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Martingales and stochastic calculus in Banach spaces

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Martingales and stochastic calculus in Banach spaces

Martingales and stochastic calculus in Banach spaces

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CONTENTS

Summary **ix**

Samenvatting **xiii**

I Introduction **1**

1 Introduction **3**

1.1 Weak differential subordination 3
1.1.1 Discrete case 4
1.1.2 Continuous-time case 5
1.2 Martingale decompositions 6
1.2.1 Meyer-Yoeurp decomposition 6
1.2.2 The canonical decomposition 7
1.3 Burkholder–Davis–Gundy inequalities. Stochastic integration 8
1.3.1 General right-hand side 9
1.3.2 Predictable right-hand side 10
1.4 Miscellanea 10
1.4.1 Fourier multipliers 11
1.4.2 Hilbert transform and orthogonal martingales 11
1.4.3 Burkholder-Rosenthal inequalities. 12
1.4.4 Random measures 13
1.4.5 Bellman functions 13
1.5 What is not in the thesis 14
1.5.1 Cylindrical continuous martingales and stochastic integration, paper [177] 14
1.5.2 Brownian representations of cylindrical continuous local martingales, paper [186] 15
1.5.3 Even Fourier multipliers and martingale transforms, paper [188] 15

2 Preliminaries **17**

2.1 Basic notions on stochastic processes 17
2.2 Martingales 18
2.2.1 Quadratic variation. 20
2.2.2 Continuous martingales 21
2.2.3 Purely discontinuous martingales. Meyer-Yoeurp decomposition 21

2.3 UMD Banach spaces . . . 23
2.4 Stopping times . . . 24
2.4.1 Predictable and totally inaccessible stopping times . . . 25
2.4.2 Quasi-left continuous martingales and martingales with accessible jumps . . . 27
2.4.3 The canonical decomposition . . . 27
2.4.4 Time-change . . . 31
2.5 Stochastic integration. . . . 32
2.6 Multidimensional Wiener process . . . 33
2.7 Brownian representation . . . 33
2.8 Random measures . . . 33
2.9 γ-radonifying operators . . . 35
2.10 Convex, concave, biconcave, zigzag-concave functions . . . 35
2.11 Corresponding dual basis . . . 36
2.12 Itô's formula . . . 37

II Weak differential subordination and the canonical decomposition of martingales **39**

3 Weak differential subordination of discrete and purely discontinuous martingales **41**
3.1 Introduction . . . 42
3.2 Preliminaries . . . 43
3.3 UMD Banach spaces and weak differential subordination . . . 45
3.3.1 Discrete case . . . 45
3.3.2 Continuous time case. . . . 53
3.4 Fourier multipliers . . . 55
3.4.1 Basic definitions and the main theorem . . . 56
3.4.2 Examples of Theorem 3.4.1. . . . 60
3.5 Hilbert transform and general conjecture . . . 61
3.5.1 Hilbert transform and Burkholder functions . . . 61
3.5.2 General conjecture . . . 65

4 L^p-estimates for weak differential subordination and for martingale decompositions **69**
4.1 Introduction . . . 70
4.2 Preliminaries . . . 72
4.3 UMD Banach spaces and martingale decompositions . . . 72
4.3.1 Meyer-Yoeurp decomposition in UMD case . . . 73
4.3.2 Yoeurp decomposition of purely discontinuous martingales. . . 81
4.3.3 Stochastic integration. . . . 83
4.4 Weak differential subordination and general martingales . . . 84

5 Existence of the canonical decomposition and weak L^1-estimates **91**
5.1 Introduction 92
5.2 Preliminaries 96
5.2.1 Martingales and càdlàg processes 96
5.2.2 Compensator and variation 97
5.3 Gundy's decomposition of continuous-time martingales 97
5.4 The canonical decomposition of local martingales 101
5.4.1 Weak differential subordination martingale transforms 102
5.4.2 Sufficiency of the UMD property 108
5.4.3 Necessity of the UMD property 116

6 Orthogonal martingales and the Hilbert transform **123**
6.1 Introduction 124
6.2 Preliminaries 126
6.2.1 Periodic Hilbert transform 126
6.2.2 Orthogonal martingales 126
6.2.3 Subharmonic and plurisubharmonic functions 128
6.2.4 Meyer-Yoeurp decomposition 130
6.3 Main theorem 131
6.4 Applications 143
6.4.1 Hilbert transforms on $\mathbb{T}$, $\mathbb{R}$, and $\mathbb{Z}$ 143
6.4.2 Decoupling constants 151
6.4.3 Necessity of the UMD property 152
6.4.4 Weak differential subordination of martingales: sharper L^p-inequalities 157
6.4.5 Weak differential subordination of harmonic functions 161
6.4.6 Inequalities for singular integral operators 165
6.4.7 Hilbert operators 166

III Stochastic integration and Burkholder–Davis–Gundy inequalities **169**

7 L^q-valued Burkholder-Rosenthal inequalities and sharp estimates for stochastic integrals **171**
7.1 Introduction 172
7.2 Preliminaries 176
7.3 L^q-valued Burkholder-Rosenthal inequalities 178
7.4 The dual of $H_p^{s_q}(X)$ 183
7.5 Sharp bounds for L^q-valued stochastic integrals 189
7.5.1 Decomposition of stochastic integrals 189
7.5.2 Purely discontinuous martingales with accessible jumps . . . 190
7.5.3 Quasi-left continuous purely discontinuous martingales 197
7.5.4 Integrals with respect to random measures 199

7.5.5 Integration with respect to continuous martingales 208
7.5.6 Integration with respect to general local martingales 209
7.A Duals of $\mathscr{S}_q^p$, $\mathscr{D}_{q,q}^p$, $\mathscr{D}_{p,q}^p$, $\hat{\mathscr{S}}_q^p$, $\hat{\mathscr{D}}_{q,q}^p$, and $\hat{\mathscr{D}}_{p,q}^p$ 213
7.A.1 $\mathscr{D}_{q,q}^p$ and $\mathscr{D}_{p,q}^p$ spaces 213
7.A.2 $\mathscr{S}_q^p$ and $\hat{\mathscr{S}}_q^p$ spaces 221

8 Burkholder–Davis–Gundy inequalities in UMD Banach function spaces 227
8.1 Introduction 228
8.2 Preliminaries 230
8.3 Lattice Doob's maximal inequality 231
8.4 Main result 234

9 Burkholder–Davis–Gundy inequalities and stochastic integration in general UMD Banach spaces 243
9.1 Introduction 244
9.2 Burkholder–Davis–Gundy inequalities: the discrete time case 247
9.3 Gaussian characteristics 249
9.3.1 Basic definitions 249
9.3.2 Basic properties of $\gamma(\cdot)$ 250
9.3.3 Finite dimensional case 254
9.4 Covariation bilinear forms 257
9.5 Burkholder–Davis–Gundy inequalities: the continuous-time case . . . 257
9.6 Ramifications of Theorem 9.5.1 261
9.6.1 Continuous and purely discontinuous martingales 261
9.6.2 Martingales with independent increments 263
9.7 Applications and miscellanea 264
9.7.1 Itô isomorphism: general martingales 264
9.7.2 Itô isomorphism: Poisson and general random measures . . . 268
9.7.3 Necessity of the UMD property 268
9.7.4 Martingale domination 269
9.7.5 Martingale approximations 270
9.7.6 The canonical decomposition 274
9.7.7 Covariation bilinear forms for pairs of martingales 274
9.8 UMD Banach function spaces 275

Index 277

References 281

Acknowledgments 297

Curriculum Vitæ 299

List of Publications 301

SUMMARY

In this thesis we study martingales and stochastic integration of processes with values in UMD Banach spaces. Recall that for a Banach space X, a stochastic process $M : \mathbb{R}_+ \times \Omega \to X$ is called a *martingale* if

$$\mathbb{E}(M_t|\mathscr{F}_s) = M_s, \quad 0 \le s \le t.$$

A Banach space X has the *UMD property* if and only if the Hilbert transform is bounded on $L^p(\mathbb{R}; X)$ for all (equivalently, for some) $1 < p < \infty$.

The thesis has three parts. Part I gives an introduction to the material covered in Part II and Part III. Part II is devoted to new properties and corresponding inequalities of martingales themselves. First in Chapter 3 and 4 we extend the notion of *differential subordination* to infinite dimensions. For two real-valued martingales M and N we say that N is *differentially subordinate* to M (we will denote this by $N \ll M$) if a.s. $|N_0| \le |M_0|$ and

$$t \mapsto [M]_t - [N_t] \text{ is nondecreasing in } t \ge 0,$$

where $[M]$ and $[N]$ are *quadratic variations* of M and N, respectively. Burkholder [33] and Wang [179] showed that the following L^p inequality holds true for any $1 < p < \infty$

$$\mathbb{E}|N_t|^p \le (p^* - 1)^p \mathbb{E}|M_t|^p, \quad t \ge 0, \tag{S.1}$$

where $p^* := \max\{p, p/(p-1)\}$. These inequalities have been widely used in harmonic analysis (see e.g. [7, 9, 10, 14, 15, 79, 140] and references therein). Note that Wang [179] extended (S.1) to the Hilbertian setting. Unfortunately, due to Kwapień's result [101] one can not prove an analogue of (S.1) for more general Banach spaces. Surprisingly, in many applications one has differential subordination of its *weak form* (i.e. under actions of linear functionals). Therefore, we define *weak differential subordination*: for a given Banach space X an X-valued martingale N is *weakly differentially subordinate* to an X-valued martingale M (we will denote this by $N \overset{w}{\ll} M$) if $\langle N, x^* \rangle \ll \langle M, x^* \rangle$ for all $x^* \in X^*$. In Chapter 3 and 4 we show that for any $1 < p < \infty$, L^p-estimates for weakly differentially subordinated martingales exist if and only if X has *the UMD property* and the constant $c_{p,X}$ in the corresponding inequality

$$\mathbb{E}\|N_t\|^p \le c_{p,X}^p \mathbb{E}\|M_t\|^p, \quad t \ge 0, \tag{S.2}$$

can be characterized in terms of the *UMD$_p$ constant* $\beta_{p,X}$ of X (recall that $\beta_{p,X}$ expresses the norm of a certain martingale transform and it is finite if and only if X has the UMD property).

In Chapter 6 we show that weak differential subordination together with *orthogonality* of martingales is closely related with the Hilbert transform. More specifically, we show that for any Banach space X, for any X-valued orthogonal martingales M and N with $N \overset{w}{\ll} M$, and for any convex functions $\Phi, \Psi : X \to \mathbb{R}_+$ with $\Psi(0) = 0$ the following inequality holds true

$$\mathbb{E}\Psi(N_t) \le C_{\Phi,\Psi,X}\mathbb{E}\Phi(M_t), \quad t \ge 0, \tag{S.3}$$

where the sharp constant $C_{\Phi,\Psi,X} \in [0,\infty]$ coincides with the Φ,Ψ-norm of the periodic Hilbert transform $\mathcal{H}^{\mathbb{T}}$

$$|\mathcal{H}^{\mathbb{T}}|_{\Phi,\Psi} := \sup_{f:\mathbb{T}\to X \text{ step}} \frac{\int_{\mathbb{T}} \Psi(\mathcal{H}^{\mathbb{T}} f(s))\, \mathrm{d}s}{\int_{\mathbb{T}} \Phi(f(s))\, \mathrm{d}s}.$$

Inequality (S.3) has several applications outlined in Section 6.4. In particular, it is shown that the optimal $c_{p,X}$ in (S.2) is of the order $\max\{\beta_{p,X}, \hbar_{p,X}\}$, where $\hbar_{p,X}$ is the norm of $\mathcal{H}^{\mathbb{T}}$ on $L^p(\mathbb{T};X)$.

Another topic described in Part II is the *canonical decomposition* of local martingales. The canonical decomposition as a natural extension of Lévy-Itô decomposition first appeared in the paper [190] by Yoeurp, and it has the following form. A local martingale M is said to have a canonical decomposition if there exist a continuous local martingale M^c (a Wiener-like part), a purely discontinuous quasi-left continuous local martingale M^q (a Poisson-like part, which jumps at non-predictable stopping times), and a purely discontinuous local martingale M^a with accessible jumps (a discrete-like part, which jumps only at certain predictable stopping times) such that $M_0^c = M_0^q = 0$ and $M = M^c + M^q + M^a$. In the same paper [190] Yoeurp showed existence and uniqueness of the canonical decomposition for any real-valued martingale. In Chapter 4 and 5 we show that for a Banach space X the following are equivalent

- X is UMD;
- any X-valued local martingale admits the canonical decomposition.

Moreover, if X is UMD, then the following estimates hold for any $i \in \{c, q, a\}$

$$\mathbb{E}\|M_t^i\|^p \le \beta_{p,X}^p \mathbb{E}\|M_t\|^p, \quad t \ge 0, \ 1 < p < \infty,$$

$$\lambda\mathbb{P}\big((M^i)_t^* > \lambda\big) \lesssim_X \mathbb{E}\|M_t\|, \quad t \ge 0, \ \lambda > 0.$$

Note that the canonical decomposition is exceptionally important for stochastic integration (see Chapter 7).

Part III is devoted to sharp bounds for stochastic integrals and Burkholder–Davis–Gundy inequalities. Namely, we try to find an answer to the following question. Given a (UMD) Banach space X, a real-valued martingale M, an elementary predictable X-valued process Φ, and $p > 0$. How do sharp bounds for $\sup_{t\geq 0}\|\int_0^t \Phi \,\mathrm{d}M\|^p$ look like?

First the answer for this question was given by van Neerven, Veraar, and Weis in [126] in the case $M = W$ is a standard Brownian motion. In this setting one has that

$$\sup_{t\geq 0}\Big\|\int_0^t \Phi \,\mathrm{d}W\Big\|^p \eqsim_{p,X} \mathbb{E}\|\Phi\|^p_{\gamma(L^2(\mathbb{R}_+),X)}, \tag{S.4}$$

where $\|\Phi\|_{\gamma(L^2(\mathbb{R}_+),X)}$ is the *γ-norm* of Φ which e.g. coincides with the Hilbert–Schmidt norm if X is a Hilbert space. Later in [175, 177] (S.4) was extended to stochastic integrals with respect to *continuous* martingales.

In Part III we extend (S.4) in two ways. First, in Chapter 7 in the case $X = L^q(S)$, $1 < q < \infty$, for a *general* real-valued martingale M we find a *predictable* norm $|\!|\!|\cdot|\!|\!|_{M,p,q}$ (i.e. the process $t \mapsto \big|\!\big|\!\big|\Phi\mathbf{1}_{[0,t]}\big|\!\big|\!\big|_{M,p,q}$, $t \geq 0$, is predictable for any elementary predictable X-valued Φ) such that for any $1 < p < \infty$

$$\sup_{t\geq 0}\Big\|\int_0^t \Phi \,\mathrm{d}M\Big\|^p \eqsim_{p,q} \mathbb{E}|\!|\!|\Phi|\!|\!|^p_{M,p,q}.$$

Though the norm $|\!|\!|\cdot|\!|\!|_{M,p,q}$ has a complicated form (which depends on the mutual positions of p, q, and 2), the latter inequalities have two major features: they are sharp (since they are two-sided) and their right-hand side as a predictable process is locally bonded by any a priori given number (up to a stopping time), which is useful in SPDE's for a fixed point argument. It remains open how an analogue of $|\!|\!|\cdot|\!|\!|_{M,p,q}$ for more general Banach spaces looks like.

If we omit the predictability assumption, then we end up with *Burkholder–Davis–Gundy* inequalities. Recall that Burkholder, Davis, and Gundy proved in [40] that for any real-valued martingale N and for any $1 \leq p < \infty$ one has that

$$\mathbb{E}\sup_{t\geq 0}|N_t|^p \eqsim_p \mathbb{E}[N]_\infty^{p/2}. \tag{S.5}$$

Thus for any real-valued martingale M and for any real-valued elementary predictable process Φ one has the following two-sided inequalities

$$\mathbb{E}\sup_{t\geq 0}\Big|\int_0^t \Phi(s)\,\mathrm{d}M_s\Big|^p \eqsim_p \mathbb{E}\int_0^\infty \Phi(s)^2\,\mathrm{d}[M]_s. \tag{S.6}$$

In order to extend (S.6) to general Banach spaces we extend (S.5) to general Banach spaces. First in Chapter 8 we show that if X is a UMD Banach *function* space over a measure space (S,Σ,μ) (i.e. a Banach space consisting of measurable functions on S), then for any X-valued martingale N and for any $1 < p < \infty$

$$\mathbb{E}\sup_{t\geq 0}\|N_t\|^p \eqsim_{p,X} \mathbb{E}\big\|[N]_\infty^{1/2}\big\|^p, \tag{S.7}$$

where the quadratic variation $[N]_\infty$ is taken pointwise on S. In Chapter 9 we present a more general, but a more complicated version of (S.7). We prove that for any UMD Banach space X and for any $t \geq 0$, any X-valued martingale N has a *covariation bilinear form* $[[N]]_t$ satisfying a.s.

$$[[N]]_t(x^*, x^*) = [\langle N, x^*\rangle]_t, \quad x^* \in X^*$$

Moreover, a.e. in Ω there exists an X-valued centred Gaussian random variable $\xi_{[[N]]_t}$ having $[[N]]_t$ as its covariance bilinear form:

$$[[N]]_t(x^*, x^*) = \mathbb{E}_\xi |\langle \xi_{[[N]]_t}, x^*\rangle|^2, \quad x^* \in X^*,$$

and if one denotes $(\mathbb{E}_\xi \|\xi_{[[N]]_t}\|^2)^{1/2}$ by $\gamma([[N]]_t)$, then the following holds true for any $1 \leq p < \infty$

$$\mathbb{E} \sup_{0\leq s\leq t} \|N_s\|^p \eqsim_{p,X} \mathbb{E}\gamma([[N]]_t)^p. \tag{S.8}$$

In particular, if $N = \int \Phi \, \mathrm{d}M$ for some real-valued martingale M and for some elementary predictable X-valued Φ, then (S.8) implies that for any $1 \leq p < \infty$

$$\mathbb{E} \sup_{t\geq 0} \Big\| \int_0^t \Phi \, \mathrm{d}M \Big\|^p \eqsim_{p,X} \mathbb{E} \|\Phi\|^p_{\gamma(L^2(\mathbb{R}_+, [M]), X)},$$

which fully extends (S.4).

SAMENVATTING

In dit proefschrift bestuderen we martingalen en stochastische integralen van processen met waarden in UMD Banachruimten. Voor een Banachruimte X wordt een stochastisch proces $M : \mathbb{R}_+ \times \Omega \to X$ een martingaal genoemd indien

$$\mathbb{E}(M_t|\mathscr{F}_s) = M_s, \quad 0 \leq s \leq t.$$

Een Banachruimte X heeft de UMD eigenschap dan en slechts dan als de Hilberttransformatie begrensd is op $L^p(\mathbb{R}; X)$ voor iedere (equivalent, voor een) $1 < p < \infty$.

Het proefschrift heeft twee hoofddelen: Deel II en Deel III. Deel II gaat over nieuwe eigenschappen van martingalen en de bijbehorende ongelijkheden. Eerst in Hoofdstuk 3 en later in 4 breiden we het begrip differentiële subordinatie uit naar oneindige dimensies. Voor twee reëel-waardige martingalen M en N zeggen we dat N *differentieel gesubordineerd* wordt door M (dit noteren we met $N \ll M$) als b.z. $|N_0| \leq |M_0|$ en

$$t \mapsto [M]_t - [N_t] \text{ is niet-dalend in } t \geq 0,$$

waarbij $[M]$ en $[N]$ de *kwadratische variatie* van M en N zijn. Burkholder [33] en Wang [179] hebben laten zien dat de volgende L^p ongelijkheden gelden voor iedere $1 < p < \infty$

$$\mathbb{E}|N_t|^p \leq (p^* - 1)^p \mathbb{E}|M_t|^p, \quad t \geq 0, \tag{S.1}$$

waarbij $p^* := \max\{p, p/(p-1)\}$. Deze ongelijkheden worden veel gebruikt in de harmonische analyse (zie bijv. [7, 9, 10, 14, 15, 79, 140] en de referenties daarin). Merk op dat Wang [179] (S.1) naar de Hilbertwaardige setting heeft uitgebreid. Helaas, volgt uit Kwapień's resultaat [101] dat het analagon van (S.1) niet geldt voor algemenre Banachruimten. Het is verrassend dat in veel toepassingen we differentiële subordinatie in *zwakke vorm* hebben (d.w.z. na toepassing van een lineaire functionaal). Daarom definiëren we *zwakke differentiële subordinatie*: voor een gegeven Banachruimte X noemen we een X-waardige martingaal N is *zwak differentieel gesubordineerd* ten aanzien van een X-waardige martingaal M (notatie $N \overset{w}{\ll} M$) als $\langle N, x^*\rangle \ll \langle M, x^*\rangle$ voor alle $x^* \in X^*$. In Hoofdstuk 3 en 4 laten we zien dat er voor elke $1 < p < \infty$, L^p-afschattingen voor zwak differentieel gesubordineerde martingalen gelden dan en slechts dan als X voldoet aan de *UMD eigenschap* en de constanten $c_{p,X}$ in de ongelijkheid

$$\mathbb{E}\|N_t\|^p \leq c_{p,X}^p \mathbb{E}\|M_t\|^p, \quad t \geq 0, \tag{S.2}$$

kunnen worden gekarakteriseerd in termen van de *UMD$_p$ constante* $\beta_{p,X}$ van X (herinner dat $\beta_{p,X}$ is de norm van een bepaalde martingaaltransformatie en is eindig dan en slechts dan als X voldoet aan de UMD eigenschap).

In Hoofdstuk 6 laten we zien dat zwakke differentiële subordinatie en *orthogonaliteit* van martingalen sterk gerelateerd is aan de begrensdheid van de Hilberttransformatie. Preciezer laten we zien dat voor iedere Banachruimte X, voor alle X-waardige orthogonale martingalen M en N met $N \overset{w}{\ll} M$, en voor iedere convexe functie $\Phi, \Psi : X \to \mathbb{R}_+$ met $\Psi(0) = 0$ de volgende ongelijkheid geldt

$$\mathbb{E}\Psi(N_t) \leq C_{\Phi,\Psi,X}\mathbb{E}\Phi(M_t), \quad t \geq 0, \tag{S.3}$$

waarbij de optimale constante $C_{\Phi,\Psi,X} \in [0,\infty]$ overeenkomt met de Φ,Ψ-norm van de periodieke Hilberttransformatie $\mathscr{H}^{\mathbb{T}}$

$$|\mathscr{H}^{\mathbb{T}}|_{\Phi,\Psi} := \sup_{f:\mathbb{T}\to X \text{ stap}} \frac{\int_{\mathbb{T}} \Psi(\mathscr{H}^{\mathbb{T}} f(s))\,\mathrm{d}s}{\int_{\mathbb{T}} \Phi(f(s))\,\mathrm{d}s}.$$

Ongelijkheid (S.3) heeft verschillende toepassingen zoals uitgelegd in Sectie 6.4. In het bijzonder wordt daar bewezen dat de optimale constante $c_{p,X}$ in (S.2) van de orde $\max\{\beta_{p,X}, \hbar_{p,X}\}$ is, waarbij $\hbar_{p,X}$ de norm van $\mathscr{H}^{\mathbb{T}}$ op $L^p(\mathbb{T};X)$ is.

Een ander onderwerp in Part II is de *canonieke decompositie* van lokale martingalen. De canonieke decompositie als uitbreiding van de Lévy-Itô decompositie verscheen voor het eerst in het artikel [190] van Yoeurp, en heeft de volgende vorm. Een lokale martingaal heeft een canonieke decompositie als er een continue lokale martingaal M^c bestaat (een Wiener-achtig deel), een puur discontinue quasi-links continue lokale martingaal M^q (een Poisson-achtig deel dat springt op niet-voorspelbare stoptijden), en een puur discontinue lokale martingaal M^a met toegankelijke sprongen (een discreet-achtig deel, met sprongen op voorspelbare stoptijden) zó dat $M^c_0 = M^q_0 = 0$ en $M = M^c + M^q + M^a$. In hetzelfde artikel [190] heeft Yoeurp existentie en eenduidigheid van de canonieke decompositie voor een willekeurige reëel-waardige martingaal laten zien. In Hoofdstuk 4 en 5 laten we zien dat voor een Banachruimte X de volgende eigenschappen equivalent zijn:

- X is UMD;
- iedere X-waardige lokale martingaal heeft een canonieke decompositie.

Bovendien geldt dat als X UMD is en $i \in \{c,q,a\}$, de volgende afschattingen gelden:

$$\mathbb{E}\|M^i_t\|^p \leq \beta^p_{p,X}\mathbb{E}\|M_t\|^p, \quad t \geq 0, \ 1 < p < \infty,$$

$$\lambda\mathbb{P}\big((M^i)^*_t > \lambda\big) \lesssim_X \mathbb{E}\|M_t\|, \quad t \geq 0, \ \lambda > 0.$$

De canonieke decompositie is extreem belangrijk voor stochastische integratie (zie Hoofdstuk 7).

Deel III is gewijd aan scherpe afschattingen voor stochastische integralen en Burkholder–Davis–Gundy ongelijkheden. We proberen namelijk om de volgende vraag te beantwoorden. Gegegeven een (UMD) Banachruimte X, een reëel-waardige martingaal M, een elementair voorspelbaar X-waardig proces Φ, en $p > 0$. Hoe zien twee-zijdige afschattingen voor $\sup_{t\geq 0}\|\int_0^t \Phi\,\mathrm{d}M\|^p$ er uit?

Allereerst was deze vraag beantwoord door Neerven, Veraar, en Weis in [126] in het geval $M = W$ een standaard Brownse beweging is. In deze setting geldt dat

$$\sup_{t\geq 0}\Big\|\int_0^t \Phi\,\mathrm{d}W\Big\|^p \eqsim_{p,X} \mathbb{E}\|\Phi\|^p_{\gamma(L^2(\mathbb{R}_+),X)}, \tag{S.4}$$

waarbij $\|\Phi\|_{\gamma(L^2(\mathbb{R}_+),X)}$ de *γ-norm* van Φ is, welke bijv. overeenkomt met de Hilbert–Schmidt norm als X een Hilbertruimte is. Daarna is (S.4) in [175, 177] uitgebreid naar stochastische integralen ten aanzien van *continue* martingalen.

In Deel III breiden we (S.4) uit op twee manieren. Ten eerste in Hoofdstuk 7 in het geval $X = L^q(S)$, $1 < q < \infty$, voor een *algemene* reëel-waardige martingaal M vinden we een *voorspelbare* norm $|||\cdot|||_{M,p,q}$ (d.w.z. het proces $t \mapsto |||\Phi \mathbf{1}_{[0,t]}|||_{M,p,q}$, $t \geq 0$, is voorspelbaar voor iedere elementaire voorspelbare X-waardige Φ) zó dat voor elke $1 < p < \infty$

$$\sup_{t\geq 0}\Big\|\int_0^t \Phi\,\mathrm{d}M\Big\|^p \eqsim_{p,q} \mathbb{E}|||\Phi|||^p_{M,p,q}.$$

Hoewel de norm $|||\cdot|||_{M,p,q}$ een gecompliceerde vorm heeft (die afhangt van de wederzijde posities van p, q, en 2), hebben de genoemde ongelijkheden twee belangrijke kenmerken: ze zijn optimaal (want twee-zijdig) en de rechterzijde is als voorspelbaar proces lokaal begrensd door een willekeurig getal (tot en met een stoptijd), wat handig is in dekpuntargumenten voor SPDV's. Het blijft een open probleem hoe $|||\cdot|||_{M,p,q}$ eruit ziet voor algemenere Banachruimten.

Indien we de voorspelbaarheidseis weglaten, dan kunnen we de *Burkholder–Davis–Gundy* ongelijkheden gebruiken. Herinner dat Burkholder, Davis, en Gundy in [40] hebben bewezen dat voor iedere reëel-waardige martingaal N en voor elke $1 \leq p < \infty$ geldt dat

$$\mathbb{E}\sup_{t\geq 0}|N_t|^p \eqsim_p \mathbb{E}[N]_\infty^{p/2}. \tag{S.5}$$

Dus voor elke reëel-waardige martingaal M en voor elke reëel-waardig elementair voorpelbaar proces Φ geldt de volgende twee-zijdige afschatting

$$\mathbb{E}\sup_{t\geq 0}\Big|\int_0^t \Phi(s)\,\mathrm{d}M_s\Big|^p \eqsim_p \mathbb{E}\int_0^\infty \Phi(s)^2\,\mathrm{d}[M]_s. \tag{S.6}$$

Om (S.6) uit te breiden naar algemenere Banachruimten, breiden we (S.5) uit naar algemenere Banachruimten. Eerst laten we in Hoofdstuk 8 zien dat als X een UMD *Banachfunctieruimte* over een maatruimte (S,Σ,μ) is (d.w.z. een Banachruimte bestaande uit meetbare functies op S), dan geldt voor iedere X-waardige martingaal N en voor iedere $1 < p < \infty$ dat

$$\mathbb{E}\sup_{t\geq 0}\|N_t\|^p \eqsim_{p,X} \mathbb{E}\big\|[N]_\infty^{1/2}\big\|^p, \tag{S.7}$$

waarbij de kwadratische variatie $[N]_\infty$ puntsgewijs op S genomen wordt. In Hoofdstuk 9 presenteren we een algemenere, maar ook ingewikkeldere versie van (S.7). We bewijzen voor elke UMD Banachruimte X en voor elke $t \geq 0$ dat voor iedere X-waardige martingaal N een *covariatie bilineaire vorm* $[[N]]_t$ bestaat zó dat b.z.

$$[[N]]_t(x^*, x^*) = [\langle N, x^*\rangle]_t, \quad x^* \in X^*$$

Bovendien geldt dat er b.o. in Ω een X-waardige gecentreerde Gaussische stochast $\xi_{[[N]]_t}$ bestaat zó dat de covariantie bilineare vorm $[[N]]_t$ voldoet aan :

$$[[N]]_t(x^*, x^*) = \mathbb{E}_\xi |\langle \xi_{[[N]]_t}, x^*\rangle|^2, \quad x^* \in X^*,$$

en als we $(\mathbb{E}_\xi \|\xi_{[[N]]_t}\|^2)^{1/2}$ schrijven als $\gamma([[N]]_t)$, dan geldt het volgende voor iedere $1 \leq p < \infty$

$$\mathbb{E} \sup_{0\leq s\leq t} \|N_s\|^p \eqsim_{p,X} \mathbb{E}\gamma([[N]]_t)^p. \tag{S.8}$$

In het bijzonder als $N = \int \Phi \, \mathrm{d}M$ waarbij M een reëel-waardige martingaal en Φ een elementair voorspelbaar X-waardig proces, dan volgt uit (S.8) dat voor alle $1 \leq p < \infty$

$$\mathbb{E} \sup_{t\geq 0} \Big\| \int_0^t \Phi \, \mathrm{d}M \Big\|^p \eqsim_{p,X} \mathbb{E}\|\Phi\|^p_{\gamma(L^2(\mathbb{R}_+,[M]),X)},$$

wat (S.4) volledig generaliseerd.

The translation is provided by Prof. dr. ir. M.C. Veraar.

I

Introduction

1

INTRODUCTION

Let X be a Banach space, $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$. A stochastic process $M : \mathbb{R}_+ \times \Omega \to X$ is called a *martingale* if $\mathbb{E}(M_t|\mathscr{F}_s) = M_s$ for all $0 \leq s \leq t$ (see Section 2.2).

The notion of martingale was introduced by Paul Lévy in 1934, and nowadays it plays an important rôle in probability theory, stochastic analysis, functional analysis, harmonic analysis, complex analysis, and in such applied areas as physics and finance, where martingales are often used as a natural model of a noise. Even though real-valued martingales are of bigger interest, Banach space-valued martingales appear naturally and are of exceptional importance while one needs to extend a theoretical result involving martingales to an infinite-dimensional setting.

The present thesis is devoted to new properties of and new methods while working with Banach space-valued martingales, and it combines papers [54, 146, 178, 184, 185, 187, 189].

Let us outline the main results of the thesis. It is worth noticing that almost all the presented results assume the so-called *UMD*[1] *property*. This property is very natural for Banach spaces when one works with martingales. In particular, due to Bourgain [23] and Burkholder [32] having the UMD property for a Banach space X is equivalent to the boundedness of the Hilbert transform on $L^p(\mathbb{R}; X)$ for all (equivalently for some) $1 < p < \infty$. We refer the reader to Section 2.3 for details on UMD Banach spaces.

1.1. WEAK DIFFERENTIAL SUBORDINATION

Differential subordination of martingales was introduced by Burkholder in [33] as a natural way of martingale domination. It turned out that real-valued differentially subordinated martingales appear inherently in harmonic analysis (see e.g. [9, 10, 12, 13, 133, 139, 140, 145]). Due to the aforementioned references sharp L^p-bounds for differentially subordinated martingales (also under different types of additional assumptions) are of great interest. Here we extend differential subordination to infinite dimensions (this extension is called *weak differential subordination*), and provide L^p-estimates for weakly differentially subordinated martingales. First

[1] UMD stands for *unconditional martingale differences*

let us explain the discrete setting as a demonstration, and then we will turn to the continuous-time setting (note that the continuous-time case is more important for applications).

1.1.1. Discrete case

Let $(d_n)_{n\geq 0}$, $(e_n)_{n\geq 0}$ be two X-valued martingale difference sequences. Then $(e_n)_{n\geq 0}$ is called to be *differentially subordinate* to $(d_n)_{n\geq 0}$ if a.s.

$$\|e_n\| \leq \|d_n\|, \quad n \geq 0. \tag{1.1.1}$$

As we already mentioned, L^p-bounds for differentially subordinated martingales are of importance. In [33] Burkholder showed the following theorem.

Theorem 1.1.1. *Let $(d_n)_{n\geq 0}$, $(e_n)_{n\geq 0}$ be two $\mathbb{R}$-valued martingale difference sequences such that $(e_n)_{n\geq 0}$ is called to be differentially subordinate to $(d_n)_{n\geq 0}$. Then for each $p \in (1,\infty)$,*

$$\mathbb{E}\Big|\sum_{n\geq 0} e_n\Big|^p \leq (p^*-1)^p \mathbb{E}\Big|\sum_{n\geq 0} d_n\Big|^p,$$

where $p^* = \max\{p, p/(p-1)\}$, and p^*-1 is sharp.

Unfortunately, if one wants to broaden the applications of Theorem 1.1.1 to infinite dimensions, one can not apply Theorem 1.1.1 anymore. Therefore we have the following natural question. *Can one extend Theorem 1.1.1 to the general Banach space-valued setting?* Unluckily, due to the following result by Osękowski (see [140, Theorem 3.24(i)]), which is heavily based on Kwapień's paper [101], one can not leave the Hilbertian setting.

Theorem 1.1.2. *A Banach space X is isomorphic to a Hilbert space if and only if for some (equivalently, for all) $1 < p < \infty$ there exists a constant $\alpha_{p,X} > 0$ such that for any pair of X-valued martingale difference sequences $(d_n)_{n\geq 0}$ and $(e_n)_{n\geq 0}$ with $(e_n)_{n\geq 0}$ being differentially subordinate to $(d_n)_{n\geq 0}$ one has that*

$$\mathbb{E}\Big\|\sum_{n\geq 0} e_n\Big\|^p \leq \alpha_{p,X}^p \mathbb{E}\Big\|\sum_{n\geq 0} d_n\Big\|^p.$$

Thus in order to extend Theorem 1.1.1 to more general Banach spaces one needs to weaken the assumption (1.1.1). We will do this in the following way, which shortly can be explained as "differential subordination under action of any linear functional".

Definition 1.1.3. Let X be a Banach space. Then $(e_n)_{n\geq 0}$ is called to be *weakly differentially subordinate* to $(d_n)_{n\geq 0}$ if for any $x^* \in X^*$ a.s.

$$|\langle e_n, x^*\rangle| \leq |\langle d_n, x^*\rangle|, \quad n \geq 0. \tag{1.1.2}$$

Notice that L^p-bounds for weakly differentially subordinated martingale difference sequences imply X having the UMD property thanks to its definition (see Section 2.3). In Chapter 3 we show the converse, i.e. we prove that the UMD property yields the desired L^p-bounds, and that the UMD_p constant $\beta_{p,X}$, the one characterizing the UMD property, is sharp for weak differential subordination.

Theorem 1.1.4. *A Banach space X is a UMD space if and only if for some (equivalently, for all) $1 < p < \infty$ there exists a constant $\beta > 0$ such that for all X-valued martingale different sequences $(d_n)_{n\geq 0}$ and $(e_n)_{n\geq 0}$ such that $(e_n)_{n\geq 0}$ is weakly differentially subordinate to $(d_n)_{n\geq 0}$ one has*

$$\mathbb{E}\Big\|\sum_{n\geq 0} e_n\Big\|^p \leq \beta^p \mathbb{E}\Big\|\sum_{n\geq 0} d_n\Big\|^p.$$

If this is the case then the smallest admissible β is the UMD constant $\beta_{p,X}$.

1.1.2. Continuous-time case

The continuous-time case is a bit more complicated then the discrete case. The first question is how to define differential subordination for continuous-time martingales. To this end we will need the notion of *quadratic variation* (see Section 2.2.1). Recall that any martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ has a quadratic variation

$$[M]_t := \mathbb{P} - \lim_{\text{mesh}\to 0} \sum_{n=1}^{N} |M(t_n) - M(t_{n-1})|^2, \quad t \geq 0,$$

where the limit in probability is taken over partitions $0 = t_0 < \ldots < t_N = t$. Quadratic variation is remarkably important for the martingale theory at least because of Burkholder–Davis–Gundy inequalities (see (1.3.2)). Using quadratic variation one can define differential subordination of continuous-time martingales.

Definition 1.1.5. Let $M, N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be martingales. Then N is *differentially subordinate* to M (we will often write $N \ll M$) if $|N_0| \leq |M_0|$ a.s. and for all $0 \leq s \leq t$ a.s. $[N]_t - [N]_s \leq [M]_t - [M]_s$.

This definition is a natural extension of the discrete one. Moreover, due to Wang [179] the following generalization of Theorem 1.1.1 holds.

Theorem 1.1.6. *Let $M, N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be martingales such that N is differentially subordinate to M. Then for any $1 < p < \infty$*

$$\mathbb{E}|M_t|^p \leq (p^* - 1)^p \mathbb{E}|N_t|^p, \quad t \geq 0.$$

Note that Wang actually proved the Hilbert space-valued version of Theorem 1.1.6, where differential subordination is defined analogously Definition 1.1.5 with using quadratic variations of Hilbert space-valued martingales (see (2.2.4)). In order to extend Theorem 1.1.6 we need first to extend Definition 1.1.5. This extension is fully analogous to Definition 1.1.3.

Definition 1.1.7. Let X be a Banach space, $M, N : \mathbb{R}_+ \times \Omega \to X$ be martingales. Then N is *weakly differentially subordinate* to M (we will often write $N \stackrel{w}{\ll} M$) if $\langle N, x^*\rangle$ is differentially subordinate to $\langle M, x^*\rangle$ for all $x^* \in X^*$.

It turns out that L^p-estimates hold for weakly differentially subordinated martingales only in UMD Banach spaces and the following theorem holds true (see Chapter 3, 4, and 6). Recall that $\beta_{p,X}$ is the UMD constant and its boundedness characterizes the UMD property (see Section 2.3).

Theorem 1.1.8. *Let X be a Banach space, $1 < p < \infty$. Then for any martingales M, N : $\mathbb{R}_+ \times \Omega \to X$ such that $N \stackrel{w}{\ll} M$ one has that*

$$\mathbb{E}\|N_t\|^p \leq c_{p,X}^p \mathbb{E}\|M_t\|^p, \quad t \geq 0, \tag{1.1.3}$$

where the sharp constant $c_{p,X}$ is within the interval $[\beta_{p,X}, \beta_{p,X} + \beta_{p,X}^2]$.

Notice that sharp bounds of $c_{p,X}$ in terms of $\beta_{p,X}$ is of big interest due to the open problem concerning bounds of the norm of the Hilbert transform on $L^p(\mathbb{R}; X)$ in terms of the UMD$_p$-constant of X (see e.g. Subsection 1.4.2), even though one can provide such sharp bounds of $c_{p,X}$ in terms of $\beta_{p,X}$ and the Hilbert transform norm (see Subsection 1.4.2 and Chapter 6).

In addition to L^p-estimates one can show weak L^1-estimates for weakly differentially subordinated martingales, which we will not present here (see the forthcoming paper [183]).

1.2. MARTINGALE DECOMPOSITIONS

A significant part of the present thesis is devoted to different types of *martingale decompositions*.

1.2.1. Meyer-Yoeurp decomposition

Throughout the history continuous martingales used to be much better understood than general martingales. This has several reasons: a continuous martingale is always locally uniformly bounded, its quadratic variation is continuous and hence locally uniformly bounded as well, and after a certain time-change procedure a continuous martingale can be represented as either a stopped Brownian motion (in the one-dimensional case) or as a stochastic integral with respect to a Brownian motion (in the multidimensional case). If one wants to move from continuous to general martingales, then the following reasonable question can be asked. *Is there a linear space of martingales "orthogonal" to continuous martingales?* The definitive answer to this question in the real-valued case was given by Meyer in [122] and Yoeurp in [190]. They proved that any local real-valued martingale M has a *unique* decomposition into a sum of a continuous local martingale M^c with

$M_0^c = 0$ and a *purely discontinuous* local martingale M^d, i.e. a local martingale M^d such that its quadratic variation $[M^d]$ is pure jump.

In Chapter 4 and 5 we extend the result of Meyer and Yoeurp to general UMD Banach spaces. First notice that for any Banach space X a local martingale $M : \mathbb{R}_+ \times \Omega \to X$ is called *purely discontinuous* if $\langle M, x^* \rangle$ is purely discontinuous for all $x^* \in X^*$. Then the following theorem holds true (see Subsection 4.3.1 and Section 5.4).

Theorem 1.2.1. *Let X be a Banach space. Then X has the UMD property if and only if any local martingale $M : \mathbb{R}_+ \times \Omega \to X$ has the Meyer-Yoeurp decomposition, i.e. there exist an X-valued continuous local martingale M^c with $M_0^c = 0$ and an X-valued purely discontinuous local martingale M^d such that $M = M^c + M^d$. Moreover, if this is the case, then for any $1 < p < \infty$*

$$\mathbb{E}\|M_t^c\|^p, \mathbb{E}\|M_t^d\|^p \leq c_{p,X}^p \mathbb{E}\|M_t\|^p, \quad t \geq 0, \tag{1.2.1}$$

$$\lambda \mathbb{P}(M_t^{c*} > \lambda), \lambda \mathbb{P}(M_t^{d*} > \lambda) \lesssim_X \mathbb{E}\|M_t\|, \quad t \geq 0, \; \lambda > 0,$$

where sharp $c_{p,X}$ is within the interval $[\frac{\beta_{p,X}-1}{2}, \beta_{p,X}]$.

Note that the sharp constant $c_{p,X}$ in (1.2.1) is known and equals $\mathrm{UMD}_p^{\{0,1\}}$-constant of X (see Subsection 1.5.3 and Remark 4.4.6).

1.2.2. The canonical decomposition

Historically there were three main separate types of martingales: continuous martingales, discrete martingales, and integrals with respect to random measures. Continuous martingales enjoy such properties as local L^p-integrability for any $1 \leq p \leq \infty$, a rather simple time-change argument due to Kazamaki [94], Lévy's characterization of a Brownian motion (see [89, Theorem 18.3]), and Brownian representation (see [93, Theorem 3.4.2]). Discrete martingales are suitable to work with since the filtration is at most countable and in many applications even can be considered finite, so it is often easier to prove a statement in the discrete setting rather than in the general continuous-time one. The theory of quasi-left continuous random measures (or just *random measures*) was discovered by Novikov in [131] and is of particular interest from the practical point of view since this is a logical generalization of Poisson measures. Somehow all these three "martingale worlds" used to be separated and there were no direct connection between them (though discrete martingales have been heavily applied for proving assertions concerning continuous martingales and random measures).

Due to the work [190] of Yoeurp it turned out that all these "martingale worlds" comprise all the martingales. First we give a couple of useful definitions. A process is said to have *accessible jumps* if it jumps only at a certain countable set of predictable stopping times (i.e. stopping times that can be *announced* by other stopping times, see Subsection 2.4.1). A process is called quasi-left continuous if it does not jump at any predictable stopping time. A classical example of a process with accessible jumps is a process that jumps only at natural points, i.e. at $\{1,2,3,\dots\}$, for instance a discrete martingale. A representative example of a quasi-left continuous process is a Poisson process (literally, one can not predict *when* it will jump). It turns out that any quasi-left continuous purely discontinuous martingale can be naturally represented as a stochastic integral with respect to a random measure, while any purely discontinuous martingale with accessible jumps after a proper approximation and a time-change argument can be represented as a discrete martingale with the same value of jumps. Moreover, thanks to Yoeurp [190] the following theorem holds.

Theorem 1.2.2 (Canonical decomposition). *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a local martingale. Then there exist* unique *local martingales M^c, M^q, and M^a such that M^c is continuous, M^q is purely discontinuous quasi-left continuous, M^a is purely discontinuous with accessible jumps, $M^c_0 = M^q_0 = 0$ a.s., and $M = M^c + M^q + M^a$.*

The decomposition in Theorem 1.2.2 is called *canonical* though it would be more correct to call it *Yoeurp*. But historically *Yoeurp decomposition* is a decomposition of a purely discontinuous local martingale into a quasi-left continuous part and a part with accessible jumps (see e.g. [89]).

In Chapter 4 and 5 we show that Theorem 1.2.2 can be extended to UMD Banach space-valued local martingales, and the UMD property here is not only sufficient but necessary. More precisely, a full analogue of Theorem 1.2.1 (with the same type of estimates) for the canonical decomposition holds.

1.3. BURKHOLDER–DAVIS–GUNDY INEQUALITIES. STOCHASTIC INTEGRATION

Stochastic integration appears naturally while working with stochastic PDEs. In particular, Banach space-valued stochastic integration is of special interest and it has been widely developed during the past decades (see [18, 25, 27, 51, 76, 126, 129, 130, 132, 162]). The first sharp inequalities for Banach space-valued stochastic integrals have been obtained in the paper [126] by van Neerven, Veraar, and Weis. They showed that for any UMD Banach space X, for a Brownian motion W, for any elementary predictable process $\Phi : \mathbb{R}_+ \times \Omega \to X$, and for any $0 < p < \infty$

$$\mathbb{E} \sup_{t\geq 0} \Big\| \int_0^t \Phi \,\mathrm{d}W \Big\|^p \eqsim_{p,X} \mathbb{E}\|\Phi\|^p_{\gamma(L^2(\mathbb{R}_+),X)}, \tag{1.3.1}$$

where $\|\cdot\|_{\gamma(L^2(\mathbb{R}_+),X)}$ is a *γ-norm* which e.g. coincides with the Hilbert-Schmidt norm if X is Hilbert (see Section 2.9). Later this inequality was extended to stochastic integrals with respect to a general continuous martingale by Veraar in [175], and to stochastic integrals with respect to a cylindrical continuous martingale noise by Veraar and the author (see [177]).

Our goal is to find sharp bounds for vector-valued stochastic integrals with respect to *general martingales*. We will consider two cases depending on whether the right-hand side of the desired inequality is predictable or not, which both extend (1.3.1) since its right-hand side is already predictable.

1.3.1. General right-hand side

Stochastic integration is very closely related to *Burkholder–Davis–Gundy inequalities*. Those inequalities connect a martingale M with its quadratic variation $[M]$ and classically due to Burkholder, Davis, and Gundy [40] have the following form: for any $\mathbb{R}$-valued martingale M and for any $1 \leq p < \infty$

$$\mathbb{E}\sup_{t\geq 0} |M_t|^p \eqsim_p \mathbb{E}[M]_{\infty}^{p/2}. \tag{1.3.2}$$

This yields sharp bounds for real-valued stochastic integrals. Indeed, for any real-valued martingale M, for any elementary predictable $\Phi : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, and for any $1 \leq p < \infty$ one has that

$$\mathbb{E}\sup_{t\geq 0}\Big|\int_0^t \Phi \,\mathrm{d}M\Big|^p \eqsim_p \mathbb{E}\Big[\int_0^{\cdot} \Phi \,\mathrm{d}M\Big]_{\infty}^{p/2} = \mathbb{E}\Big(\int_0^{\infty} \Phi^2(t)\,\mathrm{d}[M]_t\Big)^{p/2}.$$

In Chapter 8 and 9 we extend (1.3.2) to Banach function spaces and to general Banach spaces. First in Chapter 8 we show that for any UMD Banach function space X, for any X-valued martingale M, and for any $1 < p < \infty$ one has that

$$\mathbb{E}\sup_{t\geq 0} \|M_t\|^p \eqsim_{p,X} \mathbb{E}\|[M]_{\infty}^{1/2}\|^p. \tag{1.3.3}$$

Further in Chapter 9 we present a more complicated, but much more general form of (1.3.3). More specifically, we show that for any UMD Banach space X and for any $t \geq 0$ any X-valued martingale M has a *covariation bilinear form* $[[M]]_t$ satisfying the following a.s.

$$[[M]]_t(x^*, x^*) = [\langle M, x^*\rangle]_t, \quad x^* \in X^*$$

Moreover, a.s. there exists an X-valued centered Gaussian random variable $\xi_{[[M]]_t}$ having $[[M]]_t$ as its covariance bilinear form:

$$[[M]]_t(x^*, x^*) = \mathbb{E}_{\xi}|\langle \xi_{[[M]]_t}, x^*\rangle|^2, \quad x^* \in X^*,$$

and if one denotes $(\mathbb{E}\|\xi_{[[M]]_t}\|^2)^{1/2}$ by $\gamma([[M]]_t)$, then the following holds true for any $1 \leq p < \infty$

$$\mathbb{E}\sup_{0\leq s\leq t} \|M_s\|^p \eqsim_{p,X} \mathbb{E}\gamma([[M]]_t)^p. \tag{1.3.4}$$

(1.3.4) extends (1.3.3) to the case $p = 1$, and it is a natural extension of (1.3.2). Furthermore, both (1.3.3) and (1.3.4) characterize the UMD property.

The estimate (1.3.4) will allow us to extend (1.3.1) to full generality. Namely, we show that for any real-valued local martingale M, for any Banach space X and for any elementary predictable $\Phi : \mathbb{R}_+ \times \Omega \to X$ we have that for any $1 \leq p < \infty$

$$\mathbb{E}\sup_{t\geq 0}\Big\|\int_0^t \Phi \,\mathrm{d}M\Big\|^p \eqsim_{p,X} \mathbb{E}\|\Phi\|^p_{\gamma(L^2(\mathbb{R}_+,[M]),X)}. \tag{1.3.5}$$

By assuming $p = 1$ and extending the definition of a stochastic integral to general predictable functions we show that general predictable $\Phi : \mathbb{R}_+ \times \Omega \to X$ is stochastically integrable if it is locally in $L^1(\Omega; \gamma(L^2(\mathbb{R}_+, [M]), X))$, which is a natural generalization of the real-valued case [89, p. 526].

1.3.2. *Predictable right-hand side*

The sharp estimates (1.3.5) have one serious disadvantage: their right-hand side is not predictable in general. Since it is not predictable, one can not use a stopping time argument in order to bound it locally and therefore make it useful for solving SPDEs (where local boundedness of a stochastic integral plays a significant rôle for fixed point arguments) even with a Poisson noise. In Chapter 7 we find a predictable right-hand side in the case $X = L^q(S)$ for any $1 < q < \infty$. These estimates for the Poisson case appeared first in the paper [51] by Dirksen. Even in this simple case the predictable right-hand side has six different possibilities depending on the order of p, q, and 2, and in each of this cases the right-hand side has a complicated structure. In Chapter 7 we extend this result to a general martingale noise with the same six cases involved. We will not present the main result of Chapter 7 – Theorem 7.5.30 – here, but just notice that it heavily exploits the following techniques

- Burkholder-Rosenthal inequalities (the discrete analogue of Burkholder–Davis–Gundy inequalities with the predictable right-hand side, see Subsection 1.4.3),
- the canonical decomposition,
- random measure theory (see Subsection 1.4.4),
- stochastic integration with respect to continuous martingales (see Subsection 1.5.1).

1.4. MISCELLANEA

While proving the primary results of the thesis we needed some powerful tools, or we had some meaningful applications. We want to outline some of these topics here.

1.4.1. Fourier multipliers

The first motivation for considering weak differential subordination (at first it was considered only for discrete and purely discontinuous martingales) cones from *Fourier multipliers*, i.e. operators acting on $L^2(\mathbb{R}^d)$ of the form

$$T_m f := \mathscr{F}^{-1}(m\mathscr{F}(f)), \quad f \in L^2(\mathbb{R}^d),$$

where $m \in L^\infty(\mathbb{R}^d)$ is bounded by 1. Such operators appear naturally in Harmonic analysis (see e.g. [69, 79, 168, 169]). There is a natural question whether one can extend T_m to $L^p(\mathbb{R}^d)$ for a general $1 < p < \infty$, or even to $L^p(\mathbb{R}^d; X)$ for a general Banach space X. In order to answer this question, theories as theory of Mihlin, Marcinkiewicz, even homogenous, and Lévy (also known as Bañuelos-Bogdan) multipliers have been created, and for many of them it has been shown that T_m is bounded not only on $L^p(\mathbb{R}^d)$, but even on $L^p(\mathbb{R}^d; X)$ given X has UMD. In particular, in Chapter 3 we show that the so-called *Lévy multipliers* are bounded on $L^p(\mathbb{R}^d; X)$ for any $1 < p < \infty$ and any UMD Banach space X, and provide sharp upper bound for the norm of T_m in terms of the UMD constant. Recall that Bañuelos and Bogdan in [10] and Bañuelos, Bielaszewski, and Bogdan in [9] had shown that Lévy multipliers are bounded on $L^p(\mathbb{R}^d)$ by using differential subordination. In Chapter 3 we extend their result to infinite dimensions using weak differential subordination.

1.4.2. Hilbert transform and orthogonal martingales

Let X be a Banach space, $\mathbb{T} \simeq [-\pi, \pi)$ be a torus equipped with the Lebesque measure, $f : \mathbb{T} \to X$ be a step function. We define *the periodic Hilbert transform* of f in the following way

$$\mathscr{H}_X^{\mathbb{T}} f(\theta) := \frac{1}{2\pi} \mathrm{p.v.} \int_{-\pi}^{\pi} f(s) \cot \frac{\theta - s}{2} \,\mathrm{d}s, \qquad \theta \in [-\pi, \pi).$$

Recall that the periodic Hilbert transform is closely related to the UMD property since if we denote the L^p-norm of $\mathscr{H}_X^{\mathbb{T}}$ by $\hbar_{p,X}$, then thanks to Bourgain [23] and Burkholder [32] $\hbar_{p,X}$ is finite if and only if the UMD constant $\beta_{p,X}$ is finite. Moreover, by Bourgain [23] and Garling [61] the following estimate holds

$$\sqrt{\beta_{p,X}} \leq \hbar_{p,X} \leq \beta_{p,X}^2.$$

Due to a classical Doob's argument it is known that the periodic Hilbert transform has a representation in terms of stochastic integrals, which turn out to be weakly differentially subordinated orthogonal martingales. Remind that we call two X-valued martingales M and N *orthogonal* if $[\langle M, x^*\rangle, \langle N, x^*\rangle] = 0$ and $\langle M_0, x^*\rangle \cdot \langle N_0, x^*\rangle = 0$ for all $x^* \in X^*$.

Section 6 is devoted to showing the converse connection. Namely, we prove there that for any convex continuous functions $\Phi, \Psi : X \to \mathbb{R}_+$ with $\Psi(0) = 0$ and for any pair of X-valued orthogonal martingales M and N such that N is weakly differentially subordinate to M one has that

$$\mathbb{E}\Psi(N_t) \leq C_{\Phi,\Psi,X}\mathbb{E}\Phi(M_t), \quad t \geq 0,$$

where $C_{\Phi,\Psi,X}$ (finite or infinite) coincides with

$$|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} := \sup_{f:\mathbb{T}\to X \text{ step}} \frac{\int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f(s))\,\mathrm{d}s}{\int_{\mathbb{T}} \Phi(f(s))\,\mathrm{d}s}. \tag{1.4.1}$$

This fact has a number of useful applications which we will shortly outline here and which can be found in Section 6.4.

- If Φ is symmetric and $\Phi(0) = 0$, then Φ, Ψ-norms of the periodic Hilbert transform, the discrete Hilbert transform, and the nonperiodic Hilbert transform (these norms are defined similarly to (1.4.1)) are the same.
- $\hbar_{p,X}$ dominates linearly the Wiener decoupling constants of the Banach space X.
- Finiteness of the Φ, Ψ-norm $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$ of the periodic Hilbert transform together with some natural broad assumptions on Φ and Ψ yields that X has the UMD property.
- Sharp L^p-bounds for weakly differentially subordinated martingales and L^p-bounds for weakly differentially subordinated harmonic functions. In particular, it is shown that sharp $c_{p,X}$ in (1.1.3) satisfies

$$\max\{\beta_{p,X}, \hbar_{p,X}\} \leq c_{p,X} \leq \beta_{p,X} + \hbar_{p,X}.$$

1.4.3. Burkholder-Rosenthal inequalities

In [161] Rosenthal proved that for any sequence of independent mean-zero random variables $(d_i)_{i\geq 1}^n$ and of any $p \geq 2$

$$\Big(\mathbb{E}\Big|\sum_{i=1}^n d_i\Big|^p\Big)^{\frac{1}{p}} \eqsim_p \max\Big\{\Big(\sum_{i=1}^n \mathbb{E}|d_i|^p\Big)^{\frac{1}{p}}, \Big(\mathbb{E}\Big(\sum_{i=1}^n \mathbb{E}_{i-1}|d_i|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big\}. \tag{1.4.2}$$

Later in [29] Burkholder extended (1.4.2) to a general martingale difference sequence. Note that the right-hand side of (1.4.2) is predictable. Therefore it is natural to ask: let X be a Banach space and let $1 < p < \infty$. Is there a norm $|||\cdot|||_{p,X}$ on all X-valued martingale difference sequences depending only on *predictable moments* of the individual differences such that for any X-valued martingale difference sequence $(d_i)_{i\geq 1}$

$$c_{p,X}\,|||(d_i)|||_{p,X} \leq \Big(\mathbb{E}\Big\|\sum_i d_i\Big\|_X^p\Big)^{\frac{1}{p}} \leq C_{p,X}\,|||(d_i)|||_{p,X}, \tag{1.4.3}$$

In Chapter 7 we present the explicit formula of $\|\!|\cdot\|\!|_{p,X}$ for the case $X = L^q(S)$, $1 < q < \infty$. We also show that Burkholder-Rosenthal inequalities lead to sharp estimates for integrals with respect to random measures and sharp predictable estimates for stochastic integrals with respect to general martingales, which in particular are presented in Theorem 7.5.30 in the L^q-valued case. Thus Burkholder-Rosenthal inequalities for more general Banach spaces are of exceptional interest since they might yield sharp estimates for corresponding stochastic integrals.

1.4.4. Random measures

Random measure theory appeared in 1970's in works of Grigelionis and Novikov as a natural extension of Poisson random measures. A random measure μ is defined as a measure $\mu(\omega)$ on $\mathbb{R}_+ \times J$ for some measurable space $(J, \mathscr{J})$ (which is called the *jump space*) that depends on $\omega \in \Omega$ in an optional way. Any random measure μ has a *compensator* random measure ν which is predictable such that integral of an elementary predictable function with respect to $\bar{\mu} := \mu - \nu$ is a local martingale. Thanks to Novikov [131] the following inequality holds for any $p \geq 2$ and for any predictable $f : \mathbb{R}_+ \times \Omega \times J \to \mathbb{R}$

$$\mathbb{E}\Big|\int_0^t f \,\mathrm{d}\bar{\mu}\Big|^p \eqsim_p \Big(\mathbb{E}\int_0^t |f|^2 \,\mathrm{d}\nu\Big)^{p/2} + \mathbb{E}\int_0^t |f|^p \,\mathrm{d}\nu, \quad t \geq 0.$$

Note that the process on the right-hand side of the latter inequality is predictable in $t \geq 0$ since both f and ν are predictable. In Subsection 7.5.4 we extend Novikov's inequality to L^q-valued integrals with respect to a random measure. Moreover, we prove that for any Banach space X, for any $1 < p < \infty$, and for any elementary predictable $f : \mathbb{R}_+ \times \Omega \times J \to X$

$$\mathbb{E}\Big\|\int_0^t f \,\mathrm{d}\bar{\mu}\Big\|^p \eqsim_p \mathbb{E}\int_0^t \|f\|^p \,\mathrm{d}\nu, \quad t \geq 0,$$

if $\nu(\mathbb{R}_+ \times J) \leq 1$ a.s.

1.4.5. Bellman functions

For a Banach space X and a function $V : X \times X \to \mathbb{R}$ a function $U : X \times X \to \mathbb{R}$ is called *Bellman* if

- U has *nice* properties,
- $U(x, y) \leq V(x, y)$ for all $x, y \in X$, and
- $U(x, y) \geq 0$ if $x, y \in X$ are from a certain *good* subset A of $X \times X$ (e.g. $A = \{(0,0)\}$ or $A = \{(x,0), x \in X\}$).

Bellman functions are widely used in stochastic analysis (see numerous papers by Bañuelos, Burkholder, Nazarov, Osękowski, Volberg, etc.) and usually their application has the following form: in order to show that for a pair of X-valued martingales M and N under some natural assumptions $\mathbb{E}V(M_t,N_t)\geq 0$ one proves the following

$$EV(M_t,N_t) \overset{(i)}{\geq} EU(M_t,N_t) \overset{(ii)}{\geq} EU(M_0,N_0) \overset{(iii)}{\geq} 0, \tag{1.4.4}$$

where in (i) one uses the fact that $V\geq U$ on $X\times X$, (ii) follows from Itô's formula and nice properties of U, and (iii) holds by the fact that $(M_0,N_0)\in A$ a.s. Often in the literature X is taken to be $\mathbb{R}^d$ for some $d\geq 1$, so in the overwhelming majority of all the papers concerning Bellman function approach to martingale inequalities the corresponding Bellman function has a precise expression. The only exceptions when the Bellman function is given in an abstract nonconstructive way known to the author can be found in [13, 31, 35]. Here in Chapter 3, 4, and 6, as well as in papers [183, 188] we apply and even invent Bellman functions for general UMD Banach spaces X with an abstract construction. It turned out that in order to work with a Bellman function one does not need to know what the function looks like, but just the necessary properties, which often could be figured out if one needs (ii) from (1.4.4) to hold.

1.5. WHAT IS NOT IN THE THESIS

Unfortunately, due to the lack of space not all results obtained during the PhD period are presented in the thesis. Let us sketch the content of the papers which are treated here.

1.5.1. Cylindrical continuous martingales and stochastic integration, paper [177]

In the paper [177] Veraar and the author have studied cylindrical continuous martingales and stochastic integration with respect to a cylindrical continuous martingale. Namely, a wider version of (1.3.1) was proved: let X be UMD, M be a cylindrical *continuous* martingale on a Hilbert space H, $\Phi:\mathbb{R}_+\times\Omega\to\mathscr{L}(H,X)$ be elementary predictable. Then

$$\mathbb{E}\sup_{t\geq 0}\Big\|\int_0^t \Phi\,\mathrm{d}M\Big\|^p \eqsim_{p,X} \mathbb{E}\|\Phi Q_M^{1/2}\|^p_{\gamma(L^2(\mathbb{R}_+,[[M]]),X)}, \quad 0<p<\infty, \tag{1.5.1}$$

where $[[M]]:\mathbb{R}_+\times\Omega\to\mathbb{R}$ is the quadratic variation of M and $Q_M:\mathbb{R}_+\times\Omega\to\mathscr{L}(H)$ is a quadratic variation derivative (for the precise definitions of a cylindrical continuous martingale, $[[M]]$, and Q_M please have a look at [177]).

Even though the inequality (1.5.1) follows directly from (1.3.4), at that time (1.5.1) was new and important e.g. for obtaining Theorem 7.5.30, the main result of Chapter 7. Also notice that this work was for the author an introduction to stochastic analysis in Banach spaces; in particular, it led to deeper understanding of the vector-valued stochastic integration phenomenon.

1.5.2. Brownian representations of cylindrical continuous local martingales, paper [186]

The paper [186] is a spin-off of the paper [177]. Many questions concerning cylindrical continuous martingales remained open after [177]; in particular, does any cylindrical continuous martingale have a *Brownian representation*, i.e. can any cylindrical continuous martingale be represented as a stochastic integral with respect to a cylindrical Brownian motion after a certain time-change? The paper [186] contains the answer to as well as counterexamples concerning this question.

1.5.3. Even Fourier multipliers and martingale transforms, paper [188]

It turns out (see Remark 4.4.6) that the sharp L^p-estimate for the canonical decomposition has the following form. For any UMD Banach space X, for any $1 < p < \infty$, and for any martingale $M : \mathbb{R}_+ \times \Omega \to X$ one has that for $i \in \{c, q, a\}$

$$\mathbb{E}\|M_t^i\|^p \leq c_{p,X}^p \mathbb{E}\|M_t\|^p, \quad t \geq 0,$$

where $M = M^c + M^q + M^a$ is the canonical decomposition and the sharp constant $c_{p,X}$ equals the *UMD*$^{\{0,1\}}$*-constant* $\beta_{p,X}^{\{0,1\}}$ of X, i.e. the least constant $\beta \geq 0$ such that for any $n > 0$, for any X-valued martingale difference sequence $(d_i)_{i=1}^n$, and for any $\{0,1\}$-valued sequence $(\varepsilon_i)_{i=1}^n$ one has that

$$\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i d_i\Big\|^p \leq \beta^p \mathbb{E}\Big\|\sum_{i=1}^n d_i\Big\|^p.$$

Such type of martingale transforms and the corresponding sharp constants were discovered only in the real-valued case by Choi [42] and by Bañuelos and Osękowski [13]. In the paper [188] we consider the vector-valued case and extend many statements from [13] to Banach spaces including sharp bounds for even Fourier multipliers. In particular, it is shown that $\beta_{p,X}^{\{0,1\}}$ equals the norm of the second order Riesz transform.

2

PRELIMINARIES

Before presenting the results that will be used throughout this thesis, we introduce some basic notation. We denote the set of natural numbers by $\mathbb{N} = \{1,2,3,\ldots\}$. We denote the half-line $\mathbb{R}_+ = [0,+\infty)$ and $\overline{\mathbb{R}}_+ = [0,+\infty]$. Throughout this thesis we assume the scalar field $\mathbb{K}$ to be $\mathbb{R}$ or $\mathbb{C}$ unless otherwise is stated. We will use the *Kronecker symbol* δ_{ij}, which is defined in the following way: $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. For any numbers $a, b \in \mathbb{R}$ we will often denote $\min\{a,b\}$ by $a \wedge b$ and $\max\{a,b\}$ by $a \vee b$.

For each $p \in (1,\infty)$ we set $p' \in (1,\infty)$ and $p^* \in [2,\infty)$ to be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $p^* = \max\{p,p'\}$.

We write $a \lesssim_A b$ if there exists a constant c depending only on A such that $a \leq cb$. $\gtrsim_A$ is defined analogously. We write $a \eqsim_A b$ if $a \lesssim_A b$ and $a \gtrsim_A b$ simultaneously.

The letters X and Y are used to denote Banach spaces, and we write X^* for the dual of X. We denote by $\mathscr{L}(X,Y)$ the space of all bounded linear operators, with norm $\|\cdot\|_{\mathscr{L}(X,Y)}$.

Let (S,μ,Σ) be a measure space. A function $f : S \to X$ is called *strongly measurable* if it is the a.e. limit of a sequence of simple functions. For any $1 \leq p \leq \infty$ we denote by $L^p(S;X)$ the Banach space of all strongly measurable functions $f : S \to X$ such that

$$\|f\|_{L^p(S;X)} := \Big|\int_S \|f\|^p \,\mathrm{d}\mu\Big|^{1/p} < \infty, \text{ if } p < \infty,$$

$$\|f\|_{L^\infty(S;X)} := \text{ess.}\sup_{s\in S} \|f(s)\| < \infty, \text{ if } p = \infty.$$

Note that if X^* has the *Radon-Nikodym property* (e.g. X is reflexive, see [79, Section 1.3]), then for all $1 \leq p < \infty$, $\big(L^p(S;X)\big)^* = L^{p'}(S;X^*)$. Let $\mathscr{A}$ be a sub-σ-algebra of Σ. Then for any $f \in L^p(S;X)$ there exists a *conditional expectation with respect to* $\mathscr{A}$, which we will denote by $\mathbb{E}(f|\mathscr{A})$, such that $\mathbb{E}(f|\mathscr{A})$ is $\mathscr{A}$-measurable, and

$$\langle \mathbb{E}(f|\mathscr{A}), x^*\rangle = \mathbb{E}(\langle f, x^*\rangle|\mathscr{A}), \quad x^* \in X^*.$$

The reader can find more information in [79, Section 2.6].

2.1. BASIC NOTIONS ON STOCHASTIC PROCESSES

Let $I \subset \mathbb{R}$ be a closed interval (perhaps, infinite), X be a Banach space. A function $F : I \to X$ is called *càdlàg* (from a French acronym "continue à droite, limite à gauche") if F is right-continuous and has left limits. Definitions of a *càglàd*, *càd*, *càg*, *làd*, and *làg* function are analogous.

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space with a filtration $\mathbb{F}=(\mathscr{F}_t)_{t\geq 0}$ which satisfies the *usual conditions*, i.e. $\mathbb{F}$ is right-continuous and $\mathscr{F}_0$ contains all sets of $\mathbb{P}$-measure zero (see e.g. [93, Definition 1.2.25] and [155]). A process $F:\mathbb{R}_+\times\Omega\to X$ is called *adapted* if F_t is $\mathscr{F}_t$-measurable for any $t\geq 0$. We denote by $\mathscr{P}$ the *predictable σ-algebra* on $\mathbb{R}_+\times\Omega$, the σ-algebra generated by all càg adapted processes. We use $\mathscr{O}$ to denote the *optional σ-algebra* $\mathbb{R}_+\times\Omega$, the σ-algebra generated by all càdlàg adapted processes.

2.2. MARTINGALES

Let X be a Banach space. A process $M:\mathbb{R}_+\times\Omega\to X$ is called a *martingale* if M is adapted, $M_t\in L^1(\Omega;X)$ for all $t\geq 0$, and $\mathbb{E}(M_t|\mathscr{F}_s)=M_s$ for all $t\geq s\geq 0$. M is called a *local martingale* if there exists a sequence $(\tau_n)_{n\geq 1}$ of stopping times (see Section 2.4 for the definition) such that $\tau_n\to\infty$ a.s. as $n\to\infty$ and $(M_t^{\tau_n})_{t\geq 0}:=(M_{t\wedge\tau_n})_{t\geq 0}$ is a martingale for all $n\geq 1$.

Since $\mathbb{F}=(\mathscr{F}_t)_{t\geq 0}$ satisfies the usual conditions, $\mathbb{F}$ is right-continuous and the following proposition holds:

Proposition 2.2.1. *Let X be a Banach space. Then any martingale $M:\mathbb{R}_+\times\Omega\to X$ admits a càdlàg version.*

For proving the proposition we will need the following lemma. Recall that for a Banach space X and for a closed (perhaps, infinite) interval $I\subset\mathbb{R}$ we define a *Skorohod space* $\mathscr{D}(I;X)$ as a linear space consisting of all càdlàg functions $f:I\to X$. We denote the linear space of all bounded càdlàg functions $f:I\to X$ by $\mathscr{D}_b(I;X)$.

Lemma 2.2.2. *$\mathscr{D}_b(I;X)$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space.*

Proof. The proof is analogous to the proof of the same statement for continuous functions (see [154, Problem V.6.1] and [167]). □

Proof of Proposition 2.2.1. One can find the proof in [174, Proposition 2.2.2], but we will repeat it here for the convenience of the reader. Without loss of generality suppose that $M_\infty:=\lim_{t\to\infty}M_t$ exists a.s. and is in $L^1(\Omega;X)$. Also we can assume that there exists $t>0$ such that $M_t=M_\infty$. Let $(\xi^n)_{n\geq 1}$ be a sequence of simple functions in $L^1(\Omega;X)$ such that $\xi^n\to M_t$ in $L^1(\Omega;X)$ as $n\to\infty$. For each $n\geq 1$ define a martingale $M^n:\mathbb{R}_+\times\Omega\to X$ such that $M^n_s=\mathbb{E}(\xi^n|\mathscr{F}_s)$ for each $s\geq 0$. Fix $n\geq 1$. Since ξ^n takes its values in a finite dimensional subspace of X, M^n takes its values in the same finite dimensional subspace as well, and therefore by [49] (or [155, p.8]) it has a càdlàg version. But $M^n_t=\xi^n\to M_t$ in $L^1(\Omega;X)$ as $n\to\infty$, so by the (2.2.1), $M^n\to M$ in the ucp topology (the topology of the uniform convergence on compacts in probability). By taking an appropriate subsequence we can assume that $M^n \to M$ a.s. uniformly on $[0,t]$, and consequently, uniformly on $\mathbb{R}_+$. Therefore, by Lemma 2.2.2 M has a càdlàg version. □

Thanks to Proposition 2.2.1 we can define ΔM_t and M_{t-} for each $t \geq 0$,

$$\Delta M_t := M_t - \lim_{\varepsilon\to 0} M_{(t-\varepsilon)\vee 0},$$
$$M_{t-} := \lim_{\varepsilon\to 0} M_{t-\varepsilon}, \quad M_{0-} := M_0.$$

Let $1 \leq p \leq \infty$. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ is called an *L^p-bounded martingale* if $M_t \in L^p(\Omega; X)$ for each $t \geq 0$ and there exists a limit $M_\infty := \lim_{t\to\infty} M_t \in L^p(\Omega; X)$ in $L^p(\Omega; X)$-sense. We will denote the space of all X-valued L^p-bounded martingales on $\mathbb{F}$ by $\mathcal{M}_X^p(\mathbb{F})$. For brevity we will use $\mathcal{M}_X^p$ instead. Notice that $\mathcal{M}_X^p$ is a Banach space with the given norm: $\|M\|_{\mathcal{M}_X^p} := \|M_\infty\|_{L^p(\Omega;X)}$ (see [84, 89] and [79, Chapter 1]). We also denote all the X-valued locally L^p-bounded martingales by $\mathcal{M}_X^{p,\mathrm{loc}}$.

Proposition 2.2.3. *Let X be a Banach space with X^* having the Radon-Nikodým property (e.g. reflexive), $1 < p < \infty$. Then $(\mathcal{M}_X^p)^* = \mathcal{M}_{X^*}^{p'}$, and $\|M\|_{(\mathcal{M}_X^p)^*} = \|M\|_{\mathcal{M}_{X^*}^{p'}}$ for each $M \in \mathcal{M}_{X^*}^{p'}$.*

Proof. Since $\|M\|_{\mathcal{M}_X^p} = \|M_\infty\|_{L^p(\Omega;X)}$ for each $M \in \mathcal{M}_X^p$, and since for each $\xi \in L^p(\Omega; X)$ we can construct a martingale $M = (M_t)_{t\geq 0} = (\mathbb{E}(\xi|\mathscr{F}_t))_{t\geq 0}$ satisfying $\|M\|_{\mathcal{M}_X^p} = \|\xi\|_{L^p(\Omega;X)}$, $\mathcal{M}_X^p$ is isometric to $L^p(\Omega; X)$, and therefore the proposition follows from [79, Proposition 1.3.3]. □

Since $\|\cdot\| : X \to \mathbb{R}_+$ is a convex function, and M is a martingale, $\|M\|$ is a submartingale by Jensen's inequality (see [89, Lemma 7.11]), and hence by Doob's inequality (see e.g. [93, Theorem 1.3.8(i)]) we have that for all $1 < p \leq \infty$

$$\mathbb{E} \sup_{0\leq s\leq t} \|M_s\|^p \eqsim_p \mathbb{E}\|M_t\|^p, \quad t \geq 0. \tag{2.2.1}$$

Moreover, by [93, Theorem 1.3.8(i)] we have that for each $t \geq 0$, $p \geq 1$ and $\lambda > 0$

$$\mathbb{P}(M_t^* > \lambda) \leq \frac{\mathbb{E}\|M_t\|^p}{\lambda^p}, \tag{2.2.2}$$

where $M_t^* := \sup_{0\leq s\leq t} \|M_s\|$ for all $t \geq 0$.

In the sequel we will need a definition of a Paley-Walsh martingale.

Definition 2.2.4 (Rademacher random variable)**.** Let $\xi : \Omega \to \mathbb{R}$ be a random variable. Then ξ has the *Rademacher distribution* (or simply ξ is *Rademacher*) if $\mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = \frac{1}{2}$.

Definition 2.2.5 (Paley-Walsh martingale)**.** Let X be a Banach space. A discrete X-valued martingale $(f_n)_{n\geq 0}$ is called a *Paley-Walsh martingale* if there exist a sequence of independent Rademacher variables $(r_n)_{n\geq 1}$, a function $\phi_n : \{-1,1\}^{n-1} \to X$ for each $n \geq 2$, and $\phi_1 \in X$ such that $f_n - f_{n-1} = r_n\phi_n(r_1,\ldots,r_{n-1})$ for each $n \geq 2$, $f_1 = r_1\phi_1$, and $f_0 = 0$.

For a discrete X-valued martingale $(f_n)_{n\geq 0}$ we define $df_n := f_n - f_{n-1}$ for $n \geq 1$ and $df_0 := f_0$.

2.2.1. *Quadratic variation*

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ that satisfies the usual conditions. Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a local martingale. We define a *quadratic variation* of M in the following way:

$$[M]_t := \mathbb{P} - \lim_{\text{mesh}\to 0} \sum_{n=1}^{N} |M(t_n) - M(t_{n-1})|^2, \tag{2.2.3}$$

where the limit in probability is taken over all partitions $0 = t_0 < \ldots < t_N = t$. Note that $[M]$ exists and is nondecreasing a.s. For any martingales $M, N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ we can define a *covariation* $[M,N] : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ as $[M,N] := \frac{1}{4}([M+N] - [M-N])$. Since M and N have càdlàg versions, $[M,N]$ has a càdlàg version as well (see [85, Theorem I.4.47] and [120]).

Remark 2.2.6 ([120]). The process $\langle M,N\rangle - [M,N]$ is a local martingale.

Let H be a Hilbert space, $M : \mathbb{R}_+ \times \Omega \to H$ be a local martingale. We define a *quadratic variation* of M in the following way:

$$[M]_t := \mathbb{P} - \lim_{\text{mesh}\to 0} \sum_{n=1}^{N} \|M(t_n) - M(t_{n-1})\|^2, \tag{2.2.4}$$

where the limit in probability is taken over partitions $0 = t_0 < \ldots < t_N = t$. Note that $[M]$ exists and is nondecreasing a.s. and that for any orthogonal basis $(h_n)_{n\geq 1}$ of H, for any $t \geq 0$ a.s.

$$[M]_t = \sum_{n\geq 1} [\langle M, h_n\rangle]_t. \tag{2.2.5}$$

The reader can find more on quadratic variations in [120, 121, 177] for the vector-valued setting, and in [49, 89, 121, 155] for the real-valued setting.

As it was shown in [123, Proposition 1] (see also [163, Theorem 2.13] and [177, Example 3.19] for the continuous case), for any H-valued martingale M there exists an adapted process $q_M : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H)$ which we will call a *quadratic variation derivative*, such that the trace of q_M does not exceed 1 on $\mathbb{R}_+ \times \Omega$, q_M is self-adjoint nonnegative on $\mathbb{R}_+ \times \Omega$, and for any $h, g \in H$ a.s.

$$[\langle M,h\rangle, \langle M,g\rangle]_t = \int_0^t \langle q_M^{1/2}(s)h, q_M^{1/2}(s)g\rangle \,\mathrm{d}[M]_s, \quad t \geq 0. \tag{2.2.6}$$

For any martingales $M, N : \mathbb{R}_+ \times \Omega \to H$ we can define a *covariation* $[M,N] : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ as $[M,N] := \frac{1}{4}([M+N]-[M-N])$. Since M and N have càdlàg versions, $[M,N]$ has a càdlàg version as well (see [85, Theorem I.4.47] and [120]). Moreover, $\langle M,N\rangle - [M,N]$ is a local martingale.

We will frequently use the *Burkholder–Davis–Gundy inequality*: for any $1 \le p < \infty$, for any local martingale $M : \mathbb{R}_+ \times \Omega \to H$ with $M_0 = 0$, and for any stopping time τ one has that

$$\big(\mathbb{E} \sup_{0\le t\le\tau} \|M_t\|^p\big)^{1/p} \eqsim_p (\mathbb{E}[M]_\tau^{p/2})^{1/p}. \tag{2.2.7}$$

We refer to [115] for a self-contained proof.

2.2.2. *Continuous martingales*

Let X be a Banach space. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ is called *continuous* if M has continuous paths.

Remark 2.2.7 ([89, 121]). If X is a Hilbert space, $M, N : \mathbb{R}_+ \times \Omega \to X$ are continuous martingales, then $[M,N]$ has a continuous version.

Let $1 \le p \le \infty$. We will denote the linear space of all continuous X-valued L^p-bounded martingales on $\mathbb{F}$ which start at zero by $\mathcal{M}_X^{p,c}(\mathbb{F})$. For brevity we will write $\mathcal{M}_X^{p,c}$ instead of $\mathcal{M}_X^{p,c}(\mathbb{F})$ since $\mathbb{F}$ is fixed. Analogously to [89, Lemma 17.4] by applying (2.2.1) one can show the following proposition.

Proposition 2.2.8. *Let X be a Banach space, $p \in (1,\infty)$. Then $\mathcal{M}_X^{p,c}$ equipped with the norm $\|M\|_{\mathcal{M}_X^{p,c}} := \|M_\infty\|_{L^p(\Omega;X)}$ is a Banach space.*

2.2.3. *Purely discontinuous martingales. Meyer-Yoeurp decomposition*

An increasing càdlàg process $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called *pure jump* if a.s. for each $t \ge 0$, $A_t = A_0 + \sum_{s=0}^t \Delta A_s$. A local martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called *purely discontinuous* if $[M]$ is a pure jump process. We leave the following evident lemma without proof.

Lemma 2.2.9. *Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be an increasing adapted càdlàg process such that $A_0 = 0$. Then there exist unique up to indistinguishability increasing adapted càdlàg processes $A^c, A^d : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that A^c is continuous a.s., A^d is pure jump a.s., $A_0^c = A_0^d = 0$ and $A = A^c + A^d$.*

The following decomposition theorem is known due to Meyer and Yoeurp (see [122, 190] and [89, Theorem 26.14]).

Theorem 2.2.10 (Meyer-Yoeurp decomposition). *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a local martingale. Then there exist a unique continuous local martingale M^c and a unique purely discontinuous local martingale M^d such that $M_0^c = 0$ and $M = M^c + M^d$. Moreover, in this case $[M]^c = [M^c]$ and $[M]^d = [M^d]$, where $[M]^c$ and $[M]^d$ are defined as in Lemma 2.2.9.*

Corollary 2.2.11. *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a martingale which is both continuous and purely discontinuous. Then $M = M_0$ a.s.*

Proof. Let $M = M^c + M^d$ be the Meyer-Yoeurp decomposition. Since M is continuous, then $M^d = M_0$, and since M is purely discontinuous, then $M^c = 0$, so the desired holds true. □

Later we will need the following proposition.

Proposition 2.2.12. *A martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is purely discontinuous if and only if MN is a martingale for any continuous bounded martingale $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ with $N_0 = 0$.*

Note that some authors take this equivalent condition as the definition of a purely discontinuous martingale, see e.g. [85, Definition I.4.11] and [84, Chapter I].

Proof of Proposition 2.2.12. One direction follows from [89, Corollary 26.15]. Indeed, if M is purely discontinuous, then a.s. $[M,N] = 0$. Therefore by Remark 2.2.6, MN is a local martingale, and due to integrability it is a martingale.

For the other direction we apply Theorem 2.2.10. Let $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a continuous martingale such that $N_0 = 0$ and $M - N$ is purely discontinuous. Then there exists an increasing sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n \nearrow \infty$ as $n \to \infty$ and N^{τ_n} is a bounded continuous martingale for each $n \geq 1$. Therefore MN^{τ_n} and $(M-N)N^{\tau_n}$ are martingales for any $n \geq 1$, and hence $(N^{\tau_n})^2 = (MN^{\tau_n} - (M-N)N^{\tau_n})^{\tau_n}$ is a martingale that starts at zero. On the other hand it is a nonnegative martingale, so it is the zero martingale. By letting n to infinity we prove that $N = 0$ a.s., so M is purely discontinuous. □

Let us now move to the vector-valued case.

Definition 2.2.13. Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a local martingale. Then M is called *purely discontinuous* if for each $x^* \in X^*$ the local martingale $\langle M, x^*\rangle$ is purely discontinuous.

Remark 2.2.14. Let X be finite dimensional. Then similarly to Theorem 2.2.10 any martingale $M : \mathbb{R}_+ \times \Omega \to X$ can be uniquely decomposed into a sum of a purely discontinuous local martingale M^d and a continuous local martingale M^c such that $M^c_0 = 0$.

Remark 2.2.15. Analogously to Proposition 2.2.12, a martingale $M : \mathbb{R}_+ \times \Omega \to X$ is purely discontinuous if and only if $\langle M, x^*\rangle N$ is a martingale for any $x^* \in X^*$ and any continuous bounded martingale $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that $N_0 = 0$.

Let $p \in [1,\infty]$. We will denote the linear space of all purely discontinuous X-valued L^p-bounded martingales on $\mathbb{F}$ by $\mathcal{M}^{p,d}_X(\mathbb{F})$. Since $\mathbb{F}$ is fixed, we will use $\mathcal{M}^{p,d}_X$ instead. The scalar case of the next result have been presented in [84, Lemme I.2.12].

Proposition 2.2.16. *Let X be a Banach space, $p \in (1,\infty)$. Then $\mathcal{M}_X^{p,d}$ equipped with the norm $\|M\|_{\mathcal{M}_X^{p,d}} := \|M_\infty\|_{L^p(\Omega;X)}$ is a Banach space.*

Proof. Let $(M^n)_{n\geq 1}$ be a sequence of purely discontinuous X-valued L^p-bounded martingales such that $(M^n_\infty)_{n\geq 1}$ is a Cauchy sequence in $L^p(\Omega;X)$. Let $\xi \in L^p(\Omega;X)$ be such that $\lim_{n\to\infty} M^n_\infty = \xi$. Define a martingale $M : \mathbb{R}_+ \times \Omega \to X$ as follows: $M = (M_s)_{s\geq 0} = (\mathbb{E}(\xi|\mathscr{F}_s))_{s\geq 0}$. Let us show that $M \in \mathcal{M}_X^{p,d}$. First notice that $\|M_\infty\|_{L^p(\Omega;X)} = \|\xi\|_{L^p(\Omega;X)} < \infty$. Further for each $x^* \in X^*$ by [84, Lemme I.2.12] we have that $\langle M, x^*\rangle$ as a limit of real-valued purely discontinuous martingales $(\langle M^n, x^*\rangle)_{n\geq 1}$ in $\mathcal{M}^p_{\mathbb{R}}$ is purely discontinuous. Therefore M is purely discontinuous by the definition. □

In the sequel we will use the following lemma.

Lemma 2.2.17. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a martingale such that M is both continuous and purely discontinuous. Then $M = M_0$ a.s.*

Proof. Follows analogously to Corollary 2.2.11. □

Definition 2.2.18. A local martingale $M : \mathbb{R}_+ \times \Omega \to X$ is called to have the *Meyer-Yoeurp decomposition* if there exist local martingales $M^c, M^d : \mathbb{R}_+ \times \Omega \to X$ such that M^c is continuous, M^d is purely discontinuous, $M^c_0 = 0$, and $M = M^c + M^d$.

Remark 2.2.19. Note that if $M = M^c + M^d$ is the Meyer-Yoeurp decomposition, then $\langle M^c, x^*\rangle$ is continuous and $\langle M^d, x^*\rangle$ is purely discontinuous for any $x^* \in X^*$; therefore this decomposition is unique by the uniqueness of the Meyer-Yoeurp decomposition of a real-valued local martingale (see [89, Theorem 26.14] for details).

The reader can find more on purely discontinuous martingales in [84, 85, 89].

2.3. UMD BANACH SPACES

Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ is a nonatomic probability space. A Banach space X is called a *UMD Banach space* if for some (or equivalently, for all) $p \in (1,\infty)$ there exists a finite constant β such that the following holds. If $(d_n)_{n=1}^\infty$ is any X-valued martingale difference sequence (relative to some discrete-time filtration) contained in $L^p(\Omega;X)$ and $(\varepsilon_n)_{n=1}^\infty$ is any deterministic $\{-1,1\}$-valued sequence, then

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|^p\Big)^{\frac{1}{p}} \leq \beta\Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}}.$$

The least admissible constant β above is denoted by $\beta_{p,X}$ and is called the *UMD$_p$ constant* of X, or, if the value of p is understood, the *UMD constant* of X. It is well-known that UMD spaces enjoy a large number of useful properties, such as being reflexive. Examples of UMD spaces include all finite dimensional spaces, Hilbert spaces (then $\beta_{p,X} = p^* - 1$ with $p^* = \max\{p, p/(p-1)\}$), the reflexive range of L^q-spaces, Sobolev spaces, Schatten class spaces, Orlicz, and Musielak–Orlicz spaces.

On the other hand, all nonreflexive Banach spaces, e.g. $L^1(0,1)$ and $C([0,1])$, are not UMD. We refer the reader to [39, 79, 80, 153, 164] for further details.

The following proposition is a vector-valued version of [42, Theorem 4.1].

Proposition 2.3.1. *Let X be a Banach space, $p \in (1,\infty)$. Then X has the UMD property if and only if there exists $C > 0$ such that for each $n \geq 1$, for every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega;X)$, and every sequence $(\varepsilon_j)_{j=1}^n$ such that $\varepsilon_j \in \{0,1\}$ for each $j = 1,\dots,n$ we have*

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \varepsilon_j d_j\Big\|^p\Big)^{\frac{1}{p}} \leq C\Big(\mathbb{E}\Big\|\sum_{j=1}^n d_j\Big\|^p\Big)^{\frac{1}{p}}.$$

If this is the case, then the least admissible C is in the interval $[\frac{\beta_{p,X}-1}{2}, \beta_{p,X}]$

Remark 2.3.2. UMD Banach spaces form a natural environment for the L^p-boundedness of the periodic Hilbert transform (see Subsection 6.2.1). It follows from [23, 32] that for every $1 < p < \infty$ we have

$$\sqrt{\beta_{p,X}} \leq \|\mathscr{H}_X^{\mathbb{T}}\|_{L^p(\mathbb{T},X)\to L^p(\mathbb{T},X)} \leq \beta_{p,X}^2. \tag{2.3.1}$$

It is not known whether the quadratic dependence can be improved on either of the sides (see e.g. [39, 66, 79]). Notice that if $X = \mathbb{R}$, then the dependence becomes linear: indeed,

$$\frac{2}{\pi}\beta_{p,\mathbb{R}} = \frac{2}{\pi}(p^*-1) \leq \cot\Big(\frac{\pi}{2p^*}\Big) = \|\mathscr{H}_X^{\mathbb{T}}\|_{\mathscr{L}(L^p(\mathbb{T},X))} \leq p^* - 1 = \beta_{p,\mathbb{R}},$$

where, as above, $p^* := \max\{p, p/(p-1)\}$.

2.4. STOPPING TIMES

A random variable $\tau : \Omega \to \mathbb{R}_+$ is called an *optional stopping time* (or just a *stopping time*) if $\{\tau \leq t\} \in \mathscr{F}_t$ for each $t \geq 0$. With an optional stopping time τ we associate a σ-field $\mathscr{F}_\tau := \{A \in \mathscr{F}_\infty : A \cap \{\tau \leq t\} \in \mathscr{F}_t, t \in \mathbb{R}_+\}$. Note that M_τ is strongly $\mathscr{F}_\tau$-measurable for any local martingale M. For any stopping time τ we define σ-field $\mathscr{F}_{\tau-}$ in the following way

$$\mathscr{F}_{\tau-} := \sigma\big\{\mathscr{F}_0 \cup (\mathscr{F}_t \cap \{t < \tau\}), t > 0\big\} \tag{2.4.1}$$

(see [89, p. 491]). Note that for any stopping time τ and σ both $\tau\wedge\sigma$ and $\tau\vee\sigma$ are stopping times as well. We refer the reader to [89, Chapter 7] for details on stopping times.

Due to the existence of a càdlàg version of a martingale $M : \mathbb{R}_+ \times \Omega \to X$, we can define an X-valued random variables $M_{\tau-}$ and ΔM_τ for any stopping time τ in the following way: $M_{\tau-} = \lim_{\varepsilon\to 0} M_{(\tau-\varepsilon)\vee 0}$, $\Delta M_\tau = M_\tau - M_{\tau-}$.

2.4.1. Predictable and totally inaccessible stopping times

Definition 2.4.1. Let τ be a stopping time. Then τ is called *predictable* if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n < \tau$ a.s. on $\{\tau > 0\}$ for each $n \geq 1$ and $\tau_n \nearrow \tau$ a.s.

For a predictable stopping time τ we define $\mathcal{F}_{\tau-}$ in the following way analogous to (2.4.1) (see [89, Chapter 25])

$$\mathcal{F}_{\tau-} = \sigma(\mathcal{F}_{\tau_n})_{n\geq 1}.$$

Due to the equivalent form (2.4.1), $F_{\tau-}$ does not depend on the choice of the announcing sequence $(\tau_n)_{n\geq 0}$ (see also [89, Lemma 25.2(iii)]).

Definition 2.4.2. Let τ be a stopping time. Then τ is called *totally inaccessible* if $\mathbb{P}\{\tau = \sigma < \infty\} = 0$ for each predictable stopping time σ.

The reader can find more information on predictable and totally inaccessible stopping times in [85, Definition I.2.7] and [89, Chapter 25].

Lemma 2.4.3. *Let X be a Banach space, $V : \mathbb{R}_+ \times \Omega \to X$ be a predictable càdlàg process. Let τ be a totally inaccessible stopping time. Then $\Delta V_\tau = 0$ a.s.*

Proof. It is sufficient to show that $\langle \Delta V_\tau, x^*\rangle = 0$ a.s. for any $x^* \in X^*$. Then the statement follows from [85, Proposition I.2.24]. $\square$

Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a local martingale. Then M has a càdlàg version (see Proposition 2.2.1), and therefore we can define an adapted càdlàg process $M^{\tau-} = (M^{\tau-}_t)_{t\geq 0}$ in the following way

$$M^{\tau-}_t := \lim_{\varepsilon\to 0} M_{(\tau-\varepsilon)\wedge t}, \quad t \geq 0, \tag{2.4.2}$$

where we set $M_t = 0$ for $t < 0$. Notice that $M^{\tau-}$ is not necessarily a local martingale. For instance if $X = \mathbb{R}$ and M is a compensated Poisson process, $\tau := \inf_{t\geq 0}\{\Delta M_t > 0\}$, then $M^{\tau-}_t = -(t \wedge \tau)$ a.s. for each $t \geq 0$, so it is a supermartingale which is not even a local martingale. Nevertheless, if τ is a predictable stopping time, then the following lemma holds.

