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On a generalization of Power Algorithms over Max-Plus Algebra

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Abstract In this paper we discuss a generalization of power algorithms over max-plus algebra. We are interested in finding such a generalization starting from various existing power algorithms. The resulting algorithm can be used to determine the so-called generalized eigenmode of any square regular matrix over max-plus algebra. In particular, the algorithm can be applied in the case of regular reducible matrices in which the existing power algorithms can not be used to compute eigenvalues and corresponding eigenvectors.

Keywords max-plus algebra · generalized eigenmode · power algorithm · cycle time vector

1 Introduction

Many problems in operation research, performance analysis, manufacturing, communication network, etc., can be modeled as discrete event systems with maximum timing constraints. An algebra underlying such systems is based on two operations, maximization and addition, and is called max-plus algebra. For a paper on the scheduling of transportation systems using max-plus algebra, specifically in supply chain scheduling, see [11]. Eigenvalues and corresponding

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eigenvectors of square matrices over max-plus algebra have a very important role in applications. Particularly, they can be used to solve periodic scheduling problems. Periodicity of a timetable is often desired by users. Some results on eigenvalues and corresponding eigenvectors over max-plus algebra have been applied to scheduling problems in [7–10]. Moreover, eigenvalue problems for Latin squares in max-plus algebra and bipartite (min, max, +)-systems have been discussed in [12,13].

The power algorithm over max-plus algebra was initiated by Olsder in [1]. In that paper the algorithm is stated and an example is given. However, there is no further theory developed in [1]. Braker and Olsder proved that under certain conditions, the power algorithm can be used to find the eigenvalue and eigenvector of a max-algebra system. If the conditions are not satisfied, the eigenvector can be found using an extension of the power algorithm [3]. The work on power algorithms was continued by Subiono and van der Woude in [5]. They developed another power algorithm which is simpler than the previously mentioned algorithms.

In this paper we construct a generalization of the above mentioned power algorithms. The result is a generalization of Algorithm 2.5.a in [5], and of Algorithm 3.1 and 4.3 in [3], and can be used to determine the so-called generalized eigenmode of any square regular matrix over max-plus algebra. In particular, the algorithm can be applied in the case of regular reducible matrices in which the existing power algorithms can not be used to compute eigenvalues and corresponding eigenvectors. In addition, based on the results of numerical experiments, it can be concluded that the generalized power algorithm may compute the generalized eigenmode of any square regular matrix over max-plus algebra faster than the policy iteration algorithm given in [4]. However, this statement is relative to used initial condition and structure of the matrices. Note that we can consider here square regular matrices over max-plus algebra, whereas the existing power algorithms require the matrices to be irreducible and may fail to determine eigenvalues and corresponding eigenvectors in case they are applied to reducible square matrices.

The outline of this paper is as follows. In Section 2 we give a brief introduction to max-plus algebra and some related notations. In Section 3 we discuss some lemmas and theorems. In the last part of the section we give a generalized power algorithm, being the main result of this paper. We illustrate our main result by means of four examples. We end the paper with Section 4 in which we present some concluding remarks.

2 Max-Plus Algebra

In this section we briefly introduce the notion of max-plus algebra and some related notations used in the following discussion. A detailed discussion about the max-plus algebra can be found in [2,6].

The max-plus algebra $\mathbb{R}_{\max}$ is the set $\mathbb{R}\cup\{\varepsilon\}$, equipped with two operations, addition ($\oplus$) and multiplication ($\otimes$), where $\mathbb{R}$ is the set of real numbers, $\varepsilon \stackrel{\text{def}}{=}$

$-\infty$, $x \oplus y \stackrel{\text{def}}{=} \max\{x, y\}$ and $x \otimes y \stackrel{\text{def}}{=} x + y$ for every $x, y \in \mathbb{R}_{\max}$. Furthermore, in the context of max-plus algebra $a^{\otimes b} = ba$, the conventional multiplication of b and a. The algebraic structure of $\mathbb{R}_{\max}$ is an idempotent semifield, i.e., an idempotent commutative semiring where every element $x \in \mathbb{R}_{\max}$, with $x \neq \varepsilon$, has an inverse under the $\otimes$ operation, denoted $-x$. For example, in $\mathbb{R}_{\max}$ we have $3 \oplus 7 = \max\{3, 7\} = 7$ and $3 \otimes 7 = 3 + 7 = 10$. Also $\varepsilon \oplus x = \max\{-\infty, x\} = x = \max\{x, -\infty\} = x \oplus \varepsilon$ and $0 \otimes x = 0 + x = x = x + 0 = x \otimes 0$ for every x in $\mathbb{R}_{\max}$. Further, the following holds $3^{\otimes 2} = 2(3) = 6$.

2.1 Matrices over $\mathbb{R}_{\max}$

In this subsection we introduce matrices over $\mathbb{R}_{\max}$. The set of all matrices of size $m \times n$ over the max-plus algebra is denoted by $\mathbb{R}_{\max}^{m \times n}$. For $n \in \mathbb{N}$ with $n \neq 0$, we define $\underline{n} \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$. An element in row i and column j of a matrix $A \in \mathbb{R}_{\max}^{m \times n}$ is denoted by $a_{i,j}$, for $i \in \underline{m}$ and $j \in \underline{n}$. Sometimes the element $a_{i,j}$ is also denoted by $[A]_{i,j}$, $i \in \underline{m}$, $j \in \underline{n}$. A **regular** matrix is a matrix where every row contains at least one finite entry.

The identity matrix of size $n \times n$ over $\mathbb{R}_{\max}$ is denoted by E, i.e., the elements on the main diagonal of the matrix are equal to e, where $e \stackrel{\text{def}}{=} 0$, and the other elements are equal to ε. A zero matrix of size $m \times n$ in $\mathbb{R}_{\max}^{m \times n}$ is denoted by $\mathcal{E}(m, n)$, i.e., all elements of the matrix are equal to ε. For matrices $A, B \in \mathbb{R}_{\max}^{m \times n}$, the sum $A \oplus B$ is defined by $[A \oplus B]_{i,j} \stackrel{\text{def}}{=} a_{i,j} \oplus b_{i,j} = \max\{a_{i,j}, b_{i,j}\}$, for $i \in \underline{m}$ and $j \in \underline{n}$.

For a matrix $A \in \mathbb{R}_{\max}^{m \times n}$ and a scalar $\alpha \in \mathbb{R}_{\max}$, the scalar multiplication $\alpha \otimes A$ is defined by $[\alpha \otimes A]_{i,j} \stackrel{\text{def}}{=} \alpha \otimes a_{i,j}$, for $i \in \underline{m}$ and $j \in \underline{n}$. For matrices $A \in \mathbb{R}_{\max}^{m \times p}$ and $B \in \mathbb{R}_{\max}^{p \times n}$, the matrix multiplication $A \otimes B$ is defined by $[A \otimes B]_{i,j} \stackrel{\text{def}}{=} \bigoplus_{k=1}^{p} a_{i,k} \otimes b_{k,j} = \max_{k \in \underline{p}}\{a_{i,k} + b_{k,j}\}$, for $i \in \underline{m}$ and $j \in \underline{n}$. Matrix multiplication is similar to conventional matrix multiplication, where $+$ and $\times$ are replaced by $\oplus$ and $\otimes$, respectively. For a matrix $A \in \mathbb{R}_{\max}^{n \times n}$ and a positive integer k, the k-th power of A is denoted by $A^{\otimes k}$ and defined as $A^{\otimes k} \stackrel{\text{def}}{=} \underbrace{A \otimes A \otimes \ldots \otimes A}_{k}$. For completeness, $A^{\otimes 0} \stackrel{\text{def}}{=} E$. For ease of notations, we write A^+ for $\bigoplus_{k=1}^{\infty} A^{\otimes k}$, and A^* for $E \oplus A^+ = \bigoplus_{k \geq 0} A^{\otimes k}$, where A is a square matrix in $\mathbb{R}_{\max}^{n \times n}$.

2.2 Max-Plus and Graphs

In this subsection, we briefly introduce some definitions and max-plus algebra notations on graphs. We collect some basic definitions from [6].

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{D})$, where $\mathcal{V}$ is the set of vertices (nodes) of the graph $\mathcal{G}$ and $\mathcal{D} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges of $\mathcal{G}$. Here (i, j) denotes an edge from node i to node j. Node i is called the begin node of edge (i, j), and node j is called its end node. A directed graph $\mathcal{G}(A)$ associated with a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ is the graph with vertices in $\mathcal{V} = \{1, 2, ..., n\}$ and edges in $\mathcal{D} = \{(i, j) : a_{j,i} \neq \varepsilon\}$. We say that $a_{j,i}$ is the weight of edge $(i, j) \in \mathcal{D}$. In this paper we consider matrices A that are regular. It implies that every node in $\mathcal{G}$ is the end node of at least one edge.

A path p from node i to node j in a graph is a sequence of nodes $p = (i_1, i_2, ..., i_{s+1})$ with $i_1 = i$ and $i_{s+1} = j$ such that each (i_k, i_{k+1}), $1 \leq k \leq s$, is an edge of the graph. We say the path has length s and denote by $||p||_l = s$ the length of the path. The weight of the path is defined as the sum of the weights of all edges in the path, i.e., $\sum_{k=1}^{s} a_{i_{k+1},i_k}$. If this sum is t, we denote by $||p||_w = t$ the weight of the path. The average weight (or mean) of the path is then defined as $\frac{t}{s}$. The set of paths from i to j of length k will be denoted by $P(i, j, k)$. A vertex j is said to be reachable from a vertex i, denoted by $i\mathcal{R}j$, if there exists a path from i to j. A strongly connected graph is a graph such that every vertex is reachable from every other vertex. A matrix A in $\mathbb{R}_{\max}^{n\times n}$ is called irreducible if the corresponding graph $\mathcal{G}(A)$ is strongly connected. If $\mathcal{G}(A)$ is not strongly connected, i.e., if $\mathcal{G}(A)$ contains nodes that are not reachable from each other, then the matrix A is called reducible.

