subspace  $L_1$  of  $B(\mathcal{H}_{\Gamma}^f)$  by the closed linear span

$$L_1 = \overline{\text{Span}\{P_v^f \lambda_v^f(a) P_v^{f\perp} | v \in \Gamma, a \in \mathbf{A}_v^f\}}, \quad K_1 = L_1^*. \quad (6.15)$$

For a Hilbert space  $\mathcal{H}$  denote  $\mathcal{H}_C, \mathcal{H}_R$  respectively for the column and row Hilbert space, see [Pis03]. Recall that  $\mathcal{H}_C$  and  $\mathcal{H}_R$  can be seen as the subspaces of  $B(\mathbb{C} \oplus \mathcal{H})$  given by  $\mathcal{H}_C = \{x_{\xi} : \xi \in \mathcal{H}\}$  and  $\mathcal{H}_R = \{y_{\xi} : \xi \in \mathcal{H}\}$  where  $x_{\xi}$  and  $y_{\xi}$  are the operators given by  $x_{\xi}(z \oplus \eta) = z\xi$  and  $y_{\xi}(z \oplus \eta) = \langle \eta, \xi \rangle$ . In [RX06, Lemma 2.3 and Corollary 2.4] it is shown that

$$L_1 \simeq \left( \bigoplus_{v \in \Gamma} \mathcal{H}_v^{\circ} \right)_C, \quad K_1 \simeq \left( \bigoplus_{v \in \Gamma} \mathcal{H}_v^{\text{op}} \right)_R \quad (6.16)$$

completely isometrically, and that the maps  $\theta_1 : A_{\Gamma,1}^f \rightarrow L_1$  and  $\rho_1 : A_{\Gamma,1}^f \rightarrow K_1$  given for  $a \in \mathbf{A}_v$  by  $\theta_1(\lambda_v^f(a)) = P_v^f \lambda_v^f(a) P_v^{f\perp}$  and  $\rho_1(\lambda_v^f(a)) = P_v^{f\perp} \lambda_v^f(a) P_v^f$  are completely contractive. We denote  $\otimes_h$  for the Haagerup tensor product, see [ER00, Chapter 9]. We denote  $L_d = L_1^{\otimes_h d}$  and  $K_d = K_1^{\otimes_h d}$  for the  $d$ -fold tensor product and we write  $\theta_1^{\otimes d}$  for the map  $A_{\Gamma,d}^f \rightarrow L_d$  defined for  $b = b_1 \otimes \dots \otimes b_d \in A_{\Gamma,d}$  by

$$\theta_1^{\otimes d}(\lambda^f(b)) = \theta_1(\lambda^f(b_1)) \otimes_h \dots \otimes_h \theta_1(\lambda^f(b_d)).$$

Similarly, we write  $\rho_1^{\otimes d}$  for the map  $A_{\Gamma,d}^f \rightarrow K_d$  defined analogously.

We introduce notation similar to [CKL21, Section 2]. Let  $\mathbf{w} \in \mathcal{W}_{\Gamma}^f$  s.t. in the graph product  $\mathbf{w}$  is equivalent to some clique word  $\mathbf{v}_{\Gamma_0}$  for some clique  $\Gamma_0 \subseteq \Gamma$  (which we will denote by  $\mathbf{w} \equiv \mathbf{v}_{\Gamma_0}$ ). Let  $a = a_1 \otimes \dots \otimes a_d \in \mathbf{A}_{\mathbf{w}}^f$ . We define an operator  $\text{Diag}_{\mathbf{w}}(a) : \mathcal{H}_{\Gamma}^f \rightarrow \mathcal{H}_{\Gamma}^f$  on  $\mathcal{H}_{\mathbf{v}}^f$  for  $\mathbf{v} \in \mathcal{W}_{\Gamma}^f$  with  $|\mathbf{v}| = |\mathbf{w}| + |\mathbf{w}^{-1}\mathbf{v}|$  as

$$\text{Diag}_{\mathbf{w}}(a)|_{\mathcal{H}_{\mathbf{v}}^f} = P_{v_1} a_1 P_{v_1} \otimes \dots \otimes P_{v_d} a_d P_{v_d} \otimes \text{Id}_{\mathcal{H}_{v_{d+1}}} \otimes \dots \otimes \text{Id}_{\mathcal{H}_{v_{|\mathbf{v}|}}} \quad (6.17)$$

and we define  $\text{Diag}_{\mathbf{w}}(a)|_{\mathcal{H}_{\mathbf{v}}^f} = 0$  if  $\mathbf{v} \in \mathcal{W}^f$  is not of the given form. Extending this, we obtain a linear map  $\text{Diag}_{\mathbf{w}} : \mathbf{A}_{\mathbf{w}}^f \rightarrow \mathcal{B}(\mathcal{H}_{\Gamma}^f)$ . For a clique  $\Gamma_0$  in  $\Gamma$ , we now define the operator space

$$A_{\Gamma_0}^{\text{Diag}} = \text{Span}\{\text{Diag}_{\mathbf{w}}(\mathbf{A}_{\mathbf{w}}^f) | \mathbf{w} \in \mathcal{W}_{\Gamma}^f, \mathbf{w} \equiv \mathbf{v}_{\Gamma_0}\}.$$

Also, for  $\mathbf{w} \in \mathcal{W}_{\Gamma}^f$  we consider  $\mathbf{A}_{\mathbf{w}}^f$  as an operator space by embedding  $\mathbf{A}_{\mathbf{w}}^f \subseteq \mathcal{B}(\mathcal{H}_{\mathbf{w}}^f)$ .

**Proposition 6.4.1.** *For a clique  $\Gamma_0$  and a word  $\mathbf{w} \in \mathcal{W}_{\Gamma}^f$  with  $\mathbf{w} \equiv \mathbf{v}_{\Gamma_0}$  we have that the map  $\text{Diag}_{\mathbf{w}} : \mathbf{A}_{\mathbf{w}}^f \rightarrow A_{\Gamma_0}^{\text{Diag}}$  is completely contractive.*

*Proof.* We define a map  $V_{\mathbf{w}} : \mathcal{H}_{\Gamma}^f \rightarrow \mathcal{H}_{\mathbf{w}}^f \otimes \mathcal{H}_{\Gamma}^f$  as

$$V_{\mathbf{w}}|_{\mathcal{H}_{\mathbf{v}}^f} := \mathcal{Q}_{(\mathbf{w}, \mathbf{w}^{-1}\mathbf{v})}^{f*} \quad (6.18)$$

whenever  $\mathbf{v} \in \mathcal{W}_{\Gamma}^f$  is s.t.  $|\mathbf{v}| = |\mathbf{w}| + |\mathbf{w}^{-1}\mathbf{v}|$  and set  $V_{\mathbf{w}}|_{\mathcal{H}_{\mathbf{v}}^f} = 0$  when  $\mathbf{v}$  is not of this form. We then obtain that

$$\text{Diag}_{\mathbf{w}}(a) = V_{\mathbf{w}}^*(a \otimes \text{Id}_{\mathcal{H}_{\Gamma}^f}) V_{\mathbf{w}} \quad (6.19)$$

which shows the statement.  $\square$

As in [RX06] and [CKL21] we define operator spaces  $X_d$  and additionally we will define other operator spaces  $\tilde{X}_d$ . For  $\mathbf{t} \in \mathcal{W}_{\Gamma}$  a clique word, denote  $\Gamma_{\mathbf{t}}$  for the clique in  $\Gamma$ . We now set

$$X_d = \bigoplus_{\substack{n_l, n_r \geq 0, \\ (\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_{\Gamma}, \\ n_l + |\mathbf{u}_l \mathbf{t} \mathbf{u}_r| + n_r = d}} L_{n_l + |\mathbf{u}_l|} \otimes_h A_{\Gamma_{\mathbf{t}}}^{\text{Diag}} \otimes_h K_{n_r + |\mathbf{u}_r|} \quad (6.20)$$

$$\tilde{X}_d = \bigoplus_{\substack{n_l, n_r \geq 0, \\ (\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_{\Gamma}, \\ n_l + |\mathbf{u}_l \mathbf{t} \mathbf{u}_r| + n_r = d}} L_{n_l + |\mathbf{u}_l|} \otimes_h \mathbf{A}_{\mathbf{t}} \otimes_h K_{n_r + |\mathbf{u}_r|} \quad (6.21)$$

equipped with the sup-norm. We remark here that the operator space structure on  $\mathbf{A}_{\mathbf{t}}$  is given by the inclusion  $\mathbf{A}_{\mathbf{t}} = \mathbf{A}_{\mathbf{t}'}^f \subseteq \mathcal{B}(\mathcal{H}_{\mathbf{t}'}^f)$  where  $\mathbf{t}' \in \mathcal{W}_{\Gamma}^f$  is the representant of  $\mathbf{t}$ . Also, recall that  $\mathcal{T}_{\Gamma}$  was defined in Definition 6.2.2 and that in Definition 6.2.7 for a tuple  $\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$  with  $n_l, n_r \geq 0$ ,  $(\mathbf{u}_l, \mathbf{u}_r, \mathbf{t}) \in \mathcal{T}_{\Gamma}$  we defined  $|\rho| = n_l + |\mathbf{u}_l| + |\mathbf{t}| + |\mathbf{u}_r| + n_r$ . By the above, we can find a completely contractive map  $D_d : \tilde{X}_d \rightarrow X_d$  by defining  $D_d = (D_{\rho})_{\rho, |\rho|=d}$  where  $D_{\rho} = (\text{Id}_{L_{n_l + |\mathbf{u}_l|}} \otimes \text{Diag}_{\mathbf{t}'} \otimes \text{Id}_{K_{n_r + |\mathbf{u}_r|}})$  for  $\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$ .

We now define two linear maps  $\tilde{\Theta}_d : A_{\Gamma, d} \rightarrow \tilde{X}_d$  and  $j_d : \mathbf{A}_{\Gamma, d} \rightarrow X_d$  as follows. Fix a tuple  $\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$ ,  $|\rho| = d$ . We denote  $\tilde{n}_l = n_l + |\mathbf{u}_l|$  and  $\tilde{n}_r = n_r + |\mathbf{u}_r|$ . Let  $a \in \mathbf{A}_{\mathbf{w}}$  be a pure tensor with  $\mathbf{w} \in \mathcal{W}_{\Gamma}$ . Suppose that  $\mathbf{w} = \mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$  for some  $\mathbf{v}_l \in \widehat{\mathcal{W}}_{n_l}^f(\mathbf{u}_l \mathbf{t})$  and

$\mathbf{v}_r \in \widetilde{\mathcal{W}}_{n_r}'(\mathbf{u}_r \mathbf{t})$ . We can then write  $a = \mathcal{Q}_{(\mathbf{v}_l \mathbf{u}_l, \mathbf{t}, \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_1 \otimes a_2 \otimes a_3)$  for some  $a_1 \in \mathring{\mathbf{A}}_{\mathbf{v}_l \mathbf{u}_l}$ ,  $a_2 \in \mathring{\mathbf{A}}_{\mathbf{t}}$  and  $a_3 \in \mathring{\mathbf{A}}_{\mathbf{u}_r^{-1} \mathbf{v}_r^{-1}}$ . We then defined

$$\widetilde{\Theta}_d(\lambda(a))_\rho = \theta_1^{\otimes \widetilde{n}_l}(\lambda^f(a_1)) \otimes a_2 \otimes \rho_1^{\otimes \widetilde{n}_r}(\lambda^f(a_3)) \quad (6.22)$$

$$j_d(a)_\rho = \theta_1^{\otimes \widetilde{n}_l}(\lambda^f(a_1)) \otimes \text{Diag}_{\ell'}(a_2) \otimes \rho_1^{\otimes \widetilde{n}_r}(\lambda^f(a_3)). \quad (6.23)$$

In the case that  $\mathbf{w}$  is not of the given form we define  $\widetilde{\Theta}_d(\lambda(a))_\rho = 0$  and  $j_d(a)_\rho = 0$ . This is extended linearly and we set  $\widetilde{\Theta}_d(\lambda(a)) = (\widetilde{\Theta}_d(\lambda(a))_\rho)_\rho$  and  $j_d(a) = (j_d(a)_\rho)_\rho$ . We moreover define the map  $\Theta_d := D_d \circ \widetilde{\Theta}_d$  and see that  $j_d = \Theta_d \circ \lambda|_{A_{\Gamma,d}}$ . We note that the definition of  $j_d$  agrees with that in [CKL21, Equation (2.16)], and that, in the case of dealing with free products, the map  $\Theta_d$  compares with a restriction of the map  $\Theta_d$  in [RX06]. In [CKL21, Equation (2.24)] a completely bounded map  $\pi_d : E_d \rightarrow B(\mathcal{H}_\Gamma)$  was defined, where  $E_d := j_d(A_{\Gamma,d}) \subseteq X_d$ , and that satisfied  $\pi_d \circ j_d = \lambda|_{A_{\Gamma,d}}$ . For  $d \geq 1$  the norm bound  $\|\pi_d\|_{\text{cb}} \leq (\#\text{Clique}(\Gamma))^3 d$  holds by [CKL21, Theorem 2.9], where  $\#\text{Clique}(\Gamma)$  denotes the number of cliques in the graph  $\Gamma$ . We get the following commuting diagram:

$$\begin{array}{ccccc} A_{\Gamma,d} & \subseteq & A_{\Gamma,d}^f & & \\ \downarrow \lambda & \searrow j_d & & & \\ A_{\Gamma,d} & \xleftarrow{\pi_d} & E_d & \subseteq & X_d \\ & \searrow \widetilde{\Theta}_d & \downarrow D_d & & \\ & & \widetilde{X}_d & & \end{array}$$

For a clique word  $\mathbf{t} \in \mathcal{W}_\Gamma$  with representative  $(t_1, \dots, t_{|\mathbf{t}|})$  we define a unitary  $U : \mathcal{H}_{\mathbf{t}} \rightarrow \bigoplus_{\mathbf{r} \subseteq \mathbf{t}} \mathring{\mathcal{H}}_{\mathbf{r}}$  in a natural way. Let  $\eta = \eta_1 \otimes \dots \otimes \eta_{|\mathbf{t}|} \in \mathcal{H}_{\mathbf{t}}$  be a tensor with either  $\eta_i \in \mathring{\mathcal{H}}_{t_i}$  or  $\eta_i \in \mathbb{C}\xi_{t_i}$ . For  $1 \leq i \leq |\mathbf{t}|$  set  $r_i := t_i$  when  $\eta_i \in \mathring{\mathcal{H}}_{t_i}$  and  $r_i = e$  when  $\eta_i \in \mathbb{C}\xi_{t_i}$ . Then  $\mathbf{r} := r_1 \dots r_{|\mathbf{t}|}$  is a subword of  $\mathbf{t}$  since  $\mathbf{t}$  is a clique word. Using the identification  $\mathbb{C}\xi_{t_i} \simeq \mathring{\mathcal{H}}_e$  given by  $\xi_{t_i} \rightarrow \Omega$  we can define  $U(\eta) = \mathcal{Q}_{(r_1, \dots, r_{|\mathbf{t}|})}(\eta) \in \mathring{\mathcal{H}}_{\mathbf{r}}$ . This extends linearly to a unitary. We remark that for  $a \in \mathring{\mathbf{A}}_{\mathbf{t}}$  we have  $U^* \lambda(a) U = a$ . Indeed, it can be checked that for  $a_i \in \mathring{\mathbf{A}}_{t_i}$  we have  $U^* \lambda(a_i) U = \text{Id}_{\mathcal{H}_{t_1}} \otimes \dots \otimes \text{Id}_{\mathcal{H}_{t_{i-1}}} \otimes a_i \otimes \text{Id}_{\mathcal{H}_{t_{i+1}}} \otimes \dots \otimes \text{Id}_{\mathcal{H}_{t_{|\mathbf{t}|}}}$  so that the statement follows as  $\lambda(a_1 \otimes \dots \otimes a_n) = \lambda(a_1) \dots \lambda(a_n)$ .

**Theorem 6.4.2.** *The map  $\widetilde{\Theta}_d$  is completely contractive.*

*Proof.* Choose  $d \geq 0$ . Fix a tuple  $\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$  with  $|\rho| = d$  and write  $\widetilde{n}_l = n_l + |\mathbf{u}_l|$ ,  $\widetilde{n}_r = n_r + |\mathbf{u}_r|$ . We define two partial isometries

$$J_\rho : \mathcal{H}_\Gamma^{f \otimes \widetilde{n}_l} \otimes \mathcal{H}_{\mathbf{t}} \rightarrow \mathcal{H}_\Gamma^{f \otimes \widetilde{n}_l} \otimes \mathcal{H}_\Gamma \quad (6.24)$$

$$J'_\rho : \mathcal{H}_{\mathbf{t}} \otimes \mathcal{H}_\Gamma^{f \otimes \widetilde{n}_r} \rightarrow \mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma^{f \otimes \widetilde{n}_r} \quad (6.25)$$

as follows. Let  $\mathbf{r}_l \subseteq \mathbf{t}$ , let  $\eta = \eta_1 \otimes \dots \otimes \eta_{\widetilde{n}_l} \otimes \eta_0 \in \mathcal{H}_\Gamma^{f \otimes \widetilde{n}_l} \otimes (U^* \mathring{\mathcal{H}}_{\mathbf{r}_l})$  be a pure tensor and denote  $\eta'_0 := U\eta_0 \in \mathring{\mathcal{H}}_{\mathbf{r}_l}$ . If for  $i \geq 1$  we can write  $\eta_i = \eta'_i \otimes \widetilde{\eta}_i$  for some  $\eta'_i \in \mathring{\mathcal{H}}_{v_i}$  and

$\tilde{\eta}_i \in \mathcal{H}_\Gamma^f$  for which  $(v_1, \dots, v_n)$  is the representative of  $\mathbf{v}_l \mathbf{u}_l$  for some  $\mathbf{v}_l \in \widetilde{\mathcal{W}}'_{n_l}(\mathbf{u}_l \mathbf{t})$ , then we define

$$J_\rho \eta = \tilde{\eta}_1 \otimes \dots \otimes \tilde{\eta}_{\tilde{n}_l} \otimes \mathcal{D}_{(v_1, \dots, v_{\tilde{n}_l}, \mathbf{r}_l)}(\eta'_1 \otimes \dots \otimes \eta'_{\tilde{n}_l} \otimes \eta'_0) \in \mathcal{H}_\Gamma^{f \otimes \tilde{n}_l} \otimes \mathcal{H}_{\mathbf{v}_l \mathbf{u}_l \mathbf{r}_l}^{\circ} \quad (6.26)$$

and we define  $J_\rho$  as 0 on the complement of all such tensors. Similarly, let  $\mathbf{r}_r \subseteq \mathbf{t}$  let  $\eta = \eta_0 \otimes \eta_1 \otimes \dots \otimes \eta_{\tilde{n}_r} \in (U^* \mathcal{H}_{\mathbf{r}_r}^{\circ}) \otimes \mathcal{H}_\Gamma^{f \otimes \tilde{n}_r}$ , denote  $\eta'_0 := U\eta_0 \in \mathcal{H}_{\mathbf{r}_r}^{\circ}$  and suppose that for  $i \geq 1$  we can write  $\eta_i = \eta'_i \otimes \tilde{\eta}_i$  for some  $\eta'_i \in \mathcal{H}_{v_i}$  and  $\tilde{\eta}_i \in \mathcal{H}_\Gamma^f$  for which  $(v_1, \dots, v_n)$  is the representative of  $\mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$  for some  $\mathbf{v}_r \in \widetilde{\mathcal{W}}'_{n_r}(\mathbf{u}_r \mathbf{t})$  we define

$$J'_\rho \eta = \mathcal{D}_{(v_{\tilde{n}_r}, \dots, v_1, \mathbf{r}_r)}(\eta'_{\tilde{n}_r} \otimes \dots \otimes \eta'_1 \otimes \eta'_0) \otimes \tilde{\eta}_1 \otimes \dots \otimes \tilde{\eta}_{\tilde{n}_r} \in \mathcal{H}_{\mathbf{v}_r \mathbf{u}_r \mathbf{r}_r}^{\circ} \otimes \mathcal{H}_\Gamma^{f \otimes \tilde{n}_r} \quad (6.27)$$

and we define  $J'_\rho$  as 0 on the complement of all such tensors.

We shall show that

$$\tilde{\Theta}_d(\lambda(a))_\rho = (J_\rho^* \otimes \text{Id}_{\mathcal{H}_\Gamma^{f \otimes \tilde{n}_r}})(\text{Id}_{\mathcal{H}_\Gamma^{f \otimes \tilde{n}_l}} \otimes \lambda(a) \otimes \text{Id}_{\mathcal{H}_\Gamma^{f \otimes \tilde{n}_r}})(\text{Id}_{\mathcal{H}_\Gamma^{f \otimes \tilde{n}_l}} \otimes J'_\rho) \quad (6.28)$$

which then shows the statement.

Let  $\mathbf{w} \in \mathcal{W}_\Gamma$ ,  $|\mathbf{w}| = d$ , let  $a \in \mathring{\mathbf{A}}_{\mathbf{w}}$  be a pure tensor, let  $\omega = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathcal{S}_{\mathbf{w}}$ ,  $\mathbf{v}_l \in \widetilde{\mathcal{W}}'_{n_l}(\mathbf{u}_l \mathbf{t})$ ,  $\mathbf{v}_r \in \widetilde{\mathcal{W}}'_{n_r}(\mathbf{u}_r \mathbf{t})$  and  $\mathbf{r}_l, \mathbf{r}_r \subseteq \mathbf{t}$ . Now let  $\eta \in \mathcal{H}_{\mathbf{v}_l \mathbf{u}_l \mathbf{r}_l}^{\circ}$  be a pure tensor, in which case  $\lambda_\omega(a)\eta$  is also a pure tensor. Suppose that  $\lambda_\omega(a)\eta \in \mathcal{H}_{\mathbf{v}_l \mathbf{u}_l \mathbf{r}_l}^{\circ}$  and that it is non-zero, so that  $\mathbf{v}_l \mathbf{u}_l \mathbf{r}_r$  and  $\mathbf{v}_r \mathbf{u}_r \mathbf{r}_r$  start with  $\mathbf{w}_1 \mathbf{w}_2$  and  $\mathbf{w}_3^{-1} \mathbf{w}_2$  respectively and so that  $\mathbf{w}_1 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{r}_r = \mathbf{v}_l \mathbf{u}_l \mathbf{r}_l$ . Then put  $\mathbf{w}_{tail} := \mathbf{w}_2 \mathbf{w}_3 \mathbf{v}_r \mathbf{u}_r \mathbf{r}_r = \mathbf{w}_2 \mathbf{w}_1^{-1} \mathbf{v}_l \mathbf{u}_l \mathbf{r}_l$  so that  $\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_{tail}$  and  $\mathbf{w}_3^{-1} \mathbf{w}_2 \mathbf{w}_{tail}$  are reduced expressions for  $\mathbf{v}_l \mathbf{u}_l \mathbf{r}_l$  and  $\mathbf{v}_r \mathbf{u}_r \mathbf{r}_r$  respectively. We claim that  $\mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_{tail}) \supseteq \mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_{tail}) \cap \mathbf{s}_r(\mathbf{w}_3^{-1} \mathbf{w}_2 \mathbf{w}_{tail})$ . Indeed, let  $v$  be a letter in  $\mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_{tail})$  that is not in  $\mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_{tail})$ . Then  $v$  is a letter at the end of  $\mathbf{w}_1$  that commutes with  $\mathbf{w}_2$ . If  $v$  is at the same time a letter in  $\mathbf{s}_r(\mathbf{w}_3^{-1} \mathbf{w}_2 \mathbf{w}_{tail})$  then  $v$  is also a letter at the end of  $\mathbf{w}_3^{-1}$ , i.e. a letter at the start of  $\mathbf{w}_3$ . But this would contradict the fact that  $\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3$  is reduced. Thus we established the inclusion and obtain

$$\mathbf{s}_r(\mathbf{w}_2 \mathbf{w}_{tail}) \supseteq \mathbf{s}_r(\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_{tail}) \cap \mathbf{s}_r(\mathbf{w}_3^{-1} \mathbf{w}_2 \mathbf{w}_{tail}) = \mathbf{s}_r(\mathbf{v}_l \mathbf{u}_l \mathbf{r}_l) \cap \mathbf{s}_r(\mathbf{v}_r \mathbf{u}_r \mathbf{r}_r) \supseteq \mathbf{r}_l \cap \mathbf{r}_r$$

so that  $|\mathbf{w}_2 \mathbf{w}_{tail}| \geq |\mathbf{r}_l \cap \mathbf{r}_r|$ . Now, combining all this, we find

$$d + |\mathbf{r}_l \cap \mathbf{r}_r| + |\mathbf{w}_{tail}| \leq |\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3| + |\mathbf{w}_2 \mathbf{w}_{tail}| + |\mathbf{w}_{tail}| \quad (6.29)$$

$$= |\mathbf{w}_1| + 2|\mathbf{w}_2| + 2|\mathbf{w}_{tail}| + |\mathbf{w}_3| \quad (6.30)$$

$$= |\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_{tail}| + |\mathbf{w}_3^{-1} \mathbf{w}_2 \mathbf{w}_{tail}| \quad (6.31)$$

$$= |\mathbf{v}_l| + |\mathbf{u}_l| + |\mathbf{r}_l| + |\mathbf{r}_r| + |\mathbf{u}_r| + |\mathbf{v}_r| \quad (6.32)$$

$$= d + |\mathbf{r}_l| + |\mathbf{r}_r| - |\mathbf{t}| \quad (6.33)$$

$$\leq d + |\mathbf{r}_l| + |\mathbf{r}_r| - |\mathbf{r}_l \cup \mathbf{r}_r| \quad (6.34)$$

$$= d + |\mathbf{r}_l \cap \mathbf{r}_r| \quad (6.35)$$

We conclude that all the above inequalities must be equalities, in particular  $|\mathbf{w}_{tail}| = 0$ ,  $|\mathbf{t}| = |\mathbf{r}_l \cup \mathbf{r}_r|$  and  $|\mathbf{w}_2 \mathbf{w}_{tail}| = |\mathbf{r}_l \cap \mathbf{r}_r|$ . This means  $\mathbf{t} = \mathbf{r}_l \cup \mathbf{r}_r$  and  $\mathbf{w}_2 = \mathbf{w}_2 \mathbf{w}_{tail}$ . Now as also

$\mathbf{w}_2 = \mathbf{w}_2 \mathbf{w}_{tail} \supseteq \mathbf{r}_l \cap \mathbf{r}_r$  we conclude that  $\mathbf{w}_2 = \mathbf{r}_l \cap \mathbf{r}_r$ . Set  $\mathbf{t}_l := \mathbf{r}_l \mathbf{w}_2 = (\mathbf{r}_l \cap \mathbf{t} \mathbf{r}_r)$ ,  $\mathbf{t}_m := \mathbf{r}_l \cap \mathbf{r}_r$  and  $\mathbf{t}_r := \mathbf{w}_2 \mathbf{r}_r = (\mathbf{t} \mathbf{r}_l \cap \mathbf{r}_r)$ . Then, as we know  $\mathbf{v}_l \mathbf{u}_l \mathbf{r}_l = \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_{tail} = \mathbf{w}_1 \mathbf{w}_2$  and  $\mathbf{v}_r \mathbf{u}_r \mathbf{r}_r = \mathbf{w}_3^{-1} \mathbf{w}_2 \mathbf{w}_{tail} = \mathbf{w}_3^{-1} \mathbf{w}_2$ , we then obtain that  $\mathbf{w}_1 = \mathbf{v}_l \mathbf{u}_l \mathbf{t}_l$  and  $\mathbf{w}_3 = \mathbf{t}_r \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$ . Hence,  $\omega$  is of the form  $\omega = (\mathbf{v}_l \mathbf{u}_l \mathbf{t}_l, \mathbf{t}_m, \mathbf{t}_r \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})$ . We note that  $\mathbf{t}_l, \mathbf{t}_m, \mathbf{t}_r$  are disjoint subcliques of  $\mathbf{t}$  with  $\mathbf{t}_l \mathbf{t}_m \mathbf{t}_r = \mathbf{t}$ . In particular we find that the assumption implies  $\mathbf{w} = \mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$ . For a closed subspace  $\mathcal{K} \subseteq \mathcal{H}_\Gamma$  denote  $P_{\mathcal{K}}$  for the orthogonal projection on  $\mathcal{K}$ . We conclude that

$$P_{\mathcal{H}_{\mathbf{v}_l \mathbf{u}_l \mathbf{r}_l}} \lambda(a) P_{\mathcal{H}_{\mathbf{v}_r \mathbf{u}_r \mathbf{r}_r}} = \lambda_{(\mathbf{v}_l \mathbf{u}_l \mathbf{t}_l, \mathbf{t}_m, \mathbf{t}_r \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a) P_{\mathcal{H}_{\mathbf{v}_r \mathbf{u}_r \mathbf{r}_r}} \quad (6.36)$$

and moreover that this expression is zero whenever  $a \notin \mathring{\mathbf{A}}_{\mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}}$ . This shows that for  $a \in \mathring{\mathbf{A}}_{\mathbf{w}}$  with  $\mathbf{w}$  not of the form  $\mathbf{w} = \mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$  for any  $\mathbf{v}_l \in \widetilde{\mathcal{W}}'_{n_l}(\mathbf{u}_l \mathbf{t})$ ,  $\mathbf{v}_r \in \widetilde{\mathcal{W}}'_{n_r}(\mathbf{u}_r \mathbf{t})$ , the right-hand side of (6.28) is zero. In this case also the left-hand side is zero by definition of  $\Theta_d(\lambda(a))_\rho$  so that we get equality.