Lemma 2.4.4. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a local martingale, τ be a predictable stopping time. Then $M^{\tau-}$ defined as in* (2.4.2) *is a local martingale. Moreover, if M is an L^1-bounded martingale, then $M^{\tau-}$ is an L^1-bounded martingale as well.*

Proof. Without loss of generality we can let $M_0 = 0$ a.s. First assume that M is an L^∞-bounded martingale. Let $(\tau_n)_{n\geq 1}$ be an announcing to τ sequence of stopping times, i.e. $\tau_n < \tau$ a.s. on $\{\tau > 0\}$ and $\tau_n \nearrow \tau$ a.s. as $n \to \infty$. Then M^{τ_n} is an L^1-bounded martingale for each $n \geq 1$. Moreover, $M^{\tau_n}_t \to M^{\tau-}_t$ a.s. as $n \to \infty$ for each $t \geq 0$. On the other hand, $M_t^{\tau_n} = \mathbb{E}(M_t|\mathscr{F}_{\tau_n}) \to \mathbb{E}(M_t|\mathscr{F}_{\tau-})$ a.s. as $n\to\infty$ by [79, Theorem 3.3.8] and [89, Lemma 25.2(iii)], and hence in L^1 by the uniform boundedness due to the boundedness of M_∞. Therefore for each $t\geq 0$ we have that $M_t^{\tau-} = \mathbb{E}(M_t|\mathscr{F}_{\tau-})$ is integrable, hence for all $0\leq s\leq t$

$$\mathbb{E}(M_t^{\tau-}|\mathscr{F}_s) = \mathbb{E}\big(\lim_{n\to\infty} M_t^{\tau_n}|\mathscr{F}_s\big) = \lim_{n\to\infty}\mathbb{E}(M_t^{\tau_n}|\mathscr{F}_s) = \lim_{n\to\infty} M_s^{\tau_n} = M_s^{\tau-},$$

where all the limits are taken in $L^1(\Omega;X)$. Hence $(M_t^{\tau-})_{t\geq 0}$ is a martingale. Moreover, by [79, Corollary 2.6.30]

$$\mathbb{E}\|M_t^{\tau-}\| = \mathbb{E}\|\mathbb{E}(M_t|\mathscr{F}_{\tau-})\| \leq \mathbb{E}\|M_t\| \leq \mathbb{E}\|M_\infty\|, \quad t\geq 0. \tag{2.4.3}$$

Now we treat the general case. Without loss of generality using a stopping time argument assume that M is an L^1-bounded martingale. Let $(M^m)_{m\geq 1}$ be a sequence of X-valued L^∞-bounded martingales such that $M^m_\infty \to M_\infty$ in $L^1(\Omega;X)$ as $m\to\infty$. Analogously the first part of the proof $M_t^{\tau-} = \mathbb{E}(M_t|\mathscr{F}_{\tau-})$ for each $t\geq 0$; moreover, by (2.4.3) $\big((M^m)_t^{\tau-}\big)_{m\geq 1}$ is a Cauchy sequence in $L^1(\Omega;X)$. Therefore by [79, Corollary 2.6.30], $(M^m)_t^{\tau-} \to M_t^{\tau-}$ in $L^1(\Omega;X)$ for each $t\geq 0$, hence for each $t\geq s\geq 0$ by [79, Corollary 2.6.30]

$$\begin{aligned}\mathbb{E}(M_t^{\tau-}|\mathscr{F}_s) = \mathbb{E}\big(\lim_{m\to\infty}(M^m)_t^{\tau-}|\mathscr{F}_s\big) &= \lim_{m\to\infty}\mathbb{E}((M^m)_t^{\tau-}|\mathscr{F}_s)\\ &= \lim_{m\to\infty}(M^m)_s^{\tau-} = M_s^{\tau-},\end{aligned}$$

where all the limits are again taken in $L^1(\Omega;X)$. Therefore $(M_t^{\tau-})_{t\geq 0}$ is an L^1-martingale. □

Lemma 2.4.5. *Let X be a Banach space, $1\leq p<\infty$, $M:\mathbb{R}_+\times\Omega\to X$ be an L^p-bounded martingale, τ be a predictable stopping time. Then $(\Delta M_\tau \mathbf{1}_{[0,t]}(\tau))_{t\geq 0}$ is an L^p-bounded martingale as well.*

Proof. By the definition of a predictable stopping time there exists an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that $\tau_n<\tau$ a.s. for each $n\geq 0$ on $\{\tau>0\}$ and $\tau_n\nearrow\tau$ a.s. as $n\to\infty$. Then $M^\tau, M^{\tau_1},\dots,M^{\tau_n},\dots$ are L^p-bounded martingales. Moreover, $M_t^\tau - M_t^{\tau_n}\to\Delta M_\tau\mathbf{1}_{[0,t]}(\tau)$ is in $L^p(\Omega;X)$ for each $t\geq 0$ due to the fact that $\Delta M_\tau = \mathbb{E}(M_\infty|\mathscr{F}_\tau) - \mathbb{E}(M_\infty|\mathscr{F}_{\tau-})$ and [79, Corollary 2.6.30]. Consequently, $(\Delta M_\tau\mathbf{1}_{[0,t]}(\tau))_{t\geq 0}$ is an L^p-bounded martingale. □

Lemma 2.4.6. *Let $F:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ be a locally integrable càdlàg adapted process, τ be a predictable stopping time. Let $G,H:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ be such that $G_t = F_\tau\mathbf{1}_{[0,t]}(\tau)$, $H_t = \mathbf{1}_{[0,t]}(\tau)\mathbb{E}_{\mathscr{F}_{\tau-}}F_\tau$ for each $t\geq 0$. Then $G-H$ is a local martingale.*

Proof. Without loss of generality suppose that F is integrable. First of all notice that H is a predictable process thanks to [89, Lemma 25.3(ii)], and G is adapted due to the fact that $G_t = F_{\tau\wedge t}\mathbf{1}_{[0,t]}(\tau)$. Fix $t > s \geq 0$. By [89, Lemma 25.2(i)], $\mathscr{F}_s \cap \{s < \tau\} \subset \mathscr{F}_{\tau-}$ and $\mathscr{F}_s \cap \{t < \tau\} \subset \mathscr{F}_t \cap \{t < \tau\} \subset \mathscr{F}_{\tau-}$. Hence,

$$\mathscr{F}_s \cap \{s < \tau \leq t\} \subset \mathscr{F}_{\tau-}$$

and so

$$\begin{aligned}
\mathbb{E}(G_t - H_t|\mathscr{F}_s) &= \mathbb{E}(F_\tau \mathbf{1}_{\{\tau\leq t\}} - \mathbf{1}_{\{\tau\leq t\}}\mathbb{E}_{\mathscr{F}_{\tau-}}F_\tau|\mathscr{F}_s)\\
&= \mathbb{E}(F_\tau \mathbf{1}_{\{\tau\leq s\}} - \mathbf{1}_{\{\tau\leq s\}}\mathbb{E}_{\mathscr{F}_{\tau-}}F_\tau|\mathscr{F}_s)\\
&\quad + \mathbb{E}(F_\tau \mathbf{1}_{\{s<\tau\leq t\}} - \mathbf{1}_{\{s<\tau\leq t\}}\mathbb{E}_{\mathscr{F}_{\tau-}}F_\tau|\mathscr{F}_s)\\
&= G_s - H_s + \mathbb{E}(\mathbb{E}(F_\tau - \mathbb{E}_{\mathscr{F}_{\tau-}}F_\tau|\mathscr{F}_{\tau-})\mathbf{1}_{\{s<\tau\leq t\}}|\mathscr{F}_s \cap \{s < \tau \leq t\}) = G_s - H_s.
\end{aligned}$$

□

Corollary 2.4.7. *Let X be a Banach space, τ be a predictable stopping time, $\xi \in L^1(\Omega; X)$ be $\mathscr{F}_\tau$-measurable such that $\mathbb{E}_{\mathscr{F}_{\tau-}}\xi = 0$. Let $M : \mathbb{R}_+ \times \Omega \to X$ be such that $M_t = \xi\mathbf{1}_{[0,t]}(\tau)$. Then M is a martingale.*

Proof. The case $X = \mathbb{R}$ follows from Lemma 2.4.6 and the fact that $\xi\mathbf{1}_{\tau\leq t}$ is $\mathscr{F}_t$-measurable for each $t \geq 0$ by the definition of $\mathscr{F}_\tau$. For the general case we notice that $\langle M, x^*\rangle$ is a martingale for each $x^* \in X$ and since M is integrable it follows that M is a martingale. □

2.4.2. *Quasi-left continuous martingales and martingales with accessible jumps*

Let X be a Banach space. An X-valued local martingale is called *quasi-left continuous* if $\Delta M_\tau = 0$ a.s. on the set $\{\tau < \infty\}$ for each predictable stopping time τ (see [85, Chapter I.2] for more information).

We call

$$[\tau] = \{((\omega, t) \in \Omega \times \mathbb{R}_+ \ : \ t = \tau(\omega)\}$$

the *graph* of τ (although it is strictly speaking, the restriction of the graph of τ to $\Omega \times \mathbb{R}_+$). An X-valued local martingale is said to have *accessible jumps* if there exists a sequence of predictable stopping times $(\tau_n)_{n\geq 0}$ with disjoint graphs such that a.s.

$$\{t \geq 0 : \Delta M_t \neq 0\} \subset \{\tau_1, \tau_2, \ldots, \tau_n, \ldots\}. \tag{2.4.4}$$

(see [89, p.499] and [89, Corollary 26.16]).

The reader can find more information on quasi-left continuous martingales and martingales with accessible jumps in [54, 85, 89, 184, 185]

2.4.3. *The canonical decomposition*

Definition 2.4.8. Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be an adapted càdlàg process. A has *accessible jumps* if $\Delta A_\tau = 0$ a.s. for any totally inaccessible stopping time τ. A is called *quasi-left continuous* if $\Delta A_\tau = 0$ a.s. for any predictable stopping time τ.

Remark 2.4.9. According to [89, Proposition 25.17] one can show that for any pure jump increasing adapted càdlàg process $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ there exist unique increasing adapted càdlàg processes $A^a, A^q : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that A^a has accessible jumps, A^q is quasi-left continuous, $A_0^q = 0$ and $A = A^a + A^q$.

Because of Remark 2.4.9 the following lemma makes sense.

Lemma 2.4.10. *Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be an increasing adapted càdlàg process, $A_0 = 0$ a.s. Then there exist unique increasing adapted càdlàg $A^c, A^q, A^a : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that $A_0^c = A_0^q = A_0^a = 0$, A^c is continuous a.s., A^q and A^a are pure jump a.s., A^q is quasi-left continuous, A^a has accessible jumps, and $A = A^c + A^q + A^a$.*

Proof. The statement follows from [89, Proposition 25.17] and Lemma 2.2.9. □

The following decomposition theorem was shown by Yoeurp in [190] and follows from [89, Theorem 26.14 and Corollary 26.16].

Proposition 2.4.11 (Decomposition of local martingales, Yoeurp, Meyer)**.** *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a local martingale. Then there exists a unique decomposition $M = M^c + M^q + M^a$, where $M^c : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a continuous local martingale, $M^q, M^a : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ are purely discontinuous local martingales, M^q is quasi-left continuous, M^a has accessible jumps, $M_0^c = M_0^q = 0$, and then $[M^c] = [M]^c$, $[M^q] = [M]^q$ and $[M^a] = [M]^a$, with $[M]^c$, $[M]^q$ and $[M]^a$ are defined as in Lemma 2.4.10.*

We will refer to the decomposition in Proposition 2.4.11 as the *canonical* decomposition of M.

Corollary 2.4.12 (Yoeurp decomposition)**.** *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a purely discontinuous martingale. Then there exist unique purely discontinuous martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that M^a is has accessible jumps, M^q is quasi-left continuous, $M_0^q = 0$ and $M = M^a + M^q$. Moreover, then $[M^a] = [M]^a$ and $[M^q] = [M]^q$.*

Corollary 2.4.13. *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a purely discontinuous martingale which is both with accessible jumps and quasi-left continuous. Then $M = M_0$ a.s.*

Proof. Without loss of generality we can set $M_0 = 0$. Then $M = M + 0 = 0 + M$ are decompositions of M into a sum of a martingale with accessible jumps and a quasi-left continuous martingale. Since by Corollary 2.4.12 this decomposition is unique, $M = 0$ a.s. □

In the sequel we will need the following proposition.

Proposition 2.4.14. *Let $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a purely discontinuous L^p-martingale. Let $(M^n)_{n\geq 1}$ be a sequence of purely discontinuous martingales such that $M_\infty^n \to M_\infty$ in $L^p(\Omega)$. Then the following assertions hold*

(a) if $(M^n)_{n\geq 1}$ have accessible jumps, then M has accessible jumps as well;

(b) if $(M^n)_{n\geq 1}$ are quasi-left continuous martingales, then M is quasi-left continuous as well.

Proof. We will only show (a), (b) can be proven in the same way. Without loss of generality suppose that $M_0 = 0$ and $M_0^n = 0$ for each $n \geq 1$. Let $M^a, M^q : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be purely discontinuous martingales such that M^a has accessible jumps, M^q is quasi-left continuous, $M_0^a = M_0^q = 0$ and $M = M^a + M^q$ (see Corollary 2.4.12). Then by Corollary 2.4.12, the Doob maximal inequality [93, Theorem 1.3.8(iv)] and the fact the a quadratic variation is a.s. nonnegative

$$\mathbb{E}|M_\infty - M_\infty^n|^p \eqsim_p \mathbb{E}[M-M^n]_\infty^{\frac{p}{2}} = \mathbb{E}\Big([M^a - M^n]_\infty + [M^q]_\infty\Big)^{\frac{p}{2}} \geq \mathbb{E}[M^q]_\infty^{\frac{p}{2}},$$

and since $\mathbb{E}|M_\infty - M_\infty^n|^p \to 0$ as $n \to \infty$, $\mathbb{E}[M^q]_\infty^{\frac{p}{2}} = 0$. Therefore $M^q = 0$ a.s., so M has accessible jumps. □

Let us turn to the infinite dimensional case.

Definition 2.4.15. Let X be a Banach space. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ has *accessible jumps* if $\Delta M_\tau = 0$ a.s. for any totally inaccessible stopping time τ. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ is called *quasi-left continuous* if $\Delta M_\tau = 0$ a.s. for any predictable stopping time τ.

Lemma 2.4.16. *Let X be a reflexive Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous martingale.*

(i) M has accessible jumps if and only if for each $x^* \in X^*$ the martingale $\langle M, x^*\rangle$ has accessible jumps;

(ii) M is quasi-left continuous if and only if for each $x^* \in X^*$ the martingale $\langle M, x^*\rangle$ is quasi-left continuous.

Proof. Without loss of generality we can assume that X is a separable Banach space. We will show only (i), while (ii) can be proven analogously.

(i): The "only if" part is obvious. For "if" part we fix a dense subset $(x_m^*)_{m\geq 1}$ of X^*. Let τ be a totally inaccessible stopping time. Then $\Delta\langle M_\tau, x_m^*\rangle = \langle \Delta M_\tau, x_m^*\rangle = 0$ a.s. for each $m \geq 1$. Hence $\Delta M_\tau = 0$ a.s., and the "if" part is proven. □

Definition 2.4.17. Let X be a Banach space, $p \in (1,\infty)$. Then we define $\mathcal{M}_X^{p,q} \subset \mathcal{M}_X^{p,d}$ as the linear space of all X-valued purely discontinuous quasi-left continuous L^p-bounded martingales which start at 0. We define $\mathcal{M}_X^{p,a} \subset \mathcal{M}_X^{p,d}$ as the linear space of all X-valued purely discontinuous L^p-bounded martingales with accessible jumps.

Proposition 2.4.18. *Let X be a Banach space, $1 < p < \infty$. Then $\mathcal{M}_X^{p,q}$ and $\mathcal{M}_X^{p,a}$ are closed subspaces of $\mathcal{M}_X^{p,d}$.*

Proof. We only will show the case of $\mathcal{M}_X^{p,q}$, the proof for $\mathcal{M}_X^{p,a}$ is analogous. Let $(M^n)_{n\geq 1} \in \mathcal{M}_X^{p,q}$ be such that $(M^n_\infty)_{n\geq 1}$ is a Cauchy sequence in $L^p(\Omega; X)$. Let $\xi = \lim_{n\to\infty} M^n_\infty$ in $L^p(\Omega; X)$. Define an X-valued martingale M as follows: $M_t = \mathbb{E}(\xi|\mathscr{F}_t)$, $t \geq 0$. Then since conditional expectation is a contraction in $L^p(\Omega; X)$, $M_0 = \lim_{n\to\infty} M_0^n = 0$. Now let us show that M is quasi-left continuous. By Lemma 2.4.16 it is sufficient to show that $\langle M, x^*\rangle$ is quasi-left continuous for each $x^* \in X^*$. Fix $x^* \in X^*$. Define $N := \langle M, x^*\rangle$ and $N^n := \langle M^n, x^*\rangle$ for each $n \geq 1$. Then

$$\begin{aligned}\mathbb{E}\|N_\infty - N^n_\infty\|^p \eqsim_p \mathbb{E}[N - N^n]_\infty^{\frac{p}{2}} &= \mathbb{E}\big([N-N^n]^c_\infty + [N-N^n]^q_\infty + [N-N^n]^a_\infty\big)^{\frac{p}{2}}\\ &= \mathbb{E}\big([N]^c_\infty + [N-N^n]^q_\infty + [N]^a_\infty\big)^{\frac{p}{2}} \geq \mathbb{E}\big([N]^c_\infty + [N]^a_\infty\big)^{\frac{p}{2}},\end{aligned}$$

and since the first expression vanishes as $n \to \infty$, $[N]^c_\infty = [N]^a_\infty = 0$ a.s., so N is quasi-left continuous. Since $x^* \in X^*$ was arbitrary, $M \in \mathcal{M}_X^{p,q}$. □

The following lemma follows from Corollary 2.4.13.

Lemma 2.4.19. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous martingale. Let M be both with accessible jumps and quasi-left continuous. Then $M = M_0$ a.s. In other words, $\mathcal{M}_X^{p,q} \cap \mathcal{M}_X^{p,a} = 0$.*

Proof. Without loss of generality set $M_0 = 0$. Suppose that $\mathbb{P}(M \neq 0) > 0$. Then there exists $x^* \in X^*$ such that $\mathbb{P}(\langle M, x^*\rangle \neq 0) > 0$. Let $N = \langle M, x^*\rangle$. Then N is both with accessible jumps and quasi-left continuous. Hence by Corollary 2.4.13, $N = 0$ a.s., and therefore $M = 0$ a.s. □

Definition 2.4.20. A purely discontinuous local martingale $M^d : \mathbb{R}_+ \times \Omega \to X$ is called to have the *Yoeurp decomposition* if there exist purely discontinuous local martingales $M^q, M^a : \mathbb{R}_+ \times \Omega \to X$ such that M^q is quasi-left continuous, M^a has accessible jumps, $M_0^q = 0$, and $M^d = M^q + M^a$.

Remark 2.4.21. Analogously to Remark 2.2.19 it follows from [89, Corollary 26.16] that the Yoeurp decomposition is unique.

Composing Definition 2.2.18 and 2.4.20 we get the canonical decomposition.

Definition 2.4.22. A local martingale $M : \mathbb{R}_+ \times \Omega \to X$ is called to have the *canonical decomposition* if there exist local martingales $M^c, M^q, M^a : \mathbb{R}_+ \times \Omega \to X$ such that M^c is continuous, M^q and M^a are purely discontinuous, M^q is quasi-left continuous, M^a has accessible jumps, $M_0^c = M_0^q = 0$, and $M = M^c + M^q + M^a$.

Remark 2.4.23. Notice that if $M = M^c + M^q + M^a$ is the canonical decomposition, then $\Delta M^q_\tau = \Delta M_\tau$ for any totally inaccessible stopping time τ since in this case $\Delta M^c_\tau = \Delta M^a_\tau = 0$ by the definition of a continuous local martingale and a local martingale with accessible jumps. Analogously, $\Delta M^a_\tau = \Delta M_\tau$ for any predictable stopping time τ.

The reader can find further details on the martingale decomposition discussed above in [54, 85, 89, 122, 184, 190].

Remark 2.4.24. Note that if a local martingale M has some canonical decomposition, then this decomposition is unique (see Remark 2.2.19 and [89, 184, 185, 190]).

2.4.4. Time-change

A nondecreasing, right-continuous family of stopping times $\tau = (\tau_s)_{s\geq 0}$ is called a *random time-change*. Since $\mathbb{F}$ is right-continuous, according to [89, Lemma 7.3] the same holds true for the *induced filtration* $\mathbb{G} = (\mathscr{G}_s)_{s\geq 0} = (\mathscr{F}_{\tau_s})_{s\geq 0}$ (see more in [89, Chapter 7]). Let X be a Banach space. A martingale $M : \mathbb{R}_+ \times \Omega \to X$ is said to be τ*-continuous* if M is an a.s. constant on every interval $[\tau_{s-}, \tau_s]$, $s \geq 0$, where we let $\tau_{0-} = 0$. In the sequel we will frequently apply the following theorem.

Theorem 2.4.25. *Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be a strictly increasing continuous predictable process such that $A_0 = 0$ and $A_t \to \infty$ as $t \to \infty$ a.s. Let $\tau = (\tau_s)_{s\geq 0}$ be a random time-change defined as $\tau_s := \{t : A_t = s\}$, $s \geq 0$. Then $(A\circ\tau)(t) = (\tau\circ A)(t) = t$ a.s. for each $t \geq 0$. Let $\mathbb{G} = (\mathscr{G}_s)_{s\geq 0} = (\mathscr{F}_{\tau_s})_{s\geq 0}$ be the induced filtration. Then $(A_t)_{t\geq 0}$ is a random time-change with respect to $\mathbb{G}$ and for any $\mathbb{F}$-bounded martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ the following holds*

(i) $M \circ \tau$ is a continuous $\mathbb{G}$-bounded martingale if and only if M is continuous, and

(ii) $M \circ \tau$ is a purely discontinuous $\mathbb{G}$-bounded martingale if and only if M is purely discontinuous.

Proof. Let us first show that $(A\circ\tau)(t) = (\tau\circ A)(t) = t$ a.s. for each $t \geq 0$. Fix $t \geq 0$. Then a.s.

$$(\tau \circ A)(t) = \tau_{A_t} = \{s : A_s = A_t\} = t. \tag{2.4.5}$$

Since A is strictly increasing continuous and starts at zero, there exists $S_t : \Omega \to \mathbb{R}_+$ such that $A_{S_t} = t$ a.s. Then by (2.4.5) and the definition of S_t a.s.

$$(A\circ\tau)(t) = (A\circ\tau)(A_{S_t}) = (A\circ(\tau\circ A))(S_t) = A_{S_t} = t.$$

Now we turn to the second part of the theorem. Notice that $s \mapsto \tau_s$, $s \geq 0$, is a continuous strictly increasing $\mathbb{G}$-predictable process which starts at zero. Then for each $t \geq 0$ one has that $A_t = \{s : \tau_s = t\}$, so $(A_t)_{t\geq 0}$ is a random time-change with respect to the filtration $\mathbb{G}$. Since $(A\circ\tau)(t) = (\tau\circ A)(t) = t$ a.s. for each $t \geq 0$, it is sufficient to show only "if" parts of both (i) and (ii).

(i) follows from the fact that $\tau_{s-} = \tau_s$ (so M is τ-continuous), and the Kazamaki theorem [89, Theorem 17.24]. Let us now show (ii). Thanks to [89, Theorem 7.12] $M\circ\tau$ is a martingale. Let $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a continuous bounded $\mathbb{G}$-bounded martingale such that $N_0 = 0$. Then by (i), $N \circ A$ is a continuous bounded $\mathbb{F}$-bounded martingale, and therefore by Proposition 2.2.12 the process $M \cdot (N\circ A)$ is a martingale. Consequently due to [89, Theorem 7.12], $(M\circ\tau)N = (M\cdot(N\circ A))\circ\tau$ is a martingale. Since N is taken arbitrary and due to Proposition 2.2.12, $M\circ\tau$ is purely discontinuous. □

2.5. STOCHASTIC INTEGRATION

Let X be a Banach space, H be a Hilbert space. For each $h \in H$, $x \in X$ we denote a linear operator $g \mapsto \langle g,h\rangle x$, $g \in H$, by $h\otimes x$. The process $\Phi:\mathbb{R}_+\times\Omega\to\mathscr{L}(H,X)$ is called *elementary predictable* with respect to the filtration $\mathbb{F}=(\mathscr{F}_t)_{t\geq 0}$ if it is of the form

$$\Phi(t,\omega)=\sum_{k=1}^{K}\sum_{m=1}^{M}\mathbf{1}_{(t_{k-1},t_k]\times B_{mk}}(t,\omega)\sum_{n=1}^{N}h_n\otimes x_{kmn},\quad t\geq 0,\omega\in\Omega, \tag{2.5.1}$$

where $0\leq t_0<\ldots<t_K<\infty$, for each $k=1,\ldots,K$ the sets $B_{1k},\ldots,B_{Mk}$ are in $F_{t_{k-1}}$, and vectors $h_1,\ldots,h_N$ are orthogonal.

Let $M:\mathbb{R}_+\times\Omega\to H$ be a martingale. Then we define the *stochastic integral* of Φ with respect to M in the following way:

$$\int_0^t\Phi(s)\,\mathrm{d}M(s)=\sum_{k=1}^{K}\sum_{m=1}^{M}\mathbf{1}_{B_{mk}}\sum_{n=1}^{N}\langle(M(t_k\wedge t)-M(t_{k-1}\wedge t)),h_n\rangle x_{kmn},\quad t\geq 0. \tag{2.5.2}$$

We will often write $\Phi\cdot M$ for the process $\int_0^{\cdot}\Phi(s)\,\mathrm{d}M(s)$. The reader can find more on stochastic integration in the finite dimensional case in [89].

Later we will need the following proposition on the canonical decomposition of a stochastic integral.

Proposition 2.5.1. *Let H be a Hilbert space, X be a Banach space, $M:\mathbb{R}_+\times\Omega\to H$ be a martingale, $\Phi:\mathbb{R}_+\times\Omega\to\mathscr{L}(H,X)$ be elementary progressive. Then*

(i) if M is continuous, then $\Phi\cdot M$ is continuous;

(ii) if M is purely discontinuous, then $\Phi\cdot M$ is purely discontinuous;

(iii) if M has accessible jumps, then $\Phi\cdot M$ has accessible jumps;

(iv) if M is quasi-left continuous, then $\Phi\cdot M$ is quasi-left continuous.

Proof. (i): If M is continuous, then by the construction of a stochastic integral (2.5.2), $\Phi\cdot M$ is a finite sum of continuous martingales, so it is continuous as well.

(ii): Notice that according to Remark 2.2.15 the space of purely discontinuous martingales is linear, so again as in (i) by Proposition 2.2.12 and (2.5.2), $\Phi\cdot M$ is a finite sum of purely discontinuous martingales, so it is purely discontinuous as well.

(iii) and (iv): By (2.5.2) we have that for any stopping time τ a.s. $\Delta(\Phi\cdot M)_\tau\neq 0$ implies $\Delta M_\tau\neq 0$. Therefore by Definition 2.4.8 if M has accessible jumps, then $\Phi\cdot M$ has them as well, and if M is quasi-left continuous, then $\Phi\cdot M$ is quasi-left continuous as well. □

2.6. MULTIDIMENSIONAL WIENER PROCESS

Let d be a natural number. $W:\mathbb{R}_+\times\Omega\to\mathbb{R}^d$ is called a *standard d-dimensional Wiener process* if $\langle W,h\rangle$ is a standard Wiener process for each $h\in\mathbb{R}^d$ such that $\|h\|=1$. The following lemma is a multidimensional variation of [93, (3.2.19)].

Lemma 2.6.1. *Let $X=\mathbb{R}$, $d\geq 1$, W be a standard d-dimensional Wiener process, $\Phi,\Psi:\mathbb{R}_+\times\Omega\to\mathscr{L}(\mathbb{R}^d,\mathbb{R})$ be elementary predictable. Then for all $t\geq 0$ a.s.*

$$[\Phi\cdot W,\Psi\cdot W]_t=\int_0^t\langle\Phi^*(s),\Psi^*(s)\rangle\,\mathrm{d}s.$$

The reader can find more on stochastic integration with respect to a Wiener process in the Hilbert space case in [48], in the case of Banach spaces with a martingale type 2 in [25], and in the UMD case in [126]. Notice that the last mentioned work provides sharp L^p-estimates for stochastic integrals for the broadest till now known class of spaces.

2.7. BROWNIAN REPRESENTATION

The following theorem can be found in [93, Theorem 3.4.2] (see also [170, 186]).

Theorem 2.7.1. *Let $d\geq 1$, $M:\mathbb{R}_+\times\Omega\to\mathbb{R}^d$ be a continuous martingale such that $[M]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Then there exist an enlarged probability space $(\widetilde{\Omega},\widetilde{\mathscr{F}},\widetilde{\mathbb{P}})$ with an enlarged filtration $\widetilde{\mathbb{F}}=(\widetilde{F}_t)_{t\geq 0}$, a d-dimensional standard Wiener process $W:\mathbb{R}_+\times\widetilde{\Omega}\to\mathbb{R}^d$ which is defined on the filtration $\widetilde{\mathbb{F}}$, and predictable $\Phi:\mathbb{R}_+\times\widetilde{\Omega}\to\mathscr{L}(\mathbb{R}^d)$ such that $M=\Phi\cdot W$.*

2.8. RANDOM MEASURES

Throughout, H always denotes a Hilbert space. We let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space and let $\mathbb{F}=(\mathscr{F}_t)_{t\geq 0}$ be a filtration that satisfies the usual conditions. Let $(J,\mathscr{J})$ be a measurable space. We write $\widetilde{\mathscr{P}}=\mathscr{P}\otimes\mathscr{J}$ and $\widetilde{\mathscr{O}}:=\mathscr{O}\otimes\mathscr{J}$ for the induced σ-algebras on $\widetilde{\Omega}=\mathbb{R}_+\times\Omega\times J$.

A family $\mu=\{\mu(\omega;\mathrm{d}t,\mathrm{d}x),\omega\in\Omega\}$ of nonnegative measures on $(\mathbb{R}_+\times J;\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{J})$ is called a *random measure*. A random measure μ is called *integer-valued* if it takes values in $\mathbb{N}\cup\{\infty\}$, i.e. for each $A\in\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$ one has that $\mu(A)\in\mathbb{N}\cup\{\infty\}$ a.s., and if $\mu(\{t\}\times J)\in\{0,1\}$ a.s. for all $t\geq 0$. We say that μ is *non-atomic in time* if $\mu(\{t\}\times J)=0$ a.s. for all $t\geq 0$.

A process $F:\mathbb{R}_+\times\Omega\to\mathbb{R}$ is called *optional* if it is $\mathscr{O}$-measurable. A random measure μ is called *optional* (resp. *predictable*) if for any $\widetilde{\mathscr{O}}$-measurable (resp. $\widetilde{\mathscr{P}}$-measurable) nonnegative $F:\mathbb{R}_+\times\Omega\times J\to\mathbb{R}_+$ the stochastic integral

$$(F\star\mu)_t(\omega):=\int_{\mathbb{R}_+\times J}\mathbf{1}_{[0,t]}(s)F(s,\omega,x)\mu(\omega;\mathrm{d}s,\mathrm{d}x),\quad t\geq 0,\ \omega\in\Omega,$$

as a function from $\mathbb{R}_+ \times \Omega$ to $\overline{\mathbb{R}}_+$ is optional (resp. predictable).

Let X be a Banach space. Then we can extend stochastic integration to X-valued processes in the following way. Let $F : \mathbb{R}_+ \times \Omega \times J \to X$, μ be a random measure. The integral

$$(F \star \mu)_t := \int_{\mathbb{R}_+ \times J} F(s, \cdot, x) \mathbf{1}_{[0,t]}(s) \mu(\cdot; \mathrm{d}s, \mathrm{d}x), \quad t \geq 0,$$

is well-defined and optional (resp. predictable) if μ is optional (resp. predictable), F is $\widetilde{\mathscr{O}}$-strongly-measurable (resp. $\widetilde{\mathscr{P}}$-strongly-measurable), and $\int_{\mathbb{R}_+ \times J} \|F\| \,\mathrm{d}\mu$ is a.s. bounded.

A random measure μ is called $\widetilde{\mathscr{P}}$-σ-finite if there exists an increasing sequence of sets $(A_n)_{n \geq 1} \subset \widetilde{\mathscr{P}}$ such that $\int_{\mathbb{R}_+ \times J} \mathbf{1}_{A_n}(s, \omega, x) \mu(\omega; \mathrm{d}s, \mathrm{d}x)$ is finite a.s. and $\cup_n A_n = \mathbb{R}_+ \times \Omega \times J$. According to [85, Theorem II.1.8] every $\widetilde{\mathscr{P}}$-σ-finite optional random measure μ has a *compensator*: a unique $\widetilde{\mathscr{P}}$-σ-finite predictable random measure ν such that $\mathbb{E} \int_{\mathbb{R}_+ \times J} F \,\mathrm{d}\mu = \mathbb{E} \int_{\mathbb{R}_+ \times J} F \,\mathrm{d}\nu$ for each $\widetilde{\mathscr{P}}$-measurable real-valued nonnegative F. We refer the reader to [85, Chapter II.1] for more details on random measures. For any optional $\widetilde{\mathscr{P}}$-σ-finite measure μ we define the associated compensated random measure by $\bar{\mu} = \mu - \nu$.

For each $\widetilde{\mathscr{P}}$-strongly-measurable $F : \mathbb{R}_+ \times \Omega \times J \to X$ such that $\mathbb{E}(\|F\| \star \mu)_\infty < \infty$ (or, equivalently, $\mathbb{E}(\|F\| \star \nu)_\infty < \infty$, see the definition of a compensator above) we can define a process $F \star \bar{\mu}$ by $F \star \mu - F \star \nu$. The reader should be warned that in the literature $F \star \bar{\mu}$ is often used to denote the integral of F over the whole $\mathbb{R}_+$ (i.e. $(F \star \bar{\mu})_\infty$ in our notation). The following lemma is a vector-valued version of [85, Definition 1.27].

Lemma 2.8.1. *Let X be a Banach space, μ be a $\widetilde{\mathscr{P}}$-σ-finite optional random measure, $F : \mathbb{R}_+ \times \Omega \times J \to X$ be $\widetilde{\mathscr{P}}$-strongly-measurable such that $\mathbb{E} \int_{\mathbb{R}_+ \times J} \|F\| \,\mathrm{d}\mu < \infty$. Then $\big(\int_{[0,t] \times J} F \,\mathrm{d}\bar{\mu}\big)_{t \geq 0}$ is a purely discontinuous X-valued martingale.*

Proof. It is sufficient to show that

$$t \mapsto \Big\langle \int_{[0,t] \times J} F \,\mathrm{d}\bar{\mu}, x^* \Big\rangle = \int_{[0,t] \times J} \langle F, x^* \rangle \,\mathrm{d}\bar{\mu}, \quad t \geq 0,$$

is a purely discontinuous martingale for each $x^* \in X^*$, which can be shown similarly the discussion right below [85, Definition 1.27]. □

We will also need the following lemma.

Lemma 2.8.2. *Let $A \in \widetilde{\mathscr{P}}$, μ_1 be a $\widetilde{\mathscr{P}}$-σ-finite random measure with a compensator ν_1. Then $\mu_2 = \mu_1 \mathbf{1}_A$ is a $\widetilde{\mathscr{P}}$-σ-finite random measure and $\nu_2 = \nu_1 \mathbf{1}_A$ is a compensator for μ_2.*

Proof. μ_2 is $\widetilde{\mathscr{P}}$-σ-finite since $\mu_2 \leq \mu_1$ a.s. Moreover, μ_2 is optional. Indeed, let $F : \mathbb{R}_+ \times \Omega \times J \to \mathbb{R}_+$ be $\widetilde{\mathscr{O}}$-measurable. Then

$$F \star \mu_2 = F \star (\mu_1 \mathbf{1}_A) = (F \mathbf{1}_A) \star \mu_1,$$

and the last process is obviously optional.

Now let us show that $\nu_2 = \nu_1 \mathbf{1}_A$. Let $F : \mathbb{R}_+ \times \Omega \times J \to \mathbb{R}$ be simple $\widetilde{\mathscr{P}}$-measurable. Since μ_1 is $\widetilde{\mathscr{P}}$-σ-finite, so are ν_1, μ_1, ν_2. Hence, we can assume without loss of generality that $F \star \mu_1$ exists and is integrable. Then $F \star \mu_2 = F \star (\mu_1 \mathbf{1}_A) = (F\mathbf{1}_A) \star \mu_1$ exists and is integrable. Moreover,

$$\mathbb{E}(F \star \mu_2)_\infty = \mathbb{E}((F\mathbf{1}_A) \star \mu_1)_\infty = \mathbb{E}((F\mathbf{1}_A) \star \nu_1)_\infty = \mathbb{E}(F \star \nu_2)_\infty,$$

so ν_2 is a compensator of μ_2. □

The reader can find more information on random measures in [85, 89, 110, 114, 131].

2.9. γ-RADONIFYING OPERATORS

Let $(\gamma'_n)_{n\geq 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega', \mathscr{F}', \mathbb{P}')$ (we reserve the notation $(\Omega, \mathscr{F}, \mathbb{P})$ for the probability space on which our processes live) and let H be a separable Hilbert space. A bounded operator $R \in \mathscr{L}(H, X)$ is said to be *γ-radonifying* if for some (and then for each) orthonormal basis $(h_n)_{n\geq 1}$ of H the Gaussian series $\sum_{n\geq 1} \gamma'_n R h_n$ converges in $L^2(\Omega'; X)$. We then define

$$\|R\|_{\gamma(H,X)} := \Big(\mathbb{E}'\Big\| \sum_{n\geq 1} \gamma'_n R h_n \Big\|_X^2\Big)^{\frac{1}{2}}. \tag{2.9.1}$$

Often we will call $\|R\|_{\gamma(H,X)}$ the *γ-norm* of R. This number does not depend on the sequence $(\gamma'_n)_{n\geq 1}$ and the basis $(h_n)_{n\geq 1}$, and defines a norm on the space $\gamma(H, X)$ of all γ-radonifying operators from H into X. Endowed with this norm, $\gamma(H, X)$ is a Banach space, which is separable if X is separable. Moreover, if $X = L^q(S)$ for some separable measure space (S, Σ, ρ), then thanks to the Trace Duality that is presented e.g. in [80] we have that

$$(\gamma(H, X))^* \simeq \gamma(H^*, X^*). \tag{2.9.2}$$

We refer to [80, Section 9.2] and [125] for further details on γ-radonifying operators.

2.10. CONVEX, CONCAVE, BICONCAVE, ZIGZAG-CONCAVE FUNCTIONS

Definition 2.10.1. Let E be a linear space over the scalar field $\mathbb{K}$.

(i) A function $f : E \to \mathbb{R}$ is called *convex* if for each $x, y \in E$, $\lambda \in [0,1]$ one has that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$.

(ii) A function $f : E \to \mathbb{R}$ is called *concave* if for each $x, y \in E$, $\lambda \in [0,1]$ one has that $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$.

(iii) A function $f : E \times E \to \mathbb{R}$ is called *biconcave* if for each $x, y \in E$ one has that the mappings $e \mapsto f(x,e)$ and $e \mapsto f(e,y)$ are concave.

(iv) A function $f : E \times E \to \mathbb{R}$ is called *zigzag-concave* if for each $x, y \in E$ and $\varepsilon \in \mathbb{K}$, $|\varepsilon| \le 1$ the function $z \mapsto f(x+z, y+\varepsilon z)$ is concave.

Note that our definition of zigzag-concavity is a bit different from the classical one (e.g. as in [79]): usually one sets in the definition $|\varepsilon| = 1$. The reader should pay attention to this extension: thanks to this additional property Theorem 3.3.7 later will be more general than [79, Theorem 4.5.6].

2.11. CORRESPONDING DUAL BASIS

Definition 2.11.1. Let d be a natural number, E be a d-dimensional linear space, $(e_n)_{n=1}^d$ be a basis of E. Then $(e_n^*)_{n=1}^d \subset E^*$ is called the *corresponding dual basis* of $(e_n)_{n=1}^d$ if $\langle e_n, e_m^*\rangle = \delta_{nm}$ for each $m, n = 1, \dots, d$.

Note that the corresponding dual basis is uniquely determined. Moreover, if $(e_n^*)_{n=1}^d$ is the corresponding dual basis of $(e_n)_{n=1}^d$, then, the other way around, $(e_n)_{n=1}^d$ is the corresponding dual basis of $(e_n^*)_{n=1}^d$ (here we identify E^{**} with E in the natural way). The following lemma shows that a trace of bilinear forms does not depend on the choice of basis.

Lemma 2.11.2. *Let d be a natural number, E be a d-dimensional linear space. Let $V : E \times E \to \mathbb{R}$ and $W : E^* \times E^* \to \mathbb{R}$ be two bilinear functions. Then the expression*

$$\sum_{n,m=1}^{d} V(e_n, e_m) W(e_n^*, e_m^*) \tag{2.11.1}$$

does not depend on the choice of basis $(e_n)_{n=1}^d$ of E (here $(e_n^*)_{n=1}^d$ is the corresponding dual basis of $(e_n)_{n=1}^d$).

Proof. Let $(e_n)_{n=1}^d$ be a basis of E, $(e_n^*)_{n=1}^d$ be the corresponding dual basis. Fix another basis $(\tilde{e}_n)_{n=1}^d$ of E. Let $(\tilde{e}_n^*)_{n=1}^d$ be the corresponding dual basis of E^*. Let matrices $A = (a_{ij})_{i,j=1}^d$ and $B = (b_{ij})_{i,j=1}^d$ be such that $\tilde{e}_n = \sum_{i=1}^d a_{ni} e_i$, $\tilde{e}_n^* = \sum_{i=1}^d b_{ni} e_i^*$ for each $n = 1, \dots, d$. Then for each $n, m = 1, \dots, d$

$$\delta_{nm} = \langle \tilde{e}_n, \tilde{e}_m^* \rangle = \Big\langle \sum_{i=1}^d a_{ni} e_i, \sum_{j=1}^d b_{mj} e_j^* \Big\rangle = \sum_{i=1}^d a_{ni} b_{mi}.$$

Hence $A^T B = I$, and thus also $AB^T = I$ is the identical matrix as well, and therefore $\sum_{i=1}^d a_{in} b_{im} = \delta_{nm}$ for each $n, m = 1 \dots, d$. Consequently, if we paste $(\tilde{e}_n)_{n=1}^d$ and $(\tilde{e}_n^*)_{n=1}^d$ in (2.11.1), due to the bilinearity of V and W

$$\sum_{n,m=1}^{d} V(\tilde{e}_n, \tilde{e}_m) W(\tilde{e}_n^*, \tilde{e}_m^*) = \sum_{i,j,k,l,n,m=1}^{d} V(a_{ni} e_i, a_{mj} e_j) W(b_{nk} e_k^*, b_{ml} e_l^*)$$

$$= \sum_{i,j,k,l=1}^{d} \sum_{n=1}^{d} a_{ni} b_{nk} \sum_{m=1}^{d} a_{mj} b_{ml} V(e_i, e_j) W(e_k^*, e_l^*)$$
$$= \sum_{i,j,k,l=1}^{d} \delta_{ik}\delta_{jl} V(e_i, e_j) W(e_k^*, e_l^*)$$
$$= \sum_{i,j=1}^{d} V(e_i, e_j) W(e_i^*, e_j^*).$$

□

Corollary 2.11.3. *Let d be a natural number, E be a d-dimensional linear space. Let $V: E \times E \to \mathbb{R}$ and $W_1, W_2: E^* \times E^* \to \mathbb{R}$ be bilinear functions. Assume additionally that V is symmetric nonnegative (i.e. $V(x,x) \geq 0$ for all $x \in E$) and that $W_1(x^*, x^*) \leq W_2(x^*, x^*)$ for all $x^* \in X^*$. Then*

$$\sum_{n,m=1}^{d} V(e_n, e_m) W_1(e_n^*, e_m^*) \leq \sum_{n,m=1}^{d} V(e_n, e_m) W_2(e_n^*, e_m^*)$$

for any basis $(e_n)_{n=1}^d$ of E (here $(e_n^*)_{n=1}^d$ is the corresponding dual basis of $(e_n)_{n=1}^d$).

Proof. Since V is symmetric and nonnegative it defines an inner product on $E \times E$. Let $(\tilde{e}_n)_{n=1}^d$ be an orthogonal basis of E under the inner product V (i.e. $V(\tilde{e}_n, \tilde{e}_m) = 0$ for all $n \neq m$, and $V(\tilde{e}_n, \tilde{e}_n) \geq 0$ for all $n = 1, \dots, d$). Then we have that

$$\begin{aligned} \sum_{n,m=1}^{d} V(\tilde{e}_n, \tilde{e}_m) W_1(\tilde{e}_n^*, \tilde{e}_m^*) &= \sum_{n=1}^{d} V(\tilde{e}_n, \tilde{e}_n) W_1(\tilde{e}_n^*, \tilde{e}_n^*) \\ &\leq \sum_{n=1}^{d} V(\tilde{e}_n, \tilde{e}_n) W_2(\tilde{e}_n^*, \tilde{e}_n^*) = \sum_{n,m=1}^{d} V(\tilde{e}_n, \tilde{e}_m) W_2(\tilde{e}_n^*, \tilde{e}_m^*), \end{aligned} \tag{2.11.2}$$

where $(\tilde{e}_n^*)_{n=1}^d$ is the corresponding dual basis of $(\tilde{e}_n)_{n=1}^d$. Consequently, the desired follows from (2.11.2) and Lemma 2.11.2. □

2.12. ITÔ'S FORMULA

The following theorem is a variation of [89, Theorem 26.7] which does not use the Hilbert space structure of a finite dimensional space.

Theorem 2.12.1 (Itô's formula). *Let d be a natural number, X be a d-dimensional Banach space, $f \in C^2(X)$, $M: \mathbb{R}_+ \times \Omega \to X$ be a martingale. Let $(x_n)_{n=1}^d$ be a basis of X, $(x_n^*)_{n=1}^d$ be the corresponding dual basis. Then for each $t \geq 0$*

$$\begin{aligned} f(M_t) = f(M_0) &+ \int_0^t \langle \partial_x f(M_{s-}), \mathrm{d}M_s \rangle \\ &+ \frac{1}{2} \int_0^t \sum_{n,m=1}^{d} f_{x_n, x_m}(M_{s-}) \, \mathrm{d}[\langle M, x_n^* \rangle, \langle M, x_m^* \rangle]_s^c \\ &+ \sum_{s \leq t} (\Delta f(M_s) - \langle \partial_x f(M_{s-}), \Delta M_s \rangle). \end{aligned} \tag{2.12.1}$$

Here $\partial_x f(y) \in X^*$ is the Fréchet derivative of f in point $y \in X$.

Proof. To apply [89, Theorem 26.7] one needs only to endow X with a proper Euclidean norm $|||\cdot|||$. Define $|||x||| = (\sum_{n=1}^d |\langle x, x_n^*\rangle|^2)^{1/2}$ for each $x \in X$. Then $(x_n)_{n=1}^d$ is an orthonormal basis of $(X, |||\cdot|||)$, $M = \sum_{n=1}^d \langle M, x_n^*\rangle x_n$ is a decomposition of M in this orthonormal basis, and therefore (2.12.1) is equivalent to the formula in [89, Theorem 26.7]. □

II

Weak differential subordination and the canonical decomposition of martingales

3

WEAK DIFFERENTIAL SUBORDINATION OF PURELY DISCONTINUOUS MARTINGALES

This chapter is based on the paper *Fourier multipliers and weak differential subordination of martingales in UMD Banach spaces* by Ivan Yaroslavtsev, see [189].

In this chapter we introduce the notion of weak differential subordination for martingales and show that a Banach space X is a UMD Banach space if and only if for all $p \in (1,\infty)$ and all purely discontinuous X-valued martingales M and N such that N is weakly differentially subordinate to M, one has the estimate $\mathbb{E}\|N_\infty\|^p \leq C_p \mathbb{E}\|M_\infty\|^p$. As a corollary we derive the sharp estimate for the norms of a broad class of even Fourier multipliers, which includes e.g. the second order Riesz transforms.

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3.1. INTRODUCTION

Applying stochastic techniques to Fourier multiplier theory has a long history (see e.g. [9, 10, 15, 23, 32, 61, 66, 118]). It turns out that the boundedness of certain Fourier multipliers with values in a Banach space X is equivalent to this Banach space being in a special class, namely in the class of UMD Banach spaces. Burkholder in [32] and Bourgain in [23] showed that the Hilbert transform is bounded on $L^p(\mathbb{R};X)$ for $p\in(1,\infty)$ if and only if X is UMD. The same type of assertion can be proven for the Beurling-Ahlfors transform, see the paper [66] by Geiss, Montgomery-Smith and Saksman. Examples of UMD spaces include the reflexive range of L^q-, Sobolev and Besov spaces.

A more general class of Fourier multiplier has been considered in recent works of Bañuelos and Bogdan [10] and Bañuelos, Bielaszewski and Bogdan [9]. They derive sharp estimates for the norm of a Fourier multiplier with symbol

$$m(\xi)=\frac{\int_{\mathbb{R}^d}(1-\cos\xi\cdot z)\phi(z)V(\,\mathrm{d}z)+\frac{1}{2}\int_{S^{d-1}}(\xi\cdot\theta)^2\psi(\theta)\mu(\,\mathrm{d}\theta)}{\int_{\mathbb{R}^d}(1-\cos\xi\cdot z)V(\,\mathrm{d}z)+\frac{1}{2}\int_{S^{d-1}}(\xi\cdot\theta)^2\mu(\,\mathrm{d}\theta)},\ \xi\in\mathbb{R}^d, \tag{3.1.1}$$

on $L^p(\mathbb{R}^d)$. Here we will extend their result to $L^p(\mathbb{R}^d;X)$ for UMD spaces X. More precisely, we will show that a Fourier multiplier T_m with a symbol of the form (3.1.1) is bounded on $L^p(\mathbb{R}^d;X)$ if V is a Lévy measure, μ is a Borel positive measure, $|\phi|,|\psi|\le 1$, and that then the norm of T_m does not exceed the UMD$_p$ constant of X. In Subsection 3.4.2, several examples of symbols m of the form (3.1.1) are given, and we will see that for some particular symbols m the norm of T_m equals the UMD constant.

To prove the generalization of the results in [9, 10] we will need additional geometric properties of a UMD Banach space. In the fundamental paper [35], Burkholder showed that a Banach space X is UMD if and only if for some $\beta>0$ there exists a zigzag-concave function $U: X\times X\to\mathbb{R}$ (i.e., a function U such that $U(x+z,y+\varepsilon z)$ is concave in z for any sign ε and for any $x,y\in X$) such that $U(x,y)\ge \|y\|^p-\beta^p\|x\|^p$ for all $x,y\in X$. Such a function U is called a *Burkholder function*. In this situation, we can in fact take β equal to the UMD$_p$ constant of X (see Section 2.3 and Theorem 3.3.7). By exploiting appropriate Burkholder functions U one can prove a wide variety of interesting results (see [11, 14, 15, 16, 33, 34, 179] and the works [133, 134, 135, 138, 139, 140, 141, 142, 143, 144] by Osękowski). For our purposes the following result due to Burkholder [33] (for the scalar case) and Wang [179] (for the Hilbert space case) is of special importance:

Theorem 3.1.1. *Let H be a Hilbert space, $(d_n)_{n\ge0}$, $(e_n)_{n\ge0}$ be two H-valued martingale difference sequences such that $\|e_n\|\le\|d_n\|$ a.s. for all $n\ge0$. Then for each $p\in(1,\infty)$,*

$$\mathbb{E}\Big\|\sum_{n\ge0}e_n\Big\|^p\le(p^*-1)^p\,\mathbb{E}\Big\|\sum_{n\ge0}d_n\Big\|^p.$$

Here and in the sequel $p^* = \max(p, p')$, where $\frac{1}{p} + \frac{1}{p'} = 1$. This result cannot be generalized beyond the Hilbertian setting; see [140, Theorem 3.24(i)] and [79, Example 4.5.17]. In the present chapter we will show the following UMD variant of Theorem 3.1.1:

Theorem 3.1.2. *Let X be a UMD space, $(d_n)_{n\geq 0}$, $(e_n)_{n\geq 0}$ be two X-valued martingale difference sequences, $(a_n)_{n\geq 0}$ be a scalar-valued adapted sequence such that $|a_n| \leq 1$ and $e_n = a_n d_n$ for all $n \geq 0$. Then for each $p \in (1,\infty)$*

$$\mathbb{E}\Big\|\sum_{n\geq 0} e_n\Big\|^p \leq \beta_{p,X}^p \mathbb{E}\Big\|\sum_{n\geq 0} d_n\Big\|^p,$$

where $\beta_{p,X}$ is the UMD$_p$-constant of X (notice that Burkholder proved the identity $\beta_{p,H} = p^* - 1$ for a Hilbert space H, see [33]). Theorem 3.1.2 generalizes a famous Burkholder's result [30, Theorem 2.2] on martingale transforms, where $(a_n)_{n\geq 0}$ was supposed to be predictable. The main tool for proving Theorem 3.1.2 is a Burkholder function with a stricter zigzag-concavity: now we also require $U(x+z, y+\varepsilon z)$ to be concave in z for any ε such that $|\varepsilon| \leq 1$. In the finite dimensional case one gets it for free thanks to the existence of an explicit formula of U (see Remark 3.5.4 and [179]). Here we show the existence of such a Burkholder function in infinite dimension.

For the applications of our abstract results to the theory of Fourier multipliers we extend Theorem 3.1.2 to the continuous time setting. Namely, we show an analogue of Theorem 3.1.2 for purely discontinuous martingales (i.e. martingales which quadratic variations are pure jump processes, see Subsection 3.3.2).

An extension of Theorem 3.1.2 to general continuous-time martingales is shown in the paper [184]. Nevertheless, the sharp estimate in this extension for the case of continuous martingales remains an open problem. This problem is in fact of interest in Harmonic Analysis. If true, this sharp estimate can be used to study a larger class of multipliers, including the Hilbert transform $\mathcal{H}_X$. Garling in [61] proved that

$$\|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R};X))} \leq \beta_{p,X}^2,$$

and it is a long-standing open problem (see [79, pp.496–497]) to prove a linear estimate of the form

$$\|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R};X))} \leq C\beta_{p,X}$$

for some constant C. Here we will show that the latter estimate would indeed follow if one can show the existence of a Burkholder function with certain additional properties. At present, the existence of such Burkholder functions is known only in the Hilbert space case (see Remark 3.5.4).

3.2. PRELIMINARIES

The following lemma is a multidimensional version of [89, Theorem 26.6(v)].

Lemma 3.2.1. *Let d be a natural number, H be a d-dimensional Hilbert space, $M:\mathbb{R}_+\times\Omega\to H$ be a martingale, $\Phi:\mathbb{R}_+\times\Omega\to\mathscr{L}(H,\mathbb{R})$ be elementary progressive. Then $[\Phi\cdot M]\lesssim_d \|\Phi\|^2\cdot[M]$ a.s.*

Proof. Let $(h_n)_{n=1}^d$ be an orthogonal basis of H, $\Phi_1,\dots,\Phi_d:\mathbb{R}_+\times\Omega\to\mathbb{R}$ be such that $\Phi=\sum_{n=1}^d\Phi_n h_n$, and $M_1,\dots,M_d:\mathbb{R}_+\times\Omega\to\mathbb{R}$ be martingales such that $M=\sum_{n=1}^d M_n h_n$. Notice that thanks to the definition of a quadratic variation (2.2.4) one has that $[M]=[M_1]+\dots+[M_d]$. Then since a quadratic variation is a positive-definite quadratic form (see [89, Theorem 26.6]), thanks to [89, Theorem 26.6(v)] one has for each $t\geq 0$ a.s.,

$$\begin{aligned}[\Phi\cdot M]_t=[\Phi_1\cdot M_1+\dots+\Phi_d\cdot M_d]_t &\lesssim_d [\Phi_1\cdot M_1]_t+\dots+[\Phi_d\cdot M_d]_t\\ &=(\|\Phi_1\|^2\cdot[M_1])_t+\dots+(\|\Phi_d\|^2\cdot[M_d])_t\\ &\lesssim_d (\|\Phi\|^2\cdot[M])_t.\end{aligned}$$

□

Using Lemma 3.2.1 one can extend stochastic integral to the case of general Φ. In particular, the following lemma on stochastic integration can be shown.

Lemma 3.2.2. *Let d be a natural number, H be a d-dimensional Hilbert space, $p\in(1,\infty)$, $M,N:\mathbb{R}_+\times\Omega\to H$ be L^p-martingales, $F:H\to H$ be a measurable function such that $\|F(h)\|\leq C\|h\|^{p-1}$ for each $h\in H$ and some $C>0$. Let $N_-:\mathbb{R}_+\times\Omega\to H$ be such that $(N_-)_t=N_{t-}$ for each $t\geq 0$. Then $F(N_-)\cdot M$ is a martingale and for each $t\geq 0$,*

$$\mathbb{E}|(F(N_-)\cdot M)_t|\lesssim_{p,d} C(\mathbb{E}\|N_t\|^p)^{\frac{p-1}{p}}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{3.2.1}$$

Proof. First notice that $F(N_-)$ is predictable. Therefore, thanks to Lemma 3.2.1 and [89, Theorem 26.12], in order to prove that $F(N_-)$ is stochastically integrable with respect to M and that $F(N_-)\cdot M$ is a martingale it is sufficient to show that $\mathbb{E}(\|F(N_-)\|^2\cdot[M])_t^{\frac12}<\infty$. Without loss of generality suppose that $M_0=N_0=0$ a.s. and $C=1$. Then

$$\begin{aligned}\mathbb{E}(\|F(N_-)\|^2\cdot[M])_t^{\frac12}\leq\mathbb{E}(\|N_{t-}\|^{2(p-1)}\cdot[M]_t)^{\frac12}&\leq\mathbb{E}\Big(\sup_{0\leq s\leq t}\|N_s\|^{p-1}[M]_t^{\frac12}\Big)\\ &\overset{(i)}{\leq}(\mathbb{E}\sup_{0\leq s\leq t}\|N_s\|^p)^{\frac{p-1}{p}}(\mathbb{E}[M]_t^{\frac{p}{2}})^{\frac1p}\\ &\overset{(ii)}{\lesssim}_p(\mathbb{E}\|N_t\|^p)^{\frac{p-1}{p}}(\mathbb{E}\|M_t\|^p)^{\frac1p}<\infty,\end{aligned} \tag{3.2.2}$$

where (i) follows from the Hölder inequality, and (ii) holds thanks to [89, Theorem 26.12] and [93, Theorem 1.3.8(iv)].

Now let us show (3.2.1):

$$\mathbb{E}|(F(N_-)\cdot M)_t|\overset{(i)}{\lesssim}_p\mathbb{E}[F(N_-)\cdot M]_t^{\frac12}\overset{(ii)}{\lesssim}_d\mathbb{E}(\|F(N_-)\|^2\cdot[M])_t^{\frac12}$$

$$\overset{(iii)}{\lesssim}_p (\mathbb{E}\|N_t\|^p)^{\frac{p-1}{p}} (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}.$$

Here (i) follows from [89, Theorem 26.12], (ii) holds thanks to Lemma 3.2.1, and (iii) follows from (3.2.2). □

3.3. UMD BANACH SPACES AND WEAK DIFFERENTIAL SUBORDINATION

From now on the scalar field $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$.

3.3.1. *Discrete case*

In this subsection we assume that X is a Banach space over the scalar field $\mathbb{K}$ and with a separable dual X^*. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space with filtration $\mathbb{F} := (\mathscr{F}_n)_{n\geq 0}$, $\mathscr{F}_0 = \{\varnothing,\Omega\}$.

Definition 3.3.1. Let $(f_n)_{n\geq 0}$, $(g_n)_{n\geq 0}$ be X-valued local martingales. For each $n \geq 1$ we define $df_n := f_n - f_{n-1}$, $dg_n := g_n - g_{n-1}$.

(i) g is *differentially subordinate* to f (we will often write $g \ll f$) if one has that $\|dg_n\| \leq \|df_n\|$ a.s. for all $n \geq 1$ and $\|g_0\| \leq \|f_0\|$ a.s.

(ii) g is *weakly differentially subordinate* to f (we will often write $g \overset{w}{\ll} f$) if for each $x^* \in X^*$ one has that $|\langle dg_n, x^*\rangle| \leq |\langle df_n, x^*\rangle|$ a.s. for all $n \geq 1$ and $|\langle g_0, x^*\rangle| \leq |\langle f_0, x^*\rangle|$ a.s.

The following characterization of Hilbert spaces can be found in [140, Theorem 3.24(i)]:

Theorem 3.3.2. *A Banach space X is isomorphic to a Hilbert space if and only if for some (equivalently, for all) $1 < p < \infty$ there exists a constant $\alpha_{p,X} > 0$ such that for any pair of X-valued local martingales $(f_n)_{n\geq 0}$, $(g_n)_{n\geq 0}$ such that g is differentially subordinate to f one has that*

$$\mathbb{E}\|g_n\|^p \leq \alpha_{p,X}^p \mathbb{E}\|f_n\|^p \tag{3.3.1}$$

for each $n \geq 1$.

By the Pettis measurability theorem [79, Theorem 1.1.20], we may assume that X is separable. Then weak differential subordination implies differential subordination. Indeed, let $(x_k)_{k\geq 1}$ be a dense subset of X, $(x_k^*)_{k\geq 1}$ be a sequence of linear functionals on X such that $\langle x_k, x_k^*\rangle = \|x_k\|$ and $\|x_k^*\| = 1$ for each $k \geq 1$ (such a sequence exists by the Hahn-Banach theorem). Let $(g_n)_{n\geq 0}$ be weakly differentially subordinate to $(f_n)_{n\geq 0}$. Then for each $n \geq 1$ a.s.

$$\|dg_n\| = \sup_{k\geq 1} |\langle dg_n, x_k^*\rangle| \leq \sup_{k\geq 1} |\langle df_n, x_k^*\rangle| = \|df_n\|.$$

By the same reasoning $\|g_0\| \le \|f_0\|$ a.s. This means that the weak differential subordination property is more restrictive than the differential subordination property. Therefore, under the weak differential subordination, one could expect that the assertions of the type (3.3.1) characterize a broader class of Banach spaces X. Actually we will prove the following theorem, which extends [34, Theorem 2] to the UMD case.

Theorem 3.3.3. *A Banach space X is a UMD space if and only if for some (equivalently, for all) $1 < p < \infty$ there exists a constant $\beta > 0$ such that for all X-valued local martingales $(f_n)_{n\ge 0}$ and $(g_n)_{n\ge 0}$ such that g is weakly differentially subordinate to f one has*

$$\mathbb{E}\|g_n\|^p \le \beta^p \mathbb{E}\|f_n\|^p, \quad n \ge 1. \tag{3.3.2}$$

If this is the case then the smallest admissible β is the UMD constant $\beta_{p,X}$.

Theorem 3.1.2 is contained in this result as a special case.

The proof of Theorem 3.3.3 consists of several steps.

Proposition 3.3.4. *Let X be a Banach space. Let $(f_n)_{n\ge 0}$, $(g_n)_{n\ge 0}$ be two X-valued local martingales. Then g is weakly differentially subordinate to f if and only if there exists an adapted scalar-valued process $(a_n)_{n\ge 0}$ such that $|a_n| \le 1$ a.s. for all $n \ge 1$, $dg_n = a_n df_n$ a.s. and $g_0 = a_0 f_0$ a.s.*

For the proof we will need two lemmas.

Lemma 3.3.5. *Let X be a Banach space, $\ell_1, \ell_2 \in X^*$ be such that $\ker(\ell_1) \subset \ker(\ell_2)$. Then there exists $a \in \mathbb{K}$ such that $\ell_2 = a\ell_1$.*

Proof. If $\ell_2 = 0$, then the assertion is obvious and one can take $a = 0$. Suppose that $\ell_2 \neq 0$. Then $\operatorname{codim}(\ker(\ell_2)) = 1$ (see [96, p.80]), and there exists $x_0 \in X \setminus \ker(\ell_2)$ such that $x_0 \oplus \ker(\ell_2) = X$. Notice that since $\operatorname{codim}(\ker(\ell_1)) \le 1$ and $\ker(\ell_1) \subset \ker(\ell_2)$, one can easily conclude that $\ker(\ell_1) = \ker(\ell_2)$. Let $a = \ell_2(x_0)/\ell_1(x_0)$. Fix $y \in X$. Then there exists $\lambda \in \mathbb{K}$ such that $y - \lambda x_0 \in \ker(\ell_1) = \ker(\ell_2)$. Therefore

$$\ell_2(y) = \ell_2(\lambda x_0) + \ell_2(y - \lambda x_0) = a\ell_1(\lambda x_0) + a\ell_1(y - \lambda x_0) = a\ell_1(y),$$

hence $\ell_2 = a\ell_1$. □

Lemma 3.3.6. *Let X be a Banach space, (S, Σ, μ) be a measure space. Let $f, g : S \to X$ be strongly measurable such that $|\langle g, x^*\rangle| \le |\langle f, x^*\rangle|$ μ-a.s. for each $x^* \in X^*$. Then there exists a measurable function $a : S \to \mathbb{K}$ such that $\|a\|_\infty \le 1$ and $g = af$.*

Proof. By the Pettis measurability theorem [79, Theorem 1.1.20] we can assume X to be separable. Let $(x_m)_{m\ge 1}$ be a dense subset of X. By the Hahn-Banach theorem we can find a sequence $(x_m^*)_{m\ge 1}$ of linear functionals on X such that $\langle x_m, x_m^*\rangle = \|x_m\|$ and $\|x_m^*\| = 1$ for each $m \ge 1$. Let $Y_0 = \mathbb{Q}-\operatorname{span}(x_1^*, x_2^*, \dots)$, and let $Y = \overline{\operatorname{span}(x_1^*, x_2^*, \dots)}$

be a separable closed subspace of X^*. Then $X \hookrightarrow Y^*$ isometrically. Fix a set of full measure S_0 such that for all $x^* \in Y_0$, $|\langle g, x^*\rangle| \le |\langle f, x^*\rangle|$ on S_0. Fix $x^* \in Y$. Let $(y_k)_{k\ge1}$ be a sequence in Y_0 such that $y_k \to x^*$ in Y as $k \to \infty$. Then on S_0 we have that $|\langle g, y_k^*\rangle| \to |\langle g, x^*\rangle|$ and $|\langle f, y_k\rangle| \to |\langle f, x^*\rangle|$. Consequently for each $s \in S_0$,

$$|\langle g(s), x^*\rangle| \le |\langle f(s), x^*\rangle|, \quad x^* \in Y. \tag{3.3.3}$$

Therefore the linear functionals $f(s), g(s) \in X \hookrightarrow Y^*$ are such that $\ker g(s) \subset \ker f(s)$, and hence by Lemma 3.3.5 there exist $a(s)$ defined for each fixed $s \in S_0$ such that $g(s) = a(s)f(s)$. By (3.3.3) one has that $|a(s)| \le 1$.

Let us construct a measurable version of a. Y_0 is countable since it is a $\mathbb{Q}$−span of a countable set. Let $Y_0 = (y_m)_{m\ge1}$. For each $m > 1$ construct $A_m \in \Sigma$ as follows:

$$A_m = \{s \in S : \langle g(s), y_m\rangle \ne 0, \langle g(s), y_{m-1}\rangle = 0, \dots, \langle g(s), y_1\rangle = 0\}$$

and put $A_1 = \{s \in S : \langle g(s), y_1\rangle \ne 0\}$. Obviously on the set $S \setminus \cup_{m=1}^{\infty} A_m$ one has that $g = 0$, so one can redefine $a := 0$ on $S \setminus \cup_{m=1}^{\infty} A_m$. For each $m \ge 1$ we redefine $a := \frac{\langle g, y_m\rangle}{\langle f, y_m\rangle}$ on A_m. Then a constructed in such a way is Σ-measurable. □

Proof of Proposition 3.3.4. The proposition follows from Lemma 3.3.6: the assumption of this lemma holds for df_n and dg_n for any $n \ge 1$, and for f_0 and g_0. So according to Lemma 3.3.6 there exists a sequence $(a_n)_{n\ge0}$ which is a.s. bounded by 1, such that $dg_n = a_n df_n$ for each $n \ge 1$ and $g_0 = a_0 f_0$ a.s. Moreover, again thanks to Lemma 3.3.6, a_n is $\mathscr{F}_n$-measurable, so $(a_n)_{n\ge0}$ is adapted. □

In [35] Burkholder showed that the UMD property is equivalent to the existence of a certain biconcave function $V : X \times X \to \mathbb{R}$. With a slight variation of his argument (see Remark 3.3.10) one can also show the equivalence with the existence of a certain zigzag-concave function with a better structure.

Theorem 3.3.7 (Burkholder). *For a Banach space X the following are equivalent*

1. *X is a UMD Banach space;*
2. *for each $p \in (1,\infty)$ there exists a constant $\beta > 0$ and a zigzag-concave function $U : X \times X \to \mathbb{R}$ such that*

$$U(x, y) \ge \|y\|^p - \beta^p \|x\|^p, \quad x, y \in X. \tag{3.3.4}$$

The smallest admissible β for which such U exists is $\beta_{p,X}$.

Proof. The proof is essentially the same as the one given in [79, Theorem 4.5.6], but the construction of U is a bit different. The only difference is allowing $|\varepsilon| \le 1$ instead of $|\varepsilon| = 1$ for the appropriate scalars ε.

For each $x, y \in X$ we define $\mathbb{S}(x, y)$ as a set of all pairs (f, g) of discrete martingales such that

1. $f_0 \equiv x$, $g_0 \equiv y$;
2. there exists $N \geq 0$ such that $df_n \equiv 0$, $dg_n \equiv 0$ for $n \geq N$;
3. $(dg_n)_{n\geq 1} = (\varepsilon_n df_n)_{n\geq 1}$ for some sequence of scalars $(\varepsilon_n)_{n\geq 1}$ such that $|\varepsilon_n| \leq 1$ for each $n \geq 1$.

Then we define $U : X \times X \to \mathbb{R} \cup \{\infty\}$ as follows:

$$U(x,y) := \sup\big\{\mathbb{E}(\|g_\infty\|^p - \beta^p \|f_\infty\|^p) : (f,g) \in \mathbb{S}(x,y)\big\}. \tag{3.3.5}$$

The rest of the proof repeats the one given in [79, Theorem 4.5.6]. □

Remark 3.3.8. We will call the function U constructed above *the Burkholder function*. Notice that this function coincides with the one in the proof of [79, Theorem 4.5.6]. This is due to the fact that the function

$$(\varepsilon_n)_{n=1}^N \mapsto \Big(\mathbb{E}\Big\|g_0 + \sum_{n=1}^N \varepsilon_n df_n\Big\|^p\Big)^{\frac{1}{p}}$$

is convex on the $\mathbb{K}$-cube $\{(\varepsilon_n)_{n=1}^N : |\varepsilon_1|, \ldots, |\varepsilon_N| \leq 1\}$ because of the triangle inequality, therefore it takes its supremum on the set of the domain endpoints, namely on the set $\{(\varepsilon_n)_{n=1}^N : |\varepsilon_1|, \ldots, |\varepsilon_N| = 1\}$.

Remark 3.3.9. Analogously to [79, (4.31)] by (3.3.5) we have that $U(\alpha x, \alpha y) = |\alpha|^p U(x,y)$ for each $x, y \in X$, $\alpha \in \mathbb{K}$. Therefore $U(0,0) = 0$, and hence for each $x \in X$ and each scalar ε such that $|\varepsilon| \leq 1$, by the zigzag-concavity of U in the point $(0,0)$

$$U(x,\varepsilon x) = \frac{1}{2}U(0+x, 0+\varepsilon x) + \frac{1}{2}U(0-x, 0-\varepsilon x) \leq U(0,0) = 0. \tag{3.3.6}$$

Let $\xi, \eta \in L^0(\Omega; X)$ be such that $|\langle \eta, x^*\rangle| \leq |\langle \xi, x^*\rangle|$ for each $x^* \in X^*$ a.s. Then thanks to Lemma 3.3.6 and (3.3.6), $U(\xi, \eta) \leq 0$ a.s.

Remark 3.3.10. For each zigzag-concave function $U : X \times X \to \mathbb{R}$ one can construct a biconcave function $V : X \times X \to \mathbb{R}$ as follows:

$$V(x,y) = U\Big(\frac{x-y}{2}, \frac{x+y}{2}\Big), \quad x, y \in X. \tag{3.3.7}$$

Indeed, by the definition of U, for each $x, y \in X$ the functions

$$z \mapsto V(x+z, y) = U\Big(\frac{x-y}{2} + \frac{z}{2}, \frac{x+y}{2} + \frac{z}{2}\Big),$$
$$z \mapsto V(x, y+z) = U\Big(\frac{x-y}{2} - \frac{z}{2}, \frac{x+y}{2} + \frac{z}{2}\Big)$$

are concave. Moreover, for each $x, y \in X$ and $a, b \in \mathbb{K}$ such that $|a+b| \leq |a-b|$ one has that the function

$$z \mapsto V(x+az, y+bz) = U\Big(\frac{x-y}{2} + \frac{(a-b)z}{2}, \frac{x+y}{2} + \frac{(a+b)z}{2}\Big)$$

is concave since $\big|\frac{a+b}{a-b}\big| \leq 1$.

Remark 3.3.11. Due to the explicit representation (3.3.5) of U we can show that for each $x_1, x_2, y_1, y_2 \in X$,

$$|U(x_1,y_1) - U(x_2,y_2)| \le \|x_1 - x_2\|^p + \beta_{p,X}^p \|y_1 - y_2\|^p.$$

Therefore U is continuous, and consequently V is continuous as well.

Remark 3.3.12. Notice that if X is finite dimensional then by Theorem 2.20 and Proposition 2.21 in [59] there exists a unique translation-invariant measure λ_X on X such that $\lambda_X(\mathbb{B}_X) = 1$ for the unit ball $\mathbb{B}_X$ of X. We will call λ_X a *Lebesgue measure*. Thanks to the Alexandrov theorem [57, Theorem 6.4.1] $x \mapsto V(x,y)$ and $y \mapsto V(x,y)$ are a.s. Fréchet differentiable with respect to λ_X, and by [86, Proposition 3.1] and Remark 3.3.11 for a.a. $(x,y) \in X \times X$ for each $u, v \in X$ there exists the directional derivative $\frac{\partial V(x+tu,y+tv)}{\partial t}$. Moreover,

$$\frac{\partial V(x+tu,y+tv)}{\partial t} = \langle \partial_x V(x,y), u\rangle + \langle \partial_y V(x,y), v\rangle, \tag{3.3.8}$$

where $\partial_x V$ and $\partial_y V$ are the corresponding Fréchet derivatives with respect to the first and the second variable. Thanks to (3.3.8) and Remark 3.3.10 one obtains that for a.e. $(x,y) \in X \times X$, for all $z \in X$ and $a, b \in \mathbb{K}$ such that $|a+b| \le |a-b|$,

$$\begin{aligned} V(x+az, y+bz) &\le V(x,y) + \frac{\partial V(x+atz, y+btz)}{\partial t} \\ &= V(x,y) + a\langle \partial_x V(x,y), z\rangle + b\langle \partial_y V(x,y), z\rangle. \end{aligned} \tag{3.3.9}$$

Lemma 3.3.13. *Let X be a finite dimensional Banach space, $V : X \times X \to \mathbb{R}$ be as defined in* (3.3.7)*. Then there exists $C > 0$ which depends only on V such that for a.e. pair $x, y \in X$,*

$$\|\partial_x V(x,y)\|, \|\partial_y V(x,y)\| \le C(\|x\|^{p-1} + \|y\|^{p-1}).$$

Proof. We show the inequality only for $\partial_x V$, the proof for $\partial_y V$ being analogous. First we prove that there exists $C > 0$ such that $\|\partial_x V(x,y)\| \le C$ for a.e. $x, y \in X$ such that $\|x\|, \|y\| \le 1$. Let us show this by contradiction. Suppose that such C does not exist. Since V is continuous by Remark 3.3.11, and since a unit ball in X is a compact set, there exists $K > 0$ such that $|V(x,y)| < K$ for all $x, y \in X$ such that $\|x\|, \|y\| \le 2$. Let $x_0, y_0 \in X$ be such that $\|x_0\|, \|y_0\| \le 1$ and $\|\partial_x V(x_0,y_0)\| > 3K$. Therefore there exists $z \in X$ such that $\|z\| = 1$ and $\langle \partial_x V(x_0,y_0), z\rangle < -3K$. Hence we have that $\|x_0 + z\| \le 2$ and because of the concavity of V in the first variable

$$V(x_0+z, y_0) \le V(x_0,y_0) + \langle \partial_x V(x_0,y_0), z\rangle \le K - 3K \le -2K.$$

Consequently, $|V(x_0+z,y_0)| > K$, which contradicts with our suggestion.

Now fix $C > 0$ such that $|\partial_x V(x,y)| \le C$ for all $x, y \in X$ such that $\|x\|, \|y\| \le 1$. Fix $x, y \in X$. Without loss of generality assume that $\|x\| \ge \|y\|$. Let $L = \|x\|$. Then $\|\partial_x V\big(\frac{x}{L}, \frac{y}{L}\big)\| \le C$. Let $z \in X$ be such that $\|z\| = 1$. Then by Remark 3.3.9,

$$|\langle \partial_x V(x,y), z\rangle| = \Big|\lim_{t\to 0} \frac{V(x+tz,y) - V(x,y)}{t}\Big| = \Big|\lim_{t\to 0} \frac{L^p V(\frac{x}{L} + \frac{t}{L}z, \frac{y}{L}) - L^p V(\frac{x}{L}, \frac{y}{L})}{L\frac{t}{L}}\Big|$$

$$
\begin{aligned}
&= L^{p-1}\Big|\lim_{t\to 0}\frac{V(\frac{x}{L}+tz,\frac{y}{L})-V(\frac{x}{L},\frac{y}{L})}{t}\Big| = L^{p-1}\Big|\Big\langle \partial_x V\Big(\frac{x}{L},\frac{y}{L}\Big), z\Big\rangle\Big|\\
&\leq L^{p-1}C \leq C(\|x\|^{p-1}+\|y\|^{p-1}).
\end{aligned}
$$

Therefore since z was arbitrary, $\|\partial_x V(x,y)\| \leq C(\|x\|^{p-1}+\|y\|^{p-1})$. The case $\|x\| < \|y\|$ can be done in the same way. □

Lemma 3.3.14. *Let X be a finite dimensional Banach space, $1<p<\infty$, $(f_n)_{n\geq 0}$, $(g_n)_{n\geq 0}$ be X-valued martingales on a probability space $(\Omega,\mathscr{F},\mathbb{P})$ with a filtration $\mathbb{F}=(\mathscr{F}_n)_{n\geq 0}$ and assume that $(g_n)_{n\geq 0}$ is weakly differentially subordinate to $(f_n)_{n\geq 0}$. Let $Y = X\oplus\mathbb{R}$ be the Banach space with the norm as follows:*

$$
\|(x,r)\|_Y := (\|x\|_X^p + |r|^p)^{\frac{1}{p}},\quad x\in X, r\in\mathbb{R}.
$$

Then there exist two sequences $(f^m)_{m\geq 1}$ and $(g^m)_{m\geq 1}$ of Y-valued martingales on an enlarged probability space $(\overline{\Omega},\overline{\mathscr{F}},\overline{\mathbb{P}})$ with an enlarged filtration $\overline{\mathbb{F}}=(\overline{\mathscr{F}}_n)_{n\geq 0}$ such that

1. *f_n^m, g_n^m have absolutely continuous distributions with respect to the Lebesgue measure on Y for each $m\geq 1$ and $n\geq 0$;*
2. *$f_n^m \to (f_n,0)$, $g_n^m\to(g_n,0)$ pointwise as $m\to\infty$ for each $n\geq 0$;*
3. *if for some $n\geq 0$ $\mathbb{E}\|f_n\|^p<\infty$, then for each $m\geq 1$ one has that $\mathbb{E}\|f_n^m\|^p<\infty$ and $\mathbb{E}\|f_n^m-(f_n,0)\|^p\to 0$ as $m\to\infty$;*
4. *if for some $n\geq 0$ $\mathbb{E}\|g_n\|^p<\infty$, then for each $m\geq 1$ one has that $\mathbb{E}\|g_n^m\|^p<\infty$ and $\mathbb{E}\|g_n^m-(g_n,0)\|^p\to 0$ as $m\to\infty$;*
5. *for each $m\geq 1$ we have that $(g_n^m)_{n\geq 0}$ is weakly differentially subordinate to $(f_n^m)_{n\geq 0}$.*

Proof. First of all let us show that we may assume that f_0 and g_0 are nonzero a.s. For this purpose we can modify f_0 and g_0 as follows:

$$
\begin{aligned}
f_0^\varepsilon &= f_0 + \varepsilon x \mathbf{1}_{f_0=0},\\
g_0^\varepsilon &= g_0 + \varepsilon x \mathbf{1}_{f_0=0} + \varepsilon f_0 \mathbf{1}_{g_0=0, f_0\neq 0},
\end{aligned}
$$

where $\varepsilon>0$ is arbitrary and $x\in X$ is fixed. This small perturbation does not destroy the weak differential subordination property. Moreover, if we let $f_n^\varepsilon := f_0^\varepsilon + \sum_{k=1}^n df_k$, $g_n^\varepsilon := g_0^\varepsilon+\sum_{k=1}^n dg_k$ for any $n\geq 1$, then $f_n^\varepsilon\to f_n$ and $g_n^\varepsilon\to g_n$ a.s., and $f_n^\varepsilon - f_n\to 0$ and $g_n^\varepsilon - g_n\to 0$ in $L^p(\Omega;X)$ as $\varepsilon\to 0$.

From now we assume that f_0 and g_0 are nonzero a.s. This in fact means that random variable a_0 from Proposition 3.3.4 is nonzero a.s. as well. Let $\mathbb{B}_Y$ be the unit ball of Y, $(\mathbb{B}_Y,\mathscr{B}(\mathbb{B}_Y),\hat{\mathbb{P}})$ be a probability space such that $\hat{\mathbb{P}} := \lambda_Y|_{\mathbb{B}_Y}$ has the uniform Lebesgue distribution on $\mathbb{B}_Y$ (see Remark 3.3.12). Fix some scalar product $\langle\cdot,\cdot\rangle : Y\times Y \to \mathbb{R}$ in Y. We will construct a random operator $T:\mathbb{B}_Y \to \mathscr{L}(Y)$ as follows:

$$T(b,y) := \langle b,y\rangle b \quad b\in \mathbb{B}_Y, y\in Y.$$

Note that for each fixed $b\in\mathbb{B}_Y$ the mapping $y\mapsto \langle b,y\rangle b$ is a linear operator on Y. Moreover,

$$\sup_{b\in\mathbb{B}_Y} \|T(b,\cdot)\|_{\mathscr{L}(Y)} < \infty. \tag{3.3.10}$$

Now let $(\overline{\Omega},\overline{\mathscr{F}},\overline{\mathbb{P}}) := (\Omega\times\mathbb{B}_Y, \mathscr{F}\otimes\mathscr{B}(\mathbb{B}_Y), \mathbb{P}\otimes\hat{\mathbb{P}})$. For each $m\geq 1$ define an operator-valued function $Q_m:\overline{\Omega}\to\mathscr{L}(Y)$ as follows: $Q_m := I+\frac{1}{m}T$.

Fix $\varepsilon>0$. For each $n\geq 0$ define $\tilde f_n^\varepsilon := (f_n,\varepsilon)$, $\tilde g_n^\varepsilon := (g_n,\varepsilon a_0)$. Then $(\tilde f_n^\varepsilon)_{n\geq 0}$ and $(\tilde g_n^\varepsilon)_{n\geq 0}$ are Y-valued martingales which are nonzero a.s. for each $n\geq 0$ and are such that $(\tilde g_n^\varepsilon)_{n\geq 0}$ is weakly differentially subordinate to $(\tilde f_n^\varepsilon)_{n\geq 0}$. For each $m\geq 1$ define Y-valued martingales f^m and g^m in the following way:

$$f_n^m := Q_m\tilde f_n^\varepsilon,\ m\geq 1, n\geq 0,$$
$$g_n^m := Q_m\tilde g_n^\varepsilon,\ m\geq 1, n\geq 0.$$

Let us illustrate that for each $m\geq 1$, f^m and g^m are martingales with respect to the filtration $\overline{\mathbb{F}} = (\overline{\mathscr{F}}_n)_{n\geq 0} := (\mathscr{F}_n\otimes\mathscr{B}(\mathbb{B}_Y))_{t\geq 0}$: for each $n\geq 1$ we have

$$\begin{aligned}\mathbb{E}(f_n^m|\overline{\mathscr{F}}_{n-1}) = \mathbb{E}(Q_m\tilde f_n^\varepsilon|\mathscr{F}_{n-1}\otimes\mathscr{B}(\mathbb{B}_Y)) &\overset{(i)}{=} Q_m\mathbb{E}(\tilde f_n^\varepsilon|\mathscr{F}_{n-1}\otimes\mathscr{B}(\mathbb{B}_Y))\\ &\overset{(ii)}{=} Q_m\tilde f_{n-1}^\varepsilon = f_{n-1}^m,\end{aligned}$$

where (i) holds since Q_m is $\mathscr{B}(\mathbb{B}_Y)$-measurable, and (ii) holds since $\tilde f_n^\varepsilon$ is independent of $\mathscr{B}(\mathbb{B}_Y)$. The same can be proven for g^m. Thanks to (3.3.10) one has that $\lim_{m\to\infty}\sup_{b\in\mathbb{B}_Y}\|Q_m - I\|_{\mathscr{L}(Y)} = 0$ and hence (2), (3) and (4) hold for $\tilde f^\varepsilon$ and $\tilde g^\varepsilon$.

Let us prove (5). For each $m\geq 1$ and $n\geq 1$ one has:

$$dg_n^m = dQ_m\tilde g_n^\varepsilon = dQ_m a_n\tilde f_n^\varepsilon = a_n dQ_m\tilde f_n^\varepsilon = a_n df_n^m.$$

The same holds for g_0^m and f_0^m.

Now we will show (1). Let us fix a set $A\subset Y$ of Lebesgue measure zero. Then for each fixed $n\geq 0$ and $m\geq 1$,

$$\begin{aligned}\mathbb{E}\mathbf{1}_{f_n^m\in A} &= \int_\Omega\int_{\mathbb{B}_Y}\mathbf{1}_{\tilde f_n^\varepsilon+\frac{1}{m}\langle b,\tilde f_n^\varepsilon\rangle b\in A}\,\mathrm{d}\hat{\mathbb{P}}(b)\,\mathrm{d}\mathbb{P}\\ &= \int_\Omega\int_{\mathbb{B}_Y}\mathbf{1}_{\frac{1}{m}\langle b,\tilde f_n^\varepsilon\rangle b\in A-\tilde f_n^\varepsilon}\,\mathrm{d}\hat{\mathbb{P}}(b)\,\mathrm{d}\mathbb{P},\end{aligned} \tag{3.3.11}$$

where $F-y$ is a translation of a set $F\subset Y$ by a vector $y\in Y$. For each fixed $y\in Y\setminus\{0\}$ the distribution of a Y-valued random variable $b\mapsto\langle b,y\rangle b$ is absolutely continuous with respect to λ_Y. Since $\hat{\mathbb{P}}(A-y)=0$ for each $y\in Y\setminus\{0\}$, one has

$$\int_{\mathbb{B}_Y}\mathbf{1}_{\frac{1}{m}\langle b,y\rangle b\in A-y}\,\mathrm{d}\hat{\mathbb{P}}(b) = 0. \tag{3.3.12}$$

Recall that $\mathbb{P}\{\tilde f_n^\varepsilon = 0\} = 0$, therefore due to (3.3.12) a.s.

$$\int_{\mathbb{B}_Y} \mathbf{1}_{\frac{1}{m}\langle b, \tilde f_n^\varepsilon\rangle b \in A - \tilde f_n^\varepsilon} \,\mathrm{d}\hat{\mathbb{P}}(b) = 0.$$

Consequently the last double integral in (3.3.11) vanishes. The same works for g^m.

Now to construct such a sequence for $((f_n,0))_{n\geq 0}$ and $((g_n,0))_{n\geq 0}$ one needs to construct it for different ε and take an appropriate subsequence. □

Proof of Theorem 3.3.3. The "if" part is obvious thanks to the definition of a UMD Banach space. Let us prove the "only if" part. As in the proof of the lemma above, without loss of generality suppose that X is separable and that the set $\bigcup_n(\{f_n = 0\}\cup\{g_n = 0\})$ is of $\mathbb{P}$-measure 0. If it does not hold, we consider $Y := X \oplus \mathbb{R}$ instead of X with the norm of $(x,r)\in Y$ given by $\|(x,r)\|_Y = (\|x\|_X^p + |r|^p)^{1/p}$. Notice that then $\beta_{p,Y} = \beta_{p,X}$. We can suppose that a_0 is nonzero a.s., so we consider $(f_n^\varepsilon)_{n\geq 0} := (f_n\oplus\varepsilon)_{n\geq 0}$ and $(g_n^\varepsilon)_{n\geq 0} := (g_n\oplus \varepsilon a_0)_{n\geq 0}$ with $\varepsilon>0$, and let ε go to zero.

One can also restrict to a finite dimensional case. Indeed, since X is a separable reflexive space, X^* is separable as well. Let $(Y_m)_{m\geq 1}$ be an increasing sequence of finite-dimensional subspaces of X^* such that $\overline{\bigcup_m Y_m} = X^*$ and $\|\cdot\|_{Y_m} = \|\cdot\|_{X^*}$ for each $m\geq 1$. Then for each fixed $m\geq 1$ there exists a linear operator $P_m : X \to Y_m^*$ of norm 1 defined as follows: $\langle P_m x, y\rangle = \langle x, y\rangle$ for each $x\in X, y\in Y_m$. Then since Y_m is a closed subspace of X^*, [79, Proposition 4.33] yields $\beta_{p',Y_m}\leq \beta_{p',X^*}$, consequently again by [79, Proposition 4.33] $\beta_{p,Y_m^*}\leq\beta_{p,X^{**}} = \beta_{p,X}$. So if we prove the finite dimensional version, then

$$\mathbb{E}\|P_m g_n\|^p \leq \beta_{p,X}^p \mathbb{E}\|P_m f_n\|^p, \quad n\geq 0,$$

for each $m\geq 1$, and due to the fact that $\|P_m x\|_{Y_m^*}\nearrow \|x\|_X$ for each $x\in X$ as $m\to\infty$, we would obtain (3.3.2) in the general case.

Let β be the UMD constant of X, and let $U, V : X\times X\to\mathbb{R}$ be as defined in Theorem 3.3.7 and in (3.3.7) respectively, $(a_n)_{n\geq 0}$ be as defined in Proposition 3.3.4. By Lemma 3.3.14 we can suppose that f_n and g_n have distributions which are absolutely continuous with respect to the Lebesgue measure. Then

$$\begin{aligned}
\mathbb{E}(\|g_n\|^p - \beta\|f_n\|^p) &\overset{(i)}{\leq} \mathbb{E}U(f_n,g_n) = \mathbb{E}U(f_{n-1}+df_n, g_{n-1}+a_n df_n)\\
&\overset{(ii)}{=} \mathbb{E}V\big(g_{n-1}+f_{n-1}+(a_n+1)df_n, g_{n-1}-f_{n-1}+(a_n-1)df_n\big)\\
&\overset{(iii)}{\leq} \mathbb{E}V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big)\\
&\quad + \mathbb{E}\big\langle \partial_x V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big), (a_n+1)df_n\big\rangle\\
&\quad + \mathbb{E}\big\langle \partial_y V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big), (a_n-1)df_n\big\rangle\\
&\overset{(iv)}{=} \mathbb{E}V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big)\\
&\overset{(v)}{=} \mathbb{E}U(f_{n-1},g_{n-1}).
\end{aligned} \tag{3.3.13}$$

Here (i) and (iii) hold by Theorem 3.3.7 and (3.3.9) respectively, (ii) and (v) follow from the definition of V. Let us prove (iv). We will show that

$$\mathbb{E}\big\langle \partial_x V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big), (a_n+1)df_n\big\rangle = 0. \tag{3.3.14}$$

Since both f_n and $a_n f_n$ are martingale differences, $(a_n+1)df_n$ is a martingale difference as well. Therefore $\mathbb{E}\big((a_n-1)df_n|\mathscr{F}_{n-1}\big) = 0$. Note that according to Lemma 3.3.13 a.s.

$$\|\partial_x V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big)\| \lesssim_V \|f_n\|^{p-1} + \|g_n\|^{p-1}.$$

Therefore by the Hölder inequality $\langle\partial_x V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big), (a_n+1)df_n\rangle$ is integrable. Since $\partial_x V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big)$ is $\mathscr{F}_{n-1}$-measurable,

$$\begin{aligned}
&\mathbb{E}\Big(\big\langle \partial_x V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big), (a_n+1)df_n\big\rangle\big|\mathscr{F}_{n-1}\Big)\\
&= \Big\langle \partial_x V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big), \mathbb{E}\big((a_n+1)df_n\big|\mathscr{F}_{n-1}\big)\Big\rangle\\
&= \big\langle \partial_x V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big), 0\big\rangle = 0,
\end{aligned}$$

so (3.3.14) holds. By the same reason

$$\mathbb{E}\big\langle \partial_y V\big(g_{n-1}+f_{n-1}, g_{n-1}-f_{n-1}\big), (a_n-1)df_n\big\rangle = 0,$$

and (iv) follows.

Notice that thanks to Remark 3.3.9 $\mathbb{E}(f_0, g_0) \le 0$. Therefore from the inequality (3.3.13) by an induction argument we get

$$\mathbb{E}(\|g_n\|^p - \beta^p\|f_n\|^p) \le \mathbb{E}U(f_n, g_n) \le \mathbb{E}U(f_{n-1}, g_{n-1}) \le \ldots \le \mathbb{E}U(f_0, g_0) \le 0.$$

This terminates the proof. □

3.3.2. Continuous time case

Now we turn to continuous time martingales. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\ge 0}$ that satisfies the usual conditions.

Definition 3.3.15. Let $M, N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be local martingales. Then we say that N is *differentially subordinate* to M (we will often write $N \ll M$) if for each $x^* \in X^*$ one has that $[M] - [N]$ is an a.s. nondecreasing function and $|N_0| \le |M_0|$ a.s.

Definition 3.3.16. Let $M, N : \mathbb{R}_+ \times \Omega \to X$ be local martingales. Then we say that N is *weakly differentially subordinate* to M (we will often write $N \overset{w}{\ll} M$) if $\langle N, x^*\rangle \ll \langle M, x^*\rangle$ for each $x^* \in X^*$.

The following theorem is a natural extension of Proposition 3.3.4.

Theorem 3.3.17. *Let X be a Banach space. Then X is a UMD space if and only if for some (equivalently, for all) $1 < p < \infty$ there exists $\beta > 0$ such that for each purely discontinuous X-valued local martingales $M, N : \mathbb{R}_+ \times \Omega \to X$ such that N is weakly differentially subordinate to M one has*

$$\mathbb{E}\|N_t\|^p \le \beta^p \mathbb{E}\|M_t\|^p. \tag{3.3.15}$$

If this is the case then the smallest admissible β equals the UMD constant $\beta_{p,X}$.