A circuit of length s is a closed path i.e., a path $p = (i_1, i_2, ..., i_{s+1})$ such that $i_1 = i_{s+1}$. A circuit consisting of one edge is also called a loop. An elementary circuit is one in which $i_1, i_2, ..., i_s$ are distinct. For circuits, the notions of length, weight and average weight (or mean) are defined in the same way as for paths.

Node j *communicates* with node i, denoted by $i\mathcal{C}j$, if either $i = j$, or vertex j is reachable from vertex i and conversely. Hence, $i\mathcal{C}j \iff i = j$ or ($i\mathcal{R}j$ and $j\mathcal{R}i$). Note that the relation "communicate with" is an equivalence relation. We denote by $[i] \stackrel{\text{def}}{=} \{j \in \mathcal{V} : j\mathcal{C}i\}$ the set of nodes containing node i that communicate with each other. Then the set $\mathcal{V}$ can be partitioned as $[i_1] \cup [i_2] \cup \cdots \cup [i_q]$, where $[i_r]$, $r \in \underline{q}$, denotes a subset of nodes that communicate with each other, but not with other nodes of $\mathcal{V}$.

Given the above partitioning of $\mathcal{V}$, it is possible to focus on subgraphs of $\mathcal{G}$, denoted by $\mathcal{G}_r = ([i_r], \mathcal{D}_r)$, $r \in \underline{q}$, where $\mathcal{D}_r$ denotes the subset of $\mathcal{D}$ of edges that have both the begin node and the end node in $[i_r]$. The subgraph $\mathcal{G}_r$ is known as a **maximum strongly connected subgraph** (m.s.c.s.) of $\mathcal{G}$. By definition, nodes in $[i_r]$ do not communicate with nodes outside $[i_r]$.

Let $A_{r,r}$ denote the matrix obtained by restricting $A \in \mathbb{R}_{\max}^{n\times n}$ to nodes in $[i_r]$, for all $r \in \underline{q}$, i.e., $[A_{r,r}]_{k,l} = a_{k,l}$ for all $k, l \in [i_r]$. Notice that for all $r \in \underline{q}$ either, $A_{r,r}$ is irreducible or $A_{r,r} = \mathcal{E}$. The original reducible matrix A, possibly after a relabeling of nodes in $\mathcal{G}(A)$, can be written in the form

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & \cdots & A_{1,q} \\ \mathcal{E} & A_{2,2} & \cdots & \cdots & A_{2,q} \\ \mathcal{E} & \mathcal{E} & A_{3,3} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} & A_{q,q} \end{pmatrix}, \tag{1}$$

where $A_{s,r}$, $1 \leq s < r \leq q$, are matrices of appropriate sizes. Each finite entry in $A_{s,r}$ corresponds to an edge from a node in $[i_r]$ to a node in $[i_s]$. The upper triangular block form, shown above, is said to be a *normal form* of matrix A.

Throughout this paper we assume that matrix A is regular, i.e., A has a finite entry in each row. However, A is allowed to have columns without any finite entry. Also then it is possible to determine the above normal form. For an example, see Example 2.1.3 in [6].

2.3 Asymptotic Behaviour of $A^{\otimes k}$

In this subsection, we recall some definitions and results about the asymptotic behaviour of $A^{\otimes k}$ from [2] and [6]. Let $\mathcal{G}(A)$ be the graph of a matrix $A \in \mathbb{R}_{\max}^{n\times n}$. We can restrict ourselves to the case that $\mathcal{G}(A)$ has at least one circuit. More specifically, as indicated above, we restrict ourselves in this study to matrices A that are regular.

Definition 1 Let A be a square matrix and assume that the maximum cycle mean of $\mathcal{G}(A)$ is equal to $e = 0$. The following notions can then be defined:
Critical circuit: A circuit η of the graph $\mathcal{G}(A)$ is called critical if it has maximum weight, that is, $||\eta||_w = e$.
Critical graph: The critical graph $\mathcal{G}^c(A)$ consists of those nodes and edges of $\mathcal{G}(A)$ which belong to a critical circuit of $\mathcal{G}(A)$. Let the nodes constitute the set $\mathcal{V}^c$.
Cyclicity of a graph: The cyclicity of a m.s.c.s. is the greatest common divisor (g.c.d.) of the lengths of all its circuits. The cyclicity $c(\mathcal{G})$ of a graph $\mathcal{G}$ is the least common multiple (l.c.m.) of the cyclicities of all its m.s.c.s.'s.
Cyclicity of A: Let $A \in \mathbb{R}_{\max}^{n\times n}$ be such that its communication graph contains at least one circuit. The cyclicity of A, denoted by $\sigma(A)$, is the cyclicity of the critical graph A.

Let us give now some simple results about these graphs that will be useful in the next discussion.

Proposition 1 (cf. [6]) *If the critical graph of matrix A has cyclicity σ and A is irreducible, then the critical graph of matrix $A^{\otimes \sigma}$ has cyclicity one.*

Proposition 1 is used in the proof of Lemma 32.

Definition 2 Let $A \in \mathbb{R}_{\max}^{n\times n}$ and define the $n \times n$ (projection) matrix

$$Q = \bigoplus_{i \in \mathcal{V}^c} A^+_{\bullet,i} \otimes A^+_{i,\bullet},$$

where $A^+_{\bullet,i}$ and $A^+_{i,\bullet}$ stand for the i-th column and the i-th row of matrix A^+, respectively.

Proposition 2 (cf. [2]) *Let a matrix $A \in \mathbb{R}_{\max}^{n\times n}$. If the critical graph has cyclicity one and the maximum cycle mean (m.c.m.) of $\mathcal{G}(A)$ is equal to $e = 0$, then*

$$\lim_{k\to\infty} A^{\otimes k} = Q, \text{ or element-wise, } \lim_{k\to\infty} [A^{\otimes k}]_{j,l} = Q_{j,l} = \bigoplus_{i\in\mathcal{V}^c} [A^+]_{j,i} \otimes [A^+]_{i,l}.$$

Note that Proposition 2 assumes that the m.c.m. is equal to $e = 0$. We will explain the general case in Lemma 31.

From the proof of the Proposition 2 in [2], it follows that

Corollary 1 *Under the conditions of Proposition 2, if there is a path in $\mathcal{G}(A)$ from l to j that passes through a node of the critical graph, then there exists a finite q such that $[A^{\otimes k}]_{j,l} = Q_{j,l}$ for all $k \geq q$. Conversely, if no path from l to j in $\mathcal{G}(A)$ passes through a node of the critical graph, then $\lim_{k\to\infty} [A^{\otimes k}]_{j,l} = \varepsilon$.*

Corollary 1 is important to prove Lemma 31.

2.4 Eigenvalues and Eigenvectors of Square Matrix and The Power Algorithm

In this subsection we give the notion of eigenvalue and corresponding eigenvector of a square matrix A in $\mathbb{R}_{\max}^{n\times n}$. We also give an existing power algorithm to compute the eigenvalues and corresponding eigenvectors.

Definition 3 Let a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ be given. If $\lambda \in \mathbb{R}$ is a scalar and $\boldsymbol{v} \in \mathbb{R}_{\max}^{n}$ is a vector that contains at least one finite element, such that

$$A \otimes \boldsymbol{v} = \lambda \otimes \boldsymbol{v},$$

then λ is called an eigenvalue of A and $\boldsymbol{v}$ is a corresponding eigenvector of A.

In the existing power algorithms the eigenvalues and the corresponding eigenvectors of a square matrix A in $\mathbb{R}_{\max}^{n\times n}$ are determined by using the following recurrence equation:

$$\boldsymbol{x}(k+1) = A \otimes \boldsymbol{x}(k), \; k = 0, 1, 2, \ldots, \tag{2}$$

with a finite initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$. The j-th component of $\boldsymbol{x}(k)$ is denoted by $x_j(k)$ for $j \in \underline{n}$. In the following we give a known power algorithm.

Algorithm 21 (cf. [5])

1. *Choose an arbitrary initial vector $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$.*
2. *Iterate equation (2) until there are integers $p > q \geq 0$ and a real number c such that a periodic behavior occurs, i.e., $\boldsymbol{x}(p) = c \otimes \boldsymbol{x}(q)$.*
3. *Compute the eigenvalue $\lambda = \frac{c}{p-q}$*
4. *Compute a vector as follows*

$$\boldsymbol{v} = \bigoplus_{i=1}^{p-q} \left(\lambda^{\otimes(p-q-i)} \otimes \boldsymbol{x}(q+i-1)\right).$$

5. *Then* $\boldsymbol{v}$ *is an eigenvector of matrix* A *for the eigenvalue* λ.

According to step 2 of Algorithm 21, we can conclude that $\boldsymbol{x}(k) = c \otimes \boldsymbol{x}(k + (q - p))$ for $k \geq p$. Furthermore, when Algorithm 21 is successfully applied to a regular square matrix A its cycle time vector, see below for the definition, must contain equal values. In other words, the algorithm cannot be used to regular square matrices A for which the cycle time vector contain different values. Therefore, in this paper we discuss a generalization of Algorithm 21 which can be used to calculate a so-called generalized eigenmode of any regular square matrix over max-plus algebra by using the recurrence relation in Equation (2).