Let  $\mathbf{v} \in \mathcal{W}_\Gamma$ . We define.

$$\mathcal{K}_{\rho, \mathbf{v}} = \bigoplus_{\mathbf{r}_l \subseteq \mathbf{t}} \mathcal{H}_{\mathbf{v} \mathbf{u}_l \mathbf{r}_l} \quad \mathcal{K}'_{\rho, \mathbf{v}} = \bigoplus_{\mathbf{r}_r \subseteq \mathbf{t}} \mathcal{H}_{\mathbf{v} \mathbf{u}_r \mathbf{r}_r} \quad (6.37)$$

$$\mathcal{K}_\rho = \bigoplus_{\mathbf{v}_l \in \widetilde{\mathcal{W}}'_{n_l}(\mathbf{u}_l \mathbf{t})} \mathcal{K}_{\rho, \mathbf{v}_l} \quad \mathcal{K}'_\rho = \bigoplus_{\mathbf{v}_r \in \widetilde{\mathcal{W}}'_{n_r}(\mathbf{u}_r \mathbf{t})} \mathcal{K}_{\rho, \mathbf{v}_r}^R. \quad (6.38)$$

Let us now assume  $a \in \mathring{\mathbf{A}}_{\mathbf{w}}$  with  $\mathbf{w} = \mathbf{v}_l \mathbf{u}_l \mathbf{t} \mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$  for some  $\mathbf{v}_l \in \widetilde{\mathcal{W}}'_{n_l}(\mathbf{u}_l \mathbf{t})$ ,  $\mathbf{v}_r \in \widetilde{\mathcal{W}}'_{n_r}(\mathbf{u}_r \mathbf{t})$  and write  $a = \mathcal{Q}_{(\mathbf{v}_l \mathbf{u}_l \mathbf{t}, \mathbf{t} \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_1 \otimes a_2 \otimes a_3)$  for some  $a_1 \in \mathring{\mathbf{A}}_{\mathbf{v}_l \mathbf{u}_l}$ ,  $a_2 \in \mathring{\mathbf{A}}_{\mathbf{t}}$  and  $a_3 \in \mathring{\mathbf{A}}_{\mathbf{u}_r^{-1} \mathbf{v}_r^{-1}}$ . Note that in such case the words  $\mathbf{v}_l, \mathbf{v}_r$  are uniquely determined. By the above, we now find

$$P_{\mathcal{K}_\rho} \lambda(a) P_{\mathcal{K}'_\rho} = \quad (6.39)$$

$$= P_{\mathcal{K}_{\rho, \mathbf{v}_l}} \lambda(a) P_{\mathcal{K}'_{\rho, \mathbf{v}_r}} \quad (6.40)$$

$$= P_{\mathcal{K}_{\rho, \mathbf{v}_l}} \left( \sum_{\substack{\mathbf{t}_l, \mathbf{t}_m, \mathbf{t}_r \\ \text{partition of } \mathbf{t}}} \lambda_{(\mathbf{v}_l \mathbf{u}_l \mathbf{t}_l, \mathbf{t}_m, \mathbf{t}_r \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a) \right) P_{\mathcal{K}'_{\rho, \mathbf{v}_r}} \quad (6.41)$$

$$= P_{\mathcal{K}_{\rho, \mathbf{v}_l}} \left( \sum_{\substack{\mathbf{t}_l, \mathbf{t}_m, \mathbf{t}_r \\ \text{partition of } \mathbf{t}}} \lambda_{(\mathbf{v}_l \mathbf{u}_l, e, e)}(a_1) \lambda_{(\mathbf{t}_l, \mathbf{t}_m, \mathbf{t}_r)}(a_2) \lambda_{(e, e, \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_3) \right) P_{\mathcal{K}'_{\rho, \mathbf{v}_r}} \quad (6.42)$$

$$\stackrel{\text{Lemma 3.1.7}}{=} P_{\mathcal{K}_{\rho, \mathbf{v}_l}} \lambda_{(\mathbf{v}_l \mathbf{u}_l, e, e)}(a_1) \lambda(a_2) \lambda_{(e, e, \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_3) P_{\mathcal{K}'_{\rho, \mathbf{v}_r}} \quad (6.43)$$

$$= P_{\mathcal{K}_{\rho, \mathbf{v}_l}} \lambda_{(\mathbf{v}_l \mathbf{u}_l, e, e)}(a_1) (U a_2 U^*) \lambda_{(e, e, \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_3) P_{\mathcal{K}'_{\rho, \mathbf{v}_r}} \quad (6.44)$$

where we use that  $\lambda(a_2)|_{\mathcal{H}_{\mathbf{t}}} = U a_2 U^*$  for  $\mathbf{r} \subseteq \mathbf{t}$ . Now, a calculation shows that

$$(U^* \lambda_{(e, e, \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_3) P_{\mathcal{K}'_{\rho, \mathbf{v}_r}} \otimes \text{Id}) J'_\rho = (\text{Id}_{\mathcal{H}_{\mathbf{t}}} \otimes \rho_1^{\otimes \widetilde{n}_r}(\lambda^f(a_3))) \quad (6.45)$$

$$J_\rho^* (\text{Id} \otimes P_{\mathcal{K}_{\rho, \mathbf{v}_l}} \lambda_{(\mathbf{v}_l \mathbf{u}_l, e, e)}(a_1) U) = (\theta_1^{\otimes \widetilde{n}_l}(\lambda^f(a_1)) \otimes \text{Id}_{\mathcal{H}_{\mathbf{t}}}) \quad (6.46)$$

We describe the calculation for (6.45) (the calculation for (6.46) is similar by taking adjoints and using that  $\theta_1^{\otimes \tilde{n}_l}(\lambda^f(a_1))^* = \rho_1^{\otimes \tilde{n}_l}(\lambda^f(a_1^*))$ ). Let  $\eta = \eta_0 \otimes \eta_1 \otimes \cdots \otimes \eta_{\tilde{n}_r} \in (U^* \mathcal{H}_{\mathbf{r}_r}) \otimes \mathcal{H}_\Gamma^{\otimes \tilde{n}_r}$  for some  $\mathbf{r}_r \subseteq \mathbf{t}$  and so that  $\eta_i$  is a pure tensor for  $i = 0, \dots, \tilde{n}_r$ . Assume that for  $i = 1, \dots, \tilde{n}_r$  we can write  $\eta_i = \eta'_i \otimes \tilde{\eta}_i$  with  $\eta'_i \in \mathcal{H}_{v_i}^{\otimes}$  and  $\tilde{\eta}_i \in \mathcal{H}_\Gamma^f$  for which  $(v_1, \dots, v_{\tilde{n}_r})$  is the representative of  $\mathbf{u}_r^{-1} \mathbf{v}_r^{-1}$ . Indeed, if  $\eta$  is not of this form then both  $(P_{\mathcal{H}_{\rho, \mathbf{v}_r}} \otimes \text{Id}) J'_\rho \eta = 0$  and  $(\text{Id}_{\mathcal{H}_\Gamma} \otimes \rho_1^{\otimes \tilde{n}_r}(\lambda^f(a_3))) \eta = 0$  which gives the equality. Now by definition  $J'_\rho \eta = \zeta_1 \otimes \zeta_2$  where  $\zeta_1 := \mathcal{Q}_{(v_{\tilde{n}_r}, \dots, v_1, \mathbf{r}_r)}(\eta'_{\tilde{n}_r} \otimes \cdots \otimes \eta'_1 \otimes U\eta_0) \in \mathcal{H}_{\mathbf{v}_r, \mathbf{u}_r, \mathbf{r}_r}^{\otimes}$  and  $\zeta_2 := \tilde{\eta}_1 \otimes \cdots \otimes \tilde{\eta}_{\tilde{n}_r}$ . Now

$$(\lambda_{(e, e, \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_3) P_{\mathcal{H}_{\rho, \mathbf{v}_r}} \otimes \text{Id}) J'_\rho \eta = (\lambda_{(e, e, \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_3) P_{\mathcal{H}_{\rho, \mathbf{v}_r}} \zeta_1) \otimes \zeta_2 \quad (6.47)$$

$$= (\lambda_{(e, e, \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_3) \zeta_1) \otimes \zeta_2 \quad (6.48)$$

$$= \varphi(\lambda_{(e, e, \mathbf{u}_r^{-1} \mathbf{v}_r^{-1})}(a_3) \mathcal{Q}_{(v_{\tilde{n}_r}, \dots, v_1)}(\eta'_{\tilde{n}_r} \otimes \cdots \otimes \eta'_1))(U\eta_0) \otimes \zeta_2 \quad (6.49)$$

$$= (U\eta_0) \otimes (\rho_1^{\otimes \tilde{n}_r}(\lambda^f(a_3)) \eta_1 \otimes \cdots \otimes \eta_{\tilde{n}_r}) \quad (6.50)$$

$$= (U \otimes \rho_1^{\otimes \tilde{n}_r}(\lambda^f(a_3))) \eta \quad (6.51)$$

This shows equality (6.45). Hence, combining (6.45) and (6.46) we obtain

$$\tilde{\Theta}_d(\lambda(a))_\rho = \theta_1^{\otimes \tilde{n}_l}(\lambda^f(a_1)) \otimes a_2 \otimes \rho_1^{\otimes \tilde{n}_r}(\lambda^f(a_3)) \quad (6.52)$$

$$= (J_\rho^* \otimes \text{Id})(\text{Id} \otimes P_{\mathcal{H}_{\rho, \mathbf{v}_l}} \lambda_{(\mathbf{v}_l \mathbf{u}_l, e, e)}(a_1) U a_2 \otimes \rho_1^{\otimes \tilde{n}_r}(\lambda^f(a_3))) \quad (6.53)$$

$$= (J_\rho^* \otimes \text{Id})(\text{Id} \otimes P_{\mathcal{H}_\rho^L} \lambda(a) P_{\mathcal{H}_\rho^R} \otimes \text{Id})(\text{Id} \otimes J'_\rho) \quad (6.54)$$

$$= (J_\rho^* \otimes \text{Id})(\text{Id} \otimes \lambda(a) \otimes \text{Id})(\text{Id} \otimes J'_\rho) \quad (6.55)$$

This shows the equality holds for all  $a \in \mathbf{A}_{\Gamma, d}$ , and hence, by density it holds on  $A_{\Gamma, d}$ . This completes the proof.  $\square$

**Theorem 6.4.3.** *For  $v \in \Gamma$  let  $(A_v, \varphi_v)$  be a unital  $C^*$ -algebra with a GNS-faithful state. Let  $T_v : A_v \rightarrow A_v$  be a state-preserving completely bounded map and assume it naturally extends to a bounded map on  $L^2(A_v, \varphi_v)$  and on  $L^2(A_v^{op}, \varphi_v)$ . Fix  $d \geq 1$ . Then, for the reduced graph product, the map  $T_d : A_{\Gamma, d} \rightarrow A_{\Gamma, d}$  given for  $a = a_1 \otimes \cdots \otimes a_d \in \mathbf{A}_{\mathbf{v}} \subseteq \mathbf{A}_{\Gamma, d}$  by*

$$T_d(\lambda(a_1 \otimes \cdots \otimes a_d)) = \lambda(T_{v_1}(a_1) \otimes \cdots \otimes T_{v_d}(a_d)) \quad (6.56)$$

*admits a completely bounded extension on  $A_{\Gamma, d}$  with*

$$\|T_d\|_{\text{cb}} \leq (\#\text{Cliq}(\Gamma))^3 d \cdot \left( \max_v C(T_v) \right)^d. \quad (6.57)$$

*where*

$$C(T_v) := \max\{\|T_v\|_{\text{cb}}, \|T_v\|_{B(L^2(\mathbf{A}_v, \varphi_v))}, \|T_v\|_{B(L^2(\mathbf{A}_v^{op}, \varphi_v))}\}. \quad (6.58)$$

*We will denote this map as  $T_d := *_{v \in \Gamma} T_v$ . Moreover, if  $(S_v)_{v \in \Gamma}$  are maps satisfying the same conditions as  $(T_v)_{v \in \Gamma}$  then*

$$\|T_d - S_d\|_{\text{cb}} \leq (\#\text{Cliq}(\Gamma))^3 d^2 \left( \max_v \max\{C(T_v), C(S_v)\} \right)^{d-1} \max_v C(T_v - S_v). \quad (6.59)$$

*Proof.* Fix  $d \geq 1$  and suppose first that for all  $1 \leq i \leq d$  we are given maps  $E_{v,i} : A_v \rightarrow A_v$  satisfying the assumptions of the theorem for  $T_v$ . Now for  $1 \leq i \leq d$  the direct sum  $\bigoplus_{v \in \Gamma} E_{v,i}$  extends to a bounded map on  $(\bigoplus_{v \in \Gamma} \mathcal{H}_v)_C$ . Moreover, by [ER00, Theorem 3.4.1] this map is in fact completely bounded with the same norm. Hence by (6.16) the map  $E_{L,i} := (\bigoplus_{v \in \Gamma} E_{v,i})$  is completely bounded on  $L_1$  with norm

$$\|E_{L,i}\|_{\text{cb}} \leq \max_{v \in \Gamma} \|E_{v,i}\|_{B(L^2(A_v, \varphi_v))}.$$

Similarly we obtain that  $E_{R,i} := (\bigoplus_{v \in \Gamma} T_{v,i})$  is completely bounded on  $K_1$  with norm  $\|E_{R,i}\|_{\text{cb}} \leq \max_{v \in \Gamma} \|E_{v,i}\|_{B(L^2(A_v^{op}, \varphi_v))}$ . Now, fix a tuple  $\rho = (n_l, n_r, \mathbf{u}_l, \mathbf{u}_r, \mathbf{t})$  and denote  $\tilde{n}_l = n_l + |\mathbf{u}_l|$  and  $\tilde{n}_r = n_r + |\mathbf{u}_r|$ . Then by [ER00, Proposition 9.2.5] we obtain that

$$\Pi_\rho[(E_{v,i})_{v,i}] := E_{L,1} \otimes \cdots \otimes E_{L,\tilde{n}_l} \otimes E_{t_1,\tilde{n}_l+1} \otimes \cdots \otimes E_{t_{|\mathbf{t}|},\tilde{n}_l+|\mathbf{t}|} \otimes E_{R,\tilde{n}_l+|\mathbf{t}|+1} \otimes \cdots \otimes E_{R,d}$$

is a completely bounded map on  $L_{\tilde{n}_l} \otimes_h \mathbf{A}_{\mathbf{t}} \otimes_h K_{\tilde{n}_r}$  with norm

$$\|\Pi_\rho[(E_{v,i})_{v,i}]\|_{\text{cb}} \leq \prod_{i=1}^{\tilde{n}_l} \|E_{L,i}\|_{\text{cb}} \prod_{i=1}^{|\mathbf{t}|} \|E_{t_i,\tilde{n}_l+i}\|_{\text{cb}} \prod_{i=\tilde{n}_l+|\mathbf{t}|+1}^d \|E_{R,i}\|_{\text{cb}} \leq \prod_{i=1}^d \max_v C(E_{v,i}). \quad (6.60)$$

Now let the maps  $(T_v)$  be given and set  $T_\rho = \Pi_\rho[(T_v)_{v,i}]$  (i.e. taking  $E_{v,i} = T_v$  for all  $i$ ). Hence, we get a completely bounded map  $\tilde{T}_d = (T_\rho)_\rho$  on  $\tilde{X}_d$ . Denote  $T'_d$  for the natural product map on  $\mathbf{A}_{\Gamma,d}$  that is given by  $T_{v_1} \otimes \cdots \otimes T_{v_d}$  on  $\mathbf{A}_{\mathbf{v}}$  for  $\mathbf{v} = v_1 \cdots v_d$ . We then find

$$T_d \circ \lambda|_{\mathbf{A}_{\Gamma,d}} = \lambda \circ T'_d|_{\mathbf{A}_{\Gamma,d}} = \pi_d \circ j_d \circ T'_d|_{\mathbf{A}_{\Gamma,d}} = \pi_d \circ D_d \circ \tilde{T}_d \circ \tilde{\Theta}_d \circ \lambda|_{\mathbf{A}_{\Gamma,d}}. \quad (6.61)$$

This shows that  $T_d$  extends to a completely bounded map on  $A_{\Gamma,d}$ . The norm-bound now follows from the bound  $\|\pi_d\|_{\text{cb}} \leq (\#\text{Cliq}(\Gamma))^3 d$ , the bound on  $\|\tilde{T}_d\|_{\text{cb}}$  and the fact that  $D_d$  and  $\tilde{\Theta}_d$  are completely contractive.

Now suppose we are given maps  $(T_v)_{v \in V}$  and  $(S_v)_{v \in V}$  satisfying the assumptions of the theorem. Set  $S_\rho := \Pi_\rho[(S_v)]$  and  $\tilde{S}_d := (S_\rho)_\rho$ . Set  $E_{v,i,j} = T_v$  for  $i < j$ , set  $E_{v,i,j} = T_v - S_v$  for  $i = j$  and set  $E_{v,i,j} = S_v$  for  $i > j$ . Then by cancellation it follows that  $\Pi_\rho[(T_v)] - \Pi_\rho[(S_v)] = \sum_{j=1}^d \Pi_\rho[(E_{v,i,j})_{v,i}]$ . Thus it follows that

$$\|T_\rho - S_\rho\|_{\text{cb}} \leq \sum_{j=1}^d \|\Pi_\rho[(E_{v,i,j})_{v,i}]\|_{\text{cb}} \leq \sum_{j=1}^d \prod_{i=1}^d \max_v C(E_{v,i,j}) \quad (6.62)$$

$$\leq d \left( \max_v \max\{C(T_v), C(S_v)\} \right)^{d-1} \max_v C(T_v - S_v). \quad (6.63)$$

Then as  $(T_d - S_d) \circ \lambda|_{\mathbf{A}_{\Gamma,d}} = \pi_d \circ D_d \circ (\tilde{T}_d - \tilde{S}_d) \circ \tilde{\Theta}_d \circ \lambda|_{\mathbf{A}_{\Gamma,d}}$  we obtain  $\|T_d - S_d\|_{\text{cb}} \leq \|\pi_d\|_{\text{cb}} \max_\rho \|T_\rho - S_\rho\|_{\text{cb}}$  which proves the bound.  $\square$

Additionally we prove an analogue of Theorem 6.4.3 for the Hilbert spaces.

**Theorem 6.4.4.** *Let  $\Gamma$  be a finite graph and for  $v \in \Gamma$  let  $(A_v, \varphi_v)$  and  $(B_v, \psi_v)$  be unital  $C^*$ -algebras with GNS-faithful states and consider the reduced graph products  $A_\Gamma$  and  $B_\Gamma$  respectively. For  $v \in \Gamma$ , let  $T_v : A_v \rightarrow B_v$  be state-preserving maps that extend to bounded*

maps from  $L^2(A_v, \varphi_v) (= \mathcal{H}_v^A)$  to  $L^2(B_v, \psi_v) (= \mathcal{H}_v^B)$ . Fix  $d \geq 1$ . Then the map  $T_d : \mathcal{H}_{\Gamma, d}^A \rightarrow \mathcal{H}_{\Gamma, d}^B$  defined for  $\eta = \eta_1 \otimes \cdots \otimes \eta_d \in \mathcal{H}_{\mathbf{v}}^A \subseteq \mathcal{H}_{\Gamma, d}^A$  as

$$T_d(\eta) = T_{v_1}(\eta_1) \otimes \cdots \otimes T_{v_d}(\eta_d) \quad (6.64)$$

extends to a bounded map. Moreover, if  $(S_v)_{v \in \Gamma}$  are maps satisfying the same conditions as  $(T_v)_{v \in \Gamma}$  then

$$\|T_d - S_d\|_{B(\mathcal{H}_{\Gamma, d}^A, \mathcal{H}_{\Gamma, d}^B)} \quad (6.65)$$

$$\leq d(\max_v \max\{\|T_v\|_{B(\mathcal{H}_v^A, \mathcal{H}_v^B)}, \|S_v\|_{B(\mathcal{H}_v^A, \mathcal{H}_v^B)}\})^{d-1} \max_v \|T_v - S_v\|_{B(\mathcal{H}_v^A, \mathcal{H}_v^B)} \quad (6.66)$$

*Proof.* Fix  $d \geq 1$  and for  $v \in \Gamma$  and  $1 \leq i \leq d$  let  $E_{v,i} : A_v \rightarrow B_v$  be state-preserving that extend to a map in  $B(\mathcal{H}_v^A, \mathcal{H}_v^B)$ . Then as  $E_{v,i}$  is state-preserving we have  $E_{v,i}(\mathcal{H}_v^A) \subseteq \mathcal{H}_v^B$  so that the map  $\Pi[(E_{v,i})] : \mathcal{H}_{\Gamma, d}^A \rightarrow \mathcal{H}_{\Gamma, d}^B$  defined for  $\eta = \eta_1 \otimes \cdots \otimes \eta_d \in \mathcal{H}_{\mathbf{v}}^A \subseteq \mathcal{H}_{\Gamma, d}^A$  as

$$\Pi[(E_{v,i})_{v,i}](\eta) = E_{v_1,1}(\eta_1) \otimes \cdots \otimes E_{v_d,d}(\eta_d) \quad (6.67)$$

is well-defined algebraically and maps  $\mathcal{H}_{\mathbf{v}}^A$  to  $\mathcal{H}_{\mathbf{v}}^B$  for  $\mathbf{v} \in \mathcal{W}_{\Gamma}$ . Hence, since these subspaces are mutually orthogonal for  $\mathbf{v} \in \mathcal{W}_{\Gamma}$  we obtain

$$\|\Pi[(E_{v,i})]\|_{B(\mathcal{H}_{\Gamma, d}^A, \mathcal{H}_{\Gamma, d}^B)} = \max_{\mathbf{v} \in \mathcal{W}_{\Gamma}, |\mathbf{v}|=d} \|\Pi[(E_{v,i})]\|_{B(\mathcal{H}_{\mathbf{v}}^A, \mathcal{H}_{\mathbf{v}}^B)} \quad (6.68)$$

$$= \max_{\mathbf{v} \in \mathcal{W}_{\Gamma}, |\mathbf{v}|=d} \prod_{i=1}^d \|E_{v_i,i}\|_{B(\mathcal{H}_{v_i}^A, \mathcal{H}_{v_i}^B)} \quad (6.69)$$

$$\leq \prod_{i=1}^d \max_v \|E_{v,i}\|_{B(\mathcal{H}_v^A, \mathcal{H}_v^B)} \quad (6.70)$$

Now let  $(T_v)$  and  $(S_v)$  be maps satisfying the conditions from the theorem. We see that  $T_d = \Pi[(T_{v,i})_{v,i}]$  (i.e. taking  $E_{v,i} = T_v$  for all  $1 \leq i \leq d$ ) and  $S_d = \Pi[(S_{v,i})_{v,i}]$  so these maps are indeed bounded. Now set  $E_{v,i,j} = T_v$  for  $i < j$ , set  $E_{v,i,j} = T_v - S_v$  for  $i = j$  and set  $E_{v,i,j} = S_v$  for  $i > j$ . It follows from cancellation that

$$\Pi[(T_{v,i})_{v,i}] - \Pi[(S_{v,i})_{v,i}] = \sum_{j=1}^d \Pi[(E_{v,i,j})_{v,i}] \quad (6.71)$$

Hence  $\|T_d - S_d\|_{B(\mathcal{H}_{\Gamma, d}^A, \mathcal{H}_{\Gamma, d}^B)} \leq \sum_{j=1}^d \|\Pi[(E_{v,i,j})_{v,i}]\|_{B(\mathcal{H}_{\Gamma, d}^A, \mathcal{H}_{\Gamma, d}^B)}$  from which (6.65) follows.  $\square$

## 6.5. A U.C.P EXTENSION FOR CCAP IS PRESERVED UNDER GRAPH PRODUCTS

We will introduce the following definition, originating from [RX06, Section 4].

**Definition 6.5.1.** Let  $(A, \varphi)$  be a unital  $C^*$ -algebra with GNS-faithful state  $\varphi$ . We will say that it has a u.c.p extension for the CCAP, when the following are all satisfied:

1. There is a net  $(V_j)_{j \in J}$  of finite rank state-preserving maps on  $A$  that converge to the identity pointwise and with  $\limsup_j \|V_j\|_{cb} = 1$ .
2. There is a unital  $C^*$ -algebra  $(B, \psi)$  that contains  $A$  as a unital subalgebra, and s.t.  $\psi$  is GNS-faithful and extends the state  $\varphi$ .
3. There exists a net  $(U_j)_{j \in J}$  of state-preserving, u.c.p. maps  $U_j : A \rightarrow B$  for which  $\|V_j - U_j\|_{cb}, \|V_j - U_j\|_{B(L^2(A, \varphi), L^2(B, \psi))}$  and  $\|V_j - U_j\|_{B(L^2(A^{op}, \varphi), L^2(B^{op}, \psi))}$  all converge to 0 as  $j \rightarrow \infty$ .

Note that by the first property  $(A, \varphi)$  must possess the CCAP. It is clear that any finite dimensional  $C^*$ -algebra possesses the above property. In [RX06, proof of Theorem 4.13] it was proven that the reduced group  $C^*$ -algebra of any discrete group that possess the CCAP, also satisfies above criteria. In [Fre12, proof of Theorem 4.2] it was proven that the same is true for reduced  $C^*$ -algebra of a compact quantum group with Haar state whose discrete dual quantum group is weakly amenable with Cowling-Haagerup constant 1.

We will now show in the next theorem that the property of having a u.c.p extension for the CCAP is being preserved under graph products, for finite simple graphs. The proof imitates the proof method of [RX06, Proposition 4.11]. We will use here Proposition 6.3.1, Proposition 6.3.2, Proposition 6.3.3 and Theorem 6.4.3 and Theorem 6.4.4

**Theorem 6.5.2.** *Let  $\Gamma$  be a finite simple graph and for  $v \in \Gamma$  let  $(A_v, \varphi_v)$  be unital  $C^*$ -algebras (with GNS-faithful states  $\varphi_v$ ) that have a u.c.p. extension for the CCAP. Then the reduced graph product  $(A_\Gamma, \varphi) = \ast_{v \in \Gamma}^{\min} (A_v, \varphi_v)$  has a u.c.p. extension for the CCAP.*

*Proof.* We let  $(V_{v,j})_{j \in J_v}$ ,  $(B_v, \psi_v)$  and  $(U_{v,j})_{j \in J_v}$  be a u.c.p extension for the CCAP for  $(A_v, \varphi_v)$ . As for all  $v$  the algebras  $A_v, B_v$  have GNS-faithful states, their reduced graph products  $(A_\Gamma, \varphi)$  and  $(B_\Gamma, \psi)$  respectively are well-defined, and the states  $\varphi$  and  $\psi$  are GNS-faithful as well. By [CF17, Proposition 3.12] there exists a unital  $*$ -homomorphism  $\pi : A_\Gamma \rightarrow B_\Gamma$  that intertwines the graph product states. Now for  $a \in \ker \pi$  and  $b \in \lambda(A_\Gamma)$  we have  $\varphi(b^* a^* ab) = \psi(\pi(b^*) \pi(a)^* \pi(a) \pi(b)) = 0$ . Therefore, by the faithfulness of the GNS-representation of  $A_\Gamma$ , this shows that  $a = 0$  and hence  $\pi$  is injective. We will hence consider  $\pi$  as an inclusion  $A_\Gamma \subseteq B_\Gamma$ .