Lemma 3.3.18. *Let X be a finite dimensional Banach space, $1 < p < \infty$, $M, N : \mathbb{R}_+ \times \Omega \to X$ be local martingales on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\ge 0}$ such that N is weakly differentially subordinate to M. Let $Y = X \oplus \mathbb{R}$ be a Banach space such that $\|(x,r)\|_Y = (\|x\|_X^p + |r|^p)^{\frac{1}{p}}$ for each $x \in X$, $r \in \mathbb{R}$. Then there exist two sequences $(M^m)_{m\ge 1}$ and $(N^m)_{m\ge 1}$ of Y-valued martingales on an enlarged probability space $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\mathbb{P}})$ with an enlarged filtration $\overline{\mathbb{F}} = (\overline{\mathscr{F}}_t)_{t\ge 0}$ such that*

1. *M_t^m, N_t^m have absolutely continuous distributions with respect to the Lebesgue measure on Y for each $m \ge 1$ and $t \ge 0$;*
2. *$M_t^m \to (M_t, 0)$, $N_t^m \to (N_t, 0)$ pointwise as $m \to \infty$ for each $t \ge 0$;*
3. *if for some $t \ge 0$ $\mathbb{E}\|M_t\|^p < \infty$, then for each $m \ge 1$ one has that $\mathbb{E}\|M_t^m\|^p < \infty$ and $\mathbb{E}\|M_t^m - (M_t, 0)\|^p \to 0$ as $m \to \infty$;*
4. *if for some $t \ge 0$ $\mathbb{E}\|N_t\|^p < \infty$, then for each $m \ge 1$ one has that $\mathbb{E}\|N_t^m\|^p < \infty$ and $\mathbb{E}\|N_t^m - (N_t, 0)\|^p \to 0$ as $m \to \infty$;*
5. *for each $m \ge 1$ we have that N^m is weakly differentially subordinate to M^m.*

Proof. The proof is essentially the same as one of Lemma 3.3.14. □

Proof of Theorem 3.3.17. We use a modification of the argument in [179, Theorem 1], where the Hilbert space case was considered. Thanks to the same methods as were applied in the beginning of the proof of Theorem 3.3.3 and using Lemma 3.3.18 instead of Lemma 3.3.14, one can suppose that X is finite-dimensional and M_t and N_t are nonzero a.s. for each $t \ge 0$. We know that $\mathbb{E}U(M_t, N_t) \ge \mathbb{E}(\|N_t\|^p - \beta^p\|M_t\|^p)$ for each $t \ge 0$. On the other hand, thanks to the fact that $[\langle M, x^*\rangle]$ and $[\langle N, x^*\rangle]$ are pure jump for each $x^* \in X^*$ and the finite-dimensional version of Itô formula [89, Theorem 26.7], one has

$$\begin{aligned}\mathbb{E}U(M_t, N_t) = \mathbb{E}U(M_0, N_0) &+ \mathbb{E}\int_0^t \langle \partial_x U(M_{s-}, N_{s-}), \mathrm{d}M_s\rangle \\ &+ \mathbb{E}\int_0^t \langle \partial_y U(M_{s-}, N_{s-}), \mathrm{d}N_s\rangle + \mathbb{E}I,\end{aligned} \tag{3.3.16}$$

where

$$I = \sum_{0<s\le t} [\Delta U(M_s, N_s) - \langle \partial_x U(M_{s-}, N_{s-}), \Delta M_s\rangle - \langle \partial_y U(M_{s-}, N_{s-}), \Delta N_s\rangle].$$

Note that since a.s.

$$\Delta|\langle N, x^*\rangle|^2 = \Delta[\langle N, x^*\rangle] \le \Delta[\langle M, x^*\rangle] = \Delta|\langle M, x^*\rangle|^2$$

for each $x^* \in X^*$, one has that thanks to Lemma 3.3.6 for each $s \ge 0$, for a.e. $\omega \in \Omega$ there exists $a_s(\omega)$ such that $|a_s(\omega)| \le 1$ and $\Delta N_s(\omega) = a_s(\omega)\Delta M_s(\omega)$. Therefore for each $s \ge 0$ by (3.3.9) $\mathbb{P}$-a.s.

$$\begin{aligned}
&\Delta U(M_s, N_s) - \langle \partial_x U(M_{s-}, N_{s-}), \Delta M_s\rangle - \langle \partial_y U(M_{s-}, N_{s-}), \Delta N_s\rangle\\
&= V(M_{s-} + N_{s-} + (a_s+1)\Delta M_s, N_{s-} - M_{s-} + (a_s-1)\Delta M_s)\\
&\quad - V(M_{s-} + N_{s-}, N_{s-} - M_{s-})\\
&\quad - \langle \partial_x V(M_{s-} + N_{s-}, N_{s-} - M_{s-}), (a_s+1)\Delta M_s\rangle\\
&\quad - \langle \partial_y V(M_{s-} + N_{s-}, N_{s-} - M_{s-}), (a_s-1)\Delta M_s\rangle \le 0,
\end{aligned}$$

so $I \le 0$ a.s., and $\mathbb{E}I \le 0$. Also

$$\begin{aligned}
\int_0^t \langle \partial_x U(M_{s-}, N_{s-}), \mathrm{d}M_s\rangle + \int_0^t \langle \partial_y U(M_{s-}, N_{s-}), \mathrm{d}N_s\rangle&\\
= \int_0^t \langle \partial_x V(M_{s-} + N_{s-}, N_{s-} - M_{s-}), \mathrm{d}(M_s + N_s)\rangle&\\
+ \int_0^t \langle \partial_y V(M_{s-} + N_{s-}, N_{s-} - M_{s-}), \mathrm{d}(N_s - M_s)\rangle&,
\end{aligned}$$

so by Lemma 3.2.2 and Lemma 3.3.13 it is a martingale that starts at zero, and therefore its expectation is zero as well. Consequently, thanks to (3.3.4), (3.3.16) and Remark 3.3.9,

$$\mathbb{E}\|N_t\|^p - \beta_{p,X}^p \mathbb{E}\|M_t\|^p \le \mathbb{E}U(M_t, N_t) \le \mathbb{E}U(M_0, N_0) \le 0,$$

and therefore (3.3.15) holds. □

As one can see, in our proof we did not need the second order terms of the Itô formula thanks to the nature of the quadratic variation of a purely discontinuous process. Nevertheless, Theorem 3.3.17 holds for arbitrary martingales M and N, but with worse estimates (see Chapter 4). The connection of Theorem 3.3.17 for continuous martingales with the Hilbert transform will be discussed in Section 3.5.

3.4. FOURIER MULTIPLIERS

In [10] and [9] the authors exploited the differential subordination property to show boundedness of certain Fourier multipliers in $\mathscr{L}(L^p(\mathbb{R}^d))$. It turned out that it is sufficient to use the weak differential subordination property to obtain the same assertions, but in the vector-valued situation.

3.4.1. Basic definitions and the main theorem

Let $d \geq 1$ be a natural number. Recall that $\mathscr{S}(\mathbb{R}^d)$ is a space of Schwartz functions on $\mathbb{R}^d$. For a Banach space X with a scalar field $\mathbb{C}$ we define $\mathscr{S}(\mathbb{R}^d) \otimes X$ as the space of all functions $f : \mathbb{R}^d \to X$ of the form $f = \sum_{k=1}^K f_k \otimes x_k$, where $K \geq 1$, $f_1, \dots, f_K \in \mathscr{S}(\mathbb{R}^d)$, and $x_1, \dots, x_K \in X$. Notice that for each $1 \leq p < \infty$ the space $\mathscr{S}(\mathbb{R}^d) \otimes X$ is dense in $L^p(\mathbb{R}^d; X)$.

We define the *Fourier transform* $\mathscr{F}$ and the *inverse Fourier transform* $\mathscr{F}^{-1}$ on $\mathscr{S}(\mathbb{R}^d)$ as follows:

$$\mathscr{F}(f)(t) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle t,u\rangle} f(u)\,\mathrm{d}u, \quad f \in \mathscr{S}(\mathbb{R}^d), t \in \mathbb{R}^d,$$
$$\mathscr{F}^{-1}(f)(t) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\langle t,u\rangle} f(u)\,\mathrm{d}u, \quad f \in \mathscr{S}(\mathbb{R}^d), t \in \mathbb{R}^d.$$

It is well-known that for any $f \in \mathscr{S}(\mathbb{R}^d)$ we have $\mathscr{F}(f), \mathscr{F}^{-1}(f) \in \mathscr{S}(\mathbb{R}^d)$, and $\mathscr{F}^{-1}(\mathscr{F}(f)) = f$. The reader can find more details on the Fourier transform in [69].

Let $m : \mathbb{R}^d \to \mathbb{C}$ be measurable and bounded. Then we can define a linear operator T_m on $\mathscr{S}(\mathbb{R}^d) \otimes X$ as follows:

$$T_m(f \otimes x) = \mathscr{F}^{-1}(m\mathscr{F}(f)) \cdot x, \quad f \in \mathscr{S}(\mathbb{R}^d), x \in X. \tag{3.4.1}$$

The operator T_m is called a *Fourier multiplier*, while the function m is called the *symbol* of T_m. If X is finite-dimensional then T_m can be extended to a bounded linear operator on $L^2(\mathbb{R}^d; X)$. The question is usually whether one can extend T_m to a bounded operator on $L^p(\mathbb{R}^d; X)$ for a general $1 < p < \infty$ and a given X. Here the answer will be given for m of quite a special form and X with the UMD property.

Let V be a Lévy measure on $\mathbb{R}^d$, that is $V(\{0\}) = 0$, $V \neq 0$ and

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) V(\,\mathrm{d}x) < \infty.$$

Let $\phi \in L^\infty(\mathbb{R}^d; \mathbb{C})$ be such that $\|\phi\|_{L^\infty(\mathbb{R}^d;\mathbb{C})} \leq 1$. Also let $\mu \geq 0$ be a finite Borel measure on the unit sphere $S^{d-1} \subset \mathbb{R}^d$, and $\psi \in L^\infty(S^{d-1}; \mathbb{C})$ satisfies $\|\psi\|_{L^\infty(S^{d-1};\mathbb{C})} \leq 1$.

In the sequel we set $\frac{a}{0} = 0$ for each $a \in \mathbb{C}$. The following result extends [9, Theorem 1.1] to the UMD Banach space setting.

Theorem 3.4.1. *Let X be a UMD Banach space. Then the Fourier multiplier T_m with a symbol*

$$m(\xi) = \frac{\int_{\mathbb{R}^d} (1 - \cos\xi \cdot z)\phi(z) V(\,\mathrm{d}z) + \frac{1}{2}\int_{S^{d-1}} (\xi \cdot \theta)^2 \psi(\theta)\mu(\,\mathrm{d}\theta)}{\int_{\mathbb{R}^d} (1 - \cos\xi \cdot z) V(\,\mathrm{d}z) + \frac{1}{2}\int_{S^{d-1}} (\xi \cdot \theta)^2 \mu(\,\mathrm{d}\theta)}, \quad \xi \in \mathbb{R}^d, \tag{3.4.2}$$

has a bounded extension on $L^p(\mathbb{R}^d; X)$ for $1 < p < \infty$. Moreover, then for each $f \in L^p(\mathbb{R}^d; X)$

$$\|T_m f\|_p \leq \beta_{p,X} \|f\|_p. \tag{3.4.3}$$

Remark 3.4.2. The coefficient $\frac{1}{2}$ in both numerator and denominator of (3.4.1), even though it looks wired and useless (because one can always transform μ to 2μ), exists because of the strong connection with the Lévy–Khintchin representation of Lévy processes (see e.g. [8, Part 4.1]).

The proof is a modification of the arguments given in [9] and [10], but instead of real-valued process we will work with processes that take their values in a finite dimensional space. For the convenience of the reader the proof will be given in the same form and with the same notations as the original one. However, we will need to justify here some steps, so we cannot just skip the proof. First of all as that was done in [9], we reduce to the case of symmetric V and $\mu = 0$, and proceed as in the proof of [10, Theorem 1].

In the rest of the section we may assume that X is finite dimensional, since it is sufficient to show (3.4.3) for all f with values in X_0 for each finite dimensional subspace X_0 of X.

Let ν be a positive finite symmetric measure on $\mathbb{R}^d$, $\widetilde{\nu} = \nu/|\nu|$. Let T_i and Z_i, $i = \pm1, \pm2, \pm3, \dots$, be a family of independent random variables, such that each T_i is exponentially distributed with parameter $|\nu|$ (i.e. $\mathbb{E}T_i = 1/|\nu|$), and each Z_i has $\widetilde{\nu}$ as a distribution. Let $S_i = T_1 + \dots + T_i$ for a positive i and $S_i = -(T_{-1} + \dots + T_i)$ for a negative i. For each $-\infty < s < t < \infty$ we define $X_{s,t} := \sum_{s<S_i\le t} Z_i$ and $X_{s,t-} := \sum_{s<S_i<t} Z_i$. Note that $\mathcal{N}(B) = \#\{i : (S_i, Z_i) \in B\}$ defines a Poisson measure on $\mathbb{R}\times\mathbb{R}^d$ with the intensity measure $\lambda \otimes \nu$, and $X_{s,t} = \int_{s<v\le t} x\mathcal{N}(\,\mathrm{d}v, \mathrm{d}v)$ (see e.g. [165]). Let $N(s,t) = \mathcal{N}((s,t]\times\mathbb{R}^d)$ be the number of *signals* S_i such that $s < S_i \le t$. The following Lemmas 3.4.3-3.4.6 are multidimensional versions of [10, Lemma 1–5], which can be proven in the same way as in the scalar case.

Lemma 3.4.3. *Let $f : \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d \to X$ be Borel measurable and be either nonnegative or bounded, and let $s \le t$. Then*

$$\mathbb{E}\sum_{s<S_i\le t} F(S_i, X_{s,S_i-,X_{s,S_i}}) = \mathbb{E}\int_s^t\int_{\mathbb{R}^d} F(v, X_{s,v-,X_{s,v-}+z})\nu(\,\mathrm{d}z)\,\mathrm{d}v.$$

We will consider the following filtration:

$$\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}} = \{\sigma\{X_{s,t} : s \le t\}\}_{t\in\mathbb{R}}.$$

Recall that for measures ν_1 and ν_2 on $\mathbb{R}^d$ the expression $\nu_1 * \nu_2$ means the *convolution* of these measures (we refer the reader [22, Chapter 3.9] for the details). Also for each $n \ge 1$ we define $\nu_1^{*n} := \underbrace{\nu_1 * \dots * \nu_1}_{n \text{ times}}$. For each $t \in \mathbb{R}$ define

$$p_t = e^{*t(\nu-|\nu|\delta_0)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}(\nu-|\nu|\delta_0)^{*n} = e^{-t|\nu|}\sum_{n=0}^{\infty}\frac{t^n}{n!}\nu^{*n}.$$

The series converges in the norm of absolute variation of measures. As in [10, (18)] and [9, (3.9)] p_t is symmetric, and

$$\frac{\partial}{\partial t} p_t = (\nu - |\nu|\delta_0) * p_t, \quad t \in \mathbb{R}.$$

Also $p_{t_1+t_2} = p_{t_1} * p_{t_2}$ for each $t_1, t_2 \in \mathbb{R}$. In fact for all $t \le u$ the measure p_{u-t} is the distribution of $X_{t,u}$ and $X_{t,u-}$. Put

$$\Psi(\xi) = \int_{\mathbb{R}^d} (e^{i\xi\cdot z} - 1)\nu(\,\mathrm{d}z), \quad \xi \in \mathbb{R}^d.$$

Thanks to the symmetry of ν one has as well that

$$\Psi(\xi) = \int_{\mathbb{R}^d} (\cos \xi \cdot z - 1)\nu(\,\mathrm{d}z) = \Psi(-\xi) \le 0.$$

Therefore Ψ is bounded on $\mathbb{R}^d$, and due to [9, (3.12)] we have that the characteristic function of p_t is of the following form:

$$\hat{p}_t(\xi) = e^{t\Psi(\xi)}, \quad \xi \in \mathbb{R}^d.$$

(The reader can find more on characteristic functions in [22, Chapter 3.8].)

Let $g \in L^\infty(\mathbb{R}^d; X)$. Then for $x \in \mathbb{R}^d$, $t \le u$, we define the *parabolic extension* of g by

$$P_{t,u}g(x) := \int_{\mathbb{R}^d} g(x+y)p_{u-t}(\,\mathrm{d}y) = g * p_{u-t}(x) = \mathbb{E}g(x + X_{t,u}).$$

For $s \le t \le u$ we define the *parabolic martingale* by

$$G_t = G_t(x; s, u; g) := P_{t,u}g(x + X_{s,t}).$$

Lemma 3.4.4. *We have that G_t is a bounded $\mathbb{F}$-martingale.*

Let $\phi \in L^\infty(\mathbb{R}^d; \mathbb{C})$ be symmetric. For each $x \in \mathbb{R}^d$, $s \le t \le u$, and $f \in C_c(\mathbb{R}^d; X)$ we define F_t as follows:

$$\begin{aligned} F_t = F_t(x; s, u; f, \phi) :=& \\ &\sum_{s<S_i\le t} [P_{S_i,u}f(x + X_{s,S_i}) - P_{S_i,u}f(x + X_{s,S_i-})]\phi(X_{s,S_i} - X_{s,S_i-}) \\ &- \int_s^t \int_{\mathbb{R}^d} [P_{v,u}f(x + X_{s,v-} + z) - P_{u,v}f(x + X_{s,v-})]\phi(z)\nu(\,\mathrm{d}z)\,\mathrm{d}v. \end{aligned}$$

Lemma 3.4.5. *We have that $F_t = F_t(x; s, u; f, \phi)$ is an $\mathbb{F}$-martingale for $t \in [s, u]$. Moreover, $\mathbb{E}\|F_t\|^p < \infty$ for each $p > 0$.*

Lemma 3.4.6. $G_t(x; s, u; g) = F_t(x; s, u; g, 1) + P_{s,u}g(x)$.

Analogously to [10, (21)-(22)] one has that for each $x^* \in X^*$ the quadratic variations of $\langle F_t(x;s,u;f,\phi), x^*\rangle$ and $\langle G_t(x;s,u;g), x^*\rangle$ satisfy the following a.s. identities,

$$[\langle F, x^*\rangle]_t = \sum_{s<S_i\leq t} \Big(\langle P_{S_i,u} f(x+X_{s,S_i}) - P_{S_i,u} f(x+X_{s,S_i-}), x^*\rangle\Big)^2 \phi^2(\Delta X_{s,S_i}),$$

$$[\langle G, x^*\rangle]_t = |\langle P_{s,u} g(x), x^*\rangle|^2 + \sum_{s<S_i\leq t} \Big(\langle P_{S_i,u} g(x+X_{s,S_i}) - P_{S_i,u} g(x+X_{s,S_i-}), x^*\rangle\Big)^2.$$

It follows that for each $f \in C_c(\mathbb{R}^d;X)$, $(F_t(x;s,u;f,\phi))_{t\in[s,u]}$ is weakly differentially subordinate to $(G_t(x;s,u;f))_{t\in[s,u]}$ and by Theorem 3.3.17 one has for each $t \in [s,u]$

$$\mathbb{E}\|F_t(x;s,u;f,\phi)\|^p \leq \beta_{p,X}^p \mathbb{E}\|G_t(x;s,u;f)\|^p.$$

Note that $G_u(x;s,u;f) = f(x+X_{s,u})$, so

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbb{E}\|F_u(x;s,u;f,\phi)\|^p \,\mathrm{d}x &\leq \beta_{p,X}^p \int_{\mathbb{R}^d} \mathbb{E}\|f(x+X_{s,u})\|^p \,\mathrm{d}x \\ &= \beta_{p,X}^p \|f\|_{L^p(\mathbb{R}^d;X)}^p.\end{aligned} \tag{3.4.4}$$

Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Consider the linear functional on $L^{p'}(\mathbb{R}^d;X^*)$:

$$L^{p'}(\mathbb{R}^d;X^*) \ni g \mapsto \int_{\mathbb{R}^d} \mathbb{E}\langle F_u(x;s,u;f,\phi), g(x+X_{s,u})\rangle \,\mathrm{d}x.$$

Then by Hölder's inequality and (3.4.4) one has

$$\int_{\mathbb{R}^d} \mathbb{E}|\langle F_u(x;s,u;f,\phi), g(x+X_{s,u})\rangle| \,\mathrm{d}x \leq \beta_{p,X} \|f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^{p'}(\mathbb{R}^d;X^*)}.$$

By Theorem 1.3.10 and Theorem 1.3.21 in [79], $(L^{p'}(\mathbb{R}^d;X^*))^* = L^p(\mathbb{R}^d;X)$, so there exists $h \in L^p(\mathbb{R}^d;X)$ such that for each $g \in L^{p'}(\mathbb{R}^d;X^*)$

$$\int_{\mathbb{R}^d} \mathbb{E}\langle F_u(x;s,u;f,\phi), g(x+X_{s,u})\rangle \,\mathrm{d}x = \int_{\mathbb{R}^d} \langle h(x), g(x)\rangle \,\mathrm{d}x,$$

and

$$\|h\|_{L^p(\mathbb{R}^d;X)} \leq \beta_{p,X} \|f\|_{L^p(\mathbb{R}^d;X)}. \tag{3.4.5}$$

In particular, since X is finite dimensional

$$\int_{\mathbb{R}^d} \mathbb{E}F_u(x;s,u;f,\phi) g(x+X_{s,u}) \,\mathrm{d}x = \int_{\mathbb{R}^d} h(x) g(x) \,\mathrm{d}x, \quad g \in L^{p'}(\mathbb{R}^d). \tag{3.4.6}$$

For each $s<0$ define $m_s : \mathbb{R}^d \to \mathbb{C}$ as follows

$$m_s(\xi) = \begin{cases} \Big(1 - e^{2|s|\Psi(\xi)}\Big) \frac{1}{\Psi(\xi)} \int_{\mathbb{R}^d} (e^{i\xi\cdot z} - 1)\phi(z)\nu(\,\mathrm{d}z), & \Psi(\xi) \neq 0, \\ 0, & \Psi(\xi) = 0. \end{cases}$$

Let $u = 0$. Then analogously to [10, (32)], by (3.4.6) one obtains

$$\mathscr{F}(h)(\xi) = m_s(\xi)\mathscr{F}(f)(\xi), \quad \xi \in \mathbb{R}^d.$$

Let T_{m_s} be the Fourier multiplier on $L^2(\mathbb{R}^d; X)$ with symbol m_s (that is bounded by 1). By (3.4.5) one obtains that T_{m_s} extends uniquely to a bounded operator on $L^p(\mathbb{R}^d; X)$ with $\|T_{m_s}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leq \beta_{p,X}$. Let T_m be the multiplier on $L^2(\mathbb{R}^d; X)$ with the symbol m given by

$$m(\xi) = \begin{cases} \frac{1}{\Psi(\xi)} \int_{\mathbb{R}^d} (e^{i\xi\cdot z} - 1)\phi(z)\nu(\,\mathrm{d}z), & \Psi(\xi) \neq 0, \\ 0, & \Psi(\xi) = 0. \end{cases}$$

Note that m is a pointwise limit of m_s as $s \to -\infty$. Also note that $T_{m_s} f \to T_m f$ in $L^2(\mathbb{R}^d; X)$ as $s \to -\infty$ for each $f \in C_c(\mathbb{R}^d; X)$ by Plancherel's theorem. Therefore by Fatou's lemma one has that for each $f \in C_c(\mathbb{R}^d; X)$ the following holds:

$$\|T_m f\|_{L^p(\mathbb{R}^d;X)} \leq \varliminf_{s\to-\infty} \|T_{m_s} f\|_{L^p(\mathbb{R}^d;X)} \leq \beta_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)},$$

hence T_m uniquely extends to a bounded operator on $L^p(\mathbb{R}^d; X)$ with

$$\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leq \beta_{p,X}.$$

3.4.2. Examples of Theorem 3.4.1

In this subsection X is a UMD Banach space, $p \in (1,\infty)$. The examples will be mainly the same as were given in [9, Chapter 4] with some author's remarks. Recall that we set $\frac{a}{0} = 0$ for any $a \in \mathbb{C}$.

Example 3.4.7. Let V_1, V_2 be two nonnegative Lévy measures on $\mathbb{R}^d$ such that $V_1 \leq V_2$. Let

$$m(\xi) = \frac{\int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z))V_1(\,\mathrm{d}z)}{\int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z))V_2(\,\mathrm{d}z)}, \quad \xi \in \mathbb{R}^d.$$

Then $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leq \beta_{p,X}$.

Example 3.4.8. Let μ_1, μ_2 be two nonnegative measures on S^{d-1} such that $\mu_1 \leq \mu_2$. Let

$$m(\xi) = \frac{\int_{S^{d-1}} (\xi\cdot\theta)^2 \mu_1(\,\mathrm{d}\theta)}{\int_{S^{d-1}} (\xi\cdot\theta)^2 \mu_2(\,\mathrm{d}\theta)}, \quad \xi \in \mathbb{R}^d.$$

Then $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leq \beta_{p,X}$.

Example 3.4.9 (Beurling-Ahlfors transform)**.** Let $d = 2$. Put $\mathbb{R}^2 = \mathbb{C}$. Then the Fourier multiplier T_m with a symbol $m(z) = \frac{\bar{z}^2}{|z|^2}$, $z \in \mathbb{C}$, has the norm at most $2\beta_{p,X}$ on $L^p(\mathbb{R}^d; X)$. This multiplier is also known as the *Beurling-Ahlfors transform*. It is well-known that $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^2;X))} \geq \beta_{p,X}$. There is quite an old problem whether $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^2;X))} = \beta_{p,X}$. This question was firstly posed by Iwaniec in [82] in $\mathbb{C}$. Nevertheless it was neither proved nor disproved even in the scalar-valued case. We refer the reader to [8] and [79] for further details.

Example 3.4.10. Let $\alpha \in (0,2)$, μ be a finite positive measure on S^{d-1}, ψ be a measurable function on S^{d-1} such that $|\psi| \leq 1$. Let

$$m(\xi) = \frac{\int_{S^{d-1}} |(\xi\cdot\theta)|^{\alpha}\psi(\theta)\mu(\,\mathrm{d}\theta)}{\int_{S^{d-1}} |(\xi\cdot\theta)|^{\alpha}\mu(\,\mathrm{d}\theta)},\quad \xi \in \mathbb{R}^d.$$

Then analogously to [9, (4.7)], $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leq \beta_{p,X}$.

Example 3.4.11 (Double Riesz transform)**.** Let $\alpha \in (0,2]$. Let

$$m(\xi) = \frac{|\xi_1|^{\alpha}}{|\xi_1|^{\alpha}+\cdots+|\xi_d|^{\alpha}},\quad \xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d,$$

Then according to Example 3.4.10, $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leq \beta_{p,X}$. Note that if $\alpha = 2$, then T_m is a double Riesz transform. (In the paper [188] it is shown that the norm $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}$ does not depend on α and equals the $\mathrm{UMD}_p^{\{0,1\}}$ constant of X).

Example 3.4.12. Let $\alpha \in [0,2]$, $d \geq 2$. Let

$$m(\xi) = \frac{|\xi_1|^{\alpha}-|\xi_2|^{\alpha}}{|\xi_1|^{\alpha}+\cdots+|\xi_d|^{\alpha}},\quad \xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d,$$

Then by Example 3.4.10, $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leq \beta_{p,X}$. Moreover, if $d = 2$, $\alpha \in [1,2]$, then $\max_{\xi\in\mathbb{R}^2} m(\xi) = 1$, $\min_{\xi\in\mathbb{R}^2} m(\xi) = -1$ and $m|_{S^1} \in W^{2,1}(S^1)$. Therefore due to Proposition 3.4, Proposition 3.8 and Remark 3.9 in [66] one has $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^2;X))} \geq \beta_{p,X}$. This together with Theorem 3.4.1 implies $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^2;X))} = \beta_{p,X}$, which extends [66, Theorem 1.1], where the same assertion was proven for $\alpha = 2$.

Example 3.4.13. Let μ be a nonnegative Borel measure on S^{d-1}, $\psi \in L^{\infty}(S^{d-1},\mu)$, $\|\psi\|_{\infty} \leq 1$. Let

$$m(\xi) = \frac{\int_{S^{d-1}} \ln(1+(\xi\cdot\theta)^{-2})\psi(\theta)\mu(\,\mathrm{d}\theta)}{\int_{S^{d-1}} \ln(1+(\xi\cdot\theta)^{-2})\mu(\,\mathrm{d}\theta)},\quad \xi \in \mathbb{R}^d.$$

Then $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leq \beta_{p,X}$.

3.5. HILBERT TRANSFORM AND GENERAL CONJECTURE

In this section we assume that X is a finite dimensional Banach space to avoid difficulties with stochastic integration. Many of the assertions below can be extended to the general UMD Banach space case by using the same techniques as in the proof of Theorem 3.3.3.

3.5.1. Hilbert transform and Burkholder functions

It turns out that the generalization of Theorem 3.3.17 to the case of continuous martingales is connected with the boundedness of the Hilbert transform. The Fourier multiplier $\mathscr{H} \in \mathscr{L}(L^2(\mathbb{R}))$ with the symbol $m \in L^{\infty}(\mathbb{R})$ such that $m(t) = -i\,\mathrm{sign}(t)$,

$t \in \mathbb{R}$, is called the *Hilbert transform*. This operator can be extended to a bounded operator on $L^p(\mathbb{R})$, $1 < p < \infty$ (see [158] and [79, Chapter 5.1] for the details).

Let X be a Banach space. Then one can extend the Hilbert transform $\mathscr{H}$ to $\mathscr{S}(\mathbb{R}) \otimes X$ in the same way as it was done in (3.4.1). Denote this extension by $\mathscr{H}_X$. By [23, Lemma 2] and [61, Theorem 3] the following holds true:

Theorem 3.5.1 (Bourgain, Burkholder)**.** *Let X be a Banach space. Then X is a UMD Banach space if and only if $\mathscr{H}_X$ can be extended to a bounded operator on $L^p(\mathbb{R};X)$ for each $1 < p < \infty$. Moreover, then*

$$\sqrt{\beta_{p,X}} \leq \|\mathscr{H}_X\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leq \beta_{p,X}^2. \tag{3.5.1}$$

The proof of the right-hand side of (3.5.1) is based on the following result.

Proposition 3.5.2. *Let X be a finite dimensional Banach space, B_1, B_2 be two real-valued Wiener processes, $f_1, f_2 : \mathbb{R}_+ \times \Omega \to X$ be two stochastically integrable functions. Let us define $M := f_1 \cdot B_1 + f_2 \cdot B_2$, $N := f_2 \cdot B_1 - f_1 \cdot B_2$. Then for each $T \geq 0$*

$$(\mathbb{E}\|N_T\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E}\|M_T\|^p)^{\frac{1}{p}}.$$

Proof. The theorem follows from Theorem 4.4.2. Nevertheless we wish to illustrate an easier and more specific proof. Let $\widetilde{B}_1, \widetilde{B}_2 : \widetilde{\Omega} \times \mathbb{R}_+ \to \mathbb{R}$ be two Wiener process defined on an enlarged probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$ with an enlarged filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathscr{F}}_t)_{t\geq 0}$ such that $\widetilde{B}_1$ and $\widetilde{B}_2$ are independent of $\mathscr{F}$. Then by applying the decoupling theorem [79, Theorem 4.4.1] twice (see also [119]) and the fact that $-\widetilde{B}_1$ is a Wiener process

$$\begin{aligned}
\mathbb{E}\|N_T\|^p = \mathbb{E}\|(f_2 \cdot B_1)_T - (f_1 \cdot B_2)_T\|^p &\leq \beta_{p,X}^p \mathbb{E}\|(f_2 \cdot \widetilde{B}_1)_T - (f_1 \cdot \widetilde{B}_2)_T\|^p \\
&= \beta_{p,X}^p \mathbb{E}\|(f_1 \cdot (-\widetilde{B}_2))_T + (f_2 \cdot \widetilde{B}_1)_T\|^p \\
&\leq \beta_{p,X}^{2p} \mathbb{E}\|(f_1 \cdot B_1)_T + (f_2 \cdot B_2)_T\|^p \\
&= \beta_{p,X}^{2p} \mathbb{E}\|M_T\|^p.
\end{aligned}$$

□

Let $p \in (1,\infty)$. A natural question is whether there exists a constant $C_p > 0$ such that

$$\|\mathscr{H}_X\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leq C_p \beta_{p,X}. \tag{3.5.2}$$

Then the following theorem is applicable.

Theorem 3.5.3. *Let X be a Banach space, $p \in (1,\infty)$. Then there exists $C_p \geq 1$ such that* (3.5.2) *holds if there exists* ***some*** *Burkholder function $U : X \times X \to \mathbb{R}$ such that U is continuous and a.s. twice Fréchet differentiable, $U(x,y) \geq \|y\|^p - (C_p\beta_{p,X})^p\|x\|^p$ for any $x, y \in X$, $U(\alpha x, \alpha y) = |\alpha|^p U(x,y)$ for any $\alpha \in \mathbb{R}$ and $x, y \in X$, and the function*

$$t \mapsto U(x + tz_1, y + tz_2) + U(x + tz_2, y - tz_1), \quad t \in \mathbb{R},$$

or, equivalently,

$$t \mapsto U(x + tz_1, y + tz_2) + U(x - tz_2, y + tz_1), \quad t \in \mathbb{R},$$

is concave for each $x, y, z_1, z_2 \in X$ *at* $t = 0$.

Proof. Let M and N be as in Proposition 3.5.2. By the approximation argument we can suppose that M and N have absolutely continuous distributions. Let d be the dimension of X. Then by the Itô formula in Theorem 2.12.1

$$\begin{aligned} \mathbb{E}\|N_t\|_X^p - (C_p\beta_{p,X})^p \mathbb{E}\|M_t\|_X^p \leq \mathbb{E}U(M_t, N_t) &= \mathbb{E}U(M_0, N_0) \\ &+ \mathbb{E}\int_0^t \langle \partial_x U(M_s, N_s), \mathrm{d}M_s\rangle \\ &+ \mathbb{E}\int_0^t \langle \partial_y U(M_s, N_s), \mathrm{d}N_s\rangle + \frac{1}{2}\mathbb{E}I, \end{aligned} \tag{3.5.3}$$

where

$$\begin{aligned} I = \int_0^t \sum_{i,j=1}^d (U_{x_i,x_j}(M_s,N_s)\,\mathrm{d}[\langle x_i^*, M_s\rangle, \langle x_j^*, M_s\rangle] \\ + 2U_{x_i,y_j}(M_s,N_s)\,\mathrm{d}[\langle x_i^*, M_s\rangle, \langle y_j^*, N_s\rangle] \\ + U_{y_i,y_j}(M_s,N_s)\,\mathrm{d}[\langle y_i^*, N_s\rangle, \langle y_j^*, N_s\rangle]), \end{aligned} \tag{3.5.4}$$

where $(x_i)_{i=1}^d = (y_i)_{i=1}^d \subset X$ is the same basis of X, and $(x_i^*)_{i=1}^d = (y_i^*)_{i=1}^d \subset X^*$ are the same corresponding dual bases of X^*.

Notice that by Remark 3.3.9 $\mathbb{E}U(M_0, N_0) \leq 0$ since $\|N_0\| \leq \|M_0\|$ a.s. and $C_p, \beta_{p,X} \geq 1$, and that

$$\mathbb{E}\Big(\int_0^t \langle \partial_x U(M_s, N_s), \mathrm{d}M_s\rangle + \int_0^t \langle \partial_y U(M_s, N_s), \mathrm{d}N_s\rangle\Big) = 0,$$

since due to the same type of discussion as was done in the proof of Theorem 3.3.17, $\int_0^\cdot \langle \partial_x U(M_s, N_s), \mathrm{d}M_s\rangle + \int_0^\cdot \langle \partial_y U(M_s, N_s), \mathrm{d}N_s\rangle$ is a martingale which starts at zero.

Let us now prove that $I \leq 0$. For each $i = 1, 2, \ldots, d$ we define $f_i^1 := \langle x_i^*, f_1\rangle$ and $f_i^2 := \langle x_i^*, f_2\rangle$. Then for each $i, j = 1, 2 \ldots, d$ one has that

$$\mathrm{d}[\langle x_i^*, M_s\rangle, \langle x_j^*, M_s\rangle] = \mathrm{d}[\langle y_i^*, N_s\rangle, \langle y_j^*, N_s\rangle] = (f_i^1 f_j^1 + f_i^2 f_j^2)\,\mathrm{d}t, \tag{3.5.5}$$

and

$$\mathrm{d}[\langle x_i^*, M_s\rangle, \langle y_j^*, N_s\rangle] = (f_i^1 f_j^2 - f_i^2 f_j^1)\,\mathrm{d}t. \tag{3.5.6}$$

Notice also that for each $x, y \in X$

$$\begin{aligned}
\frac{\partial^2}{\partial u^2} U(x+uf_1, y+uf_2)|_{u=0} &= \sum_{i,j=1}^{d} ((U_{x_i^*,x_j^*}(x,y) f_i^1 f_j^1 + 2U_{x_i^*,y_j^*}(x,y) f_i^1 f_j^2 \\
&\quad + U_{y_i^*,y_j^*}(x,y) f_i^2 f_j^2), \\
\frac{\partial^2}{\partial u^2} U(x+uf_2, y-uf_1)|_{u=0} &= \frac{\partial^2}{\partial u^2} U(x-uf_2, y+uf_1)|_{u=0} \\
= \sum_{i,j=1}^{d} ((U_{x_i^*,x_j^*}(x,y) f_i^2 f_j^2 &- 2U_{x_i^*,y_j^*}(x,y) f_i^2 f_j^1 + U_{y_i^*,y_j^*}(x,y) f_i^1 f_j^1).
\end{aligned} \tag{3.5.7}$$

Therefore by (3.5.4), (3.5.5), (3.5.6), and (3.5.7) we have that

$$\begin{aligned}
I &= \int_0^t \sum_{i,j=1}^{d} ((U_{x_i^*,x_j^*}(M_{s-},N_{s-})(f_i^1 f_j^1 + f_i^2 f_j^2) + 2U_{x_i^*,y_j^*}(M_{s-},N_{s-})(f_i^1 f_j^2 - f_i^2 f_j^1) \\
&+ U_{y_i^*,y_j^*}(M_{s-},N_{s-})(f_i^1 f_j^1 + f_i^2 f_j^2))\,\mathrm{d}t = \int_0^t \frac{\partial^2}{\partial u^2} U(M_{s-}+uf_1, N_{s-}+uf_2)|_{u=0} \\
&+ \frac{\partial^2}{\partial u^2} U(M_{s-}+uf_2, N_{s-}-uf_1)|_{u=0}\,\mathrm{d}s \\
&= \int_0^t \frac{\partial^2}{\partial u^2}\Big(U(M_{s-}+uf_1, N_{s-}+uf_2) + U(M_{s-}+uf_2, N_{s-}-uf_1)\Big)\Big|_{u=0}\,\mathrm{d}s,
\end{aligned}$$

and thanks to the concavity of $U(x+uf_1, y+uf_2)+U(x+uf_2, y-uf_1)$ in point $u=0$ for each $x, y \in X$ one deduces that a.s. $I \le 0$. Then thanks to (3.5.3) one has that

$$\mathbb{E}\|N_t\|_X^p - (C_p \beta_{p,X})^p \mathbb{E}\|M_t\|_X^p \le \mathbb{E}U(M_t, N_t) \le 0. \tag{3.5.8}$$

Now one can prove that (3.5.8) implies (3.5.2) in the same way as it was done for instance in [61, Theorem 3], [15, p.592] or [39, Chapter 3]. □

Remark 3.5.4. Note that if X is a finite dimensional Hilbert space, then one gets condition (iii) in Theorem 3.5.3 for free from [179]. Indeed, let $U: X \times X \to \mathbb{R}$ be as in [179, p. 527], namely

$$U(x,y) = p(1-1/p^*)^{p-1}(\|y\| - (p^*-1)\|x\|)(\|x\|+\|y\|)^{p-1}, \quad x, y \in X.$$

Then U is a.s. twice Fréchet differentiable, and thanks to the property (c) of U, which is given on [179, p. 527], for all nonzero $x, y \in X$ there exists a constant $c(x,y) \ge 0$ such that

$$\begin{aligned}
\langle \partial_{xx} U(x,y), (h,h)\rangle + 2\langle \partial_{xy} U(x,y), (h,k)\rangle + \langle \partial_{yy} U(x,y), (k,k)\rangle \\
\le -c(x,y)(\|h\|^2 - \|k\|^2), \quad h, k \in X.
\end{aligned}$$

Therefore for any $z_1, z_2 \in X$

$$\frac{\partial^2}{\partial t^2}\big[U(x+tz_1, y+tz_2) + U(x+tz_2, y-tz_1)\big]\Big|_{t=0}$$

$$
\begin{aligned}
&= \langle \partial_{xx} U(x,y),(z_1,z_1)\rangle + 2\langle \partial_{xy} U(x,y),(z_1,z_2)\rangle + \langle \partial_{yy} U(x,y),(z_2,z_2)\rangle \\
&\quad + \langle \partial_{xx} U(x,y),(z_2,z_2)\rangle - 2\langle \partial_{xy} U(x,y),(z_2,z_1)\rangle + \langle \partial_{yy} U(x,y),(z_1,z_1)\rangle \\
&\le -c(x,y)(\|z_1\|^2 - \|z_2\|^2) - c(x,y)(\|z_2\|^2 - \|z_1\|^2) = 0.
\end{aligned}
$$

3.5.2. General conjecture

By Theorem 3.5.3 the estimate (3.5.2) is a direct corollary of the following conjecture.

Conjecture 3.5.5. *Let X be a finite dimensional Banach space, $p \in (1,\infty)$. Then there exists $C_p \ge 1$ such that for each pair of continuous martingales $M, N : \mathbb{R}_+ \times \Omega \to X$ such that N is weakly differentially subordinate to M one has that for each $t \ge 0$*

$$
(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \le C_p \beta_{p,X} (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{3.5.9}
$$

Remark 3.5.6. Notice that the estimate (3.5.9) follows from Theorem 4.4.2 with the constant $\beta_{p,X}^2$ instead of $C_p\beta_{p,X}$. Moreover, it is shown in Theorem 4.4.2 that C_p can not be less then 1.

We wish to finish by pointing out some particular cases in which Conjecture 3.5.5 holds. These results are about stochastic integration with respect to a Wiener process. Recall that we assume that X is a finite dimensional space. Later we will need a couple of definitions.

Let $W^H : \mathbb{R}_+ \times H \to L^2(\Omega)$ be an *H-cylindrical Brownian motion*, i.e.

- $(W^H h_1, \dots, W^H h_d) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ is a d-dimensional Wiener process for all $d \ge 1$ and $h_1, \dots, h_d \in H$,
- $\mathbb{E} W^H(t) h\, W^H(s) g = \langle h, g\rangle \min\{t,s\}\ \forall h, g \in H,\ t, s \ge 0.$

(We refer the reader to [48, Chapter 4.1] for further details). Let X be a Banach space, $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H,X)$ be elementary progressive of the form (2.5.1). Then we define a *stochastic integral* $\Phi \cdot W^H : \mathbb{R}_+ \times \Omega \to X$ of Φ with respect to W^H in the following way:

$$
(\Phi \cdot W^H)_t = \sum_{k=1}^{K} \sum_{m=1}^{M} \mathbf{1}_{B_{mk}} \sum_{n=1}^{N} (W^H(t_k \wedge t) h_n - W^H(t_{k-1} \wedge t) h_n) x_{kmn}, \ \ t \ge 0.
$$

The following lemma is a multidimensional variant of [93, (3.2.19)] and it is closely connected with Lemma 3.2.1.

Lemma 3.5.7. *Let $X = \mathbb{R}$, $\Phi, \Psi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H,\mathbb{R})$ be elementary progressive. Then for all $t \ge 0$ a.s.*

$$
[\Phi \cdot W^H, \Psi \cdot W^H]_t = \int_0^t \langle \Phi^*(s), \Psi^*(s)\rangle \, \mathrm{d}s.
$$

The reader can find more on stochastic integration with respect to an H-cylindrical Brownian motion in the UMD case in [126]. The following theorem follows from (6.4.26).

Theorem 3.5.8. *Let X be a finite dimensional Banach space, W^H be an H-cylindrical Brownian motion, $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H,X)$ be stochastically integrable with respect to W^H function. Let $A \in \mathscr{L}(H)$ be self-adjoint. Then*

$$(\mathbb{E}\|((\Phi A)\cdot W^H)_\infty\|_X^p)^{\frac{1}{p}} \leq \beta_{p,X}\|A\|(\mathbb{E}\|(\Phi\cdot W^H)_\infty\|_X^p)^{\frac{1}{p}}. \tag{3.5.10}$$

Notice that by Lemma 3.5.7 for each $x^* \in X^*$ and $0 \leq s < t < \infty$ a.s.

$$\begin{aligned}[\langle(\Phi A)\cdot W^H, x^*\rangle]_t - [\langle(\Phi A)\cdot W^H, x^*\rangle]_s &= \int_s^t \|A\Phi^*(r)x^*\|^2\,\mathrm{d}r\\ &\leq \|A\|^2\int_s^t \|\Phi^*(r)x^*\|^2\,\mathrm{d}r\\ &= \|A\|^2\big([\langle\Phi\cdot W^H, x^*\rangle]_t - [\langle\Phi\cdot W^H, x^*\rangle]_s\big).\end{aligned}$$

Hence if $\|A\| \leq 1$, then $(\Phi A)\cdot W^H$ is weakly differentially subordinate to $\Phi\cdot W^H$, and therefore Theorem 3.5.8 provides us with a special case of Conjecture 3.5.5.

Remark 3.5.9. Theorem 3.5.8 in fact can be shown using [66, Proposition 3.7.(i)].

Remark 3.5.10. An analogue of Theorem 3.5.8 for antisymmetric A (i.e. A such that $A^* = -A$) remains open. It is important for instance for the possible estimate (3.5.2). Indeed, in Proposition 3.5.2 the Hilbert space H can be taken 2-dimensional, $A = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, and $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H,X)$ is such that $\Phi\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) = af_1 + bf_2$ for each $a, b \in \mathbb{R}$. Then $M = \Phi\cdot W^H$, $N = (\Phi A)\cdot W^H$, and if one shows (3.5.10) for an antisymmetric operator A, then one automatically gains (3.5.2).

The next theorem shows that Conjecture 3.5.5 holds for stochastic integrals with respect to a one-dimensional Wiener process.

Theorem 3.5.11. *Let X be a finite dimensional Banach space, $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a one-dimensional Wiener process, $\Phi, \Psi : \mathbb{R}_+ \times \Omega \to X$ be stochastically integrable with respect to W, $M = \Phi\cdot W$, $N = \Psi\cdot W$. Let N be weakly differentially subordinate to M. Then for each $p \in (1,\infty)$,*

$$\mathbb{E}\|N_\infty\|^p \leq \beta_{p,X}^p\mathbb{E}\|M_\infty\|^p. \tag{3.5.11}$$

Proof. Without loss of generality suppose that there exists $T \geq 0$ such that $\Phi\mathbf{1}_{[T,\infty]} = \Psi\mathbf{1}_{[T,\infty]} = 0$. Since N is weakly differentially subordinate to M, by the Itô isomorphism for each $x^* \in X^*$, $0 \leq s < t < \infty$ we have a.s.

$$\begin{aligned}[\langle x^*, N\rangle]_t - [\langle x^*, N\rangle]_s &= \int_s^t |\langle x^*, \Psi(r)\rangle|^2\,\mathrm{d}r\\ &\leq \int_s^t |\langle x^*, \Phi(r)\rangle|^2\,\mathrm{d}r = [\langle x^*, M\rangle]_t - [\langle x^*, M\rangle]_s.\end{aligned}$$

Therefore we can deduce that $|\langle x^*,\Psi\rangle| \le |\langle x^*,\Phi\rangle|$ a.s. on $\mathbb{R}_+\times\Omega$. By Lemma 3.3.6 there exists progressively measurable $a:\mathbb{R}_+\times\Omega\to\mathbb{R}$ such that $|a|\le 1$ on $\mathbb{R}_+\times\Omega$ and $\Psi = a\Phi$ a.s. on $\mathbb{R}_+\times\Omega$. Now for each $n\ge 1$ set $a_n:\mathbb{R}_+\times\Omega\to\mathbb{R}$, $\Phi_n:\mathbb{R}_+\times\Omega\to X$ be elementary progressively measurable such that $|a_n|\le 1$, $a_n\to a$ a.s. on $\mathbb{R}_+\times\Omega$ and $\mathbb{E}\int_0^T \|\Phi(t)-\Phi_n(t)\|^2\,\mathrm{d}t\to 0$ as $n\to\infty$. Then by the triangle inequality

$$\begin{aligned}\Big(\mathbb{E}\int_0^T \|\Psi(t)-a_n(t)\Phi_n(t)\|^2\,\mathrm{d}t\Big)^{\frac12} \le \Big(\mathbb{E}\int_0^T \|\Phi(t)\|^2(a(t)-a_n(t))^2\,\mathrm{d}t\Big)^{\frac12}\\ + \Big(\mathbb{E}\int_0^T \|\Phi(t)-\Phi_n(t)\|^2 a_n^2\,\mathrm{d}t\Big)^{\frac12},\end{aligned} \tag{3.5.12}$$

which vanishes as $n\to\infty$ by the dominated convergence theorem. For each $n\ge 1$ the inequality

$$(\mathbb{E}\|((a_n\Phi_n)\cdot W)_\infty\|^p)^{\frac1p} \le \beta_{p,X}(\mathbb{E}\|(\Phi_n\cdot W)_\infty\|^p)^{\frac1p}$$

holds thanks to the martingale transform theorem [79, Theorem 4.2.25]. Then (3.5.11) follows from the previous estimate and (3.5.12) when one lets n go to infinity. □

Remark 3.5.12. Let W be a one-dimensional Wiener process, $\mathbb{F}$ be a filtration which is generated by W. Let $M,N:\mathbb{R}_+\times\Omega\to X$ be $\mathbb{F}$-martingales such that $M_0=N_0=0$ and N is weakly differentially subordinate to M. Then thanks to the Itô isomorphism [126, Theorem 3.5] there exist progressively measurable $\Phi,\Psi:\mathbb{R}_+\times\Omega\to X$ such that $M=\Phi\cdot W$and $N=\Psi\cdot W$, and thanks to Theorem 3.5.11

$$\mathbb{E}\|N_\infty\|^p \le \beta_{p,X}^p\mathbb{E}\|M_\infty\|^p,\quad p\in(1,\infty).$$

This shows that on certain probability spaces the estimate (3.5.9) automatically holds with a constant $C_p=1$.

4

L^p-ESTIMATES FOR WEAK DIFFERENTIAL SUBORDINATION AND FOR MARTINGALE DECOMPOSITIONS

This chapter is based on the paper *Martingale decompositions and weak differential subordination in UMD Banach spaces* by Ivan Yaroslavtsev, see [184].

In this chapter we consider Meyer-Yoeurp decompositions for UMD Banach space-valued martingales. Namely, we prove that X is a UMD Banach space if and only if for any fixed $p \in (1,\infty)$, any X-valued L^p-bounded martingale M has a unique decomposition $M = M^d + M^c$ such that M^d is a purely discontinuous martingale, M^c is a continuous martingale, $M^c_0 = 0$ and

$$\mathbb{E}\|M^d_\infty\|^p + \mathbb{E}\|M^c_\infty\|^p \leq c_{p,X}\mathbb{E}\|M_\infty\|^p.$$

An analogous assertion is shown for the Yoeurp decomposition of a purely discontinuous martingales into a sum of a quasi-left continuous martingale and a martingale with accessible jumps.

As an application we show that X is a UMD Banach space if and only if for any fixed $p \in (1,\infty)$ and for all X-valued martingales M and N such that N is weakly differentially subordinate to M, one has the estimate

$$\mathbb{E}\|N_\infty\|^p \leq C_{p,X}\mathbb{E}\|M_\infty\|^p.$$

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4.1. INTRODUCTION

It is well-known from the fundamental paper of Itô [81] on the real-valued case, and several works [1, 5, 17, 50, 157] on the vector-valued case, that for any Banach space X, any centered X-valued Lévy process has a unique decomposition $L = W + \widetilde{N}$, where W is an X-valued Wiener process, and $\widetilde{N}$ is an X-valued weak integral with respect to a certain compensated Poisson random measure. Moreover, W and $\widetilde{N}$ are independent, and therefore since W is symmetric, for each $1 < p < \infty$ and $t \geq 0$,

$$\mathbb{E}\|\widetilde{N}_t\|^p \leq \mathbb{E}\|L_t\|^p. \tag{4.1.1}$$

The natural generalization of this result to general martingales in the real-valued setting was provided by Meyer in [122] and Yoeurp in [190]. Namely, it was shown that any real-valued martingale M can be uniquely decomposed into a sum of two martingales M^d and M^c such that M^d is purely discontinuous (i.e. the quadratic variation $[M^d]$ has a pure jump version), and M^c is continuous with $M^c_0 = 0$. The reason why they needed such a decomposition is a further decomposition of a semimartingale, and finding an exponent of a semimartingale (we refer the reader to [89] and [190] for the details on this approach). In the present article we extend Meyer-Yoeurp theorem to the vector-valued setting, and provide extension of (4.1.1) for a general martingale (see Subsection 4.3.1). Namely, we prove that for any UMD Banach space X and any $1 < p < \infty$, an X-valued L^p-bounded martingale M can be uniquely decomposed into a sum of two martingales M^d and M^c such that M^d is purely discontinuous (i.e. $\langle M^d, x^*\rangle$ is purely discontinuous for each $x^* \in X^*$), and M^c is continuous with $M^c_0 = 0$. Moreover, then for each $t \geq 0$,

$$(\mathbb{E}\|M^d_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|M^c_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \tag{4.1.2}$$

where $\beta_{p,X}$ is the UMD$_p$ constant of X (see Section 2.3). Theorem 4.3.13 shows that such a decomposition together with L^p-estimates of type (4.1.2) is possible if and only if X has the UMD property.

The purely discontinuous part can be further decomposed: in [190] Yoeurp proved that any real-valued purely discontinuous M^d can be uniquely decomposed into a sum of a purely discontinuous quasi-left continuous martingale M^q (analogous to the "compensated Poisson part", which does not jump at predictable stopping times), and a purely discontinuous martingale with accessible jumps M^a (analogous to the "discrete part", which jumps only at certain predictable stopping times). In Subsection 4.3.2 we extend this result to a UMD space-valued setting with appropriate estimates. Namely, we prove that for each $1 < p < \infty$ the same type of decomposition is possible and unique for an X-valued purely discontinuous L^p-bounded martingale M^d, and then for each $t \geq 0$,

$$(\mathbb{E}\|M^q_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M^d_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|M^a_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M^d_t\|^p)^{\frac{1}{p}}. \tag{4.1.3}$$

Again as Theorem 4.3.13 shows, the (4.1.3)-type estimates are a possible only in UMD Banach spaces.

Even though the Meyer-Yoeurp and Yoeurp decompositions can be easily extended from the real-valued case to a Hilbert space case, the author could not find the corresponding estimates of type (4.1.2)-(4.1.3) in the literature, so we wish to present this special issue here. If H is a Hilbert space, $M : \mathbb{R}_+ \times \Omega \to H$ is a martingale, then there exists a unique decomposition of M into a continuous part M^c, a purely discontinuous quasi-left continuous part M^q, and a purely discontinuous part M^a with accessible jumps. Moreover, then for each $1 < p < \infty$, and for $i = c, q, a$,

$$(\mathbb{E}\|M_t^i\|^p)^{\frac{1}{p}} \leq (p^* - 1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \tag{4.1.4}$$

where $p^* = \max\{p, \frac{p}{p-1}\}$. Notice that though (4.1.4) follows from (4.1.2)-(4.1.3) since $\beta_{p,H} = p^* - 1$, it can be easily derived from the differential subordination estimates for Hilbert space-valued martingales obtained by Wang in [179].

Both the Meyer-Yoeurp and Yoeurp decompositions play a significant rôle in stochastic integration: if $M = M^c + M^q + M^a$ is a decomposition of an H-valued martingale M into continuous, purely discontinuous quasi-left continuous and purely discontinuous with accessible jumps parts, and if $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H, X)$ is elementary predictable for some UMD Banach space X, then the decomposition $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$ of a stochastic integral $\Phi \cdot M$ is a decomposition of the martingale $\Phi \cdot M$ into continuous, purely discontinuous quasi-left continuous and purely discontinuous with accessible jumps parts, and for any $1 < p < \infty$ we have that

$$\mathbb{E}\|(\Phi \cdot M)_\infty\|^p \eqsim_{p,X} \mathbb{E}\|(\Phi \cdot M^c)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^q)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^a)_\infty\|^p.$$

The corresponding Itô isomorphism for $\Phi \cdot M^c$ for a general UMD Banach space X was derived by Veraar and the author in [177], while Itô isomorphisms for $\Phi \cdot M^q$ and $\Phi \cdot M^a$ are shown in Chapter 7 for the case $X = L^r(S)$, $1 < r < \infty$.

The major underlying techniques involved in the proofs of (4.1.2) and (4.1.3) are rather different from the original methods of Meyer in [122] and Yoeurp in [190]. They include the results on the differentiability of the Burkholder function of any finite dimensional Banach space, which have been proven recently in [189] and which allow us to use Itô formula in order to show the desired inequalities in the same way as it was demonstrated by Wang in [179].

The main application of the Meyer-Yoeurp decomposition are L^p-estimates for weakly differentially subordinated martingales. The weak differential subordination property was introduced in Chapter 3, and can be described in the following way: an X-valued martingale N is weakly differentially subordinate to an X-valued martingale M if for each $x^* \in X^*$ a.s. $|\langle N_0, x^*\rangle| \leq |\langle M_0, x^*\rangle|$ and for each $t \geq s \geq 0$

$$[\langle N, x^*\rangle]_t - [\langle N, x^*\rangle]_s \leq [\langle M, x^*\rangle]_t - [\langle M, x^*\rangle]_s.$$

If both M and N are purely discontinuous, and if X is a UMD Banach space, then by [189], for each $1 < p < \infty$ we have that $\mathbb{E}\|N_\infty\|^p \leq \beta_{p,X}^p \mathbb{E}\|M_\infty\|^p$. Section 4.4 is devoted to the generalization of this result to continuous and general martingales. There we show that if both M and N are continuous, then $\mathbb{E}\|N_\infty\|^p \leq c_{p,X}^p \mathbb{E}\|M_\infty\|^p$, where the least admissible $c_{p,X}$ is within the interval $[\beta_{p,X}, \beta_{p,X}^2]$. Furthermore, using the Meyer-Yoeurp decomposition and estimates (4.1.2) we show that for general X-valued martingales M and N such that N is weakly differentially subordinate to M the following holds

$$(\mathbb{E}\|N_\infty\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2(\beta_{p,X}+1)(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}.$$

The weak differential subordination as a stronger version of the differential subordination is of interest in Harmonic Analysis. For instance, it was shown in [189] that sharp L^p-estimates for weakly differentially subordinated purely discontinuous martingales imply sharp estimates for the norms of a broad class of Fourier multipliers on $L^p(\mathbb{R}^d; X)$. Also there is a strong connection between the weak differential subordination of continuous martingales and the norm of the Hilbert transform on $L^p(\mathbb{R}; X)$ (see [189] and Remark 4.4.4).

Alternative approaches to Fourier multipliers for functions with values in UMD spaces have been constructed from the differential subordination for purely discontinuous martingales (see Bañuelos and Bogdan [10], Bañuelos, Bogdan and Bielaszewski [9], and recent work [189]), and for continuous martingales (see McConnell [118] and Geiss, Montgomery-Smith and Saksman [66]). It remains open whether one can combine these two approaches using the general weak differential subordination theory.

4.2. PRELIMINARIES

We set the scalar field to be $\mathbb{R}$.

Let X be a finite dimensional Banach space. Then according to Theorem 2.20 and Proposition 2.21 in [59] there exists a unique translation-invariant measure λ_X on X such that $\lambda_X(\mathbb{B}_X) = 1$ for the unit ball $\mathbb{B}_X$ of X. We will call λ_X the *Lebesgue measure*.

4.3. UMD BANACH SPACES AND MARTINGALE DECOMPOSITIONS

Let X be a Banach space, $1 < p < \infty$. In this section we will show that the Meyer-Yoeurp and Yoeurp decompositions for X-valued L^p-bounded martingales take place if and only if X has the UMD property.

4.3.1. Meyer-Yoeurp decomposition in UMD case

This subsection is devoted to the generalization of Meyer-Yoeurp decomposition (see Subsection 2.2.3) to the UMD Banach space case:

Theorem 4.3.1 (Meyer-Yoeurp decomposition). *Let X be a UMD Banach space, $p \in (1,\infty)$, $M : \mathbb{R}_+ \times \Omega \to X$ be an L^p-bounded martingale. Then there exist unique martingales $M^d, M^c : \mathbb{R}_+ \times \Omega \to X$ such that M^d is purely discontinuous, M^c is continuous, $M^c_0 = 0$ and $M = M^d + M^c$. Moreover, then for all $t \geq 0$*

$$(\mathbb{E}\|M^d_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|M^c_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{4.3.1}$$

The following proposition follows from Section 3.3.

Proposition 4.3.2. *Let X be a finite dimensional Banach space, $p \in (1,\infty)$. Let $Y = X \oplus \mathbb{R}$ be a Banach space such that $\|(x,r)\|_Y = (\|x\|_X^p + |r|^p)^{\frac{1}{p}}$. Then $\beta_{p,Y} = \beta_{p,X}$. Moreover, if $M : \mathbb{R}_+ \times \Omega \to X$ is a martingale on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$, then there exists a sequence $(M^m)_{m\geq 1}$ of Y-valued martingales on an enlarged probability space $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\mathbb{P}})$ with an enlarged filtration $\overline{\mathbb{F}} = (\overline{\mathscr{F}}_t)_{t\geq 0}$ such that*

1. *M^m_t has absolutely continuous distributions with respect to the Lebesgue measure on Y for each $m \geq 1$ and $t \geq 0$;*
2. *$M^m_t \to (M_t, 0)$ pointwise as $m \to \infty$ for each $t \geq 0$;*
3. *if for some $t \geq 0$ $\mathbb{E}\|M_t\|^p < \infty$, then for each $m \geq 1$ one has that $\mathbb{E}\|M^m_t\|^p < \infty$ and $\mathbb{E}\|M^m_t - (M_t, 0)\|^p \to 0$ as $m \to \infty$;*
4. *if M is continuous, then $(M^m)_{m\geq 1}$ are continuous as well,*
5. *if M is purely discontinuous, then $(M^m)_{m\geq 1}$ are purely discontinuous as well.*

Proof. The proof of (1)-(3) follows from Lemma 3.3.18, while (4) and (5) follow from the construction of M^m given in 3.3.18. □

Remark 4.3.3. Notice that the construction in Section 3.3 also allows us to sum these approximations for different martingales. Namely, if M and N are two X-valued martingales, then we can construct the corresponding Y-valued martingales $(M^m)_{m\geq 1}$ and $(N^m)_{m\geq 1}$ as in Proposition 4.3.2 in such a way that $M^m_t + N^m_t$ has an absolutely continuous distribution for each $t \geq 0$ and $m \geq 1$.

Proof of Theorem 4.3.1. Step 1: finite dimensional case. Let X be finite dimensional. Then M^d and M^c exist due to Remark 2.2.14. Without loss of generality $\mathscr{F}_t = \mathscr{F}_\infty$, $M^d_t = M^d_\infty$ and $M^c_t = M^c_\infty$. Let d be the dimension of X.

Let $|||\cdot|||$ be a Euclidean norm on X. Then $(X, |||\cdot|||)$ is a Hilbert space, and by Remark 2.2.7 the quadratic variation $[M^c]$ exists and has a continuous version. Let us show that without loss of generality we can suppose that $[M^c]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Let $A:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ be as follows: $A_t=[M^c]_t+t$. Then A is strictly increasing continuous, $A_0=0$ and $A_\infty=\infty$ a.s. Let the time-change $\tau=(\tau_s)_{s\geq 1}$ be defined as in Theorem 2.4.25. Then by Theorem 2.4.25, $M^c\circ\tau$ is a continuous martingale, $M^d\circ\tau$ is a purely discontinuous martingale, $(M^c\circ\tau)_0=0$, $(M^d\circ\tau)_0=M_0^d$ and due to the Kazamaki theorem [89, Theorem 17.24], $[M^c\circ\tau]=[M^c]\circ\tau$. Therefore for all $t>s\geq 0$ by Theorem 2.4.25 and the fact that $\tau_t\geq\tau_s$ a.s.

$$\begin{aligned}[M^c\circ\tau]_t-[M^c\circ\tau]_s=[M^c]_{\tau_t}-[M^c]_{\tau_s}&\leq[M^c]_{\tau_t}-[M^c]_{\tau_s}+(\tau_t-\tau_s)\\&=([M^c]_{\tau_t}+\tau_t)-([M^c]_{\tau_s}+\tau_s)\\&=A_{\tau_t}-A_{\tau_s}=t-s.\end{aligned}$$

Hence $[M^c\circ\tau]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Moreover, $(M^i\circ\tau)_\infty=M^i_\infty$, $i\in\{c,d\}$, so this time-change argument does not affect (4.3.1). Hence we can redefine $M^c:=M^c\circ\tau$, $M^d:=M^d\circ\tau$, $\mathbb{F}=(\mathscr{F}_s)_{s\geq 0}:=\mathbb{G}=(\mathscr{F}_{\tau_s})_{s\geq 0}$.

Since $[M^c]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$ and thanks to Theorem 2.7.1, we can extend Ω and find a d-dimensional Wiener process $W:\mathbb{R}_+\times\Omega\to\mathbb{R}^d$ and a stochastically integrable progressively measurable function $\Phi:\mathbb{R}_+\times\Omega\to\mathscr{L}(\mathbb{R}^d,X)$ such that $M^c=\Phi\cdot W$.

Let $U:X\times X\to\mathbb{R}$ be the Burkholder function that was discussed in Section 3.3. Let us show that $\mathbb{E}U(M_t,M_t^d)\leq 0$.

Due to Proposition 4.3.2 and Remark 4.3.3 we can assume that M_s^c, M_s^d and $M_s=M_s^d+M_s^c$ have absolutely continuous distributions with respect to the Lebesgue measure λ_X on X for each $s\geq 0$. Let $(x_n)_{n=1}^d$ be a basis of X, $(x_n^*)_{n=1}^d$ be the corresponding dual basis of X^* (see Definition 2.11.1). By the Itô formula (2.12.1),

$$\begin{aligned}\mathbb{E}U(M_t,M_t^d)=\mathbb{E}U(M_0,M_0^d)&+\mathbb{E}\int_0^t\langle\partial_xU(M_{s-},M_{s-}^d),\mathrm{d}M_s\rangle\\&+\mathbb{E}\int_0^t\langle\partial_yU(M_{s-},M_{s-}^d),\mathrm{d}M_s^d\rangle+\mathbb{E}I_1+\mathbb{E}I_2,\end{aligned}\tag{4.3.2}$$

where

$$I_1=\sum_{0<s\leq t}[\Delta U(M_s,M_s^d)-\langle\partial_xU(M_{s-},M_{s-}^d),\Delta M_s\rangle-\langle\partial_yU(M_{s-},M_{s-}^d),\Delta M_s^d\rangle],$$

$$\begin{aligned}I_2&=\frac12\int_0^t\sum_{i,j=1}^dU_{x_i,x_j}(M_{s-},M_{s-}^d)\,\mathrm{d}[\langle M,x_i^*\rangle,\langle M,x_j^*\rangle]_s^c\\&=\frac12\int_0^t\sum_{i,j=1}^dU_{x_i,x_j}(M_{s-},M_{s-}^d)\langle\Phi^*(s)x_i^*,\Phi^*(s)x_j^*\rangle\,\mathrm{d}s.\end{aligned}$$

(Recall that by (3.3.7) and (3.3.8), U is Fréchet-differentiable a.s. on $X\times X$, hence ∂_xU and ∂_yU are well-defined. Moreover, U is zigzag-concave, so U is concave in the first variable, and therefore the second-order derivatives U_{x_i,x_j} in the first variable are well-defined and exist a.s. on $X \times X$ by the Alexandrov theorem [57, Theorem 6.4.1].) The last equality holds due to Theorem 2.12.1 and the fact that by Lemma 2.6.1 for all $s \geq 0$ a.s.

$$[\langle M, x_i^* \rangle, \langle M, x_j^* \rangle]_s^c = [\langle \Phi \cdot W, x_i^* \rangle, \langle \Phi \cdot W, x_j^* \rangle]_s = [(\Phi^* x_i^*) \cdot W, (\Phi^* x_j^*) \cdot W]_s$$
$$= \int_0^s \langle \Phi^*(r) x_i^*, \Phi^*(r) x_j^* \rangle \, \mathrm{d}r.$$

Let us first show that $I_1 \leq 0$ a.s. Indeed, since M^d is a purely discontinuous part of M, then by Definition 2.2.13 $\langle M^d, x^* \rangle$ is a purely discontinuous part of $\langle M, x^* \rangle$, and due to Theorem 2.2.10 a.s. for each $t \geq 0$

$$\Delta |\langle M^d, x^* \rangle|_t^2 = \Delta [\langle M^d, x^* \rangle]_t = \Delta [\langle M, x^* \rangle]_t = \Delta |\langle M, x^* \rangle|_t^2$$

for each $x^* \in X^*$. Thus for each $s \geq 0$ by (3.3.8) and (3.3.9) $\mathbb{P}$-a.s.

$$\begin{aligned}
&\Delta U(M_s, M_s^d) - \langle \partial_x U(M_{s-}, M_{s-}^d), \Delta M_s \rangle - \langle \partial_y U(M_{s-}, M_{s-}^d), \Delta M_s^d \rangle \\
&= V(M_{s-} + M_{s-}^d + 2\Delta M_s, M_{s-}^d - M_{s-}) - V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}) \\
&\quad - \langle \partial_x V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), 2\Delta M_s \rangle \leq 0,
\end{aligned}$$

so $I_1 \leq 0$ a.s., and $\mathbb{E} I_1 \leq 0$. Now we show that

$$\mathbb{E}\Big(\int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d), \mathrm{d}M_s \rangle + \int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d), \mathrm{d}M_s^d \rangle \Big) = 0.$$

Indeed,

$$\begin{aligned}
\int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d), \mathrm{d}M_s \rangle &+ \int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d), \mathrm{d}M_s^d \rangle \\
&= \int_0^t \langle \partial_x V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), \mathrm{d}(M_s + M_s^d) \rangle \\
&+ \int_0^t \langle \partial_y V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), \mathrm{d}(M_s^d - M_s) \rangle
\end{aligned}$$

so by Lemma 3.2.2 and 3.3.13 it is a martingale which starts at zero, hence its expectation is zero.

Finally let us show that $I_2 \leq 0$ a.s. Fix $s \in [0, t]$ and $\omega \in \Omega$. Then $x^* \mapsto \|\Phi^*(s, \omega) x^*\|^2$ defines a nonnegative definite quadratic form on X^*, and since any nonnegative quadratic form defines a Euclidean seminorm, there exists a basis $(\tilde{x}_n^*)_{n=1}^d$ of X^* and a $\{0,1\}$-valued sequence $(a_n)_{n=1}^d$ such that

$$\langle \Phi^*(s, \omega) \tilde{x}_n^*, \Phi^*(s, \omega) \tilde{x}_m^* \rangle = a_n \delta_{mn}, \quad m, n = 1, \ldots, d.$$

Let $(\tilde{x}_n)_{n=1}^d$ be the corresponding dual basis of X as it is defined in Definition 2.11.1. Then due to Lemma 2.11.2 and the linearity of Φ and directional derivatives of U

(we skip s and ω for the simplicity of the expressions)

$$\begin{aligned}\sum_{i,j=1}^{d} U_{x_i,x_j}(M_{s-},M^d_{s-})\langle \Phi^* x_i^*, \Phi^* x_j^*\rangle &= \sum_{i,j=1}^{d} U_{\tilde{x}_i,\tilde{x}_j}(M_{s-},M^d_{s-})\langle \Phi^* \tilde{x}_i^*, \Phi^* \tilde{x}_j^*\rangle\\ &= \sum_{i=1}^{d} U_{\tilde{x}_i,\tilde{x}_i}(M_{s-},M^d_{s-})\|\Phi^* \tilde{x}_i^*\|^2.\end{aligned}$$

Recall that U is zigzag-concave, so $t \mapsto U(x+t\tilde{x}_i, y)$ is concave for each $x,y \in X$, $i = 1,\dots,d$. Therefore $U_{\tilde{x}_i,\tilde{x}_i}(M_{s-},M^d_{s-}) \le 0$ a.s., and a.s.

$$\sum_{i=1}^{d} U_{\tilde{x}_i,\tilde{x}_i}(M_{s-}(\omega),M^d_{s-}(\omega))\|\Phi^*(s,\omega)\tilde{x}_i^*\|^2 \le 0.$$

Consequently, $I_2 \le 0$ a.s., and by (4.3.2), Remark 3.3.9 and the fact that $M^d_0 = M_0$

$$\mathbb{E}U(M_t, M^d_t) \le \mathbb{E}U(M_0, M_0) \le 0.$$

By (3.3.4), $\mathbb{E}\|M^d_t\|^p - \beta^p_{p,X}\mathbb{E}\|M_t\|^p \le \mathbb{E}U(M_t,M^d_t) \le 0$, so the first part of (4.3.1) holds.

The second part of (4.3.1) follows from the same machinery applied for V. Namely, one can analogously show that

$$\mathbb{E}\|M^c_t\|^p - \beta^p_{p,X}\mathbb{E}\|M_t\|^p \le \mathbb{E}U(M_t,M^c_t) = \mathbb{E}V(M^d+2M^c,-M^d) \le 0$$

by using a V-version of (4.3.2), inequality (3.3.9), and the fact that V is concave in the first variable a.s. on $X \times X$.

Step 2: general case. Without loss of generality we set $\mathscr{F}_\infty = \mathscr{F}_t$. Let $M_t = \xi$. If ξ is a simple function, then it takes its values in a finite dimensional subspace X_0 of X, and therefore $(M_s)_{s\ge0} = (\mathbb{E}(\xi|\mathscr{F}_s))_{s\ge0}$ takes its values in X_0 as well, so the theorem and (4.3.1) follow from Step 1.

Now let ξ be general. Let $(\xi_n)_{n\ge1}$ be a sequence of simple $\mathscr{F}_t$-measurable functions in $L^p(\Omega;X)$ such that $\xi_n \to \xi$ as $n\to\infty$ in $L^p(\Omega;X)$. For each $n \ge 1$ define $\mathscr{F}_t$-measurable ξ^d_n and ξ^c_n such that

$$\begin{aligned} M^{d,n} &= (M^{d,n}_s)_{s\ge0} = (\mathbb{E}(\xi^d_n|\mathscr{F}_s))_{s\ge0},\\ M^{c,n} &= (M^{c,n}_s)_{s\ge0} = (\mathbb{E}(\xi^c_n|\mathscr{F}_s))_{s\ge0}\end{aligned} \tag{4.3.3}$$

are the respectively purely discontinuous and continuous parts of martingale $M^n = (\mathbb{E}(\xi_n|\mathscr{F}_s))_{s\ge0}$ as in Remark 2.2.14. Then due to Step 1 and (4.3.1), $(\xi^d_n)_{n\ge1}$ and $(\xi^c_n)_{n\ge1}$ are Cauchy sequences in $L^p(\Omega;X)$. Let $\xi^c := L^p-\lim_{n\to\infty}\xi^c_n$ and $\xi^d := L^p-\lim_{n\to\infty}\xi^d_n$. Define the X-valued L^p-bounded martingales M^d and M^c by

$$M^d = (M^d_s)_{s\ge0} := (\mathbb{E}(\xi^d|\mathscr{F}_s))_{s\ge0},\quad M^c = (M^c_s)_{s\ge0} := (\mathbb{E}(\xi^c|\mathscr{F}_s))_{s\ge0}.$$

Thanks to Proposition 2.2.16, M^d is purely discontinuous, and due to Proposition 2.2.8 M^c is continuous and $M^c_0 = 0$, so $M = M^d + M^c$ is the desired decomposition.

The uniqueness of the decomposition follows from Lemma 2.2.17. For estimates (4.3.1) we note that by Step 1, (4.3.1) applied for Step 1, and [79, Proposition 4.2.17] for each $n \geq 1$

$$(\mathbb{E}\|\xi_n^d\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|\xi_n^c\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n\|^p)^{\frac{1}{p}},$$

and it remains to let $n \to \infty$. □

Remark 4.3.4. Let X be a UMD Banach space, $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be continuous (resp. purely discontinuous) L^p-bounded martingale. Then there exists a sequence $(M^n)_{n\geq 1}$ of continuous (resp. purely discontinuous) X-valued L^p-bounded martingales such that M^n takes its values is a finite dimensional subspace of X for each $n \geq 1$ and $M^n_\infty \to M_\infty$ in $L^p(\Omega; X)$ as $n \to \infty$. Such a sequence can be provided e.g. by (4.3.3).

We have proven the Meyer-Yoeurp decomposition in the UMD setting. Next we prove a converse result which shows the necessity of the UMD property.

Theorem 4.3.5. *Let X be a finite dimensional Banach space, $p \in (1,\infty)$, $\delta \in (0, (\beta_{p,X} - 1) \wedge 1)$. Then there exist a purely discontinuous martingale $M^d : \mathbb{R}_+ \times \Omega \to X$, a continuous martingale $M^c : \mathbb{R}_+ \times \Omega \to X$ such that $\mathbb{E}\|M^d_\infty\|^p, \mathbb{E}\|M^c_\infty\|^p < \infty$, $M^d_0 = M^c_0 = 0$, and for $M = M^d + M^c$ and $i \in \{c, d\}$ the following hold*

$$(\mathbb{E}\|M^i_\infty\|^p)^{\frac{1}{p}} \geq \Big(\frac{\beta_{p,X} - 1}{2} - \delta\Big)(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{4.3.4}$$

Recall that by [79, Proposition 4.2.17] $\beta_{p,X} \geq \beta_{p,\mathbb{R}} = p^* - 1 \geq 1$ for any UMD Banach space X and $1 < p < \infty$.

Definition 4.3.6. A random variable $r : \Omega \to \{-1, 1\}$ is called a *Rademacher variable* if $\mathbb{P}(r = 1) = \mathbb{P}(r = -1) = \frac{1}{2}$.

Lemma 4.3.7. *Let $\varepsilon > 0$, $p \in (1,\infty)$. Then there exists a continuous martingale $M : [0,1] \times \Omega \to [-1,1]$ with a symmetric distribution such that* $\operatorname{sign} M_1$ *is a Rademacher random variable and*

$$\|M_1 - \operatorname{sign} M_1\|_{L^p(\Omega)} < \varepsilon. \tag{4.3.5}$$

Proof. Let $W : [0,1] \times \Omega \to \mathbb{R}$ be a standard Wiener process. For each $n \geq 1$ we define a stopping time $\tau_n := \inf\{t : |W_t| > \frac{1}{n}\} \wedge 1$. Then $\tau_n \to 0$ a.s. as $n \to \infty$, and hence there exists $N \geq 1$ such that $\mathbb{P}(NW_1^{\tau_N} = \operatorname{sign} W_1^{\tau_N}) > 1 - \frac{\varepsilon^p}{2^p}$. Let $M = NW^{\tau_N}$. Then

$$\|M_1 - \operatorname{sign} M_1\|_{L^p(\Omega)} \leq \Big(\mathbb{E}\big[(|M_1| + 1)^p \mathbf{1}_{M_1 \neq \operatorname{sign} M_1}\big]\Big)^{\frac{1}{p}} < \Big(2^p \cdot \frac{\varepsilon^p}{2^p}\Big)^{\frac{1}{p}} \leq \varepsilon,$$

and (4.3.5) follows.

Notice that since W is a Wiener process, W_1 has a standard Gaussian distribution. Consequently,

$$\mathbb{P}(M_1 = 0) = \mathbb{P}(NW_1^{\tau_N} = 0) \leq \mathbb{P}(NW_1 = 0) = 0,$$

and since W^{τ_N} has a symmetric distribution, $\operatorname{sign} M_1$ is Rademacher. □

Remark 4.3.8. Let X be a UMD space, $1 < p < \infty$, $\delta > 0$. Then using Proposition 2.3.1 one can construct a martingale difference sequence $(d_j)_{j=1}^n \in L^p(\Omega; X)$ and a $\{-1,1\}$-valued sequence $(\varepsilon_j)_{j=1}^n$ such that

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \frac{\varepsilon_j \pm 1}{2} d_j\Big\|^p\Big)^{\frac{1}{p}} \geq \frac{\beta_{p,X} - \delta - 1}{2}\Big(\mathbb{E}\Big\|\sum_{j=1}^n d_j\Big\|^p\Big)^{\frac{1}{p}}.$$

Proof of Theorem 4.3.5. Denote $\frac{\beta_{p,X}-\delta-1}{2}$ by $\gamma_{p,X}^{\delta}$. By Proposition 2.3.1 there exists a natural number $N \geq 1$, a discrete X-valued martingale $(f_n)_{n=0}^N$ such that $f_0 = 0$, and a sequence of scalars $(\varepsilon_n)_{n=1}^N$ such that $\varepsilon_n \in \{0,1\}$ for each $n = 1, \ldots, N$, such that

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d f_n\Big\|^p\Big)^{\frac{1}{p}} \geq \gamma_{p,X}^{\delta}(\mathbb{E}\|f_N\|^p)^{\frac{1}{p}}. \tag{4.3.6}$$

According to [79, Theorem 3.6.1] we can assume that $(f_n)_{n=0}^N$ is a Paley-Walsh martingale. Let $(r_n)_{n=1}^N$ be a sequence of Rademacher variables and $(\phi_n)_{n=1}^N$ be a sequence of functions as in Definition 2.2.5, i.e. be such that

$$f_n = \sum_{k=2}^n r_k \phi_k(r_1, \ldots, r_{k-1}) + r_1\phi_1, \quad n = 1, \ldots, N.$$

Without loss of generality we assume that

$$(\mathbb{E}\|f_N\|^p)^{\frac{1}{p}} \geq 2. \tag{4.3.7}$$

For each $n = 1, \ldots, N$ define a continuous martingale $M^n : [0,1] \times \Omega \to [-1,1]$ as in Lemma 4.3.7, i.e. a martingale M^n with a symmetric distribution such that $\operatorname{sign} M_1^n$ is a Rademacher variable and

$$\|M_1^n - \operatorname{sign} M_1^n\|_{L^p(\Omega)} < \frac{\delta}{KL}, \tag{4.3.8}$$

where $K = \beta_{p,X} N \max\{\|\phi_1\|, \|\phi_2\|_\infty, \ldots, \|\phi_N\|_\infty\}$, and $L = 2\beta_{p,X}$. Without loss of generality suppose that $(M^n)_{n=1}^N$ are independent. For each $n = 1, \ldots, N$ set $\sigma_n = \operatorname{sign} M_1^n$. Define a martingale $M : [0, N+1] \times \Omega \to X$ in the following way:

$$M_t = \begin{cases} 0, & \text{if } 0 \leq t < 1; \\ M_{n-} + M_{t-n}^n \phi_n(\sigma_1, \ldots, \sigma_{n-1}), & \text{if } t \in [n, n+1) \text{ and } \varepsilon_n = 0; \\ M_{n-} + \sigma_n \phi_n(\sigma_1, \ldots, \sigma_{n-1}), & \text{if } t \in [n, n+1) \text{ and } \varepsilon_n = 1. \end{cases}$$

Let $M = M^d + M^c$ be the decomposition of Theorem 4.3.1. Then

$$M_{N+1}^c = \sum_{n=1}^N M_1^n \phi_n(\sigma_1, \ldots, \sigma_{n-1}) \mathbf{1}_{\varepsilon_n = 0},$$

$$M_{N+1}^d = \sum_{n=1}^N \sigma_n \phi_n(\sigma_1,\dots,\sigma_{n-1})\mathbf{1}_{\varepsilon_n=1} = \sum_{n=1}^N \varepsilon_n\sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1}).$$

Notice that $(\sigma_n)_{n=1}^N$ is a sequence of independent Rademacher variables, so by (4.3.6) and the discussion thereafter

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n\sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|^p\Big)^{\frac1p} \geq \gamma_{p,X}^{\delta}\Big(\mathbb{E}\Big\|\sum_{n=1}^N \sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|^p\Big)^{\frac1p}. \tag{4.3.9}$$

Let us first show (4.3.4) with $i=d$. Note that by the triangle inequality, (4.3.7) and (4.3.8)

$$\begin{aligned}(\mathbb{E}\|M_{N+1}\|^p)^{\frac1p} &\geq (\mathbb{E}\|f_N\|^p)^{\frac1p} - \sum_{n=1}^N\Big(\mathbb{E}\Big\|(M_1^n-\sigma_n)\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|^p\Big)^{\frac1p}\\ &\geq 2-\frac{\delta}{KL}\cdot N\cdot\max\{\|\phi_1\|,\|\phi_2\|_\infty,\dots,\|\phi_N\|_\infty\}>1.\end{aligned} \tag{4.3.10}$$

Therefore,

$$\begin{aligned}(\mathbb{E}\|M_{N+1}^d\|^p)^{\frac1p} &= \Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n\sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|^p\Big)^{\frac1p} \overset{(i)}{\geq} \gamma_{p,X}^\delta\Big(\mathbb{E}\Big\|\sum_{n=1}^N \sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|^p\Big)^{\frac1p}\\ &\overset{(ii)}{\geq} \gamma_{p,X}^\delta\Big(\mathbb{E}\Big\|\sum_{n=1}^N \mathbf{1}_{\varepsilon_n=1}\sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1}) + \sum_{n=1}^N \mathbf{1}_{\varepsilon_n=0}M_1^n\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|^p\Big)^{\frac1p}\\ &\quad -\gamma_{p,X}^\delta\sum_{n=1}^N\Big(\mathbb{E}\Big\|(M_1^n-\sigma_n)\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|^p\Big)^{\frac1p}\\ &\overset{(iii)}{\geq} \gamma_{p,X}^\delta(\mathbb{E}\|M_{N+1}\|^p)^{\frac1p} - \frac{\delta}{L} \overset{(iv)}{\geq} \Big(\frac{\beta_{p,X}-1}{2}-\delta\Big)(\mathbb{E}\|M_{N+1}\|^p)^{\frac1p},\end{aligned}$$

where (i) follows from (4.3.9), (ii) holds by the triangle inequality, (iii) holds by (4.3.8), and (iv) follows from (4.3.10). By the same reason and Remark 4.3.8, (4.3.4) holds for $i=c$. □

Let $p\in(1,\infty)$. Recall that $\mathcal{M}_X^p$ is a space of all X-valued L^p-bounded martingales, $\mathcal{M}_X^{p,d},\mathcal{M}_X^{p,c}\subset\mathcal{M}_X^p$ are its subspaces of purely discontinuous martingales and continuous martingales that start at zero respectively (see Section 2.2).

Theorem 4.3.9. *Let X be a Banach space. Then X is UMD if and only if for some (or, equivalently, for all) $p\in(1,\infty)$, for any probability space $(\Omega,\mathcal{F},\mathbb{P})$ with any filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq0}$ that satisfies the usual conditions, $\mathcal{M}_X^p=\mathcal{M}_X^{p,d}\oplus\mathcal{M}_X^{p,c}$, and there exist projections $A^d,A^c\in\mathcal{L}(\mathcal{M}_X^p)$ such that ran $A^d=\mathcal{M}_X^{p,d}$, ran $A^c=\mathcal{M}_X^{p,c}$, and for any $M\in\mathcal{M}_X^p$ the decomposition $M=A^dM+A^cM$ is the Meyer-Yoeurp decomposition from Theorem 4.3.1. If this is the case, then*

$$\|A^d\|\leq\beta_{p,X} \text{ and } \|A^c\|\leq\beta_{p,X}. \tag{4.3.11}$$

Moreover, there exist $(\Omega,\mathcal{F},\mathbb{P})$ and $\mathbb{F}=(\mathcal{F}_t)_{t\geq0}$ such that

$$\|A^d\|,\|A^c\|\geq\frac{\beta_{p,X}-1}{2}\vee 1. \tag{4.3.12}$$

Proof. The "if" part follows from (4.3.11), and the "only if" part follows from (4.3.12), so it is sufficient to show (4.3.11) and (4.3.12). (4.3.11) is equivalent to (4.3.1). The bound $\geq \frac{\beta_{p,X}-1}{2}$ in (4.3.12) follows from Theorem 4.3.5, while the bound ≥ 1 follows from the fact that both A^d and A^c are projections onto nonzero spaces $\mathcal{M}_X^{p,d}$ and $\mathcal{M}_X^{p,c}$ respectively. □

Corollary 4.3.10. *Let X be a UMD Banach space, $p \in (1,\infty)$. Let $i \in \{c,d\}$. Then $(\mathcal{M}_X^{p,i})^* \simeq \mathcal{M}_{X^*}^{p',i}$, and for each $M \in \mathcal{M}_{X^*}^{p',i}$ and $N \in \mathcal{M}_X^{p,i}$*

$$\langle M,N\rangle := \mathbb{E}\langle M_\infty, N_\infty\rangle, \quad \|M\|_{(\mathcal{M}_X^{p,i})^*} \eqsim_{p,X} \|M\|_{\mathcal{M}_{X^*}^{p',i}}.$$

To prove the corollary above we will need the following lemma.

Lemma 4.3.11. *Let X be a UMD Banach space, $p \in (1,\infty)$, $M \in \mathcal{M}_X^{p,d}$, $N \in \mathcal{M}_{X^*}^{p',c}$. Then $\mathbb{E}\langle M_\infty, N_\infty\rangle = 0$.*

Proof. First suppose that N_∞ takes it values in a finite dimensional subspace Y of X^*. Let $d \geq 1$ be the dimension of Y, $(y_k)_{k=1}^d$ be the basis of Y. Then there exist $N^1,\dots,N^d \in \mathcal{M}_{\mathbb{R}}^{p',c}$ such that $N = \sum_{k=1}^d N^k y_k$. Hence

$$E\langle M_\infty, N_\infty\rangle = E\Big\langle M_\infty, \sum_{k=1}^d N_\infty^k y_k\Big\rangle = \sum_{k=1}^d \mathbb{E}\langle M_\infty, y_k\rangle N_\infty^k \overset{(*)}{=} 0, \tag{4.3.13}$$

where $(*)$ holds due to Proposition 2.2.12.

Now turn to the general case. By Remark 4.3.4 for each $N \in \mathcal{M}_{X^*}^{p',c}$ there exists a sequence $(N^n)_{n\geq 1}$ of continuous martingales such that each of N^n is in $\mathcal{M}_{X^*}^{p',c}$ and takes its valued in a finite dimensional subspace of X^*, and $N_\infty^n \to N_\infty$ in $L^{p'}(\Omega;X^*)$ as $n \to \infty$. Then due to (4.3.13), $E\langle M_\infty, N_\infty\rangle = \lim_{n\to\infty} E\langle M_\infty, N_\infty^n\rangle = 0$, so the lemma holds. □

Proof of Corollary 4.3.10. We will show only the case $i = d$, the case $i = c$ can be shown analogously.

$\mathcal{M}_{X^*}^{p',d} \subset (\mathcal{M}_X^{p,d})^*$ and $\|M\|_{(\mathcal{M}_X^{p,d})^*} \leq \|M\|_{\mathcal{M}_{X^*}^{p',d}}$ for each $M \in \mathcal{M}_{X^*}^{p',d}$ thanks to the Hölder inequality. Now let us show the inverse. Let $f \in (\mathcal{M}_X^{p,d})^*$. Since due to Proposition 2.2.16 $\mathcal{M}_X^{p,d}$ is a closed subspace of $\mathcal{M}_X^p$, by the Hahn-Banach theorem and Proposition 2.2.3 there exists $L \in \mathcal{M}_{X^*}^{p'}$ such that $\mathbb{E}\langle L_\infty, N_\infty\rangle = f(N)$ for any $N \in \mathcal{M}_X^{p,d}$, and $\|L\|_{\mathcal{M}_{X^*}^{p'}} = \|f\|_{(\mathcal{M}_X^{p,d})^*}$. Let $L = L^d + L^c$ be the Meyer-Yoeurp decomposition of L as in Theorem 4.3.1. Then by (4.3.1)

$$\|L^d\|_{\mathcal{M}_{X^*}^{p',d}} \lesssim_{p,X} \|L\|_{\mathcal{M}_{X^*}^{p'}} = \|f\|_{(\mathcal{M}_X^{p,d})^*}$$

and $\mathbb{E}\langle L_\infty^d, N_\infty\rangle = \mathbb{E}\langle L_\infty, N_\infty\rangle$, so the theorem holds. □

4.3.2. Yoeurp decomposition of purely discontinuous martingales

As Yoeurp shown in [190] (see Subsection 2.4.3 and [89]), one can provide further decomposition of a purely discontinuous martingale into two parts: a martingale with accessible jumps and a quasi-left continuous martingale. This subsection is devoted to the generalization of this result to a UMD case.

Theorem 4.3.12. *Let X be a UMD Banach space, $M:\mathbb{R}_+\times\Omega\to X$ be a purely discontinuous L^p-bounded martingale. Then there exist unique purely discontinuous martingales $M^a, M^q:\mathbb{R}_+\times\Omega\to X$ such that M^a has accessible jumps, M^q is quasi-left continuous, $M^q_0=0$ and $M=M^a+M^q$. Moreover, if this is the case, then for $i\in\{a,q\}$*

$$(\mathbb{E}\|M^i_\infty\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{4.3.14}$$

Proof. *Step 1: finite dimensional case.* First assume that X is finite dimensional. Then M^a and M^q exist and unique due to coordinate-wise applying of Corollary 2.4.12. Let $M=M^a+M^q$, $N=M^a$. Then for any $x^*\in X^*$, $t\geq 0$ by Corollary 2.4.12 and Lemma 2.4.16 a.s.

$$[\langle M,x^*\rangle]_t = [\langle M,x^*\rangle]^a_t + [\langle M,x^*\rangle]^q_t = [\langle M^a,x^*\rangle]_t + [\langle M^q,x^*\rangle]_t,$$

and

$$[\langle N,x^*\rangle]_t = [\langle N,x^*\rangle]^a_t + [\langle N,x^*\rangle]^q_t = [\langle M^a,x^*\rangle]_t.$$

Therefore a.s.

$$[\langle N,x^*\rangle]_t - [\langle N,x^*\rangle]_s \leq [\langle M,x^*\rangle]_t - [\langle M,x^*\rangle]_s,\quad 0\leq s<t.$$

Moreover $M_0=N_0$. Hence N is weakly differentially subordinate to M (see Section 4.4), and (4.3.14) for $i=a$ follows from Theorem 3.3.17. By the same reason and since $M^q_0=0$, (4.3.14) holds true for $i=q$.

Step 2: general case. Now let X be general. Let $\xi=M_\infty$. Without loss of generality we set $\mathscr{F}_\infty=\mathscr{F}_t$. Let $(\xi_n)_{n\geq1}$ be a sequence of simple $\mathscr{F}_t$-measurable functions in $L^p(\Omega;X)$ such that $\xi_n\to\xi$ as $n\to\infty$ in $L^p(\Omega;X)$. For each $n\geq1$ define $\mathscr{F}_t$-measurable ξ^d_n and ξ^c_n such that $M^{d,n}=(\mathbb{E}(\xi^d_n|\mathscr{F}_s))_{s\geq0}$ and $M^{c,n}=(\mathbb{E}(\xi^c_n|\mathscr{F}_s))_{s\geq0}$ are respectively purely discontinuous and continuous parts of a martingale $(\mathbb{E}(\xi_n|\mathscr{F}_s))_{s\geq0}$ as in Remark 2.2.14. Then thanks to Theorem 4.3.1, $\xi^d_n\to\xi$ and $\xi^c_n\to0$ in $L^p(\Omega;X)$ as $n\to\infty$ since M is purely discontinuous.