2.5 Cycle Time Vector and The Generalized Eigenmode

In this subsection, we give the definition of cycle time vector and we recall the generalized eigenmode as introduced in [4] and [6].

Definition 4 Given a regular square matrix $A \in \mathbb{R}_{\max}^{n \times n}$. Using recurrence relation (2) we obtain a sequence $\{\boldsymbol{x}(k) : k \in \mathbb{N}\} \subseteq \mathbb{R}_{\max}^{n}$. Assume that for all $j \in \underline{n}$ the quantity η_j, defined by

$$\eta_j = \lim_{k \to \infty} \frac{x_j(k)}{k},$$

exists. The vector $\chi(A) = (\eta_1, \eta_2, \ldots, \eta_n)^T$, where ^{T} denotes transposition, is called the cycle time vector of the sequence $\{\boldsymbol{x}(k) : k \in \mathbb{N}\}$.

In fact, the cycle time vector is independent of the initial condition. Therefore, $\chi(A)$ is also referred to as the cycle time vector of matrix A. The following proposition describes this independence property. Proposition 4 describes another property of the cycle time vector. Propositions 3 and 4 are used to prove Theorem 1.

Proposition 3 (cf. [6]) *Consider the recurrence relation equation (2). If* $\boldsymbol{x}_0 \in \mathbb{R}^n$ *is a particular initial condition such that the limit* $\lim_{k \to \infty} \frac{x_j(k)}{k}$ *exists for all* $j \in \underline{n}$, *then each these limits exists and has the same value for any initial condition* $\boldsymbol{y}_0 \in \mathbb{R}^n$.

Recall that $[j]$ stands for the m.s.c.s. that node j belongs to. Hence, $\eta_{[j]}$ denotes the cycle time vector corresponding to m.s.c.s. $[j]$. Similarly, $\lambda_{[i]}$ denotes the eigenvalue corresponding to the m.s.c.s. $[i]$, i.e., the eigenvalue of $A_{[i],[i]}$, where the latter is the submatrix of A with rows and columns in $[i]$. Finally, $\pi^*(j)$ stands for the set of nodes i from which there is a path in $\mathcal{G}(A)$ to node j, including node j itself. Below we also use λ_i instead of $\lambda_{[i]}$, where λ_i indicates the maximum among the average weights of all circuits that contain node i.

Proposition 4 (cf. [6]) *Consider the recurrence relation given in* (2) *with a square regular matrix* $A \in \mathbb{R}_{\max}^{n\times n}$ *and an initial condition* $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$. *Let* $\boldsymbol{\eta} = \lim_{k\to\infty} \frac{\boldsymbol{x}(k;\boldsymbol{x}_0)}{k}$ *be the cycle time vector of* A, *where* $\boldsymbol{x}(k,\boldsymbol{x}_0)$ *denotes the value of* $\boldsymbol{x}(k)$ *generated by the recurrence relation* (2) *with* $\boldsymbol{x}(0) = \boldsymbol{x}_0$. *Then for all* $j \in \underline{n}$ *and any* $\boldsymbol{x}_0 \in \mathbb{R}^n$, *we have*

$$\lim_{k\to\infty} \frac{x_j(k,\boldsymbol{x}_0)}{k} = \bigoplus_{i\in\pi^*(j)} \lambda_{[i]},$$

where $x_j(k,\boldsymbol{x}_0)$ *is the* j*-th component of* $\boldsymbol{x}(k,\boldsymbol{x}_0)$. *Furthermore, for* $\mathcal{G}(A) = (\mathcal{V},\mathcal{D})$ *then*

$$\eta_j = \bigoplus_{(i,j)\in\mathcal{D}} \eta_i$$

Next, Definition 5 introduces the generalized eigenmode.

Definition 5 A pair of vectors $(\boldsymbol{\eta},\boldsymbol{v}) \in \mathbb{R}^n \times \mathbb{R}^n$ is called a generalized eigenmode of a regular matrix $A \in \mathbb{R}_{\max}^{n\times n}$ if

$$A \otimes (m \times \boldsymbol{\eta} + \boldsymbol{v}) = (m+1) \times \boldsymbol{\eta} + \boldsymbol{v}, \text{ for all } m \geq 0. \tag{3}$$

Moreover, we refer to the vectors $\boldsymbol{\eta}$ and $\boldsymbol{v}$ as the (generalized) eigenvalue and eigenvector, respectively, of a generalized eigenmode. Existence of a generalized eigenmode of a regular matrix $A \in \mathbb{R}_{\max}^{n\times n}$ can be found in [4]. The form of the eigenvalue of a generalized eigenmode is described below.

Lemma 21 (cf. [4]) *If a regular matrix* $A \in \mathbb{R}_{\max}^{n\times n}$ *has a generalized eigenmode* $(\boldsymbol{\eta},\boldsymbol{v})$, *then we have* $\chi(A) = \boldsymbol{\eta}$.

3 Results and Discussions

In this section we discuss some lemmas and theorems and in the last part we state the generalized power algorithm as our main result. First we discuss Lemma 31, Lemma 32, Theorem 1, and Theorem 2. They will become the basis of the second step of Algorithm 31 (our main result). Before we give the lemmas and the theorems, we investigate the example below.

Example 1 Given the matrix

$$A = \begin{bmatrix} 15 & \varepsilon & \varepsilon \\ 9 & 19 & 15 \\ 20 & 5 & 14 \end{bmatrix} \text{ with cycle time vector } \boldsymbol{\eta} = \begin{bmatrix} 15 \\ 19 \\ 19 \end{bmatrix}.$$

In this example, we choose some initial conditions to observe what we can expect for the generalized eigenmode (containing the cycle time vector) by iterating Equation (2) a finite number of times. This observation is inspired by step 2 of Algorithm 21.

- Choosing initial vector $\boldsymbol{x}(0) = [14, 27, 7]^T$, we get

$$\boldsymbol{x}(3) - \boldsymbol{x}(2) = \boldsymbol{x}(4) - \boldsymbol{x}(3) = \boldsymbol{x}(5) - \boldsymbol{x}(4) = [15, 19, 19]^T = \boldsymbol{\eta}.$$

- Choosing initial vector $\boldsymbol{x}(0) = [2, 8, 6]^T$, we get

$$\boldsymbol{x}(5) - \boldsymbol{x}(4) = \boldsymbol{x}(6) - \boldsymbol{x}(5) = \boldsymbol{x}(7) - \boldsymbol{x}(6) = [15, 19, 19]^T = \boldsymbol{\eta}.$$

- Choosing initial vector $\boldsymbol{x}(0) = [5, 19, 5]^T$, we get

$$\boldsymbol{x}(3) - \boldsymbol{x}(1) = \boldsymbol{x}(4) - \boldsymbol{x}(2) = \boldsymbol{x}(5) - \boldsymbol{x}(3) = [30, 38, 38]^T = 2 \times \boldsymbol{\eta}.$$

According to the above initial conditions, there are natural numbers N and l such that $\boldsymbol{x}(m + l) = l \times \boldsymbol{\eta} + \boldsymbol{x}(m)$, for $m = N, N + 1, N + 2$. A more general version of this statement will be proved in Theorem 1. Before we discuss the theorem, we write two lemmas which help to prove Theorem 1.

Lemma 31 *Given the recurrence relation in (2) for a regular square matrix $A \in \mathbb{R}_{\max}^{n\times n}$ such that the m.c.m. (maximum cycle mean) of $\mathcal{G}(A)$ is equal to e and the cyclicity of A is one. If for a node j there is a path from all (other) nodes to node j, then there exists a positive integer q such that*

$$x_j(k + 1) = x_j(k),$$

for all $k \geq q$ and any initial condition $\boldsymbol{x}(0)$ in $\mathbb{R}^n$. More general, if the m.c.m. of $\mathcal{G}(A)$ is equal to $\lambda \in \mathbb{R}$, then

$$x_j(k + 1) = \lambda \otimes x_j(k),$$

for all $k \geq q$ and any $\boldsymbol{x}(0)$ in $\mathbb{R}^n$.

Proof: Let $\mathcal{H}_j$ denotes the set of the nodes l from which there is a path to node j that passes through a node of the critical graph. Recall that $[A^{\otimes k}]_{j,l}$ equals the (j, l)-th element of $A^{\otimes k}$. According to Corollary 1 it follows that there exists a finite q such that

$$[A^{\otimes q}]_{j,l} \otimes x_l(0) = [A^{\otimes k}]_{j,l} \otimes x_l(0),$$

for all $l \in \mathcal{H}_j$ and for all $k \geq q$. Then also for all $k \geq q$

$$\bigoplus_{l\in\mathcal{H}_j} [A^{\otimes q}]_{j,l} \otimes x_l(0) = \bigoplus_{l\in\mathcal{H}_j} [A^{\otimes k}]_{j,l} \otimes x_l(0).$$