We construct a single directed set  $\mathcal{J} = \prod_{v \in \Gamma} J_v$  with partial order  $(j_v)_{v \in \Gamma} < (j'_v)_{v \in \Gamma}$  if and only if  $j_v < j'_v$  for all  $v \in \Gamma$ . We can now define nets  $(V_{v,j})_{j \in \mathcal{J}}$ ,  $(U_{v,j})_{j \in \mathcal{J}}$  as follows: for  $j = (j_v)_{v \in \Gamma}$  we set  $V_{v,j} := V_{v,j_v}$ , and  $U_{v,j} := U_{v,j_v}$ . Note that these nets still satisfy the assumptions of a u.c.p. extension for CCAP. For  $v \in \Gamma$ ,  $j \in \mathcal{J}$  we will set

$$\epsilon_{v,j} = \|V_{v,j} - U_{v,j}\|_{cb} + \|V_{v,j} - U_{v,j}\|_{B(L^2(A_v, \varphi), L^2(B_v, \psi))} + \|V_{v,j} - U_{v,j}\|_{B(L^2(A_v^{op}, \varphi), L^2(B_v^{op}, \psi))}$$

and by restricting to a subnet we can assume  $\epsilon_{v,j} < 1$ . Since the maps  $U_{v,j}$  are u.c.p and state-preserving we have that  $U_{v,j}$  is a contraction from  $L^2(A_v, \varphi_v)$  to  $L^2(B_v, \psi_v)$  and from  $L^2(A_v^{op}, \varphi_v)$  to  $L^2(B_v^{op}, \psi_v)$ . Hence we also obtain

$$\|V_{v,j}\|_{cb}, \|V_{v,j}\|_{B(L^2(A_v, \varphi), L^2(B_v, \psi))}, \|V_{v,j}\|_{B(L^2(A_v^{op}, \varphi_v), L^2(B_v^{op}, \psi_v))} \leq 2.$$

We can now by Theorem 6.4.3 construct for  $j \in \mathcal{J}$ , the finite rank c.b. maps  $F_{d,j} = *_{v,\Gamma} V_{v,j}$  on  $A_{\Gamma,d}$ . We then obtain completely bounded, finite rank maps

$$D_{N,j} = \sum_{d=0}^N \left(1 - \frac{1}{\sqrt{N}}\right)^d F_{d,j} P_{\Gamma,d}$$

on  $A_{\Gamma}$  that on the dense subset  $\lambda(A_{\Gamma})$  tend in norm to the identity as  $N, j \rightarrow \infty$ . We can now by Proposition 6.3.2 construct the state-preserving u.c.p maps  $U_j := *_{v,\Gamma} U_{v,j}$ , and by Proposition 6.3.3 construct the u.c.p maps  $\mathcal{T}_{1-\frac{1}{\sqrt{N}}}$  and the c.b. maps  $\mathcal{T}_{(1-\frac{1}{\sqrt{N}}),N}$  on  $A_{\Gamma}$ . This gives us state-preserving u.c.p maps  $E_{N,j} = U_j \circ \mathcal{T}_{1-\frac{1}{\sqrt{N}}}$  and state-preserving c.b. maps  $\tilde{D}_{N,j} = U_j \circ \mathcal{T}_{1-\frac{1}{\sqrt{N}},N}$ . Applying Theorem 6.4.3 and using that  $C(V_{v,j}), C(U_{v,j}) \leq 2$  and  $C(V_{v,j} - U_{v,j}) \leq \epsilon_{v,j}$  we obtain

$$\|F_{d,j} - U_j|_{A_{\Gamma,d}}\|_{\text{cb}} \leq (\#\text{Cliq}(\Gamma))^3 d^2 2^{d-1} (\max_v \epsilon_{v,j}) \rightarrow 0 \text{ as } j \rightarrow \infty. \quad (6.72)$$

Similarly, by Theorem 6.4.4 we obtain

$$\|F_{d,j} - U_j\|_{B(\mathcal{H}_{\Gamma,d}^A, \mathcal{H}_{\Gamma,d}^B)} \leq d 2^{d-1} (\max_v \epsilon_{v,j}) \rightarrow 0 \text{ as } j \rightarrow \infty. \quad (6.73)$$

Now

$$\|E_{N,j} - D_{N,j}\|_{\text{cb}} \leq \|E_{N,j} - \tilde{D}_{N,j}\|_{\text{cb}} + \|\tilde{D}_{N,j} - D_{N,j}\|_{\text{cb}} \quad (6.74)$$

$$\leq \|\mathcal{T}_{1-\frac{1}{\sqrt{N}}} - \mathcal{T}_{1-\frac{1}{\sqrt{N}},N}\|_{\text{cb}} + \sum_{d=0}^N \|U_j|_{A_{\Gamma,d}} - F_{d,j}\|_{\text{cb}} \|P_{\Gamma,d}\|_{\text{cb}} \quad (6.75)$$

and similarly

$$\|E_{N,j} - D_{N,j}\|_{B(\mathcal{H}_{\Gamma}^A, \mathcal{H}_{\Gamma}^B)} \quad (6.76)$$

$$\leq \|E_{N,j} - \tilde{D}_{N,j}\|_{B(\mathcal{H}_{\Gamma}^A, \mathcal{H}_{\Gamma}^B)} + \|\tilde{D}_{N,j} - D_{N,j}\|_{B(\mathcal{H}_{\Gamma}^A, \mathcal{H}_{\Gamma}^B)} \quad (6.77)$$

$$\leq \|U_j\|_{B(\mathcal{H}_{\Gamma}^A, \mathcal{H}_{\Gamma}^B)} \|\mathcal{T}_{1-\frac{1}{\sqrt{N}}} - \mathcal{T}_{1-\frac{1}{\sqrt{N}},N}\|_{B(\mathcal{H}_{\Gamma}^A, \mathcal{H}_{\Gamma}^A)} \quad (6.78)$$

$$+ \sum_{d=0}^N \|U_j|_{A_{\Gamma,d}} - F_{d,j}\|_{B(\mathcal{H}_{\Gamma,d}^A, \mathcal{H}_{\Gamma,d}^B)} \|P_{\Gamma,d}\|_{B(\mathcal{H}_{\Gamma}^A, \mathcal{H}_{\Gamma}^A)}. \quad (6.79)$$

Note that  $\|P_{\Gamma,d}\|_{\text{cb}} \leq C_{\Gamma} d$  (Theorem 6.2.10),  $\|P_{\Gamma,d}\|_{B(\mathcal{H}_{\Gamma}^A, \mathcal{H}_{\Gamma}^A)} \leq 1$  and  $\|U_j\|_{B(\mathcal{H}_{\Gamma}^A, \mathcal{H}_{\Gamma}^B)} = 1$ . Thus we obtain using Proposition 6.3.3 that

$$\lim_N \lim_j \|E_{N,j} - D_{N,j}\|_{\text{cb}} \leq \lim_N \|\mathcal{T}_{1-\frac{1}{\sqrt{N}}} - \mathcal{T}_{1-\frac{1}{\sqrt{N}},N}\|_{\text{cb}} \quad (6.80)$$

$$\leq \lim_N C_{\Gamma} N^2 \left(1 - \frac{1}{\sqrt{N}}\right)^N = 0 \quad (6.81)$$

so that in particular  $\lim_N \lim_j \|D_{N,j}\|_{\text{cb}} = 1$ . Similarly we obtain

$$\lim_N \lim_j \|E_{N,j} - D_{N,j}\|_{B(L^2(A_{\Gamma}, \varphi), L^2(B_{\Gamma}, \psi))} \leq \lim_N \|\mathcal{T}_{1-\frac{1}{\sqrt{N}}} - \mathcal{T}_{1-\frac{1}{\sqrt{N}},N}\|_{B(\mathcal{H}_{\Gamma}^A, \mathcal{H}_{\Gamma}^A)} \quad (6.82)$$

$$\leq \lim_N \sup_{d \geq N} \left(1 - \frac{1}{\sqrt{N}}\right)^d = 0 \quad (6.83)$$

and analogously  $\lim_N \lim_j \|E_{N,j} - D_{N,j}\|_{B(L^2(A_\Gamma^{op}, \varphi), L^2(B_\Gamma^{op}, \psi))} = 0$  can be shown. Now the construction of  $(D_{N,j})$ ,  $(B_\Gamma, \psi)$  and  $(E_{N,j})$  shows that  $(A_\Gamma, \varphi)$  has a u.c.p extension for the CCAP.  $\square$

Reasoning similarly to [CF17, Corollary 3.17] we show for arbitrary (possibly infinite) simple graphs that, under the assumptions on the algebras  $A_v$ , we have that the reduced graph product possesses the CCAP.

**Theorem 6.5.3.** *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $(A_v, \varphi_v)$  be unital  $C^*$ -algebras that have a u.c.p. extension for the CCAP. Then the reduced graph product  $(A_\Gamma, \varphi) = *_{\Gamma}^{\min}(A_v, \varphi_v)$  has the CCAP.*

*Proof.* It follows from Theorem 6.5.2 that for any finite subgraph  $\Gamma_0 \subseteq \Gamma$ , the reduced graph product  $*_{v, \Gamma_0}^{\min}(A_v, \varphi_v)$  possesses the CCAP. As the reduced graph product over  $\Gamma$  is the induced limit of all reduced graph products over finite subgraphs and as the CCAP is preserved under inductive limits, this shows the result  $\square$

**Corollary 6.5.4.** *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $A_v$  be one of the following:*

1.  $(A_v, \varphi_v)$  is a finite-dimensional  $C^*$ -algebra with GNS-faithful state  $\varphi_v$ .
2.  $(A_v, \varphi_v)$  is the reduced group  $C^*$ -algebra of a discrete group with Plancherel state  $\varphi_v$  that possesses the CCAP
3.  $(A_v, \varphi_v)$  is the reduced  $C^*$ -algebra of a compact quantum group whose discrete dual quantum group is weakly amenable with Cowling-Haagerup constant 1. Here  $\varphi_v$  denotes the Haar state.

*Then the reduced graph product  $C^*$ -algebra  $(A_\Gamma, \varphi) = *_{v, \Gamma}^{\min}(A_v, \varphi_v)$  has the CCAP.*

We recall, that for a discrete group  $G$  we have that  $G$  is weakly amenable with constant 1 if and only if the reduced group  $C^*$ -algebra  $C_r^*(G)$  possesses the CCAP, if and only if the group von Neumann algebra  $\mathcal{L}(G)$  possesses the weak- $*$  CCAP. Using this we obtain the following result for von Neumann algebras.

**Corollary 6.5.5.** *Let  $\Gamma$  be a simple graph and for  $v \in \Gamma$  let  $(M_v, \psi_v)$  be the group von Neumann algebra  $\mathcal{L}(G_v)$  of a discrete group  $G_v$ , equipped with the canonical state. If  $M_v$  has the weak- $*$  CCAP for all  $v \in \Gamma$ , then the graph product von Neumann algebra  $(M_\Gamma, \psi) = *_{v, \Gamma}(M_v, \psi_v)$  possesses the weak- $*$  CCAP as well.*

*Proof.* Note that the graph product  $M_\Gamma = *_{v, \Gamma} \mathcal{L}(G_v) = \mathcal{L}(*_{v, \Gamma} G_v)$  has the weak- $*$  CCAP if and only if  $C_r^*(\mathcal{L}(*_{v, \Gamma} G_v)) = *_{v, \Gamma}^{\min} C_r^*(G_v)$  has the CCAP. The result then follows from Corollary 6.5.4  $\square$

We note that Corollary 6.5.5 was already known by [Rec17] where using different techniques it was shown that for discrete groups weak amenability with constant 1 is preserved under graph products. However, Corollary 6.5.4 does give new examples of algebras that possess the CCAP as you can consider graph products of the form  $*_{v, \Gamma}^{\min}(A_v, \varphi_v)$  where some of the algebras  $(A_v, \varphi_v)$  satisfy condition (1), some satisfy condition (2) and some satisfy condition (3).



# 7

## COMMUTATOR ESTIMATES FOR NORMAL OPERATORS IN FACTORS

For a normal measurable operator  $a$  affiliated with a von Neumann factor  $M$  we show:

If  $M$  is infinite, then there is  $\lambda_0 \in \mathbb{C}$  so that for  $\varepsilon > 0$  there are  $u_\varepsilon = u_\varepsilon^*$ ,  $v_\varepsilon \in U(M)$  with

$$v_\varepsilon|[a, u_\varepsilon]|v_\varepsilon^* \geq (1 - \varepsilon)(|a - \lambda_0 1_M| + u_\varepsilon|a - \lambda_0 1_M|u_\varepsilon).$$

If  $M$  is finite, then there is  $\lambda_0 \in \mathbb{C}$  and  $u, v \in U(M)$  so that

$$v|[a, u]|v^* \geq \frac{\sqrt{3}}{2}(|a - \lambda_0 1_M| + u|a - \lambda_0 1_M|u^*).$$

These bounds are optimal for infinite factors,  $\text{II}_1$ -factors and some  $\text{I}_n$ -factors. Furthermore, for finite factors applying  $\|\cdot\|_1$ -norms to the inequality provides estimates on the norm of the inner derivation  $\delta_a : M \rightarrow L^1(M, \tau)$  associated to  $a$ . While by [BHS23, Theorem 1.1] it is known for finite factors and self-adjoint  $a \in L^1(M, \tau)$  that

$$\|\delta_a\|_{M \rightarrow L^1(M, \tau)} = 2 \min_{z \in \mathbb{C}} \|a - z\|_1,$$

we present concrete examples of finite factors  $M$  and normal operators  $a \in M$  for which this fails.

This chapter is based on the paper:

- **Alexei Ber, Matthijs Borst and Fedor Sukochev**, *Commutator estimates for normal operators in factors with applications to derivations*, Accepted in the Journal of Operator Algebras. Preprint: [Arxiv:2304.10775v1](https://arxiv.org/abs/2304.10775v1).

### 7.1. INTRODUCTION

Derivations are linear maps  $\delta$  that satisfy the Leibniz rule  $\delta(xy) = \delta(x)y + x\delta(y)$ . They play an essential role in the theory of Lie algebras, Cohomology, the study of Semi-groups

and in Quantum Physics, see [KL14; SS95]. A classical result on derivations is due to Stampfli [Sta70] which asserts that for  $a \in B(\mathcal{H})$ , a bounded operator on a Hilbert space  $\mathcal{H}$ , the derivation  $\delta_a : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  defined by the commutator  $\delta_a(x) = [a, x] = ax - xa$  has operator norm  $\|\delta_a\| = 2 \inf_{z \in \mathbb{C}} \|a - z1_M\|$ . Through the work of [KLR67; Gaj72; Zsi73], the result of Stampfli has been extended to derivations on arbitrary von Neumann algebras  $M$  (see also [Mag95] for more in this direction). More precisely, the result of Zsidó [Zsi73, Corollary] asserts that for  $M$  a von Neumann algebra and  $a \in M$ , the derivation  $\delta_a : M \rightarrow M$  associated to  $a$  satisfies the distance formula:

$$\|\delta_a\|_{M \rightarrow M} = 2 \min_{z \in Z(M)} \|a - z\|, \quad (7.1)$$

where  $Z(M)$  denotes the center of  $M$ .

Our research aims to obtain results similar to (7.1) for derivations that map  $M$  into the predual  $M_*$ . Indeed, the predual  $M_*$  is a  $M$ -bimodule (see Section 7.7) and therefore it is possible to consider derivations  $\delta : M \rightarrow M_*$ . Important work on such derivations was done in [BP80; Haa83; BGM12] and particularly the result of [Haa83, Theorem 4.1] showed that all these derivations are inner (i.e. of the form  $\delta = \delta_a$  for some  $a \in M_*$ , defined by  $\delta_a(x) = ax - xa$ ). These studies arose after Connes proved in [Con78] that all  $C^*$ -algebras that are amenable (as Banach  $*$ -algebra) are necessarily nuclear. Haagerup proved in [Haa83] that the reverse implication is also true.

In [BHS23] the norms of these derivations were studied and results analogous to (7.1) were found in certain cases: for  $M$  properly infinite it was shown that some form of formula (7.1) holds true and for  $M$  finite the same was proved under the condition that  $a$  is self-adjoint. The proofs of these results were based on improvements of the operator estimates obtained in [BS12b; BS12a], see below:

**Theorem 7.1.1** (Theorem 1 in [BS12b]). *Let  $M$  be a factor and let  $a = a^* \in S(M)$  (here  $S(M)$  is the algebra of measurable operators affiliated with  $M$ ).*

1. *If  $M$  is a finite factor or else a purely infinite  $\sigma$ -finite factor, then there exists  $\lambda_0 \in \mathbb{R}$  and  $u_0 = u_0^* \in U(M)$ , such that*

$$|[a, u_0]| = u_0|a - \lambda_0 1_M|u_0 + |a - \lambda_0 1_M| \quad (7.2)$$

*where  $U(M)$  is the group of all unitary operators in  $M$ ;*

2. *there exists  $\lambda_0 \in \mathbb{R}$  so that for any  $\varepsilon > 0$  there exists  $u_\varepsilon = u_\varepsilon^* \in U(M)$  such that*

$$|[a, u_\varepsilon]| \geq (1 - \varepsilon)|a - \lambda_0 1_M|. \quad (7.3)$$

This theorem was extended to arbitrary von Neumann algebras in [BS12a] with the replacement of  $\lambda_0 1_M$  by an element from the center. In [BHS23, Corollary B.3] inequality (7.3) was extended to:

$$|[a, u_\varepsilon]| \geq (1 - \varepsilon)(|a - c_a| + u_\varepsilon|a - c_a|u_\varepsilon) \quad (7.4)$$

where  $c_a \in Z_h(LS(M))$  ( $Z_h(LS(M))$  is the self-adjoint part of the center of the algebra  $LS(M)$  of locally measurable operators). The question arises: is such an inequality as

(7.4) true for arbitrary  $a \in S(M)$ ? More precisely, are there such  $\lambda_0 \in \mathbb{C}$ ,  $u, v, w \in U(M)$  and a constant  $C > 0$  such that

$$|[a, u]| \geq C(v|a - \lambda_0 1_M|v^* + w|a - \lambda_0 1_M|w^*) \quad (7.5)$$

holds true if  $a$  is not necessarily self-adjoint? In this chapter, we give an answer to this question in the case when  $a$  is a normal operator (see Theorems 7.5.6, 7.6.4). It turns out that if  $M$  is an infinite factor, then the constant  $C$  can be chosen arbitrarily close to 1, just as in the case of self-adjoint  $a$ . However, in the case when  $M$  is a finite factor, the situation changes. For  $\text{II}_1$ -factors the optimal constant  $C$  turns out to be equal to  $\frac{\sqrt{3}}{2}$  and for  $I_n$ -factors appropriate upper and lower bounds on the optimal constant are given by  $\Lambda_n \leq C \leq \frac{1}{2}\tilde{\Lambda}_n$  (see (7.12) and (7.13) for definitions of these constants and (7.14) for estimates). We summarize above results in the following theorem.

**Theorem 7.1.2** (see Theorems 7.5.6, 7.6.4). *Let  $M$  be a factor and let  $a \in S(M)$  be normal. Then there is a  $\lambda_0 \in \mathbb{C}$  and unitaries  $u, v, w \in U(M)$  such that*

$$|[a, u]| \geq C(v|a - \lambda_0 1_M|v^* + w|a - \lambda_0 1_M|w^*) \quad (7.6)$$

for some constant  $C > 0$  independent of  $a$ . Moreover

1. when  $M$  is a  $I_n$ -factor,  $n < \infty$ , the optimal constant satisfies  $\Lambda_n \leq C \leq \frac{1}{2}\tilde{\Lambda}_n$ .
2. when  $M$  is a  $\text{II}_1$ -factor, the optimal constant is  $C = \frac{\sqrt{3}}{2}$ .
3. when  $M$  is an infinite factor, we can choose  $C$  arbitrarily close to 1.

This theorem can be applied to obtain norm estimates for derivations  $\delta : M \rightarrow M_*$  and extend results of [BHS23]. Specifically, we consider the case that  $M$  is finite, and  $\tau$  is a faithful normal tracial state on  $M$ . In this case  $M_*$  is isomorphic to  $L^1(M, \tau)$  (see e.g. [Tak03a, Lemma 2.12 and Theorem 2.13]). As an application of inequality (7.4), it was proved in [BHS23, Theorem 1.1] that, for  $a = a^* \in L^1(M, \tau)$ , we have

$$\|\delta_a\|_{M \rightarrow L^1(M, \tau)} = 2 \min_{z \in Z(S(M))} \|a - z\|_1 \quad (7.7)$$

(here  $Z(S(M))$  denotes the center of  $S(M)$ ) and that the minimum is attained at a self-adjoint element  $c_a = c_a^* \in L^1(M, \tau) \cap Z(S(M))$ . In this thesis, using Theorem 7.1.2, we show that for a finite factor  $M$  and for an arbitrary normal measurable  $a \in L^1(M, \tau)$ , the estimate

$$\sqrt{3} \min_{z \in \mathbb{C}} \|a - z\|_1 \leq \|\delta_a\|_{M \rightarrow L^1(M, \tau)} \leq 2 \min_{z \in \mathbb{C}} \|a - z\|_1 \quad (7.8)$$

holds (see remark after Theorem 7.7.3). In Section 7.7 we show that the estimates given in (7.8) are sharp. In particular, in Theorem 7.7.3 we demonstrate that for any finite  $\text{II}_1$ -factor  $M$  there exists a normal  $a \in M$  such that the derivation  $\delta_a$  is non-zero and satisfies  $\|\delta_a\|_{M \rightarrow L^1(M, \tau)} = \sqrt{3} \min_{z \in \mathbb{C}} \|a - z\|_1$ , whereas it follows from Theorem 7.6.4 and [BHS23, Theorem 3.1] that for any infinite factor  $M$  formula (7.7) holds for an arbitrary normal  $a \in L^1(M, \tau)$ .

Finally, we remark that (7.8) is in fact an estimate for the  $L^1$ -diameter of the unitary orbit  $\mathcal{O}(a) = \{uau^* : u \in U(M)\}$  of  $a$  as  $\text{Diam}_{L^1(M, \tau)}(\mathcal{O}(a)) = \|\delta_a\|_{M \rightarrow L^1(M, \tau)}$ , see end of Section 7.7.

### DISCUSSION OF PROOFS AND COMPARISON TO [BS12b; BS12a; BHS23]

We first discuss our proof techniques. In the proof for  $\text{II}_1$ -factor  $M$  we construct a trace-preserving injective  $*$ -homomorphism  $F$  from  $S[0, 1]$  (space of measurable functions on  $[0, 1]$ ) to  $S(M)$  satisfying  $F(g) = a$  for some measurable function  $g$  (see Theorem 7.5.2). We then construct a point  $z_0 \in \mathbb{C}$  and a partition  $\{X_1\} \cup \{X_2^{m,i} : m \geq 1, i = 1, 2\}$  of  $[0, 1]$  satisfying:

1.  $g(X_1) \subset \{z_0\}$ .
2. For  $m \geq 1$  the sets  $X_2^{m,1}$  and  $X_2^{m,2}$  have equal measure.
3. For  $m \geq 1$  the sets  $g(X_2^{m,1})$  and  $g(X_2^{m,2})$  are  $(z_0, \frac{\pi}{3})$ -conjugate (see Definition 7.4.1).

The statement that  $g(X_2^{m,1})$  and  $g(X_2^{m,2})$  are  $(z_0, \frac{\pi}{3})$ -conjugate says that  $z_0$  in some sense lies in between the sets  $g(X_2^{m,1})$  and  $g(X_2^{m,2})$ . Using the partition we can build a measure preserving transformation  $T$  of  $[0, 1]$  with  $T(X_1) = X_1$  and  $T(X_2^{m,1}) = X_2^{m,2}$  and  $T(X_2^{m,2}) = X_2^{m,1}$  for  $m \geq 1$ . Then  $T$  will satisfy

$$|g \circ T - g| \geq \frac{\sqrt{3}}{2} (|g - z_0| + |g \circ T - z_0|). \quad (7.9)$$

Using the map  $F$  and operator inequalities we can in a similar way use the partition to obtain (7.6) for some  $u, v, w$  and  $\lambda_0 (= z_0)$ . The case of  $I_n$ -factors ( $n < \infty$ ) is somewhat analogous and uses the spectral mapping theorem and the analogue of (7.9) for functions  $g$  on the measure space  $\Omega_n = \{1, \dots, n\}$  with counting measure. Upper bounds for the optimal constant  $C$  for  $I_n$ -factors and  $\text{II}_1$  factors are obtained using Proposition 7.5.5 and Lemma 7.A.2 by constructing specific operators  $a$ .

The proof for infinite factors is based on the structure of the set  $A$  of points of densification of  $a$ . This set  $A \subseteq \mathbb{C}$  is by definition the set of all complex numbers  $\lambda$  for which the spectral projection  $e^a(V)$  is equivalent to  $1_M$  for every neighbourhood  $V$  of  $\lambda$ . The set  $A$  is non-empty and compact and we distinguish three cases:

1. There is a point  $\lambda_0 \in A$  such that  $e^a(\{\lambda_0\}) \sim 1_M$ .
2. The set  $A$  has a limit point  $\lambda_0$ .
3. The set  $A$  is finite and  $e(\{\lambda\}) \not\sim 1_M$  for all  $\lambda \in A$ .

In case (1) we are directly able to build  $u, v, w$  that fulfill (7.6) for  $C = 1$  (actually with equality), while in the cases (2) and (3) for arbitrary  $\varepsilon > 0$  we first need to inductively construct some sequences  $(p_n)_{n \geq 1}, (q_n)_{n \geq 1}$  of orthogonal projections that we then use to define  $u_\varepsilon, v_\varepsilon, w_\varepsilon$  that satisfy (7.6) for  $C = 1 - \varepsilon$ . This shows that for infinite factors the constant  $C$  from (7.6) can be chosen arbitrarily close to 1.

We now compare our techniques to those applied in [BS12b; BS12a; BHS23]. The proof of [BS12b, Theorem 1] is based on comparing the spectral projections  $e^a(-\infty, \lambda)$  and  $e^a(\lambda, \infty)$  for  $\lambda \in \mathbb{R}$  and distinguishing cases. When  $e^a(-\infty, \lambda_0) + p \sim e^a(\lambda_0, \infty) + q$  for some  $\lambda_0 \in \mathbb{R}, p, q \leq e^a(\{\lambda_0\})$  with  $pq = 0$  it is shown that (7.2) is satisfied (see [BS12b,

Lemma 5)). This is in essence not so different to our proof method of Theorem 7.1.2 (for finite factors) which requires similar comparisons for certain halfspaces in the complex plane instead of in the real line (see Lemma 7.4.5). Furthermore, in the case of infinite factors, the proof of [BS12b, Theorem 1] uses the construction of sequences  $(p_n)_{n \geq 1}$ ,  $(q_n)_{n \geq 1}$  which have similarities to those we construct in Theorem 7.6.4; though the constructions are different.

The proof of [BS12a, Theorem 1] uses additional techniques to extend the result of [BS12b, Theorem 1] to the setting of arbitrary von Neumann algebras. In particular this involves obtaining a self-adjoint central element  $c_0 \in LS(M)$  and building certain orthogonal central projections  $p_+, p_-, p_0 \in M$  and combining results for the operators  $ap_0$ ,  $(a - c_0 1_M)p_-$  and  $(a - c_0 1_M)p_+$ . It is not clear whether the applied techniques can also be used to extend our results to the setting of arbitrary von Neumann algebras.

The proof of [BHS23, Theorem 13] adapts methods from [BS12b] to obtain the generalized inequality (7.4) which is more closely related to our method for infinite factors in Theorem 7.6.4.

## STRUCTURE AND OVERVIEW

In Section 7.2 we prove Proposition 7.2.1 and Theorem 7.2.2 that extend some results to locally measurable operators. In Section 7.3 we introduce the constants  $\Lambda_n$  and  $\tilde{\Lambda}_n$  for  $n \in \mathbb{N} \cup \{\infty\}$  that will be used throughout the chapter. In Section 7.4 our main result is Theorem 7.4.3, which is closely related to the constants  $\Lambda_n$  and to the operator inequality (7.5). In Section 7.5 we use this result to obtain Theorem 7.5.6 which establishes the operator inequality of Theorem 7.1.2 for normal elements in finite factors. In Section 7.6 we obtain the inequality of Theorem 7.1.2 for normal locally measurable operators affiliated with an infinite factor, see Theorem 7.6.4. In Section 7.7 we apply our results to obtain the estimate (7.8) for the norm of derivations  $\delta_a : M \rightarrow L_1(M, \tau)$  for normal  $a \in L_1(M, \tau)$ , and we show the given bounds are optimal in some cases. In the Appendix, Section 7.A, we prove two technical results regarding the constants  $\Lambda_n$  and  $\tilde{\Lambda}_n$ . In particular, Theorem 7.A.1 determines the exact value of  $\Lambda_n$  for  $n \neq 4$ .

## 7.2. ESTIMATES FOR LOCALLY MEASURABLE OPERATORS

We prove two results, Proposition 7.2.1 and Theorem 7.2.2, which generalize a known result (a type of triangle inequality for operators) to locally measurable operators. Let  $M$  be a semifinite von Neumann algebra and let  $LS(M)$  be the space of locally measurable operators (see preliminaries). Let  $x \in LS(M)$ . Denote by  $\mathbf{l}(x)$  - the left carrier of  $x$ , by  $\mathbf{r}(x)$  - the right carrier of  $x$  and  $\mathbf{s}(x) = \mathbf{l}(x) \vee \mathbf{r}(x)$ . If  $x = u|x|$  is the polar decomposition of  $x$ , then  $\mathbf{l}(x) = uu^*$  and  $\mathbf{r}(x) = u^*u$ . We denote  $\Re(x) = \frac{x+x^*}{2}$  and  $\Im(x) = \frac{x-x^*}{2i}$  for respectively the real and imaginary part of  $x$ . For a self-adjoint  $x \in LS(M)$  we denote by  $x_+$  (respectively,  $x_-$ ) its positive (respectively negative) part, defined by  $x_+ = \frac{x+|x|}{2}$  (respectively,  $x_- = -\frac{x-|x|}{2}$ ). We note that  $x_-$  and  $x_+$  are orthogonal, that is  $x_-x_+ = 0$ .

We require Theorem 7.2.2 which states a triangle inequality for operators  $x \in LS(M)$ . The statement is similar to [AAP82, Theorem 2.2] where for operators  $x \in M$  the result was shown with partial isometries instead of isometries (see also [FK86, Lemma 4.3] and [Hia21, Lemma 4.15]). To prove Theorem 7.2.2, we will need the following statement

which is similar to [AAP82, Proposition 2.1]. Here,  $v \in M$  is called an *isometry* if  $v^*v = 1_M$ .

**Proposition 7.2.1.** *For each  $x \in \text{LS}(M)$  there is an isometry  $v \in M$  such that  $\Re(x)_+ \leq v|x|v^*$ .*

*Proof.* Let  $p = \mathbf{s}(\Re(x)_+)$ ,  $a = p(x + |x|)$ . Then clearly  $\mathbf{l}(a) \leq p$ . We show  $p = \mathbf{l}(a)$ . Put  $r = p - \mathbf{l}(a)$  so that  $0 = ra = rar = rxr + r|x|r$ . Taking the real part of this equation gives  $0 = r\Re(x)r + r|x|r$ . Since  $r \leq p$  we have  $r\Re(x)_-r = 0$  and therefore  $r\Re(x)r = r\Re(x)_+r$ . Then  $0 = r\Re(x)r + r|x|r = r\Re(x)_+r + r|x|r$  and hence  $r\Re(x)_+r = 0$ . Then as  $(\Re(x)_+^{\frac{1}{2}}r)^*(\Re(x)_+^{\frac{1}{2}}r) = r\Re(x)r = 0$ , we obtain  $\Re(x)_+^{\frac{1}{2}}r = 0$  and hence  $\Re(x)_+r = 0$ . Therefore,  $\Re(x)_+(1_M - r) = \Re(x)_+$  which shows  $(1_M - r) \geq \mathbf{s}(\Re(x)_+) = p$  and we conclude  $r = 0$ , i.e.  $p = \mathbf{l}(a)$ .

Let  $a = w|a|$  be the polar decomposition of  $a$ . Then  $ww^* = p$ . Put  $q = w^*w$  and  $s = (1_M - q) \wedge p$ . We show  $s = 0$ . Indeed  $as = aqs = 0$ , thus  $s(x + |x|)s = sas = 0$  and taking the real part of this equation gives  $s\Re(x)s + s|x|s = 0$ . As  $s \leq p$  we have  $s\Re(x)_-s = 0$  so that  $s\Re(x)s = s\Re(x)_+s$ . Again, by the same arguments as before, this implies  $s\Re(x)_+s = 0$  and subsequently  $(1_M - s) \geq p$ . Thus  $s \leq (1_M - p) \wedge p = 0$ .