Since for each $n\geq1$ the random variable ξ^d_n takes its values in a finite dimensional space, by Corollary 2.4.12 there exist $\mathscr{F}_t$-measurable $\xi^a,\xi^q\in L^p(\Omega;X)$ such that purely discontinuous martingales $M^{a,n}=(\mathbb{E}(\xi^a_n|\mathscr{F}_s))_{s\geq0}$ and $M^{q,n}=(\mathbb{E}(\xi^q_n|\mathscr{F}_s))_{s\geq0}$ are respectively with accessible jumps and quasi-left continuous, $\mathbb{E}(\xi^q_n|\mathscr{F}_0)=0$, and the decomposition $M^{d,n}=M^{a,n}+M^{q,n}$ is as in Corollary 2.4.12. Since $(\xi^d_n)_{n\geq1}$ is a Cauchy sequence in $L^p(\Omega;X)$, by Step 1 both $(\xi^a_n)_{n\geq1}$ and $(\xi^q_n)_{n\geq1}$ are Cauchy in

$L^p(\Omega; X)$ as well. Let ξ^a and ξ^q be their limits. Define martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \to X$ in the following way:

$$M_s^a := \mathbb{E}(\xi^a|\mathscr{F}_s), \;\; M_s^q := \mathbb{E}(\xi^q|\mathscr{F}_s), \quad s \geq 0.$$

By Proposition 2.4.18 M^a is a martingale with accessible jumps, M^q is quasi-left continuous, $M_0^q = 0$ a.s., and therefore $M = M^a + M^q$ is the desired decomposition. Moreover, by Step 1 for each $n \geq 1$ and $i \in \{a, q\}$, $(\mathbb{E}\|\xi_n^i\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n^d\|^p)^{\frac{1}{p}}$, and hence the estimate (4.3.14) follows by letting n to infinity.

The uniqueness of the decomposition follows from Lemma 2.4.19. □

The following theorem, as Theorem 4.3.5, illustrates that the decomposition in Theorem 4.3.12 takes place only in the UMD space case.

Theorem 4.3.13. *Let X be a finite dimensional Banach space, $p \in (1, \infty)$, $\delta \in \big(0, \frac{\beta_{p,X}-1}{2}\big)$. Then there exist purely discontinuous martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \to X$ such that M^a has accessible jumps, M^q is quasi-left continuous, $\mathbb{E}\|M_\infty^a\|^p$, $\mathbb{E}\|M_\infty^q\|^p < \infty$, $M_0^a = M_0^q = 0$, and for $M = M^a + M^q$ and $i \in \{a, q\}$ the following holds*

$$(\mathbb{E}\|M_\infty^i\|^p)^{\frac{1}{p}} \geq \Big(\frac{\beta_{p,X} - 1}{2} - \delta\Big)(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{4.3.15}$$

For the proof we will need the following lemma.

Lemma 4.3.14. *Let $\varepsilon \in \big(0, \frac{1}{2}\big)$, $p \in (1, \infty)$. Then there exist martingales $M, M^a, M^q : [0,1] \times \Omega \to [-1-\varepsilon, 1+\varepsilon]$ with symmetric distributions such that M^a is a martingale with accessible jumps, $\|M_1^a\|_{L^p(\Omega)} < \varepsilon$, M^q is a quasi-left continuous martingale, $M_0^q = 0$ a.s., $M = M^a + M^q$,* $\mathrm{sign} M_1$ *is a Rademacher random variable and*

$$\|M_1 - \mathrm{sign} M_1\|_{L^p(\Omega)} < \varepsilon. \tag{4.3.16}$$

Proof. Let $N^+, N^- : [0,1] \times \Omega \to \mathbb{R}$ be independent Poisson processes with the same intensity λ_ε such that $\mathbb{P}(N_1^+ = 0) = \mathbb{P}(N_1^- = 0) < \frac{\varepsilon^p}{2^p}$ (such λ_ε exists since N_1^+ and N_1^- have Poisson distributions, see [95]). Define a stopping time τ in the following way:

$$\tau = \inf\{t : N_t^+ \geq 1\} \wedge \inf\{t : N_t^- \geq 1\} \wedge 1.$$

Let $M_t^q := N_{t\wedge\tau}^+ - N_{t\wedge\tau}^-$, $t \in [0,1]$. Then M^q is quasi-left continuous with a symmetric distribution. Let r be an independent Rademacher variable, $M_t^a = \frac{\varepsilon}{2} r$ for each $t \in [0,1]$. Then M^a is a martingale with accessible jumps and symmetric distribution, and $\|M_1^a\|_{L^p(\Omega)} = \frac{\varepsilon}{2} < \varepsilon$. Let $M = M^a + M^q$. Then a.s.

$$M_1 \in \Big\{-1 - \frac{\varepsilon}{2}, -1 + \frac{\varepsilon}{2}, -\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2}\Big\}, \tag{4.3.17}$$

so $\mathbb{P}(M_1 = 0) = 0$, and therefore $\operatorname{sign} M_1$ is a Rademacher random variable. Let us prove (4.3.16). Notice that due to (4.3.17) if $|M_1^q| = 1$, then $|M_1 - \operatorname{sign} M_1| < \frac{\varepsilon}{2}$, and if $|M_1^q| = 0$, then $|M_1 - \operatorname{sign} M_1| < 1$. Therefore

$$\begin{aligned}\mathbb{E}|M_1 - \operatorname{sign} M_1|^p &= \mathbb{E}|M_1 - \operatorname{sign} M_1|^p \mathbf{1}_{|M_1^q|=1} + \mathbb{E}|M_1 - \operatorname{sign} M_1|^p \mathbf{1}_{|M_1^q|=0}\\ &< \frac{\varepsilon^p}{2^p} + \frac{\varepsilon^p}{2^p} < \varepsilon^p,\end{aligned}$$

so (4.3.16) holds. □

Proof of Theorem 4.3.13. The proof is analogous to the proof of Theorem 4.3.5, while one has to use Lemma 4.3.14 instead of Lemma 4.3.7. □

Theorem 4.3.13 yields the following characterization of the UMD property.

Theorem 4.3.15. *Let X be a Banach space. Then X is a UMD Banach space if and only if for some (equivalently, for all) $p \in (1,\infty)$ there exists $c_{p,X} > 0$ such that for any L^p-bounded martingale $M := \mathbb{R}_+ \times \Omega \to X$ there exist unique martingales $M^c, M^q, M^a : \mathbb{R}_+ \times \Omega \to X$ such that $M_0^c = M_0^q = 0$, M^c is continuous, M^q is purely discontinuous quasi-left continuous, M^a is purely discontinuous with accessible jumps, $M = M^c + M^q + M^a$, and*

$$(\mathbb{E}\|M_\infty^c\|^p)^{\frac{1}{p}} + (\mathbb{E}\|M_\infty^q\|^p)^{\frac{1}{p}} + (\mathbb{E}\|M_\infty^a\|^p)^{\frac{1}{p}} \leq c_{p,X}(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{4.3.18}$$

If this is the case, then the least admissible $c_{p,X}$ is in the interval $\big[\frac{3\beta_{p,X}-3}{2} \vee 1, 3\beta_{p,X}\big]$.

The decomposition $M = M^c + M^q + M^a$ is called the *canonical decomposition* of the martingale M (see Subsection 2.4.3).

Proof. The "if and only if" part follows from Theorem 4.3.9, Theorem 4.3.12 and Theorem 4.3.13. The estimate $c_{p,X} \leq 3\beta_{p,X}$ follows from (4.3.1) and (4.3.14). The estimate $c_{p,X} \geq \frac{3\beta_{p,X}-3}{2} \vee 1$ follows from (4.3.4) and (4.3.15). □

Corollary 4.3.16. *Let X be a Banach space. Then X is a UMD Banach space if and only if $\mathcal{M}_X^{p,d} = \mathcal{M}_X^{p,a} \oplus \mathcal{M}_X^{p,q}$ and $\mathcal{M}_X^p = \mathcal{M}_X^{p,c} \oplus \mathcal{M}_X^{p,q} \oplus \mathcal{M}_X^{p,a}$ for any filtration that satisfies the usual conditions.*

Proof. The corollary follows from Theorem 4.3.12, Theorem 4.3.13 and Theorem 4.3.15. □

4.3.3. Stochastic integration

The current subsection is devoted to application of Theorem 4.3.15 to stochastic integration with respect to a general martingale.

Theorem 4.3.17. *Let H be a Hilbert space, X be a UMD Banach space, $p \in (1,\infty)$, $M : \mathbb{R}_+ \times \Omega \to H$ be a local martingale, $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H,X)$ be elementary progressive. Let $M = M^c + M^q + M^a$ be the canonical decomposition from Theorem 5.1.1. Then*

$$\mathbb{E}\|(\Phi\cdot M)_\infty\|^p \eqsim_{p,X} \mathbb{E}\|(\Phi\cdot M^c)_\infty\|^p + \mathbb{E}\|(\Phi\cdot M^q)_\infty\|^p + \mathbb{E}\|(\Phi\cdot M^a)_\infty\|^p. \tag{4.3.19}$$

and if $(\Phi\cdot M)_\infty \in L^p(\Omega;X)$, then $\Phi\cdot M = \Phi\cdot M^c + \Phi\cdot M^q + \Phi\cdot M^a$ is the canonical decomposition from Theorem 4.3.15.

Proof. The statement that $\Phi\cdot M = \Phi\cdot M^c + \Phi\cdot M^q + \Phi\cdot M^a$ is the canonical decomposition follows from Proposition 2.5.1, Theorem 4.3.15 and the fact that a.s. $(\Phi\cdot M)_0 = (\Phi\cdot M^c)_0 = (\Phi\cdot M^q)_0 = 0$. (4.3.19) follows then from (4.3.18) and the triangle inequality. □

Remark 4.3.18. Notice that the Itô isomorphism for the term $\Phi\cdot M^c$ from (4.3.19) was explored in [177]. It remains open what to do with the other two terms, but positive results in this direction were obtained in the case of $X = L^q(S)$ in Chapter 7.

4.4. WEAK DIFFERENTIAL SUBORDINATION AND GENERAL MARTINGALES

This section is devoted to the generalization of Theorem 3.3.17. Namely, here we show the L^p-estimates for general X-valued weakly differentially subordinated martingales.

Theorem 4.4.1. *Let X be a UMD Banach space, $M,N : \mathbb{R}_+ \times \Omega \to X$ be two martingales such that N is weakly differentially subordinate to M. Then for each $p \in (1,\infty)$, $t \geq 0$,*

$$(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2(\beta_{p,X}+1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{4.4.1}$$

The proof will be done in several steps. First we show an analogue of Theorem 3.3.17 for continuous martingales.

Theorem 4.4.2. *Let X be a Banach space. Then X is a UMD Banach space if and only if for some (equivalently, for all) $p \in (1,\infty)$ there exists $c > 0$ such that for any continuous martingales $M,N : \mathbb{R}_+ \times \Omega \to X$ such that N is weakly differentially subordinate to M, $M_0 = N_0 = 0$, one has that*

$$(\mathbb{E}\|N_\infty\|^p)^{\frac{1}{p}} \leq c_{p,X}(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{4.4.2}$$

If this is the case, then the least admissible $c_{p,X}$ is in the segment $[\beta_{p,X}, \beta_{p,X}^2]$.

For the proof we will need the following proposition, which demonstrates that one needs a slightly weaker assumption rather then in Theorem 4.4.2 so that the estimate (4.4.2) holds in a UMD Banach space.

Proposition 4.4.3. *Let X be a UMD Banach space, $1 < p < \infty$, $M, N : \mathbb{R}_+ \times \Omega \to X$ be continuous L^p-bounded martingales s.t. $M_0 = N_0 = 0$ and for each $x^* \in X^*$ a.s. for each $t \geq 0$*

$$[\langle N, x^*\rangle]_t \leq [\langle M, x^*\rangle]_t. \tag{4.4.3}$$

Then for each $t \geq 0$

$$(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{4.4.4}$$

Proof. Without loss of generality by a stopping time argument we assume that M and N are bounded and that $M_\infty = M_t$ and $N_\infty = N_t$.

One can also restrict to a finite dimensional case. Indeed, since X is a separable reflexive space, X^* is separable as well. Let $(Y_m)_{m\geq 1}$ be an increasing sequence of finite-dimensional subspaces of X^* such that $\overline{\bigcup_m Y_m} = X^*$ and $\|\cdot\|_{Y_m} = \|\cdot\|_{X^*|_{Y_m}}$ for each $m \geq 1$. Then for each fixed $m \geq 1$ there exists a linear operator $P_m : X \to Y_m^*$ of norm 1 defined as follows: $\langle P_m x, y\rangle = \langle x, y\rangle$ for each $x \in X, y \in Y_m$. Therefore $P_m M$ and $P_m N$ are Y_m^*-valued martingales. Moreover, (4.4.3) holds for $P_m M$ and $P_m N$ since there exists $P_m^* : Y_m \to X^*$, and for each $y \in Y_m$ we have that $\langle P_m M, y\rangle = \langle M, P_m y\rangle$ and $\langle P_m N, y\rangle = \langle N, P_m y\rangle$. Since Y_m is a closed subspace of X^*, [79, Proposition 4.2.17] yields $\beta_{p',Y_m} \leq \beta_{p',X^*}$, consequently again by [79, Proposition 4.2.17] $\beta_{p,Y_m^*} \leq \beta_{p,X^{**}} = \beta_{p,X}$. So if we prove the finite dimensional version, then

$$(\mathbb{E}\|P_m N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,Y_m^*}^2 (\mathbb{E}\|P_m M_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E}\|P_m M_t\|^p)^{\frac{1}{p}},$$

and (4.4.4) with $c_{p,X} = \beta_{p,X}^2$ will follow by letting $m \to \infty$.

Let d be the dimension of X, $|||\cdot|||$ be a Euclidean norm on $X \times X$. Let $L = (M, N) : \mathbb{R}_+ \times \Omega \to X \times X$ be a continuous martingale. Since $(X \times X, |||\cdot|||)$ is a Hilbert space, L has a continuous quadratic variation $[L] : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ (see Remark 2.2.7). Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be such that $A_s = [L]_s + s$ for each $s \geq 0$. Then A is continuous strictly increasing predictable. Define a random time-change $(\tau_s)_{s\geq 0}$ as in Theorem 2.4.25. Let $\mathbb{G} = (\mathcal{G}_s)_{s\geq 0} = (\mathcal{F}_{\tau_s})_{s\geq 0}$ be the induced filtration. Then thanks to the Kazamaki theorem [89, Theorem 17.24] $\widetilde{L} = L \circ \tau$ is a G-martingale, and $[\widetilde{L}] = [L] \circ \tau$. Notice that $\widetilde{L} = (\widetilde{M}, \widetilde{N})$ with $\widetilde{M} = M \circ \tau$, $\widetilde{N} = N \circ \tau$, and since by Kazamaki theorem [89, Theorem 17.24] $[M \circ \tau] = [M] \circ \tau$, $[N \circ \tau] = [N] \circ \tau$, and $(M \circ \tau)_0 = (N \circ \tau)_0 = 0$, we have that by (4.4.3) for each $x^* \in X^*$ a.s. for each $s \geq 0$

$$[\langle \widetilde{N}, x^*\rangle]_s = [\langle N, x^*\rangle]_{\tau_s} \leq [\langle M, x^*\rangle]_{\tau_s} = [\langle \widetilde{M}, x^*\rangle]_s \tag{4.4.5}$$

Moreover, for all $0 \leq u < s$ we have that a.s.

$$\begin{aligned}[\widetilde{L}]_s - [\widetilde{L}]_u = ([L] \circ \tau)_s - ([L] \circ \tau)_u &\leq ([L] \circ \tau)_s + \tau_s - ([L] \circ \tau)_u - \tau_u \\ &= ([L]_{\tau_s} + \tau_s) - ([L]_{\tau_u} + \tau_u) = s - u.\end{aligned}$$

Therefore $[\widetilde{L}]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Consequently, due to Theorem 2.7.1, there exists an enlarged probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ with an enlarged filtration $\widetilde{\mathbb{G}} = (\widetilde{\mathcal{G}}_s)_{s\geq 0}$, a $2d$-dimensional standard Wiener process W, which is defined on $\widetilde{\mathbb{G}}$, and a stochastically integrable progressively measurable function $f:\mathbb{R}_+\times\widetilde{\Omega}\to\mathscr{L}(\mathbb{R}^{2d},X\times X)$ such that $\widetilde{L}=f\cdot W$. Let $f^M,f^N:\mathbb{R}_+\times\Omega\to\mathscr{L}(\mathbb{R}^{2d},X)$ be such that $f=(f^M,f^N)$. Then $\widetilde{M}=f^M\cdot W$ and $\widetilde{N}=f^N\cdot W$. Let $(\overline{\Omega},\overline{\mathscr{F}},\overline{\mathbb{P}})$ be an independent probability space with a filtration $\overline{\mathbb{G}}$ and a $2d$-dimensional Wiener process $\overline{W}$ on it. Denote by $\overline{\mathbb{E}}$ the expectation on $(\overline{\Omega},\overline{\mathscr{F}},\overline{\mathbb{P}})$. Then because of the decoupling theorem [79, Theorem 4.4.1], for each $s\geq 0$

$$\begin{aligned}(\mathbb{E}\|\widetilde{N}_s\|^p)^{\frac1p}&=(\mathbb{E}\|(f^N\cdot W)_s\|^p)^{\frac1p}\leq\beta_{p,X}(\mathbb{E}\overline{\mathbb{E}}\|(f^N\cdot\overline{W})_s\|^p)^{\frac1p},\\ \frac{1}{\beta_{p,X}}(\mathbb{E}\overline{\mathbb{E}}\|(f^M\cdot\overline{W})_s\|^p)^{\frac1p}&\leq(\mathbb{E}\|(f^M\cdot W)_s\|^p)^{\frac1p}=(\mathbb{E}\|\widetilde{M}_s\|^p)^{\frac1p}.\end{aligned}\tag{4.4.6}$$

Due to the multidimensional version of [89, Theorem 17.11] and (4.4.5) for each $x^*\in X^*$ we have that

$$s\mapsto[\langle\widetilde{M},x^*\rangle]_s-[\langle\widetilde{N},x^*\rangle]_s=\int_0^s(|\langle x^*,f^M(r)\rangle|^2-|\langle x^*,f^N(r)\rangle|^2)\,\mathrm{d}r\tag{4.4.7}$$

is nonnegative and absolutely continuous a.s. Since X is separable, we can fix a set $\widetilde{\Omega}_0\subset\widetilde{\Omega}$ of full measure on which the function (4.4.7) is nonnegative for each $s\geq 0$.

Now fix $\omega\in\widetilde{\Omega}_0$ and $s\geq 0$. Let us prove that

$$\overline{\mathbb{E}}\|(f^N(\omega)\cdot\overline{W})_s\|^p\leq\overline{\mathbb{E}}\|(f^M(\omega)\cdot\overline{W})_s\|^p.$$

Since $f^M(\omega)$ and $f^N(\omega)$ are deterministic on $\overline{\Omega}$, and since due to (4.4.7) for each $x^*\in X^*$

$$\begin{aligned}\overline{\mathbb{E}}|\langle(f^N(\omega)\cdot\overline{W})_s,x^*\rangle|^2&=\int_0^s|\langle x^*,f^N(r,\omega)\rangle|^2\,\mathrm{d}r\\&\leq\int_0^s|\langle x^*,f^M(r,\omega)\rangle|^2\,\mathrm{d}r=\overline{\mathbb{E}}|\langle(f^M(\omega)\cdot\overline{W})_s,x^*\rangle|^2,\end{aligned}$$

by [129, Corollary 4.4] we have that $\overline{\mathbb{E}}\|(f^N(\omega)\cdot\overline{W})_s\|^p\leq\overline{\mathbb{E}}\|(f^M(\omega)\cdot\overline{W})_s\|^p$. Consequently, due to (4.4.6) and the fact that $\widetilde{\mathbb{P}}(\Omega_0)=1$

$$(\mathbb{E}\|\widetilde{N}_s\|^p)^{\frac1p}\leq\beta_{p,X}(\mathbb{E}\overline{\mathbb{E}}\|(f^N\cdot\overline{W})_s\|^p)^{\frac1p}\leq\beta_{p,X}(\mathbb{E}\overline{\mathbb{E}}\|(f^M\cdot\overline{W})_s\|^p)^{\frac1p}\leq\beta_{p,X}^2(\mathbb{E}\|\widetilde{M}_s\|^p)^{\frac1p}.$$

Recall that $\widetilde{M}$ and $\widetilde{N}$ are bounded, so thanks to the dominated convergence theorem one gets (4.4.4) with $c_{p,X}=\beta_{p,X}^2$ by letting s to infinity. □

Proof of Theorem 4.4.2. The "only if" part & the upper bound of $c_{p,X}$: The "only if" part and the estimate $c_{p,X}\leq\beta_{p,X}^2$ follows from Proposition 4.4.3 since (4.4.3) holds for M and N because N is weakly differentially subordinate to M.

The "if" part & the lower bound of $c_{p,X}$: Let $\beta_{p,X}$ be the UMD constant of X ($\beta_{p,X}=\infty$ if X is not a UMD space). Fix $K\geq 1$. Then by [79, Theorem 4.2.5] there exists $N\geq 1$, a Paley-Walsh martingale difference sequence $(d_n)_{n=1}^N$, and a $\{-1,1\}$-valued sequence $(\varepsilon_n)_{n=1}^N$ such that

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N\varepsilon_n d_n\Big\|^p\Big)^{\frac1p}\geq\Big(\beta_{p,X}\wedge 2K-\frac{1}{2K}\Big)\Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac1p}$$

Without loss of generality we can assume that

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|^p\Big)^{\frac{1}{p}}, \Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}} \leq 1.$$

Let $(r_n)_{n=1}^N$ be a sequence of Rademacher variables and $(\phi_n)_{n=1}^N$ be a sequence of functions as in Definition 2.2.5, i.e. be such that $d_n = r_n\phi_n(r_1,\dots,r_{n-1})$ for each $n = 1,\dots,N$.

By the same techniques as were used in the proof of Theorem 4.3.5 we can find a sequence of independent continuous real-valued symmetric martingales $(M^n)_{n=1}^N$ on $[0,1]$ such that for each $n = 1,\dots,N$

$$\|(M^n - \mathrm{sign}M^n)\phi_n(\mathrm{sign}M^1,\dots,\mathrm{sign}M^{n-1})\|_{L^p(\Omega;X)} \leq \frac{1}{8NK^2}. \tag{4.4.8}$$

Let $\sigma_n = \mathrm{sign}M^n$ for each $n = 1,\dots,N$. Then we define continuous martingales $M, N : \mathbb{R}_+ \times \Omega \to X$ in the following way:

$$M_t = \begin{cases} 0, & \text{if } 0 \leq t \leq 1;\\ M_n + M^n_{t-n}\phi_n(\sigma_1,\dots,\sigma_{n-1}), & \text{if } t \in (n, n+1], n \in \{1\dots,N\},\\ M_{N+1}, & \text{if } t > N+1,\end{cases}$$

$$N_t = \begin{cases} 0, & \text{if } 0 \leq t \leq 1;\\ M_n + \varepsilon_n M^n_{t-n}\phi_n(\sigma_1,\dots,\sigma_{n-1}), & \text{if } t \in (n, n+1], n \in \{1\dots,N\},\\ N_{N+1}, & \text{if } t > N+1.\end{cases}$$

Then N is weakly differentially subordinate to M. Indeed, for each $x^* \in X^*$, $n \in \{1,\dots,N\}$ and $t \in [n, n+1]$ a.s.

$$\begin{aligned}[\langle M,x^*\rangle]_t - [\langle M,x^*\rangle]_n &= [M^n]_{t-n}|\langle \phi_n(\sigma_1,\dots,\sigma_{n-1}), x^*\rangle|^2\\ &= [M^n]_{t-n}|\langle \varepsilon_n\phi_n(\sigma_1,\dots,\sigma_{n-1}), x^*\rangle|^2\\ &= [\langle N,x^*\rangle]_t - [\langle N,x^*\rangle]_n,\end{aligned}$$

therefore, since $M_1 = N_1 = 0$ a.s., we have that for each $x^* \in X^*$ and $t \geq 0$ a.s. $[\langle M,x^*\rangle]_t = [\langle N,x^*\rangle]_t$, so N is weakly differentially subordinate to M. Then

$$\begin{aligned}(\mathbb{E}\|N_\infty\|^p)^{\frac{1}{p}} &= \Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n M^n_1\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|\Big)^{\frac{1}{p}}\\ &\overset{(i)}{\geq} \Big(\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n\sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|\Big)^{\frac{1}{p}}\\ &\quad - \sum_{n=1}^N \|(M^n - \sigma_n)\phi_n(\sigma_1,\dots,\sigma_{n-1})\|_{L^p(\Omega;X)}\\ &\overset{(ii)}{\geq} \Big(\beta_{p,X} \wedge 2K - \frac{1}{2K}\Big)\Big(\mathbb{E}\Big\|\sum_{n=1}^N \sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1})\Big\|\Big)^{\frac{1}{p}} - \frac{1}{8K^2}\end{aligned}$$

$$
\begin{aligned}
&\overset{(iii)}{\geq} \Big(\beta_{p,X} \wedge 2K - \frac{1}{2K}\Big)\Big(\mathbb{E}\Big\|\sum_{n=1}^N M_1^n \phi_n(\sigma_1,\ldots,\sigma_{n-1})\Big\|\Big)^{\frac{1}{p}}\\
&\quad - 2K\sum_{n=1}^N \|(M^n - \sigma_n)\phi_n(\sigma_1,\ldots,\sigma_{n-1})\|_{L^p(\Omega;X)} - \frac{1}{8K^2}\\
&\overset{(iv)}{\geq} \Big(\beta_{p,X} \wedge K - \frac{1}{K}\Big)\Big(\mathbb{E}\Big\|\sum_{n=1}^N M_1^n \phi_n(\sigma_1,\ldots,\sigma_{n-1})\Big\|\Big)^{\frac{1}{p}}\\
&= \Big(\beta_{p,X} \wedge K - \frac{1}{K}\Big)(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}},
\end{aligned}
$$

where (i) and (iii) follow from the triangle inequality, and (ii) and (iv) follow from (4.4.8). Hence if X is not UMD, then such $c_{p,X}$ from (4.4.2) does not exist since $\big(\beta_{p,X} \wedge K - \frac{1}{K}\big) \to \infty$ as $K \to \infty$. If X is UMD, then such $c_{p,X}$ could exist, and if this is the case, then

$$
c_{p,X} \geq \lim_{K\to\infty}\Big(\beta_{p,X} \wedge K - \frac{1}{K}\Big) = \beta_{p,X}.
$$

□

Remark 4.4.4. Let X be a Banach space. Then according to [23, 32, 61] the Hilbert transform $\mathscr{H}_X$ can be extended to $L^p(\mathbb{R};X)$ for each $1 < p < \infty$ if and only if X is a UMD Banach space. Moreover, if this is the case, then

$$
\sqrt{\beta_{p,X}} \leq \|\mathscr{H}_X\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leq \beta_{p,X}^2.
$$

As it was shown in Section 3.5, the upper bound $\beta_{p,X}^2$ can be also directly derived from the upper bound for $c_{p,X}$ in Theorem 4.4.2. The sharp upper bound for $\|\mathscr{H}_X\|_{\mathscr{L}(L^p(\mathbb{R};X))}$ remains an open question (see [79, pp. 496-497]), so the sharp upper bound for $c_{p,X}$ is of interest.

Lemma 4.4.5. *Let X be a Banach space, $M^c, N^c : \mathbb{R}_+ \times \Omega \to X$ be continuous martingales, $M^d, N^d : \mathbb{R}_+ \times \Omega \to X$ be purely discontinuous martingales, $M_0^c = N_0^c = 0$. Let $M := M^c + M^d$, $N := N^c + N^d$. Suppose that N is weakly differentially subordinate to M. Then N^c is weakly differentially subordinate to M^c, and N^d is weakly differentially subordinate to M^d.*

Proof. First notice that a.s.

$$
\|N_0^c\| = 0 \leq 0 = \|M_0^c\|,
$$
$$
\|N_0^d\| = \|N_0\| \leq \|M_0\| = \|M_0^d\|.
$$

Now fix $x^* \in X^*$. It is enough now to prove that $\langle N^c, x^*\rangle$ is differentially subordinate to $\langle M^c, x^*\rangle$, and that $\langle N^d, x^*\rangle$ is differentially subordinate to $\langle M^d, x^*\rangle$. But this follows from [179, Lemma 1], Theorem 2.2.10 and the fact that $\langle M^d, x^*\rangle$ and $\langle N^d, x^*\rangle$ are purely discontinuous processes, and $\langle M^c, x^*\rangle$ and $\langle N^c, x^*\rangle$ are continuous processes. □

Proof of Theorem 4.4.1. By Theorem 4.3.1 there exist martingales $M^d, M^c, N^d, N^c : \mathbb{R}_+ \times \Omega \to X$ such that M^d and N^d are purely discontinuous, M^c and N^c are continuous, $M_0^c = N_0^c = 0$, and $M = M^d + M^c$ and $N = N^d + N^c$. By Lemma 4.4.5, N^d is weakly differentially subordinate to M^d and N^c is weakly differentially subordinate to M^c. Therefore for each $t \geq 0$

$$(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \overset{(i)}{\leq} (\mathbb{E}\|N_t^d\|^p)^{\frac{1}{p}} + (\mathbb{E}\|N_t^c\|^p)^{\frac{1}{p}} \overset{(ii)}{\leq} \beta_{p,X}^2 (\mathbb{E}\|M_t^d\|^p)^{\frac{1}{p}} + \beta_{p,X} (\mathbb{E}\|M_t^c\|^p)^{\frac{1}{p}}$$
$$\overset{(iii)}{\leq} \beta_{p,X}^2 (\beta_{p,X} + 1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}},$$

where (i) holds thanks to the triangle inequality, (ii) follows from Theorem 3.3.17 and Theorem 4.4.2, and (iii) follows from (4.3.1). □

Remark 4.4.6. It is worth noticing that in a view of recent results the sharp constant in (4.3.1) and (4.3.14) can be derived and equals the *UMD*$_p^{\{0,1\}}$*-constant* $\beta_{p,X}^{\{0,1\}}$. In order to show that this is the right upper bound one needs to use a $\{0,1\}$*-Burkholder function* instead of the Burkholder function, while the sharpness follows analogously Theorem 4.3.5 and 4.3.13. See [188] for details.

Remark 4.4.7. In Chapter 5 the existence of the canonical decomposition of a general local martingale together with the corresponding weak L^1-estimates were shown. Again existence of the canonical decomposition of any X-valued martingale is equivalent to X having the UMD property.

5

Existence of the canonical decomposition and weak L^1-estimates

This chapter is based on the paper *On the martingale decompositions of Gundy, Meyer, and Yoeurp in infinite dimensions* by Ivan Yaroslavtsev, see [185].

In this chapter we show that the canonical decomposition (comprising both the Meyer-Yoeurp and the Yoeurp decompositions) of a general X-valued local martingale is possible if and only if X has the UMD property. More precisely, X is a UMD Banach space if and only if for any X-valued local martingale M there exist a continuous local martingale M^c, a purely discontinuous quasi-left continuous local martingale M^q, and a purely discontinuous local martingale M^a with accessible jumps such that $M = M^c + M^q + M^a$. The corresponding weak L^1-estimates are provided. Important tools used in the proof are a new version of Gundy's decomposition of continuous-time martingales and weak L^1-bounds for a certain class of vector-valued continuous-time martingale transforms.

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5.1. INTRODUCTION

It is well-known thanks to the scalar-valued stochastic integration theory that a stochastic integral $\int \Phi \, \mathrm{d}N$ of a general bounded predictable real-valued process Φ with respect to a general real-valued local martingale N exists and is well defined (see e.g. Chapter 26 in [89]). Moreover, $\int \Phi \, \mathrm{d}N$ is a local martingale, so by the Burkholder-Davis-Gundy inequalities one can show the corresponding L^p-estimates for $p \in (1,\infty)$:

$$\mathbb{E} \sup_{0\leq s\leq t} \Big|\int_0^s \Phi \, \mathrm{d}N\Big|^p \eqsim_p \mathbb{E}\Big(\int_0^t \Phi^2 \, \mathrm{d}[N]\Big)^{\frac{p}{2}}, \quad t \geq 0 \tag{5.1.1}$$

(here $[N] : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is a quadratic variation of N, see (2.2.3) for the definition). The inequality (5.1.1) together with a Banach fixed point argument play an important rôle in providing solutions to SPDE's with a general martingale noise (see e.g. [48, 72, 73, 89, 126, 177] and references therein). For this reason (5.1.1)-type inequalities for a broader class of N and Φ are of interest. In particular, one can consider H-valued N and $\mathcal{L}(H,X)$-valued Φ for some Hilbert space H and Banach space X. Building on ideas of Garling [61] and McConnell [119], van Neerven, Veraar, and Weis have shown in [126] that for a special choice of N (namely, N being a Brownian motion) and a general process Φ it is necessary and sufficient that X is in the class of so-called *UMD Banach spaces* (see Section 2.3 for the definition) in order to obtain estimates of the form (5.1.1) with the right-hand side replaces by a generalized square function. Later in the paper [175] by Veraar and in the paper [177] by Veraar and the author, inequalities of the form (5.1.1) have been extended to a general continuous martingale N, again given that X has the UMD property.

Extending (5.1.1) to a general martingale N is an open problem, which was solved only for $X = L^q(S)$ with $q \in (1,\infty)$ in the recent work [54] by Dirksen and the author. One of the key tools applied therein was the so-called *canonical decomposition* of martingales. The canonical decomposition first appeared in the work [190] by Yoeurp, and partly in the paper [122] by Meyer, and has the following form: an X-valued local martingale M is said to admit the canonical decomposition if there exists a continuous local martingale M^c, a purely discontinuous quasi-left continuous local martingale M^q (a "Poisson-like" martingale which does not jump at predictable stopping times), and a purely discontinuous local martingale M^a with accessible jumps (a "discrete-like" martingale which jumps only at a certain countable set of predictable stopping times) such that $M_0^c = M_0^q = 0$ a.s. and $M = M^c + M^q + M^a$. The canonical decomposition (if it exists) is unique due to the uniqueness in the case $X = \mathbb{R}$ (see Remark 2.2.19 and 2.4.21). Moreover, when X is UMD one has by [184] that for all $p \in (1,\infty)$,

$$\mathbb{E}\|M_t\|^p \eqsim_{p,X} \mathbb{E}\|M_t^c\|^p + \mathbb{E}\|M_t^q\|^p + \mathbb{E}\|M_t^a\|^p, \ t \geq 0. \tag{5.1.2}$$

In particular, if N is H-valued and Φ is $\mathscr{L}(H,X)$-valued, then

$$\int \Phi \, \mathrm{d}N = \int \Phi \, \mathrm{d}N^c + \int \Phi \, \mathrm{d}N^q + \int \Phi \, \mathrm{d}N^a$$

is the canonical decomposition given that $N = N^c + N^q + N^a$ is the canonical decomposition, so

$$\mathbb{E}\Big\|\int_0^t \Phi \, \mathrm{d}N\Big\|^p \eqsim_{p,X} \mathbb{E}\Big\|\int_0^t \Phi \, \mathrm{d}N^c\Big\|^p + \mathbb{E}\Big\|\int_0^t \Phi \, \mathrm{d}N^q\Big\|^p + \mathbb{E}\Big\|\int_0^t \Phi \, \mathrm{d}N^a\Big\|^p, \quad t \geq 0,$$

which together with Doob's maximal inequality reduces the problem of extending (5.1.1) to the separate cases of N^c, N^q and N^a. Possible approaches of how to work with $\int \Phi \, \mathrm{d}N^c$, $\int \Phi \, \mathrm{d}N^q$, and $\int \Phi \, \mathrm{d}N^a$ have been provided by [54]: sharp estimates for the first were already obtained in [175, 177] and follow from the similar estimates for a Brownian motion from [126]; the second can be treated by using random measure theory (see Section 2.8), which is an extension of Poisson random measure integration theory (see [51] and [52]); finally, the latter one can be transformed to a discrete martingale by an approximation argument, so the desired L^p-estimates are nothing more but the *Burkholder-Rosenthal inequalities* (see [29, 54, 161] for details).

The canonical decomposition also plays a significant rôle in obtaining L^p-estimates for *weakly differentially subordinated* martingales. The weak differential subordination property as a vector-valued generalization of Burkholder's differential subordination property (see [33, 79, 102, 140]) was introduced by the author in [189], and can be described in the following way: an X-valued local martingale $\widetilde{M}$ is weakly differentially subordinate to an X-valued local martingale M if for each $x^* \in X^*$ and for each $t \geq s \geq 0$ a.s.

$$|\langle \widetilde{M}_0, x^*\rangle| \leq |\langle M_0, x^*\rangle|,$$
$$[\langle \widetilde{M}, x^*\rangle]_t - [\langle \widetilde{M}, x^*\rangle]_s \leq [\langle M, x^*\rangle]_t - [\langle M, x^*\rangle]_s.$$

If X is a UMD Banach space and $p \in (1,\infty)$, then applying L^p-bounds (5.1.2) for the terms of the canonical decomposition together with L^p-bounds for purely discontinuous (see [189]) and continuous (see [184]) weakly differentially subordinated martingales yields

$$(\mathbb{E}\|\widetilde{M}_\infty\|^p)^{\frac{1}{p}} \leq c_{p,X} (\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}, \tag{5.1.3}$$

where the best known constant $c_{p,X}$ equals $\beta_{p,X}^2(\beta_{p,X}+1)$ (here $\beta_{p,X}$ is the *UMD$_p$ constant of X*, see Section 2.3 for the definition). Sharp estimates for $c_{p,X}$ in (5.1.3) remain unknown. Moreover, it is an open problem whether one can prove weak L^1-estimates of the form

$$\lambda \mathbb{P}\big(\widetilde{M}_\infty^* > \lambda\big) \lesssim_{p,X} \mathbb{E}\|M_\infty\|, \quad \lambda > 0. \tag{5.1.4}$$

Here this question is partly solved: we show that (5.1.4) holds for $\widetilde{M}$ being one of the terms of the canonical decomposition of M (see (5.1.5) and (5.4.1)).

The discussion above demonstrates that the canonical decomposition is useful for vector-valued stochastic integration and weak differential subordination, so the following natural question arises: *for which Banach spaces X does every X-valued local martingale have the canonical decomposition?* The paper [184] together with the estimates (5.1.2) provides the answer for L^p-bounded martingales given $p \in (1,\infty)$. Then X being a UMD Banach space guarantees such a decomposition.

The present chapter is devoted to providing the definitive answer to this question (see Section 5.4):

Theorem 5.1.1. *Let X be a Banach space. Then the following are equivalent:*

(i) X is a UMD Banach space;

(ii) every local martingale $M : \mathbb{R}_+ \times \Omega \to X$ admits the canonical decomposition $M = M^c + M^q + M^a$.

Moreover, if this is the case, then for all $t \geq 0$ and $\lambda > 0$

$$\begin{aligned} \lambda \mathbb{P}((M^c)^*_t > \lambda) &\lesssim_X \mathbb{E}\|M_t\|, \\ \lambda \mathbb{P}((M^q)^*_t > \lambda) &\lesssim_X \mathbb{E}\|M_t\|, \\ \lambda \mathbb{P}((M^a)^*_t > \lambda) &\lesssim_X \mathbb{E}\|M_t\|. \end{aligned} \tag{5.1.5}$$

Notice that the inequalities (5.1.5) are new even in the real-valued case, even though in that case they are direct consequences of the sharp weak $(1,1)$-estimates for differentially subordinated martingales proven by Burkholder in [36, 37] (see also [133, 140] for details), from which one can show the following estimates

$$\begin{aligned} \lambda \mathbb{P}((M^c)^*_t > \lambda) &\leq 2\mathbb{E}|M_t|, \\ \lambda \mathbb{P}((M^q)^*_t > \lambda) &\leq 2\mathbb{E}|M_t|, \\ \lambda \mathbb{P}((M^a)^*_t > \lambda) &\leq 2\mathbb{E}|M_t|. \end{aligned}$$

The main instrument for proving $(ii) \Rightarrow (i)$ in Theorem 5.1.1 is Burkholder's characterization of UMD Banach spaces from [30]: X is a UMD Banach space if and only if there exists a constant $C > 0$ such that for any X-valued discrete martingale $(f_n)_{n\geq 0}$, for any sequence $(a_n)_{n\geq 0}$ with values in $\{-1, 1\}$ one has that

$$g^*_\infty > 1 \text{ a.s.} \implies \mathbb{E}\|f_\infty\| > C,$$

where $(g_n)_{n\geq 0}$ is an X-valued discrete martingale such that

$$\begin{aligned} g_n - g_{n-1} &= a_n(f_n - f_{n-1}), \quad n \geq 1, \\ g_0 &= a_0 f_0, \end{aligned} \tag{5.1.6}$$

and where $g^*_\infty := \sup_{n\geq 0} \|g_n\|$. Using this characterization for a given non-UMD Banach space X we construct a martingale $M : \mathbb{R}_+ \times \Omega \to X$ which does not have the canonical decomposition (see Subsection 5.4.3).

In order to obtain weak L^1-estimates of the form (5.1.5) together with $(i) \Rightarrow (ii)$ in Theorem 5.1.1 one needs to use two techniques. The first is the so-called *Gundy decomposition* of martingales. This decomposition was first obtained by Gundy in [71] for discrete real-valued martingales. Later in [41, 79, 116, 147] a more general version of this decomposition for vector-valued discrete martingales was obtained. In Section 5.3 we will present a continuous-time analogue of Gundy's decomposition, which has the following form: an X-valued martingale M can be decomposed into a sum of three martingales M^1, M^2, and M^3, depending on $\lambda > 0$, such that for each $t \geq 0$

(i) $\|M^1_t\|_{L^\infty(\Omega;X)} \leq 2\lambda$, $\mathbb{E}\|M^1_t\| \leq 5\mathbb{E}\|M_t\|$,

(ii) $\lambda\mathbb{P}((M^2)^*_t > 0) \leq 4\mathbb{E}\|M_t\|$,

(iii) $\mathbb{E}(\operatorname{Var} M^3)_t \leq 7\mathbb{E}\|M_t\|$,

where $\operatorname{Var} M$ is a variation of the path of M.

The second important tool is *weak differential subordination martingale transforms*. Discrete martingale transforms were pioneered by Burkholder in [28], where he considered a transform $(f_n)_{n\geq 0} \mapsto (g_n)_{n\geq 0}$ of a real-valued martingale $(f_n)_{n\geq 0}$ such that

$$\begin{aligned} g_n - g_{n-1} &= a_n(f_n - f_{n-1}), \quad n \geq 1, \\ g_0 &= a_0 f_0 \end{aligned}$$

for some $\{0,1\}$-valued deterministic sequence $(a_n)_{n\geq 0}$. Later in [30, 41, 67, 75, 79, 116] several approaches and generalizations to the vector-valued setting and operator-valued predictable sequence $(a_n)_{n\geq 0}$ have been discovered, while the martingale $(f_n)_{n\geq 0}$ remained discrete. In particular for a very broad class of discrete martingale transforms it was shown that L^p-boundedness of the transform implies weak L^1-bounds. In Subsection 5.4.1 (see Theorem 5.4.2) we prove the same assertion for a weak differential subordination martingale transform, i.e. for an operator T acting on continuous-time X-valued local martingales such that TM is weakly differentially subordinate to M and $\{M^*_\infty = 0\} \subset \{(TM)^*_\infty = 0\}$ for any X-valued local martingale M. A particular example of such a martingale transform T is $M \mapsto TM = M^c$, where M^c is the continuous part of M in the canonical decomposition. Due to (5.1.2) this operator is bounded as an operator acting on L^p-bounded martingales if X is UMD, so by Theorem 5.4.2 the first inequality of (5.1.5) follows. Even though in the case of a discrete filtration such an operator has a classical Burkholder's form (5.1.6) from [30] (with $(a_n)_{n\geq 0}$ being predictable instead of deterministic, see Proposition 5.4.6 and the remark thereafter), such transforms are of interest since they act on continuous-time martingales, which was not considered before.

5.2. PRELIMINARIES

In the sequel the scalar field is assumed to be $\mathbb{R}$, unless stated otherwise.

5.2.1. *Martingales and càdlàg processes*

We will denote by $\mathcal{M}_X^{1,\infty}(\mathbb{F})$ the set of all X-valued local $\mathbb{F}$-bounded martingales M such that

$$\sup_{\lambda>0} \lambda \mathbb{P}(M_\infty^* > \lambda) < \infty.$$

In the sequel we will omit $\mathbb{F}$ from the notations $\mathcal{M}_X^p(\mathbb{F})$, $\mathcal{M}_X^{p,\mathrm{loc}}(\mathbb{F})$, and $\mathcal{M}_X^{1,\infty}(\mathbb{F})$.

Let τ be a stopping time, $V : \mathbb{R}_+ \times \Omega \to X$ be càdlàg. Throughout this chapter we define $\Delta V_\tau : \Omega \to X$ in the following way:

$$\Delta V_\tau = \begin{cases} V_0, & \tau = 0, \\ V_\tau - \lim_{\varepsilon \to 0} V_{0\vee(\tau-\varepsilon)}, & 0 < \tau < \infty, \\ 0, & \tau = \infty, \end{cases}$$

where $\lim_{\varepsilon\to 0} V_{0\vee(\tau-\varepsilon)}$ exists since V has paths with left-hand limits.

One can define the so-called *ucp topology* (uniform convergence on compact sets in probability) on the linear space of all càdlàg adapted X-valued processes; convergence in this topology can be characterized in the following way: a sequence $(V^n)_{n\geq 1}$ of càdlàg adapted X-valued processes converges to $V : \mathbb{R}_+ \times \Omega \to X$ in the ucp topology if for any $t \geq 0$ and $K > 0$ we have that

$$\mathbb{P}\big(\sup_{0\leq s\leq t} \|V_s - V_s^n\| > K\big) \to 0 \quad n \to \infty. \tag{5.2.1}$$

Then the following proposition holds.

Proposition 5.2.1. *The linear space of all càdlàg adapted X-valued processes endowed with the ucp topology is complete.*

Proof. This is just the vector-valued analogue of [155, Theorem 62], for which one needs to apply the vector-valued variation of [154, Problem V.1]. □

We state without proof the following elementary but useful statement.

Lemma 5.2.2. *Let X be a Banach space, $(f_n)_{n\geq 1}$, f be continuous X-valued functions on $[0,1]$ such that $f_n \to f$ in $C([0,1];X)$ as $n \to \infty$. Then the function $F : [0,1] \to \mathbb{R}_+$ defined as follows*

$$F(t) = \sup_n \|f_n(t)\|, \quad t \in [0,1],$$

is continuous.

5.2.2. *Compensator and variation*

Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be an adapted càdlàg process. Then a predictable process $V : \mathbb{R}_+ \times \Omega$ is called a *predictable compensator* of M (or just a *compensator* of M) if $V_0 = 0$ a.s. and if $M - V$ is a local martingale.

The *variation* $\mathrm{Var}\, M : \mathbb{R}_+ \times \Omega \to \overline{\mathbb{R}}_+$ of a càdlàg process $M : \mathbb{R}_+ \times \Omega \to X$ is defined in the following way:

$$(\mathrm{Var}\, M)_t := \|M_0\| + \limsup_{\mathrm{mesh}\to 0} \sum_{n=1}^{N} \|M(t_n) - M(t_{n-1})\|, \tag{5.2.2}$$

where the limit superior is taken over all the partitions $0 = t_0 < \ldots < t_N = t$.

Let $V : \mathbb{R}_+ \times \Omega \to X$ be a càdlàg adapted process. Analogously to the scalar-valued situation we can define a càdlàg adapted process $V^* : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ of the following form

$$V_t^* := \sup_{s\in[0,t]} \|V_s\|, \quad t \geq 0.$$

5.3. GUNDY'S DECOMPOSITION OF CONTINUOUS-TIME MARTINGALES

For the proof of our main results, Theorem 5.4.1 and Theorem 5.4.2, we will need Gundy's decomposition of continuous-time martingales, which is a generalization of Gundy's decomposition of discrete martingales (see [71] and [79, Theorem 3.4.1] for the details).

Theorem 5.3.1 (Gundy's decomposition)**.** *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a martingale. Then for each $\lambda > 0$ there exist martingales M^1, M^2 $M^3 : \mathbb{R}_+ \times \Omega \to X$ such that $M = M^1 + M^2 + M^3$ and*

(i) $\|M_t^1\|_{L^\infty(\Omega;X)} \leq 2\lambda$, $\mathbb{E}\|M_t^1\| \leq 5\mathbb{E}\|M_t\|$ *for each* $t \geq 0$,

(ii) $\lambda\mathbb{P}((M^2)_t^* > 0) \leq 4\mathbb{E}\|M_t\|$ *for each* $t \geq 0$,

(iii) $\mathbb{E}(\mathrm{Var}\, M^3)_t \leq 7\mathbb{E}\|M_t\|$ *for each* $t \geq 0$.

Remark 5.3.2. Notice that if M is a discrete martingale (i.e. $M_t = M_{[t]}$ for any $t \geq 0$), then the decomposition in Theorem 5.3.1 turns to the classical discrete one from [79, Theorem 3.4.1].

For the proof we will need the following intermediate steps.

Lemma 5.3.3. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a càdlàg adapted process such that $\mathbb{E}(\mathrm{Var}\, M)_t < \infty$ for each $t \geq 0$ and a.s.*

$$M_t = \sum_{0\leq s\leq t} \Delta M_s, \quad t \geq 0.$$

Then M has a càdlàg predictable compensator $V:\mathbb{R}_+\times\Omega\to X$ such that for each $t\geq 0$

$$\mathbb{E}\|V_t\|\leq\mathbb{E}(\operatorname{Var}V)_t\leq\mathbb{E}(\operatorname{Var}M)_t. \tag{5.3.1}$$

In particular, if M has a.s. at most one jump, then

$$\mathbb{E}\|V_t\|\leq\mathbb{E}(\operatorname{Var}V)_t\leq\mathbb{E}(\operatorname{Var}M)_t=\mathbb{E}\|M_t\|. \tag{5.3.2}$$

Proof. Let μ^M be a random measure defined on $\mathbb{R}_+\times X$ pointwise in $\omega\in\Omega$ in the following way:

$$\mu^M(\omega;B\times A):=\sum_{u\in B}\mathbf{1}_{A\setminus\{0\}}(\Delta M_u(\omega)),\quad \omega\in\Omega, B\in\mathscr{B}(\mathbb{R}_+), A\in\mathscr{B}(X). \tag{5.3.3}$$

Notice that $(\operatorname{Var}M)_t=\sum_{0\leq s\leq t}\|\Delta M_s\|$ a.s. for each $t\geq 0$, so in particular a.s.

$$(\operatorname{Var}M)_t=\int_{[0,t]\times X}\|x\|\,\mathrm{d}\mu^M(x,s),\quad t\geq 0. \tag{5.3.4}$$

Also note that μ^M is $\mathscr{P}$-σ-finite: for each $0\leq u\leq v$ and $t\geq 0$ one has that

$$\begin{aligned}\mathbb{E}\int_{[0,t]\times X}\mathbf{1}_{\|x\|\in[u,v]}\,\mathrm{d}\mu^M &\eqsim_{u,v}\mathbb{E}\int_{[0,t]\times X}\|x\|\mathbf{1}_{\|x\|\in[u,v]}\,\mathrm{d}\mu^M\\ &\leq\mathbb{E}\int_{[0,t]\times X}\|x\|\,\mathrm{d}\mu^M\\ &=\mathbb{E}(\operatorname{Var}M)_t<\infty.\end{aligned}$$

Since μ^M is an integer-valued optional $\mathscr{P}$-σ-finite random measure, it has a predictable compensator ν^M (see Section 2.8 and [85, Theorem II.1.8]), and therefore since by (5.3.4)

$$\mathbb{E}\int_{[0,t]\times X}\|x\|\,\mathrm{d}\mu^M(x,s)=\mathbb{E}(\operatorname{Var}M)_t<\infty,$$

we have that

$$t\mapsto V_t:=\int_{[0,t]\times X}x\,\mathrm{d}\nu^M(x,s),\quad t\geq 0,$$

is integrable and càdlàg in time due to the fact that it is an integral with respect to the measure ν^M a.s. Moreover, by the definition of variation (5.2.2) we have that $\|V_t\|\leq(\operatorname{Var}V)_t$ a.s. for each $t\geq 0$, and hence

$$\begin{aligned}\mathbb{E}\|V_t\|\leq\mathbb{E}(\operatorname{Var}V)_t\leq\mathbb{E}\int_{[0,t]\times X}\|x\|\,\mathrm{d}\nu^M(x,s)&\overset{(*)}{=}\mathbb{E}\int_{[0,t]\times X}\|x\|\,\mathrm{d}\mu^M(x,s)\\ &\overset{(**)}{=}\mathbb{E}(\operatorname{Var}M)_t,\end{aligned}$$

where $(*)$ holds due to the definition of a compensator, and $(**)$ follows from (5.3.4). To show (5.3.2) it is sufficient to notice that if M has at most one jump then $(\operatorname{Var}M)_t=\|M_t\|$ a.s. for each $t\geq 0$. □

The following lemma is folklore, but the author could not find an appropriate reference, so we present it with the proof here.

Lemma 5.3.4. *Let X be a Banach space, $V : \mathbb{R}_+ \times \Omega \to X$ be a right-continuous predictable process, $V_0 = 0$ a.s. Then V is locally bounded.*

Proof. For each $n \geq 0$ define a stopping time $\tau_n := \inf\{t \geq 0 : \|V_t\| \geq n\}$. Then a sequence $(\tau_n)_{n\geq 1}$ of stopping times is increasing a.s. and tends to infinity as $n \to \infty$. Moreover, $(\tau_n)_{n\geq 1}$ are predictable by [89, Theorem 25.14] and the fact that for each $n \geq 1$

$$\{\tau \leq t\} = \{\sup_{0\leq s\leq t} \|V_s\| \geq n\} \in \mathscr{P}. \tag{5.3.5}$$

Therefore for each $n \geq 1$ there exists an announcing sequence $(\tau_{m,n})_{m\geq 1}$ of stopping times. Choose m_n so that $\mathbb{P}(\tau_n - \tau_{m_n,n} > \frac{1}{2^n}) < \frac{1}{2^n}$. Then $(\tau_{m_n,n})_{n\geq 1}$ is such that $\tau_{m_n,n} \to \infty$ a.s. as $n \to \infty$, and for each $n \geq 0$ we have that a.s. $\sup_{0\leq s\leq \tau_{m_n,n}} \|V_s\| \leq \sup_{0\leq s<\tau_n} \|V_s\| \leq n$. ☐

Let τ and σ be stopping times. Then we can set

$$\tau - \wedge \sigma - := (\tau \wedge \sigma)-. \tag{5.3.6}$$

Notice that if $M : \mathbb{R}_+ \times \Omega \to X$ is a càdlàg process, then $(M^{\tau-})^{\sigma-} = M^{\tau-\wedge\sigma-}$.

Proof of Theorem 5.3.1. By a stopping time argument we can assume that M is an L^1-martingale. Define a stopping time τ is the following way:

$$\tau = \inf\Big\{t \geq 0 : \|M_t\| \geq \frac{\lambda}{2}\Big\}.$$

Let $M^{2,1} := M - M^\tau$ and let $M^{3,1}(\cdot) = \Delta M_\tau \mathbf{1}_{[0,\cdot]}(\tau) + M_0^{\tau-}$, where by (2.4.2) we can conclude that a.s.

$$M_0^{\tau-} := \begin{cases} M_0, & \tau > 0, \\ 0, & \tau = 0. \end{cases} \tag{5.3.7}$$

Let $N : \mathbb{R}_+ \times \Omega \to X$ be such that $N_t = \Delta M_\tau \mathbf{1}_{[0,t]}(\tau)$, $t \geq 0$. Then due to the fact that $M_\tau = \mathbb{E}(M_\infty|\mathscr{F}_\tau)$ by [89, Theorem 7.29], [79, Corollary 2.6.30], and the fact that $\|M_{\tau-}\| \leq \frac{\lambda}{2}$ a.s., we get

$$\begin{aligned}\mathbb{E}(\operatorname{Var} N)_\infty = \mathbb{E}\|\Delta M_\tau\| = \mathbb{E}\|M_\tau - M_{\tau-}\| &\leq \mathbb{E}\|M_\tau\| + \mathbb{E}(\|M_{\tau-}\|\mathbf{1}_{\tau<\infty}) \\ &\leq \mathbb{E}\|M_\infty\| + \frac{\lambda}{2} < \infty.\end{aligned} \tag{5.3.8}$$

Therefore by Lemma 5.3.3, N has a compensator V. Let

$$\sigma := \inf\{t \geq 0 : \|V_t\| \geq \lambda\}$$

be a stopping time. Then by (5.3.5) σ is a predictable stopping time. Define now $M^1 = M^{\sigma-\wedge\tau-} + V^{\sigma-} - M_0^{\tau-}$, $M^{2,2} = (M^{\tau-} + V) - (M^{\sigma-\wedge\tau-} + V^{\sigma-})$, $M^{3,2} = N - V$ where

$\sigma-\wedge\tau-$ is defined as in (5.3.6). Define $M^2 := M^{2,1}+M^{2,2}$ and $M^3 := M^{3,1}+M^{3,2}$. Then $M = M^1+M^2+M^3$. Now let us describe why this is the right choice.

Step 1: M^1. First show that M^1 is a martingale. Indeed, for each $t\geq 0$

$$\begin{aligned} M^1_t &= M_t^{\sigma-\wedge\tau-} + V_t^{\sigma-} - M_0^{\tau-} = (M_t^{\tau-}+V_t - M_0^{\tau-})^{\sigma-}\\ &= \big(M_t^{\tau} - \mathbf{1}_{\tau\in[0,t]}\Delta M_\tau + V_t - M_0^{\tau_2-}\big)^{\sigma-}\\ &= \Big((M_t^{\tau}-M_0^{\tau-}) - (N_t - V_t)\Big)^{\sigma-}, \end{aligned} \tag{5.3.9}$$

and the last expression is a martingale due to the fact that M^τ is a martingale by [89, Theorem 7.12], the fact that $N-V$ is a martingale by the definition of a compensator, Lemma 2.4.4, and the fact that by (5.3.8)

$$\mathbb{E}\|N_\infty\| \leq \mathbb{E}(\operatorname{Var} N)_\infty \leq \mathbb{E}\|M_\infty\| + \frac{\lambda}{2} < \infty.$$

Now let us check (i): $\|M_\infty^{\sigma-\wedge\tau-}\|$, $\|M_0^{\tau-}\| \leq \frac{\lambda}{2}$ a.s. by the definition of τ, and $\|V_\infty^{\sigma-}\|\leq\lambda$ by the definition of σ, so $\|M^1_\infty\|\leq 2\lambda$ a.s.

Further, to prove the second part of (i) we will use the representation of M^1 from the last line of (5.3.9). Notice that by [89, Theorem 7.12] and [79, Corollary 2.6.30] for each fixed $t\geq 0$

$$\mathbb{E}\|M_t^\tau\| \leq \mathbb{E}\|M_t\|. \tag{5.3.10}$$

Moreover,

$$\begin{aligned} \mathbb{E}\|N_t\| = \mathbb{E}\|M_t^\tau - M_t^{\tau-}\| &\leq \mathbb{E}\|M_t^\tau\| + \mathbb{E}(\|M_t^{\tau-}\|\mathbf{1}_{\tau<\infty})\\ &\leq \mathbb{E}\|M_t^\tau\| + \mathbb{E}\Big(\frac{\lambda}{2}\mathbf{1}_{\tau<\infty}\Big) \leq 2\mathbb{E}\|M_t^\tau\| \overset{(*)}{\leq} 2\mathbb{E}\|M_t\|, \end{aligned}$$

where $\|M_t^{\tau-}\|\leq\frac{\lambda}{2}\leq\|M_t^\tau\|$ on $\{\tau<\infty\}$ by the definition of τ, and $(*)$ follows from [89, Theorem 7.12] and [79, Corollary 2.6.30]. Therefore by (5.3.2)

$$\mathbb{E}\|V_t\| \leq \mathbb{E}\|N_t\| \leq 2\mathbb{E}\|M_t\| \tag{5.3.11}$$

as well. Finally, $\mathbb{E}\|M_0^{\tau-}\|\leq\mathbb{E}\|M_0\|\leq\mathbb{E}\|M_t\|$ by (5.3.7) and [79, Corollary 2.6.30]. Consequently, the second part of (i) holds by the estimates above and by the triangle inequality.

Step 2: M^2. First note that

$$M^2 = M - M^\tau + (M^{\tau-}+V) - (M^{\tau-}+V)^{\sigma-}. \tag{5.3.12}$$

Let us check that M^2 is a martingale. $M-M^\tau$ is a martingale by [89, Theorem 7.12]. Furthermore,

$$M^{\tau-}+V = M^\tau - (N-V)$$

is a martingale as well due to [89, Theorem 7.12] and the fact that V is a compensator of N. Finally, $(M^{\tau-}+V)^{\sigma-}$ is a martingale by Lemma 2.4.4.

Let us now prove (ii). Notice that by (5.3.12)

$$\mathbb{P}((M^2)^*_t > 0) \le \mathbb{P}((M - M^\tau)^*_t > 0) + \mathbb{P}(((M^{\tau-} + V) - (M^{\tau-} + V)^{\sigma-})^*_t > 0).$$

First estimate $\mathbb{P}((M - M^\tau)^*_t > 0)$:

$$\mathbb{P}((M - M^\tau)^*_t > 0) \le \mathbb{P}(\tau \le t) \le \mathbb{P}\Big(M^*_t \ge \frac{\lambda}{2}\Big) \le \frac{2\mathbb{E}\|M_t\|}{\lambda},$$

where the latter inequality holds by (2.2.2). Using the same machinery we get

$$\begin{aligned}\mathbb{P}(((M^{\tau-} + V) - (M^{\tau-} + V)^{\sigma-})^*_t > 0) &\le \mathbb{P}(\sigma \le t)\\ &= \mathbb{P}(\|V_t\| \ge \lambda) \overset{(i)}{\le} \frac{\mathbb{E}\|V_t\|}{\lambda} \overset{(ii)}{\le} \frac{2\mathbb{E}\|M_t\|}{\lambda},\end{aligned}$$

where (i) follows from the Chebyshev inequality, and (ii) follows from (5.3.11). This terminates the proof of (ii).

Step 3: M^3. Recall that

$$M^3 = M_0^{\tau-} + N - V.$$

Therefore by the triangle inequality a.s. for each $t \ge 0$

$$\begin{aligned}\mathbb{E}(\operatorname{Var} M^3)_t &\le \mathbb{E}\|M_0^{\tau-}\| + \mathbb{E}(\operatorname{Var} N)_t + \mathbb{E}(\operatorname{Var} V)_t\\ &\le \mathbb{E}\|M_t\| + 2\mathbb{E}\|N_t\| \le 5\mathbb{E}\|M_t\|,\end{aligned} \tag{5.3.13}$$

where the latter inequality holds by (5.3.11), while the rest follows from (5.3.1) and the fact that $\mathbb{E}\|M_0^{\tau-}\| \le \mathbb{E}\|M_0\| \le \mathbb{E}\|M_t\|$. □

Remark 5.3.5. Let $p \in (1,\infty)$, M be an L^p-bounded martingale, $\lambda > 0$, $M = M^1 + M^2 + M^3$ be Gundy's decomposition (see the theorem above). Then M^1 is an L^p martingale since $\|M^1_t\|_{L^\infty(\Omega;X)} \le 2\lambda$ for all $t \ge 0$; M^3 is a local L^p-bounded martingale since $M^3 = M_0^{\tau-} + N - V$, where both $M_0^{\tau-}$ and $N_\infty = \Delta M_\tau$ are L^p-bounded (the latter is L^p-bounded by the argument similar to (5.3.8)), and V is locally L^p-bounded by Lemma 5.3.4; finally, M^2 is a local L^p-bounded martingale since $M^2 = M - M^1 - M^3$. Therefore all the martingales in Gundy's decomposition are locally L^p-bounded given M is an L^p-bounded martingale.

5.4. THE CANONICAL DECOMPOSITION OF LOCAL MARTINGALES

The current section is devoted to the proof of the fact that the canonical decomposition (as well as the Meyer-Yoeurp and the Yoeurp decompositions) of any X-valued local martingale exists if and only if X has the UMD Banach property. Recall that the Meyer-Yoeurp decomposition split a local martingale M into a sum $M = M^c + M^d$ of a continuous local martingale M^c and a purely discontinuous local martingale M^d, while the Yoeurp decomposition split a purely discontinuous local martingale M^d into a sum $M^d = M^q + M^a$ of a quasi-left continuous local martingale M^q and a local martingale M^a with accessible jumps (see Chapter 2 and 4).

Due to Theorem 4.3.15 any UMD space-valued L^p-martingale enjoys the canonical decomposition given $p > 1$. It is a natural question whether the canonical decomposition is possible and whether one can extend (4.3.18) in the case $p = 1$. It turns out that the UMD property is necessary and sufficient for the canonical decomposition of a general local martingale, while instead of (4.3.18) one gets weak-type estimates:

Theorem 5.4.1 (Canonical decomposition of local martingales)**.** *Let X be a Banach space. Then X has the UMD property if and only if any local martingale $M : \mathbb{R}_+ \times \Omega \to X$ has the canonical decomposition $M = M^c + M^q + M^a$. If this is the case, then for any $\lambda > 0$ and $t \geq 0$*

$$\begin{aligned} \lambda\mathbb{P}((M^c)^*_t > \lambda) &\lesssim_X \mathbb{E}\|M_t\|,\\ \lambda\mathbb{P}((M^q)^*_t > \lambda) &\lesssim_X \mathbb{E}\|M_t\|, \\ \lambda\mathbb{P}((M^a)^*_t > \lambda) &\lesssim_X \mathbb{E}\|M_t\|. \end{aligned} \tag{5.4.1}$$

For the proof of the main theorem we will need a considerable amount of machinery, which will be provided in Subsection 5.4.1-5.4.3.

5.4.1. Weak differential subordination martingale transforms

The current subsection is devoted to the proof of the fact that boundedness of a continuous-time martingale transform from a certain specific class acting on L^p-bounded martingales implies the corresponding weak L^1-estimates. Such type of assertions for special discrete martingale transforms was first obtained by Burkholder in [28]. Later the Burkholder's original statement was widely generalized in different directions (see [30, 41, 67, 75, 79, 116]), even though the martingale transforms were remaining acting on discrete martingales. The propose of the current subsection is to provide new results for martingale transforms of the same spirit by considering continuous-time martingales. This will allow us to consider linear operators that map a local martingale to the continuous part of the canonical decomposition, or the part of the canonical decomposition which is purely discontinuous with accessible jumps, so weak L^1-estimates (5.4.1) will follow from L^p-estimates (4.3.18) and Theorem 5.4.2.

The following theorem will be an important tool to show Theorem 5.4.1 and it is connected with [79, Proposition 3.5.4]. Recall that $\mathcal{M}_X^p$ is a space of all L^p-bounded X-valued martingales, and $\mathcal{M}_X^{p,\mathrm{loc}}$ is a space of all locally L^p-bounded X-valued martingales (see Section 2.2).

Theorem 5.4.2. *Let X be a Banach space, $p \in (1,\infty)$, $T : \mathcal{M}_X^{p,\mathrm{loc}} \to \mathcal{M}_X^{p,\mathrm{loc}}$ be a linear operator such that $TM \overset{w}{\ll} M$ and*

$$M_\infty^* = 0 \Longrightarrow (TM)_\infty^* = 0 \ \text{a.s.} \tag{5.4.2}$$

for each $M \in \mathcal{M}_X^p$. Assume that $T \in \mathcal{L}(\mathcal{M}_X^p)$. Then for any $M \in \mathcal{M}_X^p$

$$\lambda \mathbb{P}(\|(TM)_\infty^*\| > \lambda) \le C_{p,T,X} \mathbb{E}\|M_\infty\|, \ \ \lambda > 0, \tag{5.4.3}$$

where $C_{p,T,X} = 26\|T\|_{\mathcal{L}(\mathcal{M}_X^p)} \frac{p}{p-1} + 28$.

Remark 5.4.3. Notice that if X is a UMD Banach space, then T is automatically bounded on $\mathcal{M}_X^p$ and $\|T\|_{\mathcal{L}(\mathcal{M}_X^p)} \le \beta_{p,X}^2(\beta_{p,X}+1)$ by (5.1.3) and Theorem 4.4.1 since $TM \overset{w}{\ll} M$ for any $M \in \mathcal{M}_X^p$.

For the proof we will need several lemmas.

Lemma 5.4.4. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous martingale with $M_0 = 0$ a.s. Let μ^M be the corresponding random measure defined as in* (5.3.3). *Assume that*

$$\mathbb{E}\sum_{s\ge 0} \|\Delta M_s\| = \mathbb{E}\int_{\mathbb{R}_+ \times X} \|x\| \, \mathrm{d}\mu^M < \infty. \tag{5.4.4}$$

Then $M_t = \int_{[0,t]\times X} x \, \mathrm{d}\bar{\mu}^M$ for each $t \ge 0$ a.s.

Proof. By (5.4.4) there exists $N : \mathbb{R}_+ \times \Omega \to X$ such that $N_t = \sum_{0\le s\le t} \Delta M_s$ for each $t \ge 0$. Let $V = N - M$. Then both $t \mapsto N_t - V_t = M_t$, $t \ge 0$, and

$$t \mapsto N_t - \int_{[0,t]\times X} x \, \mathrm{d}\nu^M = \int_{[0,t]\times X} x \, \mathrm{d}\mu^M - \int_{[0,t]\times X} x \, \mathrm{d}\nu^M = \int_{[0,t]\times X} x \, \mathrm{d}\bar{\mu}^M, \ \ t \ge 0,$$

are martingales. Therefore

$$t \mapsto V_t - \int_{[0,t]\times X} x \, \mathrm{d}\nu^M = M_t - \int_{[0,t]\times X} x \, \mathrm{d}\bar{\mu}^M, \ \ t \ge 0,$$

is a predictable martingale, which is purely discontinuous as a difference of two purely discontinuous martingales (see Lemma 2.8.1). On the other hand it is continuous by the predictability (see e.g. [99, Theorem 4] and [92, Corollary 2.1.42]). Hence by Lemma 2.2.17 this martingale equals zero since it starts at zero, so $M = N - V = \int_{[0,\cdot]\times X} x \, \mathrm{d}\bar{\mu}^M$. □

Lemma 5.4.5. *Let X be a Banach space, $M, N : \mathbb{R}_+ \times \Omega \to X$ be purely discontinuous martingales such that $N \overset{w}{\ll} M$. Then $\mathbb{E}(\operatorname{Var} N)_t \le 2\mathbb{E}(\operatorname{Var} M)_t$ for each $t \ge 0$.*

Proof. Without loss of generality $\mathbb{E}(\operatorname{Var} M)_\infty < \infty$. Notice that since $N \overset{w}{\ll} M$, for a.e. $(t,\omega) \in \mathbb{R}_+ \times \Omega$ there exists $a(t,\omega) \in [-1,1]$ such that $\Delta N_t(\omega) = a(t,\omega)\Delta M_t(\omega)$ (see Subsection 3.3.2). Therefore a.s. for each $t \geq 0$

$$\begin{aligned}\int_{[0,t]\times X} \|x\| \,\mathrm{d}\mu^N(x,s) &= \sum_{0\leq s\leq t} \|\Delta N_s\| = \sum_{0\leq s\leq t} |a(s,\cdot)|\|\Delta M_s\| \\ &\leq \sum_{0\leq s\leq t} \|\Delta M_s\| \leq (\operatorname{Var} M)_t.\end{aligned} \tag{5.4.5}$$

So by Lemma 5.4.4 $N = \int_{[0,\cdot]\times X} x \,\mathrm{d}\bar{\mu}^N$, hence

$$\begin{aligned}(\operatorname{Var} N)_t &= \Big(\operatorname{Var} \int_{[0,\cdot]\times X} x \,\mathrm{d}\bar{\mu}^N(x,s)\Big)_t \\ &= \Big(\operatorname{Var}\Big(\int_{[0,\cdot]\times X} x \,\mathrm{d}\mu^N(x,s) - \int_{[0,\cdot]\times X} x \,\mathrm{d}\nu^N(x,s)\Big)\Big)_t \\ &\leq \Big(\operatorname{Var} \int_{[0,\cdot]\times X} x \,\mathrm{d}\mu^N(x,s)\Big)_t + \Big(\operatorname{Var} \int_{[0,\cdot]\times X} x \,\mathrm{d}\nu^N(x,s)\Big)_t \\ &\leq \int_{[0,t]\times X} \|x\| \,\mathrm{d}\mu^N(x,s) + \int_{[0,t]\times X} \|x\| \,\mathrm{d}\nu^N(x,s) \\ &= 2\int_{[0,t]\times X} \|x\| \,\mathrm{d}\mu^N(x,s) \overset{(*)}{\leq} 2(\operatorname{Var} M)_t,\end{aligned}$$

where $(*)$ holds by (5.4.5). □

Proof of Theorem 5.4.2. The proof has the same structure as the proof of [79, Proposition 3.5.16]. Fix $M \in \mathcal{M}_X^p$ and $\lambda > 0$. Let $K := \|T\|_{\mathcal{L}(\mathcal{M}_X^p)}$, $M = M^1 + M^2 + M^3$ be Gundy's decomposition of M from Theorem 5.3.1 at the level $\alpha\lambda$ for some $\alpha > 0$ which we will fix later. Notice that all M^1, M^2 and M^3 are local L^p-bounded martingales by Remark 5.3.5. Then

$$\begin{aligned}&\mathbb{P}(\|(TM)^*_\infty\| > \lambda) \\ &\qquad\leq \mathbb{P}(\|(TM^1)^*_\infty\| > \tfrac{\lambda}{2}) + \mathbb{P}(\|(TM^2)^*_\infty\| > 0) + \mathbb{P}(\|(TM^3)^*_\infty\| > \tfrac{\lambda}{2}).\end{aligned} \tag{5.4.6}$$

Let us estimate each of these three terms separately. First,

$$\begin{aligned}\mathbb{P}(\|(TM^1)^*_\infty\| > \tfrac{\lambda}{2}) &\overset{(i)}{\leq} \Big(\frac{2}{\lambda}\Big)^p \mathbb{E}\|(TM^1)^*_\infty\|^p \overset{(*)}{\leq} \Big(\frac{2}{\lambda}\frac{p}{p-1}\Big)^p \mathbb{E}\|(TM^1)_\infty\|^p \\ &\overset{(ii)}{\leq} \Big(\frac{2K}{\lambda}\frac{p}{p-1}\Big)^p \mathbb{E}\|M^1_\infty\|^p \leq \Big(\frac{2K}{\lambda}\frac{p}{p-1}\Big)^p \|M^1_\infty\|_\infty^{p-1}\mathbb{E}\|M^1_\infty\| \\ &\overset{(iii)}{\leq} \Big(\frac{2K}{\lambda}\frac{p}{p-1}\Big)^p (2\alpha\lambda)^{p-1} 5\mathbb{E}\|M_\infty\| = \frac{5\big(4\alpha K\frac{p}{p-1}\big)^p}{2\alpha\lambda}\mathbb{E}\|M_\infty\|,\end{aligned}$$

where (i) follows from (2.2.2), $(*)$ follows from Doob's maximal inequality [93, Theorem 1.3.8(iv)], (ii) holds by the definition of K, and (iii) follows from Gundy's decomposition.

Now turn to M^2. By (5.4.2)

$$\mathbb{P}((TM^2)^*_\infty > 0) \leq \mathbb{P}((M^2)^*_\infty > 0) \leq \frac{4}{\alpha\lambda}\mathbb{E}\|M_\infty\|. \tag{5.4.7}$$

Finally, by Lemma 5.4.5 and the fact that $TM^3 \overset{w}{\ll} M^3$ we have that

$$\mathbb{E}(\operatorname{Var} TM^3)_\infty \leq 2\mathbb{E}(\operatorname{Var} M^3)_\infty,$$

hence

$$\begin{aligned}\mathbb{P}(\|(TM^3)^*_\infty\| > \tfrac{\lambda}{2}) &\overset{(i)}{\leq} \frac{2}{\lambda}\mathbb{E}\|(TM^3)^*_\infty\| \leq \frac{2}{\lambda}\mathbb{E}(\operatorname{Var} TM^3)_\infty\\ &\overset{(ii)}{\leq} \frac{4}{\lambda}\mathbb{E}(\operatorname{Var} M^3)_\infty \overset{(*)}{\leq} \frac{28}{\lambda}\mathbb{E}\|M_\infty\|,\end{aligned}$$

where (i) follows from (2.2.2), (ii) holds by (5.4.7), and $(*)$ holds by Theorem 5.3.1(iii). Therefore by (5.4.6)

$$\begin{aligned}\lambda\mathbb{P}(\|(TM)^*_\infty\| > \lambda) &\leq \lambda\Big(\frac{5\big(4\alpha K\frac{p}{p-1}\big)^p}{2\alpha\lambda} + \frac{4}{\alpha\lambda} + \frac{28}{\lambda}\Big)\mathbb{E}\|M_\infty\|\\ &= \Big(\frac{5\big(4\alpha K\frac{p}{p-1}\big)^p}{2\alpha} + \frac{4}{\alpha} + 28\Big)\mathbb{E}\|M_\infty\|,\end{aligned}$$

and by choosing $\alpha = \frac{p-1}{4Kp}$ we get

$$\begin{aligned}\lambda\mathbb{P}(\|(TM)^*_\infty\| > \lambda) &\leq \Big(10K\frac{p}{p-1} + 16K\frac{p}{p-1} + 28\Big)\mathbb{E}\|M_\infty\|\\ &= \Big(26K\frac{p}{p-1} + 28\Big)\mathbb{E}\|M_\infty\|,\end{aligned}$$

which is exactly (5.4.3). □

The following proposition shows that the operator T from Theorem 5.4.2 has a special structure given the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ is generated by $(\mathscr{F}_n)_{n\geq 0}$: such martingale transforms are the same as those considered in [79, Proposition 3.5.4] and [30].

Proposition 5.4.6. *Let X be a separable Banach space. Let the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ be of the following form: $\mathscr{F}_t = \mathscr{F}_{\lfloor t\rfloor}$ for each $t \geq 0$, T be as in Theorem 5.4.2. Then there exists an $(\mathscr{F}_n)_{n\geq 0}$-predictable sequence $(a_n)_{n\geq 0}$ with values in $[-1,1]$ such that $\Delta(TM)_n = a_n\Delta M_n$ a.s. for each $n \geq 0$ for any $M \in \mathcal{M}^p_X$.*

Proof. Let $\mathbb{G} = (\mathscr{G}_n)_{n\geq 0} := (\mathscr{F}_n)_{n\geq 0}$ be a discrete filtration. Due to the construction of $\mathbb{F}$ and the fact that $\mathbb{G}$ is discrete we have that any $\mathbb{F}$-bounded martingale M is in fact discrete (i.e. $M_t = M_{\lfloor t\rfloor}$ a.s. for each $t \geq 0$), hence any martingale has accessible jumps, so by Lemma 5.4.13 it is sufficient to use the fact that $TM \overset{w}{\ll} M$ for any

$M \in \mathcal{M}_X^p$ in order to apply Theorem 5.4.2. Let us show that there exists a $\mathbb{G}$-adapted $[-1,1]$-valued sequence $(a_n)_{n\geq 1}$ such that $\Delta(TM)_n = a_n \Delta M_n$ a.s. for each $n \geq 0$. Since X is separable, $L^p(\Omega;X)$ is separable by [79, Proposition 1.2.29]. Let $(\xi^m)_{m\geq 1}$ be a dense subset of $L^p(\Omega;X)$. For each $m \geq 1$ we construct a martingale M^m in the following way: $M_t^m := \mathbb{E}(\xi^m|\mathcal{F}_t)$, $t \geq 0$. Then we have that $((TM)_n^m)_{n\geq 0} \overset{w}{\ll} (M_n^m)_{n\geq 0}$ for each $m \geq 1$, so by Subsection 3.3.2 for each $m \geq 1$ there exists a $\mathbb{G}$-adapted $[-1,1]$-valued sequence $(a_n^m)_{n\geq 0}$ such that $\Delta(TM^m)_n = a_n^m \Delta M_n^m$ for each $n \geq 0$. Let us show that for each $m_1 \neq m_2$ and $n \geq 0$ we have that

$$a_n^{m_1} = a_n^{m_2} \ \text{ a.s. on } A_n^{m_1,m_2}, \tag{5.4.8}$$

where $A_n^{m_1,m_2} := \{\Delta M_n^{m_1} \neq 0\} \cap \{\Delta M_n^{m_2} \neq 0\}$. Let $\big((c_1^k, c_2^k)\big)_{k\geq 1}$ be a dense subset of $\mathbb{R}^2$ such that for each $k \geq 1$

$$c_1^k \Delta M_n^{m_1} + c_2^k \Delta M_n^{m_2} \neq 0 \ \text{ a.s. on } A_n^{m_1,m_2}.$$

Then $T(c_1^k M^{m_1} + c_2^k M^{m_2}) \overset{w}{\ll} c_1^k M^{m_1} + c_2^k M^{m_2}$ for each $k \geq 1$, and hence by the linearity of T we have that for each $k \geq 1$ a.s. $c_1^k a_n^{m_1} \Delta M_n^{m_1} + c_2^k a_n^{m_2} \Delta M_n^{m_2}$ and $c_1^k \Delta M_n^{m_1} + c_2^k \Delta M_n^{m_2}$ are collinear vectors in X, and

$$\Big|\frac{c_1^k a_n^{m_1} \Delta M_n^{m_1} + c_2^k a_n^{m_2} \Delta M_n^{m_2}}{c_1^k \Delta M_n^{m_1} + c_2^k \Delta M_n^{m_2}}\Big| \leq 1 \ \text{ a.s. on } A_n^{m_1,m_2},$$

by the weak differential subordination. Therefore we can redefine $A_n^{m_1,m_2}$ up to a negligible set in the following way:

$$\begin{aligned} A_n^{m_1,m_2} := A_n^{m_1,m_2} &\bigcap_{k\geq 1} \{c_1^k \Delta M_n^{m_1} + c_2^k \Delta M_n^{m_2} \neq 0\} \\ &\bigcap_{k\geq 1} \Big\{\Big|\frac{c_1^k a_n^{m_1} \Delta M_n^{m_1} + c_2^k a_n^{m_2} \Delta M_n^{m_2}}{c_1^k \Delta M_n^{m_1} + c_2^k \Delta M_n^{m_2}}\Big| \leq 1\Big\}. \end{aligned}$$

Let us now fix any $\omega \in A_n^{m_1,m_2}$ and $\varepsilon > 0$. Let $x^* \in X^*$ be such that $\langle \Delta M_n^{m_1}(\omega), x^*\rangle \neq 0$ and $\langle \Delta M_n^{m_2}(\omega), x^*\rangle \neq 0$ (such x^* exists by the Hahn-Banach theorem and the definition of $A_n^{m_1,m_2}$). Then we can find $k \geq 1$ such that

$$0 < \frac{\langle c_1^k \Delta M_n^{m_1}(\omega) + c_2^k \Delta M_n^{m_2}(\omega), x^*\rangle}{|c_1^k| + |c_2^k|} < \varepsilon \tag{5.4.9}$$

since $\big((c_1^k, c_2^k)\big)_{k\geq 1}$ is dense in $\mathbb{R}^2$ (i.e. $k \geq 0$ such that (c_1^k, c_2^k) is almost orthogonal to $(\langle \Delta M_n^{m_1}(\omega) x^*\rangle, \langle \Delta M_n^{m_2}(\omega), x^*\rangle)$). But on the other hand (we will omit ω for the convenience of the reader)

$$1 \geq \Big|\frac{c_1^k a_n^{m_1} \Delta M_n^{m_1} + c_2^k a_n^{m_2} \Delta M_n^{m_2}}{c_1^k \Delta M_n^{m_1} + c_2^k \Delta M_n^{m_2}}\Big| = \frac{|\langle c_1^k a_n^{m_1} \Delta M_n^{m_1} + c_2^k a_n^{m_2} \Delta M_n^{m_2}, x^*\rangle|}{\langle c_1^k \Delta M_n^{m_1} + c_2^k \Delta M_n^{m_2}, x^*\rangle}$$

$$= \frac{|\langle c_2^k(a_n^{m_2} - a_n^{m_1})\Delta M_n^{m_2}, x^*\rangle| - |\langle c_1^k a_n^{m_1}\Delta M_n^{m_1} + c_2^k a_n^{m_1}\Delta M_n^{m_2}, x^*\rangle|}{\langle c_1^k \Delta M_n^{m_1} + c_2^k \Delta M_n^{m_2}, x^*\rangle} \tag{5.4.10}$$
$$\overset{(*)}{\geq} |a_n^{m_2} - a_n^{m_1}||\langle \Delta M_n^{m_2}, x^*\rangle|\frac{1}{\varepsilon} - 1,$$

where $(*)$ holds by the triangle inequality, (5.4.9), and the fact that $|a_n^{m_1}| \leq 1$. Since ε was arbitrary, (5.4.10) holds true if and only if $a_n^{m_2}(\omega) - a_n^{m_1}(\omega) = 0$. Now since $\omega \in A_n^{m_1,m_2}$ was arbitrary, $a_n^{m_1} = a_n^{m_2}$ on $A_n^{m_1,m_2}$.

Now we define for each $n \geq 0$ and $m \geq 1$:

$$B_n^1 = \{\Delta M_n^1 \neq 0\},$$
$$B_n^m = \{\Delta M_n^m \neq 0\} \setminus B_n^{m-1}, \quad m \geq 2,$$
$$B_n^0 = \Omega \setminus \bigcup_{m\geq 1} B_n^m,$$

and define a_n in the following way:

$$\begin{aligned} a_n(\omega) &:= a_n^m, \quad \omega \in B_n^m,\ m \geq 1, \\ a_n(\omega) &:= 0, \quad \omega \in B_0^m. \end{aligned} \tag{5.4.11}$$

Then by (5.4.8) $a_n = a_n^m$ a.s. on $\{\Delta M_n^m \neq 0\}$ for all $m \geq 1$. Therefore $\Delta(TM^m)_n = a_n \Delta M_n^m$ a.s. for all $m \geq 1$. Now let M be a general L^p-bounded martingale. Let $(M^{m_k})_{k\geq 1}$ be a sequence which converges to M in $\mathcal{M}_X^p$. Fix $n \geq 0$. Then by [79, Corollary 2.6.30] $\Delta M_n^{m_k}$ converges to ΔM_n in $L^p(\Omega; X)$ as $k \to \infty$, so by boundedness of a_n we have that $a_n \Delta M_n^{m_k} \to a_n \Delta M_n$ in $L^p(\Omega; X)$. On the other hand by boundedness of T and by [79, Corollary 2.6.30]

$$\lim_{k\to\infty} a_n \Delta M_m^{n_k} = \lim_{k\to\infty} \Delta(TM_n^{m_k})_n = \Delta(TM)_n,$$

where the limit is taken in $L^p(\Omega; X)$. Hence $\Delta(TM)_n = a_n \Delta M_n$ a.s.

It follows from (5.4.11) and Subsection 3.3.2 that $(a_n)_{n\geq 0}$ is $\mathbb{G}$-adapted and bounded by 1. Now let us show that $(a_n)_{n\geq 0}$ is $\mathbb{G}$-predictable. Assume the opposite. Then there exists $N \geq 0$ such that a_N is $\mathcal{F}_N$-measurable, but not $\mathcal{F}_{N-1}$-measurable (here we set $\mathcal{F}_{-1}$ to be the σ-algebra generated by all negligible sets). Fix $x \in X \setminus \{0\}$. Then we can construct the following L^p-bounded martingale $M : \mathbb{R}_+ \times \Omega \to X$: $\Delta M_n = 0$ if $n \neq N$ and $\Delta M_N = (a_N - \mathbb{E}(a_N|\mathcal{F}_{N-1}))x$. This is an L^p-bounded martingale since by the triangle inequality and [19, Theorem 34.2]

$$\begin{aligned} \|a_N - \mathbb{E}(a_N|\mathcal{F}_{N-1})\|_\infty &\leq \|a_N\|_\infty + \|\mathbb{E}(a_N|\mathcal{F}_{N-1})\|_\infty \leq 1 + \|\mathbb{E}(|a_N||\mathcal{F}_{N-1})\|_\infty \\ &\leq 1 + \|\mathbb{E}(1|\mathcal{F}_{N-1})\|_\infty \leq 2. \end{aligned}$$

Then we have that $\Delta(TM)_N = a_N(a_N - \mathbb{E}(a_N|\mathcal{F}_{N-1}))x$, and since TM is a martingale,

$$0 = \mathbb{E}(\Delta(TM)_N|\mathcal{F}_{N-1}) = \mathbb{E}(a_N(a_N - \mathbb{E}(a_N|\mathcal{F}_{N-1}))x|\mathcal{F}_{N-1})$$

$$
\begin{aligned}
&= x\mathbb{E}(a_N^2 - a_N\mathbb{E}(a_N|\mathscr{F}_{N-1})|\mathscr{F}_{N-1})\\
&= x\Big(\mathbb{E}(a_N^2|\mathscr{F}_{N-1}) - \big(\mathbb{E}(a_N|\mathscr{F}_{N-1})\big)^2\Big)\\
&= x\mathbb{E}\Big(\big(a_N - \mathbb{E}(a_N|\mathscr{F}_{N-1})\big)^2\Big|\mathscr{F}_{N-1}\Big),
\end{aligned}
$$

so since $x \neq 0$ and the fact that $\big(a_N - \mathbb{E}(a_N|\mathscr{F}_{N-1})\big)^2$ is nonnegative we get that $a_N - \mathbb{E}(a_N|\mathscr{F}_{N-1}) = 0$ a.s., hence a_N is $\mathscr{F}_{N-1}$-measurable. □

Remark 5.4.7. One can extend Proposition 5.4.6 to the case of a Banach space X being over the scalar field $\mathbb{C}$. The point is that because of the structure of the filtration $\mathbb{F}$ any $\mathbb{F}$-bounded martingale is purely discontinuous, so one can extend the definition of weak differential subordination in the way presented in [188]; namely, $N \overset{w}{\ll} M$ if $|\langle \Delta N_t, x^*\rangle| \leq |\langle \Delta M_t, x^*\rangle|$ a.s. for all $t \geq 0$ and $x^* \in X^*$. Then by applying the same proof one can show that the sequence $(a_n)_{n\geq 0}$ from Proposition 5.4.6 exists and is still $(\mathscr{F}_n)_{n\geq 0}$-predictable, but it takes values in the unit disk $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

5.4.2. Sufficiency of the UMD property

Now we will consider two examples of an operator T from Theorem 5.4.2, which will provide us with the Meyer-Yoeurp and the Yoeurp decompositions of any UMD space-valued local martingale.

Theorem 5.4.8 (Meyer-Yoeurp decomposition of local martingales)**.** *Let X be a UMD Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a local martingale. Then there exist unique local martingales $M^c, M^d : \mathbb{R}_+ \times \Omega \to X$ such that M^c is continuous, M^d is purely discontinuous, $M_0^c = 0$, and $M = M^c + M^d$. Moreover, for any $\lambda > 0$ and $t \geq 0$*

$$
\begin{aligned}
\lambda\mathbb{P}((M^c)_t^* > \lambda) &\lesssim_X \mathbb{E}\|M_t\|,\\
\lambda\mathbb{P}((M^d)_t^* > \lambda) &\lesssim_X \mathbb{E}\|M_t\|.
\end{aligned}
\tag{5.4.12}
$$

For the proof we will need the following lemma.

Lemma 5.4.9. *Let $M : \mathbb{R}_+ \times \Omega \to X$ be an L^1-bounded martingale, $(M^n)_{n\geq 1}$ be a sequence of purely discontinuous X-valued L^1-bounded martingales such that $M^n_\infty \to M_\infty$ in $L^1(\Omega; X)$. Then M is purely discontinuous.*

Proof. Without loss of generality $M_0 = 0$ and $M_0^n = 0$ a.s. for each $n \geq 1$. By Proposition 2.2.12 it is sufficient to check that MN is a martingale for any bounded continuous real-valued martingale N with $N_0 = 0$ a.s. Fix such N. Then due to Proposition 2.2.12 M^nN is a martingale for each $n \geq 0$. Moreover, since N_t is bounded for each $t \geq 0$, $(M^nN)_t \to (MN)_t$ in $L^1(\Omega; X)$. Therefore by the boundedness of a conditional expectation operator (see [79, Corollary 2.6.30]) for each $0 \leq s \leq t$

$$
\mathbb{E}((MN)_t|\mathscr{F}_s) = \mathbb{E}\big(\lim_{n\to\infty}(M^nN)_t|\mathscr{F}_s\big) = \lim_{n\to\infty}\mathbb{E}((M^nN)_t|\mathscr{F}_s)
$$

$$= \lim_{n\to\infty} (M^n N)_s = (MN)_s.$$

Hence, MN is a martingale. Since N was arbitrary, M is a purely discontinuous martingale. □

Proof of Theorem 5.4.8. By a stopping time argument we can assume that M is an L^1-bounded martingale. Fix $p\in(1,\infty)$. Let $(M^n)_{n\geq 1}$ be a sequence of X-valued L^p-bounded martingales such that $M^n_\infty \to M_\infty$ in $L^1(\Omega;X)$. Without loss of generality assume that $\mathbb{E}\|M_\infty - M^n_\infty\| < \frac{1}{2^{n+1}}$ for each $n\geq 1$. Let $T\in\mathscr{L}(\mathcal{M}^p_X)$ be such that T maps an L^p-bounded martingale $N:\mathbb{R}_+\times\Omega\to X$ to its continuous part N^c (such an operator exists and bounded by Theorem 4.3.15). For each $n\geq 1$ we denote TM^n by $M^{n,c}$. Then we know that by Theorem 5.4.2 for each $m\geq n\geq 1$ and any $K>0$

$$\mathbb{P}((M^{n,c}-M^{m,c})^*_\infty > K) \lesssim_{p,X} \frac{1}{K}\mathbb{E}\|M^{n,c}_\infty - M^{m,c}_\infty\| \leq \frac{1}{2^n K}, \tag{5.4.13}$$

hence $(M^{n,c})_{n\geq 1}$ is a Cauchy sequence in the ucp topology by (5.2.1). Notice that all the $M^{n,c}$'s are continuous local martingales, which are complete in the ucp topology (see [177, pp. 7–8] and Lemma 5.2.2). Hence there exists a local martingale $M^c:\mathbb{R}_+\times\Omega\to X$ which is the limit of $(M^{n,c})_{n\geq 1}$ in the ucp topology. Now it is sufficient to prove that $M^c_0 = 0$ and that $\langle M - M^c, x^*\rangle$ is a purely discontinuous local martingale for any $x^*\in X^*$ in order to show that M^c is the desired continuous local martingale. Firstly, $M^c_0 = \mathbb{P}-\lim_{n\to\infty} M^{n,c}_0 = 0$ since M^c is the limit of $(M^{n,c})_{n\geq 1}$ in the ucp topology and since $M^{n,c}_0 = 0$ a.s. for each $n\geq 1$. Secondly, since $M^{n,c}\to M^c$ in the ucp topology and $M^n\to M$ in $L^1(\Omega;X)$, $\langle M^n - M^{n,c}, x^*\rangle \to \langle M - M^c, x^*\rangle$ in the ucp topology for each fixed $x^*\in X^*$. Without loss of generality set that $\mathbb{E}\|M_\infty\|, \mathbb{E}\|M^n_\infty\| \leq 1$ for each $n\geq 1$. Also by choosing a subsequence we can assume that $M^{c,n}\to M^c$ a.s. uniformly on compacts. Therefore by Lemma 5.2.2 the process $t\mapsto \sup_{0\leq s\leq t}\sup_n \|M^{c,n}\|$ exists and continuous, and for each $m\geq 1$ we can define a stopping time τ_m in the following way

$$\tau_m := \inf\{t\geq 0: \sup_{0\leq s\leq t}\sup_n \|M^{c,n}\| \geq m\}.$$

Notice that a.s. $\tau_m\to\infty$ as $m\to\infty$. First show that $\langle (M-M^c)^{\tau_m}, x^*\rangle$ is purely discontinuous for each $m\geq 1$. Note that $(M^{c,n})^{\tau_m}_\infty \to (M^c)^{\tau_m}_\infty$ and $(M^n)^{\tau_m}_\infty \to M^{\tau_m}_\infty$ in $L^1(\Omega;X)$ as $n\to\infty$. Therefore

$$\langle (M^n - M^{c,n})^{\tau_m}, x^*\rangle \to \langle (M-M^c)^{\tau_m}, x^*\rangle$$

in $L^1(\Omega)$, so by Lemma 5.4.9 $\langle (M-M^c)^{\tau_m}, x^*\rangle$ is purely discontinuous. Notice that by letting m to infinity we get that $\langle M-M^c, x^*\rangle$ is a purely discontinuous local martingale for any $x^*\in X^*$, hence $M-M^c$ is a purely discontinuous local martingale.