Moreover, the integer q can be chosen such that

$$[A^{\otimes k}]_{j,m} \otimes x_m(0) < \bigoplus_{l\in\mathcal{H}_j} [A^{\otimes q}]_{j,l} \otimes x_l(0) = \bigoplus_{l\in\mathcal{H}_j} [A^{\otimes k}]_{j,l} \otimes x_l(0),$$

for all $m \notin \mathcal{H}_j$ and for all $k \geq q$. Hence, there exists a finite q such that for all $k \geq q$

$$\begin{aligned} x_j(k) &= \bigoplus_{l=1}^{n} [A^{\otimes k}]_{j,l} \otimes x_l(0) = \bigoplus_{l \in \mathcal{H}_j} [A^{\otimes k}]_{j,l} \otimes x_l(0) \\ &= \bigoplus_{l \in \mathcal{H}_j} [A^{\otimes q}]_{j,l} \otimes x_l(0) = \bigoplus_{l=1}^{n} [A^{\otimes q}]_{j,l} \otimes x_l(0), \end{aligned}$$

implying that for all $k \geq q$

$$x_j(k+1) = x_j(k).$$

More general, if the m.c.m. of $\mathcal{G}(A)$ is equal to λ and $B = \lambda^{\otimes -1} \otimes A$, then the m.c.m. of $\mathcal{G}(B)$ is equal to zero. Given the recurrence relation $\boldsymbol{y}(k+1) = B \otimes \boldsymbol{y}(k)$ for $k \geq 0$ and $\boldsymbol{y}(0) = \boldsymbol{x}(0)$, there exists a positive integer q_1 such that $y_j(k+1) = y_j(k)$ for all $k \geq q_1$. Hence, for all $k \geq q_1$

$$\begin{aligned} \bigoplus_{l=1}^{n} [B^{\otimes k+1}]_{j,l} \otimes y_l(0) &= \bigoplus_{l=1}^{n} [B^{\otimes k}]_{j,l} \otimes y_l(0), \\ \bigoplus_{l=1}^{n} [B^{\otimes k+1}]_{j,l} \otimes x_l(0) &= \bigoplus_{l=1}^{n} [B^{\otimes k}]_{j,l} \otimes x_l(0), \\ \bigoplus_{l=1}^{n} \lambda^{\otimes -k-1} \otimes [A^{\otimes k+1}]_{j,l} \otimes x_l(0) &= \bigoplus_{l=1}^{n} \lambda^{\otimes -k} \otimes [A^{\otimes k}]_{j,l} \otimes x_l(0), \\ \lambda^{\otimes -k-1} \otimes \left(\bigoplus_{l=1}^{n} [A^{\otimes k+1}]_{j,l} \otimes x_l(0) \right) &= \lambda^{\otimes -k} \otimes \left(\bigoplus_{l=1}^{n} [A^{\otimes k}]_{j,l} \otimes x_l(0) \right), \\ \lambda^{\otimes -k-1} \otimes x_j(k+1) &= \lambda^{\otimes -k} \otimes x_j(k), \\ x_j(k+1) &= \lambda \otimes x_j(k). \end{aligned}$$

□

One of the conditions in Lemma 31 is that the cyclicity of the regular square matrix A is one. In fact, the cyclicity of the matrix A can be greater one. To handle this problem, we derive the lemma below which is an extension of Proposition 1.

Lemma 32 *Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a square regular matrix and suppose that its cyclicity is equal to τ, i.e., $\sigma(A) = \tau$. Then the cyclicity of matrix $A^{\otimes \tau}$ is equal to one.*

Proof: Recall that by renumbering the nodes in the graph $\mathcal{G}(A)$, matrix A can be transformed into an upper triangular block form, called a normal form of A, given by the matrix in (1) with the condition that for $i \in \underline{q}$,

– either $A_{i,i}$ is an irreducible matrix with cyclicity $\sigma(A_{i,i})$, or

– $A_{i,i} = \varepsilon$ with cyclicity $\sigma(A_{i,i}) = 1$.

By Definition 1, given

$$\tau = \text{l.c.m.}(\sigma(A_{1,1}), \sigma(A_{2,2}), \ldots, \sigma(A_{q,q})).$$

Therefore, we obtain

$$B = A^{\otimes\tau} = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & \cdots & B_{1,q} \\ \mathcal{E} & B_{2,2} & \cdots & \cdots & B_{2,q} \\ \mathcal{E} & \mathcal{E} & B_{3,3} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} & B_{q,q} \end{pmatrix}.$$

It follows easily that $B_{i,i} = A_{ii}^{\otimes\tau}$. Since $\mathcal{G}(A_{i,i})$ is strongly connected (=irreducible), it follows from Proposition 1 that $\sigma(B_{i,i}) = 1$. Then we conclude that

$$\sigma(A^{\otimes\,\tau}) = \sigma(B) = \text{l.c.m.}(\sigma(B_{1,1}), \sigma(B_{2,2}), \ldots, \sigma(B_{q,q})) = 1.$$

□

Theorem 1 *Let a square regular matrix $A \in \mathbb{R}_{\max}^{n\times n}$ be given and consider the recurrence relation in Equation* (2)*. Let $\boldsymbol{x}(0) \in \mathbb{R}^n$ be a given arbitrary vector, then there exist natural numbers N, l such that $\boldsymbol{x}(m+l) = l \times \boldsymbol{\eta} + \boldsymbol{x}(m)$ for all $m \geq N$, where $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)^T$ is the cycle time vector of A.*

Proof: Let $\sigma = \sigma(A)$ and write $B = A^{\otimes\sigma}$. By Lemma 32, the cyclicity of B is one. Take an arbitrary $t \in \{0, 1, 2, \ldots, \sigma-1\}$ and consider the recurrence relation $\boldsymbol{y}(k+1) = B \otimes \boldsymbol{y}(k)$, where $\boldsymbol{y}(0) = \boldsymbol{x}(t)$. If node $i \in \underline{n}$ is contained in some circuit in $\mathcal{G}(B)$, write λ_i for the maximum among the average weights of all circuits that contain node i. If this node is not contained in any circuit in $\mathcal{G}(B)$, then write $\lambda_i = \varepsilon$. According to Proposition 4, it follows that

$$\overline{\eta}_j = \bigoplus_{i\in\pi^*(j)} \lambda_i, \quad \forall j \in \underline{n},$$

where $\overline{\eta}_j$ is the cycle time vector of B and $\lambda_{[i]}$ is replaced by λ_i (see the remark below Proposition 3). Hence, it follows that $\overline{\eta}_j \geq \lambda_i$ for all $i \in \pi^*(j)$, and that equality holds for at least one $i \in \pi^*(j)$, i.e., $\overline{\eta}_j = \lambda_i$ for at least one $i \in \pi^*(j)$.

Clearly, $\overline{\eta}_j$ equals the maximum of the average weight of all circuits in $\mathcal{G}(B)$ that are upstream of node j. Since by the regularity of A there is at least one circuit upstream of node j, it follows that $\overline{\eta}_j$ is finite.

Now, take some $j \in \underline{n}$ and restrict the graph $\mathcal{G}(B)$ to the subgraph made up of all nodes in $\pi^*(j)$ and the corresponding edges. Denote this subgraph by $\mathcal{G}(\overline{B})$, where $\overline{B}$ is the matrix made up of the corresponding rows and columns of B. Note that j is a node of $\mathcal{G}(\overline{B})$ since $j \in \pi^*(j)$. Then it follows that there is a path from every node in $\mathcal{G}(\overline{B})$ to node j. It follows from Proposition 4 that the maximal average circuit weight in $\mathcal{G}(\overline{B})$ is equal to $\overline{\eta}_j$.

By Lemma 31, there exists a finite positive integer q_j such that

$$y_j(k+1) = \overline{\eta}_j \otimes y_j(k), \text{ for all } k \geq q_j. \tag{4}$$

Hence, given any $j \in \underline{n}$, we have obtained a real number $\overline{\eta}_j$ and finite positive integer q_j such that equation (4) is true for all $k \geq q_j$. Repeating the previous steps for all $j \in \underline{n}$ and taking $q = \max\{q_1, q_2, \ldots, q_n\}$, it follows (easily) that for all $k \geq q$

$$\boldsymbol{y}(k+1) = \overline{\boldsymbol{\eta}} + \boldsymbol{y}(k),$$

where $\overline{\boldsymbol{\eta}} = (\overline{\eta}_1, \overline{\eta}_2, \ldots, \overline{\eta}_n)^T$. Then we also obtain that for all $k \geq q$

$$\begin{aligned}
\boldsymbol{x}(k\sigma + t + \sigma) &= x(t + (k+1)\sigma) \\
&= A^{\otimes (k+1)\sigma} \otimes \boldsymbol{x}(t) \\
&= B^{\otimes (k+1)} \otimes \boldsymbol{y}(0) \\
&= \boldsymbol{y}(k+1) \\
&= \overline{\boldsymbol{\eta}} + \boldsymbol{y}(k) \\
&= \overline{\boldsymbol{\eta}} + B^{\otimes k} \otimes \boldsymbol{y}(0) \\
&= \overline{\boldsymbol{\eta}} + A^{\otimes k\sigma} \otimes \boldsymbol{y}(0) \\
&= \overline{\boldsymbol{\eta}} + A^{\otimes k\sigma} \otimes \boldsymbol{x}(t) \\
&= \overline{\boldsymbol{\eta}} + \boldsymbol{x}(k\sigma + t).
\end{aligned}$$

It is easy to check that

$$\boldsymbol{\eta} = \lim_{k \to \infty} \frac{\boldsymbol{x}(k)}{k} = \frac{1}{\sigma} \overline{\boldsymbol{\eta}}.$$

Finally, we can conclude that there exist integers $N = q\sigma, l = \sigma$ such that $\boldsymbol{x}(m+l) = l \times \boldsymbol{\eta} + \boldsymbol{x}(m)$, for all $m \geq N$. □

Theorem 1 is not sufficient to obtain the eigenvalue (part) of the generalized eigenmode, because we can not check it for all $m \geq N$. To solve this problem, we derive Theorem 2 in which $\boldsymbol{x}(q)$ is written as $\boldsymbol{v}$, where q is a natural number.