Let  $(1_M - p)(1_M - q) = w_0|(1_M - p)(1_M - q)|$  be the polar decomposition of  $(1_M - p)(1_M - q)$ . Then  $w_0w_0^* \leq 1_M - p$  and  $w_0^*w_0 \leq 1_M - q$ . Moreover, if  $t = 1 - q - w_0^*w_0 = 1 - q - \mathbf{r}((1_M - p)(1_M - q))$  then we see  $(1_M - q)t = t$  and

$$(1_M - p)t = ((1_M - p)(1_M - q))t = 0 \Rightarrow t \leq p \Rightarrow t = 0.$$

So we obtain the equality  $w_0^*w_0 = 1_M - q$  and thus  $v = w + w_0$  is an isometry in  $M$ .

The inequality  $\Re(x)_+ \leq v|x|v^*$  is proved in the same way as in the proof of [AAP82, Proposition 2.1] (the monotonicity of the square root function follows from [DPS22, Corollary 2.2.28]).  $\square$

The proof of Theorem 7.2.2 is exactly the same as the proof of [AAP82, Theorem 2.2], but instead of [AAP82, Proposition 2.1] we use Proposition 7.2.1 above. We include the proof for completeness.

**Theorem 7.2.2.** *For any  $x, y \in \text{LS}(M)$  there are isometries  $v, w \in M$  such that*

$$|x + y| \leq v|x|v^* + w|y|w^*.$$

*Proof.* We write the polar decomposition  $x + y = u|x + y|$ . Then

$$|x + y| = \frac{1}{2}(u^*(x + y) + (x + y)^*u) = \Re(u^*x) + \Re(u^*y) \quad (7.10)$$

Furthermore,  $|u^*x| = (x^*u^*ux)^{\frac{1}{2}} \leq \|u\|(x^*x)^{\frac{1}{2}} \leq |x|$  and similarly  $|u^*y| \leq |y|$ . Now apply Proposition 7.2.1 to  $u^*x$  and to  $u^*y$  to obtain isometries  $v, w \in M$  so that

$$|x + y| = \Re(u^*x) + \Re(u^*y) \leq v|u^*x|v^* + w|u^*y|w^* \leq v|x|v^* + w|y|w^* \quad (7.11)$$

$\square$

### 7.3. CONSTANTS $\Lambda_n$ AND $\tilde{\Lambda}_n$

For  $n \in \mathbb{N}$  we denote by  $(\Omega_n, \mu_n)$  the set  $\{1, 2, \dots, n\}$  equipped with the normalized counting measure, and by  $(\Omega_\infty, \mu_\infty)$  we denote the interval  $[0, 1]$  equipped with Lebesgue measure. We will moreover write  $S(\Omega_n)$  for the set of complex measurable functions on  $\Omega_n$ , which is simply the collection of all  $n$ -tuples of complex numbers. We write  $\text{Aut}_n$  for the automorphism group of  $(\Omega_n, \mu_n)$ ,  $n \in \mathbb{N} \cup \{\infty\}$ , where automorphism is defined as follows:

**Definition 7.3.1.** Let  $(X_1, \mu_1)$  and  $(X_2, \mu_2)$  be measure spaces. We will say that a map  $T$  is an isomorphism between  $X_1$  and  $X_2$  if  $T$  is a measurable bijective map  $T : N_1 \rightarrow N_2$  between two sets  $N_1 \subseteq X_1$  and  $N_2 \subseteq X_2$  of full measure, and such that moreover  $T^{-1}$  is also measurable, and  $\mu_1 \circ T^{-1} = \mu_2$ . Whenever  $(X_1, \mu_1) = (X_2, \mu_2)$  we will call  $T$  an automorphism.

Let  $n \in \mathbb{N} \cup \{\infty\}$ . We now introduce two constant  $\Lambda_n$  and  $\tilde{\Lambda}_n$  as follows. Let  $g \in S(\Omega_n)$ ,  $T \in \text{Aut}_n$ ,  $z \in \mathbb{C}$ , and put

$$\Lambda(g, T, z) = \text{ess inf} \frac{|g - g \circ T|}{|g - z| + |g \circ T - z|},$$

where we assume  $\frac{0}{0} = 1$ . By the triangle inequality we have  $|g - g \circ T| \leq |g - z| + |g \circ T - z|$  which shows  $\Lambda(g, T, z) \leq 1$  for all  $g, T, z$ . We put

$$\Lambda(g) = \sup\{\Lambda(g, T, z) : T \in \text{Aut}_n, z \in \mathbb{C}\}$$

and define  $\Lambda_n$  by

$$\Lambda_n = \inf_{g \in S(\Omega_n)} \Lambda(g). \quad (7.12)$$

For  $n > 1$  we define  $\tilde{\Lambda}_n$  by setting

$$\tilde{\Lambda}_n = \begin{cases} 2 & \text{if } n = 2, n = 4 \\ \sqrt{3} & \text{if } n = 3k, \\ \frac{2\sqrt{3}}{\sqrt{\frac{3k-3}{3k+1} + \frac{3k+3}{3k+1}}} & \text{if } n = 3k+1, n \neq 4 \\ \frac{2\sqrt{3}}{\sqrt{\frac{3k+6}{3k+2} + \frac{3k}{3k+2}}} & \text{if } n = 3k+2, \\ \sqrt{3} & \text{if } n = \infty. \end{cases} \quad (7.13)$$

In Section 7.A we will prove two results on the constants  $\Lambda_n$  and  $\tilde{\Lambda}_n$ . In Theorem 7.A.1 we will precisely determine  $\Lambda_n$  for all values except for  $n = 4$ . It turns out that

$$\Lambda_1 = \Lambda_2 = 1, \quad \text{and} \quad \frac{\sqrt{3}}{2} \leq \Lambda_4 \leq 1, \quad \text{and} \quad \Lambda_n = \frac{\sqrt{3}}{2} \text{ for } n \notin \{1, 2, 4\}. \quad (7.14)$$

We observe that this implies that  $2\Lambda_n \leq \tilde{\Lambda}_n$  for  $n > 1$  with equality when  $n \equiv 0 \pmod{3}$  or  $n = \infty$  and that moreover  $\lim_{n \rightarrow \infty} 2\Lambda_n = \sqrt{3} = \lim_{n \rightarrow \infty} \tilde{\Lambda}_n$ .

We denote the diameter of a set  $A \subseteq \mathbb{C}$  by  $\text{Diam}(A) := \sup_{z, w \in A} |z - w|$ . In Lemma 7.A.2 we will show for  $n > 1$  that there exists  $g \in L^\infty(\Omega_n)$  with  $\text{Diam}(g(\Omega_n)) = 1$  and  $\tilde{\Lambda}_n = \sup_{z \in \mathbb{C}} \frac{1}{\|g - z\|_1}$ , which will be used throughout the text.

## 7.4. TECHNICAL RESULT

This section is devoted to the proof of Theorem 7.4.3, which is closely connected to the operator inequality (7.5) and to the constants  $\Lambda_n$ . To fully state the result we first give the following definition:

**Definition 7.4.1.** Let  $z \in \mathbb{C}$ ,  $0 \leq \alpha \leq \pi$ . The sets  $A, B \subset \mathbb{C}$  will be called  $(z, \alpha)$ -conjugate if there are two lines in  $\mathbb{C}$  that intersect at the point  $z$  at an angle  $\alpha$ , such that the sets  $A$  and  $B$  lie in opposite closed corners with the vertex  $z$  and the magnitude  $\alpha$  (see Fig. 7.1)

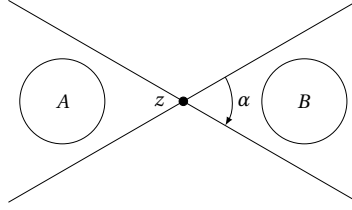


Figure 7.1: Two  $(z, \alpha)$ -conjugate sets  $A$  and  $B$  are depicted.

**Remark 7.4.2.** Let the sets  $A, B$  be  $(z, \alpha)$ -conjugate,  $a \in A$ ,  $b \in B$ . It is easy to see that

$$|a - b| \geq (|a - z| + |b - z|) \cos \frac{\alpha}{2}.$$

Indeed, it is enough to consider the projections of points  $a, b$  on the bisector of the angle  $\alpha$ .

**Theorem 7.4.3.** Let  $g \in S(\Omega_n)$ ,  $n \in \mathbb{N} \cup \{\infty\}$ . Then there exists a  $z_0 \in \mathbb{C}$  and an automorphism  $T$  of  $\Omega_n$  such that

$$|g \circ T - g| \geq \frac{\sqrt{3}}{2} (|g - z_0| + |g \circ T - z_0|). \quad (7.15)$$

i.e.

$$\Lambda(g) \geq \frac{\sqrt{3}}{2}. \quad (7.16)$$

Moreover, the set  $\Omega_n$  can be partitioned into disjoint measurable sets as follows:

1. if  $n$  is even or  $n = \infty$  then there is a partition  $\{X_1\} \cup \{X_2^{m,i} : 1 \leq m, 1 \leq i \leq 2\}$  so that  $g(X_1) \subset \{z_0\}$ ,  $\mu_n(X_2^{m,1}) = \mu_n(X_2^{m,2})$  and the sets  $g(X_2^{m,1})$ ,  $g(X_2^{m,2})$  are  $(z_0, \frac{\pi}{3})$ -conjugate for  $m = 1, 2, \dots$ ; Moreover, denoting  $X_2 = \Omega_n \setminus X_1$  we have that  $T^k|_{X_k} = \text{Id}_{X_k}$  for  $k = 1, 2$ .
2. if  $n$  is odd then there is a partition  $X_1, X_2, X_3, X_5$ , so that  $T^k|_{X_k} = \text{Id}_{X_k}$ ,  $k = 1, 2, 3, 5$ .

If  $n < \infty$  then there exists  $z_0 \in \mathbb{C}$  and  $T \in \text{Aut}_n$  so that

$$\Lambda(g, T, z_0) = \Lambda(g). \quad (7.17)$$

The above theorem relates to the operator inequality (7.5) through functional calculus. This is best visible in the case of finite-dimensional factors, see Theorem 7.5.1. Furthermore, we note that Theorem 7.4.3 provides a lower bound on the constants  $\Lambda_n$ . Indeed, given  $g \in S(\Omega_n)$  the obtained  $z_0, T$  are such that  $\Lambda(g, T, z_0) \geq \frac{\sqrt{3}}{2}$ . Hence  $\Lambda_n \geq \frac{\sqrt{3}}{2}$  for all  $n \in \mathbb{N} \cup \{\infty\}$ . In the Appendix, Theorem 7.A.1, it is proved that in fact  $\Lambda_n = \frac{\sqrt{3}}{2}$  for  $n = 3$  and  $n \geq 5$ . Therefore, for these values of  $n$ , the constant  $\frac{\sqrt{3}}{2}$  in the above theorem is best possible (i.e. maximal so that for all  $g \in S(\Omega_n)$  there exist  $z_0, T$  satisfying (7.15)).

The proof of Theorem 7.4.3 is somewhat technical and requires two other results: Lemma 7.4.4 and Lemma 7.4.5. We give a sketch of the proof. Given a measurable function  $g : \Omega_n \rightarrow \mathbb{C}$  we first use Lemma 7.4.5 to locate a point  $z_0 \in \mathbb{C}$ , and divide the plane into 6 components by drawing 3 lines intersecting in  $z_0$  making angles of  $\frac{2\pi}{6}$ . The way we do this is such that the measure of the inverse image of  $g$  of opposing components is equal. We can then construct an automorphism  $T$  by just mapping the inverse image of  $g$  of each component to the inverse image of its opposing component. For all  $\omega \in \Omega$ , we then obtain the estimate  $\angle g(\omega), z_0, g(T(\omega)) \geq \frac{2\pi}{3}$  for the angle. Lemma 7.4.4 will then imply that (7.15) holds true. In the actual proof of Theorem 7.4.3 some difficulties arise with the boundaries of the components, and particularly for the case that we are dealing with the measure space  $\Omega_n$  with  $n$  odd. Because of this reason, it is necessary to consider multiple cases in the proof.

The following lemma gives for complex numbers  $z_0, z_1, z_2$  a sufficient condition for

$$|z_1 - z_2| \geq \frac{\sqrt{3}}{2}(|z_1 - z_0| + |z_2 - z_0|) \quad (7.18)$$

to hold, namely when the angle satisfies  $\angle z_1 z_0 z_2 \geq \frac{2\pi}{3}$ . Equation (7.18) can also be described geometrically as saying that the point  $z_0$  lies in the ellipse with foci  $z_1$  and  $z_2$  and eccentricity  $\frac{\sqrt{3}}{2}$ .

**Lemma 7.4.4.** *Let  $z_0, z_1, z_2 \in \mathbb{C}$  be points in the plane, and consider the triangle  $\triangle z_0 z_1 z_2$ . Denote  $a = |z_1 - z_2|$ ,  $b = |z_1 - z_0|$ ,  $c = |z_2 - z_0|$ , and  $\alpha = \angle z_1 z_0 z_2$ . If  $\alpha \geq \frac{2\pi}{3}$  then*

$$a \geq \frac{\sqrt{3}}{2}(b + c).$$

*Proof.* According to the cosine theorem we have

$$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$

Since  $\cos \alpha \leq -\frac{1}{2}$  and  $b^2 + c^2 \geq 2bc$  we obtain

$$4a^2 \geq 4(b^2 + c^2 + bc) \geq 3b^2 + 3c^2 + 6bc = 3(b + c)^2$$

which shows the result. □

The following lemma is used, for a given function  $g \in S(\Omega_n)$ , to choose the point  $z_0 \in \mathbb{C}$  adequately such that (7.15) holds for some automorphism  $T$  that we will later determine. The point  $z_0 \in \mathbb{C}$  should be thought of as the center (or rather a center) of the image of  $g$ . In Lemma 7.4.5 we have identified  $\mathbb{C}$  with  $\mathbb{R}^2$  and the point  $z_0 \in \mathbb{C}$  is represented as a vector  $\mathbf{z}_0 \in \mathbb{R}^2$ . This vector  $\mathbf{z}_0$  is chosen together with three affine hyperplanes (i.e. lines) through  $\mathbf{z}_0$  that are represented by unit vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  orthogonal to those affine hyperplanes. The unit vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  moreover make angles  $\angle \mathbf{v}_i \mathbf{0} \mathbf{v}_j$  for  $i \neq j$  of  $\frac{2\pi}{3}$  (this means that the affine hyperplanes intersect at angles of  $\frac{2\pi}{6}$ ). To each of the affine hyperplanes correspond two closed halfspaces. The lemma tells us that  $\mathbf{z}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  can be chosen in such a way that the inverse image of  $g$  of each of these closed halfspaces has measure larger or equal to  $\frac{1}{2}$ . This explains why we think of  $\mathbf{z}_0$  as a center of the image of  $g$ . Namely, for all three affine hyperplanes it must hold that an equal portion of the domain is mapped to each side (or possibly on the affine hyperplane). However, we remark that such a ‘center point’  $\mathbf{z}_0$  with the above properties does not need to be unique.

**Lemma 7.4.5.** *Let  $(\Omega, \mu)$  be a probability space and let  $g$  be a measurable  $\mathbb{R}^2$ -valued function. Then, there exists a point  $\mathbf{z}_0 \in \mathbb{R}^2$ , unit vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$  with angles  $\angle \mathbf{v}_1 \mathbf{0} \mathbf{v}_2 = \angle \mathbf{v}_2 \mathbf{0} \mathbf{v}_3 = \angle \mathbf{v}_3 \mathbf{0} \mathbf{v}_1 = \frac{2\pi}{3}$  so that for  $i = 1, 2, 3$ , denoting  $a_i := \langle \mathbf{z}_0, \mathbf{v}_i \rangle$ , we have*

$$m_i^L := \mu\left(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}_i \rangle \leq a_i\}\right) \geq \frac{1}{2}, \quad m_i^R := \mu\left(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}_i \rangle \geq a_i\}\right) \geq \frac{1}{2}.$$

For  $i = 1, 2, 3$  we point out that  $m_i^L + m_i^R = 1$  holds if and only if  $\mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}_i \rangle = a_i\}) = 0$ .

*Proof.* We first prove the result for the case that  $g$  is bounded. Denote  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$  and for  $t \in \mathbb{T}$  set  $\mathbf{v}(t) = (\cos(t), \sin(t))$  and define

$$\Omega(t, r) = \{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \leq r\}, \quad r \in \mathbb{R},$$

$$A(t) = \left\{ r \in \mathbb{R} : \frac{1}{2} \leq \mu(\Omega(t, r)) \right\},$$

$$a(t) = \inf A(t).$$

If  $r_n \downarrow a(t)$  and  $\frac{1}{2} \leq \mu(\Omega(t, r_n))$  then  $\Omega(t, r_1) \supset \Omega(t, r_2) \supset \dots$  and  $\Omega(t, a(t)) = \bigcap_n \Omega(t, r_n)$ . Hence,

$$\frac{1}{2} \leq \mu(\Omega(t, a(t))). \quad (7.19)$$

If  $r_n \uparrow a(t)$  then  $\frac{1}{2} \geq \mu(\Omega(t, r_n))$  and  $\Omega(t, r_1) \subset \Omega(t, r_2) \subset \dots$  and  $\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle < a(t)\} = \bigcup_n \Omega(t, r_n)$ . Hence,

$$\mu\left(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle < a(t)\}\right) \leq \frac{1}{2} \leq \mu\left(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \leq a(t)\}\right) \quad (7.20)$$

and therefore

$$\mu\left(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \geq a(t)\}\right) \geq \frac{1}{2}. \quad (7.21)$$

We note that it follows from the definition of  $a$  that

$$a(t + \pi) = -\sup \left\{ r \in \mathbb{R} : \frac{1}{2} \leq \mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \geq r\}) \right\} \quad (7.22)$$

since

$$\Omega(t + \pi, r) = \{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \geq -r\}, \quad r \in \mathbb{R}.$$

Hence, we obtain by (7.21), (7.22) and by properties of the supremum that  $a(t) \leq -a(t + \pi)$  for all  $t \in \mathbb{T}$  since  $a(t) \in \left\{ r \in \mathbb{R} : \frac{1}{2} \leq \mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \geq r\}) \right\}$ . Moreover, in the second inequality of (7.20), replacing  $t$  by  $t + \pi$  we obtain

$$\frac{1}{2} \leq \mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \geq -a(t + \pi)\}). \quad (7.23)$$

Hence, for any  $t \in \mathbb{T}$ , and any  $b \in [a(t), -a(t + \pi)]$  we obtain using (7.20) and (7.23) that

$$\frac{1}{2} \leq \mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \leq b\}), \quad \frac{1}{2} \leq \mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \geq b\}). \quad (7.24)$$

We show that the function  $a$  is continuous. Indeed, let  $\varepsilon > 0$ , and choose  $\delta > 0$  such that  $\|\mathbf{v}(t) - \mathbf{v}(s)\|_2 < \varepsilon$  for all  $t, s \in \mathbb{T}$  with  $\text{Dist}(s, t) < \delta$ . Now, fix  $t, s \in \mathbb{T}$  with  $\text{Dist}(t, s) < \delta$ . Then for  $\omega \in \Omega$  we have

$$|\langle g(\omega), \mathbf{v}(t) \rangle - \langle g(\omega), \mathbf{v}(s) \rangle| \leq \|g\|_\infty \|\mathbf{v}(t) - \mathbf{v}(s)\|_2 < \varepsilon \|g\|_\infty.$$

But this means for  $r \in \mathbb{R}$  that

$$\{\omega \in \Omega : \langle g(\omega), \mathbf{v}(t) \rangle \leq r\} \subseteq \{\omega \in \Omega : \langle g(\omega), \mathbf{v}(s) \rangle \leq r + \varepsilon \|g\|_\infty\}.$$

This implies in particular that

$$\frac{1}{2} \leq \mu(\{\omega : \langle g(\omega), \mathbf{v}(t) \rangle \leq a(t)\}) \leq \mu(\{\omega : \langle g(\omega), \mathbf{v}(s) \rangle \leq a(t) + \varepsilon \|g\|_\infty\})$$

so that  $a(s) \leq a(t) + \varepsilon \|g\|_\infty$ . By symmetry of  $s$  and  $t$  we obtain similarly  $a(t) \leq a(s) + \varepsilon \|g\|_\infty$ , which implies  $|a(t) - a(s)| < \varepsilon \|g\|_\infty$  and shows the continuity of  $a$ .

Now, for  $t \in \mathbb{T}$  and  $b \in \mathbb{R}$  consider the line

$$L(t, b) = \{\mathbf{w} \in \mathbb{R}^2 : \langle \mathbf{w}, \mathbf{v}(t) \rangle = b\} = b\mathbf{v}(t) + \mathbb{R}\mathbf{v}(t + \frac{\pi}{2}).$$

For  $s \neq t \pmod{\pi}$ , the lines  $L(s)$  and  $L(t)$  intersect at a unique point  $\mathbf{w}(L(s, b), L(t, c))$ . In particular there is a  $r \in \mathbb{R}$  such that

$$\mathbf{w}(L(s, b), L(t, c)) = b\mathbf{v}(s) + r\mathbf{v}(s + \frac{\pi}{2}).$$

Therefore  $c := \langle \mathbf{w}(L(s, b), L(t, c)), \mathbf{v}(t) \rangle = b\langle \mathbf{v}(s), \mathbf{v}(t) \rangle + r\langle \mathbf{v}(s + \frac{\pi}{2}), \mathbf{v}(t) \rangle$  so that  $r = \frac{c - b\langle \mathbf{v}(s), \mathbf{v}(t) \rangle}{\langle \mathbf{v}(s + \frac{\pi}{2}), \mathbf{v}(t) \rangle}$  and thus

$$\mathbf{w}(L(s, b), L(t, c)) = b\mathbf{v}(s) + \frac{c - b\langle \mathbf{v}(s), \mathbf{v}(t) \rangle}{\langle \mathbf{v}(s + \frac{\pi}{2}), \mathbf{v}(t) \rangle} \mathbf{v}(s + \frac{\pi}{2}).$$

Let  $t \in \mathbb{T}$ . We are interested in finding values  $b_1, b_2, b_3 \in \mathbb{R}$  such that the lines  $L(t - \frac{2\pi}{3}, b_1)$ ,  $L(t + \frac{2\pi}{3}, b_2)$  and  $L(t, b_3)$  intersect at a single point. This is to say that the intersection point  $\mathbf{w}(L(t - \frac{2\pi}{3}, b_1), L(t + \frac{2\pi}{3}, b_2))$  must lie on the line  $L(t, b_3)$ . From this we obtain the expression for  $b_3$ , namely:

$$\begin{aligned} b_3 &:= \langle \mathbf{w}(L(t - \frac{2\pi}{3}, b_1), L(t + \frac{2\pi}{3}, b_2)), \mathbf{v}(t) \rangle \\ &= b_1 \langle \mathbf{v}(t - \frac{2\pi}{3}), \mathbf{v}(t) \rangle + \frac{b_2 - b_1 \langle \mathbf{v}(t - \frac{2\pi}{3}), \mathbf{v}(t + \frac{2\pi}{3}) \rangle}{\langle \mathbf{v}(t - \frac{\pi}{6}), \mathbf{v}(t + \frac{2\pi}{3}) \rangle} \langle \mathbf{v}(t - \frac{\pi}{6}), \mathbf{v}(t) \rangle \\ &= b_1 \cos(\frac{2\pi}{3}) + \frac{b_2 - b_1 \cos(\frac{4\pi}{3})}{\cos(\frac{5\pi}{6})} \cos(\frac{\pi}{6}) \\ &= b_1 \cos(\frac{2\pi}{3}) - \left( b_2 - b_1 \cos(\frac{4\pi}{3}) \right) \\ &= -b_1 - b_2. \end{aligned}$$

This shows that the lines  $L(t - \frac{2\pi}{3}, b_1)$ ,  $L(t + \frac{2\pi}{3}, b_2)$  and  $L(t, b_3)$  intersect precisely when  $b_1 + b_2 + b_3 = 0$ .

Define  $c : \mathbb{T} \rightarrow \mathbb{R}$  as  $c(t) = a(t - \frac{2\pi}{3}) + a(t) + a(t + \frac{2\pi}{3})$ , which is a continuous function. Now, we note that, similar to  $a$ , we have  $c(t) \leq -c(t + \pi)$  for all  $t$ , so that  $\int_{\mathbb{T}} c(t) dt \leq -\int_{\mathbb{T}} c(t + \pi) dt = -\int_{\mathbb{T}} c(t) dt$ , and hence that  $\int_{\mathbb{T}} c(t) dt \leq 0$ . We can thus find a  $t_1$  such that  $c(t_1) \leq 0$ . If also  $0 \leq -c(t_1 + \pi)$  then we set  $t_0 := t_1$ . If instead  $-c(t_1 + \pi) < 0$ , we set  $t_2 := t_1 + \pi$  and obtain  $-c(t_2 + \pi) = -c(t_1) \geq c(t_1 + \pi) > 0$ . By the intermediate value theorem, we then find a  $t_0 \in \mathbb{T}$  such that  $-c(t_0 + \pi) = 0$ . Then  $c(t_0) \leq -c(t_0 + \pi) = 0$ .

In both cases, we found  $t_0 \in \mathbb{T}$  with  $c(t_0) \leq 0 \leq -c(t_0 + \pi)$ . Now, as moreover  $a(t) \leq -a(t + \pi)$  for all  $t \in \mathbb{T}$ , we can determine

$$\begin{aligned} b_1 &\in [a(t_0 - \frac{2\pi}{3}), -a(t_0 + \frac{\pi}{3})], \\ b_2 &\in [a(t_0 + \frac{2\pi}{3}), -a(t_0 - \frac{\pi}{3})], \\ b_3 &\in [a(t_0), -a(t_0 + \pi)] \end{aligned}$$

such that  $b_1 + b_2 + b_3 = 0$ . Indeed, this is possible as the sum of the left-endpoints of the intervals equals  $c(t_0)$ , whereas the sum of the right-endpoints of the intervals equals  $-c(t_0 + \pi)$ . We now set  $\mathbf{v}_1 := \mathbf{v}(t_0 - \frac{2\pi}{3})$ ,  $\mathbf{v}_2 := \mathbf{v}(t_0 + \frac{2\pi}{3})$  and  $\mathbf{v}_3 := \mathbf{v}(t_0)$  and let  $\mathbf{z}_0$  be the unique intersection point of the lines  $L(t_0 - \frac{2\pi}{3}, b_1)$ ,  $L(t_0 + \frac{2\pi}{3}, b_2)$  and  $L(t_0, b_3)$ . As  $\mathbf{z}_0$  lies on each of the three lines, we obtain that  $a_i := \langle \mathbf{z}_0, \mathbf{v}_i \rangle = b_i$  for  $i = 1, 2, 3$ . By the choice of the  $b_i$ 's in the intervals, it (see (7.24)) now follows that the properties of the lemma are fulfilled. The last line of the lemma follows from the fact that  $m_i^L + m_i^R = \mu(\Omega) + \mu(\{\omega \in \Omega : \langle g(\omega), \mathbf{v}_i \rangle = a_i\})$ .

The result for unbounded  $g$  follows by the following reduction to the case of bounded functions. For  $j \in \mathbb{N}$  let  $\Omega_j \subseteq \Omega$  be a measurable subset for which  $g\chi_{\Omega_j}$  is bounded and with  $\Omega_j \uparrow \Omega$ . Denote  $\mu_j := \frac{1}{\mu(\Omega_j)}\mu$  and  $g_j := g|_{\Omega_j} \in L^\infty(\Omega_j, \mu_j)$ . Applying the result of the lemma to  $g_j$ , we find  $\mathbf{z}_{0,j}$  and  $\mathbf{v}_{i,j}$  and  $a_{i,j} = \langle \mathbf{z}_{0,j}, \mathbf{v}_{i,j} \rangle$  with the stated properties.

The sequence  $\mathbf{z}_{0,j}$  must be bounded. Indeed, otherwise there is an  $i \in \{1, 2, 3\}$  such that for a subsequence of  $(a_{i,j})_{j \geq 1}$  we have  $a_{i,j} \rightarrow +\infty$ . However, this would contradict  $\frac{1}{2} \leq \mu_j \left( \{\omega \in \Omega_j : \langle g_j(\omega), \mathbf{v}_{i,j} \rangle \geq a_{i,j}\} \right)$ . Thus, by boundedness of the sequences  $(\mathbf{z}_{0,j})_{j \geq 1}$  and  $(\mathbf{v}_{i,j})_{j \geq 1}$ , we have for some strictly increasing sequence  $(j_k)_{k \geq 1}$  in  $\mathbb{N}$ , that the limits  $\mathbf{z}_0 := \lim_{k \rightarrow \infty} \mathbf{z}_{0,j_k}$  and  $\mathbf{v}_i := \lim_{k \rightarrow \infty} \mathbf{v}_{i,j_k}$  exist. Setting  $a_i := \langle \mathbf{z}_0, \mathbf{v}_i \rangle$  we also have  $a_i = \lim_{k \rightarrow \infty} a_{i,j_k}$ . Using (reversed) Fatou's lemma, we now obtain for  $i = 1, 2, 3$  that

$$\begin{aligned} \mu \left( \{\omega \in \Omega : \langle g(\omega), \mathbf{v}_i \rangle \leq a_i\} \right) &\geq \mu \left( \bigcap_{K=1}^{\infty} \bigcup_{k \geq K} \{\omega \in \Omega_{j_k} : \langle g(\omega), \mathbf{v}_{i,j_k} \rangle \leq a_{i,j_k}\} \right) \\ &\geq \limsup_{k \rightarrow \infty} \mu \left( \{\omega \in \Omega_{j_k} : \langle g(\omega), \mathbf{v}_{i,j_k} \rangle \leq a_{i,j_k}\} \right) \\ &\geq \limsup_{k \rightarrow \infty} \mu_{j_k} \left( \{\omega \in \Omega_{j_k} : \langle g_{j_k}(\omega), \mathbf{v}_{i,j_k} \rangle \leq a_{i,j_k}\} \right) \\ &\geq \frac{1}{2}. \end{aligned}$$

In the same way  $\mu \left( \{\omega \in \Omega : \langle g(\omega), \mathbf{v}_i \rangle \geq a_i\} \right) \geq \frac{1}{2}$  can be shown. The last line of the lemma follows as before. This proves the lemma.  $\square$

We are now fully equipped to prove Theorem 7.4.3.