The uniqueness of the decomposition follows from Remark 2.2.19, while (5.4.12) holds due to the limiting argument, (5.4.13), and the completeness of $L^{1,\infty}$-spaces provided by (1.1.11) and Theorem 1.4.11 in [69]. □

Let us turn to the Yoeurp decomposition.

Theorem 5.4.10 (Yoeurp decomposition of local martingales)**.** *Let X be a UMD Banach space, $M^d : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous local martingale. Then there exist unique purely discontinuous local martingales $M^q, M^a : \mathbb{R}_+ \times \Omega \to X$ such that M^q is quasi-left continuous, M^a has accessible jumps, $M_0^q = 0$, and $M^d = M^q + M^a$. Moreover, for any $\lambda > 0$ and $t \geq 0$*

$$\begin{aligned}\lambda \mathbb{P}((M^q)_t^* > \lambda) &\lesssim_X \mathbb{E}\|M_t^d\|,\\ \lambda \mathbb{P}((M^a)_t^* > \lambda) &\lesssim_X \mathbb{E}\|M_t^d\|.\end{aligned} \tag{5.4.14}$$

For the proof we will need the following lemmas.

Lemma 5.4.11. *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a local martingale with accessible jumps, $M_0 = 0$ a.s. Then $\{M_\infty^* = 0\} = \{[M]_\infty = 0\}$ up to a negligible set.*

Proof. Let $M = M^c + M^q + M^a$ be the canonical decomposition of M (see Subsection 2.4.3). Then $M^q = 0$ since M has accessible jumps. By [89, Exercise 17.3] $\{(M^c)_\infty^* = 0\} = \{[M^c]_\infty = 0\}$ up to a negligible set. Let us show the same for M^a. Let $\tau := \inf\{t \geq 0 : \Delta M_t^a \neq 0\}$ be a stopping time. Notice that a.s.

$$\begin{aligned}\{\tau < \infty\} &\subset \Big\{\sum_{t\geq 0} |\Delta M_t^a| > 0\Big\} \subset \{(M^a)_\infty^* > 0\},\\ \{\tau < \infty\} &\subset \Big\{\sum_{t\geq 0} |\Delta M_t^a|^2 > 0\Big\} = \{[M^a]_\infty > 0\},\end{aligned}$$

so we can redefine $M^a := (M^a)^\tau$. By the definition of τ we have that for each $t \geq 0$ a.s. $\sum_{0\leq s\leq t} |\Delta M_s^a| = |\Delta M_\tau^a| \mathbf{1}_{\tau \leq t}$, hence by [190, Theoreme (1-6).3] a.s.

$$M_t^a = \Delta M_\tau^a \mathbf{1}_{\tau\leq t}, \quad t \geq 0. \tag{5.4.15}$$

Therefore since $[M^a]_t = |\Delta M_\tau^a|^2 \mathbf{1}_{\tau\leq t}$ we have that $\{(M^a)_\infty^* = 0\} = \{[M^a]_\infty = 0\}$ up to a negligible set.

Let us now show the desired. First notice that by [89, Corollary 26.16] a.s.

$$\{[M]_\infty = 0\} = \{[M^c]_\infty + [M^a]_\infty = 0\} = \{[M^c]_\infty = 0\} \cap \{[M^a]_\infty = 0\}. \tag{5.4.16}$$

On the other hand a.s.

$$\begin{aligned}\{M_\infty^* = 0\} = \{M_\infty^* = 0\} \cap \{\Delta M_t = 0\ \forall t \geq 0\} &\overset{(i)}{=} \{M_\infty^* = 0\} \cap \{(M^a)_\infty^* = 0\}\\ &\overset{(ii)}{=} \{(M^c)_\infty^* = 0\} \cap \{(M^a)_\infty^* = 0\} \overset{(iii)}{=} \{[M^c]_\infty = 0\} \cap \{[M^a]_\infty = 0\}\\ &\overset{(iv)}{=} \{[M]_\infty = 0\},\end{aligned}$$

where (i) holds by (5.4.15), (ii) follows from the fact that $M^c = M - M^a$, (iii) follows from the first half of the proof, and finally (iv) follows from (5.4.16). □

Lemma 5.4.12. *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a local martingale, $M = M^c + M^q + M^a$ be the canonical decomposition. Then up to a negligible set*

$$\{M^*_\infty = 0\} = \{(M^c)^*_\infty = 0\} \cap \{(M^q)^*_\infty = 0\} \cap \{(M^a)^*_\infty = 0\}. \tag{5.4.17}$$

Proof. Let $N := M^c + M^a$. First notice that by Lemma 5.4.11 and [89, Corollary 26.16] a.s.

$$\begin{aligned}\{N^*_\infty = 0\} = \{[N]_\infty = 0\} &= \{[M^c]_\infty + [M^a]_\infty = 0\}\\ &= \{[M^c]_\infty = 0\} \cap \{[M^a]_\infty = 0\}\\ &= \{(M^c)^*_\infty = 0\} \cap \{(M^a)^*_\infty = 0\}.\end{aligned} \tag{5.4.18}$$

Let $\tau := \inf\{t \geq 0 : \Delta M_t \neq 0\}$ be a stopping time. Then a.s.

$$\{\tau < \infty\} \subset \{M^*_\infty > 0\} \subset \{N^*_\infty > 0\} \cup \{(M^q)^*_\infty > 0\}$$

since $M = N + M^d$. Let $A = \{M^*_\infty = 0\} \subset \Omega$. Then $[M]_\infty = [N + M^q]_\infty = 0$ a.s. on A, and consequently $[N]_\infty = 0$ a.s. on A by [89, Corollary 26.16]. Therefore by Lemma 5.4.11 $N^*_\infty = 0$ a.s. on A, so $(M^q)^*_\infty = 0$ a.s. on A, and therefore by (5.4.18)

$$\begin{aligned}\{M^*_\infty = 0\} = A \subset \{N^*_\infty = 0\} &\cap \{(M^q)^*_\infty = 0\}\\ = \{(M^c)^*_\infty = 0\} &\cap \{(M^q)^*_\infty = 0\} \cap \{(M^a)^*_\infty = 0\}.\end{aligned}$$

The converse inclusion follows from the fact that $M = N + M^q$ and (5.4.18). □

Lemma 5.4.13. *Let X be a Banach space, $M, N : \mathbb{R}_+ \times \Omega \to X$ be local martingales such that N has accessible jumps and $N \overset{w}{\ll} M$. Then*

$$\mathbb{P}(N^*_t > 0) \leq \mathbb{P}(M^*_t > 0), \quad t \geq 0. \tag{5.4.19}$$

Proof. (5.4.19) follows from the fact that $\{M^*_t = 0\} \subset \{N^*_t = 0\}$. Let $(x^*_n)_{n\geq 0} \subset X^*$ be a separating set. Then up to a negligible set

$$\{M^*_t = 0\} = \bigcap_{n\geq 0} \{(\langle M, x^*_n\rangle)^*_t = 0\},$$
$$\{N^*_t = 0\} = \bigcap_{n\geq 0} \{(\langle N, x^*_n\rangle)^*_t = 0\},$$

therefore it is sufficient to consider $X = \mathbb{R}$. Let $M = M^c + M^d + M^a$ be the canonical decomposition of M (see Subsection 2.4.3). By Lemma 5.4.12 and (5.4.18)

$$\{M^*_t = 0\} \subset \{(M^c + M^a)^*_t = 0\}.$$

Moreover, by Lemma 5.4.11

$$\{\ (M^c + M^a)^*_t = 0\} = \{[M^c + M^a]_t = 0\} \subset \{[M]_t = 0\},$$

$$\{N_t^* = 0\} = \{[N]_t = 0\},$$

and hence since $N \ll M$,

$$\{M_t^* = 0\} \subset \{[M]_t = 0\} \subset \{[N]_t = 0\} = \{N_t^* = 0\}.$$

□

Proof of Theorem 5.4.10. Without loss of generality assume that M^d is an L^1-martingale and $M_0^d = 0$ a.s. We will divide the proof into two steps.

Step 1. Define a stopping time $\tau = \{t \geq 0 : \|M_t^d\| > \frac{1}{2}\}$. In this step we assume that $M^d = (M^d)^\tau$ (i.e. the martingale stops moving after reaching $\frac{1}{2}$, in particular after the first jump of absolute value bigger than 1). Let μ^M be the random measure defined by (5.3.3), ν^M be the corresponding compensator (see Section 2.8). For each $n \geq 1$ define a stopping time

$$\tau_n = \inf\Big\{t \geq 0 : \int_{[0,t]\times X} \|x\|\mathbf{1}_{\|x\|>n}\,\mathrm{d}\nu^{M^d} > 1\Big\}, \tag{5.4.20}$$

and a process $M^{d,n} : \mathbb{R}_+ \times \Omega \to X$ in the following way

$$M_t^{d,n} = \Big((M^d)_t^{\tau-} + \Delta M_\tau^d \mathbf{1}_{\|\Delta M_\tau^d\|\leq n}\mathbf{1}_{\tau\leq t} + \int_{[0,t]\times X} x\mathbf{1}_{\|x\|>n}\,\mathrm{d}\nu^{M^d}\Big)^{\tau_n-},\ \ t \geq 0, \tag{5.4.21}$$

where we define $M^{\sigma-}$ for a stopping time σ in the same way as in (2.4.2). First of all show that $\tau_n \to \infty$ a.s. as $n \to \infty$. Notice that by due to Section 2.8

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}_+\times X} \|x\|\mathbf{1}_{\|x\|>1}\,\mathrm{d}\nu^{M^d} &= \mathbb{E}\int_{\mathbb{R}_+\times X} \|x\|\mathbf{1}_{\|x\|>1}\,\mathrm{d}\mu^{M^d} \leq \mathbb{E}\|\Delta M_\tau^d\|\\ &\leq \mathbb{E}\|M_\tau^d\| + \mathbb{E}\|M_{\tau-}^d\| \overset{(*)}{\leq} \mathbb{E}\|M_\infty^d\| + \frac{1}{2} \overset{(**)}{<} \infty,\end{aligned} \tag{5.4.22}$$

where $(*)$ follows from the fact that $M_\tau = M_\infty$ and the fact that $\|M_{\tau-}\| \leq \frac{1}{2}$ a.s., and $(**)$ holds due to the fact that M is an L^1-bounded martingale. Therefore

$$\int_{\mathbb{R}_+\times X} \|x\|\mathbf{1}_{\|x\|>1}\,\mathrm{d}\nu^{M^d} < \infty \ \ \text{a.s.},$$

so by the monotone convergence theorem a.s.

$$\int_{\mathbb{R}_+\times X} \|x\|\mathbf{1}_{\|x\|>n}\,\mathrm{d}\nu^{M^d} \to 0,\ \ n \to \infty,$$

and hence $\tau_n \to \infty$ as $n \to \infty$.

We need to show that $M^{d,n}$ is an L^∞-bounded martingale for each $n \geq 1$. Clearly $M^{d,n}$ is adapted and càdlàg. It is also a local martingale since it can be rewritten in the following form:

$$M_t^{d,n} = (M^d)_t^{\tau_n-} - \int_{[0,t]\times X} x\mathbf{1}_{\|x\|>n}\mathbf{1}_{s<\tau_n}\,\mathrm{d}\bar{\mu}^{M^d},\ \ t \geq 0,$$

where the first term is a martingale by Lemma 2.4.4, and the second term is a local martingale by Lemma 2.8.1 and the fact that the process $s \mapsto \mathbf{1}_{s<\tau_n}$ is predictable by [89, Theorem 25.14] and the predictability of τ_n (the latter follows from (5.4.20) and the predictability of ν^{M^d}, see Section 2.8). Moreover, for each fixed $t \geq 0$ we have that a.s.

$$\begin{aligned}\|M_t^{d,n}\| &\leq \|(M^d)_t^{\tau-\wedge\tau_n-}\| + \|\Delta M_\tau^d \mathbf{1}_{\|\Delta M_\tau^d\|\leq n}\| + \int_{[0,\tau_n)\times X} \|x\|\mathbf{1}_{\|x\|>n}\nu^{M^d}\\ &\leq 1+n+1 = n+2.\end{aligned}$$

(Recall that $\tau - \wedge\tau_n - := (\tau\wedge\tau_n)-$, see (5.3.6)). Therefore $(M^{d,n})_{n\geq 1}$ are bounded martingales.

Now let us now show that $M_\infty^{d,n} \to M_\infty^d$ in $L^1(\Omega;X)$. First, $M_\infty^{d,n} = M_{\tau_n-}^{d,n}$ a.s., so by the triangle inequality

$$\mathbb{E}\|M_\infty^d - M_\infty^{d,n}\| \leq \mathbb{E}\|M_\infty^d - M_{\tau_n-}^d\| + \mathbb{E}\|M_{\tau_n-}^d - M_{\tau_n-}^{d,n}\|.$$

Notice that the first term vanishes as $n\to\infty$ by the fact that $\|M_\infty^d - M_{\tau_n-}^d\| \leq 1 + \|\Delta M_\tau\|$ a.s., the fact that $\tau_n\to\infty$ a.s., and the dominated convergence theorem. Let us consider the second term:

$$\begin{aligned}&\mathbb{E}\|M_{\tau_n-}^d - M_{\tau_n-}^{d,n}\|\\ &= \mathbb{E}\Big\|M_{\tau_n-}^d - (M^d)_{\tau_n-}^{\tau-} - \Delta M_\tau^d\mathbf{1}_{\|\Delta M_\tau^d\|\leq n}\mathbf{1}_{\tau<\tau_n} - \int_{[0,\tau_n)\times X} x\mathbf{1}_{\|x\|>n}\,\mathrm{d}\nu^{M^d}\Big\|\\ &= \mathbb{E}\Big\|\Delta M_\tau^d\mathbf{1}_{\tau<\tau_n} - \Delta M_\tau^d\mathbf{1}_{\|\Delta M_\tau^d\|\leq n}\mathbf{1}_{\tau<\tau_n} - \int_{[0,\tau_n)\times X} x\mathbf{1}_{\|x\|>n}\,\mathrm{d}\nu^{M^d}\Big\|\\ &= \mathbb{E}\Big\|\Delta M_\tau^d\mathbf{1}_{\|\Delta M_\tau^d\|>n}\mathbf{1}_{\tau<\tau_n} - \int_{[0,\tau_n)\times X} x\mathbf{1}_{\|x\|>n}\,\mathrm{d}\nu^{M^d}\Big\|\\ &= \mathbb{E}\Big\|\int_{[0,\tau_n)\times X} x\mathbf{1}_{\|x\|>n}\,\mathrm{d}\mu^{M^d} - \int_{[0,\tau_n)\times X} x\mathbf{1}_{\|x\|>n}\,\mathrm{d}\nu^{M^d}\Big\|\\ &\leq \mathbb{E}\int_{[0,\tau_n)\times X}\|x\|\mathbf{1}_{\|x\|>n}\,\mathrm{d}\mu^{M^d} + \mathbb{E}\int_{[0,\tau_n)\times X}\|x\|\mathbf{1}_{\|x\|>n}\,\mathrm{d}\nu^{M^d}\\ &\overset{(*)}{=} 2\mathbb{E}\int_{[0,\tau_n)\times X}\|x\|\mathbf{1}_{\|x\|>n}\,\mathrm{d}\mu^{M^d} \overset{(**)}{=} 2\mathbb{E}\|\Delta M_\tau^d\|\mathbf{1}_{\|\Delta M_\tau^d\|>n},\end{aligned}$$

and the last expression vanishes as $n\to\infty$ by the monotone convergence theorem. (Notice that $(*)$ follows from the definition of a compensator and from (5.4.22), while $(**)$ follows from the fact that $\|\Delta M_t\| \geq 1$ only if $t=\tau$ by the assumptions on M.)

Since each of $M^{d,n}$'s is an L^p-bounded martingale for each $p\in(1,\infty)$, by Theorem 4.3.15 for each $n\geq 1$ there exists the Yoeurp decomposition $M^{d,n} = M^{q,n} + M^{a,n}$ of a martingale $M^{d,n}$ into a sum of two purely discontinuous martingales $M^{q,n}, M^{a,n} : \mathbb{R}_+\times\Omega\to X$ such that $M^{q,n}$ is quasi-left continuous, $M^{a,n}$ has accessible jumps, and $M_0^{q,n} = M_0^{a,n} = 0$ a.s. (recall that $M_0^{d,n} = 0$ a.s.). Fix some $p\in(1,\infty)$.

Since an operator T^q that maps an L^p-bounded martingale $M : \mathbb{R}_+ \times \Omega \to X$ to its purely discontinuous quasi-left continuous part M^q of the canonical decomposition is continuous on $\mathscr{L}(\mathscr{M}_X^p)$ by Theorem 4.3.15, Theorem 5.4.2 together with Lemma 5.4.13 yields that for each $m, n \geq 1$ and $K > 0$

$$\begin{aligned}\mathbb{P}\big((M^{q,n} - M^{q,m})^*_\infty > K\big) &\lesssim_p \frac{1}{K}\mathbb{E}\|M^{d,n}_\infty - M^{d,m}_\infty\|\\ &\leq \frac{1}{K}(\mathbb{E}\|M^{d,n}_\infty - M^d_\infty\| + \mathbb{E}\|M^{d,m}_\infty - M^d_\infty\|),\end{aligned}$$

so $(M^{q,n})_{n\geq 1}$ is a Cauchy sequence in the ucp topology. By Proposition 5.2.1 it has a càdlàg adapted limit. Denote this limit by M^q. Let us show that M^q is a purely discontinuous quasi-left continuous local martingale. Let σ be a predictable time. Then $\Delta M^{q,n}_\sigma = 0$ a.s., and for any $t \geq 0$ a.s.

$$\begin{aligned}\sup_{0\leq s\leq t}\|M^{q,n}_s - M^q_s\| &\geq \mathbf{1}_{\sigma\leq t}\sup_{0\leq s\leq \sigma}\|M^{q,n}_s - M^q_s\|\\ &\geq \mathbf{1}_{\sigma\leq t}\big(\sup_{m\geq 1}\|M^{q,n}_{0\vee\sigma-\frac{1}{m}} - M^q_{0\vee\sigma-\frac{1}{m}}\| \vee \|M^{q,n}_\sigma - M^q_\sigma\|\big)\\ &\geq \frac{1}{2}\mathbf{1}_{\sigma\leq t}\big(\limsup_{m\geq 1}\|M^{q,n}_{0\vee\sigma-\frac{1}{m}} - M^q_{0\vee\sigma-\frac{1}{m}}\| + \|M^{q,n}_\sigma - M^q_\sigma\|\big)\\ &= \frac{1}{2}\mathbf{1}_{\sigma\leq t}\big(\|M^{q,n}_{\sigma-} - M^q_{\sigma-}\| + \|M^{q,n}_\sigma - M^q_\sigma\|\big)\\ &\overset{(*)}{\geq} \frac{1}{2}\mathbf{1}_{\sigma\leq t}\|M^{q,n}_{\sigma-} - M^q_{\sigma-} - M^{q,n}_\sigma + M^q_\sigma\|\\ &\geq \frac{1}{2}\mathbf{1}_{\sigma\leq t}\|\Delta M^q_{\sigma-} - \Delta M^{q,n}_\sigma\| = \frac{1}{2}\mathbf{1}_{\sigma\leq t}\|\Delta M^q_{\sigma-}\|,\end{aligned} \tag{5.4.23}$$

where $(*)$ follows from the triangle inequality. Since

$$\mathbb{P} - \lim_{n\to\infty}\sup_{0\leq s\leq t}\|M^{q,n}_s - M^q_s\| = 0,$$

we have that for each $t \geq 0$

$$\mathbb{P} - \lim_{n\to\infty}\mathbf{1}_{\sigma\leq t}\|\Delta M^q_{\sigma-}\| = 0.$$

But the expression under the limit in probability does not depend on n. Hence $\mathbf{1}_{\sigma\leq t}\|\Delta M^q_{\sigma-}\| = 0$ a.s. By letting $t \to \infty$ we get that a.s. $\|\Delta M^q_\sigma\| = 0$, and since σ was arbitrary predictable, M^q is quasi-left continuous.

Let now σ be a totally inaccessible stopping time. Let us show that a.s.

$$\Delta M^q_\sigma = \Delta M^d_\sigma. \tag{5.4.24}$$

First notice that for each fixed $m \geq n \geq 1$

$$\begin{aligned}\Delta M^{q,m}_\sigma \mathbf{1}_{\sigma<\tau\wedge\tau_n} &\overset{(*)}{=} \Delta M^{d,m}_\sigma \mathbf{1}_{\sigma<\tau\wedge\tau_n} \overset{(**)}{=} \Delta M^d_\sigma \mathbf{1}_{\sigma<\tau\wedge\tau_n},\\ \Delta M^{q,m}_\sigma \mathbf{1}_{\sigma=\tau<\tau_n}\mathbf{1}_{\|\Delta M^d_\tau\|\leq n} &\overset{(*)}{=} \Delta M^{d,m}_\sigma \mathbf{1}_{\sigma=\tau<\tau_n}\mathbf{1}_{\|\Delta M^d_\tau\|\leq n} \overset{(**)}{=} \Delta M^d_\sigma \mathbf{1}_{\sigma=\tau<\tau_n}\mathbf{1}_{\|\Delta M^d_\tau\|\leq n},\end{aligned} \tag{5.4.25}$$

where $(*)$ follows from Remark 2.4.23, and $(**)$ follows from the definition (5.4.21) of $M^{d,m}$ and Lemma 2.4.3. Therefore by (5.4.23) applied for our σ a.s. for each $n \geq 1$

$$\begin{aligned} \Delta M^d_\sigma \mathbf{1}_{\sigma<\tau\wedge\tau_n} &= \Delta M^q_\sigma \mathbf{1}_{\sigma<\tau\wedge\tau_n}, \\ \Delta M^d_\sigma \mathbf{1}_{\sigma=\tau<\tau_n}\mathbf{1}_{\|\Delta M^d_\tau\|\leq n} &= \Delta M^q_\sigma \mathbf{1}_{\sigma=\tau<\tau_n}\mathbf{1}_{\|\Delta M^d_\tau\|\leq n}. \end{aligned} \tag{5.4.26}$$

By letting $n \to \infty$ we get (5.4.24).

Let us show that M^q is locally integrable. For each $l \geq 1$ set $\rho_l := \inf\{t \geq 0 : \|M^q_t\| \geq l\}$. Then a.s. for each $t \geq 0$

$$\begin{aligned} \|(M^q)^{\rho_l}_t\| \leq \|(M^q)^{\rho_l}_{t-}\| + \|\Delta(M^q)^{\rho_l}_t\| &\leq l + \|\Delta(M^q)^{\rho_l}_t\|\mathbf{1}_{t=\tau} + \|\Delta(M^q)^{\rho_l}_t\|\mathbf{1}_{t<\tau} \\ &\leq l + \|\Delta M^d_\tau\| + 1. \end{aligned}$$

Therefore

$$\mathbb{E}\|(M^q)^{\rho_l}_t\| \leq l + 1 + \mathbb{E}\|\Delta M^d_\tau\| < \infty,$$

where $\mathbb{E}\|\Delta M_\tau\| < \infty$ by (5.4.22). Since M^q is càdlàg, by [154, Problem V.1] we have that $\rho_l \to \infty$ as $l \to \infty$, so M^q is locally integrable.

Now let us show that M^q is a local martingale. Let $(M^{q,n_k})_{k\geq 1}$ be a subsequence of $(M^{q,n})_{n\geq 1}$ such that $M^{q,n_k} \to M^q$ uniformly on compacts a.s. (existence of such a subsequence can be shown e.g. as in the proof of [155, Theorem 62]). It is sufficient to show that $M^{\rho_l\wedge\tau_{n_k}-}$ is a local martingale for each $l,k \geq 1$ since $\rho_l \to \infty$ and $\tau_{n_k} \to \infty$ a.s. as $l,k \to \infty$. Fix $K > 0$. Then by (5.4.25) and (5.4.26) for each $k \geq K$ we have that a.s. for each $t \geq 0$

$$\Delta(M^{q,n_k})^{\tau_{n_K}-\wedge\tau-}_t = \Delta(M^q)^{\tau_{n_K}-\wedge\tau-}_t.$$

Therefore by Lemma 5.2.2 there exists a continuous adapted process $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that a.s.

$$N_t = \sup_{k\geq K}\left\|(M^{q,n_k})^{\tau_{n_K}-\wedge\tau-}_t - (M^q)^{\tau_{n_K}-\wedge\tau-}_t\right\|, \quad t \geq 0.$$

Now for each $j \geq 1$ define a stopping time $\sigma_j = \inf\{t \geq 0 : N_t \geq j\}$. Fix $j \geq 1$. Then for each $t \geq 0$ we have that for any $k \geq K$ a.s.

$$\left\|(M^{q,n_k})^{\rho_l\wedge\tau_{n_K}-\wedge\sigma_j}_t - (M^q)^{\rho_l\wedge\tau_{n_K}-\wedge\sigma_j}_t\right\| \leq j + l + 2\|\Delta M^d_\tau\|$$

and that $(M^{q,n_k})^{\rho_l\wedge\tau_{n_K}-\wedge\sigma_j}_t - (M^q)^{\rho_l\wedge\tau_{n_K}-\wedge\sigma_j}_t \to 0$ a.s. as $k \to \infty$. Hence by the dominated convergence theory

$$(M^{q,n_k})^{\rho_l\wedge\tau_{n_K}-\wedge\sigma_j}_t \to (M^q)^{\rho_l\wedge\tau_{n_K}-\wedge\sigma_j}_t \text{ in } L^1(\Omega; X) \text{ as } k \to \infty.$$

Consequently, $\big((M^q)^{\rho_l\wedge\tau_{n_K}-\wedge\sigma_j}_t\big)_{t\geq 0}$ is an L^1-bounded martingale, which is moreover purely discontinuous by Lemma 5.4.9. By letting $l, K, j \to \infty$ we get that M^q is a purely discontinuous quasi-left continuous local martingale.

M^a can be constructed in the same way. The identity $M^d = M^q + M^a$ follows from the following limiting argument:

$$M^d = ucp - \lim_{n\to\infty} M^{d,n},$$
$$M^q = ucp - \lim_{n\to\infty} M^{q,n},$$
$$M^a = ucp - \lim_{n\to\infty} M^{a,n},$$

and the fact that $M^{d,n} = M^{q,n} + M^{a,n}$ for each $n \geq 1$.

Step 2. For a general martingale M^d we construct a sequence of stopping times $\tau_n = \inf\{t \geq 0 : \|M_t^d\| \geq \frac{n}{2}\}$. For each $M^{d,n} := (M^d)^{\tau_n}$ we construct the corresponding $M^{q,n}$ by Step 1. Then for each $m \geq n \geq 1$ we get that $(M^{q,n})^{\tau_m} = M^{q,m}$ since for any $x^* \in X^*$ a.s.

$$\langle (M^{q,n})^{\tau_m}, x^* \rangle = \langle M^{q,m}, x^* \rangle$$

due to the uniqueness of the Yoeurp decomposition in the real-valued case. Then we just set $M_0^q := 0$ and

$$M_t^q := \sum_{n\geq 1} M_t^{q,n} \mathbf{1}_{t\in(\tau_{n-1},\tau_n]}, \quad t \geq 0,$$

where $\tau_0 \equiv 0$. The obtained M^q will be the desired purely discontinuous quasi-left continuous local martingale.

We can construct M^a in the same way and show that then $M^d = M^q + M^a$ similarly to how it was shown in step 1.

The uniqueness of the decomposition follows from Remark 2.4.21, while (5.4.14) follows analogously (5.4.12). □

Proof of Theorem 5.4.1 (sufficiency of UMD and (5.4.1)*).* Sufficiency of the UMD property follows from Theorem 5.4.8 and Theorem 5.4.10, while (5.4.1) follows in the same way as (5.4.12) and (5.4.14). □

5.4.3. *Necessity of the UMD property*

In the current subsection we show that the UMD property is necessary in Theorem 5.4.8 and Theorem 5.4.10, and hence it is necessary for the canonical decomposition of a local martingale.

Theorem 5.4.14. *Let X be a Banach space that does not have the UMD property. Then there exists a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ and an $\mathbb{F}$-bounded martingale $M : \mathbb{R}_+ \times \Omega \to X$ such that M provides neither the Meyer-Yoeurp nor the canonical decomposition.*

For the proof we will need the following lemma which is a modification of the statements from p. 1001 and p. 1004 of [30]. Recall that if $(f_n)_{n\geq 0}$ is an X-valued martingale, the we define $df_n := f_n - f_{n-1}$ for $n \geq 1$ and $df_0 := f_0$.

Lemma 5.4.15. *Let X be a Banach space. Then X is a UMD Banach space if and only if there exists a constant $C>0$ such that for any X-valued discrete martingale $(f_n)_{n\geq 0}$, for any $\{0,1\}$-valued sequence $(a_n)_{n\geq 0}$ one has that*

$$g^*_\infty > 1 \text{ a.s.} \implies \mathbb{E}\|f_\infty\| > C,$$

where $(g_n)_{n\geq 0}$ is an X-valued discrete martingale such that $dg_n = a_n df_n$ for each $n\geq 0$, $g^*_\infty := \sup_{n\geq 0}\|g_n\|$.

Proof. One needs to modify [30, Theorem 2.1] in such a way that $dg_n = a_n df_n$ for some $a_n\in\{0,1\}$ for each $n\geq 0$. Then the proof is the same, and the desired statement follows from the equivalence of [30, (2.3)] and [30, (2.4)]. □

Corollary 5.4.16. *Let X be a Banach space that does not have the UMD property. Then there exists an X-valued Paley-Walsh L^1-bounded martingale $(f_n)_{n\geq 0}$ and a $\{0,1\}$-valued sequence $(a_n)_{n\geq 0}$ such that $\mathbb{P}(g^*_\infty = \infty) = 1$, where $(g_n)_{n\geq 0}$ is an X-valued martingale such that $dg_n = a_n df_n$ for each $n\geq 0$.*

Proof. Without loss of generality all the martingales used below are Paley-Walsh (see [79, Theorem 3.6.1]), so the resulting martingale will be Paley-Walsh as well. By Lemma 5.4.15 we can find $N_1>0$, an X-valued martingale $f^1 = (f^1_n)_{n=0}^{N_1}$ and a $\{0,1\}$-valued sequence $(a^1_n)_{n=0}^{N_1}$ such that $\mathbb{E}\|f^1_{N_1}\| < \frac{1}{2}$ and

$$\mathbb{P}((g^1)^*_{N_1} > 1) > \frac{1}{2},$$

where $g^1 = (g^1_n)_{n=0}^{N_1}$ is such that $dg^1_n = a^1_n df^1_n$ for each $n = 0,\dots,N_1$. Now inductively for each $k>1$ we find $N_k>0$ and an X-valued Paley-Walsh martingale $f^k = (f^k_n)_{n=0}^{N_k}$ independent of $f^1,\dots,f^{k-1}$ such that $\mathbb{E}\|f^k_{N_k}\| < \frac{1}{2^k}$ and

$$\mathbb{P}((g^k)^*_{N_k} > 2C_k) > 1 - \frac{1}{2^k},$$

where $g^k = (g^k_n)_{n=0}^{N_1}$ is such that $dg^k_n = a^k_n df^k_n$ for each $n = 0,\dots,N_k$, and $C_k > 2^k$ is such that

$$\mathbb{P}((g^1)^*_{N_1} + \ldots + (g^{k-1})^*_{N_{k-1}} > C_k) < \frac{1}{2^k}.$$

Without loss of generality assume that $f^k_0 = 0$ a.s. for each $k\geq 1$. Now construct a martingale $(f_n)_{n\geq 0}$ and a $\{0,1\}$-valued sequence $(a_n)_{n\geq 0}$ in the following way: $f_0 = a_0 = 0$ a.s., $df_n = df^k_m$ and $a_n = a^k_m$ if $n = N_1 + \cdots + N_{k-1} + m$ for some $k\geq 1$ and $1\leq m\leq N_k$. Then $(f_n)_{n\geq 0}$ is well-defined,

$$\lim_{n\to\infty}\mathbb{E}\|f_n\| = \mathbb{E}\|f_\infty\| \leq \sum_{k\geq 1}\mathbb{E}\|f^k_{N_k}\| \leq 1$$

by the triangle inequality, and for an X-valued martingale $(g_n)_{n\geq 0}$ with $dg_n = a_n df_n$ for each $n\geq 0$, for each $k\geq 2$

$$\mathbb{P}(g^*_{N_1+\cdots+N_k} > C_k) \geq \mathbb{P}((g^k)^*_{N_k} > 2C_k, (g^1)^*_{N_1} + \ldots + (g^{k-1})^*_{N_{k-1}} \leq C_k) > 1 - \frac{1}{2^{k-1}},$$

hence $g^*_\infty = \infty$ a.s. □

Proof of Theorem 5.4.14. By Corollary 5.4.16 we can construct a discrete filtration $\mathbb{G} = (\mathscr{G}_n)_{n\geq 0}$ and an X-valued L^1-bounded Paley-Walsh $\mathbb{G}$-bounded martingale $(f_n)_{n\geq 0}$ such that

$$\mathbb{E}\|f_\infty\| = \lim_{n\to\infty} \mathbb{E}\|f_n\| \leq 1, \tag{5.4.27}$$

and such that there exists $\{0,1\}$-valued sequence $(a_n)_{n\geq 0}$ so that

$$\mathbb{P}(g^*_\infty = \infty) = 1,$$

where $(g_n)_{n\geq 0}$ is an X-valued martingale with $dg_n = a_n df_n$ for each $n \geq 0$.

Since $(f_n)_{n\geq 0}$ is Paley-Walsh, there exist a sequence $(r_n)_{n\geq 0}$ of independent Rademacher variables, a sequence of functions $(\phi_n)_{n\geq 1}$ with $\phi_1 \in X$ and $\phi_n : \{-1,1\}^{n-1} \to X$ for each $n \geq 2$, so that $df_n = r_n\phi_n(r_1,\dots,r_{n-1})$ a.s. for each $n \geq 1$.

Now our goal is to construct a continuous-time X-valued martingale M which does not have the Meyer-Yoeurp decomposition (and hence the canonical decomposition) using $(f_n)_{n\geq 0}$. Let us first construct a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ on $\mathbb{R}_+$ in the following way. By Lemma 4.3.7 for each $n \geq 0$ we can find a continuous martingale $M^n : [0, \frac{1}{2^{n+1}}] \times \Omega \to \mathbb{R}$ with a symmetric distribution such that $M^n_0 = 0$ a.s., $\big|M^n_{\frac{1}{2^{n+1}}}\big| \leq 1$ a.s.,

$$\mathbb{P}\big(M^n_{\frac{1}{2^{n+1}}} = 0\big) = 0, \tag{5.4.28}$$

and

$$\mathbb{P}\Big(M^n_{\frac{1}{2^{n+1}}} \neq \operatorname{sign} M^n_{\frac{1}{2^{n+1}}}\Big) < \frac{1}{2^n(\|\phi_n\|_\infty + 1)}. \tag{5.4.29}$$

Let $(\tilde r_n)_{n\geq 0}$ be a sequence of independent Rademacher random variables. Without loss of generality assume that all $(\tilde r_n)_{n\geq 0}$ and $(M^n)_{n\geq 0}$ are independent. Then set $\mathscr{F}_0$ to be the σ-algebra generated by all negligible sets, and set

$$\mathscr{F}_t := \begin{cases} \mathscr{F}_{1-\frac{1}{2^n}}, & t \in (1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}), a_n = 0, n \geq 0,\\ \sigma(\mathscr{F}_{1-\frac{1}{2^n}}, \tilde r_n), & t = 1-\frac{1}{2^{n+1}}, a_n = 0, n \geq 0,\\ \sigma(\mathscr{F}_{1-\frac{1}{2^n}}, (M^n_s)_{s\in[0,t-1-\frac{1}{2^n}]}), & t \in (1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}], a_n = 1, n \geq 0,\\ \sigma(\mathscr{F}_s : s \in [0,1)), & t \geq 1. \end{cases}$$

Let $(\sigma_n)_{n\geq 0}$ be a sequence of independent Rademacher variables such that $\sigma_n = \tilde r_n$ if $a_n = 0$ and $\sigma_n = \operatorname{sign} M^n_{\frac{1}{2^{n+1}}}$ if $a_n = 1$ (in the latter case σ_n has the Rademacher distribution by (5.4.28) and the fact that $M^n_{\frac{1}{2^{n+1}}}$ is symmetric). Now construct M :

$\mathbb{R}_+ \times \Omega \to X$ in the following way:

$$M_t = \begin{cases} 0, & t = 0, \\ M_{1-\frac{1}{2^n}}, & t \in (1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}), a_n = 0, n \geq 0, \\ M_{1-\frac{1}{2^n}} + \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}), & t = 1-\frac{1}{2^{n+1}}, a_n = 0, n \geq 0, \\ M_{1-\frac{1}{2^n}} + M^n_{t-1-\frac{1}{2^n}} \phi_n(\sigma_1, \dots, \sigma_{n-1}), & t \in (1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}], a_n = 1, n \geq 0, \\ \lim_{n\to\infty} M_{1-\frac{1}{2^n}}, & t \geq 1. \end{cases} \tag{5.4.30}$$

First we show that $\lim_{n\to\infty} M_{1-\frac{1}{2^n}}$ exists a.s., hence M is well-defined. By [79, Theorem 3.3.8] it is sufficient to show that there exists $\xi \in L^1(\Omega; X)$ such that $M_{1-\frac{1}{2^n}} = \mathbb{E}(\xi|\mathscr{F}_{1-\frac{1}{2^n}})$ for all $n \geq 1$. Notice that $\big(M_{1-\frac{1}{2^n}}\big)_{n\geq 0}$ is a martingale since $M_{1-\frac{1}{2^{n+1}}} - M_{1-\frac{1}{2^n}}$ equals either $\sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1})$ (if $a_n = 0$) or $M^n_{\frac{1}{2^{n+1}}}\phi_n(\sigma_1,\dots,\sigma_{n-1})$ (if $a_n = 1$). Both random variables are bounded, and in both cases the conditional expectation with respect to $\mathscr{F}_{1-\frac{1}{2^n}}$ gives zero. Now let us show integrability. Let $(\tilde{f}_n)_{n\geq 0}$ be an X-valued martingale such that $\tilde{f}_0 = 0$ a.s. and

$$d\tilde{f}_n = \sigma_n\phi_n(\sigma_1,\dots,\sigma_{n-1}), \quad n \geq 1. \tag{5.4.31}$$

Then $(\tilde{f}_n)_{n\geq 0}$ has the same distribution as $(f_n)_{n\geq 0}$, so it is L^1-bounded. Now fix $n \geq 1$ and let us estimate $\mathbb{E}\|\tilde{f}_n - M_{1-\frac{1}{2^n}}\|$:

$$\begin{aligned} \mathbb{E}\big\|\tilde{f}_n - M_{1-\frac{1}{2^n}}\big\| &\overset{(i)}{=} \mathbb{E}\Big\|\sum_{k=1}^n \sigma_k\phi_k(\sigma_1,\dots,\sigma_{k-1}) \\ &\quad - \sum_{k=1}^n \big(\sigma_k\phi_k(\sigma_1,\dots,\sigma_{k-1})\mathbf{1}_{a_k=0} + M^k_{\frac{1}{2^{k+1}}}\phi_k(\sigma_1,\dots,\sigma_{k-1})\mathbf{1}_{a_k=1}\big)\Big\| \\ &= \mathbb{E}\Big\|\sum_{k=1}^n \big(\sigma_k - M^k_{\frac{1}{2^{k+1}}}\big)\phi_k(\sigma_1,\dots,\sigma_{k-1})\mathbf{1}_{a_k=1}\Big\| \\ &\overset{(ii)}{\leq} \sum_{k=1}^n \mathbb{E}\big\|\big(\sigma_k - M^k_{\frac{1}{2^{k+1}}}\big)\phi_k(\sigma_1,\dots,\sigma_{k-1})\big\| \\ &\overset{(iii)}{\leq} 2\sum_{k=1}^n \mathbb{P}\big(\sigma_k \neq M^k_{\frac{1}{2^{k+1}}}\big)\|\phi_k\|_\infty \overset{(iv)}{\leq} 2\sum_{k=1}^n \frac{1}{2^k} \leq 2, \end{aligned} \tag{5.4.32}$$

where (i) follows from (5.4.31) and the definition of M from (5.4.30), (ii) holds by the triangle inequality, (iii) follows from the fact that a.s. for each $n \geq 1$

$$\big|\sigma_n - M^n_{\frac{1}{2^{n+1}}}\big| \leq |\sigma_n| + \big|M^n_{\frac{1}{2^{n+1}}}\big| \leq 2;$$

finally, (iv) follows from (5.4.29). Let us show that there exists $\mathscr{F}_1$-measurable $\xi \in L^1(\Omega; X)$ such that $M_{1-\frac{1}{2^n}} = \mathbb{E}(\xi|\mathscr{F}_{1-\frac{1}{2^n}})$ for each $n \geq 1$. First notice that $\mathbb{E}(\tilde{f}_\infty|\mathscr{F}_{1-\frac{1}{2^n}}) =$

$\tilde{f}_n$ for each $n \geq 1$. Moreover, by (5.4.32) the series

$$\eta := \sum_{k=1}^{\infty} (\sigma_k - M^k_{\frac{1}{2^{k+1}}})\phi_k(\sigma_1, \dots, \sigma_{k-1})\mathbf{1}_{a_k=1}$$

converges in $L^1(\Omega; X)$. Therefore, if we define $\xi := \tilde{f}_\infty - \eta$, then

$$\begin{aligned}
\mathbb{E}(\xi|\mathscr{F}_{1-\frac{1}{2^n}}) &= \mathbb{E}(\tilde{f}_\infty - \eta|\mathscr{F}_{1-\frac{1}{2^n}})\\
&= \tilde{f}_n - \mathbb{E}\Big(\sum_{k=1}^{\infty} (\sigma_k - M^k_{\frac{1}{2^{k+1}}})\phi_k(\sigma_1, \dots, \sigma_{k-1})\mathbf{1}_{a_k=1}|\mathscr{F}_{1-\frac{1}{2^n}}\Big)\\
&= \tilde{f}_n - \sum_{k=1}^{\infty} \mathbb{E}\Big((\sigma_k - M^k_{\frac{1}{2^{k+1}}})\phi_k(\sigma_1, \dots, \sigma_{k-1})\mathbf{1}_{a_k=1}|\mathscr{F}_{1-\frac{1}{2^n}}\Big)\\
&= \tilde{f}_n - \sum_{k=1}^{n} (\sigma_k - M^k_{\frac{1}{2^{k+1}}})\phi_k(\sigma_1, \dots, \sigma_{k-1})\mathbf{1}_{a_k=1} = M_{1-\frac{1}{2^n}},
\end{aligned}$$

so one has an a.s. convergence by the martingale convergence theorem [79, Theorem 3.3.8].

Now let us show that M is a martingale that does not have the Meyer-Yoeurp decomposition. Assume the contrary: let $M = M^d + M^c$ be the Meyer-Yoeurp decomposition. Then one can show that for each $n \geq 1$

$$\begin{aligned}
M^d_{1-\frac{1}{2^n}} &= \sum_{k=1}^{n} \sigma_k \phi_k(\sigma_1, \dots, \sigma_{k-1})\mathbf{1}_{a_k=0},\\
M^c_{1-\frac{1}{2^n}} &= \sum_{k=1}^{n} M^k_{\frac{1}{2^{k+1}}} \phi_k(\sigma_1, \dots, \sigma_{k-1})\mathbf{1}_{a_k=1},
\end{aligned}$$

by applying $x^* \in X^*$ and showing that the corresponding processes $\langle M^d_{1-\frac{1}{2^n}}, x^*\rangle$ and $\langle M^c_{1-\frac{1}{2^n}}, x^*\rangle$ are purely discontinuous and continuous local martingales respectively (see Remark 2.2.19). Now let us show that M^c is not an X-valued local martingale. If it is a local martingale, then

$$\mathbb{P}((M^c)^*_\infty = \infty) = \mathbb{P}((M^c)^*_1 = \infty) = 0.$$

since M^c as a local martingale should have càdlàg paths (even continuous since M^c assume to be continuous). But for each fixed $n \geq 1$

$$\mathbb{P}((M^c)^*_1 = \infty) = \mathbb{P}\big((M^c - M^c_{\frac{1}{2^n}})^*_1 = \infty\big) \geq \mathbb{P}\big((\tilde{g} - \tilde{g}_n)^*_\infty = (M^c - M^c_{\frac{1}{2^n}})^*_1\big)$$

where $(\tilde{g}_n)_{n\geq 0}$ is an X-valued martingale such that $d\tilde{g}_n = a_n d\tilde{f}_n$ a.s. for each $n \geq 0$, and hence by the construction in Lemma 5.4.15 $\tilde{g}^*_\infty = \infty$ a.s. Further,

$$\mathbb{P}\big((\tilde{g} - \tilde{g}_n)^*_\infty = (M^c - M^c_{\frac{1}{2^n}})^*_1\big) = 1 - \mathbb{P}\big((\tilde{g} - \tilde{g}_n)^*_\infty \neq (M^c - M^c_{\frac{1}{2^n}})^*_1\big)$$

$$
\begin{aligned}
&\geq 1 - \sum_{k=n}^{\infty} \mathbf{1}_{a_k=1} \mathbb{P}\Big(M^k_{\frac{1}{2^{k+1}}} \neq \sigma_k\Big) \\
&\geq 1 - \sum_{k=n}^{\infty} \mathbb{P}\Big(M^k_{\frac{1}{2^{k+1}}} \neq \operatorname{sign} M^k_{\frac{1}{2^{k+1}}}\Big) \\
&\overset{(*)}{\geq} 1 - \frac{1}{2^{n-1}},
\end{aligned}
$$

where $(*)$ follows from (5.4.29). Since n was arbitrary, $(M^c)^*_1 = (M^c)^*_\infty = \infty$ a.s., so M^c can not be a local martingale. □

Proof of Theorem 5.4.1 (necessity of UMD). Necessity of the UMD property follows from Theorem 5.4.14. □

Remark 5.4.17. One can also show that existence of the Yoeurp decomposition of an arbitrary X-valued purely discontinuous local martingale is equivalent to the UMD property. We will not repeat the argument here, but just notice that one needs to modify the proof of Theorem 5.4.14 in a way which was demonstrated in Subsection 4.3.2.

Remark 5.4.18. The reader might assume that one can weaken the Meyer-Yoeurp decomposition and consider a decomposition of an X-valued local martingale M into a sum of a continuous X-valued *semi*martingale N^c and a purely discontinuous X-valued *semi*martingale N^d, which perhaps may happen in a broader (rather than UMD) class of Banach spaces. Then for any reasonable definition of an X-valued semimartingale we get that $N^c = M^c + A$ for some continuous local martingale M^c and an adapted process of (weakly) bounded variation A. Hence $M = N^c + N^d = M^c + (N^d + A)$, where $N^d + A = M - M^c$ is a local martingale, which is purely discontinuous, so M should have the Meyer-Yoeurp decomposition as well in this setting, which means that the UMD property is crucial.

6

Orthogonal martingales and the Hilbert transform

This chapter is based on the paper *The Hilbert transform and orthogonal martingales in Banach spaces* by Adam Osękowski and Ivan Yaroslavtsev, see [146].

Let X be a given Banach space and let M, N be two orthogonal X-valued local martingales such that N is weakly differentially subordinate to M. This chapter contains the proof of the estimate

$$\mathbb{E}\Psi(N_t) \leq C_{\Phi,\Psi,X}\mathbb{E}\Phi(M_t), \quad t \geq 0,$$

where $\Phi, \Psi : X \to \mathbb{R}_+$ are convex continuous functions and the least admissible constant $C_{\Phi,\Psi,X}$ coincides with the Φ,Ψ-norm of the periodic Hilbert transform. As a corollary, it is shown that the Φ,Ψ-norms of the periodic Hilbert transform, the Hilbert transform on the real line, and the discrete Hilbert transform are the same if Φ is symmetric. We also prove that under certain natural assumptions on Φ and Ψ, the condition $C_{\Phi,\Psi,X} < \infty$ yields the UMD property of the space X. As an application, we provide comparison of L^p-norms of the periodic Hilbert transform to Wiener and Paley-Walsh decoupling constants. We also study the norms of the periodic, nonperiodic and discrete Hilbert transforms, present the corresponding estimates in the context of differentially subordinate harmonic functions and more general singular integral operators.

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6.1. INTRODUCTION

The purpose of this chapter is to study a certain class of estimates for singular integral operators acting on Banach-space-valued functions. Let us start with a related classical problem which has served as a motivation for many mathematicians for almost a century. The question is: how does the size of a periodic function control the size of its conjugate? Formally, assume that f is a trigonometric polynomial of the form

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{N} \big(a_k \cos(k\theta) + b_k \sin(k\theta)\big), \qquad \theta \in \mathbb{T} \simeq [-\pi, \pi),$$

with real coefficients $a_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N$, and define the conjugate to f as

$$g(\theta) = \sum_{k=1}^{N} \big(a_k \sin(k\theta) - b_k \cos(k\theta)\big), \qquad \theta \in [-\pi, \pi).$$

Alternatively, the conjugate function can be defined as $g = \mathscr{H}_{\mathbb{R}}^{\mathbb{T}} f$, where $\mathscr{H}_{\mathbb{R}}^{\mathbb{T}}$ is the periodic Hilbert transform given by

$$\mathscr{H}_{\mathbb{R}}^{\mathbb{T}} f(\theta) = \frac{1}{2\pi} \text{p.v.} \int_{-\pi}^{\pi} f(s) \cot \frac{\theta - s}{2} \mathrm{d}s, \qquad \theta \in [-\pi, \pi), \tag{6.1.1}$$

and the symbol $\mathbb{R}$ in the lower index of $\mathscr{H}^{\mathbb{T}}$ indicates that the operator acts on real-valued functions. We can state the problem as follows. For a given $1 \le p \le \infty$, does there exist a universal constant C_p (that is, not depending on the coefficients or the number N) such that

$$\left(\int_{[-\pi,\pi)} |g(\theta)|^p \mathrm{d}\theta \right)^{1/p} \le C_p \left(\int_{[-\pi,\pi)} |f(\theta)|^p \mathrm{d}\theta \right)^{1/p} ?$$

Furthermore, if the answer is yes, what is the optimal value of C_p (i.e., what is the L^p-norm of $\mathscr{H}_{\mathbb{R}}^{\mathbb{T}}$)? The first question was answered by M. Riesz in [158]: the inequality does hold if and only if $1 < p < \infty$. The best value of C_p was determined by Pichorides [149] and Cole (unpublished): the constant $\cot(\pi/(2p^*))$ is the best possible, where $p^* = \max\{p, p/(p-1)\}$. There is a natural further question concerning the version of the above result for Banach-space-valued functions (it is not difficult to see that the formula (6.1.1) makes perfect sense in the vector setting, at least for some special f, see Section 6.2 below). Few years after the results of Riesz, it was realized that not all spaces are well-behaved: Bochner and Taylor [20] showed that $\|\mathscr{H}_{\ell_1}^{\mathbb{T}}\|_{L_p \to L_p} = \infty$ for all p. The problem of characterizing the 'good' Banach spaces was solved over forty years later: Burkholder [30] and Bourgain [23] showed that the so-called UMD spaces form a natural environment to the study of the L^p-boundedness ($1 < p < \infty$) of the periodic Hilbert transform, and more generally, for the L^p-boundedness of a wider class of singular integral operators.

UMD stands for "unconditional martingale differences"

The above problems, though expressed in an analytic language, have a very strong connection with probability theory, especially with the theory of martingales (see e.g. [12, 15, 16, 23, 32, 61, 66, 79, 144, 145]). Let us provide some necessary definitions. Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, equipped with a continuous-time filtration $(\mathscr{F}_t)_{t\geq 0}$. Let $M = (M_t)_{t\geq 0}$, $N = (N_t)_{t\geq 0}$ be two adapted real-valued local martingales, whose trajectories are right-continuous and have limits from the left. Let $[M]$, $[N]$ stand for the associated quadratic variation (square brackets) of M and N, see (2.2.3). Furthermore, $M^* = \sup_{t\geq 0}|M_t|$, $N^* = \sup_{t\geq 0}|N_t|$ denote the corresponding maximal functions. We say that M and N are *orthogonal*, if $[M,N] := \frac{[M+N]-[M-N]}{4} = 0$ and $M_0 N_0 = 0$ almost surely.

One of the remarkable examples of the aforementioned connection between the theory of singular integral operators and martingale theory was provided by Bañuelos and Wang in [15]. They have shown that the L^p-norm of $\mathscr{H}^{\mathbb{T}}$ acting on real-valued functions is equal to the sharp constant in the corresponding L^p-inequality

$$(\mathbb{E}|N_t|^p)^{\frac{1}{p}} \leq C_p (\mathbb{E}|M_t|^p)^{\frac{1}{p}}, \quad t \geq 0, \tag{6.1.2}$$

where N is assumed to be differentially subordinate and orthogonal to M. The goal of the current article is to show that this interplay between the norm of $\mathscr{H}^{\mathbb{T}}$ and the martingale inequality (6.1.2) can be extended to i) more general Φ, Ψ-norms (see the beginning of Section 6.3 for the definition) and ii) more general Banach spaces in which the functions and processes take values.

Let us say a few words about the structure of the chapter. The next section is devoted to the introduction of the background which is needed for our further study. In particular, we define an appropriate analogue of Banach-space-valued orthogonality of martingales and provide some basic information about plurisubharmonic functions, fundamental objects in the complex analysis of several variables. Section 6.3 contains the main result of the chapter, connecting the best constants in certain Φ, Ψ-estimates for the periodic Hilbert transform and their counterparts in martingale theory. Though the rough idea of the proof can be tracked back to the classical works [15, 36, 77, 149] (the validity of a given estimate for the Hilbert transform / orthogonal differentially subordinate martingales is equivalent to the existence of a certain special plurisubharmonic function), there are several serious technical problems to be overcome, due to the fact that we work in the Banach-space-valued setting. Section 6.4 is devoted to some applications. The first and the most notable one connects together the Φ, Ψ-norms of the periodic Hilbert transform $\mathscr{H}_X^{\mathbb{T}}$, the Hilbert transform $\mathscr{H}_X^{\mathbb{R}}$ defined on a real line, and the discrete Hilbert transform $\mathscr{H}_X^{\mathrm{dis}}$ (for the definition of the latter object, consult Definition 6.4.1 and 6.4.2 below). It turns out that all these norms coincide for quite general class of Φ and Ψ. This in particular generalizes the recent result of Bañuelos and Kwaśnicki

[12] on the discrete Hilbert transform $\mathcal{H}_{\mathbb{R}}^{\mathrm{dis}}$, which asserts that

$$\|\mathcal{H}_{\mathbb{R}}^{\mathrm{dis}}\|_{L^p(\mathbb{Z})\to L^p(\mathbb{Z})} = \|\mathcal{H}_{\mathbb{R}}^{\mathbb{T}}\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} = \cot\Big(\frac{\pi}{2p^*}\Big), \qquad 1 < p < \infty.$$

This used to be an open problem for 90 years (see [12, 103, 172]). Subsection 6.4.2 is devoted to the comparison of L^p-norms of the periodic Hilbert transform to Wiener and Paley-Walsh decoupling constants. Application in Subsection 6.4.3 is concerned with UMD Banach spaces and can be regarded as an extension of Bourgain's result [23]: we show that under some mild assumption on Φ and Ψ, the validity of the corresponding Φ,Ψ-estimate (with some finite constant) implies the UMD property of X. In Subsection 6.4.4 we prove that the results obtained in this chapter can be applied to obtain sharper estimates for weakly differentially subordinate martingales (not necessarily satisfying the orthogonality assumption). Subsection 6.4.5 contains the study of related estimates in the context of harmonic functions on Euclidean domains. In Subsection 6.4.6 we present the possibility of extending the estimates to the more general class of singular integral operators. Our final application, described in Subsection 6.4.7, discusses the vector-valued extension of the classical results of Hardy concerning Hilbert operators.

6.2. PRELIMINARIES

This section contains the definitions of some basic notions and facts used later. Here and below, the scalar field is assumed to be $\mathbb{R}$, unless stated otherwise.

6.2.1. *Periodic Hilbert transform*

In what follows, the symbol $\mathbb{T}$ will stand for the torus $(\{z \in \mathbb{C} : |z| = 1\}, \cdot)$ equipped with the natural multiplication. Sometimes, passing to the argument of a complex number, we will identify $\mathbb{T}$ with the interval $[-\pi,\pi)$. Let X be a Banach space. A function $f : \mathbb{T} \to X$ is called a *step function*, if it is of the form

$$f = \sum_{k=1}^{N} x_k \mathbf{1}_{A_k}(s), \quad -\pi \le s < \pi,$$

where N is finite, $x_k \in X$ and A_k are intervals in $\mathbb{T}$. The *periodic Hilbert transform* $\mathcal{H}_X^{\mathbb{T}}$ of a step function $f : \mathbb{T} \to X$ is given by the singular integral

$$\mathcal{H}_X^{\mathbb{T}} f(t) = \frac{1}{2\pi}\mathrm{p.v.}\int_{-\pi}^{\pi} f(s)\cot\frac{t-s}{2}\,\mathrm{d}s, \qquad -\pi \le t < \pi. \tag{6.2.1}$$

6.2.2. *Orthogonal martingales*

We have defined the notion of orthogonality of real-valued local martingales in the introductory section. We turn our attention to its vector-valued analogue.

Definition 6.2.1. Let M, N be local martingales taking values in a given Banach space X. Then M, N are said to be *orthogonal*, if $\langle M_0, x^*\rangle \cdot \langle N_0, x^*\rangle = 0$ and $[\langle M, x^*\rangle, \langle N, x^*\rangle] = 0$ almost surely for all functionals $x^* \in X^*$.

Remark 6.2.2. Assume that M, N are local martingales taking values in some Banach space X. If M, N are orthogonal and N is weakly differentially subordinate to M, then $N_0 = 0$ almost surely (which follows immediately from the above definitions). Moreover, under these assumptions, N must have continuous trajectories with probability 1. Indeed, in such a case for any fixed $x^* \in X^*$ the real-valued local martingales $\langle M, x^*\rangle$, $\langle N, x^*\rangle$ are orthogonal and we have $\langle N, x^*\rangle \ll \langle M, x^*\rangle$. Therefore, $\langle N, x^*\rangle$ has a continuous version for each $x^* \in X^*$ by [136, Lemma 3.1] (see also [16, Lemma 1]), which in turn implies that N is continuous: any X-valued local martingale has a càdlàg version (see Proposition 2.2.1).

Remark 6.2.3. The requirement $\langle M_0, x^*\rangle \cdot \langle N_0, x^*\rangle = 0$ for all $x^* \in X^*$ in Definition 6.2.1 is usually omitted (see e.g. [15, 16, 85]). Nevertheless we need this requirement in order to simplify all the statements in the sequel concerning orthogonal martingales.

Weakly differentially subordinate orthogonal martingales appear naturally while working with the periodic Hilbert transform, which can be seen by exploiting the classical argument of Doob (the composition of a harmonic function with a Brownian motion is a martingale). Indeed, suppose that X is a given Banach space. Suppose that f is a simple function and put $g = \mathscr{H}_X^{\mathbb{T}} f$. Let u_f, u_g denote the harmonic extensions of f and g to the unit disc, obtained by the convolution with the Poisson kernel. In particular, the equality $g = \mathscr{H}^{\mathbb{T}} f$ implies that $u_g(0,0) = 0$ and for any functional $x^* \in X^*$, the function $\langle u_f, x^*\rangle + i\langle u_g, x^*\rangle$ is holomorphic on the disc.

Next, suppose that $W = (W^1, W^2)$ is a planar Brownian motion started from $(0,0)$ and stopped upon leaving the unit disc. Then the processes $M = (M_t)_{t\geq 0} = (u_f(W_t))_{t\geq 0}$, $N = (N_t)_{t\geq 0} = (u_g(W_t))_{t\geq 0}$ are X-valued martingales such that $N_0 = 0$. For any functional $x^* \in X^*$, we apply the standard, one-dimensional Itô's formula to obtain, for any $t \geq 0$,

$$\langle M_t, x^*\rangle = \langle M_0, x^*\rangle + \int_0^t \nabla\langle u_f(W_s), x^*\rangle \mathrm{d}W_s$$

and

$$\langle N_t, x^*\rangle = \langle N_0, x^*\rangle + \int_0^t \nabla\langle u_g(W_s), x^*\rangle \mathrm{d}W_s.$$

By the aforementioned connection to analytic functions, the gradients $\nabla\langle u_f, x^*\rangle$, $\nabla\langle u_g, x^*\rangle$ are orthogonal and of equal length, so

$$[\langle M, x^*\rangle, \langle N, x^*\rangle]_t = \int_0^t \nabla\langle u_f(W_s), x^*\rangle \cdot \nabla\langle u_g(W_s), x^*\rangle \,\mathrm{d}s = 0,$$

and

$$[\langle M, x^*\rangle]_t - [\langle N, x^*\rangle]_t = \int_0^t \nabla\langle u_f(W_s), x^*\rangle^2 - \nabla\langle u_g(W_s), x^*\rangle^2 \,\mathrm{d}s = 0.$$

Hence M, N are orthogonal and satisfy the weak differential subordination $N \overset{w}{\ll} M$. Since the distribution of W_∞ is uniform on the unit circle $\mathbb{T}$, essentially any estimate of the form

$$\mathbb{E}V(M_t, N_t) \le 0, \qquad t \ge 0,$$

for weakly differentially subordinate orthogonal martingales leads to the analogous bound

$$\int_{\mathbb{T}} V(f, \mathcal{H}_X^{\mathbb{T}} f)\mathrm{d}x \le 0$$

for the periodic Hilbert transform, at least when restricted to the class of simple functions. (Later in Theorem 6.3.1 we will show that the reverse holds true).

For more information and examples concerning orthogonal martingales, we refer the reader to [15, 16, 33, 179].

6.2.3. Subharmonic and plurisubharmonic functions

A function $f : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ is called *subharmonic* if for any ball $B \subset \mathbb{R}^d$ and any harmonic function $g : B \to \mathbb{R}$ such that $f \le g$ on ∂B one has the inequality $f \le g$ on the whole B. The following lemma follows from [105, Proposition I.9].

Lemma 6.2.4. *Let $d \ge 1$ and let $f : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function. Then either $f \equiv -\infty$, or f is locally integrable.*

Let X be a Banach space. The function $F : X + iX \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic* if for any $x, y \in X + iX$ the restriction $z \mapsto F(x + yz)$ is subharmonic in $z \in \mathbb{C}$.

Remark 6.2.5. Notice that $X + iX$ is a Banach space equipped with the norm

$$\|x + iy\|_{X+iX} := \sup_{x^* \in X^*, \|x^*\| \le 1} (|\langle x, x^*\rangle|^2 + |\langle y, x^*\rangle|^2)^{\frac{1}{2}}, \quad x, y \in X$$

(see [79, Subsection B.4]).

Remark 6.2.6. Let X be finite-dimensional. Then any plurisubharmonic function defined on $X + iX$ is subharmonic (see [105, Proposition I.9] and [64, Theorem 39]). Therefore, by Lemma 6.2.4, a plurisubharmonic function either identically equals $-\infty$, or is locally integrable.

Let $F : X + iX \to \mathbb{R}$ be k-times differentiable, $u_1, \ldots, u_k \in X + iX$. Then we denote

$$\frac{\partial^k F(v)}{\partial u_1 \cdots \partial u_k} := \frac{\partial^k}{\partial t_1 \cdots \partial t_k} F(v + t_1 u_1 + \cdots + t_k u_k)\Big|_{t_1, \ldots, t_k = 0}, \quad v \in X + iX.$$

In particular, for any $u \in X + iX$,

$$\frac{\partial^k F(v)}{\partial u^k} := \frac{\partial^k}{\partial t^k} F(v + tu)\Big|_{t=0}, \quad v \in X + iX.$$

Remark 6.2.7. Note that if X is finite-dimensional, F is plurisubharmonic and twice differentiable, then for all $z_0 \in X + iX$ and $x \in X$ we have

$$\frac{\partial^2 F(z_0)}{\partial x^2} + \frac{\partial^2 F(z_0)}{\partial i x^2} = \Big(\frac{\partial^2 F(z_0+zx)}{\partial \mathrm{Re}\, z^2} + \frac{\partial^2 F(z_0+zx)}{\partial \mathrm{Im}\, z^2}\Big)\Big|_{z=0}$$
$$= \Delta_z F(z_0+zx)|_{z=0} \geq 0.$$

Later on we will need the following result.

Proposition 6.2.8. *Let X be a Banach space and let $F : X + iX \to \mathbb{R} \cup \{-\infty\}$ be plurisubharmonic. Assume further that $y \mapsto F(x+iy)$ is concave in $y \in X$ for any fixed $x \in X$. Then $x \mapsto F(x+iy)$ is convex in $x \in X$ for any $y \in X$, and F is continuous.*

For the proof we will need the following lemma.

Lemma 6.2.9. *Let X be a finite-dimensional Banach space and let $V : X + iX \to \mathbb{R}$ be a continuous twice differentiable plurisubharmonic function. Let $y \mapsto V(x+iy)$ be concave in $y \in X$ for all $x \in X$. Then $t \mapsto V(tx+z)$ is convex in $t \in \mathbb{R}$ for all $x \in X$ and $z \in X + iX$. In particular, $t \mapsto V(tx+z)$ is differentiable, so*

$$V(tx+z) \geq V(sx+z) + \partial_s V(sx+z)(t-s), \quad t,s \in \mathbb{R}. \tag{6.2.2}$$

Proof. The first part follows from the fact that V is plurisubharmonic and twice differentiable. Indeed, we have

$$\frac{\partial^2 V(tx+z)}{\partial t^2}$$
$$= \Big(\frac{\partial^2 V(tx+z+isx)}{\partial t^2} + \frac{\partial^2 V(tx+z+isx)}{\partial s^2}\Big)\Big|_{s=0} - \frac{\partial^2 V(tx+z+isx)}{\partial s^2}\Big|_{s=0} \geq 0$$

since

$$\Big(\frac{\partial^2 V(tx+z+isx)}{\partial t^2} + \frac{\partial^2 V(tx+z+isx)}{\partial s^2}\Big)\Big|_{s=0} \geq 0$$

by plurisubharmonicity and $\frac{\partial^2 V(tx+z+isx)}{\partial s^2} \leq 0$ by concavity of $y \mapsto V(x+z+iy)$. The inequality (6.2.2) follows immediately from the convexity of $t \mapsto V(tx+iy)$ and twice differentiability of V. □

For the proof we will need the following observation which will allow us to integrate over a Banach space.

Remark 6.2.10. Let X be a finite dimensional Banach space. Then due to [59, Theorem 2.20 and Proposition 2.21] there exists a unique translation-invariant measure λ_X on X such that $\lambda_X(\mathbb{B}_X) = 1$ for the unit ball $\mathbb{B}_X$ of X. We will call λ_X the *Lebesgue measure*. In the sequel we will omit the Lebesgue measure notation while integrating over X (i.e. we will write $\int_X F(s)\,\mathrm{d}s$ instead of $\int_X F(s)\lambda_X(\,\mathrm{d}s)$).

Proof of Proposition 6.2.8. Without loss of generality we can assume that X is finite-dimensional and that $f \not\equiv -\infty$. Let $\phi : X + iX \to \mathbb{R}_+$ be a C^∞ function with bounded support such that

$$\int_{X+iX} \phi(s)\,\mathrm{d}s = 1.$$

(This integral is well-defined due to Remark 6.2.5 and 6.2.10). For each $\varepsilon > 0$ we define $F_\varepsilon : X + iX \to \mathbb{R}$ in the following way:

$$F_\varepsilon(s) = \int_{X+iX} F(s - \varepsilon t)\phi(t)\,\mathrm{d}t, \quad s \in X + iX. \tag{6.2.3}$$

Then F_ε is plurisubharmonic due to [78, Theorem 4.1.4]. Moreover, again by [78, Theorem 4.1.4], we have $F_\varepsilon \searrow F$ as $\varepsilon \searrow 0$. On the other hand, F_ε is well-defined and of class C^∞. Furthermore, the function $y \mapsto F_\varepsilon(x + iy)$ is concave in $y \in X$ for any $x \in X$ by (6.2.3): here we use the fact that F is locally integrable (see Remark 6.2.6) and the concavity of $y \mapsto F(x + iy)$ for any fixed $x \in X$. Therefore by Lemma 6.2.9, the function $x \mapsto F_\varepsilon(x + iy)$ is convex for any fixed $y \in X$; hence so is F, being the pointwise limit of $(F_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$.

Let us now show that $F > -\infty$. Assume that there exists $x_0, y_0 \in X$ such that $F(x_0 + iy_0) = -\infty$. Since the function $y \mapsto F(x_0 + iy)$ is concave, the set $A = \{y \in X : F(x_0 + iy) > \infty\} \subset X$ is convex and open; moreover, $y_0 \notin A$, so $X \setminus A$ is of positive measure. Now fix $(x, y) \in X \times (X \setminus A)$. Notice that $F(x_0 + iy) = -\infty$. On the other hand $x \mapsto F(x + iy)$ is convex, so $F(x + iy) = -\infty$ as well (if a convex function equals $-\infty$ in one point, it equals $-\infty$ on the whole X). Therefore $F = -\infty$ in the set $X \times (X \setminus A)$ of positive measure; hence $F \equiv -\infty$ by Remark 6.2.6, which leads to a contradiction.

Finally, note that $F < \infty$: we have $F \le F_1$ with F_1 defined in (6.2.3). Therefore F is continuous as a finite concave-convex function (see [171, Proposition 3.3] and [86, Corollary 4.5]). □

For further material on subharmonic and plurisubharmonic functions, we recommend the works [64, 78, 105, 159, 160].

6.2.4. *Meyer-Yoeurp decomposition*

The following result shows the connection between the Meyer-Yoeurp decomposition and the weak differential subordination.

Proposition 6.2.11. *Let X be a Banach space and let M, N be local X-valued martingales possessing the Meyer-Yoeurp decompositions $M = M^c + M^d$, $N = N^c + N^d$. Then $N \overset{w}{\ll} M$ if and only if $N^c \overset{w}{\ll} M^c$ and $N^d \overset{w}{\ll} M^d$. Moreover, if M and N are orthogonal, then M^c and N^c, M^d and N^d are pairwise orthogonal.*

Proof. The first part follows from Lemma 4.4.5 (see also [179, Lemma 1]). Due to Remark 6.2.2 we know that $N^d = 0$, so it is sufficient to show that M^c and N^c

are orthogonal. The latter is equivalent to the fact that $\langle M^c, x^*\rangle$ and $\langle N^c, x^*\rangle$ are orthogonal for any $x^* \in X^*$, which holds true by [16, Lemma 1]. □

6.3. MAIN THEOREM

Having introduced all the necessary notions, we turn to the study of our new results. For given two nonnegative and continuous functions $\Phi, \Psi : X \to \mathbb{R}_+$, we define the associated 'Φ, Ψ-norm' of $\mathcal{H}_X^{\mathbb{T}}$ by the formula

$$|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} := \inf\Big\{ c \in [0,\infty] : \int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f(s))\,\mathrm{d}s \le c \int_{\mathbb{T}} \Phi(f(s))\,\mathrm{d}s \quad \text{for all step functions } f : \mathbb{T} \to X \Big\}.$$

Notice that if $\Psi \equiv 0$, then $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} = 0$, and if $\Phi \equiv 0$, then $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} \in \{0, +\infty\}$. Throughout the chapter we exclude these trivial cases: we will assume that both Φ and Ψ are not identically zero. Furthermore, for any $1 < p < \infty$, we will denote the L^p-norm of $\mathcal{H}_X^{\mathbb{T}}$ by $\hbar_{p,X}$ (in the language of Φ, Ψ-norms, we have $\hbar_{p,X}^p = |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$ with $\Phi(x) = \Psi(x) = ||x||^p$).

The following theorem is the main result of this section.

Theorem 6.3.1. *Let X be a separable Banach space and let $\Phi, \Psi : X \to \mathbb{R}_+$ be continuous convex functions such that $\Psi(0) = 0$ and $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} < \infty$. Let M, N be two orthogonal X-valued local martingales such that $N \overset{w}{\ll} M$. Then*

$$\mathbb{E}\Psi(N_t) \le C_{\Phi,\Psi,X}\mathbb{E}\Phi(M_t), \qquad t \ge 0, \tag{6.3.1}$$

and the least admissible $C_{\Phi,\Psi,X}$ equals $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$.

The idea behind the proof of (6.3.1) can be roughly described as follows. First, we will show that the condition $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} < \infty$ (i.e., the validity of a Φ, Ψ-estimate for the periodic Hilbert transform) implies the existence of a certain special function on $X + iX$, enjoying appropriate size conditions and concavity. Next, we will compose this function with $M + iN$ and prove, using the concavity and Itô's formula from the previous section, that the resulting process has nonnegative expectation. This in turn will give the desired bound, in the light of the size condition of the special function. Though this reasoning is typical for this kind of martingale inequalities, there are two essential differences. First, we will see that the special function will not have any explicit form: in particular, this makes the exploitation of its properties much harder, as one can get them only from some abstract (and restricted) reasoning. The second difference is related to the fact that we work will Banach-space-valued processes: this enforces us to study some additional, structural properties of the local martingales involved. Moreover, since we will work in infinite-dimensional Banach spaces, the approximation to finite dimensions exploited in the proof should be especially delicate because we do not want to ruin weak differential subordination and orthogonality of the corresponding martingales.

Having described our plan, we turn to its realization. We will need several intermediate facts. The following theorem links the quantity $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$ with a certain special plurisubharmonic function.

Theorem 6.3.2. *Let X be a separable Banach space and let $\Phi, \Psi : X \to \mathbb{R}_+$ be continuous functions such that $\Psi(0) = 0$ and $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} < \infty$. Then there exists a plurisubharmonic function $U_{\Phi,\Psi} : X + iX \to \mathbb{R}$ such that $U_{\Phi,\Psi}(x) \geq 0$ for all $x \in X$ and*

$$U_{\Phi,\Psi}(x + iy) \leq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}\Phi(x) - \Psi(y), \quad x, y \in X.$$

Moreover, if Ψ is convex, then $y \mapsto U_{\Phi,\Psi}(x + iy)$ is concave in $y \in X$ for all $x \in X$.

Proof (sketch). We repeat the reasoning presented in [77, Theorem 2.3] (the separability of X is a key part of the construction $U_{\Phi,\Psi}$). The last property follows from the construction of $U_{\Phi,\Psi}$, the fact that $y \mapsto |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}\Phi(x) - \Psi(y)$ is a concave function in $y \in X$, and the fact that a minimum of concave functions is a concave function as well. □

Corollary 6.3.3. *Let X be a Banach space, $1 < p < \infty$. Then X is a UMD Banach space if and only if there exists a plurisubharmonic function $U_{p,X} : X + iX \to \mathbb{R}$ such that $U_{p,X}(x) \geq 0$ for all $x \in X$ and*

$$U_{p,X}(x + iy) \leq \hbar_{p,X}^p \|x\|^p - \|y\|^p, \quad x, y \in X.$$

Moreover, if this is the case, then $y \mapsto U_{p,X}(x + iy)$ is concave in $y \in X$ for all $x \in X$.

Proof. It is sufficient to take $\Phi(x) = \Psi(x) = \|x\|^p$, $x \in X$, and apply Theorem 6.3.2 and the fact that $\hbar_{p,X} < \infty$ if and only if X is a UMD Banach space (see [23, 30]). □

Lemma 6.3.4. *Let X be a Banach space, let M be an X-valued local martingale and let $(\tau_n)_{n\geq 1}$ be a sequence of stopping times increasing to infinity almost surely. Let $\Phi : X \to \mathbb{R}_+$ be a convex function such that $\mathbb{E}\Phi(M_t) < \infty$ for some $t \geq 0$. Then $\mathbb{E}\Phi(M_{t\wedge\tau_n}) \nearrow \mathbb{E}\Phi(M_t)$ as $n \to \infty$.*

Proof. Notice that $(\mathbb{E}\Phi(M_{t\wedge\tau_n}))_{n\geq 1}$ is an increasing sequence which is less then $\mathbb{E}\Phi(M_t)$ by the conditional Jensen's inequality, [89, Theorem 7.12], and [89, Lemma 7.1(iii)]. On the other other hand $\Phi(M_{t\wedge\tau_n}) \to \Phi(M_t)$ a.s. since $\tau_n \to \infty$ as $n \to \infty$. It suffices to apply Fatou's lemma to get the assertion. □

The next statement contains the proof of a structural property of orthogonal martingales. We need an additional notion. A linear operator T acting on a Hilbert space H is called *skew-symmetric* (or *antisymmetric*) if $\langle Th, h\rangle = 0$ for all $h \in H$.

Proposition 6.3.5. *Let $d \geq 1$, W be a d-dimensional standard Brownian motion, let X be a finite-dimensional Banach space and let $\phi, \psi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(\mathbb{R}^d, X)$ be progressively measurable processes such that $M := \phi \cdot W$ and $N := \psi \cdot W$ are well-defined orthogonal martingales. Assume further that $N \overset{w}{\ll} M$. Then there exists a operator-valued progressively-measurable process $A : \mathbb{R}_+ \times \Omega \to \mathcal{L}(\mathbb{R}^d)$ such that $\|A\| \leq 1$, $\psi^* = A\phi^*$ a.s. on $\mathbb{R}_+ \times \Omega$, and $P_{\mathrm{Ran}(\phi^*)}(s,\omega)A(s,\omega)$ is skew-symmetric for all $s \geq 0$ and $\omega \in \Omega$, where $P_{\mathrm{Ran}(\phi^*)} \in \mathcal{L}(\mathbb{R}^d)$ is the orthoprojection on* $\mathrm{Ran}(\phi^*)$.

Proof. Let $(x_n^*)_{n\geq 1}$ be a dense sequence in X^*. Then by the orthogonality of M, N and the condition $N \overset{w}{\ll} M$, we have

$$\|\psi^*(t,\omega)x_n^*\| \leq \|\phi^*(t,\omega)x_n^*\|,$$

$$\langle \psi^*(t,\omega)x_n^*, \phi^*(t,\omega)x_n^* \rangle = 0$$

for almost all $\omega \in \Omega$, all $t \in \mathbb{R}_+$ and all $n \geq 1$. Hence by the density argument, for any $x^* \in X^*$, almost all $\omega \in \Omega$ and all $t \in \mathbb{R}_+$,

$$\|\psi^*(t,\omega)x^*\| \leq \|\phi^*(t,\omega)x^*\|, \tag{6.3.2}$$

$$\langle \psi^*(t,\omega)x^*, \phi^*(t,\omega)x^* \rangle = 0. \tag{6.3.3}$$

Fix $t \in \mathbb{R}_+$ and $\omega \in \Omega$ such that (6.3.2) and (6.3.3) hold for any $x^* \in X^*$. Define $A(t,\omega) : H \to H$ in the following way (we omit (t,ω) for the convenience of the reader):

$$Ah := \begin{cases} \psi^* x^*, & \text{if } \exists x^* \in X^* \text{ such that } h = \phi^* x^*; \\ 0, & \text{if } h \perp \mathrm{Ran}(\phi^*). \end{cases} \tag{6.3.4}$$

Then A is well-defined since if $h = \phi^*(t,\omega)x_1^* = \phi^*(t,\omega)x_2^*$ for some different $x_1^*, x_2^* \in X^*$, then by (6.3.2),

$$\begin{aligned} \|\psi^*(t,\omega)x_1^* - \psi^*(t,\omega)x_2^*\| &= \|\psi^*(t,\omega)(x_1^* - x_2^*)\| \\ &\leq \|\phi^*(t,\omega)(x_1^* - x_2^*)\| \\ &= \|\phi^*(t,\omega)x_1^* - \phi^*(t,\omega)x_2^*\| = \|h - h\| = 0. \end{aligned}$$

Moreover, A is linear on both $\mathrm{Ran}(\phi^*)$ and $(\mathrm{Ran}(\phi^*))^\perp$, so it can be extended to a linear operator $A \in \mathcal{L}(H)$. Notice that then we have $\psi^* = A\phi^*$. Furthermore, the conditions (6.3.2) and (6.3.4) imply that $\|A\| \leq 1$, while (6.3.3) and (6.3.4) give that $P_{\mathrm{Ran}(\phi^*)}A$ is skew-symmetric ($P_{\mathrm{Ran}(\phi^*)}$ being the orthoprojection on $\mathrm{Ran}(\phi^*)$). □

In our later considerations, we will also need the following technical result.

Proposition 6.3.6. *Let X be a finite-dimensional Banach space and let $\Phi, \Psi : X \to \mathbb{R}_+$ be continuous functions such that Ψ is convex, $\Psi(0) = 0$ and $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} < \infty$. Let $U_{\Phi,\Psi}$:*

$X+iX\to\mathbb{R}$ be the special function from Theorem 6.3.2. Assume additionally that $U_{\Phi,\Psi}$ is twice differentiable. Then for any $x,y\in X$, $z_0\in X+iX$ and any $\lambda\in[-1,1]$ we have

$$\begin{aligned}\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial x^2}+\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial y^2}+2\lambda\Big(\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial x\partial iy}-\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial y\partial ix}\Big)\\ +\lambda^2\Big(\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial ix^2}+\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial iy^2}\Big)\geq 0.\end{aligned}\tag{6.3.5}$$

Proof. Notice that the function

$$\begin{aligned}\lambda\mapsto\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial x^2}+\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial y^2}+2\lambda\Big(\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial x\partial iy}-\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial y\partial ix}\Big)\\ +\lambda^2\Big(\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial ix^2}+\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial iy^2}\Big)\end{aligned}$$

is concave due to the fact that $\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial ix^2}$, $\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial iy^2}\leq 0$ by the last part of Theorem 6.3.2. Therefore it is sufficient to show (6.3.5) for $\lambda=1$ and $\lambda=-1$. We will consider the first possibility only, the second can be handled analogously. We have

$$\begin{aligned}\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial x^2}+\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial y^2}+2\Big(\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial x\partial iy}-\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial y\partial ix}\Big)&\\ +\Big(\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial ix^2}+\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial iy^2}\Big)&\\ =\frac{\partial^2 U_{\Phi,\Psi}(z_0+t(x+iy))}{\partial t^2}+\frac{\partial^2 U_{\Phi,\Psi}(z_0+t(y-ix))}{\partial t^2}&\\ =\Delta_z U_{\Phi,\Psi}(z_0+z(y-ix))\geq 0,&\end{aligned}$$

since $U_{\Phi,\Psi}$ is plurisubharmonic (here Δ_z is the Laplace operator acting with respect to the z-variable). □

Corollary 6.3.7. *Under the assumptions of the previous Proposition, for any $x,y\in X$, $z_0\in X+iX$, $\lambda\in[-1,1]$ and any $\mu\in[-|\lambda|,|\lambda|]$ we have*

$$\begin{aligned}\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial x^2}+\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial y^2}+2\mu\Big(\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial x\partial iy}-\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial y\partial ix}\Big)\\ +\lambda^2\Big(\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial ix^2}+\frac{\partial^2 U_{\Phi,\Psi}(z_0)}{\partial iy^2}\Big)\geq 0.\end{aligned}\tag{6.3.6}$$

Proof. The left-hand side of (6.3.6) is linear in μ, so it is sufficient to check the estimate for $\mu=\pm\lambda$. □

The next few statements aim at establishing an appropriate "localization" procedure: we will prove how to deduce the general, possibly infinite-dimensional context from its finite-dimensional counterpart. We need some additional notation. Let X be a Banach space with a dual X^*, $Y\subset X^*$ be a linear subspace. Let

$P: Y \hookrightarrow X^*$ be the continuous embedding operator. Then P^* is a well-defined bounded linear operator from X^{**} to $X_Y := Y^*$ such that $\mathrm{Ran}(P^*) = X_Y$. Moreover, if Y is finite-dimensional, then $\mathrm{Ran}(P^*|_X) = X_Y$, where $P^*|_X : X \to X_Y$ is a well-defined restriction of P^* on X due to the natural embedding $X \hookrightarrow X^{**}$. For any function $\phi : X \to \mathbb{R}_+$, we can define $\phi_Y : X_Y \to \mathbb{R}_+$ by the formula

$$\phi_Y(\tilde{x}) = \inf\{\phi(x) : x \in X, P^* x = \tilde{x}\}, \quad \tilde{x} \in X_Y. \tag{6.3.7}$$

Lemma 6.3.8. *Let X be a Banach space with a dual X^* and let $Y \subset X^*$ be a linear subspace. Let $\phi : X \to \mathbb{R}_+$ be a convex function. Then $\phi_Y : X_Y \to \mathbb{R}_+$ defined by* (6.3.7) *is convex and we have $\phi_Y(P^* x) \le \phi(x)$ for all $x \in X$.*

Proof. Fix $\tilde{x}_1, \tilde{x}_2 \in X_Y$, $\lambda \in [0,1]$ and set $\tilde{x} = \lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2$. Then

$$\begin{aligned}
\phi_Y(\tilde{x}) = \inf_{\substack{x \in X \\ P^* x = \tilde{x}}} \phi(x) &= \inf_{\substack{x_1 \in X, P^* x_1 = \tilde{x}_1 \\ x_2 \in X, P^* x_2 = \tilde{x}_2}} \phi(\lambda x_1 + (1-\lambda)x_2) \\
&\le \inf_{\substack{x_1 \in X, P^* x_1 = \tilde{x}_1 \\ x_2 \in X, P^* x_2 = \tilde{x}_2}} \lambda\phi(x_1) + (1-\lambda)\phi(x_2) \\
&= \lambda \inf_{x_1 \in X, P^* x_1 = \tilde{x}_1} \phi(x_1) + (1-\lambda) \inf_{x_2 \in X, P^* x_2 = \tilde{x}_2} \phi(x_2) \\
&= \lambda\phi_Y(\tilde{x}_1) + (1-\lambda)\phi_Y(\tilde{x}_2),
\end{aligned}$$

so ϕ_Y is convex. The last part of the lemma follows from the definition of ϕ_Y. □

Lemma 6.3.9. *Let X be a separable Banach space, $\phi : X \to \mathbb{R}_+$ be convex lower semi-continuous. Then there exists an increasing sequence of finite-dimensional subspaces $(Y_n)_{n\ge1}$ of X^* such that the following holds. If $P_n : Y_n \hookrightarrow X^*$ is the corresponding embedding for each $n \ge 1$ and $\phi_n : Y_n^* \to \mathbb{R}_+$ satisfies*

$$\phi_n(\tilde{x}) = \inf\{\phi(x) : x \in X, P_n^* x = \tilde{x}\}, \quad \tilde{x} \in Y_n^*, \tag{6.3.8}$$

then for each $x \in X$ the sequence $(\phi_n(P_n^* x))_{n\ge1}$ increases to $\phi(x)$ as $n \to \infty$.

Proof. By [79, Lemma 1.2.10] there exist a sequence $(x_n^*)_{n\ge1}$ in X^* and a sequence $(a_n)_{n\ge1}$ of real numbers such that

$$\phi(x) = \sup_n \langle x, x_n^*\rangle + a_n, \quad x \in X. \tag{6.3.9}$$

Let $Y_n := \mathrm{span}(x_1^*, \ldots, x_n^*)$ for each $n \ge 1$. Fix $x \in X$. First notice that $\phi_n(P_n^* x) \le \phi(x)$ by Lemma 6.3.8. Moreover, $\phi_n(P_n^* x) \le \phi_{n+1}(P_{n+1}^* x)$ for each $n \ge 1$ since $Y_n \subset Y_{n+1}$ (see (6.3.8)). Fix $n \ge 1$. Then for any $y \in X$ such that $P_n^* x = P_n^* y$ we have $\langle x, x_k^*\rangle = \langle y, x_k^*\rangle$ for any $k = 1, \ldots, n$, so by (6.3.9),

$$\begin{aligned}
\phi_n(P_n^* x) &= \inf\{\phi(y) : y \in X, P_n^* y = P_n^* x\} \\
&\ge \inf\{\sup_{1\le k\le n} \langle y, x_k^*\rangle + a_k : y \in X, P_n^* y = P_n^* x\}
\end{aligned}$$

$$
\begin{aligned}
&= \inf\{\sup_{1\le k\le n} \langle x, x_k^*\rangle + a_k : y\in X, P_n^* y = P_n^* x\}\\
&= \sup_{1\le k\le n} \langle x, x_k^*\rangle + a_k.
\end{aligned}
$$

Since the latter expression tends to $\phi(x)$ as $n\to\infty$, we obtain the desired monotone convergence $\phi_n(P_n^*x) \nearrow \phi(x)$. □

Proposition 6.3.10. *Let X be a Banach space with a dual X^* and let $Y\subset X^*$ be a finite-dimensional linear subspace. Assume further that $\Phi,\Psi : X\to\mathbb{R}_+$ are convex continuous functions and let $\Phi_Y,\Psi_Y : X_Y\to\mathbb{R}_+$ be defined by* (6.3.7)*. Then*

$$
|\mathcal{H}^{\mathbb{T}}_{X_Y}|_{\Phi_Y,\Psi_Y} \le |\mathcal{H}^{\mathbb{T}}_X|_{\Phi,\Psi}.
$$

Proof. Recall that

$$
|\mathcal{H}^{\mathbb{T}}_{X_Y}|_{\Phi_Y,\Psi_Y} = \sup_{f\in F^{\mathrm{step}}_{X_Y}} \frac{\int_{\mathbb{T}}\Psi_Y(\mathcal{H}^{\mathbb{T}}_{X_Y} f)\,\mathrm{d}\mu}{\int_{\mathbb{T}}\Phi_Y(f)\,\mathrm{d}\mu},
$$

where μ is the Lebesgue measure on $\mathbb{T}$. Fix $f\in F^{\mathrm{step}}_{X_Y}$ and $\varepsilon>0$. Let $(\tilde{x}_n)_{n=1}^N\subset X_Y$ be the range of f. For each $n=1,\dots,N$ we define $x_n\in X$ to be such that $P^*x_n=\tilde{x}_n$ and $\Phi(x_n)\le(1+\varepsilon)\Phi_Y(\tilde{x}_n)$ (existence of such x_n follows from the fact that $\mathrm{Ran}(P^*)=X_Y$); we define $g:\mathbb{T}\to X$ to be such that $f(s)=\tilde{x}_n$ if and only if $g(s)=x_n$, $s\in\mathbb{T}$. Then $\Phi_Y(f)=\Phi_Y(P^*g)$ and $\Psi_Y(\mathcal{H}^{\mathbb{T}}_{X_Y}f)=\Psi_Y(\mathcal{H}^{\mathbb{T}}_{X_Y}P^*g)=\Psi_Y(P^*\mathcal{H}^{\mathbb{T}}_X g)$ for any $s\in\mathbb{T}$ by the definition of the Hilbert transform on the torus. Therefore

$$
\begin{aligned}
\frac{\int_{\mathbb{T}}\Psi_Y(\mathcal{H}^{\mathbb{T}}_{X_Y} f)\,\mathrm{d}\mu}{\int_{\mathbb{T}}\Phi_Y(f)\,\mathrm{d}\mu} = \frac{\int_{\mathbb{T}}\Psi_Y(P^*\mathcal{H}^{\mathbb{T}}_X g)\,\mathrm{d}\mu}{\int_{\mathbb{T}}\Phi_Y(P^*g)\,\mathrm{d}\mu} &\overset{(*)}{\le} (1+\varepsilon)\frac{\int_{\mathbb{T}}\Psi(\mathcal{H}^{\mathbb{T}}_X g)\,\mathrm{d}\mu}{\int_{\mathbb{T}}\Phi(g)\,\mathrm{d}\mu}\\
&\overset{(**)}{\le} (1+\varepsilon)|\mathcal{H}^{\mathbb{T}}_X|_{\Phi,\Psi},
\end{aligned}
$$

where $(*)$ follows from the fact that $\Phi(g(s))\le(1+\varepsilon)\Phi_Y(f(s))$ for any $s\in\mathbb{T}$ and from the fact that $\Psi_Y(P^*\cdot)\le\Psi(\cdot)$ on X, while $(**)$ follows from the definition of $|\mathcal{H}^{\mathbb{T}}_X|_{\Phi,\Psi}$. Since $f\in F^{\mathrm{step}}_{X_Y}$ and $\varepsilon>0$ were arbitrary, the claim follows. □

The final ingredient is the following well-known statement from the theory of stochastic integration.

Lemma 6.3.11. *Let $d\ge 1$ and let M be a martingale with values in $\mathbb{R}^d$ satisfying the condition $\mathbb{E}M^*_\infty<\infty$. Let $V:\mathbb{R}_+\times\Omega\to\mathbb{R}^d$ be a predictable and bounded process. Then $V\cdot M := \int\langle V,\mathrm{d}M\rangle$ is a well-defined martingale and $\mathbb{E}(V\cdot M)^*_\infty<\infty$.*

Equipped with the above statements, we are ready for the study of our main result. We should point out that the main difficulty lies in proving the inequality (6.3.1) for finite-dimensional Banach spaces. The novelty in comparison to other results from the literature is that we work under slightly different condition of weak differential subordination and orthogonality; therefore, though at some places the arguments might look similar to, for instance, those appearing in [15], there is no apparent connection between them.

Proof of (6.3.1) *for finite-dimensional X.* We split the reasoning into several intermediate parts.

Step 1. Some reductions. First assume that the function $U_{\Phi,\Psi}$ (defined in Theorem 6.3.2) is continuous and twice differentiable. Since N has continuous paths almost surely, we may assume that N is a bounded martingale: this is due to a simple stopping time argument combined with Lemma 6.3.4. Moreover, we may assume that $\mathbb{E}\Phi(M_t) < \infty$, since otherwise there is nothing to prove. Let d be the dimension of X. Then analogously to the proof of Proposition 4.4.3 we can find a continuous time-change $\tau = (\tau_s)_{s\geq 0}$ and redefine $M := M\circ\tau$ and $N := N\circ\tau$, so that the following holds. For some $2d$-dimensional standard Brownian motion W on an extended probability space $(\widetilde{\Omega},\widetilde{\mathscr{F}},\widetilde{\mathbb{P}})$ equipped with an extended filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathscr{F}}_t)_{t\geq 0}$, there exist progressively measurable processes $\phi,\psi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(\mathbb{R}^{2d}, X)$ such that $M^c = \phi\cdot W$ and $N = \psi\cdot W$, where $M = M^c + M^d$ is the Meyer-Yoeurp decomposition of M (see Section 2.2.3). In addition, the arguments in the proof of Proposition 4.4.3 also yield the identities $[M\circ\tau] = [M]\circ\tau$, $[N\circ\tau] = [N]\circ\tau$ and $[M\circ\tau, N\circ\tau] = [M,N]\circ\tau$, so the weak differential subordination and orthogonality are not ruined under the time-change.