Theorem 2 *Let a square regular matrix $A \in \mathbb{R}_{\max}^{n \times n}$ be given and let $B \in \mathbb{R}_{\max}^{n \times n}$ be the matrix defined by*

$$[B]_{i,j} (= b_{i,j}) = \begin{cases} 0, & \text{if } [A]_{i,j} \neq \varepsilon, \\ \varepsilon, & \text{if } [A]_{i,j} = \varepsilon. \end{cases}$$

If l is a positive integer and $\boldsymbol{\psi}, \boldsymbol{v} \in \mathbb{R}^n$ are vectors such that the following equations hold

$$A^{\otimes l} \otimes \boldsymbol{v} = \boldsymbol{\psi} + \boldsymbol{v}, \tag{5}$$

$$A^{\otimes l} \otimes (A^{\otimes l} \otimes \boldsymbol{v}) = \boldsymbol{\psi} + (A^{\otimes l} \otimes \boldsymbol{v}), \tag{6}$$

$$B \otimes \boldsymbol{\psi} = \boldsymbol{\psi}, \tag{7}$$

then we obtain

$$A^{\otimes l} \otimes (m \times \boldsymbol{\psi} + (A^{\otimes l} \otimes \boldsymbol{v})) = (m+1) \times \boldsymbol{\psi} + (A^{\otimes l} \otimes \boldsymbol{v}),$$

for any real number $m \geq 0$. □

It is clear that with $\boldsymbol{v} = \boldsymbol{x}(q)$, we can rewrite Equation (5), (6), and (7) in Theorem 2 as

$$\begin{aligned} \boldsymbol{x}(q+l) &= \boldsymbol{\psi} + \boldsymbol{x}(q), \\ \boldsymbol{x}(q+2l) &= \boldsymbol{\psi} + \boldsymbol{x}(q+l) \\ B \otimes \boldsymbol{\psi} &= \boldsymbol{\psi}, \end{aligned}$$

or we can combine first and second equation, so that we obtain

$$\begin{aligned} \boldsymbol{x}(q+l) - \boldsymbol{x}(q) = \boldsymbol{x}(q+2l) - \boldsymbol{x}(q+l) &= \boldsymbol{\psi}, \\ B \otimes \boldsymbol{\psi} &= \boldsymbol{\psi}. \end{aligned}$$

Proof of Theorem 2: Note that from the definitions it follows that $B^{\otimes l} \otimes \boldsymbol{\psi} = \boldsymbol{\psi}$. By equations (5) and (6), we have

$$A^{\otimes l} \otimes (\boldsymbol{\psi} + \boldsymbol{v}) = \boldsymbol{\psi} + (A^{\otimes l} \otimes \boldsymbol{v}),$$

or, for $i \in \underline{n}$ we can write

$$\bigoplus_{j=1}^{n} ([A^{\otimes l}]_{i,j} + \psi_j + v_j) = \bigoplus_{j=1}^{n} (\psi_i + [A^{\otimes l}]_{i,j} + v_j), \tag{8}$$

where ψ_j, v_j are the j-th component of $\boldsymbol{\psi}$ and $\boldsymbol{v}$, respectively. Recall that we write $[A^{\otimes l}]_{i,j}$ for the component of matrix $A^{\otimes l}$ in i-th row and j-th column. Recall that $\pi^*(i)$ is the set of indices j such that in the graph $\mathcal{G}(A)$ there is a path from node j to node i. We can rewrite equation (8) then as

$$\bigoplus_{j \in \pi^*(i)} ([A^{\otimes l}]_{i,j} + \psi_j + v_j) = \bigoplus_{j \in \pi^*(i)} (\psi_i + [A^{\otimes l}]_{i,j} + v_j), \tag{9}$$

According to equation (7) and $B^{\otimes l} \otimes \boldsymbol{\psi} = \boldsymbol{\psi}$, we obtain

$$\psi_i = \bigoplus_{j \in \pi^*(i)} \psi_j. \tag{10}$$

Next, denote by $\overline{\pi}^*(i) = \{j \in \pi^*(i) | \psi_i = \psi_j\}$ and take arbitrary $s \notin \overline{\pi}^*(i)$, but $s \in \pi^*(i)$, then by equation (9) and (10), we obtain

$$\begin{aligned} [A^{\otimes l}]_{i,s} + \psi_s + v_s &< [A^{\otimes l}]_{i,s} + \psi_i + v_s \\ &\leq \bigoplus_{j \in \pi^*(i)} \big(\psi_i + [A^{\otimes l}]_{i,j} + v_j\big) = \bigoplus_{j \in \pi^*(i)} \big(\psi_j + [A^{\otimes l}]_{i,j} + v_j\big). \end{aligned}$$

Thus for all $i \in \underline{n}$,

$$\bigoplus_{j \in \pi^*(i)} \big([A^{\otimes l}]_{i,j} + \psi_j + v_j\big) = \bigoplus_{j \in \overline{\pi}^*(i)} \big([A^{\otimes l}]_{i,j} + \psi_j + v_j\big). \tag{11}$$

Therefore, there exists a $t \in \overline{\pi}^*(i)$ such that

$$[A^{\otimes l}]_{i,s} + \psi_s + v_s < [A^{\otimes l}]_{i,t} + \psi_t + v_t.$$

Since $m\psi_s \leq m\psi_t$ for any real number $m \geq 0$, it follows that

$$[A^{\otimes l}]_{i,s} + (m+1)\psi_s + v_s < [A^{\otimes l}]_{i,t} + (m+1)\psi_t + v_t,$$

such that in the same way as (11) is obtained

$$\begin{aligned}\bigoplus_{j\in\pi^*(i)} \left([A^{\otimes l}]_{i,j} + \psi_j^{\otimes\, m+1} + v_j\right) &= \bigoplus_{j\in\overline{\pi}^*(i)} \left([A^{\otimes l}]_{i,j} + \psi_j^{\otimes\, m+1} + v_j\right) \\ &= \bigoplus_{j\in\overline{\pi}^*(i)} \left([A^{\otimes l}]_{i,j} + \psi_i^{\otimes\, m} + \psi_j + v_j\right) \\ &= \psi_i^{\otimes\, m} + \bigoplus_{j\in\overline{\pi}^*(i)} \left([A^{\otimes l}]_{i,j} + \psi_j + v_j\right). \quad (12)\end{aligned}$$

Substituting (11) into (12), we can rewrite (12) as

$$\bigoplus_{j\in\pi^*(i)} \left([A^{\otimes l}]_{i,j} + \psi_j^{\otimes\, m+1} + v_j\right) = \psi_i^{\otimes\, m} + \bigoplus_{j\in\pi^*(i)} \left([A^{\otimes l}]_{i,j} + \psi_j + v_j\right). \quad (13)$$

Substituting (9) into (13), it follows that

$$\begin{aligned}\bigoplus_{j\in\pi^*(i)} \left([A^{\otimes l}]_{i,j} + \psi_j^{\otimes\, m+1} + v_j\right) &= \psi_i^{\otimes\, m} + \bigoplus_{j\in\pi^*(i)} \left([A^{\otimes l}]_{i,j} + \psi_i + v_j\right) \\ &= \psi_i^{\otimes\, m+1} + \bigoplus_{j\in\pi^*(i)} \left([A^{\otimes l}]_{i,j} + v_j\right).\end{aligned}$$

Finally, from the last equation, we obtain that

$$A^{\otimes\, l} \otimes \left((m+1) \times \boldsymbol{\psi} + \boldsymbol{v}\right) = (m+1) \times \boldsymbol{\psi} + (A^{\otimes\, l} \otimes \boldsymbol{v}).$$

Using (5) we get

$$A^{\otimes l} \otimes \left(m \times \boldsymbol{\psi} + (A^{\otimes\, l} \otimes \boldsymbol{v})\right) = (m+1) \times \boldsymbol{\psi} + (A^{\otimes\, l} \otimes \boldsymbol{v}). \quad (14)$$

for any real number $m \geq 0$. □

Note:

- For integer m and $\boldsymbol{v} = \boldsymbol{x}(q)$, with q a natural number, we can rewrite Equation (14) as

  $$\boldsymbol{x}((m+2)l+q) - \boldsymbol{x}((m+1)l+q) = \boldsymbol{\psi},$$

  for all integer $m \geq 0$. By (5) it follows from (14) that

  $$A^{\otimes l} \otimes ((m+1)\boldsymbol{\psi} + \boldsymbol{v}) = (m+2)\boldsymbol{\psi} + \boldsymbol{v}.$$

  It follows now by mathematical induction using (5) that

  $$x((m+1)l+q) = (m+1)\boldsymbol{\psi} + \boldsymbol{v}$$

  for any integer $m \geq 0$. With this the required identity follows.

- Matrix B in Theorem 2 is inspired by the equation

$$\psi_j = \bigoplus_{(i,j)\in\mathcal{D}} \psi_i,$$

in Proposition 4, which we can rewrite as Equation (7).

Corollary 2 *Under the condition of Theorem 2, if* $l = 1$ *then* $(\boldsymbol{\psi}, A \otimes \boldsymbol{v})$ *is a generalized eigenmode of* A*.*

Proof: According to (14), it is clear that $(\boldsymbol{\psi}, A\otimes\boldsymbol{v})$ is a generalized eigenmode of A. □

Next, we discuss Lemma 33. It will become basis of the third step of Algorithm 31 and will be used to prove Lemma 34. Note that vector $\boldsymbol{w}$ in Equation (15), see below, is inspired by vector $\boldsymbol{v}$ in Algorithm 21. Before we give the lemma, we investigate the example below.