*Proof of Theorem 7.4.3.* By identifying  $\mathbb{C}$  with  $\mathbb{R}^2$ , we can apply Lemma 7.4.5, to obtain  $\mathbf{z}_0$  and  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  and  $a_1, a_2, a_3$  which we will use to prove the result. Without loss of generality we can moreover assume that  $\mathbf{v}_1, \mathbf{v}_2$  and  $\mathbf{v}_3$  are orientated counter-clockwise. In the proof, we distinguish cases, depending on  $n$ . We prove the result separately for the cases: (1) for  $n$  even, or  $n = \infty$  and (2) for  $n$  odd.

(1)  $n$  is even, or  $n = \infty$ . First, suppose that  $n \in \mathbb{N}$  is even. Then, by the choice of the point  $\mathbf{z}_0$  and of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  (see Lemma 7.4.5) and the fact that  $n$  is even, we can for  $j = 1, 2, 3$  create partitions  $\{I_j^+, I_j^-\}$  of  $\Omega_n$  such that  $\mu_n(I_j^+) = \frac{\mu_n(\Omega_n)}{2} = \mu_n(I_j^-)$  and such that  $\langle g(\omega), \mathbf{v}_j \rangle \leq a_j$  whenever  $\omega \in I_j^-$  and  $\langle g(\omega), \mathbf{v}_j \rangle \geq a_j$  whenever  $\omega \in I_j^+$ . If instead  $n = \infty$  then the same is true, because of the fact that  $\mu_n$  is atomless in that case. We can now define the sets

$$\begin{aligned} P_1^+ &= I_1^+ \cap I_2^- \cap I_3^-, & P_1^- &= I_1^- \cap I_2^+ \cap I_3^+, \\ P_2^+ &= I_1^- \cap I_2^+ \cap I_3^-, & P_2^- &= I_1^+ \cap I_2^- \cap I_3^+, \\ P_3^+ &= I_1^- \cap I_2^- \cap I_3^+, & P_3^- &= I_1^+ \cap I_2^+ \cap I_3^-, \\ P_4^+ &= I_1^+ \cap I_2^+ \cap I_3^+, & P_4^- &= I_1^- \cap I_2^- \cap I_3^-. \end{aligned}$$

that partition  $\Omega_n$ .

We show that  $g(P_4^+ \cup P_4^-) \subseteq \{\mathbf{z}_0\}$ . We have that  $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0$  and therefore  $a_1 + a_2 + a_3 = 0$ . For  $\omega \in I_1^+ \cap I_2^+ \cap I_3^+$  we have  $\langle g(\omega), \mathbf{v}_i \rangle \geq a_i$ ,  $i = 1, 2, 3$ , and  $\sum_{i=1}^3 \langle g(\omega), \mathbf{v}_i \rangle = 0$ . Hence,  $\langle g(\omega), \mathbf{v}_i \rangle = a_i$ ,  $i = 1, 2, 3$ . But this means precisely that  $g(\omega) = \mathbf{z}_0$ . Similarly  $g(P_4^-) \subseteq \{\mathbf{z}_0\}$ . For benefit of the reader, we have visualized the partition sets in Fig. 7.2.

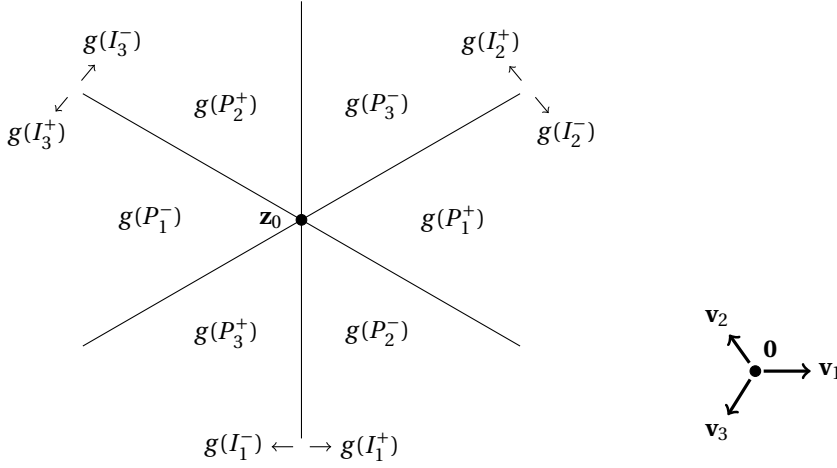


Figure 7.2: The partition sets are visualized for a universal example (any example is like this, except for shifting  $\mathbf{z}_0$  and rotating the lines). The 3 lines intersect in a single point  $\mathbf{z}_0$ . For every line, the set  $\Omega_n$  is partitioned in two sets  $I_i^+$  and  $I_i^-$ , so that  $g(I_i^+)$  and  $g(I_i^-)$  lie only on one side of this line. The partition sets  $P_j^\pm$  are then such that  $g(P_j^\pm)$  lies in one connected component (or its boundary). The sets  $g(P_4^+)$  and  $g(P_4^-)$  are not visualized. For these we must have  $g(P_4^+ \cup P_4^-) \subseteq \{\mathbf{z}_0\}$ .

We have

$$\mu_n(P_1^+ \cup P_2^- \cup P_3^- \cup P_4^+) = \mu_n(I_1^+) = \mu_n(I_1^-) = \mu_n(P_1^- \cup P_2^+ \cup P_3^+ \cup P_4^-), \quad (7.25)$$

$$\mu_n(P_1^- \cup P_2^+ \cup P_3^- \cup P_4^+) = \mu_n(I_2^+) = \mu_n(I_2^-) = \mu_n(P_1^+ \cup P_2^- \cup P_3^+ \cup P_4^-), \quad (7.26)$$

$$\mu_n(P_1^- \cup P_2^- \cup P_3^+ \cup P_4^+) = \mu_n(I_3^+) = \mu_n(I_3^-) = \mu_n(P_1^+ \cup P_2^+ \cup P_3^- \cup P_4^-), \quad (7.27)$$

(7.25)+(7.26):

$$\mu_n(P_3^-) + \mu_n(P_4^+) = \mu_n(P_3^+) + \mu_n(P_4^-),$$

(7.25)+(7.27):

$$\mu_n(P_2^-) + \mu_n(P_4^+) = \mu_n(P_2^+) + \mu_n(P_4^-),$$

(7.26)+(7.27):

$$\mu_n(P_1^-) + \mu_n(P_4^+) = \mu_n(P_1^+) + \mu_n(P_4^-).$$

We thus obtain that  $t := \mu_n(P_j^+) - \mu_n(P_j^-)$  is independent of  $j = 1, 2, 3, 4$ .

Let us assume that  $t \geq 0$  so that  $\mu_n(P_j^+) \geq t$ . Choose  $A_j \subseteq P_j^+$  with  $\mu_n(A_j) = t$ . We denote  $X_1 = (P_4^+ \cup P_4^-) \setminus A_4$  and

$$X_2^{1,1} = P_1^+ \setminus A_1, \quad X_2^{1,2} = P_1^-, \quad X_2^{2,1} = P_2^+ \setminus A_2, \quad X_2^{2,2} = P_2^-, \quad X_2^{3,1} = P_3^+ \setminus A_3, \quad X_2^{3,2} = P_3^-.$$

First, suppose that  $n \in \mathbb{N}$  is even. Then  $A_j = \{a_{j,1}, \dots, a_{j,l}\}$ ,  $j = 1, 2, 3, 4$ ,  $l = tn$ . Fix  $k = 1, \dots, l$ . In each triple  $(a_{1,k}, a_{2,k}, a_{3,k})$  there will be such  $i, j \in \{1, 2, 3\}$  (see Fig. 7.2) so that

$$\frac{2\pi}{3} \leq \angle g(a_{i,k}), \mathbf{z}_0, g(a_{j,k}) \leq \pi.$$

Let  $\{q\} = \{1, 2, 3\} \setminus \{i, j\}$ . Then  $\{g(a_{i,k})\}$  and  $\{g(a_{j,k})\}$ , and also  $\{g(a_{q,k})\}$  and  $\{g(a_{4,k})\}$  form pairs of  $(\mathbf{z}_0, \frac{\pi}{3})$ -conjugate sets. We put  $X_2^{2k+2,1} = \{a_{i,k}\}$ ,  $X_2^{2k+2,2} = \{a_{j,k}\}$ ,  $X_2^{2k+3,1} = \{a_{q,k}\}$ ,  $X_2^{2k+3,2} = \{a_{4,k}\}$  and  $X_2^{m,1} = X_2^{m,2} = \emptyset$  for  $m \geq 2l + 4$ .

We assume now that  $n = \infty$ . Let  $\Sigma_j = \{Y_j^1, Y_j^2, \dots\}$  be a maximal system of pairwise disjoint measurable subsets of  $A_j$ ,  $j = 1, 2, 3, 4$ , such that  $\mu_\infty(Y_1^k) = \mu_\infty(Y_2^k) = \mu_\infty(Y_3^k) = \mu_\infty(Y_4^k) > 0$  and the four  $(g(Y_1^k), g(Y_2^k), g(Y_3^k), g(Y_4^k))$  is divided into two pairs of  $(\mathbf{z}_0, \frac{\pi}{3})$ -conjugate sets for  $k = 1, 2, \dots$

Put  $B_j = A_j \setminus \bigcup_k Y_j^k$ . Then  $\mu_\infty(B_1) = \mu_\infty(B_2) = \mu_\infty(B_3) = \mu_\infty(B_4) = t_0$ . Suppose that  $t_0 > 0$ . If the sets  $g(B_1), g(B_2), g(B_3)$  are located on three rays emanating from  $\mathbf{z}_0$  and forming angles  $\frac{2\pi}{3}$  then  $g(B_1), g(B_2)$  are  $(\mathbf{z}_0, \frac{\pi}{3})$ -conjugate sets as well as  $g(B_3), g(B_4)$ . This contradicts the maximality of the above set systems  $\Sigma_j$ . Thus we can assume there are  $b_1 \in B_i$ ,  $b_2 \in B_j$ ,  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ , with  $\angle g(b_1), \mathbf{z}_0, g(b_2) > \frac{2\pi}{3}$  and such that  $g(b_1), g(b_2)$  are essential values of  $g|_{B_i \cup B_j}$ . Then there will be such neighborhoods  $V_1$  and  $V_2$  of the points  $g(b_1)$  and  $g(b_2)$ , respectively, that  $V_1, V_2$  are  $(\mathbf{z}_0, \frac{\pi}{3})$ -conjugate sets. Therefore there exist sets  $Y_1 \subset B_i$ ,  $Y_2 \subset B_j$  so that  $\mu_\infty(Y_1) = \mu_\infty(Y_2) > 0$  and  $g(Y_k) \subset V_k$ ,  $k = 1, 2$ . Hence,  $g(Y_1), g(Y_2)$  are  $(\mathbf{z}_0, \frac{\pi}{3})$ -conjugate sets. Let  $\{q\} = \{1, 2, 3\} \setminus \{i, j\}$ . There exists  $Y_3 \subset B_q$ ,  $Y_4 \subset B_4$ ,  $\mu_\infty(Y_3) = \mu_\infty(Y_4) = \mu_\infty(Y_1)$ . It is clear that  $g(Y_3), g(Y_4)$  are  $(\mathbf{z}_0, \frac{\pi}{3})$ -conjugate sets. The presence of sets  $Y_1, Y_2, Y_3, Y_4$  contradicts the maximality of the above systems  $\Sigma_j$ .

The contradiction obtained in both cases shows  $t_0 = 0$ . Therefore the system  $\{X_1\} \cup \{X_2^{m,i} : 1 \leq m \leq 3, 1 \leq i \leq 2\}$  can be completed using  $\Sigma_j$ ,  $j = 1, 2, 3, 4$ .

It remains to define  $T$  so that  $T_{X_1} = \text{Id}_{X_1}$ ,  $T(X_2^{m,1}) = X_2^{m,2}$ ,  $T(X_2^{m,2}) = X_2^{m,1}$  for  $m = 1, 2, \dots$  and such that  $T^2 = \text{Id}_{\Omega_n}$ . Then the inequality (7.15) follows from the Lemma 7.4.4.

The case that  $t \leq 0$  is similar, by changing the roles of  $P_j^+$  and  $P_j^-$ .

(2).  $n$  is odd. We can for  $i = 1, 2, 3$  instead build partitions  $\{I_i^+, \{\omega_i\}, I_i^-\}$  of  $\Omega_n$  with  $\mu_n(I_i^+) = \mu_n(I_i^-)$  and such that  $\langle g(\omega), \mathbf{v}_i \rangle \leq a_i$  whenever  $\omega \in I_i^-$  and  $\langle g(\omega), \mathbf{v}_i \rangle \geq a_i$  whenever  $\omega \in I_i^+$  and  $\langle g(\omega_i), \mathbf{v}_i \rangle = a_i$ . Indeed, such  $\omega_i$  exist because  $|\{\omega \in \Omega_n : \langle g(\omega), \mathbf{v}_i \rangle \leq a_i\}|, |\{\omega \in \Omega_n : \langle g(\omega), \mathbf{v}_i \rangle \geq a_i\}| \geq \frac{n+1}{2}$  and therefore  $\{\omega \in \Omega_n : \langle g(\omega), \mathbf{v}_i \rangle \leq a_i\} \cap \{\omega \in \Omega_n : \langle g(\omega), \mathbf{v}_i \rangle \geq a_i\} \neq \emptyset$ . Denote  $Y_0 = \{\omega_1, \omega_2, \omega_3\}$ .

Now, suppose that  $\mathbf{z}_0 \in g(\Omega_n)$ . Then we could have chosen  $\omega_1 = \omega_2 = \omega_3$  all equal and such that  $g(\omega_i) = \mathbf{z}_0$ . Then  $|Y_0| = 1$  and the sets  $\{I_i^+, I_i^-\}$  are all partitions of  $\Omega_n \setminus Y_0$  similar to (1), and we can build the corresponding automorphism  $T$  of  $\Omega_n \setminus Y_0$ . This completes the proof for that case by setting  $T(\omega_1) = \omega_1$ .

We can thus assume that  $\mathbf{z}_0 \notin g(\Omega_n)$  so that in particular  $g(\omega_i) \neq g(\omega_j)$  for  $i \neq j$  and  $|Y_0| = 3$ . Now suppose first that  $\mathbf{z}_0 \in \text{Conv}(g(Y_0))$ . For all  $i \in \{1, 2, 3\}$  we then have that  $Y_0 \cap I_i^+$  and  $Y_0 \cap I_i^-$  both consist of 1 element. Hence,  $\mu_n(I_i^+ \setminus Y_0) = \mu_n(I_i^- \setminus Y_0)$  and the partitions  $\{I_i^+ \setminus Y_0, I_i^- \setminus Y_0\}$  of  $\Omega_n \setminus Y_0$  satisfy the same properties as (1). We thus obtain a measure preserving automorphism  $T$  of  $\Omega_n \setminus Y_0$  with the same properties. Now we can set  $T(\omega_1) = \omega_2$ ,  $T(\omega_2) = \omega_3$  and  $T(\omega_3) = \omega_1$ , so that  $\angle g(\omega_i), \mathbf{z}_0, g(T(\omega_i)) = \frac{2\pi}{3}$ . This finishes the proof by Lemma 7.4.4

Now suppose that  $\mathbf{z}_0 \notin \text{Conv}(g(Y_0))$ . Then it can be seen geometrically (for intuition see Fig. 7.3), that there is a unique choice of (distinct) indices  $i_1, i_2, i_3 \in \{1, 2, 3\}$  such that

$$\{\omega_{i_1}\} = Y_0 \cap I_{i_2}^- \quad \{\omega_{i_3}\} = Y_0 \cap I_{i_2}^+. \quad (7.28)$$

Now, suppose that  $\omega_{i_1} \notin I_{i_3}^+$ . Then as  $\omega_{i_1} \neq \omega_{i_3}$  we get  $\omega_{i_1} \in I_{i_3}^-$ . But then as  $\omega_{i_1} \in \{\omega_{i_1}\} \cap I_{i_2}^- \cap I_{i_3}^-$  we would get  $g(\omega_{i_1}) = \mathbf{z}_0$  by the same argument as why  $g(P_4^+ \cup P_4^-) \subseteq \{\mathbf{z}_0\}$  in (1). However,  $\mathbf{z}_0 \notin g(\Omega_n)$  by our assumption so this cannot be the case. We conclude that we must have  $\omega_{i_1} \in I_{i_3}^+$ . By a same argument we find that we must have  $\omega_{i_3} \in I_{i_1}^-$  (Indeed, otherwise  $\omega_{i_3} \in I_{i_1}^+$  so that  $\omega_{i_3} \in I_{i_1}^+ \cap I_{i_2}^+ \cap \{\omega_{i_3}\}$ , which would imply  $g(\omega_{i_3}) = \mathbf{z}_0$ , which gives a contradiction). Furthermore, we claim that  $\omega_{i_2} \in I_{i_3}^+$ . Indeed, if  $\omega_{i_2} \in I_{i_3}^-$  then we could rearrange the indexes as  $i'_1 = i_2, i'_2 = i_3$  and  $i'_3 = i_1$ , so that we get  $\{\omega_{i'_1}\} = \{\omega_{i_2}\} = Y_0 \cap I_{i_3}^- = Y_0 \cap I_{i'_2}^-$  and  $\{\omega_{i'_3}\} = \{\omega_{i_1}\} = Y_0 \cap I_{i_3}^+ = Y_0 \cap I_{i'_2}^+$ . This contradicts the uniqueness of the choice  $i_1, i_2, i_3$  satisfying (7.28). We conclude that indeed  $\omega_{i_2} \in I_{i_3}^+$ . By the same argument we find  $\omega_{i_2} \in I_{i_1}^-$  (Indeed, if  $\omega_{i_2} \in I_{i_1}^+$  we could take the rearrangement  $i'_1 = i_3, i'_2 = i_1$  and  $i'_3 = i_2$  to obtain  $\{\omega_{i'_1}\} = \{\omega_{i_3}\} = Y_0 \cap I_{i_1}^- = Y_0 \cap I_{i'_2}^-$  and  $\{\omega_{i'_3}\} = \{\omega_{i_2}\} = Y_0 \cap I_{i_1}^+ = Y_0 \cap I_{i'_2}^+$ , which contradicts the uniqueness). For clarity we summarize the results:

$$\begin{aligned} \{\omega_{i_1}\} &= Y_0 \cap I_{i_2}^- & \{\omega_{i_3}\} &= Y_0 \cap I_{i_2}^+, \\ \{\omega_{i_2}, \omega_{i_3}\} &= Y_0 \cap I_{i_1}^- & \{\omega_{i_1}, \omega_{i_2}\} &= Y_0 \cap I_{i_3}^+. \end{aligned}$$

We now obtain

$$\mu_n(I_{i_1}^+ \cap I_{i_2}^-) + \mu_n(I_{i_1}^+ \cap I_{i_2}^+) = \mu_n(I_{i_1}^+ \setminus \{\omega_{i_2}\}) = \mu_n(I_{i_1}^+), \quad (7.29)$$

$$\mu_n(I_{i_1}^+ \cap I_{i_2}^-) + \mu_n(I_{i_1}^- \cap I_{i_2}^-) = \mu_n(I_{i_2}^- \setminus \{\omega_{i_1}\}) = \mu_n(I_{i_2}^-) - \frac{1}{n}, \quad (7.30)$$

$$\mu_n(I_{i_2}^+ \cap I_{i_3}^-) + \mu_n(I_{i_2}^- \cap I_{i_3}^-) = \mu_n(I_{i_3}^- \setminus \{\omega_{i_2}\}) = \mu_n(I_{i_3}^-), \quad (7.31)$$

$$\mu_n(I_{i_2}^+ \cap I_{i_3}^+) + \mu_n(I_{i_2}^+ \cap I_{i_3}^-) = \mu_n(I_{i_2}^+ \setminus \{\omega_{i_3}\}) = \mu_n(I_{i_2}^+) - \frac{1}{n}. \quad (7.32)$$

Hence, by (7.29) + (7.30) we obtain  $\mu_n(I_{i_1}^+ \cap I_{i_2}^+) = \frac{1}{n} + \mu_n(I_{i_1}^- \cap I_{i_2}^-)$  and by summing up (7.31) with (7.32) we obtain  $\mu_n(I_{i_2}^- \cap I_{i_3}^-) = \frac{1}{n} + \mu_n(I_{i_3}^+ \cap I_{i_2}^+)$ . We conclude the existences of  $\omega_4 \in I_{i_1}^+ \cap I_{i_2}^+$  and  $\omega_5 \in I_{i_2}^- \cap I_{i_3}^-$ . Now, for the sets  $P_4^+ := I_{i_1}^+ \cap I_{i_2}^+ \cap I_{i_3}^+$  and  $P_4^- := I_{i_1}^- \cap I_{i_2}^- \cap I_{i_3}^-$  we have that  $g(P_4^+ \cup P_4^-) \subseteq \{\mathbf{z}_0\}$  (same as in (1)), and hence  $P_4^+ \cup P_4^- = \emptyset$  as  $\mathbf{z}_0 \notin g(\Omega_n)$  by assumption. This means that  $\omega_4 \notin I_{i_3}^+$  and  $\omega_5 \notin I_{i_1}^-$ . Also, as  $\omega_{i_3} \in I_{i_1}^-$  and  $\omega_{i_1} \in I_{i_3}^+$  we get that  $\omega_4 \neq \omega_{i_3}$  and  $\omega_5 \neq \omega_{i_1}$ . As  $\{I_i^+, \{\omega_i\}, I_i^-\}$  are partitions of  $\Omega_n$ , we conclude that  $\omega_4 \in I_{i_1}^+ \cap I_{i_2}^+ \cap I_{i_3}^-$  and  $\omega_5 \in I_{i_1}^+ \cap I_{i_2}^- \cap I_{i_3}^-$ . Denote  $Y_1 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ , so that by the above we have  $|Y_1| = 5$  and moreover:

$$\begin{aligned} Y_1 \cap I_{i_1}^- &= \{\omega_{i_2}, \omega_{i_3}\}, & Y_1 \cap I_{i_2}^- &= \{\omega_{i_1}, \omega_5\}, & Y_1 \cap I_{i_3}^- &= \{\omega_4, \omega_5\}, \\ Y_1 \cap I_{i_1}^+ &= \{\omega_4, \omega_5\}, & Y_1 \cap I_{i_2}^+ &= \{\omega_{i_3}, \omega_4\}, & Y_1 \cap I_{i_3}^+ &= \{\omega_{i_1}, \omega_{i_2}\}. \end{aligned}$$

Now, as all these sets have size 2, we must have that

$$\mu_n(I_i^+ \setminus Y_1) = \mu_n(I_i^+) - \frac{2}{n} = \mu_n(I_i^-) - \frac{2}{n} = \mu_n(I_i^- \setminus Y_1).$$

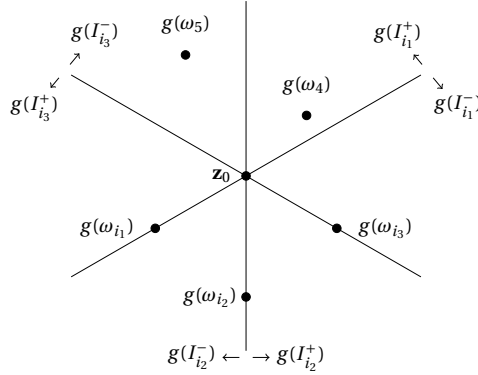


Figure 7.3: The 5 points are depicted for an example.

This means that the partitions  $\{I_i^+ \setminus Y_1, I_i^- \setminus Y_1\}$  of  $\Omega_n \setminus Y_1$  satisfy the same properties as in (1). We can therefore find a transformation  $T$  of  $\Omega_n \setminus Y_1$  with the same properties. We can now define  $T$  on  $Y_1$  by setting  $T(\omega_{i_1}) = \omega_4$ ,  $T(\omega_4) = \omega_{i_2}$ ,  $T(\omega_{i_2}) = \omega_5$ ,  $T(\omega_5) = \omega_{i_3}$  and  $T(\omega_{i_3}) = \omega_{i_1}$ . Then  $\angle g(\omega_i), z_0, g(T(\omega_i)) \geq \frac{2\pi}{3}$  for all  $i$ . Appealing to Lemma 7.4.4 this implies  $|g(w) - g(T(w))| \leq \frac{\sqrt{3}}{2}(|g(w) - z_0| + |g(T(w)) - z_0|)$  for all  $w \in \Omega$ , which shows that (7.15) holds true. The inequality (7.16) follows from it. Furthermore, in each of the considered cases it clear how to split  $\Omega_n$  into the parts  $X_1, X_2, X_3, X_5$  (note that by construction we have  $T^k(\omega) = \omega$  for some  $k \in \{1, 2, 3, 5\}$  for  $\omega \in \Omega$ ). We prove the final statement.

Let  $n < \infty$  and let  $(T_m), (z_m)$  be sequences such that  $0 < \Lambda(g, T_m, z_m) \uparrow \Lambda(g)$ . Then

$$\Lambda(g, T_m, z_m)^{-1} |g(\omega) - g(T_m(\omega))| \geq |g(\omega) - z_m| + |g(T_m(\omega)) - z_m| \geq |g(\omega) - z_m|$$

for any  $\omega \in \Omega_n$ . Let  $\omega_0 \in \Omega_n$ . Then

$$|g(\omega_0) - z_m| \leq \Lambda(g, T_m, z_m)^{-1} \text{Diam}(g(\Omega_n)) \leq \Lambda(g, T_1, z_1)^{-1} \text{Diam}(g(\Omega_n)).$$

Since  $\text{Aut}_n$  has cardinality  $|\text{Aut}_n| = n! < \infty$  and since the set

$$\{z \in \mathbb{C} : |g(\omega_0) - z| \leq \Lambda(g, T_1, z_1)^{-1} \text{Diam}(g(\Omega_n))\}$$

is compact, there exists an increasing sequence  $(m_k)$  in  $\mathbb{N}$  and a  $z_0 \in \mathbb{C}$  such that

$$z_0 = \lim_k z_{m_k}, \quad T_0 := T_{m_1} = T_{m_2} = \cdots = T_{m_k} = \cdots$$

Then  $\Lambda(g, T_0, z_0) = \Lambda(g)$ . □

## 7.5. COMMUTATOR ESTIMATES FOR NORMAL OPERATORS IN FINITE FACTORS

The main result of this section, Theorem 7.5.6 below, establishes the commutator inequality (7.5) for normal element  $a \in S(M)$ , where  $M$  is a finite factor, and provides upper

and lower bounds on the optimal constant  $C_M$ . This yields a version of [BHS23, Theorem 1.1], suitable for normal elements. We consider the case of  $I_n$ -factors ( $n < \infty$ ) in Theorem 7.5.1 and the case of  $II_1$ -factors in Theorem 7.5.4 and show that the commutator inequality holds for the constant  $\frac{\sqrt{3}}{2}$ . The proof for  $II_1$ -factors requires two additional results, Theorem 7.5.2 and Lemma 7.5.3. Furthermore, in order to prove the upper bounds in Theorem 7.5.6 we provide Proposition 7.5.5.

**Theorem 7.5.1.** *Let  $M = B(\mathcal{H})$  be an  $I_n$ -factor for  $n \in \mathbb{N}$ . For an arbitrary normal operator  $a \in M$  there is a unitary  $u \in U(M)$  and a  $z_0 \in \mathbb{C}$  such that*

$$|[a, u]| \geq \frac{\sqrt{3}}{2} (|a - z_0 1_M| + u|a - z_0 1_M|u^*). \quad (7.33)$$

Moreover,  $u$  can be chosen so that

- when  $n$  is even there are projections  $p_1, p_2$  such that  $p_1 + p_2 = 1_M$
- when  $n$  is odd there are projections  $p_1, p_2, p_3, p_5$  such that  $p_1 + p_2 + p_3 + p_5 = 1_M$

so that

$$p_k u = u p_k, \quad u^k p_k = p_k, \quad k = 1, 2, 3, 5.$$

If  $a \in M$  is such that its spectrum  $\sigma(a)$  lies on a straight line, then we can obtain true equality:

$$|[a, u]| = |a - z_0 1_M| + u|a - z_0 1_M|u, \quad \text{for some } u^* = u \in U(M), z_0 \in \mathbb{C}. \quad (7.34)$$

We remark that when  $n = 1, 2$  every normal  $a \in M$  satisfies this extra condition.

*Proof.* Since  $a$  is a normal element on an  $n$ -dimensional Hilbert space, it follows from the spectral mapping theorem that there is a unitary  $U : \mathcal{H} \rightarrow L^2(\Omega_n)$  such that  $a = U^* M_g U$ , where  $M_g$  is the multiplication operator on  $L^2(\Omega_n)$  for some  $g \in L^\infty(\Omega_n)$ . Applying Theorem 7.4.3 to  $g$ , we find a transformation  $T$  and a  $z_0 \in \mathbb{C}$  such that

$$|g \circ T - g| \geq \frac{\sqrt{3}}{2} (|g - z_0| + |g \circ T - z_0|) \quad (7.35)$$

together with the given partition of  $\Omega_n$  consisting of the sets  $X_1, X_2$  (when  $n$  is even) or  $X_1, X_2, X_3, X_5$  (when  $n$  is odd) and that satisfy  $T^k|_{X_k} = \text{Id}_{X_k}$ . Now let  $u_T$  be the Koopman operator on  $L^2(\Omega_n)$  corresponding to  $T$ , i.e.  $u_T f = f \circ T$ . Denote  $u = U^* u_T U$ . Then

$$\begin{aligned} |[a, u]| &= |u(u^* a u - a)| \\ &= |u a u^* - a| \\ &= U^* |u_T M_g u_T^* - M_g| U \\ &= U^* |M_{g \circ T} - M_g| U \\ &= U^* M_{|g \circ T - g|} U \\ &\geq \Lambda_n (U^* |M_g - z_0| U + U^* |M_{g \circ T} - z_0| U) \\ &= \Lambda_n (U^* |M_g - z_0| U + U^* u_T |M_g - z_0| u_T^* U) \\ &= \Lambda_n (|a - z_0| + u|a - z_0|u^*) \end{aligned}$$

We now define the projections by setting  $p_k = U^* \chi_{X_k} U$  which clearly satisfy the state-ments.