Now, for each $n\geq 1$, introduce the stopping time

$$\sigma_n := \inf\{t\geq 0 : M_t > n\}. \tag{6.3.10}$$

By Lemma 6.3.4 it is sufficient to show that

$$\mathbb{E}\Psi(N_{t\wedge\sigma_n}) \leq |\mathscr{H}_X^{\mathbb{T}}|_{\Phi,\Psi}\mathbb{E}\Phi(M_{t\wedge\sigma_n}) \tag{6.3.11}$$

for any $n \geq 1$. Actually, passing to M/n, N/n, we see that it is enough to show the above estimate for $n = 1$. For the sake of notational convenience, we redefine $M := M^{\sigma_1}$ and $N := N^{\sigma_1}$ and observe that it suffices to show $\mathbb{E}U_{\Phi,\Psi}(M_t + iN_t) \geq 0$, since then (6.3.11) follows at once from the majorization property of $U_{\Phi,\Psi}$.

Step 2. Application of Itô's formula. Let $(e_n)_{n=1}^d$ be a basis of X, and $(e_n^*)_{n=1}^d$ be the corresponding dual basis. Then by the Itô formula (2.12.1), we get

$$\begin{aligned}\mathbb{E}U_{\Phi,\Psi}(M_t + iN_t) = \mathbb{E}U_{\Phi,\Psi}(M_0 + iN_0) &+ \mathbb{E}\int_0^t \langle \partial_x U_{\Phi,\Psi}(M_{s-} + iN_{s-}), \mathrm{d}M_s\rangle\\ &+ \mathbb{E}\int_0^t \langle \partial_{ix} U_{\Phi,\Psi}(M_{s-} + iN_{s-}), \mathrm{d}N_s\rangle\\ &+ \mathbb{E}I_1 + \mathbb{E}I_2,\end{aligned} \tag{6.3.12}$$

where $\partial_x U_{\Phi,\Psi}(\cdot), \partial_{ix}U_{\Phi,\Psi}(\cdot) \in X^*$ are the corresponding Fréchet derivatives of $U_{\Phi,\Psi}$ in the real and the imaginary subspaces of $X + iX$ respectively,

$$I_1 = \sum_{0\leq s\leq t} (\Delta U_{\Phi,\Psi}(M_s + iN_s) - \langle \partial_x U_{\Phi,\Psi}(M_{s-} + iN_{s-}), \Delta M_s\rangle),$$

and

$$
\begin{aligned}
I_2 &= \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial e_j}\,\mathrm{d}[\langle M^c, e_i^*\rangle, \langle M^c, e_j^*\rangle]_s \\
&\quad + \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_j}\,\mathrm{d}[\langle N, e_i^*\rangle, \langle N, e_j^*\rangle]_s \\
&\quad + \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial ie_j}\,\mathrm{d}[\langle M^c, e_i^*\rangle, \langle N, e_j^*\rangle]_s \\
&= \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial e_j}\langle \phi^*(s)e_i^*, \phi^*(s)e_j^*\rangle\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_j}\langle \psi^*(s)e_i^*, \psi^*(s)e_j^*\rangle\,\mathrm{d}s \\
&\quad + \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial ie_j}\langle \phi^*(s)e_i^*, \psi^*(s)e_j^*\rangle\,\mathrm{d}s.
\end{aligned}
$$

Step 3. Analysis of the terms on the right of (6.3.12). Let us first show that

$$
\mathbb{E}\int_0^t \langle \partial_x U_{\Phi,\Psi}(M_{s-}+iN_{s-}), \mathrm{d}M_s\rangle + \mathbb{E}\int_0^t \langle \partial_{ix} U_{\Phi,\Psi}(M_{s-}+iN_{s-}), \mathrm{d}N_s\rangle
$$

exists and equals zero. First notice that since $M = M^{\sigma_1}$, the variable M_{s-} is bounded by 1 for any $0 \le s \le \sigma_1$. Furthermore, as we have assumed above, the process N is also bounded. Since $U_{\Phi,\Psi}$ is twice differentiable, both $\partial_x U_{\Phi,\Psi}(\cdot)$ and $\partial_{ix} U_{\Phi,\Psi}(\cdot)$ are continuous functions, so $s \mapsto \partial_x U_{\Phi,\Psi}(M_{s-}+iN_{s-})$ and $s \mapsto \partial_{ix} U_{\Phi,\Psi}(M_{s-}+iN_{s-})$ define bounded processes on $0 \le s \le \sigma_1$. Furthermore, it is easy to see that

$$
\mathbb{E}M_t^* = \mathbb{E}M_{t\wedge\sigma_1}^* \le \mathbb{E}\|M_{t\wedge\sigma_1}\| + 1 \le \mathbb{E}\|M_t\| + 1 < \infty,
$$

and hence by Lemma 6.3.11,

$$
\begin{aligned}
t &\mapsto \int_0^t \langle \partial_x U_{\Phi,\Psi}(M_{s-}+iN_{s-})\mathbf{1}_{s\in[0,\sigma_1]}, \mathrm{d}M_s\rangle, \quad t \ge 0, \\
t &\mapsto \int_0^t \langle \partial_{ix} U_{\Phi,\Psi}(M_{s-}+iN_{s-})\mathbf{1}_{s\in[0,\sigma_1]}, \mathrm{d}N_s\rangle, \quad t \ge 0,
\end{aligned}
\tag{6.3.13}
$$

define martingales. Moreover, with probability 1,

$$
\begin{aligned}
\int_0^t \langle \partial_x U_{\Phi,\Psi}(M_{s-}+iN_{s-})\mathbf{1}_{s\in[0,\sigma_1]}, \mathrm{d}M_s\rangle &= \int_0^t \langle \partial_x U_{\Phi,\Psi}(M_{s-}+iN_{s-}), \mathrm{d}M_s\rangle, \\
\int_0^t \langle \partial_{ix} U_{\Phi,\Psi}(M_{s-}+iN_{s-})\mathbf{1}_{s\in[0,\sigma_1]}, \mathrm{d}N_s\rangle &= \int_0^t \langle \partial_{ix} U_{\Phi,\Psi}(M_{s-}+iN_{s-}), \mathrm{d}N_s\rangle,
\end{aligned}
$$

since $M = M^{\sigma_1}$ and $N = N^{\sigma_1}$, and consequently the expectations of the above integrals vanish. Let us now show that $I_1, I_2 \ge 0$ almost surely. For the first term, the argument is simple: by (6.2.2), each summand in I_1 is nonnegative. The analysis of I_2 is slightly more complex. By Proposition 6.2.11, we get that $N \overset{w}{\ll} M^c$ and M^c, N are orthogonal, so Proposition 6.3.5 implies the existence of a progressively measurable operator-valued process $A : \mathbb{R}_+ \times \Omega \to \mathscr{L}(\mathbb{R}^d)$ such that $\|A\| \le 1$, $\psi^* = A\phi^*$, and $P_{\mathrm{Ran}(\phi^*)}A$ is skew-symmetric on $\mathbb{R}_+ \times \Omega$ (here $P_{\mathrm{Ran}(\phi^*)}$ is an orthoprojection on $\mathrm{Ran}(\phi^*)$). Thus it is enough to show that

$$\begin{aligned}\sum_{i,j=1}^{d} &\frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial e_j}\langle \phi^*(s)e_i^*, \phi^*(s)e_j^*\rangle \,\mathrm{d}s\\ &+\sum_{i,j=1}^{d} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_j}\langle \psi^*(s)e_i^*, \psi^*(s)e_j^*\rangle \,\mathrm{d}s\\ &+2\sum_{i,j=1}^{d} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial ie_j}\langle \phi^*(s)e_i^*, P_{\mathrm{Ran}(\phi^*)}A\phi^*(s)e_j^*\rangle \,\mathrm{d}s \ge 0.\end{aligned} \tag{6.3.14}$$

By the spectral theory of skew-symmetric matrices (see e.g. [191, Corollary 2]) there exist $L \ge 0$, positive numbers $(\lambda_n)_{n=1}^L$ and an orthonormal basis $(h_n)_{n=1}^{2d}$ of $\mathbb{R}^{2d}$ such that $P_{\mathrm{Ran}(\phi^*)}Ah_{2n-1} = \lambda_n h_{2n}$ and $P_{\mathrm{Ran}(\phi^*)}Ah_{2n} = -\lambda_n h_{2n-1}$ for all $n = 1,\dots,L$, and $P_{\mathrm{Ran}(\phi^*)}Ah_n = 0$ for all $2L < n \le d$. Moreover, the condition $\|A\| \le 1$ implies that $|\lambda_1|,\dots,|\lambda_L| \le 1$, and since $(\mathrm{Ran}(\phi^*))^\perp$ is a zero eigenspace of $P_{\mathrm{Ran}(\phi^*)}A$ (see the construction of A in the proof of Proposition 6.3.5), we conclude that $h_n \in \mathrm{Ran}(\phi^*)$ for $n = 1,2,\dots,2L$. By a usual orthogonalization procedure, we may assume that there exists $K \ge 2L$ such that $h_n \in \mathrm{Ran}(\phi^*)$ for $2L < n \le K$ and $h_n \perp \mathrm{Ran}(\phi^*)$ for $K < n \le 2d$ (then K is the dimension of $\mathrm{Ran}(\phi^*)$). Notice that X^* is d-dimensional, so $\mathrm{Ran}(\phi^*)$ is at most d-dimensional and hence obviously $K \le d$. Due to Lemma 2.11.2, the expression (6.3.14) does not depend on the basis $(e_n)_{n=1}^d$ (and the corresponding dual basis $(e_n^*)_{n=1}^d$), so we can choose a basis $(e_n)_{n=1}^d$ such that $\phi^* e_n^* = h_n$ for all $n = 1,\dots,K$ and $\phi^* e_n^* = 0$ for all $K < n \le d$ (such a basis exists since $\mathrm{span}\{h_1,\dots,h_K\} = \mathrm{Ran}(\phi^*)$). Then (6.3.14) becomes

$$\begin{aligned}\sum_{i,j=1}^{K} &\frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial e_j}\langle h_i, h_j\rangle\\ &+\sum_{i,j=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_j}\langle \psi^* e_i^*, \psi^* e_j^*\rangle\\ &+2\sum_{i,j=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial ie_j}\langle h_i, P_{\mathrm{Ran}(\phi^*)}Ah_j\rangle \ge 0\end{aligned} \tag{6.3.15}$$

(The second sum is up to K due to the fact that $\phi^* x^* = 0$ implies $\psi^* x^* = 0$ for any $x^* \in X^*$, see (6.3.2)). Notice that the bilinear form $V : X \times X \to \mathbb{R}$ defined by

$$V(x,y) := -\frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ix \partial iy}, \quad x, y \in X,$$

is nonnegative by Theorem 6.3.2 and symmetric by the definition. Moreover, by (6.3.2),

$$\langle \psi^* x^*, \psi^* x^* \rangle = \|\psi^* x^*\|^2 \le \|\phi^* x^*\|^2 = \langle \phi^* x^*, \phi^* x^* \rangle, \quad \text{for } x^* \in X^*.$$

Therefore Corollary 2.11.3 yields

$$\begin{aligned}\sum_{i,j=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_j} \langle \psi^* e_i^*, \psi^* e_j^* \rangle &\ge \sum_{i,j=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_j} \langle \phi^* e_i^*, \phi^* e_j^* \rangle \\ &= \sum_{i,j=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_j} \langle h_i, h_j \rangle,\end{aligned}$$

so (6.3.15) is not less than

$$\begin{aligned}&\sum_{i,j=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial e_j} \langle h_i, h_j \rangle + \sum_{i,j=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_j} \langle h_i, h_j \rangle \\ &\qquad + 2\sum_{i,j=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial ie_j} \langle h_i, P_{\mathrm{Ran}(\phi^*)} A h_j \rangle \\ &= \sum_{i=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial e_i} \langle h_i, h_i \rangle + \sum_{i=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_i} \langle h_i, h_i \rangle \\ &\qquad + 2\sum_{i,j=1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial ie_j} \langle h_i, P_{\mathrm{Ran}(\phi^*)} A h_j \rangle.\end{aligned}$$

The latter expression consists of two parts:

$$\begin{aligned}&\sum_{i=1}^{2L} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial e_i} + \sum_{i=1}^{2L} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_i} \\ &\quad + 2\sum_{n=1}^{L} \lambda_n \Big(\frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_{2n-1} \partial ie_{2n}} - \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_{2n} \partial ie_{2n-1}} \Big) \\ &= \sum_{n=1}^{L} \Big\{ \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_{2n-1} \partial e_{2n-1}} + \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_{2n} \partial e_{2n}} \\ &\quad + 2\lambda_n \Big(\frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_{2n-1} \partial ie_{2n}} - \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_{2n} \partial ie_{2n-1}} \Big) \\ &\quad + \Big(\frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_{2n-1} \partial ie_{2n-1}} + \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_{2n} \partial ie_{2n}} \Big) \Big\}\end{aligned} \tag{6.3.16}$$

and

$$\begin{aligned}&\sum_{i=2L+1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial e_i} + \sum_{i=2L+1}^{K} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_i} \\ &= \sum_{i=2L+1}^{K} \Big(\frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial e_i \partial e_i} + \frac{\partial^2 U_{\Phi,\Psi}(M_{s-}+iN_{s-})}{\partial ie_i \partial ie_i} \Big).\end{aligned} \tag{6.3.17}$$

Now, the expression (6.3.16) is nonnegative by Corollary 6.3.7 and (6.3.17) is nonnegative by Remark 6.2.7. This gives $I_2 \geq 0$. Putting all the above facts together, we obtain

$$\mathbb{E}U_{\Phi,\Psi}(M_t + iN_t) \geq \mathbb{E}U_{\Phi,\Psi}(M_0 + iN_0).$$

However, by Remark 6.2.2, we have $N_0 = 0$ almost surely, so Theorem 6.3.2 implies

$$\mathbb{E}U_{\Phi,\Psi}(M_0 + iN_0) = \mathbb{E}U_{\Phi,\Psi}(M_0) \geq 0,$$

which completes the proof.

Step 4. Now we assume that $U_{\Phi,\Psi}$ is general (i.e., not necessarily twice integrable). We will use a standard mollification argument. Let $\phi : X + iX \to \mathbb{R}_+$ be a C^∞ radial function with compact support such that $\int_{X+iX} \phi(s)\,\mathrm{d}s = 1$. For each $\varepsilon > 0$, define $U^\varepsilon_{\Phi,\Psi} : X + iX \to \mathbb{R}$ via the convolution

$$U^\varepsilon_{\Phi,\Psi}(x + iy) := \int_{X+iX} U_{\Phi,\Psi}(x + iy - \varepsilon s)\phi(s)\,\mathrm{d}s, \quad x, y \in X.$$

Then $U^\varepsilon_{\Phi,\Psi}$ is of class C^∞ and for any $x \in X$ we have

$$U^\varepsilon_{\Phi,\Psi}(x) = \int_{X+iX} U_{\Phi,\Psi}(x - \varepsilon s)\phi(s)\,\mathrm{d}s \geq U_{\Phi,\Psi}(x) \geq 0, \tag{6.3.18}$$

since $U_{\Phi,\Psi}$ is subharmonic (see Remark 6.2.6). Therefore, repeating the arguments from the above steps, we get

$$\begin{aligned}\mathbb{E}\int_{X+iX} \Big[|\mathcal{H}^{\mathbb{T}}_X|_{\Phi,\Psi}\Phi(M_t - \varepsilon r) - \Psi(N_t - \varepsilon u)\Big]\phi(r + iu)\,\mathrm{d}s&\\ \geq \mathbb{E}U^\varepsilon_{\Phi,\Psi}(M_t + iN_t) \geq \mathbb{E}U^\varepsilon_{\Phi,\Psi}(M_0) \geq 0,&\end{aligned} \tag{6.3.19}$$

where the latter bound follows from (6.3.18). Note that $\Psi(N_t + \varepsilon u)$ is uniformly bounded (when $r + iu$ runs over the support of ϕ) and notice that for any x, $\varepsilon \mapsto \frac{\Phi(x-\varepsilon)+\Phi(x+\varepsilon)}{2}$ is an increasing function of $\varepsilon > 0$. Furthermore, we have $\phi(r + iu) = \phi(-r + iu) \geq 0$ and hence

$$\begin{aligned}\varepsilon \mapsto &\int_{X+iX} \Phi(M_t - \varepsilon r)\phi(r + iu)\,\mathrm{d}(r + iu)\\ &= \int_{X+iX} \frac{\Phi(M_t - \varepsilon r) + \Phi(M_t + \varepsilon r)}{2}\phi(r + iu)\,\mathrm{d}(r + iu),\end{aligned} \tag{6.3.20}$$

decreases as $\varepsilon \downarrow 0$. Combining these observations with standard limiting theorems, we deduce the desired claim. □

Now we prove our main result in full generality. Of course, we will exploit an appropriate limiting procedure, which enables us to deduce the claim from its finite-dimensional version just established above.

Proof of (6.3.1) *for infinite-dimensional* X. We may assume that $\mathbb{E}\Phi(M_t) < \infty$, since otherwise the claim is obvious. Suppose that $(Y_n)_{n\geq 1}$ is a sequence of finite-dimensional subspaces of X^* such that $Y_n \subset Y_{n+1}$ for any $n \geq 1$ and $\overline{\cup_{n\geq 1} Y_n} = X^*$. For each $n \geq 1$ define $X_n := Y_n^*$, let $P_n : Y_n \hookrightarrow X^*$ be the corresponding embedding operator and let $P_n^* : X \to X_n$ be its adjoint (recall that X is reflexive). Finally, define $\Phi_n, \Psi_n : X_n \to \mathbb{R}_+$ by the formulae

$$\Phi_n(\tilde{x}) = \inf\{\Phi(x) : x \in X, P_n^* x = \tilde{x}\}, \quad \Psi_n(\tilde{x}) = \inf\{\Psi(x) : x \in X, P_n^* x = \tilde{x}\},$$

for $\tilde{x} \in X_n$. In the light of Lemma 6.3.8, both Φ_n and Ψ_n are convex functions. Moreover, by Proposition 6.3.10,

$$|\mathcal{H}^{\mathbb{T}}_{X_n}|_{\Phi_n,\Psi_n} \leq |\mathcal{H}^{\mathbb{T}}_X|_{\Phi,\Psi}. \tag{6.3.21}$$

Let us show that the processes $P_n^* M$ and $P_n^* N$ are orthogonal for each $n \geq 1$. By the very definition, we must prove that for a fixed functional $x^* \in X_n^*$, the local martingales $\langle P_n^* M, x^*\rangle$ and $\langle P_n^* N, x^*\rangle$ are orthogonal. This follows at once from orthogonality of M, N and the identities

$$\langle P_n^* M, x^*\rangle = \langle M, P_n x^*\rangle, \qquad \langle P_n^* N, x^*\rangle = \langle N, P_n x^*\rangle. \tag{6.3.22}$$

These identities also immediately give the weak differential subordination $P_n^* N \overset{w}{\ll} P_n^* M$, since M, N enjoy this condition. Finally, observe that by Lemma 6.3.8, we have $\mathbb{E}\Phi_n(P_n^* M_t) \leq \mathbb{E}\Phi(M_t) < \infty$. Therefore, applying the finite-dimensional version of (6.3.1), we see that for each $n \geq 1$,

$$\mathbb{E}\Psi_n(P_n^* N_t) \leq |\mathcal{H}^{\mathbb{T}}_{X_n}|_{\Phi_n,\Psi_n} \mathbb{E}\Phi_n(P_n^* M_t) \leq |\mathcal{H}^{\mathbb{T}}_X|_{\Phi,\Psi} \mathbb{E}\Phi_n(P_n^* M_t), \tag{6.3.23}$$

where the second passage is due to (6.3.21). Note that with probability 1 we have $\Phi_n(P_n^* M_t) \nearrow \Phi(M_t)$ and $\Psi_n(P_n^* N_t) \nearrow \Psi(N_t)$ monotonically as $n \to \infty$ by Lemma 6.3.9. This establishes the desired estimate, by Lebesgue's monotone convergence theorem. □

It remains to handle the sharpness of (6.3.1).

Proof of the estimate $|\mathcal{H}^{\mathbb{T}}_X|_{\Phi,\Psi} \leq C_{\Phi,\Psi,X}$. This follows immediately from the reasoning presented in Section 6.2.2: indeed, (6.3.1) implies the corresponding bound

$$\int_{\mathbb{T}} \Psi\big(\mathcal{H}^{\mathbb{T}}_X f\big) \mathrm{d}x \leq C_{\Phi,\Psi,X} \int_{\mathbb{T}} \Phi(f) \mathrm{d}x$$

for any step function $f : \mathbb{T} \to X$. □

Remark 6.3.12. It is easy to see that if X is finite dimensional, then there is no need for Φ to be convex. The limiting argument presented in the above proof does not need this requirement. (The only place where the convexity of Φ is used is (6.3.20); we leave to the reader the question how to avoid this issue).

6.4. APPLICATIONS

6.4.1. Hilbert transforms on $\mathbb{T}$, $\mathbb{R}$, and $\mathbb{Z}$

Let X be a Banach space and let $\Phi, \Psi : X \to \mathbb{R}_+$ be continuous functions. Let (S, Σ, μ) be a measure space, with S equal to $\mathbb{T}$, $\mathbb{R}$, or $\mathbb{Z}$. A function $f : S \to X$ is called a *step function*, if it is of the form

$$f(t) = \sum_{k=1}^{N} x_k \mathbf{1}_{A_k}(t), \quad t \in S,$$

where N is finite, $x_k \in X$ and A_k are intervals in S of a finite measure.

Definition 6.4.1. The *Hilbert transform* $\mathscr{H}_X^{\mathbb{R}}$ is a linear operator that maps a step function $f : \mathbb{R} \to X$ to the function

$$(\mathscr{H}_X^{\mathbb{R}} f)(t) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{t-s} \, \mathrm{d}s, \quad t \in \mathbb{R}. \tag{6.4.1}$$

The associated Φ, Ψ-norms $|\mathscr{H}_X^{\mathbb{R}}|_{\Phi,\Psi}$ are given by a formula similar to that used previously:

$$|\mathscr{H}_X^{\mathbb{R}}|_{\Phi,\Psi} := \inf\Big\{ c \in [0,\infty] : \int_{\mathbb{R}} \Psi(\mathscr{H}_X^{\mathbb{R}} f(s)) \, \mathrm{d}s \le c \int_{\mathbb{R}} \Phi(f(s)) \, \mathrm{d}s \text{ for all step functions } f : \mathbb{R} \to X \Big\}.$$

Definition 6.4.2. The *discrete Hilbert transform* $\mathscr{H}_X^{\mathrm{dis}}$ is a linear operator that maps a step function $f : \mathbb{Z} \to X$ to the function

$$(\mathscr{H}_X^{\mathrm{dis}} f)(t) := \frac{1}{\pi} \sum_{s \in \mathbb{Z} \setminus \{t\}} \frac{f(s)}{t-s}, \quad t \in \mathbb{Z}.$$

The associated Φ, Ψ-norms $|\mathscr{H}_X^{\mathrm{dis}}|_{\Phi,\Psi}$ are given by

$$|\mathscr{H}_X^{\mathrm{dis}}|_{\Phi,\Psi} := \inf\Big\{ c \in [0,\infty] : \sum_{s \in \mathbb{Z}} \Psi(\mathscr{H}_X^{\mathrm{dis}} f(s)) \le c \sum_{s \in \mathbb{Z}} \Phi(f(s)) \text{ for all step functions } f : \mathbb{Z} \to X \Big\}.$$

We will also need a certain variant of Φ, Ψ-norm in the periodic setting. Namely, define $|\mathscr{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$ by

$$|\mathscr{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} := \inf\Big\{ c \in [0,\infty] : \int_{\mathbb{T}} \Psi(\mathscr{H}_X^{\mathbb{T}} f(s)) \mathrm{d}s \le c \int_{\mathbb{T}} \Phi(f(s)) \mathrm{d}s \text{ for all step functions } f : \mathbb{T} \to X \text{ with } \int_{\mathbb{T}} f(s) \mathrm{d}s = 0 \Big\}.$$

The following theorem demonstrates that the norm of the Hilbert transform does not depend whether it is defined on $\mathbb{T}$, $\mathbb{R}$, or $\mathbb{Z}$.

Theorem 6.4.3. *Let X be a Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be continuous convex functions such that $\Phi(0) = 0$. Then*

$$|\mathscr{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} = |\mathscr{H}_X^{\mathbb{R}}|_{\Phi,\Psi} \le |\mathscr{H}_X^{\mathrm{dis}}|_{\Phi,\Psi} \le |\mathscr{H}_X^{\mathbb{T}}|_{\Phi,\Psi}.$$

Moreover, if Φ is symmetric, then

$$|\mathscr{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} = |\mathscr{H}_X^{\mathbb{R}}|_{\Phi,\Psi} = |\mathscr{H}_X^{\mathrm{dis}}|_{\Phi,\Psi} = |\mathscr{H}_X^{\mathbb{T}}|_{\Phi,\Psi}.$$

The proof will consist of several steps.

Proposition 6.4.4. *Let X be a Banach space and let $\Phi, \Psi : X \to \mathbb{R}_+$ be convex functions. Then we have*

$$|\mathscr{H}_X^{\mathbb{R}}|_{\Phi,\Psi} \le |\mathscr{H}_X^{\mathrm{dis}}|_{\Phi,\Psi} \le |\mathscr{H}_X^{\mathbb{T}}|_{\Phi,\Psi}.$$

Proof. Introduce yet another Hilbert-type operator acting on step functions $f : \mathbb{R} \to \mathbb{R}$ by

$$(\mathscr{H}_X^{\mathbb{R},\mathrm{dis}} f)(t) := \frac{1}{\pi} \sum_{s \in \mathbb{Z}\setminus\{0\}} \frac{f(t-s)}{s}, \quad t \in \mathbb{R},$$

and define its Φ, Ψ-norm analogously. We will first prove that $|\mathscr{H}_X^{\mathbb{R}}|_{\Phi,\Psi} \le |\mathscr{H}_X^{\mathbb{R},\mathrm{dis}}|_{\Phi,\Psi}$. To this end, fix a step function f on $\mathbb{R}$ and define its ε-dilation by $f_\varepsilon(\cdot) := f(\varepsilon\cdot)$. Then similarly to [103, Theorem 4.3], we have

$$\begin{aligned}
\frac{\int_{\mathbb{R}} \Psi((\mathscr{H}_X^{\mathbb{R},\mathrm{dis}} f_\varepsilon)(s))\,\mathrm{d}s}{\int_{\mathbb{R}} \Phi(f_\varepsilon(s))\,\mathrm{d}s} &= \frac{\int_{\mathbb{R}} \Psi(\pi^{-1}\sum_{k\in\mathbb{Z}\setminus\{0\}} f_\varepsilon(s-k)/k)\,\mathrm{d}s}{\int_{\mathbb{R}} \Phi(f_\varepsilon(s))\,\mathrm{d}s}\\
&= \frac{\int_{\mathbb{R}} \Psi(\pi^{-1}\sum_{k\in\mathbb{Z}\setminus\{0\}} \varepsilon f(\varepsilon s-\varepsilon k)/(\varepsilon k))\,\mathrm{d}(\varepsilon s)}{\int_{\mathbb{R}} \Phi(f(\varepsilon s))\,\mathrm{d}(\varepsilon s)}\\
&= \frac{\int_{\mathbb{R}} \Psi(\pi^{-1}\sum_{k\in\mathbb{Z}\setminus\{0\}} \varepsilon f(s-\varepsilon k)/(\varepsilon k))\,\mathrm{d}s}{\int_{\mathbb{R}} \Phi(f(s))\,\mathrm{d}s}.
\end{aligned}$$

Since $\frac{1}{\pi}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{f(s-\varepsilon k)}{\varepsilon k}\varepsilon \to \mathscr{H}_X^{\mathbb{R}} f(s)$ for a.e. $s \in \mathbb{R}$, Fatou's lemma yields

$$\begin{aligned}
|\mathscr{H}_X^{\mathbb{R}}|_{\Phi,\Psi} = \sup_{f \in F_X^{\mathrm{step}}} \frac{\int_{\mathbb{R}} \Psi(\mathscr{H}_X^{\mathbb{R}} f(s))\,\mathrm{d}s}{\int_{\mathbb{R}} \Phi(f(s))\,\mathrm{d}s} &\le \sup_{f \in F_X^{\mathrm{step}}} \liminf_{\varepsilon\to 0} \frac{\int_{\mathbb{R}} \Psi((\mathscr{H}_X^{\mathbb{R},\mathrm{dis}} f_\varepsilon)(s))\,\mathrm{d}s}{\int_{\mathbb{R}} \Phi(f_\varepsilon(s))\,\mathrm{d}s}\\
&\le |\mathscr{H}_X^{\mathbb{R},\mathrm{dis}}|_{\Phi,\Psi} = |\mathscr{H}_X^{\mathrm{dis}}|_{\Phi,\Psi}.
\end{aligned}$$

where the latter equality follows from the direct repetition of the arguments from [103, Theorem 4.2]. This gives us the first inequality of the assertion. The proof of the fact that $|\mathscr{H}_X^{\mathrm{dis}}|_{\Phi,\Psi} \le |\mathscr{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$ follows word-by-word from the infinite-dimensional analogue of the recent approach of Bañuelos and Kwaśnicki [12] combined with the estimate (6.3.1). □

Theorem 6.4.5. *Let X be a Banach space and let $\Phi, \Psi : X \to \mathbb{R}_+$ be continuous functions. Then $|\mathscr{H}_X^{\mathbb{R}}|_{\Phi,\Psi} \le |\mathscr{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$.*

Proof. Fix a step function $f : \mathbb{R} \to X$. It takes only a finite number of values, so we may assume that X is finite dimensional (which will guarantee the validity of the reasoning below). For any $n \geq 1$, introduce the function $g_n : \mathbb{R} \to X$ by

$$g_n(x) = \frac{1}{2\pi n} \int_{-\pi n}^{\pi n} f(t) \cot \frac{x-t}{2n} \mathrm{d}t, \quad x \in \mathbb{R}.$$

It follows from the observation of Zygmund [194, p. 256] that $g_n \to \mathcal{H}_X^{\mathbb{R}} f$ a.e. as $n \to \infty$. On the other hand, the function $x \mapsto g_n(nx)$, $|x| \leq \pi$, is precisely the periodic Hilbert transform of the function $x \mapsto f(nx)$, $|x| \leq \pi$ (see (6.2.1)). Therefore, it is also the periodic Hilbert transform of the *centered* function

$$x \mapsto f(nx) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(ns) \mathrm{d}s, \qquad |x| \leq \pi.$$

Clearly, the latter is a step function. Consequently, by Fatou's lemma and the definition of $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$,

$$\begin{aligned}
\int_{\mathbb{R}} \Psi(\mathcal{H}_X^{\mathbb{R}} f) \mathrm{d}x &\leq \liminf_{n\to\infty} \int_{-\pi n}^{\pi n} \Psi(g_n) \mathrm{d}x \\
&= \liminf_{n\to\infty} \int_{-\pi}^{\pi} \Psi(g_n(nx)) n \mathrm{d}x \\
&\leq |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \liminf_{n\to\infty} \int_{-\pi}^{\pi} \Phi\left(f(nx) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(ns) \mathrm{d}s \right) n \mathrm{d}x \\
&= |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \liminf_{n\to\infty} \int_{-\pi n}^{\pi n} \Phi\left(f(x) - \frac{1}{2\pi n} \int_{-\pi n}^{\pi n} f(s) \mathrm{d}s \right) \mathrm{d}x.
\end{aligned}$$

However, $\frac{1}{2\pi n} \int_{-\pi n}^{\pi n} f(s) \mathrm{d}s \to 0$ by the fact that f is a step function. Therefore, again using this property of f and the continuity of Φ, the last expression of the above chain equals

$$|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \int_{\mathbb{R}} \Phi(f) \mathrm{d}x.$$

Since f was arbitrary, the result follows. □

Now we turn our attention to the estimate in the reverse direction. We start from the observation that it does not hold true if $\Phi(0) > 0$ and $\Psi \neq 0$. Indeed, if $\Phi(0) > 0$, then $\int_{\mathbb{R}} \Phi(f) \mathrm{d}x = \infty$ for any step function and hence $|\mathcal{H}^{\mathbb{R}}|_{\Phi,\Psi} = 0$. On the other hand, the condition $\Psi \neq 0$ implies that $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} > 0$: it is easy to construct a step function $f : \mathbb{T} \to X$ of mean zero for which $\int_{\mathbb{R}} \Psi(\mathcal{H}^{\mathbb{T}} f) \mathrm{d}x > 0$.

In other words, the inequality $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \leq |\mathcal{H}_X^{\mathbb{R}}|_{\Phi,\Psi}$ fails, because of obvious reasons, if $\Phi(0) > 0$ and $\Psi \neq 0$. If Ψ is identically 0, then the estimate holds true: the reason is even more trivial – both sides are zero. It remains to study the key possibility when $\Phi(0) = 0$ and $\Psi \neq 0$.

Theorem 6.4.6. *Let X be a Banach space and let $\Phi, \Psi : X \to \mathbb{R}_+$ be arbitrary continuous functions such that $\Phi(0) = 0$ and $\Psi \neq 0$. Then $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \leq |\mathcal{H}_X^{\mathbb{R}}|_{\Phi,\Psi}$.*

Proof. As was mentioned above, the assumption $\Psi \neq 0$ implies $|\mathscr{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} > 0$. For the sake of clarity, we split the reasoning into a few separate parts.

Step 1. Auxiliary analytic maps. Let D denote the open unit disc of $\mathbb{C}$ and let $H = R \times (0,\infty)$ be the upper halfplane. Define $K : D \cap H \to H$ by the formula $K(z) = -(1-z)^2/(4z)$. It is not difficult to verify that K is conformal and hence so is its inverse L. Let us extend L to the continuous function on $\overline{H}$. It is easy to see that $L(z) \to 0$ as $z \to \infty$. Furthermore, L maps the interval $[0,1]$ onto $\{e^{i\theta} : 0 \le \theta \le \pi\}$. More precisely, we have the following formula: if $x \in [0,1]$, then

$$L(x) = e^{i\theta}, \text{ where } \theta \in [0,\pi] \text{ is uniquely determined by } x = \sin^2(\theta/2). \tag{6.4.2}$$

In addition, L maps the set $\mathbb{R} \setminus [0,1]$ onto the open interval $(-1,1)$; precisely, we have the identity

$$L(x) = \begin{cases} 1 - 2x - 2\sqrt{x^2 - x} & \text{if } x < 0, \\ 1 - 2x + 2\sqrt{x^2 - x} & \text{if } x > 1. \end{cases} \tag{6.4.3}$$

In particular, we easily check that for any $\delta > 0$, the function L is bounded away from 1 outside any interval of the form $[-\delta, 1+\delta]$ and $|L(x)| = O(|x|^{-1})$ as $x \to \pm\infty$.

Step 2. A function on $\mathbb{T}$ and its extension to a disc. Fix a positive number ε and pick a step function $f : \mathbb{T} \to X$ of integral 0 such that

$$\int_{\mathbb{T}} \Psi(\mathscr{H}_X^{\mathbb{T}} f)\mathrm{d}x > (|\mathscr{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} - \varepsilon) \cdot \int_{\mathbb{T}} \Phi(f)\mathrm{d}x.$$

We may assume that X is finite-dimensional, restricting to the range of f if necessary. Given a big number $R > 0$, consider a continuous function $\kappa^R : X \to [0,1]$ equal to 1 on $B(0,R)$ and equal to 0 outside $B(0,2R)$. Set $\Psi^R(x) = \Psi(x) \cdot \kappa^R(x)$ for $x \in X$. Note that Ψ^R is uniformly continuous, since it is continuous and supported on the compact ball $B(0,2R)$ (recall that X is finite dimensional). By Lebesgue's monotone convergence theorem, if R is sufficiently big, we also have

$$\int_{\mathbb{T}} \Psi^R(\mathscr{H}_X^{\mathbb{T}} f)\mathrm{d}x > (|\mathscr{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} - \varepsilon) \cdot \int_{\mathbb{T}} \Phi(f)\mathrm{d}x. \tag{6.4.4}$$

There is an analytic function $F : D \to X + iX$ with the property that the radial limit $\lim_{r \to 1-} F(re^{i\theta})$ is equal to $f(e^{i\theta}) + i\mathscr{H}_X^{\mathbb{T}} f(e^{i\theta})$ for almost all $|\theta| \le \pi$. Note that we have

$$F(0) = \frac{1}{2\pi}\int_{\mathbb{T}} f\mathrm{d}x + i \cdot 0 = 0 \tag{6.4.5}$$

and that the "real part" of F is bounded (by the supremum norm of f). Consider the analytic function $M_n : H \to X + iX$ given by the composition

$$M_n(z) = F(L^{2n}(z))$$

and decompose it as $M_n(z) = \operatorname{Re} M_n(z) + i\operatorname{Im} M_n(z)$, with $\operatorname{Re} M_n$ and $\operatorname{Im} M_n$ taking values in X. Observe that for each n the function $\operatorname{Re} M_n$ is bounded by the supremum norm of f (which is directly inherited from the "real part" of the function

F). In addition, the function $h = \mathbf{1}_{[0,1]}\mathrm{Re}\, M_n$ is a step function (with the number of steps depending on n and going to infinity). Since $\lim_{z\to\infty} L(z) = 0$, we have $\lim_{z\to\infty} M_n(z) = 0$ and therefore $\mathcal{H}_X^{\mathbb{T}}\mathrm{Re}\, M_n(x) = \mathrm{Im}\, M_n(x)$ for $x \in \mathbb{R}$.

Step 3. Calculations. We compute that

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(h(x))\,\mathrm{d}x &= \int_0^1 \Phi(\mathrm{Re}\, M_n(x)))\,\mathrm{d}x \\
&= \int_0^1 \Psi\big(f(L^{2n}(x))\big)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\pi} \Phi\Big(f(e^{2in\theta})\Big)\sin\theta\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2n\pi} \Phi\Big(f(e^{i\theta})\Big)\sin\Big(\frac{\theta}{2n}\Big)\frac{\mathrm{d}\theta}{2n} \\
&= \frac{1}{2}\int_0^{2\pi} \Phi\Big(f(e^{i\theta})\Big)\sum_{k=0}^{n-1}\sin\Big(\frac{k\pi}{n}+\frac{\theta}{2n}\Big)\frac{\mathrm{d}\theta}{2n} \\
&= \frac{1}{2}\int_0^{2\pi} \Phi\Big(f(e^{i\theta})\Big)\frac{\cos\big(\frac{\theta-\pi}{n}\big)}{2n\sin\big(\frac{\pi}{2n}\big)}\mathrm{d}\theta \\
&\xrightarrow{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \Phi\Big(f(e^{i\theta})\Big)\mathrm{d}\theta.
\end{aligned} \tag{6.4.6}$$

Now, let us similarly handle the integral $\int_{\mathbb{R}} \Psi^K(\mathcal{H}^{\mathbb{R}}h)\mathrm{d}x$. We have

$$\begin{aligned}
&\int_{\mathbb{R}} \Psi^R\big(\mathcal{H}_X^{\mathbb{R}}h(x)\big)\,\mathrm{d}x \\
&\quad\geq \int_0^1 \Psi^R\big(\mathcal{H}_X^{\mathbb{R}}h(x)\big)\,\mathrm{d}x \\
&\quad= \int_0^1 \Psi^R(\mathcal{H}_X^{\mathbb{R}}\mathrm{Re}\, M_n - \mathcal{H}_X^{\mathbb{R}}(\mathbf{1}_{\mathbb{R}\setminus[0,1]}\mathrm{Re}\, M_n))\mathrm{d}x \\
&\quad= \int_0^1 \Psi^R(\mathcal{H}_X^{\mathbb{R}}\mathrm{Re}\, M_n)\mathrm{d}x \\
&\qquad+ \int_0^1 \Big[\Psi^R(\mathcal{H}_X^{\mathbb{R}}\mathrm{Re}\, M_n - \mathcal{H}_X^{\mathbb{R}}(\mathbf{1}_{\mathbb{R}\setminus[0,1]}\mathrm{Re}\, M_n)) - \Psi^R(\mathcal{H}_X^{\mathbb{R}}\mathrm{Re}\, M_n)\Big]\mathrm{d}x.
\end{aligned} \tag{6.4.7}$$

Now, we have $\mathcal{H}_X^{\mathbb{R}}\mathrm{Re}\, M_n(x) = \mathrm{Im}\, M_n(x) = \mathcal{H}_X^{\mathbb{T}} f(L^{2n}(x))$, so a calculation similar to that in (6.4.6) gives

$$\int_0^1 \Psi^R(\mathcal{H}_X^{\mathbb{R}}\mathrm{Re}\, M_n)\mathrm{d}x \xrightarrow{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \Psi^R\Big(\mathcal{H}_X^{\mathbb{T}} f(e^{i\theta})\Big)\mathrm{d}\theta.$$

To deal with the last integral in (6.4.7) we will first show that $\mathcal{H}_X^{\mathbb{R}}(\mathbf{1}_{\mathbb{R}\setminus[0,1]}\mathrm{Re}\, M_n)$ converges to 0 in L^2, as $n\to\infty$. To this end, recall that X is finite-dimensional and hence it has the UMD property. Consequently, by [79, Corollary 5.2.11]

$$\int_{\mathbb{R}} |\mathcal{H}_X^{\mathbb{R}}(\mathbf{1}_{\mathbb{R}\setminus[0,1]}\mathrm{Re}\, M_n)|^2\mathrm{d}x \leq C_X \int_{\mathbb{R}\setminus[0,1]} |\mathrm{Re}\, M_n|^2\mathrm{d}x \tag{6.4.8}$$

for some constant C_X depending only on X. Fix an arbitrary $\eta > 0$. As we have already noted above, $\operatorname{Re} M_n$ is bounded by the supremum norm of f. Setting $\delta = \eta/(C_X \sup_X \|f\|^2)$, we see that

$$\int_{(-\delta,0)} |\operatorname{Re} M_n(x)|^2 \mathrm{d}x + \int_{(1,1+\delta)} |\operatorname{Re} M_n(x)|^2 \mathrm{d}x \le 2\eta C_X^{-1}. \tag{6.4.9}$$

Furthermore, recall that L maps $\mathbb{R}\setminus[0,1]$ onto $(-1,1)$, it is bounded away from 1 outside $[-\delta, 1+\delta]$ and $|L(x)| = O(|x|^{-1})$ as $x \to \pm\infty$. Since F is analytic and vanishes at 0, we conclude that $M_n(x) = F(L^{2n}(x)) = O(|x|^{-2n})$ and hence

$$\lim_{n\to\infty} \int_{\mathbb{R}\setminus[-\delta,1+\delta]} |\operatorname{Re} M_n(x)|^2 \mathrm{d}x = 0. \tag{6.4.10}$$

Putting (6.4.8), (6.4.9) and (6.4.10) together, we see that if n is sufficiently large, then $\int_{\mathbb{R}} |\mathcal{H}_X^{\mathbb{R}}(\mathbf{1}_{\mathbb{R}\setminus[0,1]}\operatorname{Re} M_n)|^2 \mathrm{d}x \le 3\eta$ and the aforementioned convergence in L^2 holds. In particular, passing to a subsequence if necessary, we see that $\mathcal{H}_X^{\mathbb{R}}(\mathbf{1}_{\mathbb{R}\setminus[0,1]}\operatorname{Re} M_n) \to 0$ almost everywhere. However, as we have already mentioned above, the function Ψ^R is uniformly continuous, so the expression in the square brackets in the last term in (6.4.7) converges to zero almost everywhere. In addition, this expression is bounded in absolute value by $\sup \Psi^R$. Consequently, by Lebesgue's dominated convergence theorem, the last integral in (6.4.7) converges to 0 as $n \to \infty$. Putting all the above facts together, we see that if n is sufficiently large, then

$$\int_{\mathbb{R}} \Psi^R\left(\mathcal{H}_X^{\mathbb{R}} h(x)\right) \mathrm{d}x \ge (1-\varepsilon)\cdot \frac{1}{2\pi}\int_0^{2\pi} \Psi^R\left(\mathcal{H}_X^{\mathbb{T}} f(e^{i\theta})\right) \mathrm{d}\theta.$$

Combining this with (6.4.4) and (6.4.6), we obtain that for n large enough we have

$$\int_{\mathbb{R}} \Psi(\mathcal{H}_X^{\mathbb{R}} h(x)) \mathrm{d}x \ge \int_{\mathbb{R}} \Psi^R\left(\mathcal{H}_X^{\mathbb{R}} h(x)\right) \mathrm{d}x \ge (1-\varepsilon)(|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} - \varepsilon)\int_{\mathbb{R}} \Phi(h)\mathrm{d}x.$$

Since h is a step function and ε was arbitrary, the claim follows. □

Remark 6.4.7. Note that if $\Psi(0) \ne 0$ then Theorem 6.4.5 and 6.4.6 do not make any sense. Indeed, if this is the case, then there exists $\varepsilon > 0$ and R such that $\Psi(x) \ge \varepsilon$ for any $x \in X$ with $\|x\| \le R$. Since for any step function $f : \mathbb{R} \to X$ the function $\mathcal{H}_X^{\mathbb{R}} f$ is in $L^2(\mathbb{R}; X)$, the set $\{\|\mathcal{H}_X^{\mathbb{R}} f\| \le R\} \subset \mathbb{R}$ is of infinite measure, so

$$\int_{\mathbb{R}} \Psi(\mathcal{H}_X^{\mathbb{R}} f(s))\, \mathrm{d}s \ge \int_{\mathbb{R}} \mathbf{1}_{\|\mathcal{H}_X^{\mathbb{R}} f\| \le R}(s)\varepsilon\, \mathrm{d}s = \infty,$$

so $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} \ge |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} = |\mathcal{H}_X^{\mathbb{R}}|_{\Phi,\Psi} = \infty$.

Remark 6.4.8. The finiteness of $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$ implies the existence of a plurisubharmonic function $U_{\Phi,\Psi} : X + iX \to \mathbb{R}$ such that $U_{\Phi,\Psi}(0) \ge 0$. Hence, modifying the proof of Theorem 6.3.1, we see that the inequality (6.3.1) holds, with $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$ replaced with $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$, if the dominating martingale M is additionally assumed to start from 0.

Theorem 6.4.9. *Let $\Phi, \Psi : X \to \mathbb{R}_+$ be continuous such that Φ is symmetric (i.e., $\Phi(x) = \Phi(-x)$ for all $x \in X$) and Ψ is convex. Then $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} = |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$.*

Proof. It suffices to show the estimate $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \geq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$. Fix $\varepsilon > 0$. By the definition of $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$, there is a step function $f : \mathbb{T} \to X$ such that

$$\int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f)\mathrm{d}x > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon) \int_{\mathbb{T}} \Phi(f)\mathrm{d}x. \tag{6.4.11}$$

Let $F = F_1 + iF_2$ be the analytic extension of $f + i\mathcal{H}_X^{\mathbb{T}} f : \mathbb{T} \to X + iX$ to the unit disc and suppose that $B = (B^1, B^2)$ is the planar Brownian motion started at 0 and stopped upon hitting $\mathbb{T}$. Let $\tau = \inf\{t \geq 0 : |B_t| = 1\}$ be the lifetime of B. The processes $M_t = F_1(B_t)$, $N_t = F_2(B_t)$ are orthogonal martingales such that N is weakly differentially subordinate to M. By Fatou's lemma and Lebesgue's monotone convergence theorem (observe that f, being a step function, is bounded) we see that if t is sufficiently large, then

$$\mathbb{E}\Psi(N_t) > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon)\mathbb{E}\Phi(M_t).$$

If the expectation of M is zero, then by Remark 4.20 we know that

$$\mathbb{E}\Psi(N_t) \leq |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}\mathbb{E}\Phi(M_t)$$

and hence we obtain that

$$|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \geq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon. \tag{6.4.12}$$

We will show that this is also true if the expectation $\mathrm{x} = \mathbb{E}M_t$ does not vanish. To this end, consider another Brownian motion $W = (W^1, W^2)$ in $\mathbb{R}^2$ started at 0 and stopped upon reaching the boundary of the strip $S = \{(x,y) : |x| \leq 1\}$. Let $\sigma = \inf\{t : |W_t^1| = 1\}$ denote its lifetime. We may assume that W is constructed on the same probability space as B and that both processes are independent. We splice these processes as follows: set

$$\widetilde{M}_s = \begin{cases} \mathrm{x}W_s^1 & \text{if } s \leq \sigma, \\ \operatorname{sgn}(W_\sigma^1) M_{s-\sigma} & \text{if } s > \sigma \end{cases}$$

and

$$\widetilde{N}_s = \begin{cases} \mathrm{x}W_s^2 & \text{if } s \leq \sigma, \\ \mathrm{x}W_\sigma^2 + N_{s-\sigma} & \text{if } s > \sigma. \end{cases}$$

In other words, the pair $(\widetilde{M}, \widetilde{N})$ behaves like a Brownian motion evolving in the strip $S\mathrm{x}$ until its first coordinate reaches x or $-\mathrm{x}$, and then it starts behaving like the pair $(M, \widetilde{N}_\sigma + N)$ or $(-M, \widetilde{N}_\sigma + N)$, depending on which the side of the boundary of $S\mathrm{x}$ the process $\widetilde{M}$ reaches. Note that $\widetilde{M}$, $\widetilde{N}$ are orthogonal martingales such that

$\widetilde{N}$ is weakly differentially subordinate to $\widetilde{M}$ and $\widetilde{M}_0 = 0$. Consequently, by Remark 6.4.8 for any t,

$$\mathbb{E}\Psi(\widetilde{N}_t) \le |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}\mathbb{E}\Phi(\widetilde{M}_t). \tag{6.4.13}$$

Now,

$$\mathbb{E}\Psi(\widetilde{N}_t) \ge \mathbb{E}\Psi(\widetilde{N}_t)1_{\{t\ge\sigma\}} = \mathbb{E}\Psi(\mathrm{x}W_\sigma^2 + N_{t-\sigma})1_{\{t\ge\sigma\}}.$$

However, W and B are independent, and the random variable $\mathrm{x}W_\sigma^2$ is symmetric. Therefore, using the fact that Ψ is convex, we see that

$$\mathbb{E}\Psi(\widetilde{N}_t) \ge \mathbb{E}\Psi(N_{t-\sigma})1_{\{t\ge\sigma\}}.$$

Furthermore, using the symmetry of Φ, we have

$$\mathbb{E}\Phi(\widetilde{M}_t)1_{\{t\ge\sigma\}} = \mathbb{E}\Phi(\mathrm{sgn}(W_\sigma^1)M_{t-\sigma})1_{\{t\ge\sigma\}} = \mathbb{E}\Phi(M_{t-\sigma})1_{\{t\ge\sigma\}}.$$

As previously, combining (6.4.11) with Fatou's lemma and Lebesgue's dominated convergence theorem, if t is sufficiently large, then

$$\mathbb{E}\Psi(N_{t-\sigma})1_{\{t\ge\sigma\}} > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon)\mathbb{E}\Phi(M_{t-\sigma})1_{\{t\ge\sigma\}}$$

and hence also

$$\mathbb{E}\Psi(\widetilde{N}_t) > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon)\mathbb{E}\Phi(\widetilde{M}_t)1_{\{t\ge\sigma\}}.$$

But $\lim_{t\to\infty}\mathbb{E}\Phi(\widetilde{M}_t)1_{\{t<\sigma\}} = 0$, by Lebesgue's dominated convergence theorem (we have $1_{\{t<\sigma\}} \to 0$ and the norm of $\widetilde{M}_t$ is bounded by $\|x\|$ for $t \in [0,\sigma]$). Therefore, the preceding estimate gives

$$\mathbb{E}\Psi(\widetilde{N}_t) > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon)\mathbb{E}\Phi(\widetilde{M}_t)$$

if t is sufficiently big. By (6.4.13), this gives (6.4.12) and completes the proof of the theorem, since ε was arbitrary. □

Remark 6.4.10. Assume that $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} = |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$ (this holds true under some additional assumptions on Φ and Ψ, see Theorem 6.4.9). Then the plurisubharmonic function $U_{\Phi,\Psi}$ considered in Remark 6.4.8 coincides with the one considered in Theorem 6.3.2, and hence we automatically have that $U_{\Phi,\Psi}(x) \ge 0$ for all $x \in X$.

Proof of Theorem 6.4.3. The theorem follows from Proposition 6.4.4, Theorem 6.4.5, 6.4.6, 6.4.9, and the fact that $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \le |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$. □

Remark 6.4.11. Notice that Theorem 6.4.3 can not be applied to more general norms. For example, if X is a UMD Banach space, $1 < q < p < \infty$, then

$$\|\mathcal{H}_X^{\mathbb{T}}\|_{\mathcal{L}(L^p(\mathbb{T};X),L^q(\mathbb{T};X))} < \infty,$$

and

$$\|\mathcal{H}_X^{\mathbb{R}}\|_{\mathcal{L}(L^p(\mathbb{R};X),L^q(\mathbb{R};X))} = \infty.$$

6.4.2. Decoupling constants

We turn our attention to the next important application.

Definition 6.4.12. Let X be a Banach space and let $1 < p < \infty$ be a fixed parameter. Then we define $\beta^{\Delta,+}_{p,X}$ and $\beta^{\Delta,-}_{p,X}$ to be the smallest β^+ and β^- such that

$$\frac{1}{(\beta^-)^p}\mathbb{E}\Big\|\sum_{n=0}^{\infty} df_n\Big\|^p \leq \mathbb{E}\Big\|\sum_{n=0}^{\infty} r_n df_n\Big\|^p \leq (\beta^+)^p\mathbb{E}\Big\|\sum_{n=0}^{\infty} df_n\Big\|^p$$

for any finite Paley-Walsh martingale $(f_n)_{n\geq 0}$ and any independent Rademacher sequence $(r_n)_{n\geq 0}$. Furthermore, we define $\beta^{\gamma,+}_{p,X}$ and $\beta^{\gamma,-}_{p,X}$ to be the least possible values of β^+ and β^- for which

$$\frac{1}{(\beta^-)^p}\mathbb{E}\Big\|\int_0^{\infty}\phi\,\mathrm{d}W\Big\|^p \leq \mathbb{E}\Big\|\int_0^{\infty}\phi\,\mathrm{d}\widetilde{W}\Big\|^p \leq (\beta^+)^p\mathbb{E}\Big\|\int_0^{\infty}\phi\,\mathrm{d}W\Big\|^p,$$

where W is a standard Brownian motion, $\phi : \mathbb{R}_+ \times \Omega \to X$ is an elementary progressive process, and $\widetilde{W}$ is another Brownian motion independent of ϕ and W.

Decoupling constants appear naturally while working with UMD Banach spaces (see e.g. [44, 45, 61, 65, 79, 119, 176]). The following result, a natural corollary of Theorem 6.3.1 for $\Phi(x) = \Psi(x) = \|x\|^p$, exhibits the direct connection between decoupling constants and $\hbar_{p,X} := \|\mathscr{H}^{\mathbb{T}}_X\|_{\mathscr{L}(L^p(\mathbb{T};X))}$ (see Corollary 6.3.3).

Corollary 6.4.13. *Let X be a Banach space and let $1 < p < \infty$ be a fixed parameter. Then we have*

$$\hbar_{p,X} \geq \max\{\beta^{\gamma,+}_{p,X}, \beta^{\gamma,-}_{p,X}\} \tag{6.4.14}$$

and hence

$$\hbar_{p,X} \geq C\max\{\beta^{\Delta,+}_{p,X}, \beta^{\Delta,-}_{p,X}\}. \tag{6.4.15}$$

Here $C = \mathbb{E}|\gamma|\mathbb{E}\sqrt{\tau}$, where γ is a standard normal random variable and $\tau = \inf\{t \geq 0 : |W_t| = 1\}$ for a standard Brownian motion W.

Note that $\mathbb{E}\tau \leq (\mathbb{E}\sqrt{\tau})^{\frac{2}{3}}(\mathbb{E}\tau^2)^{\frac{1}{3}}$ by Hölder's inequality, so C in (6.4.15) is bounded from below by $\frac{(\mathbb{E}\tau)^{\frac{3}{2}}}{(\mathbb{E}\tau^2)^{\frac{1}{2}}}\mathbb{E}|\gamma| = \frac{\sqrt{6}}{\sqrt{5\pi}} \approx 0.618$ (since $\mathbb{E}\tau = 1$ and $\mathbb{E}\tau^2 = \frac{5}{3}$).

Proof. The inequality (6.4.14) follows directly from the definition of $\beta^{\gamma,+}_{p,X}$ and $\beta^{\gamma,-}_{p,X}$. Indeed, for any Brownian motion W, elementary progressive process ϕ, and a Brownian motion $\widetilde{W}$ independent of ϕ and W we have, for any $x^* \in X^*$,

$$\Big[\Big\langle\int_0^{\cdot}\phi\,\mathrm{d}W, x^*\Big\rangle\Big]_t = \Big[\int_0^{\cdot}\langle\phi, x^*\rangle\,\mathrm{d}W\Big]_t = \int_0^t |\langle\phi(s), x^*\rangle|^2\,\mathrm{d}s,$$

$$\Big[\Big\langle\int_0^{\cdot}\phi\,\mathrm{d}\widetilde{W}, x^*\Big\rangle\Big]_t = \Big[\int_0^{\cdot}\langle\phi, x^*\rangle\,\mathrm{d}\widetilde{W}\Big]_t = \int_0^t |\langle\phi(s), x^*\rangle|^2\,\mathrm{d}s,$$

so $\int \phi \,\mathrm{d}W \overset{w}{\ll} \int \phi \,\mathrm{d}\widetilde{W} \overset{w}{\ll} \int \phi \,\mathrm{d}W$. Moreover, by [89, Lemma 17.10],

$$\Big[\Big\langle \int_0^{\cdot} \phi \,\mathrm{d}W, x^*\Big\rangle, \Big\langle \int_0^{\cdot} \phi \,\mathrm{d}\widetilde{W}, x^*\Big\rangle\Big]_t = \Big[\int_0^{\cdot} \langle \phi, x^*\rangle \,\mathrm{d}W, \int_0^{\cdot} \langle \phi, x^*\rangle \,\mathrm{d}\widetilde{W}\Big]_t$$
$$= \int_0^t |\langle \phi(s), x^*\rangle|^2 \,\mathrm{d}[W, \widetilde{W}]_s = 0,$$

where the latter holds since W and $\widetilde{W}$ are independent. Therefore $\int \phi \,\mathrm{d}W$ and $\int \phi \,\mathrm{d}\widetilde{W}$ are orthogonal local martingales satisfying the differential subordination ("in both directions"), so by Theorem 6.3.1,

$$\frac{1}{(\hbar_{p,X})^p}\mathbb{E}\Big\|\int_{\mathbb{R}_+} \phi \,\mathrm{d}W\Big\|^p \leq \mathbb{E}\Big\|\int_{\mathbb{R}_+} \phi \,\mathrm{d}\widetilde{W}\Big\|^p \leq (\hbar_{p,X})^p \mathbb{E}\Big\|\int_{\mathbb{R}_+} \phi \,\mathrm{d}W\Big\|^p.$$

Let us now turn to the second part. First notice that $\beta^{\gamma,+}_{p,X} \geq C\beta^{\Delta,+}_{p,X}$ (see [176, (2.5)] and the discussion thereafter), so $\hbar_{p,X} \geq \beta^{\gamma,+}_{p,X} \geq C\beta^{\Delta,+}_{p,X}$. On the other hand, X can be assumed UMD (and hence reflexive), so by the discussion above we have $\hbar_{p',X^*} \geq C\beta^{\Delta,+}_{p',X^*}$. But $\hbar_{p',X^*} = \hbar_{p,X}$ (since $(\mathscr{H}^{\mathbb{T}}_X)^* = \mathscr{H}^{\mathbb{T}}_{X^*}$), and $\beta^{\Delta,+}_{p',X^*} \geq \beta^{\Delta,-}_{p,X}$ analogously to [62, Theorem 1], so $\hbar_{p,X} \geq C\beta^{\Delta,-}_{p,X}$. □

Remark 6.4.14. Notice that (6.4.14) together with [61, Theorem 3] yields the related estimate $\max\{\beta^{\gamma,+}_{p,X}, \beta^{\gamma,-}_{p,X}\} \leq \hbar_{p,X} \leq \beta^{\gamma,+}_{p,X}\beta^{\gamma,-}_{p,X}$.

Remark 6.4.15. Let X be a UMD Banach function space. Then inequality (6.4.15) together with [91] provide the lower bound for $\hbar_{p,X}$ in terms of $\beta_{p,X}$ of the same order as (2.3.1). Indeed, by [91] thanks to Banach function space techniques one can show that

$$\beta_{p,X} \lesssim_p q(c_{q,X}\beta^{\Delta,+}_{p,X})^2,$$

where q is the cotype of X and $c_{q,X}$ is the corresponding cotype constant. Therefore by applying (6.4.15) we get the following square root dependence:

$$\sqrt{\beta_{p,X}} \lesssim_p \sqrt{q}\, c_{q,X}\hbar_{p,X}.$$

6.4.3. *Necessity of the UMD property*

Our next result answers a very natural question about the link of the number $|\mathscr{H}^{\mathbb{T},0}_X|_{\Phi,\Psi}$ to the UMD property.

Theorem 6.4.16. *Let Φ, $\Psi : X \to \mathbb{R}_+$ be continuous convex functions such that $\Psi(0) = 0$. Assume in addition that there is a positive number C such that the sets $\{x \in X : \Psi(x) < C\}$ and $\Phi(B(0,C))$ are bounded. If $|\mathscr{H}^{\mathbb{T},0}_X|_{\Phi,\Psi} < \infty$, then X is UMD.*

Remark 6.4.17. It is easy to see that the assumption $\Psi(0) = 0$ combined with the boundedness of $\{\Psi < C\}$ enforces the function Ψ to explode "uniformly" in the whole space. That is, if $B(0,R)$ is the ball containing $\{\Psi < C\}$, then the convexity of

Ψ implies $\Psi(x) \geq C\|x\|/R$ for all $x \notin B(0,R)$. Some condition of this type is necessary, as the following simple example indicates. Take $X = \ell_\infty$ and set $\Phi(x) = |x_1|^2 = \Psi(x)$. Then $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} = 1 < \infty$, while X is not UMD. The reason is that the function Ψ controls only the subspace generated by the first coordinate.

Remark 6.4.18. Note that X being UMD does not imply $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} < \infty$. Indeed, if Φ and Ψ are of different homogeneity (i.e. $\Phi(ax) = a^\alpha\Phi(x)$, $\Phi(ax) = a^\beta\Phi(x)$ for any $x \in X$, $a \geq 0$, and for some fixed positive $\alpha \neq \beta$), then for any nonzero step function $f : \mathbb{T} \to X$ such that $\int_{\mathbb{T}} f(s)\,\mathrm{d}s = 0$ and for any $a \geq 0$ we have that

$$\begin{aligned}\int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f(s))\,\mathrm{d}s = \frac{1}{a^\beta}\int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}}(af)(s))\,\mathrm{d}s \leq \frac{1}{a^\beta}|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}\int_{\mathbb{T}} \Phi(af(s))\,\mathrm{d}s\\ = a^{\alpha-\beta}|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}\int_{\mathbb{T}} \Phi(f(s))\,\mathrm{d}s,\end{aligned}$$

so $a^{\alpha-\beta}|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \leq |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$ for any $a > 0$, and since $\alpha \neq \beta$, $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} = \infty$. The classical examples of such Φ and Ψ are $\Phi(x) = \|x\|^p$, $\Psi(x) = \|x\|^q$, $x \in X$ for different p and q.

The proof of Theorem 6.4.16 will exploit the following four lemmas. In what follows, $N^* = \sup_{t\geq 0}\|N_t\|$ is the maximal function of N.

Lemma 6.4.19. *Under the assumptions of Theorem 6.4.16, there exists a constant c_1 depending on Φ, Ψ and X, such that if M, N are orthogonal martingales such that N is weakly differentially subordinate to M, $M_0 = 0$ and $\|M\|_\infty \leq c_1$, then $\mathbb{P}(N^* \geq 1) < 1$.*

Proof. Let R be as in Remark 6.4.17 and suppose that $\Phi(B(0,C)) \subseteq [-R', R']$. Then for any $\lambda \geq 1$ we have, in the light of Remark 6.4.8,

$$\mathbb{P}(\|N_t\| \geq 1) = \mathbb{P}(R\lambda\|N_t\| \geq R\lambda) \leq \frac{\mathbb{E}\Psi(R\lambda N_t)}{C\lambda} \leq \frac{|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}\mathbb{E}\Phi(R\lambda M_t)}{C\lambda}.$$

It suffices to take $\lambda = \frac{2R'|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}}{C}$ and $c_1 = C/(R\lambda)$. □

Lemma 6.4.20. *Suppose that the assumptions of Theorem 6.4.16 are satisfied. Let M and N be* continuous-path *orthogonal martingales such that N is weakly differentially subordinate to M, $M_0 = 0$ and $\mathbb{P}(N^* > 1) = 1$. Then there exist continuous-path martingales $\widetilde{M}$, $\widetilde{N}$ such that $\widetilde{N}$ is weakly differentially subordinate to $\widetilde{M}$, $\widetilde{M}_0 = 0$, $\mathbb{P}(\widetilde{N}^* > 1) \geq 1/2$ and $\|\widetilde{M}\|_\infty \leq 2\|M\|_1$.*

Proof. Define $\tau = \inf\{t \geq 0 : \|M_t\| \geq 2\|M\|_1\}$ (as usual, $\inf\varnothing = +\infty$) and put $\widetilde{M} = M^\tau$, $\widetilde{N} = N^\tau$. Since M has continuous paths and starts from 0, we have $\|\widetilde{M}\|_\infty \leq 2\|M\|_1$. Furthermore, $\mathbb{P}(\widetilde{N}^* > 1) \geq \mathbb{P}(\widetilde{N} = N) \geq 1/2$, since

$$\mathbb{P}(\widetilde{N} \neq N) = \mathbb{P}(\tau < \infty) = \mathbb{P}(M^* \geq 2\|M\|_1) \leq 1/2$$

by [93, Theorem 1.3.8(i)]. □

Lemma 6.4.21. *Suppose that the assumptions of Theorem 6.4.16 are satisfied. Then there exists a constant $c > 0$ such that if M, N are continuous-path orthogonal martingales such that N is weakly differentially subordinated to M, $M_0 = 0$ and $N^* > 1$ almost surely, then $\|M\|_1 \geq c$.*

Proof. Let c_1 be the number guaranteed by Lemma 6.4.19. Suppose that such a c does not exist. Then for any positive integer j there exist a pair (M^j, N^j) of orthogonal martingales such that N^j is weakly differentially subordinate to M^j, $M_0^j = 0$, $\mathbb{P}((N^j)^* > 2) = 1$ and $\|M^j\|_1 \leq 2^{-j-1}c_1$. By Lemma 6.4.20, for each j there is a pair $(\widetilde{M}^j, \widetilde{N}^j)$ of orthogonal, weakly differentially subordinate martingales satisfying $\widetilde{M}_0^j = 0$, $\mathbb{P}((\widetilde{N}^j)^* > 2) \geq 1/2$ and $\|\widetilde{M}^j\|_\infty \leq 2^{-j}c_1$. We may assume that the underlying probability space is the same for all pairs and that all the pairs are independent. For each j there is a positive number t_j such that the event

$$A_j = \{\|\widetilde{N}_t^j\| > 2 \text{ for some } t \leq t_j\}$$

has probability greater than $1/3$. Set $t_0 = 0$ and consider the martingale pair (M, N) defined as follows: if $t \in [t_0 + t_1 + \ldots + t_n, t_0 + t_1 + \ldots + t_{n+1})$ for some n, then

$$M_t = \widetilde{M}_{t_1}^1 + \widetilde{M}_{t_2}^2 + \ldots + \widetilde{M}_{t_n}^n + \widetilde{M}_{t-t_1-t_2-\ldots-t_n}^{n+1}, \tag{6.4.16}$$

and analogously for N. Then M and N are orthogonal, N is weakly differentially subordinate to M, $M_0 = 0$ and

$$\|M\|_\infty \leq \sum_{j=1}^{\infty} \|\widetilde{M}^j\|_\infty \leq \sum_{j=1}^{\infty} 2^{-j}c_1 = c_1.$$

Furthermore, by Borel-Cantelli lemma,

$$\mathbb{P}(N^* \geq 1) \geq \mathbb{P}\left(\limsup_{j\to\infty} A_j\right) = 1,$$

since the events A_j are independent and $\sum_{j=1}^{\infty} \mathbb{P}(A_j) = \infty$. Therefore we have that $\|M\|_\infty \leq c_1$, $\mathbb{P}(N^* \geq 1) = 1$, $N \overset{w}{\ll} M$, and M and N are orthogonal, which contradicts the assertion of Lemma 6.4.19. □

Lemma 6.4.22. *Suppose that the assumptions of Theorem 6.4.16 are satisfied. Then there exists a positive constant C such that if M, N are continuous-path orthogonal martingales such that N is weakly differentially subordinate to M and $M_0 = 0$, then*

$$\mathbb{P}(N^* > 1) \leq C\|M\|_1. \tag{6.4.17}$$

Proof. Let c be the constant guaranteed by the previous lemma. Suppose that the assertion is not true. Then for any positive integer j there is a martingale pair (M^j, N^j) satisfying the usual structural properties such that

$$\mathbb{P}((N^j)^* > 2) > 2^{j+1}c^{-1}\|M^j\|_1. \tag{6.4.18}$$

We splice these martingale pairs into one pair (M,N) as previously, however, this time we allow pairs to appear several times. More precisely, denote $a_j = \mathbb{P}((N^*)^j > 2)$. Consider $\lceil 1/a_1 \rceil$ copies of (M^1,N^1), $\lceil 1/a_2 \rceil$ copies of (M^2,N^2), and so on (all the pairs are assumed to be independent). Let t_j be positive numbers such that the events $A_j = \{\|N_t^j\| > 2 \text{ for some } t \le t_j\}$ have probability greater than $a_j/2$. Splice the aforementioned independent martingale pairs (with multiplicities) into one pair (M,N) using a formula analogous to (6.4.16). Then, by (6.4.18),

$$\|M\|_1 \le \sum \|M^j\|_1 \le \sum_{j=1}^{\infty} \left\lceil \frac{1}{a_j} \right\rceil \|M^j\|_1 \le \sum_{j=1}^{\infty} \frac{2}{a_j} \cdot a_j c 2^{-j-1} = c$$

and, again by Borel-Cantelli lemma, $\mathbb{P}(N^* > 1) = 1$. Here we use the independence of the events A_j and

$$\sum \mathbb{P}(A_j) \ge \sum_{j=1}^{\infty} \frac{1}{a_j} \cdot \frac{a_j}{2} = \infty.$$

This contradicts Lemma 6.4.21. □

Proof of Theorem 6.4.16. We will prove that theorem using the well-known extrapolation technique (good-λ inequalities) of Burkholder [29].

Step 1. First we show that for any fixed $0 < \delta < 1$ and $\beta > 1$ there exists $\varepsilon > 0$ depending only on δ, β, and X such that for any orthogonal *continuous-path* martingales $M,N : \mathbb{R}_+ \times \Omega \to X$ with $M_0 = N_0 = 0$ and $N \overset{w}{\ll} M$,

$$\mathbb{P}(N^* > \beta\lambda, M^* \le \delta\lambda) \le \varepsilon \mathbb{P}(N^* > \lambda) \tag{6.4.19}$$

for any $\lambda > 0$. Without loss of generality assume that both martingales take their values in a finite-dimensional subspace of X. Define three stopping times

$$\begin{aligned} \mu &:= \inf\{t \ge 0 : \|N_t\| > \lambda\}, \\ \nu &:= \inf\{t \ge 0 : \|M_t\| > \delta\lambda\}, \\ \sigma &:= \inf\{t \ge 0 : \|N_t\| > \beta\lambda\}. \end{aligned} \tag{6.4.20}$$

All the stopping times are predictable since M and N are continuous. Therefore, the equation $U(t) = \mathbf{1}_{[\mu,\nu\wedge\sigma]}(t)$ defines a predictable process, which in turn gives rise to the martingales

$$\begin{aligned} \widetilde{M} &:= \int U \, \mathrm{d}M = M^{\mu} - M^{\nu\wedge\delta}, \\ \widetilde{N} &:= \int U \, \mathrm{d}N = N^{\mu} - N^{\nu\wedge\delta}. \end{aligned} \tag{6.4.21}$$

Notice that by (6.4.20) and (6.4.21), $\widetilde{M}^* \le 2\delta\lambda$ on $\{\mu < \infty\}$ and $\widetilde{M}^* = 0$ on $\{\mu = \infty\}$, so

$$\|\widetilde{M}\|_1 \le 2\delta\lambda \mathbb{P}(N^* > \lambda). \tag{6.4.22}$$

Since $\widetilde{N} \overset{w}{\ll} \widetilde{M}$, $\widetilde{M}_0 = \widetilde{N}_0 = 0$ and $\widetilde{M}$ and $\widetilde{N}$ are orthogonal,

$$\mathbb{P}(N^* > \beta\lambda, M^* \le \delta\lambda) \le \mathbb{P}(\widetilde{N}^* > (\beta-1)\lambda) \overset{(i)}{\le} \frac{C}{(\beta-1)\lambda}\|\widetilde{M}\|_1 \overset{(ii)}{\le} \frac{2\delta C}{(\beta-1)}\mathbb{P}(N^* > \lambda),$$

where (i) follows from (6.4.17) with the same constant C depending only on X, and (ii) follows from (6.4.22). Therefore (6.4.19) holds with $\varepsilon = 2\delta C/(\beta-1)$.

Step 2. Now a straightforward integration argument (cf. [29, Lemma 7.1]), together with Doob's maximal inequality, yield the L^p estimate

$$\sup_{t\ge 0}\|N_t\|_p \le \|N^*\|_p \le C_{p,X}\|M^*\|_p \le \frac{pC_{p,X}}{p-1}\sup_{t\ge 0}\|M_t\|_p, \qquad 1 < p < \infty,$$

for any pair of continuous, orthogonal, differentially subordinated martingales such that $M_0 = 0$. Here

$$C_{p,X}^p = \frac{\delta^{-p}\beta^p}{1-\beta^p\cdot 2\delta C/(\beta-1)}, \tag{6.4.23}$$

which, if we let $\beta = 1 + p^{-1}$ and $\delta = (10Cp)^{-1}$, depends only on p and the constant in (6.4.17). This in turn yields the corresponding L^p inequality for the periodic Hilbert transform for functions of integral 0. By Theorem 6.4.3 the assumption on the zero-average can be omitted, and hence X is UMD by [79, Corollary 5.2.11]. □

Now we will take a closer look at the classical "LlogL" estimates of Zygmund [194]. For a Banach space X and a step function $f : \mathbb{T} \to X$, we define

$$\|f\|_{L\log L(\mathbb{T};X)} := \int_{\mathbb{T}} (\|f(s)\| + 1)\log(\|f(s)\| + 1)\,\mathrm{d}s$$

and denote

$$\hbar_{L\log L,X} = |\mathcal{H}_X^{\mathbb{T}}|_{L\log L(\mathbb{T};X)\to L^1(\mathbb{T};X)} := \sup_{f:\mathbb{T}\to X \text{ step}} \frac{\|\mathcal{H}_X^{\mathbb{T}} f\|_{L^1(\mathbb{T};X)}}{\|f\|_{L\log L(\mathbb{T};X)}}.$$

Remark 6.4.23. In the light of Theorem 6.4.3, we have

$$\begin{aligned}\hbar_{L\log L,X} = |\mathcal{H}_X^{\mathbb{T},0}|_{L\log L(\mathbb{T};X)\to L^1(\mathbb{T};X)} &= |\mathcal{H}_X^{\mathbb{R}}|_{L\log L(\mathbb{R};X)\to L^1(\mathbb{R};X)}\\ &= |\mathcal{H}_X^{\mathrm{dis}}|_{L\log L(\mathbb{Z};X)\to L^1(\mathbb{Z};X)}\end{aligned}$$

for any Banach space X.

We will establish the following statement.

Theorem 6.4.24. *Let X be a Banach space. Then X has the UMD property if and only if $\hbar_{L\log L,X} < \infty$.*

For the proof we will need the following lemma.

Lemma 6.4.25. *Let X be a UMD Banach space. Then there exists a constant C_X depending only on X such that $\hbar_{p,X} \leq C_X \frac{p}{p-1}$ for all $1 < p < 2$.*

Proof. Let $M, N : \mathbb{R}_+ \times \Omega \to X$ be continuous orthogonal martingales such that $N \overset{w}{\ll} M$ and $N_0 = 0$. As we have already seen above,

$$\sup_{t\geq 0}(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq \frac{p}{p-1} C_{p,X} \sup_{t\geq 0}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}},$$

where $C_{p,X} \leq 10Cpe(1-e/5)^{-1/p}$ (see (6.4.23) and the discussion following it). Therefore, if $1 < p < 2$, we may assume that this constant depends only on C (which essentially depends only on X). The claim follows from the sharpness part of Theorem 6.3.1. □

Proof of Theorem 6.4.24. The inequality $\hbar_{L\log L,X} < \infty$ implies UMD by Theorem 6.4.16 applied to $\Phi(x) = (\|x\|+1)\log(\|x\|+1)$ and $\Psi(x) = \|x\|$, $x \in X$. The converse holds true by Lemma 6.4.25 and Yano's extrapolation argument (see e.g. [56, 182]). □

6.4.4. Weak differential subordination of martingales: sharper L^p-inequalities

As it was shown in (4.4.1), for a UMD Banach space X, any $1 < p < \infty$ and any X-valued local martingales M and N such that $N \overset{w}{\ll} M$, we have

$$\mathbb{E}\|N_t\|^p \leq c_{p,X}^p \mathbb{E}\|M_t\|^p, \quad t \geq 0,$$

with $c_{p,X} \leq \beta_{p,X}^2(\beta_{p,X}+1)$. The purpose of this subsection is to show that this upper bound can be substantially improved.

Theorem 6.4.26. *Let X be a Banach space, let $1 < p < \infty$ and assume that M, N are local martingales satisfying $N \overset{w}{\ll} M$. Then*

$$\mathbb{E}\|N_t\|^p \leq (\beta_{p,X} + \hbar_{p,X})^p \mathbb{E}\|M_t\|^p \text{ for any } t \geq 0. \tag{6.4.24}$$

Remark 6.4.27. Note that $\hbar_{p,X} \leq \beta_{p,X}^2$ (see (2.3.1)), so (6.4.24) gives

$$(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\beta_{p,X}+1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}} \quad t \geq 0,$$

which is better than (4.4.1).

For the proof of Theorem 6.4.26 we will need the Burkholder function (see p. 47).

Remark 6.4.28. Suppose that the Banach space X is finite-dimensional and let $U : X \times X \to \mathbb{R}$ be a zigzag-concave function (e.g. the Burkholder funciton). Let $\rho : X \times X \to \mathbb{R}_+$ be a compactly supported nonnegative function of class C^∞. Then the convolution $U_\rho := U * \rho : X \times X \to \mathbb{R}$ is zigzag-concave and of class C^∞ (see e.g. [13]).

While working with the Burkholder function $U : X \times X \to \mathbb{R}$ we will use the following notation: for given vectors $x, y \in X$ instead of writing

$$\frac{\partial^2 U}{\partial (x,0)^2}, \frac{\partial^2 U}{\partial (0,y)^2}, \frac{\partial^2 U}{\partial (x,0)\partial (0,y)}$$

we will write

$$\frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 U}{\partial y^2}, \frac{\partial^2 U}{\partial x \partial y}.$$

Therefore for the convenience of the reader throughout this subsection we always assume that the first coordinate of any vector in $X \times X$ is x (perhaps with a subscript), while the second coordinate is y (perhaps with a subscript). The same holds for partial derivatives.

We also will need the following lemma.

Lemma 6.4.29. *Let X be a finite-dimensional Banach space, let $F : X \times X \to \mathbb{R}$ be a zigzag-concave function and let $(x_0, y_0) \in X \times X$ be such that F is twice Fréchet differentiable at (x_0, y_0). Let $(x, y) \in X \times X$ be such that $y = x$. Then for each $\lambda \in [-1,1]$,*

$$\frac{\partial^2 F(x_0,y_0)}{\partial x^2} + 2\lambda \frac{\partial^2 F(x_0,y_0)}{\partial x \partial y} + \frac{\partial^2 F(x_0,y_0)}{\partial y^2} \le 0.$$

Proof. Since the function

$$\lambda \mapsto \frac{\partial^2 F(x_0,y_0)}{\partial x^2} + 2\lambda \frac{\partial^2 F(x_0,y_0)}{\partial x \partial y} + \frac{\partial^2 F(x_0,y_0)}{\partial y^2}$$

is linear in $\lambda \in [-1,1]$, it is sufficient to check the cases $\lambda = \pm 1$. To this end notice that

$$\frac{\partial^2 F(x_0,y_0)}{\partial x^2} \pm 2 \frac{\partial^2 F(x_0,y_0)}{\partial x \partial y} + \frac{\partial^2 F(x_0,y_0)}{\partial y^2} = \frac{\partial^2}{\partial t^2} F(x_0 + tx, y_0 \pm tx)\Big|_{t=0} \le 0,$$

where the latter follows from Definition 2.10.1. □

Proof of Theorem 6.4.26. We begin with similar reductions as in the proof of Theorem 6.3.1. First, we may assume that X is a finite-dimensional Banach space. Let $d \ge 1$ be the dimension of X. Let $M = M^c + M^d$ and $N = N^c + N^d$ be the Meyer-Yoeurp decompositions (see Subsection 6.2.4). Then by Proposition 6.2.11 $N^c \overset{w}{\ll} M^c$ and $N^d \overset{w}{\ll} M^d$. Let $\tau = (\tau_s)_{s\ge 0}$ be the time-change constructed in Step 1 of the proof of Theorem 6.3.1 (see also the proof of Proposition 4.4.3). So, there exists a $2d$-dimensional standard Brownian motion W on an extended probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$ equipped with an extended filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathscr{F}}_t)_{t\ge 0}$, and there exist two progressively measurable processes $\phi, \psi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(\mathbb{R}^{2d}, X)$ such that $M^c \circ \tau = \phi \cdot W$ and $N^c \circ \tau = \psi \cdot W$. Let us redefine $M := M \circ \tau$ and $N := N \circ \tau$ (hence $M^c := M^c \circ \tau$,

$M^d := M^d \circ \tau$, $N^c := N^c \circ \tau$, and $N^d := N^d \circ \tau$ by Theorem 2.4.25). Without loss of generality we may further assume that M and N terminate after some deterministic time: $M_t = M_{t\wedge T}$ and $N_t = N_{t\wedge T}$ for some fixed parameter $T \geq 0$. Analogously to Proposition 6.3.5 there exists a progressively measurable $A : \mathbb{R}_+ \times \Omega \to \mathscr{L}(\mathbb{R}^{2d})$ which satisfies $\|A\| \leq 1$ on $\mathbb{R}_+ \times \Omega$ and $\psi = \phi A$. Let us define $A^{\mathrm{sym}} := \frac{A+A^T}{2}$, $A^{\mathrm{asym}} := \frac{A-A^T}{2}$. If we set

$$N^{\mathrm{sym}} := N^d + (\phi A^{\mathrm{sym}}) \cdot W, \qquad N^{\mathrm{asym}} := (\phi A^{\mathrm{asym}}) \cdot W,$$

then $N^{\mathrm{sym}} \overset{w}{\ll} M$ and $N^{\mathrm{asym}} \overset{w}{\ll} M$. Indeed, if $N^{\mathrm{sym}} = N^{\mathrm{sym},c} + N^{\mathrm{sym},d}$ and $N^{\mathrm{asym}} = N^{\mathrm{asym},c} + N^{\mathrm{asym},d}$ are the corresponding Meyer-Yoeurp decompositions, then $N^{\mathrm{sym},d} = N^d \overset{w}{\ll} M^d$, $N^{\mathrm{asym},d} = 0 \overset{w}{\ll} M^d$, and for any $x^* \in X^*$ and $t \geq 0$, we have

$$\begin{aligned}[\langle N^{\mathrm{sym},c}, x^*\rangle]_t = \int_0^t \Big\| \tfrac{A(s)+A^T(s)}{2} \phi^*(s) x^* \Big\|^2 \,\mathrm{d}s \leq \int_0^t \Big\| \tfrac{A(s)+A^T(s)}{2} \Big\|^2 \|\phi^*(s)x^*\|^2 \,\mathrm{d}s \\ \leq \int_0^t \|\phi^*(s)x^*\|^2 \,\mathrm{d}s = [\langle M^c, x^*\rangle]_t.\end{aligned}$$

Here $\big\| \frac{A(s)+A^T(s)}{2} \big\| \leq 1$ by the triangle inequality. Therefore $N^{\mathrm{sym},c} \overset{w}{\ll} M^c$ and, analogously, $N^{\mathrm{asym},c} \overset{w}{\ll} M^c$, so the weak differential subordination holds by virtue of Proposition 6.2.11.

Let us now show that

$$\mathbb{E}\|N_t^{\mathrm{asym}}\|^p \leq \hbar_{p,X}^p \mathbb{E}\|M_t\|^p \quad \text{for } t \geq 0. \tag{6.4.25}$$

We have $N_0^{\mathrm{asym}} = 0$ and $N^{\mathrm{asym}} \overset{w}{\ll} M$; we will prove in addition that M and N^{asym} are orthogonal. For fixed $x^* \in X^*$ and $t \geq 0$ we may write

$$\begin{aligned}[\langle M, x^*\rangle, \langle N^{\mathrm{asym}}, x^*\rangle]_t &= [\langle M^c, x^*\rangle, \langle N^{\mathrm{asym}}, x^*\rangle]_t + [\langle M^d, x^*\rangle, \langle N^{\mathrm{asym}}, x^*\rangle]_t \\ &= [\langle M^c, x^*\rangle, \langle N^{\mathrm{asym}}, x^*\rangle]_t = [\langle \phi \cdot W, x^*\rangle, \langle (\phi A^{\mathrm{asym}}) \cdot W, x^*\rangle]_t \\ &= [\langle \phi, x^*\rangle \cdot W, \langle (\phi A^{\mathrm{asym}}), x^*\rangle \cdot W]_t \\ &= \int_0^t \langle \phi^*(s)x^*, A^{\mathrm{asym}*}(s)\phi^*(s)x^*\rangle \,\mathrm{d}s = 0,\end{aligned}$$

where the second equality is a consequence of pure discontinuity of M^d and continuity of N^{asym}, while the last equality follows from the fact that A^{asym} is antisymmetric. This gives the orthogonality of the processes and (6.4.25) follows from (6.3.1).

The next step is to show that

$$\mathbb{E}\|N_t^{\mathrm{sym}}\|^p \leq \beta_{p,X}^p \mathbb{E}\|M_t\|^p \quad \text{for } t \geq 0. \tag{6.4.26}$$

Let $U : X \times X \to \mathbb{R}$ be the Burkholder function guaranteed by Theorem 3.3.7. Using the same argument as in [13], we may assume that U is of class C^∞ (see also Remark 6.4.28). Applying Itô's formula (2.12.1) for a fixed basis $(x_i)_{i=1}^d$ of X with the dual basis $(x_i^*)_{i=1}^d$ of X^*, we get

$$\mathbb{E}U(M_t, N_t^{\mathrm{sym}}) = \mathbb{E}U(M_0, N_0^{\mathrm{sym}}) + \frac{1}{2}\mathbb{E}I_1 + \mathbb{E}I_2,$$

where

$$\begin{aligned} I_1 := &\int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\mathrm{sym}})}{\partial x_i \partial x_j} \,\mathrm{d}[\langle M, x_i^*\rangle, \langle M, x_j^*\rangle]_s^c \\ &+ \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\mathrm{sym}})}{\partial y_i \partial y_j} \,\mathrm{d}[\langle N^{\mathrm{sym}}, x_i^*\rangle, \langle N^{\mathrm{sym}}, x_j^*\rangle]_s^c \\ &+ 2\int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\mathrm{sym}})}{\partial x_i \partial y_j} \,\mathrm{d}[\langle M, x_i^*\rangle, \langle N^{\mathrm{sym}}, x_j^*\rangle]_s^c \end{aligned} \tag{6.4.27}$$

and

$$\begin{aligned} I_2 := \sum_{0\le s\le t} (\Delta U(M_s, N_s^{\mathrm{sym}}) &- \langle \partial_x U(M_{s-}, N_{s-}^{\mathrm{sym}}), \Delta M_s\rangle \\ &- \langle \partial_y U(M_{s-}, N_{s-}^{\mathrm{sym}}), \Delta N_s^{\mathrm{sym}}\rangle). \end{aligned}$$

Here $\partial_x U(\cdot), \partial_y U(\cdot) \in X^*$ are the corresponding Fréchet derivatives of U in the first and the second X-subspace of the product space $X \times X$. Let us first show that $\mathbb{E}I_1 \le 0$. Observe that

$$\begin{aligned} &\sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\mathrm{sym}})}{\partial x_i \partial x_j} \langle \phi^* x_i^*, \phi^* x_j^*\rangle \\ &\quad + \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\mathrm{sym}})}{\partial y_i \partial y_j} \langle A^{\mathrm{sym}*}\phi^* x_i^*, A^{\mathrm{sym}*}\phi^* x_j^*\rangle \\ &\quad + 2\sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\mathrm{sym}})}{\partial x_i \partial y_j} \langle \phi^* x_i^*, A^{\mathrm{sym}*}\phi^* x_j^*\rangle \le 0. \end{aligned} \tag{6.4.28}$$

Note that by Corollary 2.11.3 and convexity of U in the second variable,

$$\begin{aligned} &\sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\mathrm{sym}})}{\partial y_i \partial y_j} \langle A^{\mathrm{sym}*}\phi^* x_i^*, A^{\mathrm{sym}*}\phi^* x_j^*\rangle \\ &\qquad \le \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\mathrm{sym}})}{\partial y_i \partial y_j} \langle \phi^* x_i^*, \phi^* x_j^*\rangle. \end{aligned} \tag{6.4.29}$$

The operator $P_{\mathrm{Ran}(\phi^*)} A^{\mathrm{sym}*} P_{\mathrm{Ran}(\phi^*)}$ is symmetric and

$$\| P_{\mathrm{Ran}(\phi^*)} A^{\mathrm{sym}*} P_{\mathrm{Ran}(\phi^*)} \| \le 1.$$

Therefore by the spectral theorem there exist a $[-1,1]$-valued sequence $(\lambda_i)_{i=1}^{2d}$ and an orthonormal basis $(\tilde{h}_i)_{i=1}^{2d}$ of $(\mathbb{R}^{2d})^*$ such that $P_{\mathrm{Ran}(\phi^*)} A^{\mathrm{sym}*} P_{\mathrm{Ran}(\phi^*)} \tilde{h}_i = \lambda_i \tilde{h}_i$.

Moreover, since $\operatorname{Ran}(P_{\operatorname{Ran}(\phi^*)}A^{\mathrm{sym}*}P_{\operatorname{Ran}(\phi^*)}) \subset \operatorname{Ran}(\phi^*)$, $\tilde{h}_i \in \operatorname{Ran}(\phi^*)$ if $\lambda_i \neq 0$, so we may assume that there exists a basis $(\tilde{x}_i)_{i=1}^d$ of X with the dual basis $(\tilde{x}_i^*)_{i=1}^d$ such that $\phi^*\tilde{x}_i^* = \tilde{h}_i$ for $1 \le i \le m$ and $\phi^*\tilde{x}_i^* = 0$ for $m < i \le d$, where $m \in \{0,\dots,d\}$ is the dimension of ϕ^*. By Lemma 2.11.2 the expression on the left-hand side of (6.4.28) does not depend on the choice of the basis of X and the corresponding dual basis. Therefore, using (6.4.29), it is not bigger than

$$\sum_{i=1}^m \frac{\partial^2 U(M_{s-},N_{s-}^{\mathrm{sym}})}{\partial x_i \partial x_i} + \sum_{i=1}^m \frac{\partial^2 U(M_{s-},N_{s-}^{\mathrm{sym}})}{\partial y_i \partial y_i} + 2\sum_{i=1}^m \lambda_i \frac{\partial^2 U(M_{s-},N_{s-}^{\mathrm{sym}})}{\partial x_i \partial y_i},$$

which is bounded from above by 0 (see Lemma 6.4.29). Thus, (6.4.28) follows. Therefore by (6.4.27) and (6.4.28), we see that

$$\begin{aligned} I_1 &= \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-},N_{s-}^{\mathrm{sym}})}{\partial x_i \partial x_j} \langle \phi^* x_i^*, \phi^* x_j^* \rangle \,\mathrm{d}s \\ &\quad + \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-},N_{s-}^{\mathrm{sym}})}{\partial y_i \partial y_j} \langle A^{\mathrm{sym}*}\phi^* x_i^*, A^{\mathrm{sym}*}\phi^* x_j^* \rangle \,\mathrm{d}s \\ &\quad + 2\int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-},N_{s-}^{\mathrm{sym}})}{\partial x_i \partial y_j} \langle \phi^* x_i^*, A^{\mathrm{sym}*}\phi^* x_j^* \rangle \,\mathrm{d}s \le 0, \end{aligned}$$

and hence the expectation of I_1 is nonpositive. The inequality $I_2 \le 0$ can be proved by repeating the arguments from proof of Theorem 3.3.17, while for the estimate $U(M_0, N_0^{\mathrm{sym}}) \le 0$, consult Remark 3.3.9. Therefore, we have

$$\mathbb{E}\|N_t^{\mathrm{sym}}\|^p - \beta_{p,X}^p \mathbb{E}\|M_t\|^p \le \mathbb{E}U(M_t, N_t^{\mathrm{sym}}) \le \mathbb{E}U(M_0, N_0^{\mathrm{sym}}) \le 0,$$

so (6.4.26) holds. The general inequality (6.4.24) follows from (6.4.25), (6.4.26), and the triangle inequality. □

Remark 6.4.30. It is an open problem whether there exists a Burkholder function U such that $-U$ is plurisubharmonic (note that $X \times X \simeq X + iX$, so the plurisubharmonicity condition is well-defined). If it exists, then $\hbar_{p,X} \le \beta_{p,X}$ by Theorem 6.3.2, and so the open problem outlined in Remark 2.3.2 is solved. Unfortunately, plurisubharmonicity of $-U$ is discovered only in the Hilbert space case (see [179] and Remark 3.5.4).

6.4.5. Weak differential subordination of harmonic functions

Let X be a Banach space, let $d \ge 1$ be a fixed dimension and let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$. A function $f : \mathcal{O} \to X$ is called *harmonic* if it takes its values in a finite-dimensional subspace of X, is twice-differentiable, and

$$\Delta f(s) := \sum_{i=1}^d \partial_i^2 f(s) = 0, \quad s \in \mathcal{O}.$$

For each $s \in \mathcal{O}$, we define $\nabla f(s) \in \mathcal{L}(\mathbb{R}^d, X)$ by

$$\nabla f(s)(a_1 e_1 + \cdots a_d e_d) = \sum_{i=1}^{d} a_i \partial_i f(s), \quad a_1, \ldots, a_d \in \mathbb{R},$$

where $(e_i)_{i=1}^d$ is the basis of $\mathbb{R}^d$.