Example 2 Given matrix

$$A = \begin{bmatrix} 15 & \varepsilon & \varepsilon & 19 & \varepsilon & 13 & \varepsilon & \varepsilon & \varepsilon \\ 9 & 19 & 15 & \varepsilon & 16 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 20 & 5 & 14 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 17 & \varepsilon & \varepsilon & 10 & 13 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 2 & 5 & 7 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 & 7 & 1 & \varepsilon & \varepsilon & \varepsilon \\ 7 & \varepsilon & \varepsilon & \varepsilon & 9 & \varepsilon & 18 & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & 16 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 13 & 14 & 20 \end{bmatrix} \text{ and initial vector } \boldsymbol{x}(0) = \begin{bmatrix} 31 \\ -17 \\ 22 \\ -22 \\ -138 \\ -121 \\ 52 \\ -24 \\ 13 \end{bmatrix}.$$

Recall that matrix B can be obtained from matrix A by replacing in the latter all finite entries by zero. Iterating Equation (2), we obtain $\boldsymbol{x}(0), \boldsymbol{x}(1), \boldsymbol{x}(2), \boldsymbol{x}(3), \boldsymbol{x}(4), \boldsymbol{x}(5)$ and $\boldsymbol{x}(6)$ with $\boldsymbol{x}(4) - \boldsymbol{x}(2) \neq \boldsymbol{x}(6) - \boldsymbol{x}(4)$, and $B \otimes (\boldsymbol{x}(4) - \boldsymbol{x}(2)) \neq \boldsymbol{x}(4) - \boldsymbol{x}(2)$. Hence, this iteration does not yet satisfy the conditions in Theorem 2 and has to be continued with a next iteration. After some iterations, we obtain $q = 16$ and $l = 2$, where $\boldsymbol{\psi} = \boldsymbol{x}(16+4) - \boldsymbol{x}(16+2) = \boldsymbol{x}(16+2) - \boldsymbol{x}(16) = [36, 38, 38, 36, 36, 36, 36, 40, 40]^T$ and $B \otimes \boldsymbol{\psi} = \boldsymbol{\psi}$. According to Theorem 2, we obtain

$$\boldsymbol{x}(16 + 2(p + 1)) = \boldsymbol{x}(16) + (p + 1)\boldsymbol{\psi},$$

for all $p \geq 0$. Hence, for $j \in \underline{n}$ we obtain, with $l = 2$,

$$\begin{aligned}
\eta_j &= \lim_{k\to\infty} \frac{x_j(k)}{k} \\
&= \lim_{p\to\infty} \frac{x_j(16+2(p+1))}{16+2(p+1)} \\
&= \lim_{p\to\infty} \frac{(p+1)\psi_j + x_j(16)}{q+2(p+1)} \\
&= \lim_{p\to\infty} \frac{(p+1)\psi_j}{16+2(p+1)} + \frac{x_j(16)}{16+2(p+1)} \\
&= \frac{\psi_j}{l}, \text{since } l = 2.
\end{aligned}$$

So, $\boldsymbol{\eta} = \frac{1}{2}\boldsymbol{\psi} = [18, 19, 19, 18, 18, 18, 18, 20, 20]^T$. Then we compute $\boldsymbol{v} = (\boldsymbol{\eta} + \boldsymbol{x}(17)) \oplus \boldsymbol{x}(18)$ by (15). In Algorithm 21, vector $\boldsymbol{v}$ is eigenvector, but in this example, vector $\boldsymbol{v}$ is not eigenvector, because $(A \otimes \boldsymbol{v}) - \boldsymbol{v} \neq \boldsymbol{\eta}$. Actually, a (generalized) eigenvector of A is given by $A^{\otimes 3} \otimes \boldsymbol{v}$, because $(A^{\otimes 4} \otimes \boldsymbol{v}) - (A^{\otimes 3} \otimes \boldsymbol{v}) = (A^{\otimes 3} \otimes \boldsymbol{v}) - (A^{\otimes 2} \otimes \boldsymbol{v}) = \boldsymbol{\eta}$ (see Corollary 2). More details on the procedure to obtain the eigenvalue are given in Lemma 33. The lemma also gives a property of the vector $\boldsymbol{w}$, defined in (15), playing a similar role as vector $\boldsymbol{v}$ in Algorithm 21. The vector $\boldsymbol{w}$ is used in Lemma 34.

Lemma 33 *Let a square regular matrix $A \in \mathbb{R}_{\max}^{n\times n}$ be given and consider the recurrence relation in equation (2). Suppose that for arbitrary $\boldsymbol{x}(0) \in \mathbb{R}^n$, there exist positive integers q, l and a vector $\boldsymbol{\psi} \in \mathbb{R}^n$ such that $\boldsymbol{x}(q+2l) - \boldsymbol{x}(q+l) = \boldsymbol{x}(q+l) - \boldsymbol{x}(q)$, and $B \otimes \boldsymbol{\psi} = \boldsymbol{\psi}$, where $\boldsymbol{\psi} = \boldsymbol{x}(q+l) - \boldsymbol{x}(q)$. Then the cycle time vector of A is $\boldsymbol{\eta} = \lim_{k\to\infty} \frac{1}{k}\boldsymbol{x}(k) = \frac{1}{l}\,\boldsymbol{\psi}$. Next, define*

$$\boldsymbol{w} = \bigoplus_{i=1}^{l} [(l-i)\boldsymbol{\eta} + \boldsymbol{x}(q+i-1)] \tag{15}$$

then $A^{\otimes m} \otimes \boldsymbol{w} \leq \boldsymbol{\eta} + (A^{\otimes m-1} \otimes \boldsymbol{w})$ for all integer $m \geq 0$.

Proof: From Proposition 4 it follows that the cycle time vector $\boldsymbol{\eta} = \lim_{k\to\infty} \frac{1}{k}\boldsymbol{x}(k)$ exists, independently of the initial condition. From $\boldsymbol{x}(q+2l) - \boldsymbol{x}(q+l) = \boldsymbol{x}(q+l) - \boldsymbol{x}(q) = \boldsymbol{\psi}$ and $B \otimes \boldsymbol{\psi} = \boldsymbol{\psi}$, it follows by Theorem 2 that $\boldsymbol{x}(q+(p+1)l) =$

$\boldsymbol{x}(q) + (p+1)\boldsymbol{\psi}$ for all $p \geq 0$. Hence, for $j \in \underline{n}$ we obtain

$$\begin{aligned}
\eta_j &= \lim_{k\to\infty} \frac{x_j(k)}{k} \\
&= \lim_{p\to\infty} \frac{x_j(q+(p+1)l)}{q+(p+1)l} \\
&= \lim_{p\to\infty} \frac{(p+1)\psi_j + x_j(q)}{q+(p+1)l} \\
&= \lim_{p\to\infty} \frac{(p+1)\psi_j}{q+(p+1)l} + \frac{x_j(q)}{q+(p+1)l} \\
&= \frac{\psi_j}{l}.
\end{aligned}$$

Next, by induction on m we prove that $A^{\otimes m} \otimes \boldsymbol{w} \leq \boldsymbol{\eta} + \left(A^{\otimes m-1} \otimes \boldsymbol{w}\right)$ for all integer $m \geq 0$. First, we will prove that $A \otimes \boldsymbol{w} \leq \boldsymbol{\eta} + \boldsymbol{w}$. Recall the notation that $[B]_{i,j}$ or $b_{i,j}$ is the component of $B \in \mathbb{R}_{\max}^{n\times n}$ in the i-th row and j-th column, and $[\boldsymbol{x}]_j$ or x_j is the j-th component of vector $\boldsymbol{x} \in \mathbb{R}^n$. For $j \in \underline{n}$, we can write

$$\begin{aligned}
[A \otimes \boldsymbol{w}]_j &= \bigoplus_{t=1}^{n} a_{j,t} \otimes \Big(\bigoplus_{i=1}^{l} ((l-i)\eta_t + x_t(q+i-1)) \Big) \\
&= \bigoplus_{t=1}^{n} \bigoplus_{i=1}^{l} \big(a_{j,t} + (l-i)\eta_t + x_t(q+i-1)\big) \\
&= \bigoplus_{i=1}^{l} \bigoplus_{t=1}^{n} \big(a_{j,t} + (l-i)\eta_t + x_t(q+i-1)\big) \\
&= \bigoplus_{i=1}^{l} \bigoplus_{(t,j)\in\mathcal{D}} \big(a_{j,t} + (l-i)\eta_t + x_t(q+i-1)\big)
\end{aligned}$$

where $\mathcal{D} = \{(i,j) | [A]_{j,i} \neq \varepsilon\}$ is the set of edges of graph $\mathcal{G}(A)$. Since $\eta_j = \bigoplus_{(t,j)\in\mathcal{D}} \eta_t$ and $l - i \geq 0$, it follows that

$$\begin{aligned}
[A \otimes \boldsymbol{w}]_j &\leq \bigoplus_{i=1}^{l} \bigoplus_{(t,j)\in\mathcal{D}} \big(a_{j,t} + (l-1)\eta_j + x_t(q+i-1)\big) \\
&= \bigoplus_{i=1}^{l} \Big((l-1)\eta_j + \bigoplus_{(t,j)\in\mathcal{D}} \big(a_{j,t} + x_t(q+i-1)\big)\Big) \\
&= \bigoplus_{i=1}^{l} \Big((l-1)\eta_j + \bigoplus_{t=1}^{n} \big(a_{j,t} + x_t(q+i-1)\big)\Big)
\end{aligned}$$