If  $\sigma(a)$  lies on a straight line, then there exist scalars  $\alpha, \beta \in \mathbb{C}$ ,  $|\alpha| = 1$ , such that  $a_1 := \alpha(a - \beta 1_M) \in M$  is self-adjoint. It follows from Theorem 7.1.1 that there exist  $z_0 \in \mathbb{R}$  and  $u = u^* \in U(M)$  that

$$|[a, u]| = |[a_1, u]| = |a_1 - z_0 1_M| + u|a_1 - z_0 1_M|u = |a - (\beta + z_0 \alpha^{-1}) 1_M| + u|a - (\beta + z_0 \alpha^{-1}) 1_M|u.$$

□

We need the following result, which for a diffuse semifinite von Neumann algebra  $(M, \tau)$  and a normal measurable  $a \in S(M)$  establishes an injective  $*$ -homomorphism  $F$  between  $S[0, 1]$  and  $S(M)$  which preserves measure and is such that  $a$  lies in the image of  $F$ . Special cases of the result which follows for positive bounded elements of  $M$  and positive elements of  $L^1(M, \tau)$  can be found in [DSZ15, Lemma 9] and in [CS94, Lemma 4.1] respectively.

**Theorem 7.5.2.** *Let  $M$  be a diffuse (i.e. atomless) von Neumann algebra with a faithful normal tracial state  $\tau$ , let  $a \in S(M)$  be a normal operator. There exists such an injective  $*$ -homomorphism  $F : S[0, 1] \rightarrow S(M)$  such that  $a \in \text{Im}(F)$  and  $m(A) = \tau(F(\chi_A))$  for any measurable subset  $A \subset [0, 1]$  (here  $m$  is the Lebesgue measure on  $[0, 1]$ ).*

*Proof.* Let  $e$  be a spectral measure of the operator  $a$  defined on the  $\sigma$ -algebra  $\mathcal{B}(\sigma(a))$  of Borel subsets in  $\sigma(a)$ . Then  $\tau(e(\cdot))$  yields a probability measure on  $\mathcal{B}(\sigma(a))$ . By the spectral theorem (see [Rud91, Theorem 13.33]), we have

$$a = \int_{\sigma(a)} \lambda de(\lambda).$$

Let  $X_0$  be a set of eigenvalues of  $a$ . It is clear that  $X_0 \subset \sigma(a)$  is at most countable. Indeed, if  $t \in X_0$  then  $e(\{t\}) \neq 0$  and  $\sum_{t \in X_0} \tau(e(\{t\})) = \tau(e(X_0)) \leq 1$ . Let  $t \in X_0$ . Since  $M$  is diffuse, it follows that in  $M$  there is a chain of projections  $f_s^t \uparrow_s e(\{t\})$  such that  $\tau(f_s^t) = s$  for  $s \in Y_t := [0, \tau(e(\{t\}))]$ . Denote by  $f_t$  the spectral measure on  $\mathcal{B}(Y_t)$  given by the equality

$$f_t((s_1, s_2)) = f_{s_2}^t - f_{s_1}^t.$$

We have  $\tau(f_t(A)) = m(A)$  for any  $A \in \mathcal{B}(Y_t)$ . Let us now set

$$X = (\sigma(a) \setminus X_0) \sqcup \bigsqcup_{t \in X_0} Y_t.$$

On  $\mathcal{B}(X)$ , we define a spectral measure  $g$  such that

$$g|_{\mathcal{B}(\sigma(a) \setminus X_0)} = e|_{\mathcal{B}(\sigma(a) \setminus X_0)}, \quad g|_{\mathcal{B}(Y_t)} = f_t, \quad t \in X_0,$$

and a scalar measure

$$\mu_X(A) = \tau(e(A \cap (\sigma(a) \setminus X_0))) + \sum_{t \in X_0} \mu(A \cap Y_t).$$

It follows that  $(X, \mathcal{B}(X), \mu_X)$  is a Lebesgue space with an atomless probability measure. Hence, it is isomorphic to the segment  $[0, 1]$  equipped with Lebesgue measure  $m$ , see e.g. [Bog07, Theorem 9.5.1].

A linear mapping  $F : S(X, \mathcal{B}(X), \mu_X) \rightarrow S(M)$  is defined by

$$F(\varphi) = \int_X \varphi(x) dg(x)$$

for any  $\varphi \in S(X, \mathcal{B}(X), \mu_X)$  (see [DPS22, Definition 1.5.6]). We remark that  $F(\chi_A) = g(A)$  for measurable  $A \subseteq X$  and that  $\mu_X(A) = \tau(F(\chi_A))$ . Furthermore  $F(\chi_A \chi_B) = F(\chi_{A \cap B}) = g(A \cap B) = g(A)g(B) = F(\chi_A)F(\chi_B)$  for measurable sets  $A, B \subseteq X$ . Therefore, as  $F$  is continuous with respect to the topologies of convergence in measure in  $S(X, \mathcal{B}(X), \mu_X)$  and  $S(M, \tau)$  and since simple functions in  $S(X, \mathcal{B}(X), \mu_X)$  are dense with respect to the measure topology, it follows that  $F(\varphi\psi) = F(\varphi)F(\psi)$  for all  $\varphi, \psi \in S(X, \mathcal{B}(X), \mu_X)$ . Moreover,  $F(\overline{\varphi}) = \int_X \overline{\varphi(x)} dg(x) = \overline{F(\varphi)}$  so we find that  $F$  is a  $*$ -homomorphism. Now, suppose  $\varphi \in S(X, \mathcal{B}(X), \mu_X)$  is such that  $F(\varphi) = 0$  and  $B \subseteq X$  is such that  $\varphi(x) \neq 0$  for a.e.  $x \in B$ . Then  $g(B) = F(\chi_B) = F(\frac{1}{\varphi} \chi_B)F(\varphi) = 0$ , thus  $\mu_X(B) = \tau(g(B)) = 0$ . This shows that  $F$  is injective.

Finally, let us define the function  $f$  by setting  $f(t) = t$  for  $t \in \mathcal{B}(\sigma(a) \setminus X_0)$  or  $t \in Y_f$ . Then  $f \in S(X, \mathcal{B}(X), \mu_X)$  and  $F(f) = a$ .  $\square$

**Lemma 7.5.3.** *Let  $M$  be a finite von Neumann algebra, let  $a, b \in S(M)$  be normal operators,  $z_0 \in \mathbb{C}$ ,  $0 \leq \alpha < \pi$  and let  $\sigma(a)$ ,  $\sigma(b)$  be  $(z_0, \alpha)$ -conjugate sets. Then*

$$v|a - b|v^* \geq (|a - z_0 1_M| + |b - z_0 1_M|) \cos \frac{\alpha}{2} \quad (7.36)$$

for some  $v \in U(M)$ .

*Proof.* Since  $\sigma(a)$  and  $\sigma(b)$  are  $(z_0, \alpha)$ -conjugate, the shifted sets  $\sigma(a) - z_0$  and  $\sigma(b) - z_0$  are  $(0, \alpha)$ -conjugate. We can then obtain a pair of lines as in Fig. 7.1, intersecting at the origin with an angle  $\alpha$ . By rotating the complex plane around the origin we can assure that these lines are symmetric with respect to the real axis. This is to say that there exists a function  $f(z) = c(z - z_0)$  with  $|c| = 1$  so that

$$f(\sigma(a)) \subset \{z : -\frac{\alpha}{2} \leq \operatorname{Arg}(z) \leq \frac{\alpha}{2}\}, \quad f(\sigma(b)) \subset \{z : \pi - \frac{\alpha}{2} \leq \operatorname{Arg}(z) \leq \pi + \frac{\alpha}{2}\}.$$

Let  $a_1 = f(a)$ ,  $b_1 = f(b)$ . We have

$$|a_1| \cos \frac{\alpha}{2} \leq \Re a_1, \quad |b_1| \cos \frac{\alpha}{2} \leq -\Re b_1.$$

Therefore

$$(|a - z_0 1_M| + |b - z_0 1_M|) \cos \frac{\alpha}{2} = (|a_1| + |b_1|) \cos \frac{\alpha}{2} \leq \Re a_1 - \Re b_1 = \Re(a_1 - b_1) \leq \Re(a_1 - b_1)_+ \quad (7.37)$$

By Proposition 7.2.1, we obtain

$$\Re(a_1 - b_1)_+ \leq v|a_1 - b_1|v^* = v|a - b|v^*. \quad (7.38)$$

for some  $v \in M$  with  $v^*v = 1_M$ . Since  $1_M$  is a finite projection it follows that  $vv^* = 1_M$ , i.e.  $v \in U(M)$ . Combining (7.37) and (7.38) establishes (7.36)  $\square$

We now prove a version of Theorem 7.5.1 for  $\text{II}_1$ -factors. Equation (7.39) below is slightly different from (7.33) as it involves a second unitary  $w \in \text{U}(M)$ .

**Theorem 7.5.4.** *Let  $M$  be a factor of type  $\text{II}_1$ ,  $a \in S(M)$  be normal. Then there exists a  $z_0 \in \mathbb{C}$ ,  $u = u^* \in \text{U}(M)$  and  $w \in \text{U}(M)$  so that*

$$w|[a, u]|w^* \geq \frac{\sqrt{3}}{2} \cdot (|a - z_0 1_M| + u|a - z_0 1_M|u). \quad (7.39)$$

If  $\sigma(a)$  lies on a straight line then

$$|[a, u]| = |a - z_0 1_M| + u|a - z_0 1_M|u. \quad (7.40)$$

*Proof.* Let  $\tau$  be a faithful normal tracial state on  $M$  and let  $F: S[0, 1] \rightarrow S(M)$  be an injective  $*$ -homomorphism from Theorem 7.5.2 satisfying  $a \in \text{Im}(F)$ . Let  $g = F^{-1}(a)$ .

It follows from Theorem 7.4.3 that there exists  $z_0$  such that  $[0, 1]$  can be divided into disjoint measurable parts  $\{X_1\} \cup \{X_2^{m,i} : m \geq 1, 1 \leq i \leq 2\}$  so that  $g(X_1) \subset \{z_0\}$ ,  $\mu(X_2^{m,1}) = \mu(X_2^{m,2})$  and the sets  $g(X_2^{m,1})$ ,  $g(X_2^{m,2})$  are  $(z_0, \frac{\pi}{3})$ -conjugate for  $m = 1, 2, \dots$  (where  $\mu$  is the Lebesgue measure on  $[0, 1]$ ).

Let  $e = F(\chi_{X_1})$ ,  $p_m = F(\chi_{X_2^{m,1}})$ ,  $q_m = F(\chi_{X_2^{m,2}})$ ,  $m = 1, 2, \dots$ . Then  $p_m \sim q_m$ ,  $m = 1, 2, \dots$ , since  $\tau(p_m) = \mu(X_2^{m,1}) = \mu(X_2^{m,2}) = \tau(q_m)$ . Besides  $e + \sum_{m \geq 1} (p_m + q_m) = 1_M$ . Hence, there exists such  $u = u^* \in \text{U}(M)$  that

$$ue = e, \quad up_m = q_m u, \quad m = 1, 2, \dots$$

Note also that  $p_m u = u q_m$  since  $u$  self-adjoint. It is clear that

$$|[a, u]|e = |[a - z_0 1_M, u]|e = 0 = (|a - z_0 1_M| + u|a - z_0 1_M|u)e.$$

For any  $m = 1, 2, \dots$   $\sigma(ap_m)$  coincides with the set  $A_m$  of essential values of the function  $g|_{X_2^{m,1}}$  and  $\sigma(uaup_m) = \sigma(aq_m)$  coincides with the set  $B_m$  of essential values of the function  $g|_{X_2^{m,2}}$  (here the operators  $ap_m$  and  $uaup_m$  are considered as elements of the algebra  $p_m \mathcal{M} p_m$ ). The sets  $A_m$  and  $B_m$  are  $(z_0, \frac{\pi}{3})$ -conjugate sets. It follows from the Lemma 7.5.3 that

$$v_m |a - uau| v_m^* p_m = v_m |a - uau| p_m v_m^* \geq \frac{\sqrt{3}}{2} \cdot (|a - z_0 1_M| + u|a - z_0 1_M|u) p_m \quad (7.41)$$

for some  $v_m \in \text{U}(p_m \mathcal{M} p_m)$ .

Applying the automorphism  $u \cdot u$  to (7.41), and noting that  $u|a - uau|u = |a - uau|$ , we obtain

$$(uv_m u) |a - uau| (uv_m u)^* q_m \geq \frac{\sqrt{3}}{2} \cdot (|a - z_0 1_M| + u|a - z_0 1_M|u) q_m. \quad (7.42)$$

To complete the proof, it remains to define

$$w = e + \sum_{n=1}^{\infty} (v_n + uv_n u)$$

which is a unitary (the series converges in the strong operator topology) (note here that  $uv_m u \in U(q_m M q_m)$ ). We observe that

$$w|[a, u]|w^* p_n = w|[a, u]|p_n v_n^* p_n = w p_n |[a, u]|v_n^* p_n = v_n |a - uau|v_n^* p_n \quad (7.43)$$

and similarly,  $w|[a, u]|w^* q_n = (uv_n u)|a - uau|(uv_n u)^* q_n$  and  $w|[a, u]|w^* e = |[a, u]|e = 0$ . Summing up the inequalities (7.41) and (7.42) in the measure topology we arrive at

$$\begin{aligned} w|[a, u]|w^* &= w|[a, u]|w^* e + \sum_{n=1}^{\infty} w|[a, u]|w^* (p_n + q_n) \\ &= \sum_{n=1}^{\infty} v_n |a - uau|v_n^* p_n + (uv_n u)|a - uau|(uv_n u)^* q_n \\ &\geq \sum_{n=1}^{\infty} \frac{\sqrt{3}}{2} (|a - z_0 1_M| + u|a - z_0 1_M|u)(p_n + q_n) \\ &= \frac{\sqrt{3}}{2} (|a - z_0 1_M| + u|a - z_0 1_M|u) \end{aligned}$$

which proves (7.39). Regarding the proof of equality (7.40), see the end of the proof of the Theorem 7.5.1. □

We have now established in Theorem 7.5.1 and Theorem 7.5.4 that for finite factors the commutator estimate (7.5) holds with the constant  $\frac{\sqrt{3}}{2}$ . However, this may not be the best constant for which, for all normal  $a \in M$ , the inequality holds. We will now establish upper bounds on the best possible constant and we will in particular show that  $\frac{\sqrt{3}}{2}$  is in fact the best possible constant when  $M$  is a  $\text{II}_1$ -factor or a  $I_n$ -factor ( $n < \infty$ ) with  $n \equiv 0 \pmod{3}$ . To do this we need the following proposition, which is partly motivated by the proof of [HW53, Theorem 1]. Here, for a given algebra  $A$  we denote by  $\text{Mat}_n(A)$  the set of all  $n \times n$  matrices with entries in  $A$ .

**Proposition 7.5.5.** *Let  $N$  be a finite factor with a faithful normal tracial state  $\tau_N$ . Let  $n \in \mathbb{N}$  and put  $M = \text{Mat}_n(\mathbb{C}) \otimes N \simeq \text{Mat}_n(N)$ . Let  $\tau_M = \frac{1}{n} \text{Tr}_n \otimes \tau_N$  be the tracial state on  $M$ . Denote  $\mathcal{U}_n^{\text{per}} \subseteq \text{Mat}_n(\mathbb{C})$  for the group of permutation matrices and  $\text{Diag}_n(\mathbb{C}) \subseteq \text{Mat}_n(\mathbb{C})$  for the set of diagonal matrices. If  $a \in \text{Diag}_n(\mathbb{C}) \otimes 1_N$  then*

$$\sup_{u \in U(M)} \|a - u^* a u\|_2 = \max_{u \in \mathcal{U}_n^{\text{per}} \otimes 1_N} \|a - u^* a u\|_2$$

(The isomorphism (identification) of  $\text{Mat}_n(N) \rightarrow \text{Mat}_n(\mathbb{C}) \otimes N$  is given by the mapping  $(a_{ij})_{i,j=1}^n \rightarrow \sum_{i,j=1}^n \mathbf{e}_{ij} \otimes a_{ij}$  where  $\mathbf{e}_{ij}$  are matrix units of  $\text{Mat}_n(\mathbb{C})$ .)

*Proof.* Write  $a = \text{Diag}(a_i)_{i=1}^n \otimes 1_N$  with  $a_i \in \mathbb{C}$  and let  $u = (u_{ij})_{i,j=1}^n \in U(M)$ ,  $u_{ij} \in N$ ,  $i, j = 1, \dots, n$ . We note that

$$\|a - uau^*\|_2^2 = \tau_M((a - uau^*)(a^* - ua^*u)) = 2\tau_M(|a|^2) - 2\Re(\tau_M(aua^*u^*)).$$

We are interested in finding a unitary element  $u \in M$  for which the scalar

$$R(u) := -\Re(\tau_M(aua^*u^*)) = -\frac{1}{n} \sum_{i,j} \Re(\tau_N(a_i u_{ij} \overline{a_j} u_{ij}^*)) = -\frac{1}{n} \sum_{i,j} \Re(a_i \overline{a_j}) \tau_N(u_{ij} u_{ij}^*)$$

attains its maximum. For convenience, let  $(d_{ij}) \in \text{Mat}_n(\mathbb{C})$  be the matrix given by  $d_{ij} = -\frac{1}{n}\Re(a_i \overline{a_j})$ , so that  $R(u) = \sum_{ij} d_{ij} \tau_N(u_{ij} u_{ij}^*)$ . Define the set

$$\mathcal{W}_n = \{(\tau_N(v_{ij} v_{ij}^*))_{ij} \in \text{Mat}_n(\mathbb{C}) : v = (v_{ij}) \in \text{U}(\text{Mat}_n(N))\}.$$

We observe for  $w = (\tau_N(v_{ij} v_{ij}^*))_{ij} \in \mathcal{W}_n$  and every  $j$  such that  $1 \leq j \leq n$ , we have  $\sum_i w_{ij} = \tau_N(\sum_i v_{ij} v_{ij}^*) = \tau_N(1_N) = 1$ . Similarly, for  $1 \leq i \leq n$  we have  $\sum_j w_{ij} = \tau_N(\sum_j v_{ij} v_{ij}^*) = \tau_N(1_N) = 1$ . Furthermore, as  $v_{ij} v_{ij}^* \geq 0$  in  $N$ , it is clear that  $w_{ij} \geq 0$  for all  $i, j$ . Now, denote by  $\mathcal{X}_n$  the set of all elements  $x = (x_{ij}) \in \text{Mat}_n(\mathbb{C})$  satisfying

$$\forall j : \sum_i x_{ij} = 1, \quad \forall i : \sum_j x_{ij} = 1, \quad \forall i, j : x_{ij} \geq 0$$

so that  $\mathcal{W}_n \subseteq \mathcal{X}_n$ . Considering  $\mathcal{X}_n$  as a subset of  $\mathbb{R}^{n^2}$ , we see that  $\mathcal{X}_n$  defines a closed convex polytope. By [HW53, Lemmal], the vertices of  $\mathcal{X}_n$  are the permutation matrices. Hence the maximum of the linear form  $(x_{ij}) \rightarrow \sum_{ij} d_{ij} x_{ij}$  on  $\mathcal{X}_n$  is attained for some permutation matrix  $\tilde{u} = (\tilde{u}_{ij}) \in \mathcal{U}_n^{\text{per}}$ . As  $\tilde{u} \in \mathcal{U}_n^{\text{per}} \subseteq \text{Mat}_n(N)$  we have that  $\tau_N(\tilde{u}_{ij} \tilde{u}_{ij}^*) = \tilde{u}_{ij}$  and so

$$R(\tilde{u}) = \sum_{i,j} d_{ij} \tau_N(\tilde{u}_{ij} \tilde{u}_{ij}^*) = \sum_{i,j} d_{ij} \tilde{u}_{ij} = \max_{x \in \mathcal{X}_n} \sum_{i,j} d_{ij} x_{ij} \geq \sup_{w \in \mathcal{W}_n} \sum_{i,j} d_{ij} w_{ij} = \sup_{u \in \text{U}(M)} R(u).$$

Thus,  $\sup_{u \in \text{U}(M)} \|a - u^* a u\|_2 \leq \|a - (\tilde{u} \otimes 1_N)^* a (\tilde{u} \otimes 1_N)\|_2$  and the claim follows.  $\square$

Combining Theorem 7.5.1 and Theorem 7.5.4, we estimate the maximal constant  $C_M$  that satisfies the commutator estimate (7.5) for finite factors  $M$  in Theorem 7.5.6 below. For the definitions of the constants  $\Lambda_n$  and  $\tilde{\Lambda}_n$  we refer to (7.12) and (7.13) and for the exact values of  $\Lambda_n$  we refer to Theorem 7.A.1.

**Theorem 7.5.6.** *Let  $M$  be a finite factor with  $M \neq \mathbb{C}$ . Then there is a constant  $C > 0$  with the property that:*

(\*) *For any normal  $a \in S(M)$  there exists a complex number  $z_0 \in \mathbb{C}$  and unitaries  $u, v, w \in \text{U}(M)$  such that*

$$|[a, u]| \geq C(v|a - z_0 1_M|v^* + w|a - z_0 1_M|w^*). \quad (7.44)$$

Moreover, a maximal constant  $C_M$  with this property exists and it satisfies  $\Lambda_n \leq C_M \leq \frac{1}{2}\tilde{\Lambda}_n$  when  $M$  is a  $I_n$ -factor ( $1 < n < \infty$ ), and  $C_M$  equals  $\frac{1}{2}\sqrt{3}$  when  $M$  is a  $II_1$ -factor.

*Proof.* Combining Theorem 7.5.1 and Theorem 7.5.4 we obtain for any finite factor that the constant  $C = \frac{1}{2}\sqrt{3}$  is admissible for (\*). By Theorem 7.A.1 we have that  $\Lambda_n = \frac{1}{2}\sqrt{3}$  when  $n = 3$  or  $5 \leq n \leq \infty$ . Let  $n < \infty$ . To see that  $C = \Lambda_n$  is admissible for all  $n$  we note that by Theorem 7.4.3 we have for  $g \in S(\Omega_n)$  that there exist  $z_0 \in \mathbb{C}$ ,  $T \in \text{Aut}_n$  such that  $\Lambda(g, T, z_0) = \Lambda_n(g) \geq \Lambda_n$ , which means

$$|g \circ T - g| \geq \Lambda_n(|g - z_0| + |g \circ T - z_0|). \quad (7.45)$$

Repeating the proof of Theorem 7.5.1, replacing (7.35) with (7.45), we obtain that  $C = \Lambda_n$  is also an admissible constant for (\*). We will later see that the maximal admissible constant  $C_M$  actually exists. First we prove upper bounds on constants  $C$  satisfying (\*) for  $M$ . Let  $\tau$  be a tracial state on  $M$ .

Let  $M$  be a  $I_n$ -factor with  $1 < n < \infty$ . Let  $g \in S(\Omega_n)$  be the function from Lemma 7.A.2 and let  $a = \text{Diag}(g(1), \dots, g(n)) \in M$ . Let  $z_0 \in \mathbb{C}$ ,  $u, v, w \in U(M)$  such that (\*) is satisfied for  $a$  with constant  $C$ . It follows from Proposition 7.5.5 ( $N = \mathbb{C}$ ) that

$$\|[a, u]\|_1 \leq \|[a, u]\|_2 = \|a - u^* a u\|_2 \leq \max_{u_0 \in \mathcal{U}_n^{\text{per}}} \|a - u_0^* a u_0\|_2 \leq \text{Diam}(\sigma(a)).$$

Hence,

$$2C\|a - z_0 1_M\|_1 = C\|v|a - z_0 1_M|v^* + w|a - z_0 1_M|w^*\|_1 \leq \|[a, u]\|_1 \leq \text{Diam}(\sigma(a)).$$

Now, choosing  $g$  as in the assertion of Lemma 7.A.2 we obtain

$$1 \geq \text{Diam}(\sigma(a)) \geq 2C\|a - z_0 1_M\|_1 \geq 2C\|g - z_0\|_1 \geq 2C\tilde{\Lambda}_n^{-1}.$$

Hence,  $C \leq \frac{1}{2}\tilde{\Lambda}_n$ .

Let  $M$  be of type  $\text{II}_1$ . Then  $M \cong \text{Mat}_3(\mathbb{C}) \otimes N$  for some  $\text{II}_1$ -factor  $N$ . Let the function  $g \in S(\Omega_3)$  be as in Lemma 7.A.2 and let  $a_1 = \text{Diag}(g(1), g(2), g(3)) \in \text{Mat}_3(\mathbb{C})$  and  $a = a_1 \otimes 1_N \in M$ . Let  $z_0 \in \mathbb{C}$ ,  $u, v, w \in U(M)$  be such that (\*) holds for  $a$  with constant  $C$ . We have

$$\|[a, u]\|_1 \leq \|[a, u]\|_2 = \|a - u^* a u\|_2 \leq \max_{u_0 \in \mathcal{U}_3^{\text{per}} \otimes 1_N} \|a - u_0^* a u_0\|_2 \quad (7.46)$$

$$= \max_{u_0 \in \mathcal{U}_3^{\text{per}}} \|a_1 - u_0^* a_1 u_0\|_2 \quad (7.47)$$

$$\leq \text{Diam}(\sigma(a_1)). \quad (7.48)$$

Hence,

$$2C\|a_1 - z_0 1_M\|_1 = 2C\|a - z_0 1_M\|_1 \quad (7.49)$$

$$= C\|v|a - z_0 1_M|v^* + w|a - z_0 1_M|w^*\|_1 \quad (7.50)$$

$$\leq \|[a, u]\|_1 \quad (7.51)$$

$$\leq \text{Diam}(\sigma(a_1)) \leq 1. \quad (7.52)$$

It follows from Lemma 7.A.2 that

$$1 \geq \text{Diam}(\sigma(a_1)) \geq 2C\|g - z_0\|_1 \geq 2C\tilde{\Lambda}_3^{-1}.$$

Hence,  $C \leq \frac{1}{2}\tilde{\Lambda}_3 = \frac{\sqrt{3}}{2}$ . For  $M$  a  $\text{II}_1$ -factor, this shows that in fact  $C_M$  exists and that  $C_M = \frac{1}{2}\sqrt{3}$ .

We now show that the maximal constant  $C_M$  also exists when  $M$  is a  $I_n$ -factor ( $1 < n < \infty$ ). Let  $(C_i)_{i \geq 1}$  be an increasing sequence of positive constants admissible for (\*) and set  $C = \sup C_i \leq \frac{1}{2}\tilde{\Lambda}_n$ . For a normal  $a \in M$  there exists corresponding  $u_i \in U(M)$  and  $z_{0,i} \in \mathbb{C}$  such that the equation (7.45) holds with the constant  $C_i$ . Now by

$$2\|a\|_1 \geq \|[a, u_i]\|_1 \geq 2C_i\|a - z_{0,i} 1_M\|_1 \geq 2C_i(|z_{0,i}| - \|a\|_1)$$

we obtain  $|z_{0,i}| \leq \frac{1+C_i}{C_i} \|a\|_1 \leq \frac{1+C_1}{C_1} \|a\|_1$ . Therefore, as the sequences  $(u_i)_i$  and  $(z_{0,i})_i$  are bounded and as  $M$  is finite-dimensional, we can assume these sequences converge in norm to some  $u \in U(M)$  and some  $z_0 \in \mathbb{C}$  (otherwise restrict to a subsequence). Now the elements  $d_i := |[a, u_i]| - C_i(|a - z_{0,i}1_M| + u_i|a - z_{0,i}1_M|u_i^*)$  are all positive and converge to  $d = |[a, u]| - C(|a - z_01_M| + u|a - z_01_M|u^*)$ . As the cone of positive elements in  $M$  is closed in the norm, we obtain  $d \geq 0$ . This shows that  $|[a, u]| \geq C(|a - z_01_M| + u|a - z_01_M|u^*)$  holds, and therefore  $C$  is admissible for  $(*)$  as well. Hence, the supremum of all admissible constants (which is finite), is again admissible, and this shows that  $C_M$  exists. It now follows that  $\Lambda_n \leq C_M \leq \frac{1}{2}\tilde{\Lambda}_n$

□

## 7.6. COMMUTATOR ESTIMATES FOR NORMAL OPERATORS IN INFINITE FACTORS

We shall now obtain the commutator estimate (7.5) for normal elements in an infinite factor. We show in Theorem 7.6.4 that for such factors the constant  $C$  in this estimate can be chosen arbitrary close to 1. For infinite factors, this extends the result of [BHS23, Theorem B.1] to normal elements. The proof of Theorem 7.6.4 extensively uses the geometry of projections. Before we start its proof, we state and prove three short technical lemmas. We recall for projections  $p, q$  in a von Neumann algebra  $M$  we write  $p < q$  if  $p \leq q$  and  $p \neq q$ .