Definition 6.4.31. Let X, d, $\mathcal{O}$ be as above and assume that $f, g : \mathcal{O} \to X$ are harmonic functions. Then

1. g is said to be *weakly differentially subordinate* to f (which will be denoted by $g \overset{w}{\ll} f$) if
$$|\langle \nabla g(s), x^* \rangle| \leq |\langle \nabla f(s), x^* \rangle|, \quad s \in \mathcal{O}, x^* \in X^*; \tag{6.4.30}$$
2. f and g are said to be *orthogonal* if
$$\Big\langle \langle \nabla f(s), x^* \rangle, \langle \nabla g(s), x^* \rangle \Big\rangle = 0, \quad s \in \mathcal{O}, x^* \in X^*. \tag{6.4.31}$$

Here $|\cdot|$ in (6.4.30) is assumed to be the usual Euclidean norm in $(\mathbb{R}^d)^* \simeq \mathbb{R}^d$, and $\langle \cdot, \cdot \rangle$ in (6.4.31) is the usual scalar product in $(\mathbb{R}^d)^* \simeq \mathbb{R}^d$.

The notion of weak differential subordination of vector-valued harmonic functions extends the concept originally formulated in the one-dimensional case by Burkholder [37]. As shown in that paper, the differential subordination of harmonic functions lead to the corresponding L^p-inequalities for $1 < p < \infty$. The aim of this subsection is to show the extension of that result to general weakly differentially subordinated harmonic functions and to show more general Φ, Ψ-type estimates under the orthogonality assumption. We start with recalling the definition of a harmonic measure.

Definition 6.4.32. Let $\mathcal{O} \subset \mathbb{R}^d$ be an open set containing the origin and let $\partial\mathcal{O}$ be the *boundary* of $\mathcal{O}$. The probability measure μ on $\partial\mathcal{O}$ is called a *harmonic measure with respect to the origin*, if for any Borel subset $A \subset \partial\mathcal{O}$ we have

$$\mu(A) := \mathbb{P}\{W_\tau \in A\}.$$

Here $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ is a standard Brownian motion starting from 0 and τ is the exit-time of W from $\mathcal{O}$.

Theorem 6.4.33. *Let X be a Banach space, let $d \geq 1$ be a fixed dimension and let $\mathcal{O}$ be an open, bounded subset of $\mathbb{R}^d$ containing the origin. Assume further that $\Phi, \Psi : X \to \mathbb{R}_+$ are continuous functions such that Ψ is convex and $\Psi(0) = 0$. Then for any continuous functions $f, g : \overline{\mathcal{O}} \to X$ harmonic and orthogonal on $\mathcal{O}$ satisfying $g \overset{w}{\ll} f$ and $g(0) = 0$ we have*

$$\int_{\partial\mathcal{O}} \Psi(g(s))\,\mathrm{d}\mu(s) \leq C_{\Phi,\Psi,X} \int_{\partial\mathcal{O}} \Phi(f(s))\,\mathrm{d}\mu(s).$$

Here μ is the harmonic measure on $\partial\mathcal{O}$ with respect to the origin and the least admissible $C_{\Phi,\Psi,X}$ equals $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$.

Remark 6.4.34. We do not assume that Φ is convex because both f and g take their values in a finite-dimensional subspace of X, see Remark 6.3.12.

Proof of Theorem 6.4.33. Let $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be a standard Brownian motion and let $\tau := \inf\{t \geq 0 : W_t \notin \mathscr{O}\}$. Then both $M := f(W^\tau)$ and $N := g(W^\tau)$ are martingales since both f and g are harmonic on $\mathscr{O}$ (see e.g. [89, Theorem 18.5]). By Itô's formula and the fact that both f and g are harmonic we have

$$
\begin{aligned}
M_t = f(W_t^\tau) &= f(0) + \int_0^t \nabla f(W_s^\tau)\,\mathrm{d}W_s^\tau, \quad t \geq 0,\\
N_t = g(W_t^\tau) &= \int_0^t \nabla g(W_s^\tau)\,\mathrm{d}W_s^\tau, \quad t \geq 0,
\end{aligned}
$$

where in the second line we have used the equality $g(0) = 0$. Therefore for any $x^* \in X^*$ and any $0 \leq u \leq t$ we have

$$
\begin{aligned}
[\langle N, x^*\rangle]_t - [\langle N, x^*\rangle]_u &= \int_u^t \|\langle \nabla g(W_s^\tau), x^*\rangle\|^2\,\mathrm{d}s\\
&\leq \int_u^t \|\langle \nabla f(W_s^\tau), x^*\rangle\|^2\,\mathrm{d}s = [\langle M, x^*\rangle]_t - [\langle M, x^*\rangle]_u,
\end{aligned}
$$

and

$$
[\langle M, x^*\rangle, \langle N, x^*\rangle]_t = \int_0^t \Big\langle \langle \nabla g(W_s^\tau), x^*\rangle, \langle \nabla f(W_s^\tau), x^*\rangle \Big\rangle\,\mathrm{d}s = 0.
$$

Consequently, M and N are orthogonal and $N \overset{w}{\ll} M$, so

$$
\begin{aligned}
\int_{\partial\mathscr{O}} \Psi(g(s))\,\mathrm{d}\mu(s) &= \lim_{t\to\infty} \mathbb{E}\Psi(g(W_t^\tau))\\
&\leq \lim_{t\to\infty} |\mathscr{H}_X^{\mathbb{T}}|_{\Phi,\Psi}\mathbb{E}\Phi(f(W_t^\tau)) = |\mathscr{H}_X^{\mathbb{T}}|_{\Phi,\Psi} \int_{\partial\mathscr{O}} \Phi(f(s))\,\mathrm{d}\mu(s).
\end{aligned}
$$

Here the first and the last equality follow from the dominated convergence theorem and the definition of μ, while the middle one is due to Theorem 6.3.1.

The sharpness of the constant $C_{\Phi,\Psi,X} = |\mathscr{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$ follows from the case $d = 2$, $\mathscr{O} \subset \mathbb{R}^2$ being the unit disc, f and g being such that $g|_{\partial\mathscr{O}} = \mathscr{H}_X^{\mathbb{T}}(f|_{\partial\mathscr{O}})$ (in this case μ becomes the probability Lebesgue measure on the unit circle $\partial\mathscr{O}$). □

Remark 6.4.35. Sharpness of the estimate

$$
\int_{\partial\mathscr{O}} \Psi(g(s))\,\mathrm{d}\mu(s) \leq |\mathscr{H}_X^{\mathbb{T}}|_{\Phi,\Psi} \int_{\partial\mathscr{O}} \Phi(f(s))\,\mathrm{d}\mu(s)
$$

for a fixed domain $\mathscr{O}$ remains open. Nevertheless, in the case $d = 2$ and $\mathscr{O}$ being bounded with a Jordan boundary (e.g. polygon-shaped) the sharpness follows immediately from the Carathéodory's theorem (see e.g. [63, Subsection I.3 and Appendix F]).

Let us turn to the corresponding result for L^p-estimates for differentially subordinate harmonic functions (i.e., not necessarily orthogonal).

Theorem 6.4.36. *Let X, d and $\mathscr{O}$ be as in the previous statement. Assume further that $f, g : \overline{\mathscr{O}} \to X$ are continuous functions harmonic on $\mathscr{O}$ satisfying $g \overset{w}{\ll} f$ and $g(0) = a_0 f(0)$ for some $a_0 \in [-1,1]$. Then for any $1 < p < \infty$ we have*

$$\Big(\int_{\partial\mathscr{O}} \|g(s)\|^p \,\mathrm{d}\mu(s)\Big)^{\frac{1}{p}} \leq C_{p,X}\Big(\int_{\partial\mathscr{O}} \|f(s)\|^p \,\mathrm{d}\mu(s)\Big)^{\frac{1}{p}}, \tag{6.4.32}$$

where μ is the harmonic measure of $\partial\mathscr{O}$, and the least admissible constant $C_{p,X}$ is within the segment $[\hbar_{p,X}, \beta_{p,X} + \hbar_{p,X}]$.

Remark 6.4.37. In the scalar-valued setting it is known that the optimal $C_{p,\mathbb{R}}$ is within the range $[\cot(\frac{\pi}{2p^*}), p^* - 1]$. The precise identification of $C_{p,\mathbb{R}}$ is an open problem formulated by Burkholder in [37].

Proof of Theorem 6.4.36. This is quite similar to the proof of the latter statement, so we weill be brief and only indicate the necessary changes which need to be implemented. For the lower bound $C_{p,X} \geq \hbar_{p,X}$, modify appropriately the last sentence of the proof of Theorem 6.4.33. To show the upper bound for $C_{p,X}$, consider the martingales $M := f(W^\tau)$ and $N := g(W^\tau)$, where W and τ are as previously. Arguing as in the proof of Theorem 6.4.33, we show that $N \overset{w}{\ll} M$ and hence

$$\begin{aligned}
\Big(\int_{\partial\mathscr{O}} \|g(s)\|^p \,\mathrm{d}\mu(s)\Big)^{\frac{1}{p}} &= \lim_{t\to\infty} (\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \\
&\leq \limsup_{t\to\infty} (\beta_{p,X} + \hbar_{p,X})(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}} \\
&\leq \lim_{t\to\infty} (\beta_{p,X} + \hbar_{p,X})(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}} \\
&= (\beta_{p,X} + \hbar_{p,X})\Big(\int_{\partial\mathscr{O}} \|f(s)\|^p \,\mathrm{d}\mu(s)\Big)^{\frac{1}{p}}.
\end{aligned}$$

This completes the proof. □

Remark 6.4.38. Note that any significant improvement for the upper bound of $C_{p,X}$ in (6.4.32) could automatically solve an open problem. Let us outline two remarkable examples. If one could show that $C_{p,X} \leq C\beta_{p,X}$ for some universal constant $C > 0$, then the open problem outlined in Remark 2.3.2 will be solved. On the other hand, if one could show that $C_{p,X} = \hbar_{p,X}$, then the question of Burkholder concerning the optimal constant $C_{p,\mathbb{R}}$ in the real-valued case would be answered (see Remark 6.4.37).

6.4.6. *Inequalities for singular integral operators*

Our final application concerns the extension of Φ, Ψ-estimates from the setting of nonperiodic Hilbert transform to the case of odd-kernel singular integral operators on $\mathbb{R}^d$. We start with the notion of a *directional Hilbert transform*: given a unit vector $\theta \in \mathbb{R}^d$, we define the operator $\mathcal{H}_\theta$ by

$$\mathcal{H}_\theta f(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(x - t\theta)\frac{\mathrm{d}t}{t}, \quad x \in \mathbb{R}^d,$$

where f is a sufficiently regular real-valued function on $\mathbb{R}^d$, and call it the Hilbert transform of f in the direction θ. For example, if e_1 stands for the unit vector $(1,0,0,\dots,0) \in \mathbb{R}^d$, then $\mathcal{H}_{e_1}$ is obtained by applying the Hilbert transform in the first variable followed by the identity operator in the remaining variables. Consequently, by Fubini's theorem, we see that for any functions Φ, $\Psi : X \to [0,\infty)$ and any step function $f : \mathbb{R}^d \to X$ (finite linear combination of characteristic functions of rectangles) we have

$$\int_{\mathbb{R}^d} \Psi(\mathcal{H}_{e_1} f)\mathrm{d}x \le |\mathcal{H}_X^{\mathbb{R}}|_{\Phi,\Psi}\int_{\mathbb{R}^d}\Phi(f)\mathrm{d}x.$$

Now, if A is an arbitrary orthogonal matrix, we have

$$\mathcal{H}_{Ae_1}(f)(x) = \mathcal{H}_{e_1}(f\circ A)(A^{-1}x), \quad x \in \mathbb{R}^d,$$

so the above inequality holds true for any directional Hilbert transform $\mathcal{H}_\theta$.

Suppose that $\Omega : S^{d-1} \to \mathbb{R}$ is an odd function satisfying $||\Omega||_{L^1(S^{d-1})} = 1$ and define the associated operator

$$T_\Omega f(x) = \frac{2}{\pi}\text{p.v.}\int_{\mathbb{R}^d}\frac{\Omega(y/|y|)}{|y|^d}f(x-y)\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

Then T_Ω can be expressed as an average of directional Hilbert transforms:

$$T_\Omega f(x) = \int_{S^{d-1}}\Omega(\theta)\mathcal{H}_\theta f(x)\mathrm{d}\theta, \quad x \in \mathbb{R}^d.$$

(Sometimes this identity is referred to as the method of rotations.) Consequently, if Ψ is convex and even, we get

$$\begin{aligned}\int_{\mathbb{R}^d}\Psi(T_\Omega f)\mathrm{d}x &= \int_{\mathbb{R}^d}\Psi\left(\int_{S^{d-1}}\Omega(\theta)\mathcal{H}_\theta f(x)\mathrm{d}\theta\right)\mathrm{d}x\\ &\le \int_{S^{d-1}}|\Omega(\theta)|\int_{\mathbb{R}^d}\Psi(\mathcal{H}_\theta f(x))\mathrm{d}x\mathrm{d}\theta \le |\mathcal{H}_{\Phi,\Psi}^{\mathbb{R}}|\int_{\mathbb{R}^d}\Phi(f)\mathrm{d}x.\end{aligned}$$

In particular, if we fix d and $j \in \{1,2,\dots,d\}$, then the kernel

$$\Omega_{j,d}(\theta) = \frac{\pi\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d+1)/2}}\theta_j, \qquad \theta \in S^{d-1},$$

gives rise to the Riesz transform R_j. Therefore, we see that any Φ,Ψ-estimate for the nonperiodic Hilbert transform (where Ψ is assumed to be a convex and odd function on X) holds true, with an unchanged constant, also in the context of Riesz transforms.

The following theorem connects the Φ,Ψ-norm of an odd power of a Riesz transform with the Φ,Ψ-norm of the Hilbert transform.

Theorem 6.4.39. *Let X be a Banach space, $d \geq 1$, $j \in \{1,\dots,d\}$, $m \geq 1$ be odd. Let $R_{j,X}$ be the corresponding Riesz transform acting on X-valued step functions, $\Phi,\Psi : X \to \mathbb{R}_+$ be convex continuous such that Ψ is even. Then*

$$|R^m_{j,X}|_{\Phi,\Psi} \leq \Big|\frac{2\Gamma(\frac{m+d}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{m}{2})}\mathscr{H}^{\mathbb{R}}_X\Big|_{\Phi,\Psi}.$$

Proof. The proof follows from the discussion above, the fact that $R^m_{j,X}$ is a singular integral of the following form (see e.g. [83, p. 33]):

$$R^m_{j,X} f(x) = \frac{\Gamma(\frac{m+d}{2})}{\pi^{\frac{d}{2}}\Gamma(\frac{m}{2})}\int_{\mathbb{R}^d}\frac{f(x-y)y_j^m}{|y|^{m+d}}\,\mathrm{d}y, \quad x \in \mathbb{R}^d,$$

where $f : \mathbb{R}^d \to X$ is a step function, and the fact that the volume of S^{d-1} equals $2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$. □

Notice that if d is fixed, then $\frac{2\Gamma(\frac{m+d}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{m}{2})}$ is of the order $m^{d/2}$, so in particular we have that for all $1 < p < \infty$

$$\|R^m_{j,X}\|_{L^p(\mathbb{R}^d;X)\to L^p(\mathbb{R}^d;X)} \eqsim_d m^{d/2}\|\mathscr{H}^{\mathbb{R}}_X\|_{L^p(\mathbb{R};X)\to L^p(\mathbb{R};X)}.$$

6.4.7. Hilbert operators

Let X be a Banach space, let d be a positive integer and pick $j \in \{1,\dots,d\}$. Let $f : \mathbb{R}^d_{j+} \to X$ be locally integrable function, where $\mathbb{R}^d_{j+} = \{x \in \mathbb{R}^d : x_j > 0\}$. We define $T_j f : \mathbb{R}^d_{j+} \to X$ by the formula

$$T_j f(x) := \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}}\int_{\mathbb{R}^d_{j+}}\frac{f(y)(x_j+y_j)}{|x+y|^{d+1}}\,\mathrm{d}y, \quad x \in \mathbb{R}^d_{j+}.$$

This type of operators resembles Riesz transforms, but due to the domain restrictions the use of principal value is not necessary. Note that if $d = 1$, then T_j is the *Hilbert operator T* given by

$$Tf(x) := \frac{1}{\pi}\int_{\mathbb{R}_+}\frac{f(y)}{x+y}\,\mathrm{d}y, \quad x \in \mathbb{R}_+.$$

We have the following statement.

Theorem 6.4.40. *Let X be a Banach space, $\Phi, \Psi : X \to \mathbb{R}_+$ be convex continuous such that Ψ is even, $d \geq 1$, $j \in \{1, \dots, d\}$, $1 < p < \infty$. Then*

$$|T_j|_{\Phi,\Psi} \leq |\mathcal{H}_X^{\mathbb{R}}|_{\Phi,\Psi}. \tag{6.4.33}$$

Proof. By the discussion in Subsection 6.4.6 it is sufficient to show that

$$|T_j|_{\Phi,\Psi} \leq |R_{j,X}|_{\Phi,\Psi}.$$

Fix a step function $f : \mathbb{R}^d_{j+} \to X$. Let $\tilde{f} : \mathbb{R}^d \to X$ be such that $\tilde{f}(x_1, \dots, x_d) = 0$ if $x_j < 0$ and $\tilde{f}|_{\mathbb{R}^d_{j+}} = f$. Then $T_j f(x) = R_{j,X}\tilde{f}(-x)$ for any $x \in \mathbb{R}^d_{j+}$, and therefore

$$\begin{aligned}
\int_{\mathbb{R}^d_{j+}} \Psi(T_j f(x))\,\mathrm{d}x &= \int_{\mathbb{R}^d} \Psi(R_{j,X}\tilde{f}(-x))\mathbf{1}_{x_j>0}\,\mathrm{d}x \leq \int_{\mathbb{R}^d} \Psi(R_{j,X}\tilde{f}(-x))\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d} \Psi(R_{j,X}\tilde{f}(x))\,\mathrm{d}x \leq |R_{j,X}|_{\Phi,\Psi} \int_{\mathbb{R}^d} \Phi(\tilde{f}(x))\,\mathrm{d}x\\
&= |R_{j,X}|_{\Phi,\Psi} \int_{\mathbb{R}^d_{j+}} \Phi(f(x))\,\mathrm{d}x.
\end{aligned}$$

□

Remark 6.4.41. Notice that if Φ and Ψ are of the form $\Phi(x) = \phi(\|x\|)$, $\Psi(x) = \psi(\|x\|)$ for some convex symmetric functions $\phi, \psi : \mathbb{R} \to \mathbb{R}_+$, then one can improve (6.4.33). Indeed, one can show that $|T_j|_{\Phi,\Psi} = |T_j|_{\phi,\psi}$, which does not depend on the Banach space X: for any step function $f : \mathbb{R}^d_{j+} \to X$ one has that

$$\begin{aligned}
\int_{\mathbb{R}^d_{j+}} \Psi(T_j f(x))\,\mathrm{d}x &= \int_{\mathbb{R}^d_{j+}} \psi(\|T_j f(x)\|)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d_{j+}} \psi\Big(\Big\|\frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d_{j+}} \frac{f(y)(x_j + y_j)}{|x+y|^{d+1}}\,\mathrm{d}y\Big\|\Big)\,\mathrm{d}x\\
&\leq \int_{\mathbb{R}^d_{j+}} \psi\Big(\frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d_{j+}} \frac{g(y)(x_j + y_j)}{|x+y|^{d+1}}\,\mathrm{d}y\Big)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d_{j+}} \psi(T_j g(x))\,\mathrm{d}x \leq |T_j|_{\phi,\psi} \int_{\mathbb{R}^d_{j+}} \phi(g(x))\,\mathrm{d}x\\
&= |T_j|_{\phi,\psi} \int_{\mathbb{R}^d_{j+}} \Phi(f(x))\,\mathrm{d}x,
\end{aligned}$$

where $g : \mathbb{R}^d_{j+} \to \mathbb{R}_+$ is a step function such that $g(\cdot) = \|f(\cdot)\|$. In particular, if $\Phi(x) = \Psi(x) = \|x\|^p$ for some $1 < p < \infty$, then by [145, Theorem 1.1]

$$\|T_j\|_{L^p(\mathbb{R}^d_{j+};X) \to L^p(\mathbb{R}^d_{j+};X)} = \sin^{-1}(\pi/p).$$

III

Stochastic integration and Burkholder–Davis–Gundy inequalities

7

L^q-valued Burkholder-Rosenthal inequalities and sharp estimates for stochastic integrals

This chapter is based on the paper *L^q-valued Burkholder-Rosenthal inequalities and sharp estimates for stochastic integrals* by Sjoerd Dirksen and Ivan Yaroslavtsev, see [54].

We prove sharp maximal inequalities for L^q-valued stochastic integrals with respect to any Hilbert space-valued local martingale. Our proof relies on new Burkholder-Rosenthal type inequalities for martingales taking values in an L^q-space.

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7.1. INTRODUCTION

This work is motivated by the semigroup approach to stochastic partial differential equations. In this approach one first reformulates an SPDE as a stochastic ordinary differential equation in a suitable infinite-dimensional state space X and then establishes existence, uniqueness and regularity properties of a mild solution via a fixed point argument. An important ingredient for this argument is a maximal inequality for the X-valued stochastic convolution associated with the semigroup generated by the operator in the stochastic evolution equation. The semigroup approach for equations driven by Gaussian noise in Hilbert spaces is well-established and can be found in [47]. This theory has more recently been developed in two directions. Firstly, the theory for equations driven by Gaussian noise has been extended to the context of UMD Banach spaces, see e.g. [126, 127, 128]. In particular, the latter results cover L^q-spaces and Sobolev spaces and, as a consequence, allow to achieve better regularity results than the Hilbert space theory. Secondly, there has been increased interested in equations driven by discontinuous noise, e.g. Poisson- and Lévy-type noise [26, 58, 97, 111, 112, 113, 148]. The latter results are mostly restricted to the Hilbert space setting. The development of this theory in a non-Hilbertian setting is hindered by the fact that maximal inequalities for vector-valued stochastic convolutions with respect to discontinuous noise are not yet well-understood. In general, only some non-sharp maximal estimates based on geometric assumptions on the Banach space are available [52, 193]. In fact, even the theory for 'vanilla' stochastic integrals (corresponding to the trivial semigroup) is incomplete. Sharp maximal inequalities for L^q-valued stochastic integrals with respect to Poisson random measures were obtained only recently [51].

The main purpose of the present chapter is to contribute to the foundation of the semigroup approach by proving sharp estimates for L^q-valued stochastic integrals with respect to general Hilbert-space valued local martingales. In our main result, Theorem 7.5.30, we identify a suitable norm $\|\!|\cdot|\!\|_{M,p,q}$ so that, for any elementary predictable processes Φ with values in the bounded operators from H into $L^q(S)$,

$$c_{p,q}\,\|\!|\Phi|\!\|_{M,p,q} \leq \Bigl(\mathbb{E}\sup_{0\leq s\leq t}\Bigl\|\int_0^s \Phi\,dM\Bigr\|^p_{L^q(S)}\Bigr)^{\frac{1}{p}} \leq C_{p,q}\,\|\!|\Phi|\!\|_{M,p,q}, \tag{7.1.1}$$

with universal constants $c_{p,q}$, $C_{p,q}$ depending only on p and q. Let us emphasize two important points. Firstly, the norm $\|\!|\cdot|\!\|_{M,p,q}$ can be computed in terms of *predictable* quantities, which is important for applications. Secondly, we call the estimates in (7.1.1) 'sharp' as these inequalities are two-sided and therefore identify the largest possible class of L^p-stochastically integrable processes. We do not require the constants $c_{p,q}$ and $C_{p,q}$ to be sharp or even to depend optimally on p and q. For applications to stochastic evolution equations, the precise constants in fact do not play a role. In forthcoming work together with Marinelli [53], we show that the upper bound (7.1.1) can be transferred to a large class of stochastic convolutions and apply these new estimates to obtain improved well-posedness and regularity results for the associated stochastic evolution equations in L^q-spaces.

Let us roughly sketch our approach to (7.1.1). As a starting point, we use a classical result due to Meyer [122] and Yoeurp [190] to decompose the integrator as a sum of three local martingales $M = M^c + M^q + M^a$, where M^c is continuous, M^q is purely discontinuous and quasi-left continuous, and M^a is purely discontinuous with accessible jumps. Sharp bounds for stochastic integrals with respect to continuous local martingales were already obtained in a more general setting [177].

To estimate the integral with respect to M^a we prove, more generally, sharp bounds for an arbitrary purely discontinuous L^q-valued local martingale with accessible jumps in Theorem 7.5.8. To establish this result we first show that such a process can be represented as an essentially discrete object, namely a sum of jumps occurring at predictable times. Using an approximation argument, the problem can then be further reduced to proving *Burkholder-Rosenthal type inequalities* for L^q-valued discrete-time martingales. In general, if $1 \le p < \infty$ and X is a Banach space, we understand under Burkholder-Rosenthal inequalities estimates for X-valued martingale difference sequences (d_i) of the form

$$c_{p,X} \,|||\, (d_i) \,|||_{p,X} \le \Big(\mathbb{E}\Big\| \sum_i d_i \Big\|_X^p\Big)^{\frac{1}{p}} \le C_{p,X} \,|||\, (d_i) \,|||_{p,X}, \tag{7.1.2}$$

where $|||\cdot|||_{p,X}$ is a suitable norm on (d_i) which can be computed explicitly in terms of the *predictable moments* of the individual differences d_i. In the scalar-valued case, these type of inequalities were proven by Burkholder [29], following work of Rosenthal [161] in the independent case: for $2 \le p < \infty$

$$\Big(\mathbb{E}\Big|\sum_{i=1}^n d_i\Big|^p\Big)^{\frac{1}{p}} \eqsim_p \max\Big\{\Big(\sum_{i=1}^n \mathbb{E}|d_i|^p\Big)^{\frac{1}{p}}, \Big(\mathbb{E}\Big(\sum_{i=1}^n \mathbb{E}_{i-1}|d_i|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big\}. \tag{7.1.3}$$

Here we write $A \lesssim_\alpha B$ if there is a constant $c_\alpha > 0$ depending only on α such that $A \le c_\alpha B$ and write $A \eqsim_\alpha B$ if both $A \lesssim_\alpha B$ and $B \lesssim_\alpha A$ hold. To state our L^q-valued extension, we fix a filtration $\mathbb{F} = (\mathscr{F}_i)_{i\ge 0}$, denote by $(\mathbb{E}_i)_{i\ge 0}$ the associated sequence of conditional expectations and set $\mathbb{E}_{-1} := \mathbb{E}$. Let (S, Σ, ρ) be any measure space. Let us introduce the following norms on the linear space of all finite sequences (f_i) of random variables in $L^\infty(\Omega; L^q(S))$. Firstly, for $1 \le p, q < \infty$ we set

$$\|(f_i)\|_{S_q^p} = \Big(\mathbb{E}\Big\|\Big(\sum_i \mathbb{E}_{i-1}|f_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}, \tag{7.1.4}$$

From the work of Junge on conditional sequence spaces [87] one can deduce that this expression is a norm. We let S_q^p denote the completion with respect to this norm. Furthermore, we define

$$\begin{aligned} \|(f_i)\|_{D^p_{q,q}} &= \Big(\mathbb{E}\Big(\sum_i \mathbb{E}_{i-1}\|f_i\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}, \\ \|(f_i)\|_{D^p_{p,q}} &= \Big(\sum_i \mathbb{E}\|f_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}}. \end{aligned} \tag{7.1.5}$$

Clearly these expressions define two norms and we let $D^p_{p,q}$ and $D^p_{q,q}$ denote the completions in these norms. Although these spaces depend on the filtration $\mathbb{F}$, we will suppress this from the notation. We let $\hat{S}^p_q$, $\hat{D}^p_{q,q}$ and $\hat{D}^p_{p,q}$ denote the closed subspaces spanned by all martingale difference sequences in the above spaces.

Theorem 7.1.1. *Let $1 < p, q < \infty$ and let S be any measure space. If (d_i) is an $L^q(S)$-valued martingale difference sequence, then*

$$\Big(\mathbb{E}\Big\|\sum_i d_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} \|(d_i)\|_{\hat{s}_{p,q}}, \tag{7.1.6}$$

where $\hat{s}_{p,q}$ is given by

$$\begin{aligned} &\hat{S}^p_q \cap \hat{D}^p_{q,q} \cap \hat{D}^p_{p,q} &&\text{if } 2 \le q \le p < \infty; \\ &\hat{S}^p_q \cap (\hat{D}^p_{q,q} + \hat{D}^p_{p,q}) &&\text{if } 2 \le p \le q < \infty; \\ &(\hat{S}^p_q \cap \hat{D}^p_{q,q}) + \hat{D}^p_{p,q} &&\text{if } 1 < p < 2 \le q < \infty; \\ &(\hat{S}^p_q + \hat{D}^p_{q,q}) \cap \hat{D}^p_{p,q} &&\text{if } 1 < q < 2 \le p < \infty; \\ &\hat{S}^p_q + (\hat{D}^p_{q,q} \cap \hat{D}^p_{p,q}) &&\text{if } 1 < q \le p \le 2; \\ &\hat{S}^p_q + \hat{D}^p_{q,q} + \hat{D}^p_{p,q} &&\text{if } 1 < p \le q \le 2. \end{aligned}$$

Consequently, if $\mathscr{F} = \sigma(\cup_{i\ge 0}\mathscr{F}_i)$, then the map $f \mapsto (\mathbb{E}_i f - \mathbb{E}_{i-1} f)_{i\ge 0}$ induces an isomorphism between $L^p_0(\Omega; L^q(S))$, the subspace of mean-zero random variables in $L^p(\Omega; L^q(S))$, and $s_{p,q}$.

Let us say a few words about the proof of Theorem 7.1.1. We derive the upper bound in (7.1.6) from the known special case that the d_i are independent [51] by applying powerful decoupling techniques due to Kwapień and Woyczyński [102]. In the scalar-valued case this route was already traveled by Hitczenko [75] to deduce the optimal order of the constant in the classical Burkholder-Rosenthal inequalities (7.1.3) from the one already known for martingales with independent increments. The lower bound in (7.1.6) is derived by using a duality argument. For this purpose, we show that for $1 < p, q < \infty$ the spaces $s_{p,q}$ satisfy the duality relation

$$(s_{p,q})^* = s_{p',q'}, \ \frac{1}{p} + \frac{1}{p'} = 1, \ \frac{1}{q} + \frac{1}{q'} = 1.$$

The only non-trivial step in proving this duality is to show that $(D^p_{q,q})^* = D^{p'}_{q',q'}$. In Section 7.4 we prove a more general result: we show that if X is a reflexive separable Banach space, then for the space $H^{s_q}_p(X)$ of all adapted X-valued sequences (f_i)

such that

$$\|(f_i)\|_{H_p^{s_q}(X)} = \Big(\mathbb{E}\Big(\sum_i \mathbb{E}_{i-1}\|f_i\|_X^q\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} < \infty,$$

the identity $(H_p^{s_q}(X))^* = H_{p'}^{s_{q'}}(X^*)$ holds isomorphically with constants depending only on p and q. Somewhat surprisingly, this result only seems to be known in the literature if $X = \mathbb{R}$ and either $1 < p \leq q < \infty$ or $2 \leq q \leq p < \infty$ (see [181]).

Let us now discuss our approach to the integral of Φ with respect to M^q, the purely discontinuous quasi-left continuous part of M. We first show that this integral can be represented as an integral with respect to $\bar{\mu}^{M^q}$, the compensated version of the random measure μ^{M^q} that counts the jumps of M^q. In Theorem 7.5.22 we then prove the following sharp estimates for integrals with respect to $\bar{\mu} = \mu - \nu$, where μ is any integer-valued random measure that has a compensator ν that is non-atomic in time. This result covers μ^{M^q} as a special case. To formulate our result, let $(J, \mathscr{J})$ be a measurable space and $\widetilde{\mathscr{P}}$ be the predictable σ-algebra on $\mathbb{R}_+ \times \Omega \times J$. For $1 < p, q < \infty$ we define the spaces $\hat{\mathscr{S}}_q^p$, $\hat{\mathscr{D}}_{q,q}^p$ and $\hat{\mathscr{D}}_{p,q}^p$ as the Banach spaces of all $\widetilde{\mathscr{P}}$-measurable functions $F : \mathbb{R}_+ \times \Omega \times J \to L^q(S)$ for which the corresponding norms

$$\|F\|_{\hat{\mathscr{S}}_q^p} := \Big(\mathbb{E}\Big\|\Big(\int_{\mathbb{R}_+\times J} |F|^2 \,\mathrm{d}\nu\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}},$$

$$\|F\|_{\hat{\mathscr{D}}_{q,q}^p} := \Big(\mathbb{E}\Big(\int_{\mathbb{R}_+\times J} \|F\|_{L^q(S)}^q \,\mathrm{d}\nu\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}},$$

$$\|F\|_{\hat{\mathscr{D}}_{p,q}^p} := \Big(\mathbb{E}\int_{\mathbb{R}_+\times J} \|F\|_{L^q(S)}^p \,\mathrm{d}\nu\Big)^{\frac{1}{p}}$$

are finite.

Theorem 7.1.2. *Fix $1 < p, q < \infty$. Let μ be an optional $\widetilde{\mathscr{P}}$-σ-finite random measure on $\mathbb{R}_+ \times J$ and suppose that its compensator ν is non-atomic in time. Then for any $\widetilde{\mathscr{P}}$-measurable $F : \mathbb{R}_+ \times \Omega \times J \to L^q(S)$,*

$$\Big(\mathbb{E} \sup_{0\leq s\leq t} \Big\| \int_{[0,s]\times J} F(u,x)\bar{\mu}(\mathrm{d}u, \mathrm{d}x)\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \eqsim_{p,q} \|F\mathbf{1}_{[0,t]}\|_{\mathscr{I}_{p,q}},$$

where $\mathscr{I}_{p,q}$ is given by

$$\hat{\mathscr{S}}_q^p \cap \hat{\mathscr{D}}_{q,q}^p \cap \hat{\mathscr{D}}_{p,q}^p \text{ if } 2 \leq q \leq p < \infty,$$
$$\hat{\mathscr{S}}_q^p \cap (\hat{\mathscr{D}}_{q,q}^p + \hat{\mathscr{D}}_{p,q}^p) \text{ if } 2 \leq p \leq q < \infty,$$
$$(\hat{\mathscr{S}}_q^p \cap \hat{\mathscr{D}}_{q,q}^p) + \hat{\mathscr{D}}_{p,q}^p \text{ if } 1 < p < 2 \leq q < \infty,$$
$$(\hat{\mathscr{S}}_q^p + \hat{\mathscr{D}}_{q,q}^p) \cap \hat{\mathscr{D}}_{p,q}^p \text{ if } 1 < q < 2 \leq p < \infty,$$
$$\hat{\mathscr{S}}_q^p + (\hat{\mathscr{D}}_{q,q}^p \cap \hat{\mathscr{D}}_{p,q}^p) \text{ if } 1 < q \leq p \leq 2,$$
$$\hat{\mathscr{S}}_q^p + \hat{\mathscr{D}}_{q,q}^p + \hat{\mathscr{D}}_{p,q}^p \text{ if } 1 < p \leq q \leq 2.$$

In the scalar-valued case this result is due to A.A. Novikov [131]. In the special case that μ is a Poisson random measure, Theorem 7.5.22 was obtained in [51]. A very different proof of the upper bounds in Theorem 7.5.22, based on tools from stochastic analysis, was discovered independently of our work in [110].

The proof of the upper bounds in Theorem 7.1.2 relies on the Burkholder-Rosenthal inequalities in Theorem 7.1.1, a Banach space-valued extension of Novikov's inequality in the special case that $\nu(\mathbb{R}_+ \times J) \leq 1$ a.s. (Proposition 7.5.15), and a time-change argument. For the lower bounds, the non-trivial work is to show that

$$(\hat{\mathscr{S}}_q^p)^* = \hat{\mathscr{S}}_{q'}^{p'}, \qquad (\hat{\mathscr{D}}_{q,q}^p)^* = \hat{\mathscr{D}}_{q',q'}^{p'}$$

hold isomorphically with constants depending only on p and q. These duality statements are derived in Appendix 7.A.

The chapter is structured as follows. In Section 7.3 we prove Theorem 7.1.1. Section 7.4 contains the proof of the duality for the space $H_p^{s_q}(X)$. In Section 7.5 we prove the sharp bounds (7.1.1). In particular, Subsections 7.5.2, 7.5.3 and 7.5.5 are dedicated to integration with respect to local martingales with accessible jumps, purely discontinuous quasi-left continuous local martingales and continuous local martingales, respectively. These three parts can be read independently of each other.

7.2. PRELIMINARIES

Throughout, $(\Omega, \mathscr{F}, \mathbb{P})$ denotes a complete probability space. If X and Y are Banach spaces, then $\mathscr{L}(X, Y)$ denotes the Banach space of bounded linear operators from X into Y.

In the following, we will frequently use duality arguments for sums and intersections of Banach spaces. Let us recall some basic facts in this direction. If (X, Y) is a compatible couple of Banach spaces, i.e., X, Y are continuously embedded in a Hausdorff topological vector space, then their intersection $X \cap Y$ and sum $X + Y$ are Banach spaces under the norms

$$\|z\|_{X\cap Y} = \max\{\|z\|_X, \|z\|_Y\}$$

and

$$\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y : z = x + y,\ x \in X,\ y \in Y\}.$$

If $X \cap Y$ is dense in both X and Y, then

$$(X \cap Y)^* = X^* + Y^*, \qquad (X + Y)^* = X^* \cap Y^* \tag{7.2.1}$$

hold isometrically. The duality brackets under these identifications are given by

$$\langle x^*, x\rangle = \langle x^*|_{X\cap Y}, x\rangle \qquad (x^* \in X^* + Y^*)$$

and

$$\langle x^*, x\rangle = \langle x^*, y\rangle + \langle x^*, z\rangle \qquad (x^* \in X^* \cap Y^*,\ x = y + z \in X + Y), \tag{7.2.2}$$

respectively, see e.g. [98, Theorem I.3.1].

The following observation facilitates a duality argument that we will use repeatly below. We provide a proof for the convenience of the reader.

Lemma 7.2.1. *Let X and Y be Banach spaces, X be reflexive, U be a dense linear subspace of X, and let V be a dense linear subspace of X^*. Consider $j_0 \in \mathscr{L}(U, Y)$ and $k_0 \in \mathscr{L}(V, Y^*)$ so that* ran j_0 *is dense in Y and $\langle x^*, x\rangle = \langle k_0(x^*), j_0(x)\rangle$ for each $x \in U, x^* \in V$. Then*

(i) there exists $j \in \mathscr{L}(X, Y)$, $k \in \mathscr{L}(X^*, Y^*)$ such that $j|_U = j_0$, $k|_V = k_0$,

(ii) ran $j = Y$, ran $k = Y^*$, *in particular k and j are invertible, and*

(iii) for each $x \in X$ and $x^* \in X^*$

$$\begin{aligned} \frac{1}{\|k\|}\|x\| \le \|j(x)\| \le \|j\|\|x\|, \\ \frac{1}{\|j\|}\|x^*\| \le \|k(x^*)\| \le \|k\|\|x^*\|. \end{aligned} \tag{7.2.3}$$

Proof. (i) holds due to the continuity of j_0 and k_0, and as a consequence

$$\langle x^*, x\rangle = \langle k(x^*), j(x)\rangle,\ \ x \in X, x^* \in X^*.$$

Notice that j and k are embeddings. Indeed, $\langle x^*, x\rangle = \langle k(x^*), j(x)\rangle$ for each $x \in X$, $x^* \in X^*$, so for each nonzero $x \in X$, $x^* \in X^*$ both $j(x)$ and $k(x^*)$ define nonzero linear functionals on Y^* and Y respectively, hence they are nonzero.

For (ii), fix any $y^* \in Y^*$. Since $j \in \mathscr{L}(X, Y)$, we can define $x^* \in X^*$ by

$$\langle x^*, x\rangle := \langle y^*, j(x)\rangle, \qquad x \in X.$$

Since $\langle x^*, x\rangle = \langle k(x^*), j(x)\rangle$ and hence $\langle y^* - k(x^*), j(x)\rangle = 0$ for any $x \in X$, we conclude by density of ran j that $y^* = k(x^*)$. Thus ran $k = Y^*$ and k is invertible by the bounded inverse theorem. Using reflexivity of X one can similarly prove the statement for j. To prove (iii), we note that for each $x \in X$

$$\begin{aligned} \|j(x)\| &= \sup_{x^* \in X^*, \|k(x^*)\|=1} \langle k(x^*), j(x)\rangle = \sup_{x^* \in X^*, \|k(x^*)\|=1} \langle x^*, x\rangle \\ &\ge \sup_{x^* \in X^*, \|x^*\| = \frac{1}{\|k\|}} \langle x^*, x\rangle = \frac{1}{\|k\|}\|x\|, \end{aligned}$$

and obviously $\|j(x)\| \le \|j\|\|x\|$. The estimates for k are derived similarly. □

7.3. L^q-VALUED BURKHOLDER-ROSENTHAL INEQUALITIES

In this section we prove Theorem 7.1.1. Our starting point is the following L^q-valued version of the classical Rosenthal inequalities [161]. For $1 \le p, q < \infty$ let S_q and $D_{p,q}$ be the spaces of all sequences of $L^q(S)$-valued random variables such the respective norms

$$\begin{aligned} \|(f_i)\|_{S_q} &= \Big\|\Big(\sum_i \mathbb{E}|f_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}, \\ \|(f_i)\|_{D_{p,q}} &= \Big(\sum_i \mathbb{E}\|f_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \end{aligned} \tag{7.3.1}$$

are finite. Note that the following result corresponds to a special case of Theorem 7.1.1, in which the martingale differences d_i are independent.

Theorem 7.3.1. *[51] Let $1 < p, q < \infty$ and let (S, Σ, σ) be a measure space. If (ξ_i) is a sequence of independent, mean-zero random variables taking values in $L^q(S)$, then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} \|(\xi_i)\|_{s_{p,q}}, \tag{7.3.2}$$

where $s_{p,q}$ is given by

$$\begin{aligned} S_q \cap D_{q,q} \cap D_{p,q} \quad &\text{if } 2 \le q \le p < \infty; \\ S_q \cap (D_{q,q} + D_{p,q}) \quad &\text{if } 2 \le p \le q < \infty; \\ (S_q \cap D_{q,q}) + D_{p,q} \quad &\text{if } 1 < p < 2 \le q < \infty; \\ (S_q + D_{q,q}) \cap D_{p,q} \quad &\text{if } 1 < q < 2 \le p < \infty; \\ S_q + (D_{q,q} \cap D_{p,q}) \quad &\text{if } 1 < q \le p \le 2; \\ S_q + D_{q,q} + D_{p,q} \quad &\text{if } 1 < p \le q \le 2. \end{aligned}$$

Moreover, the estimate $\lesssim_{p,q}$ in (7.3.2) remains valid if $p = 1$, $q = 1$ or both.

To derive the upper bound in Theorem 7.1.1 we use the following decoupling techniques from [102]. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, let $(\mathscr{F}_i)_{i\ge0}$ be a filtration and let X be a (quasi-)Banach space. Two $(\mathscr{F}_i)_{i\ge1}$-adapted sequences $(d_i)_{i\ge1}$ and $(e_i)_{i\ge1}$ of X-valued random variables are called *tangent* if for every $i \ge 1$ and $A \in \mathscr{B}(X)$

$$\mathbb{P}(d_i \in A|\mathscr{F}_{i-1}) = \mathbb{P}(e_i \in A|\mathscr{F}_{i-1}). \tag{7.3.3}$$

An $(\mathscr{F}_i)_{i\ge1}$-adapted sequence $(e_i)_{i\ge1}$ of X-valued random variables is said to satisfy *condition (CI)* if, firstly, there is a sub-σ-algebra $\mathscr{G} \subset \mathscr{F}_\infty = \sigma(\cup_{i\ge0}\mathscr{F}_i)$ such that for every $i \ge 1$ and $A \in \mathscr{B}(X)$,

$$\mathbb{P}(e_i \in A|\mathscr{F}_{i-1}) = \mathbb{P}(e_i \in A|\mathscr{G}) \tag{7.3.4}$$

and, secondly, $(e_i)_{i\geq 1}$ consists of $\mathscr{G}$-independent random variables, i.e. for all $n \geq 1$ and $A_1, \ldots, A_n \in \mathscr{B}(X)$,

$$\mathbb{E}(\mathbf{1}_{e_1 \in A_1} \cdot \ldots \cdot \mathbf{1}_{e_n \in A_n} | \mathscr{G}) = \mathbb{E}(\mathbf{1}_{e_1 \in A_1} | \mathscr{G}) \cdot \ldots \cdot \mathbb{E}(\mathbf{1}_{e_n \in A_n} | \mathscr{G}).$$

It is shown in [102] that for every $(\mathscr{F}_i)_{i\geq 1}$-adapted sequence $(d_i)_{i\geq 1}$ there exists an $(\mathscr{F}_i)_{i\geq 1}$-adapted sequence $(e_i)_{i\geq 1}$ on a possibly enlarged probability space which is tangent to $(d_i)_{i\geq 1}$ and satisfies condition (CI). This sequence is called a *decoupled tangent sequence* for $(d_i)_{i\geq 1}$ and is unique in law.

To derive the upper bound in Theorem 7.1.1 for a given martingale difference sequence $(d_i)_{i\geq 1}$ we apply Theorem 7.3.1 conditionally to its decoupled tangent sequence $(e_i)_{i\geq 1}$. For this approach to work, we will need to relate various norms on $(d_i)_{i\geq 1}$ and $(e_i)_{i\geq 1}$. One of these estimates can be formulated as a Banach space property. Following [45], we say that a (quasi-)Banach space X satisfies *the p-decoupling property* if for some $0 < p < \infty$ there is a constant $C_{p,X}$ such that for any complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, any filtration $(\mathscr{F}_i)_{i\geq 0}$, and any $(\mathscr{F}_i)_{i\geq 1}$-adapted sequence $(d_i)_{i\geq 1}$ in $L^p(\Omega, X)$,

$$\Big(\mathbb{E}\Big\| \sum_{i=1}^n d_i \Big\|_X^p\Big)^{\frac{1}{p}} \leq C_{p,X} \Big(\mathbb{E}\Big\| \sum_{i=1}^n e_i \Big\|_X^p\Big)^{\frac{1}{p}}, \tag{7.3.5}$$

for all $n \geq 1$, where $(e_i)_{i\geq 1}$ is the decoupled tangent sequence of $(d_i)_{i\geq 1}$. It is shown in [45, Theorem 4.1] that this property is independent of p, so we may simply say that X satisfies the *decoupling property* if it satisfies the p-decoupling property for some (then all) $0 < p < \infty$. Known examples of spaces satisfying the decoupling property are the $L^q(S)$-spaces for any $0 < q < \infty$ and UMD Banach spaces. If X is a UMD Banach space, then one can also *recouple*, meaning that for all $1 < p < \infty$ there is a constant $c_{p,X}$ such that for any martingale difference sequence $(d_i)_{i\geq 1}$ and any associated decoupled tangent sequence $(e_i)_{i\geq 1}$,

$$\Big(\mathbb{E}\Big\| \sum_{i=1}^n e_i \Big\|_X^p\Big)^{\frac{1}{p}} \leq c_{p,X} \Big(\mathbb{E}\Big\| \sum_{i=1}^n d_i \Big\|_X^p\Big)^{\frac{1}{p}}. \tag{7.3.6}$$

Conversely, if both (7.3.5) and (7.3.6) hold for some (then all) $1 < p < \infty$, then X must be a UMD space. This equivalence is independently due to McConnell [119] and Hitczenko [74].

To further relate a sequence with its decoupled tangent sequence we use the following technical observation, which is a special case of [45, Lemma 2.7].

Lemma 7.3.2. *Let X be a (quasi-)Banach space and for every $i \geq 1$ let $h_i : X \to X$ be a Borel measurable function. Let $(d_i)_{i\geq 1}$ be an $(\mathscr{F}_i)_{i\geq 1}$-adapted sequence and $(e_i)_{i\geq 1}$ a decoupled tangent sequence. Then $(h_i(e_i))_{i\geq 1}$ is a decoupled tangent sequence for $(h_i(d_i))_{i\geq 1}$.*

We are now ready to prove the announced result.

Proof. (Of Theorem 7.1.1) Step 1: upper bounds. We will only give a proof in the case $1 \le q \le 2 \le p < \infty$. The other cases are proved analogously. Let us write $\mathbb{E}_{\mathcal{G}} = \mathbb{E}(\cdot|\mathcal{G})$ for brevity. By density we may assume that the d_i take values in $L^q(S) \cap L^\infty(S)$. Fix an arbitrary decomposition $d_i = d_{i,1} + d_{i,2}$, where $d_{i,1}, d_{i,2}$ are $L^q(S) \cap L^\infty(S)$-valued martingale difference sequences. Let $e_i = (e_{i,1}, e_{i,2})$ be the decoupled tangent sequence for the martingale difference sequence $(d_{i,1}, d_{i,2})$ which takes values in $(L^q(S) \cap L^\infty(S)) \times (L^q(S) \cap L^\infty(S))$. Lemma 7.3.2 implies that $d_{i,\alpha}$ is the decoupled tangent sequence for $e_{i,\alpha}$, $\alpha = 1,2$, and $e_{i,1} + e_{i,2}$ is the decoupled tangent sequence for d_i. By the decoupling property for $L^q(S)$,

$$\Big(\mathbb{E}\Big\|\sum_i d_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i e_{i,1} + e_{i,2}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}.$$

Since the summands $e_{i,1} + e_{i,2}$ are $\mathcal{G}$-conditionally independent and $\mathcal{G}$-mean zero, we can apply Theorem 7.3.1 conditionally to find, a.s.,

$$\begin{aligned}&\Big(\mathbb{E}_{\mathcal{G}}\Big\|\sum_i e_{i,1} + e_{i,2}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\\ &\quad\lesssim_{p,q} \max\Big\{\Big\|\Big(\sum_i \mathbb{E}_{\mathcal{G}}|e_{i,1}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}\\ &\qquad\qquad + \Big(\sum_i \mathbb{E}_{\mathcal{G}}\|e_{i,2}\|_{L^q(S)}^q\Big)^{\frac{1}{q}}, \Big(\sum_i \mathbb{E}_{\mathcal{G}}\|e_{i,1} + e_{i,2}\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\Big\}.\end{aligned}$$

Now we take L^p-norms on both sides and apply the triangle inequality to obtain

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\sum_i d_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\\ &\quad\lesssim_{p,q} \max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_i \mathbb{E}_{\mathcal{G}}|e_{i,1}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\\ &\qquad\qquad + \Big(\mathbb{E}\Big(\sum_i \mathbb{E}_{\mathcal{G}}\|e_{i,2}\|_{L^q(S)}^q\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|e_{i,1} + e_{i,2}\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\Big\}\end{aligned}$$

By the properties (7.3.4) and (7.3.3) of a decoupled tangent sequence,

$$\mathbb{E}_{\mathcal{G}}|e_{i,1}|^2 = \mathbb{E}_{i-1}|e_{i,1}|^2 = \mathbb{E}_{i-1}|d_{i,1}|^2,$$

and therefore

$$\Big(\sum_i \mathbb{E}_{\mathcal{G}}|e_{i,1}|^2\Big)^{\frac{1}{2}} = \Big(\sum_i \mathbb{E}_{i-1}|d_{i,1}|^2\Big)^{\frac{1}{2}}.$$

Similarly,

$$\mathbb{E}_{\mathcal{G}}\|e_{i,2}\|_{L^q(S)}^q = \mathbb{E}_{i-1}\|d_{i,2}\|_{L^q(S)}^q.$$

We conclude that

$$\Big(\mathbb{E}\Big\|\sum_i d_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}$$

$$\lesssim_{p,q} \max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_i \mathbb{E}_{i-1}|d_{i,1}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} + \Big(\mathbb{E}\Big(\sum_i \mathbb{E}_{i-1}\|d_{i,2}\|_{L^q(S)}^q\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|d_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\Big\}.$$

Taking the infimum over all decompositions as above yields the inequality '$\lesssim_{p,q}$' in (7.1.6).

Step 2: lower bounds. We deduce the lower bounds by duality. Since $(S_q^p)^* = S_{q'}^{p'}$ (by [87]), $(D_{p,q}^p)^* = D_{p',q'}^{p'}$, and $(D_{q,q}^p)^* = D_{q',q'}^{p'}$ (by Theorem 7.4.1 below) hold isomorphically with constants depending only on p and q, it follows from (7.2.1) that $s_{p,q}^* = s_{p',q'}$ with duality bracket

$$\langle (f_i),(g_i)\rangle = \sum_i \mathbb{E}\langle f_i, g_i\rangle \qquad ((f_i)\in s_{p,q},\ (g_i)\in s_{p',q'}).$$

Let $\hat{x}^* \in (\hat{s}_{p,q})^*$. Define the map $P: s_{p,q} \to \hat{s}_{p,q}$ by

$$P((f_i)) = (\Delta_i f_i),$$

where $\Delta_i := \mathbb{E}_i - \mathbb{E}_{i-1}$. By the triangle inequality and Jensen's inequality one readily sees that P is a bounded projection. As a consequence, we can define $x^* \in s_{p,q}^*$ by $x^* = \hat{x}^* \circ P$. Let $(g_i) \in s_{p',q'}$ be such that

$$x^*((f_i)) = \sum_i \mathbb{E}\langle f_i, g_i\rangle \qquad ((f_i)\in s_{p,q}).$$

Then, for any $(f_i) \in \hat{s}_{p,q}$,

$$\hat{x}^*((f_i)) = \sum_i \mathbb{E}\langle f_i, g_i\rangle = \sum_i \mathbb{E}\langle f_i, \Delta_i g_i\rangle = \langle (f_i), P(g_i)\rangle.$$

This shows that $(\hat{s}_{p,q})^* = \hat{s}_{p',q'}$ isomorphically. Let U and V be the dense linear subspaces spanned by all finite martingale difference sequences in $\hat{s}_{p,q}$ and $\hat{s}_{p',q'}$, respectively. Define

$$Y = \overline{\operatorname{span}}\Big\{\sum_i d_i \ :\ (d_i)\in U\Big\} \subset L^p(\Omega; L^q(S)).$$

By Step 1, we can define two maps $j_0 \in \mathscr{L}(U,Y)$, $k_0 \in \mathscr{L}(V,Y^*)$ by

$$j_0((d_i)) = \sum_i d_i, \qquad k_0((\tilde{d}_i)) = \sum_i \tilde{d}_i.$$

By the martingale difference property,

$$\langle j_0((d_i)), k_0((\tilde{d}_i))\rangle = \mathbb{E}\Big\langle \sum_i d_i, \sum_i \tilde{d}_i\Big\rangle = \sum_i \mathbb{E}\langle d_i, \tilde{d}_i\rangle = \langle (d_i), (\tilde{d}_i)\rangle. \tag{7.3.7}$$

The lower bounds now follow immediately from Lemma 7.2.1.

For the final assertion of the theorem, suppose that $\mathscr{F} = \sigma(\cup_{i\geq 0}\mathscr{F}_i)$. Let $f \in L^p_0(\Omega; L^q(S))$ and define $f_n = \mathbb{E}_n f$. Then $\lim_{n\to\infty} f_n = f$ (see e.g. [79, Theorem 3.3.2]). Conversely, let $(f_n)_{n\geq 1}$ be a martingale with $\sup_{n\geq 1} \|f_n\|_{L^p(\Omega;L^q(S))} < \infty$. By reflexivity of $L^q(S)$ we have $L^p(\Omega; L^q(S)) = (L^{p'}(\Omega; L^{q'}(S)))^*$ and hence its unit ball is weak*-compact. Let f be the weak*-limit of (f_n). It is easy to check that $f_n = \mathbb{E}_n f$. In conclusion, any martingale difference sequence $(d_i)_{i\geq 0}$ of a bounded martingale in $L^p(\Omega; L^q(S))$ corresponds uniquely to an $f \in L^p(\Omega; L^q(S))$ such that

$$f - \mathbb{E}f = \sum_i d_i, \qquad d_i = \mathbb{E}_i f - \mathbb{E}_{i-1} f.$$

The two-sided inequality (7.1.6) now implies that the map $f \mapsto (\mathbb{E}_i f - \mathbb{E}_{i-1} f)_{i\geq 0}$ is a linear isomorphism between $L^p_0(\Omega; L^q(S))$ and $\hat{s}_{p,q}$, with constants depending only on p and q. □

Remark 7.3.3. Let $1 < p, q < \infty$. Define $\hat{S}^{p,odd}_q$, $\hat{D}^{p,odd}_{q,q}$ and $\hat{D}^{p,odd}_{p,q}$ as the closed subspaces of $\hat{S}^p_q$, $\hat{D}^p_{q,q}$ and $\hat{D}^p_{p,q}$, respectively, spanned by all L^q-valued martingale difference sequences $(d_i)_{i\geq 0}$ such that $d_{2i} = 0$ for each $i \geq 0$. By the proof of Theorem 7.1.1, any L^q-valued martingale difference sequence $(d_i)_{i\geq 0}$ such that $d_{2i} = 0$ for each $i \geq 0$ satisfies

$$\Big(\mathbb{E}\Big\|\sum_i d_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} \|(d_i)\|_{\hat{s}^{odd}_{p,q}},$$

where $\hat{s}^{odd}_{p,q}$ is given by

$$\begin{array}{rl}
\hat{S}^{p,odd}_q \cap \hat{D}^{p,odd}_{q,q} \cap \hat{D}^{p,odd}_{p,q} & \text{if } 2 \leq q \leq p < \infty; \\
\hat{S}^{p,odd}_q \cap (\hat{D}^{p,odd}_{q,q} + \hat{D}^{p,odd}_{p,q}) & \text{if } 2 \leq p \leq q < \infty; \\
(\hat{S}^{p,odd}_q \cap \hat{D}^{p,odd}_{q,q}) + \hat{D}^{p,odd}_{p,q} & \text{if } 1 < p < 2 \leq q < \infty; \\
(\hat{S}^{p,odd}_q + \hat{D}^{p,odd}_{q,q}) \cap \hat{D}^{p,odd}_{p,q} & \text{if } 1 < q < 2 \leq p < \infty; \\
\hat{S}^{p,odd}_q + (\hat{D}^{p,odd}_{q,q} \cap \hat{D}^{p,odd}_{p,q}) & \text{if } 1 < q \leq p \leq 2; \\
\hat{S}^{p,odd}_q + \hat{D}^{p,odd}_{q,q} + \hat{D}^{p,odd}_{p,q} & \text{if } 1 < p \leq q \leq 2.
\end{array}$$

This fact will be used in the proof of Theorem 7.5.5.

Remark 7.3.4. Let us compare our result to the literature. As was mentioned in the introduction, the scalar-valued version of Theorem 7.1.1 is due to Burkholder [29], following work of Rosenthal [161]. A version for noncommutative martingales, as well as a version of (7.1.3) for $1 < p \leq 2$, was obtained by Junge and Xu [88]. Various upper bounds for the moments of a martingale with values in a uniformly 2-smooth (or equivalently, cf. [152], martingale type 2) Banach space were obtained by Pinelis [150], with constants of optimal order. For instance, if $2 \leq p < \infty$ then ([150], Theorem 4.1)

$$\Big(\mathbb{E}\Big\|\sum_i d_i\Big\|^p_X\Big)^{\frac{1}{p}} \lesssim p(\mathbb{E}\sup_i \|d_i\|^p_X)^{\frac{1}{p}} + \sqrt{p}\,\tau_2(X)\Big(\mathbb{E}\Big(\sum_i \mathbb{E}_{i-1}\|d_i\|^2_X\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}, \tag{7.3.8}$$

where $\tau_2(X)$ is the 2-smoothness constant of X. As already remarked in [150], due to the presence of the second term on the right hand side this type of inequality cannot hold in a Banach space which is not 2-uniformly smooth (or equivalently, has martingale type 2). On the other hand, one can show that the reverse inequality holds (with different constants) if and only if X is 2-uniformly convex (or equivalently, has martingale cotype 2). Thus, a two-sided inequality involving the norm on the right hand side of (7.3.8) can only hold in a space with both martingale type and cotype equal to 2. Such a space is necessarily isomorphic to a Hilbert space by a well-known result of Kwapień (see e.g. [2], Theorem 7.4.1).

It should be mentioned that the dependence of the implicit constants on p and q in (7.1.6) is not optimal. We leave it as an interesting open problem to determine the optimal dependence on the constants.

7.4. THE DUAL OF $H_p^{s_q}(X)$

In the proof of Theorem 7.1.1 we used the fact that $(D_{q,q}^p)^* = D_{q',q'}^{p'}$ holds isomorphically (with constants depending only on p and q) for all $1 < p, q < \infty$. In this section we will prove a more general statement.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathscr{F}_k)_{k\geq 0}$, X be a Banach space, and let $1 < p, q < \infty$. For an adapted sequence $f = (f_k)_{k\geq 0}$ of X-valued random variables we define

$$s_q^n(f) := \Big(\sum_{k=0}^{n} \mathbb{E}_{k-1}\|f_k\|^q\Big)^{1/q}, \quad s_q(f) := \Big(\sum_{k=0}^{\infty} \mathbb{E}_{k-1}\|f_k\|^q\Big)^{1/q},$$

where $\mathbb{E}_k = \mathbb{E}(\cdot|\mathscr{F}_k)$, $\mathbb{E}_{-1} = \mathbb{E}$. We let $H_p^{s_q}(X)$ be the space of all adapted sequences $f = (f_k)_{k\geq 0}$ satisfying

$$\|f\|_{H_p^{s_q}(X)} := (\mathbb{E}s_q(f)^p)^{1/p} < \infty.$$

Similarly we define $H_p^{s_q^n}(X)$. We will prove the following result, which was only known before if $X = \mathbb{R}$ and either $1 < p \leq q < \infty$ or $2 \leq q \leq p < \infty$ (see [181, Theorem 15] and the remark following it).

Theorem 7.4.1. *Let X be a reflexive separable Banach space, $1 < p, q < \infty$. Then $\big(H_p^{s_q}(X)\big)^* = H_{p'}^{s_{q'}}(X^*)$ isomorphically. The isomorphism is given by*

$$g \mapsto F_g, \quad F_g(f) = \mathbb{E}\Big(\sum_{k=0}^{\infty} \langle f_k, g_k\rangle\Big) \quad \big(f \in H_p^{s_q}(X), g \in H_{p'}^{s_{q'}}(X^*)\big), \tag{7.4.1}$$

and

$$\min\Big\{\frac{q}{p}, \frac{q'}{p'}\Big\}\|g\|_{H_{p'}^{s_{q'}}(X^*)} \leq \|F_g\|_{(H_p^{s_q}(X))^*} \leq \|g\|_{H_{p'}^{s_{q'}}(X^*)}. \tag{7.4.2}$$

In particular, $H_p^{s_q}(X)$ is a reflexive Banach space.

To prove this result, we will first extend an argument of Csörgő [46] to show that $\big(H_p^{s_q^n}(X)\big)^*$ and $H_{p'}^{s_{q'}^n}(X^*)$ are isomorphic if $1 < p, q < \infty$, with isomorphism constants depending on p, q and n. In particular, this shows that $H_p^{s_q^n}(X)$ is reflexive. In a second step, we exploit this reflexivity to show that the isomorphism constants do not depend on n. The proof of this result, Theorem 7.4.5, relies on an argument of Weisz [181]. After this step, it is straightforward to deduce Theorem 7.4.1.

We start by introducing an operator that serves as a replacement for the sign-function in a vector-valued context.

Lemma 7.4.2. *Let X be a Banach space with a separable dual. Fix $\varepsilon > 0$. Then there exists a discrete-valued Borel-measurable function $P_\varepsilon : X^* \to X$ such that $\|P_\varepsilon(x^*)\| = 1$ and*

$$(1-\varepsilon)\|x^*\| \le \langle P_\varepsilon x^*, x^*\rangle \le \|x^*\| \tag{7.4.3}$$

for each $x^* \in X^*$.

Proof. Let $(x_n^*)_{n\ge1}$ be a dense subset of the unit sphere of X^*. For each $n \ge 1$ define $U_n = U \bigcap B(x_n^*, \frac{\varepsilon}{2})$, where $B(y^*, r)$ denotes the ball in X^* with radius r and center y^*. Define $V_1 = U_1$ and

$$V_n = U_n \setminus \Big(\bigcup_{k=1}^{n-1} V_k\Big), \qquad n \ge 2.$$

For each $n \ge 1$ one can find an $x_n \in X$ such that $\|x_n\| = 1$ and $\langle x_n, x_n^*\rangle \ge 1 - \frac{\varepsilon}{2}$. Now define

$$P_\varepsilon(x^*) := \sum_{n=1}^{\infty} \mathbf{1}_{V_n}\Big(\frac{x^*}{\|x^*\|}\Big) x_n, \quad x^* \in X^*.$$

This function is Borel since the V_n are Borel sets. As the V_n form a disjoint cover of the unit sphere, for every $x^* \in X^*$ there exists a unique $n = n(x^*)$ so that $x^*/\|x^*\| \in V_n$. Hence, $\|P_\varepsilon(x^*)\| = 1$ and

$$\langle P_\varepsilon(x^*), x^*\rangle = \|x^*\| \Big\langle x_n, \frac{x^*}{\|x^*\|}\Big\rangle \ge \|x^*\| \langle x_n, x_n^*\rangle - \frac{\varepsilon}{2}\|x^*\| \ge (1-\varepsilon)\|x^*\|,$$

so (7.4.3) follows. □

Theorem 7.4.3. *Let X be a reflexive separable Banach space, $1 < p, q < \infty$, $n \ge 0$. Then $\big(H_p^{s_q^n}(X)\big)^* = H_{p'}^{s_{q'}^n}(X^*)$ isomorphically (with constants depending on p, q and n). The isomorphism is given by*

$$g \mapsto F_g, \quad F_g(f) = \mathbb{E}\Big(\sum_{k=0}^{n} \langle f_k, g_k\rangle\Big) \ \ \big(f \in H_p^{s_q^n}(X), g \in H_{p'}^{s_{q'}^n}(X^*)\big). \tag{7.4.4}$$

In particular, $H_p^{s_q^n}(X)$ is a reflexive Banach space.

Proof. The main argument is inspired by the proof of [46, Theorem 1]. By the conditional Hölder inequality and the usual version of Hölder's inequality,

$$\begin{aligned} |F_g(f)| &\le \mathbb{E}\Big(\sum_{k=0}^n \mathbb{E}_{k-1}(\|f_k\|\|g_k\|)\Big) \\ &\le \mathbb{E}\Big(\sum_{k=0}^n (\mathbb{E}_{k-1}\|f_k\|^q)^{1/q}(\mathbb{E}_{k-1}\|g_k\|^{q'})^{1/q'}\Big) \qquad (7.4.5) \\ &\le \|f\|_{H_p^{s_q^n}(X)}\|g\|_{H_{p'}^{s_{q'}^n}(X^*)}. \end{aligned}$$

Hence, the functional F_g is bounded and $\|F_g\| \le \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}$.

To prove that $\|F_g\| \gtrsim_{p,q,n} \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}$ we need to construct an appropriate $f \in H_p^{s_q^n}(X)$ with

$$\|f\|_{H_p^{s_q^n}(X)} \lesssim_{p,q,n} 1, \qquad \langle F_g, f\rangle \gtrsim_{p,q,n} \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}.$$

Fix $0 < \varepsilon < 1$. We define f by setting

$$f_k := P_\varepsilon g_k \frac{\|g_k\|^{q'-1}}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}(\mathbb{E}_{k-1}\|g_k\|^{q'})^{\frac{p'-q'}{q'}}, \qquad 0 \le k \le n$$

where P_ε is as in Lemma 7.4.2. Using $pp' = p + p'$ and $qq' = q + q'$ we find

$$\begin{aligned} \|f\|^p_{H_p^{s_q^n}(X)} &= \mathbb{E}\Big(\sum_{k=0}^n \mathbb{E}_{k-1}\|f_k\|^q\Big)^{p/q} = \frac{1}{\|g\|^{p(p'-1)}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\Big(\sum_{k=0}^n (\mathbb{E}_{k-1}\|g_k\|^{q'})^{\frac{(p'-1)q}{q'}}\Big)^{\frac{p}{q}} \\ &\eqsim_{n,p,q} \frac{1}{\|g\|^{p'}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\Big(\sum_{k=0}^n (\mathbb{E}_{k-1}\|g_k\|^{q'})\Big)^{\frac{p'}{q'}} = 1, \end{aligned}$$

so $f \in H_p^{s_q^n}(X)$. Moreover,

$$\begin{aligned} \langle F_g, f\rangle &\ge (1-\varepsilon)\frac{1}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\sum_{k=0}^n \|g_k\|^{q'}(\mathbb{E}_{k-1}\|g_k\|^{q'})^{\frac{p'-q'}{q'}} \\ &= (1-\varepsilon)\frac{1}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\sum_{k=0}^n (\mathbb{E}_{k-1}\|g_k\|^{q'})^{\frac{p'}{q'}} \\ &\eqsim_{p,q,n} (1-\varepsilon)\frac{1}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\Big(\sum_{k=0}^n \mathbb{E}_{k-1}\|g_k\|^{q'}\Big)^{\frac{p'}{q'}} = \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}, \end{aligned}$$

as desired, since ε was arbitrary and can be chosen, say, $\frac{1}{2}$.

Now we will show that every $F \in \big(H_p^{s_q^n}(X)\big)^*$ is equal to F_g for a suitable $g \in H_{p'}^{s_{q'}^n}(X^*)$. For this purpose we consider the disjoint direct sum of $(\Omega, \mathscr{F}_k, \mathbb{P})$, $k = 0, \dots, n$. Formally, we set $\Omega_k = \Omega \times \{k\}$, $\widetilde{\mathscr{F}_k} = \mathscr{F}_k \times \{k\}$ and define a probability measure $\mathbb{P}_k$ on $\widetilde{\mathscr{F}_k}$ by $\mathbb{P}_k(A \times \{k\}) = \mathbb{P}(A)$. Now the disjoint direct sum $(\Omega^n, \mathscr{F}^n, \mathbb{P}^n)$ is defined by

$$\Omega^n = \bigcup_{k=0}^n \Omega_k, \qquad \mathscr{F}^n = \{A \in \Omega^n \ : \ A \cap \Omega_k \in \widetilde{\mathscr{F}_k}, \text{ for all } 1 \leq k \leq n\}$$

and

$$\mathbb{P}^n(A) = \sum_{k=0}^N \mathbb{P}_k(A \cap \Omega_k), \qquad A \in \mathscr{F}^n.$$

Let $P_k : (\Omega, \mathscr{F}_k) \to (\Omega^n, \mathscr{F}^n)$, $P_k(\omega) = (\omega, k)$, be the measurable bijection between $(\Omega, \mathscr{F}_k)$ and its disjoint copy. We can now define an X^*-valued set function μ by

$$\langle \mu(A), x\rangle := F\big((x \cdot \mathbf{1}_{P_k^{-1}(A\cap\Omega_k)})_{k=0}^n\big), \qquad A \in \mathscr{F}^n, \ x \in X.$$

We will show that μ is σ-additive, absolutely continuous with respect to $\mathbb{P}^n$ and of finite variation. Let us first show that μ is of finite variation. Let $(A_m)_{m=1}^M \subset \mathscr{F}^n$ be disjoint such that $\cup_m A_m = \Omega^n$. Then

$$\begin{aligned}
\sum_{m=1}^M \|\mu(A_m)\| &= \sum_{m=1}^M \sup_{x_m \in X : \|x_m\|=1} F\big((x_m \cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)})_{k=0}^n\big)\\
&= \sup_{(x_m)_{m=1}^M \subset X : \|x_m\|=1} \sum_{m=1}^M F\big((x_m \cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)})_{k=0}^n\big)\\
&= \sup_{(x_m)_{m=1}^M \subset X : \|x_m\|=1} F\Big(\Big(\sum_{m=1}^M x_m \cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\Big)_{k=0}^n\Big)\\
&\leq \|F\| \sup_{(x_m)_{m=1}^M \subset X : \|x_m\|=1} \Big\|\Big(\sum_{m=1}^M x_m \cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\Big)_{k=0}^n\Big\|_{H_p^{s_q^n}(X)}\\
&= \|F\| \sup_{(x_m)_{m=1}^M \subset X : \|x_m\|=1} \Big(\mathbb{E}\Big(\sum_{k=0}^n \mathbb{E}_{k-1}\Big\|\sum_{m=1}^M x_m \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\Big\|^q\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}\\
&\leq \|F\| \Big[\mathbb{E}\Big(\sum_{k=0}^n \mathbb{E}_{k-1}\Big(\sum_{m=1}^M \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\Big)^q\Big)^{\frac{p}{q}}\Big]^{\frac{1}{p}}\\
&= \|F\| \Big(\mathbb{E}\Big(\sum_{k=0}^n \mathbb{E}_{k-1}\mathbf{1}_{\Omega}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} = \|F\|(n+1)^{\frac{1}{q}}
\end{aligned}$$

Now let us prove the σ-additivity. Obviously μ is additive. Let $(A_m)_{m\geq 0} \subset \mathscr{F}_n$ be such that $A_m \searrow \varnothing$. Then

$$\|\mu(A_m)\| = \sup_{x \in X : \|x\|=1} |F((x \cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)})_{k=0}^n)|$$

$$\leq \|F\| \sup_{x\in X:\|x\|=1} \|(x\cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)})_{k=0}^n\|_{H_p^{s_q^n}(X)}$$
$$= \|F\|\Big(\mathbb{E}\Big(\sum_{k=0}^n \mathbb{E}_{k-1}\mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} \to 0 \text{ as } m\to\infty,$$

by the monotone convergence theorem. This computation also shows that μ is absolutely continuous with respect to $\mathbb{P}^n$.

Since X is reflexive, X^* has the Radon-Nikodym property (see e.g. [79, Theorem 1.3.21]). Thus, there exists a $g \in L^1(\Omega^n; X^*)$ so that

$$\mu(A) = \int_A g\, d\mathbb{P}^n = \sum_{k=0}^n \int_{A\cap\Omega_k} g\, d\mathbb{P}_k.$$

If we now define $g_k := g\circ P_k$ then g_k is $\mathscr{F}_k$-measurable and

$$\mu(A) = \sum_{k=0}^n \int_{P_k^{-1}(A\cap\Omega_k)} g_k\, d\mathbb{P}.$$

It now follows for $f = (f_k)_{k=0}^n \in H_p^{s_q^n}(X)$ with f_k bounded for all $k = 0,\dots,n$ that

$$F(f) = F_g(f) = \mathbb{E}\sum_{k=0}^n \langle f_k, g_k\rangle. \tag{7.4.6}$$

Now fix a general $f \in H_p^{s_q^n}(X)$. Fix $0<\varepsilon<1$ and let $h := (h_k)_{k=0}^n = (\|f_k\| P_\varepsilon g_k)_{k=0}^n$. Define $h^m := (h_k^m)_{k=0}^n = (h_k \mathbf{1}_{\|h_k\|\leq m})_{k=0}^n$ for each $m\geq 1$. Then formula (7.4.6) holds for h^m. But $F(h^m)\to F(h)$ as m goes to infinity, so by the monotone convergence theorem $F(h) = \mathbb{E}\sum_{k=0}^n\langle h_k, g_k\rangle$. This shows that

$$\mathbb{E}\sum_{k=0}^n |\langle f_k, g_k\rangle| \leq \mathbb{E}\sum_{k=0}^n \|f_k\|\|g_k\| \leq (1-\varepsilon)^{-1}\mathbb{E}\sum_{k=0}^n \langle h_k, g_k\rangle < \infty. \tag{7.4.7}$$

Now consider $f^m := (f_k^m)_{k=0}^n = (f_k\mathbf{1}_{\|f_k\|\leq m})_{k=0}^n$. Since (7.4.6) holds for f^m and $F(f^m)\to F(f)$, we can use (7.4.7) and the dominated convergence theorem to conclude that f satisfies (7.4.6).

It remains to prove that $g \in H_{p'}^{s_{q'}^n}(X^*)$. For each $m\geq 1$ we consider the approximation $g^m := (g_k\mathbf{1}_{\|g_k\|\leq m})_{k=0}^n$. Then $\|g^m\|_{H_{p'}^{s_{q'}^n}(X^*)} \lesssim_{p,q,n} \|F_{g^m}\| \leq \|F\|$. Therefore by the monotone convergence theorem $\|g\|_{H_{p'}^{s_{q'}^n}(X^*)} \lesssim_{p,q,n} \|F\|$. □

One can easily show the following simple lemma.

Lemma 7.4.4. *Let X and Y be reflexive Banach spaces such that X^* is isomorphic to Y and*

$$a\|x^*\|_Y \leq \|x^*\|_{X^*} \leq b\|x^*\|_Y, \quad x^*\in X^*.$$

Then Y^* is isomorphic to $X^{**} = X$ and

$$a\|x\|_X \leq \|x\|_{Y^*} \leq b\|x\|_X, \quad x\in X.$$

Theorem 7.4.5. *Let X be a reflexive separable Banach space, $1 < p, q < \infty$, $n \geq 0$. Then*

$$\min\Big\{\frac{q}{p}, \frac{q'}{p'}\Big\}\|g\|_{H_{p'}^{s_{q'}^n}(X^*)} \leq \|F_g\|_{\left(H_p^{s_q^n}(X)\right)^*} \leq \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}. \tag{7.4.8}$$

Proof. We already proved in Theorem 7.4.3 that $H_p^{s_q^n}(X)$ is reflexive, so by Lemma 7.4.4 it is enough to show (7.4.8) for $p \leq q$. It was already noted in (7.4.5) that $\|F_g\| \leq \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}$. It is sufficient to show (7.4.8) for a bounded g. The following construction is in essence the same as in [181, Theorem 15]. Set

$$(v_k)_{k=0}^n = \left(\frac{(s_{q'}^k(g))^{p'-q'}}{\|g\|_{H_{p'}^{s_{q'}^n}(X^*)}^{p'-1}}\right)_{k=0}^n.$$

Fix $0 < \varepsilon < 1$. Let us define $h \in H_p^{s_q^n}(X)$ by setting

$$h_k = v_k \|g_k\|^{q'-1} P_\varepsilon g_k,$$

where $P_\varepsilon : X^* \to X$ is as given in Lemma 7.4.2. Then

$$(s_q^n(h))^q \leq \sum_{k=0}^n \frac{(s_{q'}^k(g))^{qp'-qq'}}{\|g\|_{H_{p'}^{s_{q'}^n}(X^*)}^{qp'-q}} \mathbb{E}_{k-1}\|g_k\|^{q'} \leq \frac{(s_{q'}^n(g))^{qp'-(q-1)q'}}{\|g\|_{H_{p'}^{s_{q'}^n}(X^*)}^{qp'-q}}.$$

and therefore

$$\mathbb{E}(s_q^n(h))^p \leq \frac{\mathbb{E}(s_{q'}^n(g))^{(qp'-(q-1)q')\frac{p}{q}}}{\|g\|_{H_{p'}^{s_{q'}^n}(X^*)}^{pp'-p}} = 1.$$

As a consequence,

$$\begin{aligned}
\|F_g\| &\geq |\langle F_g, h\rangle| \\
&\geq (1-\varepsilon)\frac{1}{\|g\|_{H_{p'}^{s_{q'}^n}(X^*)}^{p'-1}} \mathbb{E}\sum_{k=0}^n (s_{q'}^k(g))^{p'-q'} \mathbb{E}_{k-1}\|g_k\|^{q'} \\
&= (1-\varepsilon)\frac{1}{\|g\|_{H_{p'}^{s_{q'}^n}(X^*)}^{p'-1}} \mathbb{E}\sum_{k=0}^n (s_{q'}^k(g))^{p'-q'}((s_{q'}^k(g))^{q'} - (s_{q'}^{k-1}(g))^{q'}).
\end{aligned} \tag{7.4.9}$$

By the mean value theorem,

$$x^\alpha - 1 \leq \alpha(x-1)x^{\alpha-1}, \quad x, \alpha \geq 1. \tag{7.4.10}$$

Applying this for $x = \frac{(s^k_{q'}(g))^{q'}}{(s^{k-1}_{q'}(g))^{q'}} \geq 1$ and $\alpha = \frac{p'}{q'} \geq 1$ we find

$$\frac{q'}{p'}((s^k_{q'}(g))^{p'} - (s^{k-1}_{q'}(g))^{p'}) \leq ((s^k_{q'}(g))^{q'} - (s^{k-1}_{q'}(g))^{q'})(s^k_{q'}(g))^{p'-q'}.$$

Combining this with (7.4.9) and letting $\varepsilon \to 0$,

$$\|F_g\| \geq \frac{q'}{p'\|g\|^{p'-1}_{H^{s^n_{q'}}_{p'}(X^*)}} \mathbb{E}(s^n_{q'}(g))^{p'} = \frac{q'}{p'}\|g\|_{H^{s^n_{q'}}_{p'}(X^*)}.$$

□

We can now deduce the main result of this section.

Proof of Theorem 7.4.1. Let $F \in (H^{s_q}_p(X))^*$. For every $n \geq 0$ there exists an $F_n \in \big(H^{s^n_q}_p(X)\big)^*$ such that $\langle F, f\rangle = \langle F_n, (f_k)^n_{k=0}\rangle$ for each $f \in H^{s_q}_p(X)$ satisfying $f_m = 0$ for all $m > n$. Thanks to Theorem 7.4.3, for each $n \geq 0$ there exists a $g^n = (g^n_k)^n_{k=0} \in H^{s^n_{q'}}_{p'}(X)$ such that $F_n = F_{g^n}$. Obviously $g^m_k = g^n_k$ for each $m, n \geq k$, so there exists a unique $g = (g_k)^\infty_{k=0}$ such that $g^n = (g_k)^n_{k=0}$. Moreover, Theorem 7.4.5 implies

$$\min\Big\{\frac{q}{p}, \frac{q'}{p'}\Big\}\|g^n\|_{H^{s^n_{q'}}_{p'}(X)} \leq \|F_n\|_{\big(H^{s^n_q}_p(X)\big)^*} \leq \|F\|_{(H^{s_q}_p(X))^*},$$

so $g \in H^{s_{q'}}_{p'}(X)$ and

$$\min\Big\{\frac{q}{p}, \frac{q'}{p'}\Big\}\|g\|_{H^{s_{q'}}_{p'}(X)} \leq \|F\|_{(H^{s_q}_p(X))^*}.$$

Now obviously $F = F_g$, as these two functionals coincide on the dense subspace of all finitely non-zero sequences in $H^{s_q}_p(X)$, and (7.4.1) and (7.4.2) hold. □

7.5. SHARP BOUNDS FOR L^q-VALUED STOCHASTIC INTEGRALS

We now turn to proving sharp bounds for stochastic integrals.

7.5.1. Decomposition of stochastic integrals

To prove sharp bounds for the stochastic integral, we will decompose it by decomposing the integrator M into three parts. By Proposition 2.5.1, if $M = M^c + M^q + M^a$ is the canonical decomposition of M, then the canonical decomposition of $\Phi \cdot M$ is given by

$$\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a. \tag{7.5.1}$$

The following four subsections are dedicated to sharp estimates of the respective parts on the right hand side. In Subsection 7.5.6 we combine our work to estimate $\Phi \cdot M$.

7.5.2. *Purely discontinuous martingales with accessible jumps*

In this section we prove Burkholder-Rosenthal type inequalities for purely discontinuous martingales with accessible jumps. As an immediate consequence, we find sharp bounds for the accessible jump part in (7.5.1).

As a first step, we will show that we can represent a purely discontinuous martingales with accessible jumps as a sum of jumps occurring at predictable times.

The following lemma follows from Theorem 9.7.12.

Lemma 7.5.1. *Let $1 < p, q < \infty$, $M : \mathbb{R}_+ \times \Omega \to L^q(S)$ be a purely discontinuous L^p-martingale with accessible jumps. Let $\mathcal{T} = (\tau_n)_{n=0}^{\infty}$ be any sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M. Then for each $n \geq 0$*

$$M_t^n = \sum_{k=0}^{n} \Delta M_{\tau_k} \mathbf{1}_{[0,t]}(\tau_k) \tag{7.5.2}$$

defines an L^p-martingale. Moreover, for any $t \geq 0$, $\|M_t - M_t^n\|_{L^p(\Omega;L^q(S))} \to 0$ as $n \to \infty$. If $\sup_{t\geq 0} \mathbb{E}\|M_t\|^p < \infty$, then $\|M_\infty - M_\infty^n\|_{L^p(\Omega;L^q(S))} \to 0$ for $n \to \infty$.

Definition 7.5.2. For $1 < p, q < \infty$ we define $\mathcal{M}_{p,q}^{\mathrm{acc}}$ as the linear space of all $L^q(S)$-valued purely discontinuous L^p-martingales with accessible jumps, endowed with the norm $\|M\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}} := \|M_\infty\|_{L^p(\Omega;L^q(S))}$.

The following proposition follows from Proposition 2.4.18.

Proposition 7.5.3. *For any $1 < p, q < \infty$ the space $\mathcal{M}_{p,q}^{\mathrm{acc}}$ is a Banach space.*

We now turn to the Burkholder-Rosenthal inequalities. Let $1 < p, q < \infty$ and let $M : \mathbb{R}_+ \times \Omega \to L^q(S)$ be a purely discontinuous martingale with accessible jumps. Let $\mathcal{T} = (\tau_n)_{n\geq 0}$ be a sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M. We define three expressions

$$\begin{aligned}
\|M\|_{\widetilde{S}_q^p} &= \Big(\mathbb{E}\Big\|\Big(\sum_{n\geq 0} \mathbb{E}_{\mathcal{F}_{\tau_n-}} |(\Delta M(\omega)(s))_{\tau_n}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}},\\
\|M\|_{\widetilde{D}_{q,q}^p} &= \Big(\mathbb{E}\Big(\sum_{n\geq 0} \mathbb{E}_{\mathcal{F}_{\tau_n-}} \|\Delta M(\omega)_{\tau_n}\|_{L^q(S)}^q\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}},\\
\|M\|_{\widetilde{D}_{p,q}^p} &= \Big(\mathbb{E}\sum_{t\geq 0} \|\Delta M_t\|_{L^q(S)}^p\Big)^{\frac{1}{p}}.
\end{aligned} \tag{7.5.3}$$

Proposition 7.5.4. *The expressions in* (7.5.3) *do not depend on the choice of the family $\mathcal{T}$.*

Proof. Assume that $\mathcal{T}' = (\tau'_m)_{m\geq 0}$ is another family of predictable stopping times with disjoint graphs that exhausts the jumps of M. Notice that due to [85, Proposition I.2.11],

$$\mathcal{F}_{\tau-} \cap \{\tau = \sigma\} = \mathcal{F}_{\sigma-} \cap \{\tau = \sigma\}$$

for any pair of predictable stopping times τ and σ. Therefore for a.e. $s \in S$ a.s.

$$\begin{aligned}\sum_{n\geq 0} \mathbb{E}_{\mathscr{F}_{\tau_n-}} |(\Delta M(\omega)(s))_{\tau_n}|^2 &\overset{(*)}{=} \sum_{n\geq 0} \mathbb{E}_{\mathscr{F}_{\tau_n-}} \Big(\sum_{m\geq 0} |(\Delta M(\omega)(s))_{\tau_n}|^2 \mathbf{1}_{\tau_n=\tau'_m}\Big)\\ &= \sum_{n\geq 0}\sum_{m\geq 0} \mathbb{E}_{\mathscr{F}_{\tau_n-}} \big(|(\Delta M(\omega)(s))_{\tau_n}|^2 \mathbf{1}_{\tau_n=\tau'_m}\big)\\ &= \sum_{n\geq 0}\sum_{m\geq 0} \mathbb{E}_{\mathscr{F}_{\tau'_m-}} \big(|(\Delta M(\omega)(s))_{\tau_n}|^2 \mathbf{1}_{\tau_n=\tau'_m}\big)\\ &= \sum_{m\geq 0} \mathbb{E}_{\mathscr{F}_{\tau'_m-}} \Big(\sum_{n\geq 0} |(\Delta M(\omega)(s))_{\tau'_m}|^2 \mathbf{1}_{\tau_n=\tau'_m}\Big)\\ &\overset{(*)}{=} \sum_{n\geq 0} \mathbb{E}_{\mathscr{F}_{\tau'_m-}} |(\Delta M(\omega)(s))_{\tau'_m}|^2,\end{aligned}$$

where $(*)$ holds since

$$\begin{aligned}&\mathbb{P}\{\exists u \geq 0 : \Delta M_u \neq 0, u \notin \{\tau_0, \tau_1, \dots\}\}\\ &\qquad = \mathbb{P}\{\exists u \geq 0 : \Delta M_u \neq 0, u \notin \{\tau'_0, \tau'_1, \dots\}\} = 0.\end{aligned}$$

Therefore we can conclude that $\|M\|_{\widetilde{S}_q^p}$ does not depend on the choice of the exhausting family. The same holds for $\|M\|_{\widetilde{D}_{q,q}^p}$ by an analogous argument. □

We let $\widetilde{S}_q^p$, $\widetilde{D}_{q,q}^p$ and $\widetilde{D}_{p,q}^p$ denote the sets of all purely discontinuous martingales with accessible jumps for which the respective expressions in (7.5.3) are finite. We will prove shortly that the expressions in (7.5.3) are norms. For a fixed family $\mathscr{T} = (\tau_n)_{n\geq 0}$ of predictable stopping times with disjoint graphs we let $\widetilde{S}_q^{p,\mathscr{T}}$, $\widetilde{D}_{q,q}^{p,\mathscr{T}}$ and $\widetilde{D}_{p,q}^{p,\mathscr{T}}$ be the subsets of $\widetilde{S}_q^p$, $\widetilde{D}_{q,q}^p$ and $\widetilde{D}_{p,q}^p$ consisting of martingales M with $\{t \in \mathbb{R}_+ : \Delta M_t \neq 0\} \subset \{\tau_0, \tau_1, \dots\}$ a.s.

We start by proving a version of the main theorem of this subsection (Theorem 7.5.8 below) for a martingales with finitely many jumps.

Theorem 7.5.5. *Let $1 < p, q < \infty$, $N \geq 1$, $\mathscr{T} = (\tau_n)_{n=0}^N$ be a finite family of predictable stopping times with disjoint graphs. Then $\widetilde{S}_q^{p,\mathscr{T}}$, $\widetilde{D}_{q,q}^{p,\mathscr{T}}$ and $\widetilde{D}_{p,q}^{p,\mathscr{T}}$ are Banach spaces under the norms in* (7.5.3). *As a consequence, $A_{p,q}^{\mathscr{T}}$ given by*

$$\begin{aligned}&\widetilde{S}_q^{p,\mathscr{T}} \cap \widetilde{D}_{q,q}^{p,\mathscr{T}} \cap \widetilde{D}_{p,q}^{p,\mathscr{T}} \text{ if } 2 \leq q \leq p < \infty,\\ &\widetilde{S}_q^{p,\mathscr{T}} \cap (\widetilde{D}_{q,q}^{p,\mathscr{T}} + \widetilde{D}_{p,q}^{p,\mathscr{T}}) \text{ if } 2 \leq p \leq q < \infty,\\ &(\widetilde{S}_q^{p,\mathscr{T}} \cap \widetilde{D}_{q,q}^{p,\mathscr{T}}) + \widetilde{D}_{p,q}^{p,\mathscr{T}} \text{ if } 1 < p < 2 \leq q < \infty,\\ &(\widetilde{S}_q^{p,\mathscr{T}} + \widetilde{D}_{q,q}^{p,\mathscr{T}}) \cap \widetilde{D}_{p,q}^{p,\mathscr{T}} \text{ if } 1 < q < 2 \leq p < \infty,\\ &\widetilde{S}_q^{p,\mathscr{T}} + (\widetilde{D}_{q,q}^{p,\mathscr{T}} \cap \widetilde{D}_{p,q}^{p,\mathscr{T}}) \text{ if } 1 < q \leq p \leq 2,\\ &\widetilde{S}_q^{p,\mathscr{T}} + \widetilde{D}_{q,q}^{p,\mathscr{T}} + \widetilde{D}_{p,q}^{p,\mathscr{T}} \text{ if } 1 < p \leq q \leq 2.\end{aligned} \tag{7.5.4}$$

is a well-defined Banach space. Moreover, $(\mathscr{A}^{\mathcal{T}}_{p,q})^* = \mathscr{A}^{\mathcal{T}}_{p',q'}$ *with isomorphism given by*

$$\begin{aligned} &g \mapsto F_g, \quad F_g(f) = \mathbb{E} \sum_{t\in\mathcal{T}} \langle \Delta g_t, \Delta f_t\rangle \overset{(*)}{=} \mathbb{E}\langle g_\infty, f_\infty\rangle, \\ &\|F_g\|_{(\mathscr{A}^{\mathcal{T}}_{p,q})^*} \eqsim_{p,q} \|g\|_{\mathscr{A}^{\mathcal{T}}_{p',q'}}. \end{aligned} \tag{7.5.5}$$

Finally, for any purely discontinuous L^p*-martingale* $M : \mathbb{R}_+ \times \Omega \to L^q(S)$ *with accessible jumps such that* $\{t \in \mathbb{R}_+ : \Delta M_t \neq 0\} \subset \{\tau_0, \dots, \tau_N\}$ *a.s., and for all* $t \geq 0$*,*

$$\Big(\mathbb{E} \sup_{0\le s\le t} \|M_t\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} \|M\mathbf{1}_{[0,t]}\|_{\mathscr{A}^{\mathcal{T}}_{p,q}}. \tag{7.5.6}$$

Proof. The idea of the proof is to identify $\widetilde{S}^{p,\mathcal{T}}_q$, $\widetilde{D}^{p,\mathcal{T}}_{q,q}$ and $\widetilde{D}^{p,\mathcal{T}}_{p,q}$ with discrete martingale spaces S^p_q, $D^p_{q,q}$ and $D^p_{p,q}$ for an appropriate filtration. Since the τ_i have disjoint graphs, we can find predictable stopping times $\tau'_0, \dots, \tau'_N$ such that

$$\{\tau_0(\omega), \dots, \tau_N(\omega)\} = \{\tau'_0(\omega), \dots, \tau'_N(\omega)\}$$

and $\tau'_0(\omega) < \dots < \tau'_N(\omega)$ for a.e. $\omega \in \Omega$. Indeed, we can set $\tau'_0 = \min\{\tau_0, \dots, \tau_N\}$ and

$$\tau'_i = \min(\{\tau_0, \dots, \tau_N\} \setminus \{\tau'_0, \dots, \tau'_i\}), \qquad 0 \le i \le N-1.$$

Fix the sequence of σ-algebras $\mathbb{G} = (\mathscr{G}_k)_{k=0}^{2N+1} = (\mathscr{F}_{\tau'_0-}, \mathscr{F}_{\tau'_0}, \dots, \mathscr{F}_{\tau'_N-}, \mathscr{F}_{\tau'_N})$. Using [89, Lemma 25.2] and the fact that $(\tau'_n)_{n=0}^N$ is a.s. a strictly increasing sequence one can show that $\mathbb{G}$ is a filtration.