or we can rewrite

$$\begin{aligned}
A \otimes \boldsymbol{w} &\leq \bigoplus_{i=1}^{l} \big((l-i)\boldsymbol{\eta} + (A \otimes \boldsymbol{x}(q+i-1))\big) \\
&= \bigoplus_{i=1}^{l} \big((l-i)\boldsymbol{\eta} + \boldsymbol{x}(q+i)\big) \\
&= \bigoplus_{j=2}^{l+1} \big((l-j+1)\boldsymbol{\eta} + \boldsymbol{x}(q+j-1)\big) \\
&= \boldsymbol{\eta} + \Big(\bigoplus_{j=2}^{l+1} \big((l-j)\boldsymbol{\eta} + \boldsymbol{x}(q+j-1)\big)\Big) \\
&= \boldsymbol{\eta} + \Big(\bigoplus_{j=2}^{l} \big((l-j)\boldsymbol{\eta} + \boldsymbol{x}(q+j-1)\big) \oplus \big((-1)\boldsymbol{\eta} + \boldsymbol{x}(q+l)\big)\Big) \\
&= \boldsymbol{\eta} + \Big(\bigoplus_{j=2}^{l} \big((l-j)\boldsymbol{\eta} + \boldsymbol{x}(q+j-1)\big) \oplus \big((-1)\boldsymbol{\eta} + l\boldsymbol{\eta} + \boldsymbol{x}(q)\big)\Big) \\
&= \boldsymbol{\eta} + \Big(\bigoplus_{j=2}^{l} \big((l-j)\boldsymbol{\eta} + \boldsymbol{x}(q+j-1)\big) \oplus \big((l-1)\boldsymbol{\eta} + \boldsymbol{x}(q)\big)\Big) \\
&= \boldsymbol{\eta} + \Big(\bigoplus_{j=1}^{l} \big((l-j)\boldsymbol{\eta} + \boldsymbol{x}(q+j-1)\big)\Big) = \boldsymbol{\eta} + \boldsymbol{w}
\end{aligned}$$

Hence, we obtain $A \otimes \boldsymbol{w} \leq \boldsymbol{\eta} + \boldsymbol{w}$. Next, assume that m is an integer such that $A^{\otimes m} \otimes \boldsymbol{w} \leq \boldsymbol{\eta} + (A^{\otimes m-1} \otimes \boldsymbol{w})$. Then we have

$$\begin{aligned}
A^{\otimes m+1} \otimes \boldsymbol{w} &= A \otimes A^{\otimes m} \otimes \boldsymbol{w} \\
&\leq A \otimes (\boldsymbol{\eta} + (A^{\otimes m-1} \otimes \boldsymbol{w})).
\end{aligned}$$

Thus for all $j \in \underline{n}$, we can write

$$\begin{aligned}
[A^{\otimes m+1} \otimes \boldsymbol{w}]_j &\leq \bigoplus_{t=1}^{n} \big(a_{j,t} + \eta_t + [A^{\otimes m-1} \otimes \boldsymbol{w}]_t\big) \\
&= \bigoplus_{(t,j)\in\mathcal{D}} \big(a_{j,t} + \eta_t + [A^{\otimes m-1} \otimes \boldsymbol{w}]_t\big).
\end{aligned}$$

Because $\eta_j = \bigoplus_{(t,j)\in\mathcal{D}} \eta_t$, it follows that

$$\begin{aligned}
[A^{\otimes\, m+1} \otimes \boldsymbol{w}]_j &\le \bigoplus_{(t,j)\in\mathcal{D}} \big(a_{j,t} + \eta_t + [A^{\otimes\, m-1} \otimes \boldsymbol{w}]_t\big) \\
&\le \bigoplus_{(t,j)\in\mathcal{D}} \big(a_{j,t} + \eta_j + [A^{\otimes\, m-1} \otimes \boldsymbol{w}]_t\big) \\
&= \eta_j + \bigoplus_{(t,j)\in\mathcal{D}} \big(a_{j,t} + [A^{\otimes\, m-1} \otimes \boldsymbol{w}]_t\big) \\
&= \eta_j + \bigoplus_{t=1}^{n} \big(a_{j,t} + [A^{\otimes\, m-1} \otimes \boldsymbol{w}]_t\big),
\end{aligned}$$

and hence we obtain that

$$A^{\otimes\, m+1} \otimes \boldsymbol{w} \le \boldsymbol{\eta} + (A \otimes A^{\otimes\, m-1}\boldsymbol{w}) = \boldsymbol{\eta} + (A^{\otimes\, m} \otimes \boldsymbol{w}),$$

which concludes the proof. □

Next, we discuss Lemma 34. It will become the basis of Step 5 to 7 in Algorithm 31. This algorithm explains how to obtain an eigenvector of the generalized eigenmode from the initial condition $\boldsymbol{x}(0) = \boldsymbol{w}$ and iteration of Equation (2).

Lemma 34 *Let a square regular matrix $A \in \mathbb{R}_{\max}^{n\times n}$ be given and consider the recurrence relation in equation (2). Take as initial condition $\boldsymbol{x}(0) = \boldsymbol{w}$, where vector $\boldsymbol{w}$ is given by (15). Then there exists an $r \in \mathbb{N}$ such that*

$$\boldsymbol{x}(r+2) = \boldsymbol{\eta} + \boldsymbol{x}(r+1) = 2\boldsymbol{\eta} + \boldsymbol{x}(r),$$

where $\boldsymbol{\eta}$ is the cycle time vector of A. Furthermore $(\boldsymbol{\eta}, \boldsymbol{x}(r+1))$ is a generalized eigenmode of matrix A.

Proof: By Lemma 33, for all natural numbers m, we have

$$\begin{aligned}
\boldsymbol{x}(m) - \boldsymbol{x}(m-1) &= A^{\otimes\, m} \otimes \boldsymbol{x}(0) - A^{\otimes\, m-1} \otimes \boldsymbol{x}(0) \\
&= (A^{\otimes\, m} \otimes \boldsymbol{w}) - (A^{\otimes\, m-1} \otimes \boldsymbol{w}) \\
&\le (\boldsymbol{\eta} + (A^{\otimes\, m-1} \otimes \boldsymbol{w})) - (A^{\otimes\, m-1} \otimes \boldsymbol{w}) = \boldsymbol{\eta} \qquad (16)
\end{aligned}$$

and according to Theorem 1, there exist positive integers q and σ such that

$$\boldsymbol{x}(q+2\sigma) = \sigma\boldsymbol{\eta} + \boldsymbol{x}(q+\sigma) = 2\sigma\boldsymbol{\eta} + \boldsymbol{x}(q).$$

If $\sigma = 1$, then the proof is complete. If $\sigma \ne 1$, note that

$$\boldsymbol{x}(q+\sigma) - \boldsymbol{x}(q) = \sigma\boldsymbol{\eta}.$$

Applying equation (16), it follows that

$$\begin{aligned} \boldsymbol{x}(q+\sigma) - \boldsymbol{x}(q+\sigma-1) &\leq \boldsymbol{\eta}, \\ \boldsymbol{x}(q+\sigma-1) - \boldsymbol{x}(q+\sigma-2) &\leq \boldsymbol{\eta}, \\ &\vdots \\ \boldsymbol{x}(q+1) - \boldsymbol{x}(q) &\leq \boldsymbol{\eta}, \end{aligned}$$

and hence, with $\boldsymbol{x}(q+\sigma) - \boldsymbol{x}(q) = \sigma\boldsymbol{\eta}$, it follows that

$$\begin{aligned} \boldsymbol{x}(q+\sigma) - \boldsymbol{x}(q+\sigma-1) &= \boldsymbol{x}(q+\sigma-1) - \boldsymbol{x}(q+\sigma-2) \\ &= \dots \\ &= \boldsymbol{x}(q+1) - \boldsymbol{x}(q) \\ &= \boldsymbol{\eta}. \end{aligned}$$

Thus we have proved that there exists an $r = q \in \mathbb{N}$ such that

$$\boldsymbol{x}(r+2) = \boldsymbol{\eta} \otimes \boldsymbol{x}(r+1) = 2\boldsymbol{\eta} \otimes \boldsymbol{x}(r).$$

Furthermore, according to Corollary 2, we can conclude that $(\boldsymbol{\eta}, \boldsymbol{x}(r+1))$ is a generalized eigenmode of A. □

Inspired by Lemma 33 and Lemma 34, we state the following algorithm which is the main result of our paper. Step 1 until step 4, and step 5 until step 7 are based on Lemma 33 and 34, respectively.

Algorithm 31 **(Generalized Power Algorithm)**

1. *Choose an arbitrary initial vector* $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$.
2. *Iterate equation* $\boldsymbol{x}(k+1) = A \otimes \boldsymbol{x}(k)$ *until there are positive integers* σ, q *and a vector* $\boldsymbol{\psi} \in \mathbb{R}^n$ *that satisfy* $\boldsymbol{x}(q+2\sigma) - \boldsymbol{x}(q+\sigma) = \boldsymbol{x}(q+\sigma) - \boldsymbol{x}(q)$ *and* $B \otimes \boldsymbol{\psi} = \boldsymbol{\psi}$, *where* $\boldsymbol{\psi} = \boldsymbol{x}(q+\sigma) - \boldsymbol{x}(q)$ *and matrix* B *is defined in Theorem 2.*
3. *Compute the cycle time vector* $\boldsymbol{\eta} = \frac{1}{\sigma}\boldsymbol{\psi}$.
4. *Compute vector* $\boldsymbol{w} = \bigoplus_{i=1}^{l} [(\sigma - i)\boldsymbol{\eta} + \boldsymbol{x}(q+i-1)]$ *according to Lemma 33.*
5. *Define a new initial vector* $\boldsymbol{x}(0) = \boldsymbol{w}$.
6. *Iterate equation* $\boldsymbol{x}(k+1) = A \otimes \boldsymbol{x}(k)$ *until there is an integer* $r > 0$ *such that*
$$\boldsymbol{x}(r+2) - \boldsymbol{x}(r+1) = \boldsymbol{x}(r+1) - \boldsymbol{x}(r) = \boldsymbol{\eta}.$$
7. *Set the generalized eigenmode of* A *as* $(\boldsymbol{\eta}, \boldsymbol{v}) = \left(\frac{1}{\sigma}\boldsymbol{\psi}, \boldsymbol{x}(r+1)\right)$.