**Lemma 7.6.1.** *Let  $M$  be an infinite factor and  $p$  be a infinite projection from  $M$ . If  $p_1, \dots, p_n \in P(M)$  are pairwise commuting and  $p_1, \dots, p_n < p$ , then  $p_1 \vee \dots \vee p_n < p$ .*

*Proof.* Let  $q_1 = p_1$  and  $q_{k+1} = p_{k+1}(1_M - q_1 - \dots - q_k)$  for  $k = 1, \dots, n-1$ . Then  $q_i q_j = 0$  for  $i \neq j$ ,  $q_k < p$  for  $k = 1, \dots, n$  and  $p_1 \vee \dots \vee p_n = q_1 + \dots + q_n < p$  (see [BS12b, Lemma 2 (ii)]). □

**Lemma 7.6.2.** *Let  $M$  be a factor,  $a$  be a normal operator from  $S(M)$ ,  $p, q \in P(M)$ ,  $q \leq p$ . Suppose that one of the following conditions holds:*

1.  *$q$  is finite and there exists a sequence of finite projections  $(p_n)$  in  $M$  such that  $p_n \uparrow p$  and  $[a, p_n] = 0$  for all  $n \in \mathbb{N}$ ;*
2.  *$q$  is an infinite projection and  $[a, p] = 0$ .*

*Then there exists a projection  $q_1 \in M$  such that  $q_1 \sim q$ ,  $[a, q_1] = 0$  and such that  $q_1 \leq p$ .*

*Proof.* The proof follows along the lines of [BS12b, Lemma 3] and is therefore omitted. □

**Lemma 7.6.3.** *Let  $M$  be a von Neumann algebra,  $a, b \in LS(M)$ ,  $\alpha_1, \alpha_2 > 0$ , and*

$$|a| \geq \alpha_1 1_M, \quad \alpha_1 > 2\alpha_2, \quad \alpha_2 1_M \geq |b|.$$

*Then there exists  $v \in U(M)$  such that*

$$v|a - b|v^* \geq (1 - \frac{2\alpha_2}{\alpha_1})|a| + |b|.$$

*Proof.* Let  $a, b \in \text{LS}(M)$ ,  $\alpha_1, \alpha_2 > 0$  satisfy the assumption of the lemma. By Theorem 7.2.2, we have that

$$|a| \leq v|a - b|v^* + w|b|w^*$$

for some  $v, w \in M$  with  $v^*v = w^*w = 1_M$ . Then

$$\begin{aligned} v|a - b|v^* &\geq |a| - w|b|w^* \geq |a| - \alpha_2 w w^* \geq |a| - \alpha_2 1_M \\ &\geq |a| + |b| - 2\alpha_2 1_M \geq |a| + |b| - \frac{2\alpha_2}{\alpha_1} |a| = (1 - \frac{2\alpha_2}{\alpha_1})|a| + |b|. \end{aligned}$$

Since  $v|a - b|v^* \geq (1 - \frac{2\alpha_2}{\alpha_1})|a| \geq (\alpha_1 - 2\alpha_2)1_M$ , it follows

$$0 = (1_M - vv^*)v|a - b|v^*(1_M - vv^*) \geq (\alpha_1 - 2\alpha_2)(1_M - vv^*) \geq 0.$$

Therefore, we have  $1_M - vv^* = 0$ , i.e.  $v \in \mathcal{U}(M)$ . □

**Theorem 7.6.4.** *Let  $M$  be an infinite factor, and let  $a \in S(M)$  be normal. There is a  $\lambda_0 \in \mathbb{C}$  such that for any  $\varepsilon > 0$  there exist  $u_\varepsilon = u_\varepsilon^* \in \mathcal{U}(M)$ ,  $w_\varepsilon \in \mathcal{U}(M)$  so that*

$$w_\varepsilon|[a, u_\varepsilon]|w_\varepsilon^* \geq (1 - \varepsilon)(|a - \lambda_0 1_M| + u_\varepsilon|a - \lambda_0 1_M|u_\varepsilon). \quad (7.53)$$

*Proof.* Let  $e(\cdot)$  be the spectral measure of  $a$  on  $\mathbb{C}$ , in particular,  $e(X) = \chi_X(a)$  for any  $X \in \mathcal{B}(\mathbb{C})$ . Since  $a \in S(M)$  there exists a  $R > 0$  so that  $e(X_R)$  is a finite projection, where  $X_R = \{\lambda \in \mathbb{C} : |\lambda| > R\}$ . Then  $Y_R := \mathbb{C} \setminus X_R$  is compact and it follows from Lemma 7.6.1 that  $e(Y_R) \sim 1_M$ . A point  $\lambda \in \mathbb{C}$  will be called a *point of densification* for  $a$  if  $e(V) \sim 1_M$  for any neighborhood  $V$  of a point  $\lambda$ . Denote by  $A$  the set of all points of densification for  $a$ .

We claim that  $A \neq \emptyset$ . To see that the claim holds it is sufficient to show there exists a system of nested sets  $B_n = [\alpha_n, \alpha_n + \frac{5R}{2^n}] \times [\beta_n, \beta_n + \frac{5R}{2^n}]$ , with  $e(B_n) \sim 1_M$ . We put  $\alpha_1 = \beta_1 = -R$  so that clearly  $Y_R \subset B_1$  and therefore  $e(B_1) \sim 1_M$ . Now suppose  $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$  are already constructed so that  $e(B_1) \sim \dots \sim e(B_n) \sim 1_M$ . We can divide the rectangle  $B_n$  into 4 smaller rectangles by

$$B_n = \bigcup_{k,l=0}^1 [\alpha_n + k \cdot \frac{5R}{2^{n+1}}, \alpha_n + (k+1) \cdot \frac{5R}{2^{n+1}}] \times [\beta_n + l \cdot \frac{5R}{2^{n+1}}, \beta_n + (l+1) \cdot \frac{5R}{2^{n+1}}].$$

It follows from Lemma 7.6.1 that one of the sets from this union can be taken for  $B_{n+1}$  (which then defines  $\alpha_{n+1}, \beta_{n+1}$ ). This completes the induction. The point  $\lambda := (\sup_n \alpha_n) + (\sup_n \beta_n)i$  is a point of densification for  $a$  since any neighbourhood  $V$  of  $\lambda$  contains a set  $B_n$  for some  $n$ . Therefore  $A \neq \emptyset$ .

We show that  $A$  is closed. Indeed, if  $\lambda$  is a limit point of  $A$  and  $V$  is a neighborhood of  $\lambda$ , then  $V$  is also a neighborhood of some point from  $A$ . Hence  $e(V) \sim 1_M$ . This shows  $\lambda \in A$ . Thus  $A$  is closed. Obviously,  $A \subset Y_R$ . Therefore,  $A$  is a nonempty compact subset in  $\mathbb{C}$ .

Let us consider three cases covering the full picture.

- 1. *There is a point  $\lambda_0 \in \mathbb{C}$  such that  $e(\{\lambda_0\}) \sim 1_M$ . Then  $e(\mathbb{C} \setminus \{\lambda_0\}) \leq e(\{\lambda_0\})$  and therefore there is a  $v \in M$  with  $v^*v = e(\mathbb{C} \setminus \{\lambda_0\})$  and  $vv^* \leq e(\{\lambda_0\})$ . Let's put  $u =$*

$v + v^* + (e(\{\lambda_0\}) - vv^*)$ . Then  $u = u^* \in U(M)$ . Since

$$\begin{aligned} (a - \lambda_0 1_M)u(a - \lambda_0 1_M)^* &= (a - \lambda_0 1_M)u(1_M - e(\{\lambda_0\}))(a - \lambda_0 1_M)^* \\ &= (a - \lambda_0 1_M)v(1_M - vv^*)(a - \lambda_0 1_M)^* \\ &= (a - \lambda_0 1_M)e(\{\lambda_0\})u(a - \lambda_0 1_M)^* = 0 \end{aligned}$$

and, similarly,

$$(a - \lambda_0 1_M)^* u(a - \lambda_0 1_M) = 0$$

then

$$|[a, u]| = |(a - \lambda_0 1_M) - u(a - \lambda_0 1_M)u| = |a - \lambda_0 1_M| + u|a - \lambda_0 1_M|u$$

which shows the result for this case with  $w_\varepsilon = 1_M$ .

In the following two cases, the scalar  $\lambda_0 \in \mathbb{C}$  will be found and for a fixed number  $\varepsilon > 0$  a sequence of pairs of projectors  $\{(p_n, q_n)\}_{n \geq 1}$  of  $M$  will be constructed together with a sequence  $(\gamma_n)$  of positive numbers converging to zero satisfying the following conditions:

(i).  $p_n q_m = 0$ ,  $p_n p_m = \delta_{nm} p_n$ ,  $q_n q_m = \delta_{nm} q_n$ ,  $[a, p_n] = [a, q_n] = 0$ ,  $p_n \sim q_n$  for all  $n, m$ ;

(ii).  $q_n \leq e(W_n)$ ,  $p_n \leq e(V_n)$  for all  $n \geq 1$ ;

(iii).  $\bigvee_{n \geq 0} p_n \vee \bigvee_{n \geq 0} q_n = 1_M - e(\{\lambda_0\})$ ,

where  $V_n := \{\lambda : |\lambda - \lambda_0| > \gamma_n\}$  and  $W_n := \{\lambda : |\lambda - \lambda_0| < \frac{\varepsilon}{2} \gamma_n\}$ .

- 2. The set  $A$  has a limit point  $\lambda_0$ . We can assume that  $\varepsilon < \frac{1}{2}$ . We inductively construct the sequences of positive numbers  $(\gamma_n)$  (and hence the sets  $V_n$ ,  $W_n$ ), numbers  $(\lambda_n)$  from  $A$ , and sets

$$U_n = \{\lambda : |\lambda - \lambda_{2n}| < \gamma_{n+1}\} \quad (7.54)$$

in such a way that  $U_n \subseteq W_n \cap V_{n+1}$  and the set  $V_{n+1} \setminus \bigcup_{k=1}^n (U_k \cup V_k)$  is a neighborhood of the point  $\lambda_{2n+1}$ . First, let  $\lambda_1 \in A \setminus \{\lambda_0\}$  and put  $\gamma_1 = \frac{|\lambda_1 - \lambda_0|}{2}$ . Then  $V_1$  is a neighborhood of the point  $\lambda_1$ . Next, in the set  $W_1$  there will be different points  $\lambda_2, \lambda_3$  from  $A \setminus \{\lambda_0\}$ . Put  $\gamma_2 = \frac{1}{2} \min\{|\lambda_3 - \lambda_0|, |\lambda_2 - \lambda_3|, \frac{\varepsilon}{2} \gamma_1 - |\lambda_2 - \lambda_0|, |\lambda_2 - \lambda_0|\}$  and note that  $\gamma_2 < \frac{1}{2} |\lambda_3 - \lambda_0| \leq \frac{\gamma_1}{4}$ . Note also that the set  $V_2 \setminus (V_1 \cup U_1)$  is a neighborhood of the point  $\lambda_3$  and that  $U_1 \subseteq W_1 \cap V_2$ . We continue this process by induction. Let these sequences be constructed for the indices  $1, \dots, n$ . Then in the set  $W_n$  there will be different points  $\lambda_{2n}, \lambda_{2n+1}$  from  $A \setminus \{\lambda_0\}$ . Put  $\gamma_{n+1} = \frac{1}{2} \min\{|\lambda_{2n+1} - \lambda_0|, |\lambda_{2n} - \lambda_{2n+1}|, \frac{\varepsilon}{2} \gamma_n - |\lambda_{2n} - \lambda_0|, |\lambda_{2n} - \lambda_0|\}$ . Then  $\gamma_{n+1} < \frac{\gamma_n}{4}$  and  $V_{n+1} \setminus \bigcup_{k=1}^n (U_k \cup V_k)$  is a neighborhood of the point  $\lambda_{2n+1}$ , and  $U_n \subseteq W_n \cap V_{n+1}$ . Thus, the above sequences are constructed. We remark that for  $n < m$  we have  $U_n \cap U_m \subseteq W_m \cap V_{n+1} = \emptyset$ .

Put  $p_1 = e(V_1)$ ,  $q_1 = e(U_1)$ ;  $q_n = e(U_n)$ ,  $p_n = e(V_n \setminus \bigcup_{k=1}^{n-1} (U_k \cup V_k))$  for  $n > 1$ . Then we have by the construction that  $p_1, q_1, p_2, q_2, \dots$  are pairwise orthogonal and  $p_n \sim 1_M \sim q_n$  for any  $n$ . Now since  $V_n = (V_n \setminus \bigcup_{k=1}^{n-1} (U_k \cup V_k)) \cup \bigcup_{k=1}^{n-1} (U_k \cup V_k)$  and  $\bigcup_{n=1}^\infty V_n = \mathbb{C} \setminus \{\lambda_0\}$  we find  $\bigvee_{n \geq 0} p_n \vee \bigvee_{n \geq 0} q_n = 1_M - e(\{\lambda_0\})$ .

- 3. The set  $A$  is finite and  $e(\{\lambda\}) < 1_M$  for any  $\lambda \in A$ . We can by assumption write  $A = \{\lambda_0, \dots, \lambda_m\}$  for some  $m \geq 0$  (note  $A$  is non-empty). When  $|A| = 1$  put  $r = 1$  and when  $|A| > 1$  let  $r$  be the minimum distance between points in  $A$ . Consider the sets  $V(t) = \mathbb{C} \cup \bigcup_{k=0}^m \{\lambda : |\lambda - \lambda_k| < t\}$  for  $0 < t < \frac{r}{2}$ . It is clear that  $V(t) \uparrow \mathbb{C} \setminus A$  at  $t \downarrow 0$ . We show that  $e(V(t)) < 1_M$  for  $0 < t < \frac{r}{2}$ . Indeed, for any point  $z \in V(t) \setminus X_R$  there is a neighborhood  $U_z$  of  $z$  with  $e(U_z) < 1_M$ . Now as the set  $V(t) \setminus X_R$  is compact we can let  $\{U_{z_1}, \dots, U_{z_l}\}$  be a finite subcover for  $V(t) \setminus X_R$ . Then  $\{X_R, U_{z_1}, \dots, U_{z_l}\}$  is the coverage of the set  $V(t)$ . It follows from Lemma 7.6.1 that  $e(V(t)) < 1_M$ .

There are now two possible cases:

3.1. All projections  $e(V(t))$ ,  $t > 0$ , are finite. In this case, put  $\gamma_1 = \frac{r}{3}$ .

3.2. There is a  $0 < t_0 < \frac{r}{3}$  so that the projection  $e(V(t_0))$  is infinite. In this case put  $\gamma_1 = t_0$ .

Set  $\gamma_n = \frac{\gamma_1}{2^{n-1}}$ ,  $n > 1$  (and hence  $V_n, W_n$  are defined as well); We set  $p_1 := e(V(\gamma_1) \cup (A \setminus \{\lambda_0\})) \leq e(V_1)$ . It follows from Lemma 7.6.1 that  $p_1 < 1_M$  and  $p := e(W_1) \sim 1_M$ . If we put  $q = p_1$ , then for  $p, q$  the conditions Lemma 7.6.2 are met: condition (ii) is met if  $q$  is an infinite projection, and condition (i) is met in case 3.1 if  $q$  is a finite projection (in this case, the set  $W_1$  is covered by the system  $V(t)$ ,  $t > 0$ ). Therefore, there is a projection  $q_1 \leq e(W_1)$  such that  $q_1 \sim p_1$  and  $[a, q_1] = 0$ . Now, suppose the projections  $p_1, q_1, \dots, p_n, q_n < 1_M$  are constructed. We build projections  $p_{n+1}, q_{n+1}$ . We put  $p_{n+1} = e(V(\gamma_{n+1})) \cdot (1_M - \sum_{k=1}^n (p_k + q_k))$ . Then  $p_{n+1} < 1_M$  since  $p_{n+1} \leq e(V(\gamma_{n+1}))$ . Furthermore, since  $e(W_n) \sim 1_M$  and  $p_1, q_1, \dots, p_n, q_n < 1_M$  we find  $e(W_n) \cdot (1_M - \sum_{k=1}^n (p_k + q_k)) \sim 1_M$ . Again using Lemma 7.6.2, we find such a projection  $q_{n+1} \sim p_{n+1}$  that  $q_{n+1} \leq e(W_n) \cdot (1_M - \sum_{k=1}^n (p_k + q_k))$  and  $[a, q_{n+1}] = 0$  (two cases are considered again:  $p_{n+1}$  is a infinite projection;  $p_{n+1}$  is a finite projection and the condition 3.1 is met). As  $p_{n+1} + \sum_{k=1}^n (p_k + q_k) \geq e(V(\gamma_{n+1}))$  and  $p_1 \geq e(A \setminus \{\lambda_0\})$  we conclude  $\sum_{k=1}^\infty (p_k + q_k) = 1_M - e(\{\lambda_0\})$ . Therefore, the projections  $p_1, q_1, p_2, q_2, \dots$  satisfy the conditions (i)-(iii).

In the cases (2) and (3) we can now find partial isometries  $v_n \in M$  so that  $v_n^* v_n = p_n$ ,  $v_n v_n^* = q_n$ , for  $n = 1, 2, \dots$ . We put  $u_\varepsilon = e(\{\lambda_0\}) + \sum_{n=1}^\infty (v_n + v_n^*)$ . Then  $u_\varepsilon = u_\varepsilon^* \in U(M)$ ,  $u_\varepsilon e(\{\lambda_0\}) = e(\{\lambda_0\})$  and  $u_\varepsilon p_n = q_n u_\varepsilon$  for all  $n$ . We have

$$|a - \lambda_0 1_M| p_n \geq \gamma_n p_n, |a - \lambda_0 1_M| q_n \leq \frac{\varepsilon}{2} \gamma_n q_n, \forall n. \quad (7.55)$$

Therefore

$$|u_\varepsilon a u_\varepsilon - \lambda_0 1_M| p_n = u_\varepsilon |a - \lambda_0 1_M| q_n u_\varepsilon \leq \frac{\varepsilon}{2} \gamma_n u_\varepsilon q_n u_\varepsilon = \frac{\varepsilon}{2} \gamma_n p_n, \forall n. \quad (7.56)$$

Since  $[a, p_n] = [u_\varepsilon a u_\varepsilon, p_n] = 0$  then

$$|a - u_\varepsilon a u_\varepsilon| p_n = |(a - \lambda_0 1_M) p_n - (u_\varepsilon a u_\varepsilon - \lambda_0 1_M) p_n|. \quad (7.57)$$

It follows from Lemma 7.6.3 that

$$w_n |a - u_\varepsilon a u_\varepsilon| p_n w_n^* \geq ((1 - \varepsilon) |a - \lambda_0 1_M| + |u_\varepsilon a u_\varepsilon - \lambda_0 1_M|) p_n$$

for some  $w_n \in U(p_n M p_n)$ .

Therefore

$$w_n |a - u_\varepsilon a u_\varepsilon| w_n^* p_n \geq ((1 - \varepsilon)(|a - \lambda_0 1_M| + |u_\varepsilon a u_\varepsilon - \lambda_0 1_M|)) p_n. \quad (7.58)$$

Applying the automorphism  $u_\varepsilon \cdot u_\varepsilon$  to (7.58), and noting that  $u_\varepsilon |a - u_\varepsilon a u_\varepsilon| u_\varepsilon = |a - u_\varepsilon a u_\varepsilon|$ , we obtain

$$(u_\varepsilon w_n u_\varepsilon) |a - u_\varepsilon a u_\varepsilon| (u_\varepsilon w_n u_\varepsilon)^* q_n \geq ((1 - \varepsilon)(|a - \lambda_0 1_M| + |u_\varepsilon a u_\varepsilon - \lambda_0 1_M|)) q_n. \quad (7.59)$$

Recall that  $S(M) = M$  if  $M$  has type I or III. In this case, we denote by  $t$  the strong operator topology in  $M$ . If the factor  $M$  is of type II then  $S(M) = S(M, \tau)$  for any faithful semifinite normal trace  $\tau$  on  $M$ . In this case we let  $t$  stand for the measure topology  $t_\tau$  (this topology is defined in the preliminaries, the need to use this topology is due to the fact that  $a$  can be an unbounded operator).

To complete the proof, it remains to set

$$w_\varepsilon = e(\{\lambda_0\}) + \sum_{n=1}^{\infty} (w_n + u_\varepsilon w_n u_\varepsilon)$$

(the series converges in the strong operator topology) and sum up the inequalities (7.58) and (7.59) in the topology  $t$ .  $\square$

## 7.7. ESTIMATES FOR INNER DERIVATIONS ASSOCIATED TO NORMAL ELEMENTS

In this section we apply the operator estimates from Theorem 7.5.1 and Theorem 7.5.6 to extend the result of [BHS23, Theorem 1.1] and estimate the norm of inner derivations  $\delta_a : M \rightarrow L^1(M, \tau)$  in the case when  $M$  a finite factor with faithful normal trace  $\tau$  and  $a \in L^1(M, \tau)$  is normal.

We establish some notation first. Let  $M$  be a von Neumann algebra with predual  $M_*$ . The Banach space  $M_*$  can be embedded into its double dual  $(M_*)^{**} = M^*$ . In this way we identify  $M_*$  with the space of ultraweakly continuous linear functionals on  $M$ . The predual  $M_*$  is a Banach  $M$ -bimodule with the bimodule actions given by:

$$(a \cdot \omega)(x) = \omega(xa), \quad (\omega \cdot a)(x) = \omega(ax), \quad a, x \in M, \quad \omega \in M_*. \quad (7.60)$$

If there is a faithful normal semifinite trace  $\tau$  on  $M$ , then the Banach  $M$ -bimodule  $M_*$  is isomorphic to  $L^1(M, \tau)$  (see e.g. [Tak03a, Chapter IX, Lemma 2.12 and Theorem 2.13]).

A linear operator  $\delta : M \rightarrow M_*$  is called a *derivation* if

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all  $x, y \in M$ . For each  $a \in M_*$  a derivation  $\delta_a : M \rightarrow M_*$  can be defined by the equality

$$\delta_a(x) = [a, x] = ax - xa$$

(using the  $M$ -bimodule structure as defined in (7.60)). Such derivations are called *inner*. In fact it holds true that any derivation  $\delta : M \rightarrow M_*$  is inner. Moreover, there exists  $a \in M_*$

so that  $\delta = \delta_a$  and  $\|a\|_{M_*} \leq \|\delta\|_{M \rightarrow M_*}$  see [Haa83, Theorem 4.1] and [BGM12, Corollary C]. We are interested in describing the norm of the derivations  $\delta_a : M \rightarrow M_*$  for  $a \in M_*$ . Is it true that a distance formula similar to (7.1) holds true? This question has been fully settled in [BHS23, Theorem 3.1] for infinite factors. Moreover, in [BHS23] the following theorem was proved:

**Theorem 7.7.1** (Theorem 1.1 in [BHS23]). *If  $M$  is a von Neumann algebra with a faithful normal finite trace  $\tau$  and  $a = a^* \in L^1(M, \tau)$ , then there exists  $c_a = c_a^* \in L^1(M, \tau) \cap Z(S(M))$  such that*

$$\|\delta_a\|_{M \rightarrow L^1(M, \tau)} = 2 \|a - c_a\|_1 = 2 \min_{z \in Z(S(M))} \|a - z\|_1 \quad (7.61)$$

where  $Z(S(M))$  stands for the center of the algebra of all measurable operators affiliated with  $M$

We focus on the case that  $M$  is finite. For brevity, we will denote the norm  $\|\cdot\|_{M \rightarrow L^1(M, \tau)}$  by  $\|\cdot\|_{\infty, 1}$ . For general  $a \in L^1(M, \tau)$  we do not know the relationship between  $\|\delta_a\|_{\infty, 1}$  and  $\inf\{\|a - z\|_1 : z \in Z(S(M))\}$ . In Theorem 7.7.3, we shall give upper and lower estimates of this relation in the case when  $M$  is a finite factor and  $a$  is a normal operator. We will see a substantial difference with the case of inner derivations associated to self-adjoint elements. First we state Theorem 7.7.2 which is related and is used in the proof of Theorem 7.7.3. Recall that when  $n \equiv 0 \pmod{3}$  or  $n = \infty$  we have  $2\Lambda_n = \sqrt{3} = \tilde{\Lambda}_n$  and that in addition,

$$\lim_{n \rightarrow \infty} \tilde{\Lambda}_n = \sqrt{3},$$

and

$$2\Lambda_n = \sqrt{3} \text{ for } n = 3, \text{ or } n \geq 5.$$

For convenience, we define for a finite factor  $M$  the value

$$n(M) := \begin{cases} n & M \text{ is a } I_n\text{-factor} \\ \infty & M \text{ is a } II_1\text{-factor} \end{cases} \quad (7.62)$$

**Theorem 7.7.2.** *Let  $M$  be a finite factor with a faithful tracial state  $\tau$ . Assume  $M \neq \mathbb{C}$ . Then*

1. *For every derivation  $\delta_a : M \rightarrow L^1(M, \tau)$  with  $a \in M$  normal, there is a normal  $b \in M$  such that  $\delta_a = \delta_b$  and  $\|\delta_b\|_{\infty, 1} \geq 2\Lambda_{n(M)} \|b\|_1$ .*
2. *There exists a normal  $a \in M$  for which the derivation  $\delta_a : M \rightarrow L^1(M, \tau)$  is non-zero and such that for every  $b \in M$  with  $\delta_a = \delta_b$  we have  $\|\delta_b\|_{\infty, 1} \leq \tilde{\Lambda}_{n(M)} \|b\|_1$ .*

*Proof.* (1) Let  $a \in M$  be normal. By Theorem 7.5.6 there exist  $u, w \in U(M)$ ,  $z_0 \in \mathbb{C}$  satisfying the commutator estimate (7.44), hence  $\|\delta_a\|_{\infty, 1} \geq \|\delta_a(u)\|_1 \geq 2\Lambda_{n(M)} \|a - z_0 1_M\|_1$ . This shows the result since  $b := a - z_0 1_M$  is normal and  $\delta = \delta_a = \delta_b$ .

(2) Let  $M$  be a finite factor. When  $M$  is a  $I_n$ -factor, we set  $m := n$  and we can write  $M = \text{Mat}_m(N)$ , with  $N = \mathbb{C}$ . When  $M$  is a  $II_1$ -factor we set  $m = 3$  and we can write  $M = \text{Mat}_m(N)$  for some  $II_1$ -factor  $N$ . We now let  $g \in L^\infty(\Omega_m)$  be non-constant and

let  $a$  be the diagonal matrix  $a = \text{Diag}(g(1), \dots, g(m)) \otimes 1_N \in \text{Mat}_m(\mathbb{C}) \otimes N = M$ . Then  $\delta_a : M \rightarrow L_1(M, \tau)$  is a non-zero derivation. To estimate  $\|\delta_a\|_{\infty,1}$  we recall that the Russo-Dye Theorem, [RD66, Theorem 1], asserts for a unital  $C^*$ -algebra that the closed unit ball equals the closed convex hull of all the unitaries. Now, for  $x \in \text{Conv}(U(M))$  we can write  $x = \sum_{i=1}^K c_i u_i$  with  $K \in \mathbb{N}$ ,  $u_i \in U(M)$  and  $c_i \geq 0$  with  $\sum_{i=1}^K c_i = 1$ . Then clearly  $\|\delta_a(x)\|_1 \leq \sum_{i=1}^K c_i \|\delta_a(u_i)\|_1 \leq \max_{1 \leq i \leq K} \|\delta(u_i)\|_1 \leq \sup_{u \in U(M)} \|\delta_a(u)\|_1$ . By continuity of  $\delta_a$  this inequality holds for all  $x$  in the closed convex hull as well. By the Russo-Dye Theorem this shows that

$$\|\delta_a\|_{\infty,1} = \sup_{x \in M, \|x\| \leq 1} \|\delta_a(x)\|_1 = \sup_{x \in \overline{\text{Conv}(U(M))}} \|\delta_a(x)\|_1 = \sup_{u \in U(M)} \|\delta_a(u)\|_1. \quad (7.63)$$

Using this and Proposition 7.5.5 we find

$$\begin{aligned} \|\delta_a\|_{\infty,1} &= \sup_{u \in U(M)} \|\delta_a(u)\|_1 \\ &= \sup_{u \in U(\text{Mat}_m(N))} \|u^* [a, u]\|_1 \\ &= \sup_{u \in U(\text{Mat}_m(N))} \|u^* au - a\|_1 \\ &\leq \sup_{u \in U(\text{Mat}_m(N))} \|u^* au - a\|_2 \\ &= \sup_{u \in \mathcal{U}_m^{\text{per}} \otimes 1_N} \|u^* au - a\|_2 \\ &= \sup_{\substack{T: \Omega_m \rightarrow \Omega_m \\ \text{permutation}}} \|g \circ T - g\|_2. \end{aligned}$$

The last step follows from the fact that, for  $u \in \mathcal{U}_m^{\text{per}} \otimes 1_N$ , we have  $u^* au = \text{Diag}(g \circ T(1), \dots, g \circ T(n)) \otimes 1_N$  for some permutation  $T$ . By Lemma 7.A.2 we can fix a  $g$  so that  $\text{Diam}(g(\Omega_m)) = 1 \leq \tilde{\Lambda}_m \inf_{z \in \mathbb{C}} \|g - z\|_1$  (note that such  $g$  is non-constant). Take any  $b \in M$  with  $\delta_a = \delta_b$ . Then  $a - b$  lies in the center of  $M$ , so  $a - b = z_0 1_M$  for some  $z_0 \in \mathbb{C}$ . Hence,  $\|b\|_1 = \|a - z_0 1_M\|_1 = \|g - z_0\|_1$  so that  $\|\delta\|_{\infty,1} \leq \text{Diam}(g(\Omega_m)) \leq \tilde{\Lambda}_m \|b\|_1$ . The result now follows. Indeed, when  $M$  is a  $I_n$ -factor, we obtained  $\|\delta\|_{\infty,1} \leq \tilde{\Lambda}_{n(M)} \|b\|_1$  and when  $M$  is a  $\text{II}_1$ -factor we obtained  $\|\delta\|_{\infty,1} \leq \tilde{\Lambda}_3 \|b\|_1 = \tilde{\Lambda}_\infty \|b\|_1 = \tilde{\Lambda}_{n(M)} \|b\|_1$ .  $\square$

The following theorem shows that for (most) finite factors the distance formula from (7.61) does not hold for arbitrary normal  $a \in L^1(M, \tau)$ , which shows a crucial difference with the classical result of Stampfli and its generalisations describing the norm of derivations  $\delta_a : M \rightarrow M$ , as for these derivations the distance formula (7.1) holds for all  $a \in M$ . While the distance formula does not hold true, we are able to obtain constant bounds on the ratio  $\frac{\|\delta_a\|_{\infty,1}}{\min_{z \in \mathbb{C}} \|a - z 1_M\|_1}$ . In the case of  $\text{II}_1$ -factors and  $I_n$ -factors ( $1 < n < \infty$ ) with  $n \equiv 0 \pmod 3$  these constants can not be improved.