Before we give a conclusion, we give some examples to illustrate the generalized power algorithm.

Example 3 Given the matrix

$$A = \begin{bmatrix} 3 & \varepsilon & \varepsilon \\ 6 & 4 & \varepsilon \\ \varepsilon & 7 & 5 \end{bmatrix} \text{ and initial vector } \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

As in Theorem 2, we define the matrix

$$B = \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ 0 & 0 & \varepsilon \\ \varepsilon & 0 & 0 \end{bmatrix}.$$

Iterating Equation (2), we obtain $\boldsymbol{x}(0), \boldsymbol{x}(1)$ and $\boldsymbol{x}(2)$. By inspection it can be shown that $B \otimes (\boldsymbol{x}(1) - \boldsymbol{x}(0)) = \boldsymbol{x}(1) - \boldsymbol{x}(0)$, but $\boldsymbol{x}(1) - \boldsymbol{x}(0) \neq \boldsymbol{x}(2) - \boldsymbol{x}(1)$. Hence, this iteration does not yet satisfy conditions in second step of the Algorithm 31, and the algorithm has to be continued with a next iteration. After some iterations we obtain $q = 2$ and $\sigma = 1$ where $\boldsymbol{\psi} = \boldsymbol{x}(q+2\sigma) - \boldsymbol{x}(q+\sigma) = \boldsymbol{x}(q+\sigma) - \boldsymbol{x}(q)$ and $B \otimes \boldsymbol{\psi} = \boldsymbol{\psi}$, so we obtain $\boldsymbol{\eta} = \frac{1}{\sigma}\boldsymbol{\psi} = [3, 4, 5]^T$. Then we compute $\boldsymbol{w}$ according to Lemma 33. This gives $\boldsymbol{w} = \boldsymbol{x}(2)$. Next we redefine $\boldsymbol{x}(0) = \boldsymbol{w}$ and obtain directly that $r = 0$. Finally, we obtain a generalized eigenmode of A given by $(\boldsymbol{\eta}, \boldsymbol{v}) = \big([3, 4, 5]^T, [9, 14, 18]^T\big)$.

Example 4 Consider the matrix

$$A = \begin{bmatrix} \varepsilon & \varepsilon & 16 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 14 & 15 & 18 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 14 & 2 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 17 & 3 & \varepsilon & 12 & 2 & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon \\ 12 & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 8 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 7 & 19 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 13 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 10 & 7 & 12 & 2 & 5 \end{bmatrix} \text{ and initial vector } \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

According to Theorem 2 we obtain matrix

$$B = \begin{bmatrix} \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & \varepsilon & 0 & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again, iterating Equation (2), we obtain $\boldsymbol{x}(0), \boldsymbol{x}(1), \boldsymbol{x}(2), \boldsymbol{x}(3), \boldsymbol{x}(4), \boldsymbol{x}(5)$ and $\boldsymbol{x}(6)$ with $\boldsymbol{x}(4) - \boldsymbol{x}(2) = \boldsymbol{x}(6) - \boldsymbol{x}(4)$, but $B \otimes (\boldsymbol{x}(4) - \boldsymbol{x}(2)) \neq \boldsymbol{x}(4) - \boldsymbol{x}(2)$. Hence, this iteration does not yet satisfy conditions in second step of the Algorithm 31 and has to be continued with a next iteration. After some iterations, we obtain $q = 17$ and $\sigma = 2$ where $\boldsymbol{\psi} = \boldsymbol{x}(q+2\sigma) - \boldsymbol{x}(q+\sigma) = \boldsymbol{x}(q+\sigma) - \boldsymbol{x}(q)$ and $B \otimes \boldsymbol{\psi} = \boldsymbol{\psi}$, so we obtain $\boldsymbol{\eta} = \frac{1}{\sigma}\boldsymbol{\psi} = [15, 15, 15, 15, 15, 8, 7.5, 7.5, 7.5, 8]^T$. Then computing $\boldsymbol{w} = (\boldsymbol{\eta} + \boldsymbol{x}(17)) \oplus \boldsymbol{x}(18)$ and redefining $\boldsymbol{x}(0) = \boldsymbol{w}$,we obtain $r = 0$. Finally we get $\boldsymbol{v} = [286, 288, 285, 288, 283, 152, 154, 142.5, 148, 154]^T$.

Note that in both Examples 3 and 4, the cycle time vectors of matrices A contain different values. Therefore, we can not use Algorithm 21, but we can use Algorithm 31.

Numerical Experiment

In this part, we will compare the computation times between our generalized power algorithm and policy iteration. Specifications of the computer we used are: RAM 2 GB, OS 64 byte and processor Intel Celeron. We use a toolbox which contains the functions "policyIteration" (policy iteration) and "maxplusmaxalgolgeneral" (generalized power algorithm) in the scilab toolbox (can be seen at `https://atoms. scilab.org/toolboxes/maxplus_petrinet`). We performed numerical experiments with 10 random matrices of size $n \times n$ for $n = 5, 10, 15, \cdots, 200$. For every matrix of size $n \times n$, the form of the matrices was fixed to be

$$\begin{bmatrix} A_1 & A_2 \\ \mathcal{E}(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor) & A_3 \end{bmatrix}$$

where $\mathcal{E}$ is max-plus zero matrix of size $\lfloor n/2 \rfloor \times \lfloor n/2 \rfloor$, and A_1, A_2, A_3 are random matrices of which 20% of the elements are max-plus zeros ($\varepsilon = -\infty$).

For every matrix, we computed the generalized eigenmode by using policy iteration and by the generalized power algorithm (initial condition is chosen random), and record the computation times of these algorithms. Figure 1 is a plot containing computation times of the generalized power algorithm and policy iteration where each computation time is obtained as the average over 10 random matrices of similar size and form.

Based on this experiment, we can conclude that computation times of the generalized power algorithm are less than computation times of the policy iteration algorithm given in [4]. But this experiment is relative to the chosen initial condition and structure of matrices A. For example, we want obtain the generalized eigenmode of matrix

$$A = \begin{bmatrix} 19 & \varepsilon & \varepsilon \\ 18 & 7 & 5 \\ 8 & 8 & 18 \end{bmatrix},$$

using the generalized power algorithm. If we choose $[254, 201, 283]^T$ as initial condition, then computation time is 0.04 second, but if we choose the initial condition $[-1017, 1260, 1142]^T$, then computation time is 4.843 second. The second initial condition is slow because we obtain $q = 2278$, and the first initial condition is faster because $q = 40$.

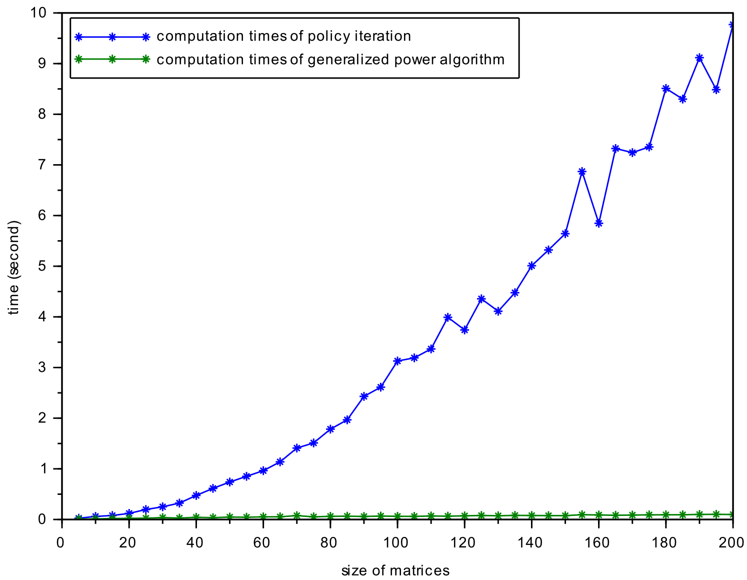

Fig. 1 computation times of the generalized power algorithm and policy iteration

4 Conclusion

In this paper we have constructed and proved a generalized power algorithm that can be used to calculate a generalized eigenmode of any regular square matrix over max-plus algebra by using the recurrence relation in Equation (2). The algorithm is presented in Algorithm 31. The algorithm is illustrated by means of simple examples.

In Subiono and van der Woude [5] a kind of power algorithm was presented that, under some mild conditions on the structure of the (max,+)-systems, determines eigenvalues and corresponding eigenvectors in an iterative way. This algorithm was the inspiration to the so-called generalized power algorithm which has been discussed in this paper. Finally, the resulting algorithm can be used to determine the so-called generalized eigenmode of any square regular matrix over max-plus algebra. In particular, the algorithm can be applied in the case of regular reducible matrices in which the existing power algorithms can not be used to compute eigenvalues and corresponding eigenvectors. The computations in the algorithm are done faster than the policy iteration algorithm given in [4] (according to the numerical experiment). But this conclusion is relative to the chosen initial condition and structure of matrices A.

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