**Theorem 7.7.3.** *Let  $M$  be a finite factor with a faithful tracial state  $\tau$  and let  $a \in L^1(M, \tau) \setminus Z(M)$  be normal and measurable. Then the derivation  $\delta_a : M \rightarrow L^1(M, \tau)$  satisfies:*

$$2\Lambda_{n(M)} \leq \frac{\|\delta_a\|_{\infty,1}}{\min_{z \in \mathbb{C}} \|a - z 1_M\|_1} \leq 2. \quad (7.64)$$

Moreover, when  $M \neq \mathbb{C}$  there exist non-zero derivations  $\delta_a, \delta_b$  corresponding to normal  $a, b \in M$  such that  $\|\delta_a\|_{\infty,1} \leq \tilde{\Lambda}_{n(M)} \min_{z \in \mathbb{C}} \|a - z1_M\|_1$  and  $\|\delta_b\|_{\infty,1} = 2 \min_{z \in \mathbb{C}} \|b - z1_M\|_1$ . We remark that

1. When  $n(M) \notin \{1, 2, 4\}$  then the distance formula of (7.61) does not extend to arbitrary normal measurable  $a \in L^1(M, \tau)$ , since  $\tilde{\Lambda}_{n(M)} < 2$  in these cases.
2. When  $M$  is a  $II_1$ -factor or a  $I_n$ -factor with  $n \equiv 0 \pmod{3}$  then the constant bounds given in (7.64) can not be improved as in these cases  $2\Lambda_{n(M)} = \sqrt{3} = \tilde{\Lambda}_{n(M)}$ .

*Proof.* Let  $a \in L^1(M, \tau) \setminus Z(M)$  be normal and measurable. By Theorem 7.5.6 there exist  $u, w \in U(M)$ ,  $z_0 \in \mathbb{C}$  satisfying (7.44) so that  $\|\delta_a\|_{\infty,1} \geq \|\delta_a(u)\|_1 \geq 2\Lambda_{n(M)} \|a - z_0 1_M\|_1$ , from which the first inequality follows. The second inequality follows from the fact that  $\|\delta_a(x)\|_1 = \|(a - z1_M)x - x(a - z1_M)\|_1 \leq 2\|a - z1_M\|_1 \|x\|$  holds for any  $x \in M$ ,  $z \in \mathbb{C}$ .

For the next statement, we note by (7.61) that  $\|\delta_b\|_{\infty,1} = 2 \inf_{z \in \mathbb{C}} \|b - z1_M\|_1$  holds for any self-adjoint  $b \in M$ , and that when  $M \neq \mathbb{C}$  we can choose  $b$  so that moreover  $b \notin Z(M)$ , ensuring that  $\delta_b$  is non-zero. Moreover, by Theorem 7.7.2(2) we obtain a normal  $a \in M$  such that  $\delta_a$  is a non-zero derivation with  $\|\delta_a\|_{\infty,1} \leq \tilde{\Lambda}_{n(M)} \|a - z1_M\|_1$  for every  $z \in \mathbb{C}$  since  $\delta_a = \delta_{a - z1_M}$ . Thus  $\|\delta_a\|_{\infty,1} \leq \tilde{\Lambda}_{n(M)} \min_{z \in \mathbb{C}} \|a - z1_M\|_1$  (it is clear the minimum exists). The last two remarks follow directly.  $\square$

We point out that Theorem 7.7.3 in particular shows the statement of (7.8) that for a finite factor  $M$  and normal measurable  $a \in L^1(M, \tau)$  we have

$$\sqrt{3} \min_{z \in \mathbb{C}} \|a - z1_M\|_1 \leq \|\delta_a\|_{\infty,1} \leq 2 \min_{z \in \mathbb{C}} \|a - z1_M\|_1$$

Indeed, for normal  $a \in L^1(M, \tau) \setminus Z(M)$  this follows by (7.64) and (7.14) while for  $a \in Z(M)$  this is trivial.

Secondly, we remark that this actually yields an estimate on the  $L^1$ -diameter of the unitary orbit  $\mathcal{O}(a) := \{uau^* : u \in U(M)\}$  of  $a$ . Indeed, as we already showed in (7.63), we obtain by the Russo-Dye Theorem [RD66, Theorem 1] that  $\|\delta_a\|_{\infty,1} = \sup_{u \in U(M)} \|\delta_a(u)\|_1$ . Therefore

$$\text{Diam}_{L^1(M, \tau)}(\mathcal{O}(a)) = \sup_{u \in U(M)} \|a - uau^*\|_1 = \sup_{u \in U(M)} \|\delta_a(u)\|_1 = \|\delta_a\|_{\infty,1}.$$

## 7.A. APPENDIX: CALCULATING CONSTANTS

We prove two technical results concerning the constants  $\Lambda_n$  and  $\tilde{\Lambda}_n$ . In Theorem 7.A.1 we will for  $n \neq 4$  determine the exact value of  $\Lambda_n$  with the help of Theorem 7.4.3. In Lemma 7.A.2 we prove a property of the constants  $\tilde{\Lambda}_n$  that we used in Theorem 7.7.2.

**Theorem 7.A.1.** *We have that  $\Lambda_1 = \Lambda_2 = 1$ ,  $\frac{\sqrt{3}}{2} \leq \Lambda_4 \leq 1$  and  $\Lambda_n = \frac{\sqrt{3}}{2}$  for any  $n \notin \{1, 2, 4\}$ . Moreover, for  $n \neq 4$  there is a  $g \in L^\infty(\Omega_n)$ ,  $T \in \text{Aut}_n$ ,  $z \in \mathbb{C}$  such that  $\Lambda(g, T, z) = \Lambda(g) = \Lambda_n$ .*

*Proof.* If  $n = 1$  then  $\Lambda(g, \text{Id}, g(1)) = 1$  for all  $g \in S(\Omega_n)$  since we agreed to count  $\frac{0}{0} = 1$ . Hence,  $\Lambda_1 = 1$ . If  $n = 2$  then  $\Lambda(g, T, \frac{g(1)+g(2)}{2}) = 1$  for all  $g \in S(\Omega_n)$  where  $T(1) = 2$ . Hence,

$\Lambda_2 = 1$ . It follows from Theorem 7.4.3 that  $\Lambda_n \geq \frac{\sqrt{3}}{2}$  for all  $n \geq 3$ . It only remains to show that this is in fact an equality whenever  $n = 3$  or  $n \geq 5$ , which we shall do now. For the given values of  $n$ , we can find a partition  $\{A_1, A_2, A_3\}$  of  $\Omega_n$  such that  $\frac{1}{5} \leq \frac{\mu_n(A_j)}{\mu_n(\Omega_n)} \leq \frac{2}{5}$  for  $j = 1, 2, 3$ . Now, denote  $w_j := e^{\frac{2\pi i j}{3}}$  for  $j = 1, 2, 3$  and construct the function  $g = \sum_{j=1}^3 w_j \chi_{A_j} \in L_\infty(\Omega_n)$ . We will show that  $\Lambda(g) \leq \frac{\sqrt{3}}{2}$ .

Suppose  $\Lambda(g) > \frac{\sqrt{3}}{2}$ . Then there exists  $T \in \text{Aut}_n$ ,  $z_0 \in \mathbb{C}$  and  $\lambda > \frac{\sqrt{3}}{2}$  so that

$$|g(T(\omega)) - g(\omega)| \geq \lambda(|g(T(\omega)) - z_0| + |g(\omega) - z_0|) \quad \text{a.e..}$$

We note that for  $k \neq l$  we have

$$|w_k - w_l| = \sqrt{3}.$$

Denote  $B_{k,j} = A_k \cap T^{-1}(A_j)$  so that  $B_{k,j} \subseteq A_k$  and  $T(B_{k,j}) \subseteq A_j$ . Moreover, since  $\{A_1, A_2, A_3\}$  is a partition of  $\Omega_n$ , we have for  $l = 1, 2, 3$  that

$$A_l = B_{l,1} \cup B_{l,2} \cup B_{l,3} \quad T^{-1}(A_l) = B_{1,l} \cup B_{2,l} \cup B_{3,l}. \quad (7.65)$$

We note that if  $\mu_n(B_{k,j} \cup B_{j,k}) > 0$  we must by the assumption have that

$$|w_k - w_j| \geq \lambda(|w_k - z_0| + |w_j - z_0|).$$

This is to say that  $z_0$  lies within the ellipse with foci  $w_k$  and  $w_j$  and eccentricity  $\lambda$ .

Now suppose  $\mu_n(B_{k,k}) > 0$  for some  $k$ . Then  $z_0 = w_k$  and for  $l, j \neq k$  we have

$$|w_l - w_j| \leq \sqrt{3} < 2\lambda < 2\lambda\sqrt{3} = \lambda(|w_l - w_k| + |w_j - w_k|) = \lambda(|w_l - z_0| + |w_j - z_0|)$$

and hence  $\mu_n(B_{l,j}) = 0$ . However, (7.65) then implies for  $j \neq k$  that

$$\mu_n(A_j) = \mu_n(B_{j,1}) + \mu_n(B_{j,2}) + \mu_n(B_{j,3}) = \mu_n(B_{j,k}).$$

Therefore, using this and (7.65) we obtain

$$\begin{aligned} 2\mu_n(A_k) &= \mu_n(A_k) + (\mu_n(B_{1,k}) + \mu_n(B_{2,k}) + \mu_n(B_{3,k})) \\ &= \mu_n(A_k) + \left( \sum_{\substack{1 \leq l \leq 3 \\ l \neq k}} \mu_n(B_{l,k}) \right) + \mu_n(B_{k,k}) \\ &= \mu_n(A_k) + \left( \sum_{\substack{1 \leq l \leq 3 \\ l \neq k}} \mu_n(A_l) \right) + \mu_n(B_{k,k}) \\ &= \mu_n(B_{k,k}) + \mu_n(A_1) + \mu_n(A_2) + \mu_n(A_3) \\ &= \mu_n(\Omega_n) + \mu_n(B_{k,k}) > \mu_n(\Omega_n). \end{aligned}$$

Hence  $\frac{\mu_n(A_k)}{\mu_n(\Omega_n)} > \frac{1}{2}$ , which is a contradiction with the choice of the partition.

We conclude that  $\mu_n(B_{k,k}) = 0$  for  $k = 1, 2, 3$ . Now suppose that for some  $1 \leq l, j \leq 3$  with  $l \neq j$  we have  $\mu_n(B_{l,j} \cup B_{j,l}) = 0$ . Let  $k \in \{1, 2, 3\}$  such that  $k \neq l, j$ . Then we obtain  $\mu_n(A_l) = \mu_n(B_{l,l}) + \mu_n(B_{j,l}) + \mu_n(B_{k,l}) = \mu_n(B_{k,l})$  and  $\mu_n(A_j) = \mu_n(B_{l,j}) + \mu_n(B_{j,j}) + \mu_n(B_{k,j}) = \mu_n(B_{k,j})$ . We thus have

$$\begin{aligned} 2\mu_n(A_k) &= \mu_n(A_k) + \mu_n(B_{k,l}) + \mu_n(B_{k,j}) + \mu_n(B_{k,k}) \\ &= \mu_n(A_k) + \mu_n(A_l) + \mu_n(A_j) = \mu_n(\Omega_n) \end{aligned}$$

and thus  $\frac{\mu_n(A_k)}{\mu_n(\Omega_n)} = \frac{1}{2}$ . This contradicts the choice of the partition sets.

Hence,  $\mu_n(B_{l,j} \cup B_{j,l}) > 0$  for all  $l, j$  with  $l \neq j$ . This means that the point  $z_0$  lies in all three ellipses (i.e. for  $l \neq j$  the point  $z_0$  has to lie inside the ellipse with foci  $w_l$  and  $w_j$  and eccentricity  $\lambda$ ). We obtain that for  $\lambda = \frac{\sqrt{3}}{2}$  the only point in the intersection of the three ellipses is 0, and that for  $\lambda > \frac{\sqrt{3}}{2}$  the intersection is empty (see Fig. 7.4). Hence,  $\Lambda(g) \leq \frac{\sqrt{3}}{2}$ . Therefore  $\Lambda_n = \frac{\sqrt{3}}{2}$ .

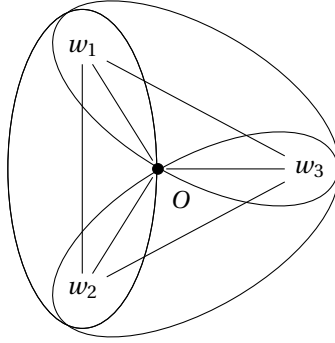


Figure 7.4: The image of the simple function  $g$  consists of the three points  $w_1, w_2$  and  $w_3$ . The three ellipses with foci  $w_l$  and  $w_j$  (for  $l$  and  $j$  different) and eccentricity  $\lambda = \frac{\sqrt{3}}{2}$  are drawn. The only point that lies in all three the ellipses is the point  $z_0 := 0$ .

□

**Lemma 7.A.2.** *Let  $1 < n \leq \infty$ . Then there is a  $g \in L^\infty(\Omega_n)$  with  $\text{Diam}(g(\Omega_n)) = 1$  and so that  $\tilde{\Lambda}_n = \sup_{z \in \mathbb{C}} \frac{1}{\|g - z\|_1}$ .*

*Proof.* The result for  $n = 2$  follows directly by taking  $g = \chi_{\{1\}}$ .

Thus, suppose  $n \geq 3$ . We can build a partition  $\{A_1, A_2, A_3\}$  of  $\Omega_n$  so that:

- If  $n = 3k$ ,  $k \in \mathbb{N}$ , or  $n = \infty$ , then  $\mu_n(A_1) = \mu_n(A_2) = \mu_n(A_3) = \frac{1}{3}$ .
- If  $n = 3k + 1$ ,  $k \in \mathbb{N}$ , then  $\mu_n(A_1) = \mu_n(A_2) = \frac{k}{n}$ ,  $\mu_n(A_3) = \frac{k+1}{n}$ .
- If  $n = 3k + 2$ ,  $k \in \mathbb{N}$ , then  $\mu_n(A_1) = \mu_n(A_2) = \frac{k+1}{n}$ ,  $\mu_n(A_3) = \frac{k}{n}$ .

For convenience let us denote

$$a = \mu_n(A_1) = \mu_n(A_2), \quad b = \mu_n(A_3), \quad w_k = e^{\frac{2\pi ki}{3}}, \quad k = 0, 1, 2.$$

Define  $g_0 \in L^\infty(\Omega_n, \mu_n)$  by

$$g_0 = \chi_{A_1} w_1 + \chi_{A_2} w_2 + \chi_{A_3} w_0.$$

Since  $\mu_n(A_1) = \mu_n(A_2)$ , it is clear that the minimum of  $\mathbb{C} \ni z \mapsto \|g_0 - z\|_1$  is attained for real-valued  $z$ , and moreover that  $-\frac{1}{2} \leq z \leq 1$ . When  $n = 4$ , it is clear from the triangle inequality that the minimum is attained at the point  $t_0 = 1$  and we have  $\|g_0 - t_0\|_1 = \frac{\sqrt{3}}{2}$ . Now assume  $n \neq 4$  so that the ratio  $\frac{b}{a}$  satisfies  $\frac{b}{a} < \sqrt{3}$  (the ratio  $\frac{b}{a}$  is maximal for  $n = 7$  in which case we have  $\frac{b}{a} = \frac{3}{7} = \frac{3}{2} < \sqrt{3}$ ). Hence  $\sqrt{3}a - b > 0$ . We have for  $t \in [-\frac{1}{2}, 1]$  that

$$\|g_0 - t\|_1 = 2a|w_1 - t| + b(1 - t).$$

Then

$$\frac{d}{dt} \|g_0 - t\|_1 = 2a \frac{t + \frac{1}{2}}{|w_1 - t|} - b.$$

As  $\frac{d}{dt} \|g_0 - t\|_1$  is negative when evaluated at  $-\frac{1}{2}$  and positive when evaluated at 1 (as  $\sqrt{3}a - b > 0$ ), the minimum of  $\|g_0 - t\|_1$  must be assumed at a point  $t_0 \in [-\frac{1}{2}, 1]$  satisfying

$$b|w_1 - t_0| = 2a(t_0 + \frac{1}{2}).$$

Then

$$b^2((t_0 + \frac{1}{2})^2 + \frac{3}{4}) = 4a^2(t_0 + \frac{1}{2})^2$$

and

$$(t_0 + \frac{1}{2})^2 = \frac{3b^2}{4(4a^2 - b^2)} = \frac{3b^2}{4(2a - b)}$$

since  $2a + b = 1$ . Therefore

$$(t_0 + \frac{1}{2})^2 + \frac{3}{4} = \frac{3b^2}{4(2a - b)} + \frac{3}{4} = \frac{3a^2}{(2a - b)}$$

and

$$\begin{aligned} \|g_0 - t_0\|_1 &= 2a|t_0 - w_1| + b(1 - t_0) \\ &= 2 \frac{\sqrt{3}a^2}{\sqrt{2a - b}} + b - b(\frac{\sqrt{3}b}{2\sqrt{2a - b}} - \frac{1}{2}) \\ &= \frac{\sqrt{3}\sqrt{2a - b}}{2} + \frac{3b}{2} \\ &= \frac{\sqrt{3 - 6b}}{2} + \frac{3b}{2}. \end{aligned}$$

- For  $n = 3k$  or  $n = \infty$  we have  $\mu_n(A_3) = \frac{1}{3}$  and find  $\|g_0 - t_0\|_1 = 1$ .
- For  $n = 3k + 1$  ( $n \neq 4$ ) we have  $\mu_n(A_3) = \frac{k+1}{3k+1}$  and find

$$\|g_0 - t_0\|_1 = \frac{1}{2} \sqrt{\frac{3k-3}{3k+1}} + \frac{1}{2} \cdot \frac{3k+3}{3k+1}.$$

- For  $n = 3k + 2$  we have  $\mu_n(A_3) = \frac{k}{3k+2}$  and find

$$\|g_0 - t_0\|_1 = \frac{1}{2} \sqrt{\frac{3k+6}{3k+2}} + \frac{1}{2} \cdot \frac{3k}{3k+2}.$$

Now, take  $g = \frac{1}{\sqrt{3}} g_0$  so that  $\text{Diam}(g(\Omega_n)) = 1$ . Then

$$\sup_{z \in \mathbb{C}} \frac{1}{\|g - z\|_1} = \sup_{z \in \mathbb{C}} \frac{\sqrt{3}}{\|g_0 - z\|_1} = \frac{\sqrt{3}}{\|g_0 - t_0\|_1} = \tilde{\Lambda}_n.$$

□

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# SAMENVATTING

Dit proefschrift bestudeert het vakgebied van operatoralgebra's, niet-commutatieve functionaalanalyse en rigiditeitstheorie. We bestuderen structureigenschappen van zowel  $C^*$ -algebra's als von Neumann-algebra's, met een focus op de laatste. Deze wiskundige structuren werden geïntroduceerd door von Neumann in [Neu30] wegens de noodzaak voor een niet-commutatief framework om kwantumsystemen te beschrijven. De theorie werd verder ontwikkeld door Murray en von Neumann in meerdere artikelen: [MN36], [MN37], [Neu39], [Neu40], [MN43], [Neu43] en [Neu49]. Tegenwoordig vormt de studie van deze operatoralgebra's zijn eigen deelgebied in de wiskunde. Over de jaren heen is geprobeerd om von Neumann-algebra's te classificeren. Er zijn veel structureigenschappen van von Neumannalgebra's geïntroduceerd en bestudeerd. In dit proefschrift bestuderen we zulke eigenschappen, waaronder: afwezigheid van Cartan-deelalgebra's, priemheid, de (zwakke-\*) CCAP, de Akemann-Ostrand eigenschap en sterke soliditeit. Verder bestuderen we operatorafschattingen voor commutatoren.

Voor een discrete groep  $G$  bestuderen we de groep-von Neumann-algebra  $\mathcal{L}(G)$ . Het doel is connecties te leggen tussen de groep  $G$  en zijn von Neumann-algebra  $\mathcal{L}(G)$ . We bestuderen *rigiditeitstheorie*, wat zich bezighoudt met de vraag welke informatie van de groep  $G$  kan worden afgeleid uit zijn von Neumann-algebra  $\mathcal{L}(G)$ . We zijn in het bijzonder geïntereerd in *Coxetergroepen*. Zo'n groep  $\mathcal{W}$  kan worden gezien als een abstracte reflectiegroep. Voor een Coxetergroep  $\mathcal{W}$  zullen we niet alleen  $\mathcal{L}(\mathcal{W})$  bestuderen, maar ook de  $\mathbf{q}$ -deformaties:  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$  genaamd *Hecke-von Neumann-algebra's*. De focus is op Coxetergroepen die rechthoekig zijn. Deze Coxetergroepen kunnen op natuurlijke wijze worden geschreven als graafproduct  $\mathcal{W} = *_{v,\Gamma} \mathcal{W}_v$  van de groepen  $\mathcal{W}_v = \mathbb{Z}/2\mathbb{Z}$ . De constructie van graafproducten van groepen was geïntroduceerd door Green in [Gre90] als een generalisatie van zowel directe sommen  $G_1 \oplus G_2$  als vrije producten  $G_1 * G_2$ . Later zijn graafproducten ook gedefinieerd in de setting van  $C^*$ -algebra's en von Neumann-algebra's in [Mlo04] en [CF17]. Hier generaliseren graafproducten zowel tensorproducten als vrije producten. Deze begrippen van graafproducten komen overeen met die voor groepen aangezien  $\mathcal{L}(*_{v,\Gamma} G_v) = *_{v,\Gamma} \mathcal{L}(G_v)$ . In het geval van rechthoekige Coxetergroepen geldt eenzelfde ontbinding ook voor Hecke-von Neumann-algebra's.

Dit proefschrift bestaat uit 7 hoofdstukken, waaronder de inleiding (Hoofdstuk 1) en de technische achtergrond (Hoofdstuk 2). In Hoofdstuk 3 voeren we berekeningen uit in graafproducten die we nodig hebben in latere hoofdstukken. In Hoofdstuk 4 is de studie gericht op (rechthoekige) Coxetergroepen, hun groep-von Neumann-algebra's  $\mathcal{L}(\mathcal{W})$  en Hecke-von Neumann-algebra's  $\mathcal{N}_{\mathbf{q}}(\mathcal{W})$ . We bestuderen wanneer deze von Neumann-algebra's *sterk solide* zijn en wanneer ze de Akemann-Ostrand eigenschap  $(AO)^+$  bezitten. Sterke soliditeit is een sterkere versie van Ozawa's eigenschap soliditeit [Oza04] en kan worden gezien als een sterke onontbindbaarheidseigenschap. Deze eigenschap

impliceert namelijk dat de von Neumann-algebra niet ontbindt als een tensorproduct  $M = M_1 \overline{\otimes} M_2$  (priemheid) noch als een groep-maatriimte  $M = L^\infty(0, 1) \rtimes G$  (afwezigheid van Cartan). Met behulp van kwantum Markov halfgroepen en de niet-commutatieve Riesz-transformatie bewijzen we nieuwe sterke soliditeitsresultaten.

In Hoofdstuk 5 bestuderen we sterke soliditeit voor algemene graafproducten van von Neumann-algebra's. We gebruiken Popa's *intertwining-by-bimodule* theorie om voor graafproducten een volledige karakterisering van sterke soliditeit te krijgen. In het bijzonder voltooit dit de karakterisering voor rechthoekige Hecke-algebra's. Voor rechthoekige Coxetergroepen geeft dit een simpele karakterisering wanneer de groep-von Neumann-algebra sterk solide is. We bestuderen ook andere aspecten van graafproducten. We geven voldoende voorwaarde voor een (gereduceerd) graafproduct om nucleair te zijn. Verder karakteriseren we volledig priemheid en vrijproduct-onontbindbaarheid voor graafproducten. We bestuderen ook rigiditeitstheorie voor graafproducten. Het doel is om de graaf  $\Gamma$  en de von Neumann-algebra's  $(M_\nu)_{\nu \in \Gamma}$  terug te halen uit de von Neumann-algebra  $M_\Gamma$ . We introduceren in dit proefschrift een klasse  $\mathcal{C}_{\text{vertex}}$  van von Neumann-algebra's en een klasse van grafen die we *rigide* noemen. We laten zien dat uit het graafproduct  $M_\Gamma = *_{\nu \in \Gamma} (M_\nu, \tau_\nu)$  we de rigide graaf  $\Gamma$  en de von Neumann algebra's  $(M_\nu)_{\nu \in \Gamma}$  kunnen terughalen (op amplificaties na). In het bijzonder verkrijgen we hiermee unieke priemfactorisaties en unieke vrijproduct-ontbindingen voor nieuwe klassen van von Neumann-algebra's. We laten ook zien dat, zonder sterke voorwaarden op de von Neumann-algebra's  $M_\nu$ , het mogelijk is om (op een constante na) de radius van de graaf  $\Gamma$  af te leiden uit het graafproduct  $M_\Gamma$ .

In Hoofdstuk 6 bestuderen we benaderingseigenschappen van graafproducten. Voor een groep  $G$  stellen benaderingseigenschappen dat we de constante functie  $1_G$  puntgewijs kunnen benaderen met *goede* functies  $m_k : G \rightarrow \mathbb{C}$ . Eensgelijks, voor een operator-algebra  $M$ , stelt een benaderingseigenschap dat we de identiteitsafbeelding  $\text{Id}_M$  puntsgewijs kunnen benaderen met *goede* afbeeldingen  $\theta_k : M \rightarrow M$ . Voor het gereduceerde graafproduct van  $C^*$ -algebra's bestuderen we de *CCAP*. Op eenzelfde wijze bestuderen we voor graafproducten van von Neumann-algebra's de *zwakke-\** *CCAP*. Deze benaderingseigenschappen vormen de operatoralgebraïsche tegenhanger van zwakke amenabiliteit met constante 1. We bestuderen stabiliteit van deze eigenschappen onder graafproducten en breiden resultaten uit van [Rec17] en [RX06].

In Hoofdstuk 7 wijken we af van het hoofdonderwerp van dit proefschrift en bestuderen we commutatorafschattingen. We breiden de operatorafschattingen van [BS12b], [BS12a] en [BHS23] voor zelf-geadjungeerde operatoren uit naar normale operatoren. Voor een normaal element  $a$  in een factor  $M$  laten we zien dat er een unitair  $u \in M$  bestaat, waarvoor we een *goede* operatorafschatting krijgen voor de commutator  $[a, u] := au - ua$ . Voor eindige factoren geeft dit een afschatting op de  $L^1$ -norm van de vorm

$$\sqrt{3} \min_{z \in \mathbb{C}} \|a - z1_M\|_{L^1(M, \tau)} \leq \|[a, u]\|_{L^1(M, \tau)}.$$

We gebruiken dit resultaat om scherpe afschattingen te krijgen op de norm  $\|\delta_a\|_{M \rightarrow L^1(M, \tau)}$  van de derivatie  $\delta_a : M \rightarrow L^1(M, \tau)$  gegeven door  $\delta_a(x) = [a, x]$ .

# CURRICULUM VITÆ

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Matthijs Borst was born in Gouda, the Netherlands on the 31th of May 1998. He followed secondary education at GSR in Rotterdam, which he completed in 2016. Hereafter, he started successively the bachelor and master Applied Mathematics at Delft University of Technology. He participated in the Honours program under supervision of dr. C. Kraaikamp. He wrote his bachelor thesis on *Kwapién's theorem on representing bounded mean zero functions* with the supervision of prof.dr.ir. M.C. Veraar. He completed his bachelor studies cum laude in 2019. As part of his master studies he did an internship of three months at the Datalab of insurance company DSW. Furthermore, he wrote his master thesis on *Gradient- $S_p$  quantum Markov semigroups, Coxeter groups and Strong solidity* under the supervision of dr. M.P.T. Caspers. He completed his master cum laude in 2021 after which he started a PhD research under the supervision of dr. M.P.T. Caspers and prof.dr.ir. M.C. Veraar at Delft University of Technology. Part of this research was conducted during a five month research visit to prof.dr. F.A. Sukochev and dr. D. Potapov at the University of New South Whales in Sydney, Australia.



# LIST OF PUBLICATIONS

8. **Matthijs Borst, Martijn Caspers and Enli Chen**, *Rigid graph products*, Preprint submitted to journal: [Arxiv:2408.06171v2](#).
7. **Matthijs Borst and Martijn Caspers**, *Classification of right-angled Coxeter groups with a strongly solid von Neumann algebra*, [Journal de Mathématiques Pures et Appliquées](#) 189 (2024) 103591.
6. **Alexei Ber, Matthijs Borst and Fedor Sukochev**, *Commutator estimates for normal operators in factors with applications to derivations*, Accepted in the Journal of Operator Theory. Preprint: [Arxiv:2304.10775v1](#).
5. **Matthijs Borst**, *The CCAP for graph products of operator algebras*, [Journal of Functional Analysis](#) 286.8 (2024) 110350.
4. **Matthijs Borst, Martijn Caspers, Mario Klisse, Mateusz Wasilewski**, *On the isomorphism classes of  $q$ -Gaussian  $C^*$ -algebras for infinite variables*, [Proceedings of American Mathematical Society](#) 151 (2023) pp 737-744.
3. **Matthijs Borst, Martijn Caspers and Mateusz Wasilewski**, *Bimodule coefficients, Riesz transforms on Coxeter groups and strong solidity*, [Groups, Geometry, and Dynamics](#) 18.2 (2023) pp. 501–549.
2. **Alexei Ber, Matthijs Borst, Sander Borst and Fedor Sukochev**, *A multidimensional solution to additive homological equations*, [Izvestiya Mathematics](#) 87.2 (2023) pp. 201-251.
1. **Alexei Ber, Matthijs Borst and Fedor Sukochev**, *Full proof of Kwapién's theorem on representing bounded mean zero functions on  $[0,1]$* , [Studia Mathematica](#) 259.3 (2021) pp. 241-270.