Consider Banach spaces $\hat{S}^{p,odd}_q$, $\hat{D}^{p,odd}_{q,q}$ and $\hat{D}^{p,odd}_{p,q}$ with respect to the filtration $\mathbb{G}$ that were defined in Remark 7.3.3. For any purely discontinuous L^q-valued martingale M with accessible jumps in $\mathcal{T}$ we can construct a $\mathbb{G}$-martingale difference sequence $(d_k)_{n=0}^{2N+1}$ by setting $d_{2n} = 0$, $d_{2n+1} = \Delta M_{\tau_n}$ for $n = 0, \dots, N$. Indeed, by [89, Lemma 26.18] (see also [85, Lemma 2.27]) for each $n = 0, \dots, N$

$$\mathbb{E}(d_{2n+1}|\mathscr{G}_{2n}) = \mathbb{E}(\Delta M_{\tau'_n}|\mathscr{F}_{\tau'_n-}) = 0.$$

By Lemma 2.4.6,

$$\|M\|_{\widetilde{S}^{p,\mathcal{T}}_q} = \|(d_n)\|_{S^{p,odd}_q}, \quad \|M\|_{\widetilde{D}^{p,\mathcal{T}}_{q,q}} = \|(d_n)\|_{D^{p,odd}_{q,q}}, \quad \|M\|_{\widetilde{D}^{p,\mathcal{T}}_{p,q}} = \|(d_n)\|_{D^{p,odd}_{p,q}}.$$

Moreover, by Corollary 2.4.7 any element $(d_k)_{k=0}^{2N+1}$ of $\hat{S}^{p,odd}_q$, $D^{p,odd}_{q,q}$, or $D^{p,odd}_{p,q}$ (so in particular, $d_{2n} = 0$ for each $n = 0, \dots, N$) can be converted back to an element M of $\widetilde{S}^{p,\mathcal{T}}_q$, $\widetilde{D}^{p,\mathcal{T}}_{q,q}$, or $\widetilde{D}^{p,\mathcal{T}}_{p,q}$, respectively, by defining

$$M_t = \sum_{n=0}^N d_{2n+1}\mathbf{1}_{[0,t]}(\tau'_n), \quad t \ge 0.$$

Using this identification, we find that $\widetilde{S}^{p,\mathcal{T}}_q$, $\widetilde{D}^{p,\mathcal{T}}_{q,q}$, and $\widetilde{D}^{p,\mathcal{T}}_{p,q}$ are Banach spaces. As a consequence, $\mathscr{A}^{\mathcal{T}}_{p,q}$ is a well-defined Banach space that is isometrically isomorphic to $\hat{s}^{odd}_{p,q}$. The duality statement now follows from the duality for $s^{odd}_{p,q}$ and $(*)$ in (7.5.5) follows from (7.3.7).

Now let us show (7.5.6). By Doob's maximal inequality,

$$\Big(\mathbb{E}\sup_{0\le s\le t}\|M_t\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_p \Big(\mathbb{E}\|M_t\|^p_{L^q(S)}\Big)^{\frac{1}{p}}.$$

Again define a $\mathbb{G}$-martingale difference sequence $(d_n)_{n=0}^{2N+1}$ by setting $d_{2n}=0$, $d_{2n+1}=\Delta M_{\tau_n}$, where $n=0,\dots,N$. Then by Remark 7.3.3

$$\|M_\infty\|_{L^p(\Omega;X)} = \Big\|\sum_{n=0}^{2N+1} d_k\Big\|_{L^p(\Omega;X)} \eqsim_{p,q} \|(d_n)_{n=0}^{2N+1}\|_{\hat{s}^{odd}_{p,q}} = \|M\|_{\mathscr{A}^{\mathscr{T}}_{p,q}}.$$

□

To treat the general case we use an approximation argument based on the following observation.

Lemma 7.5.6. *Let $1<p,q<\infty$. Let M be in $\widetilde{S}^p_q$, $\widetilde{D}^p_{q,q}$, or $\widetilde{D}^p_{p,q}$ and let $\mathscr{T}=(\tau_n)_{n\ge 0}$ be any sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M. Consider the process $M^n=M^n_{\mathscr{T}}$ defined in* (7.5.2)*. Then $M^n\to M$ in $\widetilde{S}^p_q$, $\widetilde{D}^p_{q,q}$, or $\widetilde{D}^p_{p,q}$, respectively. As a consequence, $\widetilde{S}^p_q$, $\widetilde{D}^p_{q,q}$, and $\widetilde{D}^p_{p,q}$ are normed linear spaces and $\mathscr{A}_{p,q}$ given by*

$$\begin{aligned}
&\widetilde{S}^p_q\cap\widetilde{D}^p_{q,q}\cap\widetilde{D}^p_{p,q} \text{ if } 2\le q\le p<\infty,\\
&\widetilde{S}^p_q\cap(\widetilde{D}^p_{q,q}+\widetilde{D}^p_{p,q}) \text{ if } 2\le p\le q<\infty,\\
&(\widetilde{S}^p_q\cap\widetilde{D}^p_{q,q})+\widetilde{D}^p_{p,q} \text{ if } 1<p<2\le q<\infty,\\
&(\widetilde{S}^p_q+\widetilde{D}^p_{q,q})\cap\widetilde{D}^p_{p,q} \text{ if } 1<q<2\le p<\infty,\\
&\widetilde{S}^p_q+(\widetilde{D}^p_{q,q}\cap\widetilde{D}^p_{p,q}) \text{ if } 1<q\le p\le 2,\\
&\widetilde{S}^p_q+\widetilde{D}^p_{q,q}+\widetilde{D}^p_{p,q} \text{ if } 1<p\le q\le 2.
\end{aligned} \tag{7.5.7}$$

is a well-defined normed linear space. If $M\in\mathscr{A}_{p,q}$, then there exists a sequence of predictable stopping times $\mathscr{T}$ with disjoint graphs that exhausts the jumps of M so that $M^n_{\mathscr{T}}\to M$ in $\mathscr{A}_{p,q}$.

Proof. We prove the two first statements only for $\widetilde{S}^p_q$. By the dominated convergence theorem, we obtain $M^n\to M$ in $\widetilde{S}^p_q$ as well as $\|M^n\|_{\widetilde{S}^p_q}\nearrow\|M\|_{\widetilde{S}^p_q}$. Suppose now that $M,N\in\widetilde{S}^p_q$. By [85, Lemma I.2.23], there exists a sequence $\mathscr{T}=\{\tau_n\}_{n\ge0}$ of predictable stopping times with disjoint graphs that exhausts the jumps of both M and N. Now clearly, $(M+N)^n=M^n+N^n$ and so

$$\begin{aligned}
\|M+N\|_{\widetilde{S}^p_q} &= \lim_{n\to\infty}\|M^n+N^n\|_{\widetilde{S}^p_q}\\
&= \lim_{n\to\infty}\|M^n+N^n\|_{\widetilde{S}^{p,\mathscr{T}}_q}
\end{aligned}$$

$$\leq \lim_{n\to\infty} \|M^n\|_{\widetilde{S}_q^{p,\mathcal{T}}} + \|N^n\|_{\widetilde{S}_q^{p,\mathcal{T}}} = \|M\|_{\widetilde{S}_q^p} + \|N\|_{\widetilde{S}_q^p}.$$

Let us prove the final statement if $p \leq q \leq 2$, the other cases are similar. Let $M \in \mathcal{A}_{p,q}$ and let $M_1 \in \widetilde{S}_q^p$, $M_2 \in \widetilde{D}_{q,q}^p$, $M_3 \in \widetilde{D}_{p,q}^p$ be such that $M = M_1 + M_2 + M_3$. Let $\mathcal{T} = \{\tau_n\}_{n\geq 0}$ be a sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M_1, M_2 and M_3. Then $M^n = M_1^n + M_2^n + M_3^n$ and by the above,

$$\|M - M^n\|_{\mathcal{A}_{p,q}} \leq \|M_1 - M_1^n\|_{\widetilde{S}_q^{p,\mathcal{T}}} + \|M_2 - M_2^n\|_{\widetilde{D}_{q,q}^p} + \|M_3 - M_3^n\|_{\widetilde{D}_{p,q}^p} \to 0$$

as $n \to \infty$. □

Lemma 7.5.7. *Let $1 < p, q < \infty$, $N \geq 1$, $\mathcal{T} = (\tau_n)_{n=0}^N$ be a finite family of predictable stopping times with disjoint graphs. Then $\mathcal{A}_{p,q}^{\mathcal{T}} \hookrightarrow \mathcal{A}_{p,q}$ isometrically.*

Proof. We will consider only the case $p \leq q \leq 2$, the other cases can be shown analogously. Let $M \in \mathcal{A}_{p,q}^{\mathcal{T}}$. Then automatically $M \in \mathcal{A}_{p,q}$ and $\|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}} \geq \|M\|_{\mathcal{A}_{p,q}}$. Let us show the reverse inequality. Fix $\varepsilon > 0$, and let $M^1 \in \widetilde{S}_q^p$, $M^2 \in \widetilde{D}_{q,q}^p$ and $M^3 \in \widetilde{D}_{p,q}^p$ be martingales such that $M = M^1 + M^2 + M^3$ and

$$\|M\|_{\mathcal{A}_{p,q}} \geq \|M^1\|_{\widetilde{S}_q^p} + \|M^2\|_{\widetilde{D}_{q,q}^p} + \|M^3\|_{\widetilde{D}_{p,q}^p} - \varepsilon.$$

By Lemma 2.4.5 we can define martingales $\widetilde{M}^1$, $\widetilde{M}^2$ and $\widetilde{M}^3$ by

$$\widetilde{M}_t^i = \sum_{s\in\mathcal{T}\cap[0,t]} \Delta M_s^i, \quad t \geq 0,\ i = 1,2,3. \tag{7.5.8}$$

Notice that $|\Delta\widetilde{M}_t^i(\omega)(s)| \leq |\Delta M_t^i(\omega)(s)|$ for each $t \geq 0$, $\omega \in \Omega$, $s \in S$ and $i = 1,2,3$. Therefore $\widetilde{M}^1 \in \widetilde{S}_q^p$, $\widetilde{M}^2 \in \widetilde{D}_{q,q}^p$ and $\widetilde{M}^3 \in \widetilde{D}_{p,q}^p$ and $\|\widetilde{M}^1\|_{\widetilde{S}_q^p} \leq \|M^1\|_{\widetilde{S}_q^p}$, $\|\widetilde{M}^2\|_{\widetilde{D}_{q,q}^p} \leq \|M^2\|_{\widetilde{D}_{q,q}^p}$ and $\|\widetilde{M}^3\|_{\widetilde{D}_{p,q}^p} \leq \|M^3\|_{\widetilde{D}_{p,q}^p}$. Moreover, $M = \widetilde{M}^1 + \widetilde{M}^2 + \widetilde{M}^3$. Indeed, since all the martingales here are purely discontinuous with accessible jumps, by (7.5.8) we find for each $t \geq 0$ a.s.

$$\begin{aligned} M_t = \sum_{s\in\mathcal{T}\cap[0,t]} \Delta M_s &= \sum_{s\in\mathcal{T}\cap[0,t]} \left(\Delta M_s^1 + \Delta M_s^2 + \Delta M_s^3\right) \\ &= \sum_{s\in\mathcal{T}\cap[0,t]} \Delta M_s^1 + \sum_{s\in\mathcal{T}\cap[0,t]} \Delta M_s^2 + \sum_{s\in\mathcal{T}\cap[0,t]} \Delta M_s^3 \\ &= \widetilde{M}_t^1 + \widetilde{M}_t^2 + \widetilde{M}_t^3. \end{aligned}$$

Therefore

$$\begin{aligned} \|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}} &\leq \|\widetilde{M}^1\|_{\widetilde{S}_q^p} + \|\widetilde{M}^2\|_{\widetilde{D}_{q,q}^p} + \|\widetilde{M}^3\|_{\widetilde{D}_{p,q}^p} \\ &\leq \|M^1\|_{\widetilde{S}_q^p} + \|M^2\|_{\widetilde{D}_{q,q}^p} + \|M^3\|_{\widetilde{D}_{p,q}^p} \leq \|M\|_{\mathcal{A}_{p,q}} + \varepsilon. \end{aligned}$$

Since ε was arbitrary, we conclude that $\|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}} \leq \|M\|_{\mathcal{A}_{p,q}}$, and consequently $\|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}} = \|M\|_{\mathcal{A}_{p,q}}$. □

We can now readily deduce the main theorem of this section.

Theorem 7.5.8. *Let $1 < p, q < \infty$, $M : \mathbb{R}_+ \times \Omega \to L^q(S)$ be a purely discontinuous martingale with accessible jumps. Then for all $t \geq 0$ one has that*

$$\Big(\mathbb{E} \sup_{0\le s\le t} \|M_t\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} \|M\mathbf{1}_{[0,t]}\|_{\mathscr{A}_{p,q}}, \tag{7.5.9}$$

where $\mathscr{A}_{p,q}$ is as in (7.5.7). *In particular, $\mathscr{A}_{p,q}$ is a Banach space of L^p-martingales.*

Proof. Suppose first that $M \in \mathscr{A}_{p,q}$. By Lemma 7.5.6 there exists a sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M so that $M^n_{\mathcal{T}} \to M$ in $\mathscr{A}_{p,q}$. In particular, $(M^n_{\mathcal{T}})_{n\geq 0}$ is Cauchy in $\mathscr{A}_{p,q}$. By Lemma 7.5.7 and Theorem 7.5.5 it follows that it is also Cauchy in $\mathscr{M}^{\mathrm{acc}}_{p,q}$. By Proposition 7.5.3 $(M^n_{\mathcal{T}})_{n\geq 0}$ converges and clearly the limit is M.

Suppose now that $M \in \mathscr{M}^{\mathrm{acc}}_{p,q}$. It suffices to show that $M \in \mathscr{A}_{p,q}$. Indeed, Lemma 7.5.6 then shows that there is a sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M so that $M^n_{\mathcal{T}} \to M$ in $\mathscr{A}_{p,q}$. By Lemma 7.5.1 we also have $M^n_{\mathcal{T}} \to M$ in $\mathscr{M}^{\mathrm{acc}}_{p,q}$ and so the lower bound in (7.5.9) follows from Lemma 7.5.7 and Theorem 7.5.5. We will show that $M \in \mathscr{A}_{p,q}$ in the two cases $2 \le q \le p$ and $p \le q \le 2$, the other cases can be treated analogously.

Case $2 \le q \le p$. We will show that $\|M\|_{\widetilde{S}^p_q} \lesssim_{p,q} \|M\|_{\mathscr{M}^{\mathrm{acc}}_{p,q}}$. The analogous statements for $\widetilde{D}^p_{q,q}$ and $\widetilde{D}^p_{p,q}$ can be shown in the same way. By Theorem 7.5.5,

$$\|M^n\|_{\widetilde{S}^p_q} \lesssim_{p,q} \|M^n\|_{\mathscr{M}^{\mathrm{acc}}_{p,q}}.$$

Also, by Theorem 3.3.17 we have $\|M^n\|_{\mathscr{M}^{\mathrm{acc}}_{p,q}} \lesssim_{p,q} \|M\|_{\mathscr{M}^{\mathrm{acc}}_{p,q}}$ for all $n \geq 0$. Therefore $\|M^n\|_{\widetilde{S}^p_q} \lesssim_{p,q} \|M\|_{\mathscr{M}^{\mathrm{acc}}_{p,q}}$ uniformly in n, so by monotone convergence

$$\begin{aligned}\|M\|^p_{\widetilde{S}^p_q} &= \mathbb{E}\Big\|\Big(\sum_{m\geq 0} \mathbb{E}_{\mathscr{F}_{\tau_m-}}|(\Delta M(\omega)(s))_{\tau_m}|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(S)}\\ &= \lim_{n\to\infty} \mathbb{E}\Big\|\Big(\sum_{m=0}^{n} \mathbb{E}_{\mathscr{F}_{\tau_m-}}|(\Delta M(\omega)(s))_{\tau_m}|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(S)}\\ &= \lim_{n\to\infty} \|M^n\|^p_{\widetilde{S}^p_q} \lesssim_{p,q} \|M\|^p_{\mathscr{M}^{\mathrm{acc}}_{p,q}}.\end{aligned}$$

Case $p \le q \le 2$. Observe that $\|M^n\|_{\mathscr{A}_{p,q}} \eqsim_{p,q} \|M^n\|_{\mathscr{M}^{\mathrm{acc}}_{p,q}}$ for each $n \geq 0$ by Theorem 7.5.5 and since $(M^n)_{n\geq 0}$ is a Cauchy sequence in $\mathscr{M}^{\mathrm{acc}}_{p,q}$ due to Lemma 7.5.1, it follows that $(M^n)_{n\geq 0}$ is a Cauchy sequence in $\mathscr{A}_{p,q}$. Thus there exists a subsequence $(M^{n_k})_{k\geq 0}$ such that

$$\|M^{n_{k+1}} - M^{n_k}\|_{\mathscr{A}_{p,q}} < \frac{1}{2^{k+1}}, \quad k \geq 0.$$

Let $N^k = M^{n_k} - M^{n_{k-1}}$, $k \geq 1$, $N^0 = M^{n_0}$. Set $n_{-1} = -1$. By Theorem 7.5.5, for each $k \geq 0$ there exist $N^{k,1}$, $N^{k,2}$ and $N^{k,3}$ such that $N^{k,1} \in \widetilde{S}^p_q$, $N^{k,2} \in \widetilde{D}^p_{q,q}$, $N^{k,3} \in \widetilde{D}^p_{p,q}$, $N^k = N^{k,1} + N^{k,2} + N^{k,3}$,

$$\{t : \Delta N^{k,i}_t \neq 0, i = 1,2,3\} \subset \{\tau_{n_{k-1}+1}, \ldots, \tau_{n_k}\}, \text{ a.s.,}$$

and

$$\begin{aligned}
&\|N^{k,1}\|_{\widetilde{S}_q^p} + \|N^{k,2}\|_{\widetilde{D}_{q,q}^p} + \|N^{k,3}\|_{\widetilde{D}_{p,q}^p} < \frac{1}{2^k}, \quad k \geq 1,\\
&\|N^{0,1}\|_{\widetilde{S}_q^p} + \|N^{0,2}\|_{\widetilde{D}_{q,q}^p} + \|N^{0,3}\|_{\widetilde{D}_{p,q}^p} \leq 2\|M^{n_0}\|_{\mathscr{A}_{p,q}}.
\end{aligned} \tag{7.5.10}$$

Let

$$M^{m,i} := \sum_{k=0}^{m} N^{k,i}, \quad m \geq 0, \quad i = 1,2,3.$$

Then by (7.5.10), $(M^{m,1})_{m\geq 0}$, $(M^{m,2})_{m\geq 0}$ and $(M^{m,3})_{m\geq 0}$ are Cauchy sequences in $\widetilde{S}_q^p$, $\widetilde{D}_{q,q}^p$ and $\widetilde{D}_{p,q}^p$ respectively. By construction, each of $M^{m,i}$, $m \geq 0$, $i = 1,2,3$, has finitely many jumps occurring in $\{\tau_0, \ldots, \tau_{n_m}\}$, so by Theorem 7.5.5 the sequences $(M^{m,1})_{m\geq 0}$, $(M^{m,2})_{m\geq 0}$ and $(M^{m,3})_{m\geq 0}$ are Cauchy in $\mathscr{M}_{p,q}^{\mathrm{acc}}$ as well. Due to Proposition 7.5.3 there exist $\widetilde{M}^1$, $\widetilde{M}^2$ and $\widetilde{M}^3$ such that $M^{m,i} \to \widetilde{M}^i$ in $\mathscr{M}_{p,q}^{\mathrm{acc}}$ as $m \to \infty$ for each $i = 1,2,3$. Since $M^{m,1} + M^{m,2} + M^{m,3} \to M$ in $\mathscr{M}_{p,q}^{\mathrm{acc}}$ as $m \to \infty$ by Lemma 7.5.1, it follows that $M = \widetilde{M}^1 + \widetilde{M}^2 + \widetilde{M}^3$.

Let us now show that the jumps of $\widetilde{M}^1$, $\widetilde{M}^2$ and $\widetilde{M}^3$ are exhausted by $\mathscr{T} = (\tau_n)_{n\geq 0}$. Indeed, assume that for some $i = 1,2,3$ there exists a predictable stopping time τ such that $\mathbb{P}\{\Delta\widetilde{M}_\tau^i \neq 0, \tau \notin \{\tau_0, \tau_1, \ldots\}\} > 0$. Then by separability of $X = L^q(S)$ there exists an $x^* \in X^*$ such that

$$\mathbb{P}\{\langle \Delta\widetilde{M}_\tau^i, x^*\rangle \neq 0, \tau \notin \{\tau_0, \tau_1, \ldots\}\} > 0 \tag{7.5.11}$$

and so, by the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
\mathbb{E}|\langle (M^{m,i} - \widetilde{M}^i)_\infty, x^*\rangle|^p &\eqsim_p \mathbb{E}[\langle M^{m,i} - \widetilde{M}^i, x^*\rangle]_\infty^{\frac{p}{2}}\\
&= \mathbb{E}\Big(\sum_{u\geq 0} |\langle \Delta(M^{m,i} - \widetilde{M}^i)_u, x^*\rangle|^2\Big)^{\frac{p}{2}}\\
&\geq \mathbb{E}|\langle \Delta\widetilde{M}_\tau^i, x^*\rangle|^p \mathbf{1}_{\tau\notin\{\tau_0,\tau_1,\ldots\}},
\end{aligned} \tag{7.5.12}$$

where the final inequality holds as $\mathbb{P}\{\Delta M_\tau^{m,i} \neq 0, \tau \notin \{\tau_0, \tau_1, \ldots\}\} = 0$. But the last expression in (7.5.12) does not vanish as $m \to \infty$ because of (7.5.11), which contradicts with the fact that $M^{m,i} \to \widetilde{M}^i$ in $\mathscr{M}_{p,q}^{\mathrm{acc}}$.

By monotone convergence,

$$\begin{aligned}
\|\widetilde{M}^1\|_{\widetilde{S}_q^p}^p &= \mathbb{E}\Big\|\Big(\sum_{n\geq 0} \mathbb{E}_{\mathscr{F}_{\tau_n-}} |(\Delta\widetilde{M}^1(\omega)(s))_{\tau_n}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}^p\\
&= \lim_{m\to\infty} \mathbb{E}\Big\|\Big(\sum_{n=0}^{n_m} \mathbb{E}_{\mathscr{F}_{\tau_n-}} |(\Delta\widetilde{M}^1(\omega)(s))_{\tau_n}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}^p\\
&= \lim_{m\to\infty} \|M^{m,1}\|_{\widetilde{S}_q^p}^p,
\end{aligned}$$

and the last expression is bounded due to the fact that $M^{m,1}$ is a Cauchy sequence in $\widetilde{S}_q^p$. By the same reasoning $\widetilde{M}^2 \in \widetilde{D}_{q,q}^p$ and $\widetilde{M}^3 \in \widetilde{D}_{p,q}^p$, so $M \in \mathscr{A}_{p,q}$. This completes the proof. □

Theorem 7.5.8 and Lemma 7.5.26 yield the following sharp estimates.

Corollary 7.5.9. *Let* $1 < p, q < \infty$, $M : \mathbb{R}_+ \times \Omega \to H$ *be a purely discontinuous* L^p*-martingale with accessible jumps,* $X = L^q(S)$, $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H, X)$ *be elementary predictable. Then for all* $t \geq 0$ *one has that*

$$\Big(\mathbb{E} \sup_{0\leq s\leq t} \|(\Phi \cdot M)_t\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} \|(\Phi \mathbf{1}_{[0,t]}) \cdot M\|_{\mathscr{A}_{p,q}},$$

where $\mathscr{A}_{p,q}$ *is as given in* (7.5.7).

7.5.3. Quasi-left continuous purely discontinuous martingales

We now turn to estimates for the stochastic integral $\Phi \cdot M$ in the case that M is a purely discontinuous quasi-left continuous local martingale. We will first show in Lemma 7.5.11 that one can (essentially) represent $\Phi \cdot M$ as a stochastic integral $\Phi_H \star \bar{\mu}^M$, where $\bar{\mu}^M$ is the compensated version of the jump measure μ^M of M. Afterwards, in Theorem 7.5.22 we prove sharp bounds for stochastic integrals of the form $f \star \bar{\mu}$, where μ is any integer-valued random measure with a compensator that is non-atomic in time. By combining these two observations, we immediately find sharp bounds for $\Phi \cdot M$.

To any purely discontinuous local martingale M we can associate an integer-valued random measure μ^M on $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(H)$ by setting

$$\mu^M(\omega; B \times A) := \sum_{u\in B} \mathbf{1}_{A\setminus\{0\}}(\Delta M_u(\omega)), \quad \omega \in \Omega,$$

for each $B \in \mathscr{B}(\mathbb{R}_+)$, $A \in \mathscr{B}(H)$. That is, $\mu^M(\omega; B \times A)$ counts the number of jumps within the time set B with size in A on the trajectory belonging to the sample point ω.

Recall that a process $M : \mathbb{R}_+ \times \Omega \to H$ is called quasi-left continuous if $\Delta M_\tau = 0$ a.s. on the set $\{\tau < \infty\}$ for each predictable stopping time τ (see [85, Chapter I.2] for more information). If $M : \mathbb{R}_+ \times \Omega \to H$ is a quasi-left continuous local martingale, then μ^M is $\widetilde{\mathscr{P}}$-σ-finite and there exists a compensator ν^M (see e.g. [85, Proposition II.1.16] and [89, Theorem 25.22]). If M is, in addition, purely discontinuous, then the following characterization holds thanks to [85, Corollary II.1.19].

Lemma 7.5.10. *Let* H *be a separable Hilbert space and* $M : \mathbb{R}_+ \times \Omega \to H$ *be a purely discontinuous local martingale. Let* μ^M *and* ν^M *be the associated integer-valued random measure and its compensator. Then* M *is quasi-left continuous if and only if* ν^M *is non-atomic in time.*

Let us now show that $\Phi \cdot M$ can(essentially) be represented as a stochastic integral with respect to $\bar{\mu}_M$.

Lemma 7.5.11. *Let X be a Banach space, H be a Hilbert space, $1 \le p < \infty$, $M : \mathbb{R}_+ \times \Omega \to H$ be a purely discontinuous quasi-left continuous local martingale, and $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H, X)$ be elementary predictable. Define $\Phi_H : \mathbb{R}_+ \times \Omega \times H \to X$ by*

$$\Phi_H(t,\omega,h) := \Phi(t,\omega)h, \quad t \ge 0, \omega \in \Omega, h \in H.$$

Then there exists an increasing sequence $(A_n)_{n\ge1} \in \widetilde{\mathscr{P}}$ such that $\cup_n A_n = \mathbb{R}_+ \times \Omega \times J$, $\Phi_H \mathbf{1}_{A_n}$ is integrable with respect to $\bar{\mu}^M$ for each $n \ge 1$, and

(i) if $\Phi \cdot M \in L^p(\Omega; X)$ then $(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M \to \Phi \cdot M$ in $L^p(\Omega; X)$;

(ii) if $\Phi \cdot M \notin L^p(\Omega; X)$ then $\|(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M\|_{L^p(\Omega;X)} \to \infty$ for $n \to \infty$.

Proof. For each $k, l \ge 1$ we define a stopping time $\tau_{k,l}$ by

$$\tau_{k,l} = \inf\{t \in \mathbb{R}_+ : \#\{s \in [0,t] : \|\Delta M_s\| \in [1/k, k]\} = l\}.$$

Since M has càdlàg trajectories, $\tau_{k,l}$ is a.s. well-defined and takes its values in $[0,\infty]$. Moreover, $\tau_{k,l} \to \infty$ for each $k \ge 1$ a.s. as $l \to \infty$.

Set $B_k = \{h \in H : \|h\| \in [1/k, k]\}$. For each $k, l \ge 1$ define $A_{k,l} = \mathbf{1}_{[0,\tau_{k.l}]\times B_k} \subset \widetilde{\mathscr{P}}$. Then $\Phi_H \mathbf{1}_{A_{k,l}}$ is integrable with respect to μ^M. Indeed, a.s.

$$\big((\Phi_H \mathbf{1}_{A_{k,l}}) \star \mu^M\big)_\infty \le \sup \|\Phi\| k \big(\mathbf{1}_{A_{k,l}} \star \mu^M\big)_\infty \le \sup \|\Phi\| kl.$$

Since $\tau_{k,l} \to \infty$ for each $k \ge 1$ a.s. as $l \to \infty$, we can find a subsequence $(\tau_{k_n,l_n})_{n\ge1}$ such that $k_n \ge n$ for each $n \ge 1$ and $\inf_{m\ge n} \tau_{k_m,l_m} \to \infty$ a.s. as $n \to \infty$. Let $\tau_n = \inf_{m\ge n} \tau_{k_m,l_m}$ and define $(A_n)_{n\ge1} \subset \widetilde{\mathscr{P}}$ by

$$A_n = \mathbf{1}_{[0,\tau_n]\times B_n}.$$

Then $\cup_n A_n = \mathbb{R}_+ \times \Omega \times J$ and $\Phi_H \mathbf{1}_{A_n}$ is integrable with respect to $\bar{\mu}^M$ for all $n \ge 1$.

Now prove that $(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M \to \Phi \cdot M$ in $L^p(\Omega; X)$. Since Φ is simple, it takes its values in a finite dimensional subspace of X, so we can endow X with a Euclidean norm $|||\cdot|||$. First suppose that $(\Phi \cdot M)_\infty \notin L^p(\Omega; X)$. By the Burkholder-Davis-Gundy inequality this is equivalent to the fact that $[\Phi \cdot M]_\infty^{\frac12} \notin L^p(\Omega; X)$. Notice that

$$\begin{aligned}\mathbb{E}\,|||(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M)_\infty|||^p &\eqsim_p \mathbb{E}\big[(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M\big]_\infty^{\frac{p}{2}} \\ &= \mathbb{E}\Big(\sum_{t\in[0,\tau_n]} |||\Delta(\Phi \cdot M)_t|||^2 \mathbf{1}_{\|\Delta M_t\|\in[1/n,n]}\Big)^{\frac{p}{2}},\end{aligned}$$

and the last expression monotonically goes to infinity since $\tau_n \to \infty$ a.s. and

$$\mathbb{E}\Big(\sum_{t\ge0} \|\Delta(\Phi \cdot M)_t\|^2\Big)^{\frac{p}{2}} = \mathbb{E}[\Phi \cdot M]_\infty^{\frac{p}{2}} = \infty.$$

So if $(\Phi \cdot M)_\infty \notin L^p(\Omega; X)$, then $\big\|\big((\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M\big)_\infty\big\|_{L^p(\Omega;X)} \to \infty$ as $n \to \infty$.

Now assume that $(\Phi \cdot M)_\infty \in L^p(\Omega; X)$. Then

$$\begin{aligned}\mathbb{E}\,\big|\!\big|\!\big| (\Phi\cdot M)_\infty - ((\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M)_\infty \big|\!\big|\!\big|^p &\eqsim_p \mathbb{E}[\Phi\cdot M - (\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M]_\infty^{\frac{p}{2}}\\ &= \mathbb{E}\Big(\sum_{t\in[0,\tau_n]} |\!|\!| \Delta(\Phi\cdot M)_t |\!|\!|^2 \mathbf{1}_{\|\Delta M_t\|\notin[1/n,n]}\\ &\qquad + \sum_{t\in(\tau_n,\infty)} |\!|\!| \Delta(\Phi\cdot M)_t |\!|\!|^2\Big)^{\frac{p}{2}} \to 0, \;\; n\to\infty\end{aligned}$$

by the dominated convergence theorem. □

By Lemmas 7.5.10 and 7.5.11 it now suffices to obtain sharp bounds for the stochastic integral $(F\star\bar{\mu})_\infty$, where μ is any optional integer-valued random measure whose compensator ν is non-atomic in time.

7.5.4. Integrals with respect to random measures

Throughout this subsection, *μ denotes an optional integer-valued random measure whose compensator ν is non-atomic in time*, i.e., $\nu(\{t\}\times J) = 0$ a.s. for all $t \geq 0$. The following result was first shown in [131, Theorem 1].

Lemma 7.5.12 (A.A. Novikov). *Let $f : \mathbb{R}_+ \times \Omega \times J \to \mathbb{R}$ be $\widetilde{\mathscr{P}}$-measurable. Then*

$$\begin{aligned}&\mathbb{E}|f\star\bar{\mu}|^p \lesssim_p \mathbb{E}|f|^p \star \nu \;\; \text{if} \;\; 1\le p\le 2,\\ &\mathbb{E}|f\star\bar{\mu}|^p \lesssim_p (\mathbb{E}|f|^2\star\nu)^{\frac{p}{2}} + \mathbb{E}|f|^p\star\nu \;\; \text{if} \;\; p\geq 2.\end{aligned}$$

The following lemma easily follows from [85, Theorem II.1.33] (or from [70, p.98] and [131, (6)] as well).

Lemma 7.5.13. *Let H be a Hilbert space, $f : \mathbb{R}_+ \times \Omega \times J \to H$ be $\widetilde{\mathscr{P}}$-measurable. Then*

$$\mathbb{E}\|f\star\bar{\mu}\|^2 = \mathbb{E}\|f\|^2\star\nu. \tag{7.5.13}$$

Equivalently, for each $\widetilde{\mathscr{P}}$-measurable $f, g : \mathbb{R}_+ \times \Omega \times J \to H$ such that $\mathbb{E}\|f\|^2\star\nu < \infty$ and $\mathbb{E}\|g\|^2\star\nu < \infty$

$$\mathbb{E}\langle f\star\bar{\mu}, g\star\bar{\mu}\rangle = \mathbb{E}\langle f, g\rangle\star\nu. \tag{7.5.14}$$

Proof. The case $H = \mathbb{R}$ can be deduced from [85, II.1.34] as ν is assumed to be non-atomic in time. By applying this special case coordinate-wise, we obtain the general case. □

Corollary 7.5.14. *Let X be a Banach space, $1 < p < \infty$, μ be a random measure, ν be the corresponding compensator, $F : \mathbb{R}_+ \times \Omega \times J \to X$ and $G : \mathbb{R}_+ \times \Omega \times J \to X^*$ be simple $\widetilde{\mathscr{P}}$-measurable functions. Then for each $A \in \widetilde{\mathscr{P}}$ such that $\mathbb{E}(\mathbf{1}_A\star\mu)_\infty < \infty$ the stochastic integrals $(F\mathbf{1}_A)\star\bar{\mu}$ and $(G\mathbf{1}_A)\star\bar{\mu}$ are well-defined and*

$$\mathbb{E}\langle (F\mathbf{1}_A)\star\bar{\mu}, (G\mathbf{1}_A)\star\bar{\mu}\rangle = \mathbb{E}(\langle F, G\rangle \mathbf{1}_A)\star\nu. \tag{7.5.15}$$

Proof. Without loss of generality we can assume that X is finite dimensional. By Lemma 2.8.2, we can also redefine $F := F\mathbf{1}_A$, $G := G\mathbf{1}_A$. First notice that since $\|F\|_\infty, \|G\|_\infty < \infty$ and $\mathbb{E}\mu(F \neq 0), \mathbb{E}\mu(G \neq 0) < \infty$, both integrals $F \star \bar{\mu}$ and $G \star \bar{\mu}$ exist. Moreover, every finite dimensional space is isomorphic to a Hilbert space, so by Lemma 7.5.13 both $F \star \bar{\mu}$ and $G \star \bar{\mu}$ are L^2-integrable, and therefore the left-hand side of (7.5.15) is well-defined.

Now let d be the dimension of X, $(x_k)_{k=1}^d$ and $(x_k^*)_{k=1}^d$ be bases of X and X^* respectively. Then there exist simple $\widetilde{\mathscr{P}}$-measurable $F^1, \ldots, F^d, G^1, \ldots, G^d : \mathbb{R}_+ \times \Omega \times J \to \mathbb{R}$ such that $F = F^1 x_1 + \cdots + F^d x_d$ and $G = G^1 x_1^* + \cdots + G^d x_d^*$. Now (7.5.14) implies

$$\begin{aligned}\mathbb{E}\langle F \star \bar{\mu}, G \star \bar{\mu}\rangle &= \sum_{k,l=1}^d \langle x_k, x_l^*\rangle \mathbb{E}\big(F^k \star \bar{\mu} \cdot G^l \star \bar{\mu}\big) = \sum_{k,l=1}^d \langle x_k, x_l^*\rangle \mathbb{E}(F^k G^l) \star \nu\\ &= \mathbb{E}\Big(\sum_{k,l=1}^d \langle x_k, x_l^*\rangle F^k G^l\Big) \star \nu = \mathbb{E}\langle F, G\rangle \star \nu.\end{aligned}$$

□

The following proposition extends Novikov's inequalities presented in Lemma 7.5.12 in the case that $\nu(\mathbb{R}_+ \times J) \leq 1$ a.s. If $X = L^q(S)$ this result can be seen as a special case of Theorem 7.5.22 below. In the proof we will use the measure $\mathbb{P} \times \nu$ on $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F} \otimes \mathscr{J}$ that is defined by setting

$$\mathbb{P} \times \nu\Big(\bigcup_{i=1}^n A_i \times B_i\Big) := \sum_{i=1}^n \mathbb{E}(\mathbf{1}_{A_i} \nu(B_i)),$$

for disjoint $A_i \in \mathscr{F}$ and disjoint $B_i \in \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{J}$, and extending $\mathbb{P} \times \nu$ to $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F} \otimes \mathscr{J}$ via the Carathéodory extension theorem.

Proposition 7.5.15. *Suppose that $\nu(\mathbb{R}_+ \times J) \leq 1$ a.s. Let X be a Banach space and $f : \mathbb{R}_+ \times \Omega \times J \to X$ be simple $\widetilde{\mathscr{P}}$-measurable. Then for all $1 < p < \infty$*

$$\mathbb{E}\|F \star \bar{\mu}\|^p \eqsim_p \mathbb{E}\|F\|^p \star \nu.$$

Proof. We first prove $\lesssim_p$, and later deduce $\gtrsim_p$ by a duality argument.

Step 1: upper bounds. The case $X = \mathbb{R}$ follows from Lemma 7.5.12 and the fact that $\|\cdot\|_{L^2(\mathbb{R}_+ \times \Omega \times J, \mathbb{P} \otimes \nu)} \leq \|\cdot\|_{L^p(\mathbb{R}_+ \times \Omega \times J, \mathbb{P} \otimes \nu)}$ for each $p \geq 2$ since $\mathbb{P} \otimes \nu(\mathbb{R}_+ \times \Omega \times J) \leq 1$. Now let X be a general Banach space. Then

$$\begin{aligned}\mathbb{E}\|F \star \bar{\mu}\|^p &\overset{(i)}{\lesssim_p} \mathbb{E}\|F \star \mu\|^p + \mathbb{E}\|F \star \nu\|^p \overset{(ii)}{\leq} \mathbb{E}\big|\|F\| \star \mu\big|^p + \mathbb{E}\big|\|F\| \star \nu\big|^p\\ &\overset{(iii)}{\lesssim}_p \mathbb{E}\big|\|F\| \star \bar{\mu}\big|^p + \mathbb{E}\big|\|F\| \star \nu\big|^p \overset{(iv)}{\lesssim}_p \mathbb{E}\|F\|^p \star \nu,\end{aligned}$$

where (i) and (iii) follow from the fact that $\bar{\mu} = \mu - \nu$ and the triangle inequality, (ii) follows from [79, Proposition 1.2.2], and (iv) follows from the real-valued case and the fact that a.s.

$$\|\cdot\|_{L^1(\mathbb{R}_+ \times J; \nu)} \leq \|\cdot\|_{L^p(\mathbb{R}_+ \times J; \nu)}.$$

Step 2: lower bounds. We can assume that X is finite dimensional since F is simple. Let $Y = L^p(\mathbb{R}_+ \times \Omega \times J, \mathbb{P}\otimes\nu; X)$. Recall that by [79, Proposition 1.3.3] $Y^* = L^{p'}(\mathbb{R}_+ \times \Omega \times J, \mathbb{P}\otimes\nu; X^*)$ and $(L^p(\Omega;X))^* = L^{p'}(\Omega;X^*)$. Therefore due to the upper bounds from Step 1 and Corollary 7.5.14

$$\begin{aligned}(\mathbb{E}\|F\|^p \star \nu)^{\frac{1}{p}} &= \sup_{G\in Y^*:\|G\|\le 1} \mathbb{E}\langle F, G\rangle \star \nu = \sup_{G\in Y^*:\|G\|\le 1} \mathbb{E}\langle F\star\bar{\mu}, G\star\bar{\mu}\rangle\\ &\lesssim_p \sup_{\xi\in L^{p'}(\Omega;X^*):\|\xi\|\le 1} \mathbb{E}\langle F\star\bar{\mu}, \xi\rangle = (\mathbb{E}\|F\star\bar{\mu}\|^p)^{\frac{1}{p}}.\end{aligned}$$

□

Remark 7.5.16. The condition $\nu(\mathbb{R}_+ \times J) \le 1$ a.s. is necessary in general. Indeed, let N be a Poisson process with intensity parameter λ and let μ be the random measure on $\mathbb{R}_+ \times \{0\}$ defined by $\mu([0,t]\times\{0\}) = N_t$. Then the corresponding compensator ν satisfies $\nu([0,t]\times\{0\}) = \lambda t$. In particular,

$$\mathbb{E}|\mathbf{1}_{[0,1]} \star \bar{\mu}|^4 = \mathbb{E}|N-\lambda|^4 = \sum_{k=0}^{\infty} \frac{(k-\lambda)^4\lambda^k e^{-\lambda}}{k!} = \lambda(3\lambda+1),$$

which is not comparable with $\mathbb{E}|\mathbf{1}_{[0,1]}|^4 \star \nu = \lambda$ if λ is large.

The condition $\nu(\mathbb{R}_+ \times J) \le 1$ a.s. is however not needed for the upper bounds if $1 \le p \le 2$ and X is a Hilbert space. Indeed, for $p=1$

$$\mathbb{E}\|F\star\bar{\mu}\| \le \mathbb{E}\|F\star\mu\| + \mathbb{E}\|F\star\nu\| \le \mathbb{E}\|F\|\star\mu + \mathbb{E}\|F\|\star\nu = 2\mathbb{E}\|F\|\star\nu,$$

and for case $p=2$ follows immediately from Lemma 7.5.13:

$$\mathbb{E}\|F\star\bar{\mu}\|^2 = \mathbb{E}\|F\|^2\star\nu.$$

Therefore by the vector-valued Riesz-Thorin theorem [79, Theorem 2.2.1] for each $1\le p\le 2$

$$(\mathbb{E}\|F\star\bar{\mu}\|^p)^{\frac{1}{p}} \le 2(\mathbb{E}\|F\|^p\star\nu)^{\frac{1}{p}}.$$

Corollary 7.5.17. *Suppose that $\nu(\mathbb{R}_+\times J)\le 1$ a.s. Let X be a Banach space, $f:\mathbb{R}_+\times\Omega\times J\to X$ be simple $\widetilde{\mathscr{P}}$-measurable. Then for each $p\in(1,\infty)$ a.s.*

$$(\mathbb{E}\|F\star\bar{\mu}\|^p|\mathscr{F}_0) \eqsim_p (\mathbb{E}\|F\|^p\star\nu|\mathscr{F}_0). \tag{7.5.16}$$

Proof. Fix $A\in\mathscr{F}_0$. Then by Lemma 2.8.2 and Proposition 7.5.15

$$\mathbb{E}(\|F\star\bar{\mu}\|^p\cdot\mathbf{1}_A) = \mathbb{E}\|(F\cdot\mathbf{1}_A)\star\bar{\mu}\|^p \eqsim_p \mathbb{E}\|F\cdot\mathbf{1}_A\|^p\star\nu = \mathbb{E}(\|F\|^p\star\nu\cdot\mathbf{1}_A).$$

Since A is arbitrary, (7.5.16) holds. □

For each $m\ge 1$ let $\mathscr{P}_m$ be the σ-field on $\mathbb{R}_+\times\Omega$ generated by all $\mathscr{P}$-measurable $f:\mathbb{R}_+\times\Omega\to\mathbb{R}$ such that $f\big|_{(\frac{n}{2^m},\frac{n+1}{2^m}]\times\Omega}$ is $\mathscr{B}\big((\frac{n}{2^m},\frac{n+1}{2^m}]\big)\otimes\mathscr{F}_{\frac{n}{2^m}}$-measurable for each $n\ge 0$. Then the following theorem holds.

Theorem 7.5.18. *Let $f:\mathbb{R}_+\times\Omega\to\mathbb{R}$ be bounded and $\mathscr{P}$-measurable. Then for each $m\geq 1$*

$$\mathbb{E}(f|\mathscr{P}_m)(s) = \sum_{n\geq 0}\mathbb{E}\big(f(s)\big|\mathscr{F}_{\frac{n}{2^m}}\big),\quad s\in\Big(\frac{n}{2^m},\frac{n+1}{2^m}\Big], n\geq 0. \tag{7.5.17}$$

Moreover, $\mathbb{E}(f|\mathscr{P}_m)\to f$ a.s. on $\mathbb{R}_+\times\Omega$ as $m\to\infty$.

Proof. Let us first show (7.5.17). Fix $m\geq 1$. Fix a simple $\mathscr{P}_m$-measurable process $g:\mathbb{R}_+\times\Omega\to\mathbb{R}$. Then for each $n\geq 0$ and $s\in(\frac{n}{2^m},\frac{n+1}{2^m}]$ a random variable $g(s)$ is $\mathscr{F}_{\frac{n}{2^m}}$-measurable. Define $\widetilde{f}:\mathbb{R}_+\times\Omega\to\mathbb{R}$ by

$$\widetilde{f}(s) = \sum_{n\geq 0}\mathbb{E}\big(f(s)\big|\mathscr{F}_{\frac{n}{2^m}}\big)\mathbf{1}_{s\in(\frac{n}{2^m},\frac{n+1}{2^m}]},\quad s\geq 0.$$

Then for each $n\geq 0$ and $s\in(\frac{n}{2^m},\frac{n+1}{2^m}]$

$$\begin{aligned}\mathbb{E}\big[(f(s)-\widetilde{f}(s))g(s)\big] &= \mathbb{E}\Big[\mathbb{E}\big((f(s)-\widetilde{f}(s))g(s)\big|\mathscr{F}_{\frac{n}{2^m}}\big)\Big]\\ &= \mathbb{E}\Big[\mathbb{E}\big((f(s)-\widetilde{f}(s))\big|\mathscr{F}_{\frac{n}{2^m}}\big)g(s)\Big] = 0.\end{aligned}$$

Therefore

$$\mathbb{E}\int_{\mathbb{R}_+}\big(f(s)-\widetilde{f}(s)\big)g(s)\,\mathrm{d}s = \int_{\mathbb{R}_+}\mathbb{E}\Big[\big(f(s)-\widetilde{f}(s)\big)g(s)\Big]\,\mathrm{d}s = 0,$$

and hence (7.5.17) holds. Now notice that $(\mathscr{P}_m)_{m\geq 1}$ forms a filtration on $\mathbb{R}_+\times\Omega$, and obviously $\sigma\{\cup_m\mathscr{P}_m\} = \mathscr{P}$. Therefore the second part of the theorem follows from the martingale convergence theorem (see e.g. [89, Theorem 7.23]). □

Corollary 7.5.19. *Let $F:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ be an increasing predictable function such that $F(t)-F(s)\leq C(t-s)$ a.s. for all $0\leq s\leq t$ and for some fixed constant $C\geq 0$ and $F(0)=0$ a.s. Then for each fixed $T\geq 0$*

$$F(T) = \lim_{m\to\infty}\sum_{n=0}^{[2^mT]-1}\mathbb{E}\Big[F\Big(\frac{n+1}{2^m}\Big)-F\Big(\frac{n}{2^m}\Big)\Big|\mathscr{F}_{\frac{n}{2^m}}\Big],$$

where the last limit holds a.s. and in $L^p(\Omega)$ for all $1<p<\infty$.

For the proof we will need the following lemma.

Lemma 7.5.20. *Let $F:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ be an increasing predictable function such that $F(t)-F(s)\leq C(t-s)$ a.s. for all $0\leq s\leq t$ and for some fixed constant $C\geq 0$ and $F(0)=0$ a.s. Then there exists a predictable $f:\mathbb{R}_+\times\Omega\to[0,C]$ such that $F(T)=\int_0^T f(s)\,\mathrm{d}s$ for each fixed $T\geq 0$.*

Proof. F is a.s. differentiable in t because F is Lipschitz, so there exists $f:\mathbb{R}_+\times\Omega\to[0,C]$ such that for a.e. $\omega\in\Omega$ and $t\geq 0$

$$f(t,\omega) = \lim_{\varepsilon\to 0}\frac{F(t,\omega)-F((t-\varepsilon)\vee 0,\omega)}{\varepsilon}.$$

Since F is predictable, $t\mapsto F(t)-F((t-\varepsilon)\vee 0)$ is a predictable process as well for each $\varepsilon\geq 0$, so the obtained f is predictable. □

Proof of Corollary 7.5.19. Let $f:\mathbb{R}_+\times\Omega\to[0,C]$ be as defined in Lemma 7.5.20. Then by Theorem 7.5.18, $\mathbb{E}(f|\mathscr{P}_m)$ exists and converges to f a.s. on $\mathbb{R}_+\times\Omega$. Moreover, f is bounded by C, so $\mathbb{E}(f|\mathscr{P}_m)$ is bounded by C as well. Therefore for each $m\geq 1$ we find using (7.5.17)

$$\begin{aligned}\sum_{n=0}^{[2^mT]-1}\mathbb{E}\Big[F\Big(\frac{n+1}{2^m}\Big)-F\Big(\frac{n}{2^m}\Big)\Big|\mathscr{F}_{\frac{n}{2^m}}\Big] &= \sum_{n=0}^{[2^mT]-1}\mathbb{E}\Bigg[\int_{(\frac{n}{2^m},\frac{n+1}{2^m}]} f(s)\,\mathrm{d}s\Bigg|\mathscr{F}_{\frac{n}{2^m}}\Bigg]\\ &= \sum\nolimits_{n=0}^{[2^mT]-1}\int_{(\frac{n}{2^m},\frac{n+1}{2^m}]} E\big(f(s)\big|\mathscr{F}_{\frac{n}{2^m}}\big)\,\mathrm{d}s\\ &= \int_{\left(0,\frac{[2^mT]}{2^m}\right]}\mathbb{E}(f|\mathscr{P}_m)(s)\,\mathrm{d}s,\end{aligned}$$

and since $\frac{[2^mT]}{2^m}\to T$ as $m\to\infty$,

$$\begin{aligned}\lim_{m\to\infty}\sum_{n=0}^{[2^mT]-1}\mathbb{E}\Big[F\Big(\frac{n+1}{2^m}\Big)-F\Big(\frac{n}{2^m}\Big)\Big|\mathscr{F}_{\frac{n}{2^m}}\Big] &= \lim_{m\to\infty}\int_{\left(0,\frac{[2^mT]}{2^m}\right]}\mathbb{E}(f|\mathscr{P}_m)(s)\,\mathrm{d}s\\ &= \int_{(0,T]} f(s)\,\mathrm{d}s = F(T),\end{aligned}$$

where the limit holds a.s., and since $F(T)\leq CT$ and all the functions above are bounded by CT as well, by the dominated convergence theorem the limit holds in $L^p(\Omega)$ for each $1<p<\infty$. □

In the proof of Theorem 7.5.22 we will use a time-change argument. We recall some necessary definitions and results. A nondecreasing, right-continuous family of stopping times $\tau=(\tau_s)_{s\geq 0}$ is called a *random time-change*. If $\mathbb{F}$ is right-continuous, then according to [89, Lemma 7.3] the same holds true for the *induced filtration* $\mathbb{G}=(\mathscr{G}_s)_{s\geq 0}=(\mathscr{F}_{\tau_s})_{s\geq 0}$.

For a random time-change $\tau=(\tau_s)_{s\geq 0}$ and for a random measure μ we define $\mu\circ\tau$ in the following way:

$$\mu\circ\tau((s,t]\times B)=\mu((\tau_s,\tau_t]\times A),\quad t\geq s\geq 0, A\in\mathscr{J}.$$

μ is said to be *τ-continuous* if $\mu((\tau_{s-},\tau_s]\times J)=0$ a.s. for each $s\geq 0$, where we let $\tau_{s-}:=\lim_{\varepsilon\to 0}\tau_{s-\varepsilon}$, $\tau_{0-}:=\tau_0$. Later we will need the following proposition.

Proposition 7.5.21. *Let $A:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ be a strictly increasing continuous predictable process such that $A_0=0$ and $A_t\to\infty$ as $t\to\infty$ a.s. Then*

$$\tau_s=\{t:A_t=s\},\quad s\geq 0.$$

defines a random time-change $\tau=(\tau_s)_{s\geq 0}$. It satisfies $(A\circ\tau)(t)=(\tau\circ A)(t)=t$ a.s. for each $t\geq 0$. Let $\mathbb{G}=(\mathscr{G}_s)_{s\geq 0}=(\mathscr{F}_{\tau_s})_{s\geq 0}$ be the induced filtration. Then $(A_t)_{t\geq 0}$ is a random time-change with respect to $\mathbb{G}$. Moreover, for any random measure μ the following hold:

(i) if μ is $\mathbb{F}$-optional, then $\mu\circ\tau$ is $\mathbb{G}$-optional,

(ii) if μ is $\mathbb{F}$-predictable, then $\mu\circ\tau$ is $\mathbb{G}$-predictable,

(iii) if μ is an $\mathbb{F}$-optional random measure with a compensator ν, then $\nu\circ\tau$ is a compensator of $\mu\circ\tau$, and for each $\widetilde{\mathscr{P}}$-measurable simple $F:\mathbb{R}_+\times\Omega\times J\to\mathbb{R}$ such that $\mathbb{E}(F\star\mu)_\infty<\infty$ we have $\mathbb{E}\big((F\circ\tau)\star(\mu\circ\tau)\big)_\infty<\infty$ and a.s.

$$
\begin{aligned}
(F\star\mu)_\infty &= \big((F\circ\tau)\star(\mu\circ\tau)\big)_\infty,\\
(F\star\nu)_\infty &= \big((F\circ\tau)\star(\nu\circ\tau)\big)_\infty,
\end{aligned}
\tag{7.5.18}
$$

$$
(F\star\bar{\mu})(\tau_s) = ((F\circ\tau)\star(\overline{\mu\circ\tau}))(s),\quad s\geq 0. \tag{7.5.19}
$$

Proof. First of all notice that since A is strictly increasing and continuous a.s., $s\mapsto\tau_s$ is an a.s. continuous function, so any random measure μ is τ-continuous. Therefore (i) and (ii) follow from [84, Theorem 10.27(c,d)]. Let us prove (iii). The fact that $\nu\circ\tau$ is a compensator of $\mu\circ\tau$ holds due to [84, Theorem 10.27(e)], while the rest follows from [84, Theorem 10.28], and in particular (7.5.18) follows from the definition of $\mu\circ\tau$ and $\nu\circ\tau$. □

For more information on time-changes for random measures we refer to [84, Chapter X].

Let (S,Σ,ρ) be a measure space. For $1<p,q<\infty$ we define $\hat{\mathscr{S}}_q^p$, $\hat{\mathscr{D}}_{q,q}^p$ and $\hat{\mathscr{D}}_{p,q}^p$ as the Banach spaces of all functions $F:\mathbb{R}_+\times\Omega\times J\to L^q(S)$ that are $\widetilde{\mathscr{P}}$-measurable and for which the corresponding norms are finite:

$$
\begin{aligned}
\|F\|_{\hat{\mathscr{S}}_q^p} &:= \Big(\mathbb{E}\Big\|\Big(\int_{\mathbb{R}_+\times J}|F|^2\,\mathrm{d}\nu\Big)^{\frac12}\Big\|_{L^q(S)}^p\Big)^{\frac1p},\\
\|F\|_{\hat{\mathscr{D}}_{q,q}^p} &:= \Big(\mathbb{E}\Big(\int_{\mathbb{R}_+\times J}\|F\|_{L^q(S)}^q\,\mathrm{d}\nu\Big)^{\frac pq}\Big)^{\frac1p},\\
\|F\|_{\hat{\mathscr{D}}_{p,q}^p} &:= \Big(\mathbb{E}\int_{\mathbb{R}_+\times J}\|F\|_{L^q(S)}^p\,\mathrm{d}\nu\Big)^{\frac1p}.
\end{aligned}
\tag{7.5.20}
$$

We show in Appendix 7.A that

$$
(\hat{\mathscr{S}}_q^p)^* = \hat{\mathscr{S}}_{q'}^{p'},\qquad (\hat{\mathscr{D}}_{q,q}^p)^* = \hat{\mathscr{D}}_{q',q'}^{p'},\qquad (\hat{\mathscr{D}}_{p,q}^p)^* = \hat{\mathscr{D}}_{p',q'}^{p'}
$$

hold isomorphically with constants depending only on p and q.

Theorem 7.5.22. *Fix $1<p,q<\infty$. Let μ be an optional $\widetilde{\mathscr{P}}$-σ-finite random measure on $\mathbb{R}_+\times\mathscr{J}$ and suppose that its compensator ν is non-atomic in time. Then for any simple $\widetilde{\mathscr{P}}$-measurable $F:\mathbb{R}_+\times\Omega\times J\to L^q(S)$ and for any $A\in\widetilde{\mathscr{P}}$ with $\mathbb{E}\mathbf{1}_A\star\mu<\infty$*

$$
\Big(\mathbb{E}\sup_{0\leq s\leq t}\|((F\mathbf{1}_A)\star\bar{\mu})_s\|_{L^q(S)}^p\Big)^{\frac1p}\eqsim_{p,q}\|F\mathbf{1}_A\mathbf{1}_{[0,t]}\|_{\mathscr{I}_{p,q}}, \tag{7.5.21}
$$

where $\mathscr{I}_{p,q}$ is given by

$$\begin{aligned}
&\hat{\mathscr{S}}_q^p \cap \hat{\mathscr{D}}_{q,q}^p \cap \hat{\mathscr{D}}_{p,q}^p \text{ if } 2 \le q \le p < \infty,\\
&\hat{\mathscr{S}}_q^p \cap (\hat{\mathscr{D}}_{q,q}^p + \hat{\mathscr{D}}_{p,q}^p) \text{ if } 2 \le p \le q < \infty,\\
&(\hat{\mathscr{S}}_q^p \cap \hat{\mathscr{D}}_{q,q}^p) + \hat{\mathscr{D}}_{p,q}^p \text{ if } 1 < p < 2 \le q < \infty,\\
&(\hat{\mathscr{S}}_q^p + \hat{\mathscr{D}}_{q,q}^p) \cap \hat{\mathscr{D}}_{p,q}^p \text{ if } 1 < q < 2 \le p < \infty,\\
&\hat{\mathscr{S}}_q^p + (\hat{\mathscr{D}}_{q,q}^p \cap \hat{\mathscr{D}}_{p,q}^p) \text{ if } 1 < q \le p \le 2,\\
&\hat{\mathscr{S}}_q^p + \hat{\mathscr{D}}_{q,q}^p + \hat{\mathscr{D}}_{p,q}^p \text{ if } 1 < p \le q \le 2.
\end{aligned} \tag{7.5.22}$$

Proof. By Lemma 2.8.2 we can assume without loss of generality that $F := F\mathbf{1}_A$, $\mu := \mu\mathbf{1}_A$, and that there exists a $T \ge 0$ such that $F(t) = 0$ for each $t \ge T$. Since F is simple, it is uniformly bounded on $\mathbb{R}_+ \times \Omega \times J$ and, due to the fact that $\mathbb{E}\mathbf{1}_A \star \mu = \mathbb{E}\mu(\mathbb{R}_+ \times \Omega) < \infty$, we find $\mathbb{E}\|F \star \mu\| < \infty$. Consequently $F \star \bar{\mu}$ exists and it is a local martingale. Therefore Doob's maximal inequality implies

$$(\mathbb{E}\|(F \star \bar{\mu})_t\|^p)^{\frac{1}{p}} \eqsim_p \Big(\mathbb{E} \sup_{0\le s\le t} \|(F\star\bar{\mu})_s\|_{L^q(S)}^p\Big)^{\frac{1}{p}}$$

and so it is enough to show that

$$(\mathbb{E}\|(F \star \bar{\mu})_t\|^p)^{\frac{1}{p}} \eqsim_{p,q} \|F\mathbf{1}_{[0,t]}\|_{\mathscr{I}_{p,q}}. \tag{7.5.23}$$

The proof consists of two steps. In the first step we assume that ν is absolutely continuous with respect to Lebesgue measure. In this case, we can derive the upper bounds in (7.5.23) from the Burkholder-Rosenthal inequalities, Corollary 7.5.17 and Corollary 7.5.19. The lower bounds then follow by duality. In the second step we deduce the general result via a time-change argument based on Proposition 7.5.21.

Step 1: $\nu((s,t] \times J) \le (t-s)$ for each $t \ge s \ge 0$ a.s. We will consider the cases $2 \le q \le p < \infty$ and $1 < p \le q \le 2$, the proofs in the other cases are similar.

Case $2 \le q \le p < \infty$: Fix $m \ge 1$. Let $F_n := F\mathbf{1}_{\left(\frac{n}{2^m}, \frac{n+1}{2^m}\right]}$ for each $n \ge 0$. Then

$$(d_n)_{n\ge 0} := \big((F_n \star \bar{\mu})_\infty\big)_{n\ge 0}$$

is an $L^q(S)$-valued martingale difference sequence with respect to a filtration $\big(\mathscr{F}_{\frac{n+1}{2^m}}\big)_{n\ge 0}$. Theorem 7.1.1 implies

$$\begin{aligned}
\mathbb{E}\|(F \star \bar{\mu})_\infty\|_{L^q(S)}^p = \mathbb{E}\Big\|\sum_{n\ge 0}(F_n \star \bar{\mu})_\infty\Big\|_{L^q(S)}^p = \mathbb{E}\Big\|\sum_{n\ge 0} d_n\Big\|_{L^q(S)}^p &\eqsim_{p,q} \|(d_n)\|_{s_{p,q}}^p\\
&\eqsim_p (\|(d_n)\|_{S_q^p} + \|(d_n)\|_{D_{q,q}^p} + \|(d_n)\|_{D_{p,q}^p})^p.
\end{aligned}$$

To bound $\|(d_n)\|_{S_q^p}$, observe that

$$\|(d_n)\|_{S_q^p} = \Big(\mathbb{E}\Big\|\sum_n \mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}} |d_n|^2\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} = \Big(\mathbb{E}\Big\|\sum_n \mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}} |(F_n \star \bar{\mu})_\infty|^2\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}$$

$$
\begin{aligned}
&\overset{(*)}{\eqsim}_p \Big(\mathbb{E}\Big\|\sum_n \mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}}(|F_n|^2\star\nu)_\infty\Big\|_{L^q(S)}^p\Big)^{\frac1p} \qquad (7.5.24)\\
&= \Big(\mathbb{E}\Big\|\sum_n \mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}}\big((|F|^2\star\nu)_{\frac{n+1}{2^m}}-(|F|^2\star\nu)_{\frac{n}{2^m}}\big)\Big\|_{L^q(S)}^p\Big)^{\frac1p},
\end{aligned}
$$

where $(*)$ holds by Corollary 7.5.17 and the fact that

$$
\nu\Big(\Big(\frac{n}{2^m},\frac{n+1}{2^m}\Big]\Big)\le\frac{n+1}{2^m}-\frac{n}{2^m}=\frac{1}{2^m}\le 1.
$$

Notice that for a.e. $\omega\in\Omega$, all $s\in S$, and each $t\ge u\ge 0$

$$
\begin{aligned}
(|F|^2\star\nu)_t(s,\omega)-(|F|^2\star\nu)_u(s,\omega)&\le \sup|F(s)|^2(\nu((u,t]\times J)(\omega))\\
&\le \sup|F(s)|^2(t-u),
\end{aligned}
$$

so by Corollary 7.5.19

$$
\sum_n \mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}}\big((|F|^2\star\nu)_{\frac{n+1}{2^m}}-(|F|^2\star\nu)_{\frac{n}{2^m}}\big)\to(|F|^2\star\nu)_T=(|F|^2\star\nu)_\infty
$$

a.s. as $m\to\infty$. Therefore thanks to (7.5.24)

$$
\begin{aligned}
\|(d_n)\|_{S_q^p}&\eqsim\Big(\mathbb{E}\Big\|\sum_n \mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}}\big((|F|^2\star\nu)_{\frac{n+1}{2^m}}-(|F|^2\star\nu)_{\frac{n}{2^m}}\big)\Big\|_{L^q(S)}^p\Big)^{\frac1p} \qquad (7.5.25)\\
&\overset{m\to\infty}{\longrightarrow}(\mathbb{E}\|(|F|^2\star\nu)_\infty\|_{L^q(S)}^p)^{\frac1p}=\|F\|_{\mathscr{S}_q^p}.
\end{aligned}
$$

Now let us estimate $\|(d_n)\|_{D_{q,q}^p}$. Analogously to (7.5.24)

$$
\begin{aligned}
\|(d_n)\|_{D_{q,q}^p}&=\Big(\mathbb{E}\Big(\sum_n\mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}}\|d_n\|_{L^q(S)}^q\Big)^{\frac pq}\Big)^{\frac1p}=\Big(\mathbb{E}\Big(\sum_n\mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}}\|(F_n\star\bar\mu)_\infty\|_{L^q(S)}^q\Big)^{\frac pq}\Big)^{\frac1p}\\
&\eqsim\Big(\mathbb{E}\Big(\sum_n\mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}}(\|F_n\|_{L^q(S)}^q\star\nu)_\infty\Big)^{\frac pq}\Big)^{\frac1p} \qquad (7.5.26)\\
&=\Big(\mathbb{E}\Big(\sum_n\mathbb{E}_{\mathscr{F}_{\frac{n}{2^m}}}\big((\|F\|_{L^q(S)}^q\star\nu)_{\frac{n+1}{2^m}}-(\|F\|_{L^q(S)}^q\star\nu)_{\frac{n}{2^m}}\big)\Big)^{\frac pq}\Big)^{\frac1p},
\end{aligned}
$$

and similarly to (7.5.25) the last expression converges to $\|F\|_{\mathscr{D}_{q,q}^p}$. The same can be shown for $\mathscr{D}_{p,q}^p$.

Case $1<p\le q\le 2$: Let $\mathscr{I}_{\mathrm{elem}}(\widetilde{\mathscr{P}})$ denote the linear space of all simple $\widetilde{\mathscr{P}}$-measurable $L^q(S)$-valued functions. This linear space is dense in $\hat{\mathscr{S}}_q^p$, $\hat{\mathscr{D}}_{p,q}^p$ and $\hat{\mathscr{D}}_{q,q}^p$. Let $F\in\mathscr{I}_{\mathrm{elem}}(\widetilde{\mathscr{P}})$. Fix a decomposition $F=F_1+F_2+F_3$ with $F_\alpha\in\mathscr{I}_{\mathrm{elem}}(\widetilde{\mathscr{P}})$.

Fix $m\ge1$ and set $F_{n,\alpha}=F_\alpha\mathbf{1}_{\left(\frac{n}{2^m},\frac{n+1}{2^m}\right]}$, $d_{n,\alpha}=F_{n,\alpha}\star\bar\mu$, $\alpha=1,2,3$, so that

$$
(F\star\bar\mu)_T=(F\star\bar\mu)_\infty=\sum_n d_{n,1}+d_{n,2}+d_{n,3}.
$$

Then by Theorem 7.1.1, (7.5.24), (7.5.25) and (7.5.26) we conclude that

$$
\Big(\mathbb{E}\|(F\star\bar\mu)_\infty\|_{L^q(S)}^p\Big)^{\frac1p}\lesssim_{p,q}\|F_1\|_{\mathscr{S}_q^p}+\|F_2\|_{\mathscr{D}_{p,q}^p}+\|F_3\|_{\mathscr{D}_{q,q}^p}.
$$

Since $\mathscr{I}_{\mathrm{elem}}(\widetilde{\mathscr{P}})$ is dense in $\hat{\mathscr{S}}_q^p$, $\hat{\mathscr{D}}_{p,q}^p$ and $\hat{\mathscr{D}}_{q,q}^p$, we conclude by taking the infimum over F_1, F_2, F_3 as above that

$$\Big(\mathbb{E}\|(F\star\bar{\mu})_\infty\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \gtrsim_{p,q} \|F\|_{\mathscr{I}_{p,q}}.$$

The duality argument: Fix $t<\infty$, $1<p,q<\infty$. Using the upper bounds in (7.5.23) we can obtain the stochastic integral $(F\star\bar{\mu})_t$ as an L^p-limit of the integrals of the corresponding simple approximations of F in $\mathscr{I}_{p,q}$. Let Y be the closure of the linear subspace $\cup_{F\in\mathscr{I}_{p,q}}(F\star\bar{\mu})_t$ in $L^p(\Omega;L^q(S))$ and let $X=\mathscr{I}_{p,q}$. By Corollary 7.A.8, $X^*=\mathscr{I}_{p',q'}$. Let U (resp. V) be the dense subspace of X (resp. X^*) consisting of all $\widetilde{\mathscr{P}}$-measurable simple $L^q(S)$-valued (resp. $L^{q'}(S)$-valued) functions. Define both $j_0: U\to Y$ and $k_0: V\to Y^*$ by $F\mapsto(F\star\bar{\mu})_t$. Note that k_0 maps into Y^* since each $(F\star\bar{\mu})_t$ is in $L^{p'}(\Omega;L^{q'}(S))$, so it defines a bounded linear functional on Y. By the upper bounds in (7.5.23), j_0 and k_0 are bounded. Moreover, by the definition of Y, ran j_0 is dense in Y. Finally, by Corollary 7.5.14 $\langle F^*,F\rangle=\langle k_0(F^*), j_0(F)\rangle$ for all $F\in U$ and $F^*\in V$. Now (7.5.21) follows from Lemma 7.2.1.

Step 2: general case. Recall that, due to our assumptions in the beginning of the proof, $\mathbb{E}\mu(\mathbb{R}_+\times\Omega)=\mathbb{E}\nu(\mathbb{R}_+\times\Omega)<\infty$. Since ν is non-atomic in time, we can define a continuous strictly increasing predictable process $A:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ by

$$A_t=\nu([0,t]\times J)+t,\quad t\geq 0.$$

Let $\tau=(\tau_s)_{s\geq0}$ be the time-change defined in Proposition 7.5.21. Then according to Proposition 7.5.21 the random measure $\mu_\tau:=\mu\circ\tau$ is $\mathbb{G}$-optional, where $\mathbb{G}:=(\mathscr{G}_s)_{s\geq0}=(\mathscr{F}_{\tau_s})_{s\geq0}$. Moreover, $\nu_\tau:=\nu\circ\tau$ is $\mathbb{G}$-predictable and a compensator of μ_τ. Let $G:=F\circ\tau$. Notice that for each $t\geq s\geq 0$ a.s.

$$\begin{aligned}\nu_\tau((s,t]\times J)&=\nu((\tau_s,\tau_t]\times J)=\nu((0,\tau_t]\times J)-\nu((0,\tau_s]\times J)\\&\leq\nu((0,\tau_t]\times J)-\nu((0,\tau_s]\times J)+(\tau_t-\tau_s)\\&=(\nu((0,\tau_t]\times J)+\tau_t)-(\nu((0,\tau_s]\times J)+\tau_s)=t-s.\end{aligned}\tag{7.5.27}$$

Let $\mathscr{I}_{p,q}^\tau$ be defined as $\mathscr{I}_{p,q}$ but for the random measure ν_τ. By (7.5.27) Step 1 yields $\mathbb{E}\|G\star\bar{\mu}_\tau\|^p\eqsim_{p,q}\|G\|_{\mathscr{I}_{p,q}^\tau}^p$. Indeed, by (7.5.19), $\mathbb{E}\|(G\star\bar{\mu}_\tau)_\infty\|^p=\mathbb{E}\|F\star\bar{\mu}\|^p$. Moreover, for given F_i and $G_i=F_i\circ\tau$, $i=1,2,3$, it follows from (7.5.18) that

$$\begin{aligned}\mathbb{E}\Big\|\Big(\int_{\mathbb{R}_+\times J}|G_1|^2\,\mathrm{d}\nu_\tau\Big)^{\frac12}\Big\|_{L^q(S)}^p&=\mathbb{E}\Big\|\Big(\int_{\mathbb{R}_+\times J}|F_1|^2\,\mathrm{d}\nu\Big)^{\frac12}\Big\|_{L^q(S)}^p=\|F_1\|_{\mathscr{S}_p^q}^p,\\ \mathbb{E}\Big(\int_{\mathbb{R}_+\times J}\|G_2\|_{L^q(S)}^q\,\mathrm{d}\nu_\tau\Big)^{\frac{p}{q}}&=\mathbb{E}\Big(\int_{\mathbb{R}_+\times J}\|F_2\|_{L^q(S)}^q\,\mathrm{d}\nu\Big)^{\frac{p}{q}}=\|F_2\|_{\mathscr{D}_{q,q}^p}^p,\\ \mathbb{E}\int_{\mathbb{R}_+\times J}\|G_3\|_{L^q(S)}^p\,\mathrm{d}\nu_\tau&=\mathbb{E}\int_{\mathbb{R}_+\times J}\|F_3\|_{L^q(S)}^p\,\mathrm{d}\nu=\|F_3\|_{\mathscr{D}_{p,q}^p}^p.\end{aligned}$$

Consequently, $\|G\|_{\mathscr{I}_{p,q}^\tau}=\|F\|_{\mathscr{I}_{p,q}}$. We conclude that $\mathbb{E}\|F\star\bar{\mu}\|^p\eqsim_{p,q}\|F\|_{\mathscr{I}_{p,q}}^p$. $\square$

Remark 7.5.23. Let us compare our result to the literature. The upper bounds in Theorem 7.5.22 were discovered in the scalar-valued case by A.A. Novikov in [131, Theorem 1]. By exploiting an orthonormal basis one can easily extend this result to the Hilbert-space valued integrands, see [114, Section 3.3] for details. The paper [114] contains several other proofs of the Hilbert-space valued version of Novikov's inequality. In the context of Poisson random measures, Theorem 7.5.22 was obtained in [51]. Some one-sided extensions of the latter result in the context of general Banach spaces were obtained in [52]. However, these bounds, which are based on the martingale type and cotype of the space, are only matching in the Hilbert-space case and not optimal in general (in particular for L^q-spaces). A very different proof of the upper bounds in Theorem 7.5.22, which exploits tools from stochastic analysis, was discovered independently of our work by Marinelli in [110].

As a corollary, we obtain the following sharp bounds for stochastic integrals.

Theorem 7.5.24. *Fix $1 < p, q < \infty$. Let H be a Hilbert space, (S,Σ,ρ) be a measure space and let $M : \mathbb{R}_+ \times \Omega \to H$ be a purely discontinuous quasi-left continuous local martingale. Let $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H, L^q(S))$ be elementary predictable. Then*

$$\Big(\mathbb{E} \sup_{0\le s\le t} \|(\Phi\cdot M)_s\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} \|\Phi_H \mathbf{1}_{[0,t]}\|_{\mathscr{I}_{p,q}}, \tag{7.5.28}$$

where $\Phi_H : \mathbb{R}_+ \times \Omega \times H \to L^q(S)$ is defined by

$$\Phi_H(t,\omega,h) := \Phi(t,\omega)h, \quad t \ge 0, \omega \in \Omega, h \in H,$$

and $\mathscr{I}_{p,q}$ is given as in (7.5.22) *for $\nu = \nu^M$.*

Proof. The result follows from Doob's maximal inequality, Lemma 7.5.11, Theorem 7.5.22, and the fact that $\|\Phi_H \mathbf{1}_{A_n}\|_{\mathscr{I}_{p,q}} \nearrow \|\Phi_H\|_{\mathscr{I}_{p,q}}$ as $n \to \infty$ by the monotone convergence theorem. □

7.5.5. Integration with respect to continuous martingales

Finally, let us recall the known sharp bounds for L^q-valued stochastic integrals with respect to continuous local martingales. These bounds are a special case of the main result in [177].

For $F : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing, we define a measure ρ_F on $\mathscr{B}(\mathbb{R}_+)$ by

$$\rho_F((s,t]) = F(t) - F(s), \quad 0 \le s < t < \infty.$$

If X is a Banach space and $1 \le p \le \infty$, then we write $L^p(\mathbb{R}_+, F; X)$ for the Banach space $L^p(\mathbb{R}_+, \rho_F; X)$.

Let $M : \mathbb{R}_+ \times \Omega \to H$ be a continuous local martingale. Then by Subsection 2.2.1 one can define a continuous predictable process $[M] : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and a strongly progressively measurable $q_M:\mathbb{R}_+\times\Omega\to\mathscr{L}(H)$ such that $[M]$ is a quadratic variation of M and $\int_0^{\cdot}\langle q_M(s)h,h\rangle\,\mathrm{d}[M]_s$ is a quadratic variation of $[Mh]$ for each $h\in H$. The following theorem immediately follows from [177].

Theorem 7.5.25. *Let H be a Hilbert space, $1<p,q<\infty$. Let $M:\mathbb{R}_+\times\Omega\to H$ be a continuous local martingale, $\Phi:\mathbb{R}_+\times\Omega\to\mathscr{L}(H,L^q(S))$ be elementary predictable. Then*

$$\mathbb{E}\big(\sup_{0\le s\le t}\|(\Phi\cdot M)_s\|^p\big)\eqsim_{p,q}\mathbb{E}\|\Phi q_M^{\frac12}\mathbf{1}_{[0,t]}\|^p_{\gamma(L^2(\mathbb{R}_+,[M^c];H),L^q(S))}.$$

7.5.6. Integration with respect to general local martingales

We can now combine the sharp estimates obtained for the three special type of stochastic integrals to obtain sharp estimates for $\Phi\cdot M$, where M is an arbitrary local martingale.

Lemma 7.5.26. *Let H be a Hilbert space, X be a Banach space space, $M:\mathbb{R}_+\times\Omega\to H$ be a local martingale, $\Phi:\mathbb{R}_+\times\Omega\to\mathscr{L}(H,X)$ be elementary predictable, $F:\mathbb{R}_+\times\Omega\times H\to X$ be elementary $\widetilde{\mathscr{P}}$-measurable. Then*

(i) if M is continuous, then $\Phi\cdot M$ is continuous,

(ii) if M is purely discontinuous quasi-left continuous, then $F\star\bar\mu^M$ is purely discontinuous quasi-left continuous,

(iii) if M is purely discontinuous with accessible jumps, then $\Phi\cdot M$ is purely discontinuous with accessible jumps.

Proof. Since Φ is elementary predictable, X can be assumed to be finite-dimensional.

(i) holds since if M is continuous, then the formula (2.5.2) defines an a.s. continuous process.

To prove pure discontinuity in (ii) one has to endow X with a Euclidean norm and notice that if M is purely discontinuous quasi-left continuous then by [85, Proposition II.1.28] $[F\star\bar\mu^M]_t=\sum_{0\le s\le t}\|F(\Delta M)\|^2$ a.s. for all $t\ge0$ since $F\star\nu^M$ is absolutely continuous, so it does not effect on the quadratic variation. Therefore $[F\star\bar\mu^M]$ is purely discontinuous, and so $F\star\bar\mu^M$ is purely discontinuous by [89, Theorem 26.14]. Quasi-left continuity then follows as $\Delta(F\star\bar\mu^M)_\tau=F(\Delta M_\tau)=0$ a.s. for any predictable stopping time τ.

Pure discontinuity of $\Phi\cdot M$ in (iii) follows from the same argument as in (ii), and the rest can be proven using the fact that a.s.

$$\{t\in\mathbb{R}_+:\Delta(\Phi\cdot M)_t\neq0\}\subset\{t\in\mathbb{R}_+:\Delta M_t\neq0\}.$$

□

The following observation is fundamental for the duality argument used to prove the lower bounds in Theorem 7.5.29.

Lemma 7.5.27. *Let H be a Hilbert space, X be a Banach space, $M^c, M^q : \mathbb{R}_+ \times \Omega \to H$ be continuous and purely discontinuous quasi-left continuous martingales, $M^{a,1} : \mathbb{R}_+ \times \Omega \to X$, $M^{a,2} : \mathbb{R}_+ \times \Omega \to X^*$ be purely discontinuous martingales with accessible jumps, $\Phi_1 : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H,X)$, $\Phi_2 : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H,X^*)$ be elementary predictable, $F_1 : \mathbb{R}_+ \times \Omega \times H \to X$, $F_2 : \mathbb{R}_+ \times \Omega \times H \to X^*$ be elementary $\widetilde{\mathcal{P}}$-measurable. Assume that $(\Phi_1 \cdot M^c)_\infty$, $(F_1 \star \bar{\mu}^{M^q})_\infty$, $M^{a,1}_\infty \in L^p(\Omega;X)$ and $(\Phi_2 \cdot M^c)_\infty$, $(F_2 \star \bar{\mu}^{M^q})_\infty$, $M^{a,2}_\infty \in L^{p'}(\Omega;X^*)$ for some $1<p<\infty$. Then, for all $t \geq 0$,*

$$
\begin{aligned}
&\mathbb{E}\langle(\Phi_1 \cdot M^c + F_1 \star \bar{\mu}^{M^q} + M^{a,1})_t, (\Phi_2 \cdot M^c + F_2 \star \bar{\mu}^{M^q} + M^{a,2})_t\rangle \\
&\quad= \mathbb{E}\langle(\Phi_1 \cdot M^c)_t, (\Phi_2 \cdot M^c)_t\rangle + \mathbb{E}\langle(F_1 \star \bar{\mu}^{M^q})_t, (F_2 \star \bar{\mu}^{M^q})_t\rangle + \mathbb{E}\langle M^{a,1}_t, M^{a,2}_t\rangle. \quad (7.5.29)
\end{aligned}
$$

Lemma 7.5.28. *Let X be a Banach space, $X_0 \subset X$ be a finite-dimensional subspace, $1 < p < \infty$, $M^q : \mathbb{R}_+ \times \Omega \to X_0$ be a purely discontinuous quasi-left continuous L^p-martingale, $M^q_0 = 0$, $M^a : \mathbb{R}_+ \times \Omega \to X^*$ be a purely discontinuous $L^{p'}$-martingale with accessible jumps. Then $\mathbb{E}\langle M^q_t, M^a_t\rangle = 0$ for each $t \geq 0$.*

Proof. Let d be the dimension of X_0, $x_1, \ldots, x_d$ be a basis of X_0. Then there exist purely discontinuous quasi-left continuous L^p-martingales $M^{q,1}, \ldots, M^{q,d} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that $M^q = M^{q,1}x_1 + \cdots + M^{q,d}x_d$. Thus for any $i = 1, \ldots, d$ and any purely discontinuous $L^{p'}$-martingale $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ with accessible jumps $[M^{q,i}, N] = 0$ a.s. by [89, Corollary 26.16]. Hence [85, Proposition I.4.50(a)] implies that $M^{q,i}N$ is a local martingale, and due to integrability it is a martingale. Notice also that all $M^{q,i}$ start at zero, therefore

$$
\mathbb{E}\langle M^q_t, M^a_t\rangle = \sum_{i=1}^d \mathbb{E}M^{q,i}_t \langle x_i, M^a_t\rangle = \sum_{i=1}^d \mathbb{E}M^{q,i}_0 \langle x_i, M^a_0\rangle = 0.
$$

□

Proof of Lemma 7.5.27. Since all the integrands Φ_1, Φ_2, F_1, F_2 are elementary, one can suppose that X and X^* are finite dimensional, so we can endow these spaces with Euclidean norms. Since by Lemma 7.5.26 $\Phi_1 \cdot M^c$ and $\Phi_2 \cdot M^c$ are continuous, $F_1 \star \bar{\mu}^{M^q}$, $F_1 \star \bar{\mu}^{M^q}$, $M^{a,1}$ and $M^{a,2}$ are purely discontinuous, then [85, Definition I.4.11] implies that for each $t \geq 0$

$$
\begin{aligned}
\mathbb{E}\langle(\Phi_1 \cdot M^c)_t, (F_2 \star \bar{\mu}^{M^q})_t\rangle &= \mathbb{E}[\Phi_1 \cdot M^c, F_2 \star \bar{\mu}^{M^q}]_t = 0,\\
\mathbb{E}\langle(\Phi_2 \cdot M^c)_t, (F_1 \star \bar{\mu}^{M^q})_t\rangle &= \mathbb{E}[\Phi_2 \cdot M^c, F_1 \star \bar{\mu}^{M^q}]_t = 0,\\
\mathbb{E}\langle(\Phi_1 \cdot M^c)_t, M^{a,2}_t\rangle &= \mathbb{E}[\Phi_1 \cdot M^c, M^{a,2}]_t = 0,\\
\mathbb{E}\langle(\Phi_2 \cdot M^c)_t, M^{a,1}_t\rangle &= \mathbb{E}[\Phi_2 \cdot M^c, M^{a,1}]_t = 0.
\end{aligned}
$$

Moreover, thanks to Lemma 7.5.26 and Lemma 7.5.28

$$
\mathbb{E}\langle M^{a,1}_t, (F_2 \star \bar{\mu}^{M^q})_t\rangle = \mathbb{E}\langle M^{a,2}_t, (F_1 \star \bar{\mu}^{M^q})_t\rangle = 0,
$$

so (7.5.29) easily follows. □

Theorem 7.5.29. *Let H be a Hilbert space, $1 < p, q < \infty$. Let M^c, $M^q : \mathbb{R}_+ \times \Omega \to H$ be continuous and purely discontinuous quasi-left continuous local martingales, $M^a : \mathbb{R}_+ \times \Omega \to L^q(S)$ be a purely discontinuous L^p-martingale with accessible jumps, $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H, L^q(S))$ be elementary predictable, $F : \mathbb{R}_+ \times \Omega \times H \to L^q(S)$ be elementary $\widetilde{\mathscr{P}}$-measurable. If $\Phi \cdot M^c$ and $F \star \bar{\mu}^{M^q}$ are L^p-martingales, then*

$$\Big(\mathbb{E} \sup_{0\le s\le t} \big\| \big(\Phi \cdot M^c + F \star \bar{\mu}^{M^q} + M^a\big)_\infty \big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \tag{7.5.30}$$
$$\eqsim_{p,q} \big(\mathbb{E}\|\Phi q^{\frac{1}{2}}_{M^c}\|^p_{\gamma(L^2(\mathbb{R}_+,[M^c];H),X)}\big)^{\frac{1}{p}} + \|F\|_{\mathscr{I}_{p,q}} + \|M^a\|_{\mathscr{A}_{p,q}},$$

where $\mathscr{I}_{p,q}$ is given as in (7.5.22) *for $\nu = \nu^{M^{\mathrm{q}}}$, $\mathscr{A}_{p,q}$ is given as in* (7.5.7)*.*

Proof. The estimate $\lesssim_{p,q}$ follows from the triangle inequality and Theorems 7.5.25, 7.5.22 and 7.5.8. Let us now prove $\gtrsim_{p,q}$ via duality. Without loss of generality due to the proof of Theorem 7.5.8 and due to Lemma 7.5.7 we can assume that there exists $N \ge 1$ and a sequence of predictable stopping times $\mathscr{T} = (\tau_n)_{n=0}^N$ such that M has a.s. at most N jumps and a.s. $\{t \in \mathbb{R}_+ : \Delta M_t \neq 0\} \subset \{\tau_0, \dots, \tau_N\}$. Define the Banach space

$$X := L^p(\Omega; \gamma(L^2(\mathbb{R}_+, [M^c]; H), L^q(S))) \times \mathscr{I}_{p,q} \times \mathscr{A}^{\mathscr{T}}_{p,q}$$

and let Y be the closure of the lineare subspace $\cup_{(\Phi,F,M^a)\in X}(\Phi \cdot M^c + F \star \bar{\mu}^{M^{\mathrm{d}}} + M^a)_\infty$ in $L^p(\Omega; L^q(S))$. Then by [79, Proposition 1.3.3], the Trace duality (2.9.2), Corollary 7.A.8 and the duality statement in Theorem 7.5.5

$$X^* = L^{p'}(\Omega; \gamma(L^2(\mathbb{R}_+, [M^c]; H), L^{q'}(S))) \times \mathscr{I}_{p',q'} \times \mathscr{A}^{\mathscr{T}}_{p',q'}.$$

By the upper bounds in (7.5.30), the maps $j : X \to Y$ and $k : X^* \to Y^*$ defined via $(\Phi, F, M^a) \mapsto (\Phi \cdot M^c + F \star \bar{\mu}^{M^{\mathrm{d}}} + M^a)_\infty$ are both continuous linear mappings. Let $x = (\Phi_1, F_1, M^a_1) \in X$, $x^* = (\Phi_2, F_2, M^a_2) \in X^*$ be such that Φ_1 and Φ_2 are elementary predictable, and F_1 and F_2 are elementary $\widetilde{\mathscr{P}}$-measurable. Then $\langle \tilde{x}^*, \tilde{x}\rangle = \langle k(\tilde{x}^*), j(\tilde{x})\rangle$ by Lemma 7.5.27 and (7.5.5) and so Lemma 7.2.1 yields $\gtrsim_{p,q}$ in (7.5.30). □

Theorem 7.5.30. *Let H be a Hilbert space, $1 < p, q < \infty$. Let $M : \mathbb{R}_+ \times \Omega \to H$ be a local martingale, $M^c, M^q, M^d : \mathbb{R}_+ \times \Omega \to H$ be local martingales such that $M^c_0 = M^q_0 = 0$, M^c is continuous, M^q is purely discontinuous quasi-left continuous, M^a is purely discontinuous with accessible jumps, $M = M^c + M^q + M^a$. Let $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H, L^q(S))$ be elementary predictable. Then,*

$$\Big(\mathbb{E} \sup_{0\le s\le t} \|(\Phi \cdot M)_s\|^p_{L^q(S)}\Big)^{\frac{1}{p}}$$
$$\eqsim_{p,q} \Big(\mathbb{E}\|\Phi q^{\frac{1}{2}}_{M^c} \mathbf{1}_{[0,t]}\|^p_{\gamma(L^2(\mathbb{R}_+,[M^c];H),X)}\Big)^{\frac{1}{p}}$$
$$+ \|\Phi_H \mathbf{1}_{[0,t]}\|_{\mathscr{I}_{p,q}} + \|(\Phi \mathbf{1}_{[0,t]}) \cdot M^a\|_{\mathscr{A}_{p,q}}, \tag{7.5.31}$$

where $\Phi_H : \mathbb{R}_+ \times \Omega \times H \to L^q(S)$ *is defined by*

$$\Phi_H(t,\omega,h) := \Phi(t,\omega)h, \quad t \geq 0, \omega \in \Omega, h \in H,$$

$\mathscr{I}_{p,q}$ *is given as in* (7.5.22) *for* $\nu = \nu^{M^q}$, *and* $\mathscr{A}_{p,q}$ *is as defined in* (7.5.7).

Proof. First of all notice that $\Phi \cdot M$ is an $L^q(S)$-valued local martingale, so by Doob's maximal inequality

$$\mathbb{E} \sup_{0\leq s\leq t} \|(\Phi\cdot M)_s\|^p_{L^q(S)} \eqsim_p \mathbb{E}\|(\Phi\cdot M)_t\|^p_{L^q(S)}. \tag{7.5.32}$$

Since Φ is elementary predictable, we can assume that X is finite dimensional. Consequently, (7.5.31) holds by (7.5.32), Lemma 7.5.26 and Theorem 7.5.29. □

Remark 7.5.31. Let $M = (M_n)_{n\geq 0}$ be a discrete L^q-valued martingale. Then due to the *Strong Doob maximal inequality* (also known as the *Fefferman-Stein inequality*), presented e.g. in [79, Theorem 3.2.7] and [3, Theorem 2.6],

$$\Big(\mathbb{E}\Big(\int_S |\sup_{n\geq 0} M_n(s)|^q \,\mathrm{d}s\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} \eqsim_{p,q} (\mathbb{E}\sup_{n\geq 0}\|M_n\|^p_{L^q(S)})^{\frac{1}{p}}.$$

As a consequence, for any continuous time martingale $M : \mathbb{R}_+ \times \Omega \to L^q(S)$

$$\Big(\mathbb{E}\|\sup_{t\geq 0} M_t\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} (\mathbb{E}\sup_{t\geq 0}\|M_t\|^p_{L^q(S)})^{\frac{1}{p}}.$$

Indeed, this follows by the existence of a pointwise càdlàg version of M and by approximating M by a discrete-time martingale. Thus, all the sharp bounds for stochastic integrals proved in this section, in particular Theorems 7.5.8, 7.5.22, 7.5.24, 7.5.25, and, finally, Theorems 7.5.29 and 7.5.30, remain valid if we move the supremum over time inside the L^q-norm.

7.A. DUALS OF $\mathscr{S}_q^p$, $\mathscr{D}_{q,q}^p$, $\mathscr{D}_{p,q}^p$, $\hat{\mathscr{S}}_q^p$, $\hat{\mathscr{D}}_{q,q}^p$, AND $\hat{\mathscr{D}}_{p,q}^p$

In this section we will find the duals of $\mathscr{S}_q^p$, $\mathscr{D}_{q,q}^p$, $\mathscr{D}_{p,q}^p$, $\hat{\mathscr{S}}_q^p$, $\hat{\mathscr{D}}_{q,q}^p$, and $\hat{\mathscr{D}}_{p,q}^p$ for all $1 < p, q < \infty$. As a consequence, we show the duality for the space $\mathscr{I}_{p,q}$ that was used to prove the lower bounds in Theorem 7.5.22.

7.A.1. $\mathscr{D}_{q,q}^p$ and $\mathscr{D}_{p,q}^p$ spaces

Let X be a Banach space and consider any random measure ν on $\mathbb{R}_+ \times J$. In sequel we will assume that $\int_{\mathbb{R}_+\times J} \mathbf{1}_A \,\mathrm{d}\nu$ is an $\overline{\mathbb{R}}_+$-valued random variable for each $\mathscr{B}(\mathbb{R}_+)\otimes \mathscr{J}$-measurable $A \subset \mathbb{R}_+ \times J$. Notice that this condition always holds for any optional random measure ν. Indeed, without loss of generality we may assume that there exist $A_{\mathbb{R}_+} \in \mathscr{B}(\mathbb{R}_+)$ and $A_J \in \mathscr{J}$ such that $A = A_{\mathbb{R}_+} \times A_J$. Let $\widetilde{A} = A \times \Omega$. Then $\widetilde{A} \in \widetilde{\mathscr{O}}$ (since $A_{\mathbb{R}_+} \times \Omega \in \mathscr{O}$), therefore $\mathbf{1}_{\widetilde{A}} \star \nu$ is an optional process, and

$$\int_{\mathbb{R}_+\times J} \mathbf{1}_A \,\mathrm{d}\nu = \lim_{t\to\infty} (\mathbf{1}_{\widetilde{A}} \star \nu)_t$$

is an $\overline{\mathbb{R}}_+$-valued $\mathscr{F}$-measurable function as a monotone limit of $\overline{\mathbb{R}}_+$-valued $\mathscr{F}$-measurable functions.

We define $\mathscr{D}_q^p(X)$ to be the space of all $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$-strongly measurable functions $f : \mathbb{R}_+ \times \Omega \times J \to X$ such that

$$\|f\|_{\mathscr{D}_q^p(X)} := \Big(\mathbb{E}\Big(\int_{\mathbb{R}_+\times J} \|f\|_X^q \,\mathrm{d}\nu\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} < \infty.$$

Recall that the measure $\mathbb{P}\otimes\nu$ on $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$ is defined by setting

$$\mathbb{P}\otimes\nu\Big(\bigcup_{i=1}^n A_i \times B_i\Big) := \sum_{i=1}^n \mathbb{E}(\mathbf{1}_{A_i}\nu(B_i)),$$

for disjoint $A_i \in \mathscr{F}$ and disjoint $B_i \in \mathscr{B}(\mathbb{R}_+)\otimes\mathscr{J}$ and extending $\mathbb{P}\times\nu$ to $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$ via the Carathéodory extension theorem.

The following result is well-known if ν is a deterministic measure. The argument for random measures is similar and provided for the reader's convenience.

Theorem 7.A.1. *Let $1 < p, q < \infty$, X be reflexive. Then $(\mathscr{D}_q^p(X))^* = \mathscr{D}_{q'}^{p'}(X^*)$. Moreover*

$$\|\phi\|_{\mathscr{D}_{q'}^{p'}(X^*)} = \|\phi\|_{(\mathscr{D}_q^p(X))^*}, \quad \phi \in \mathscr{D}_{q'}^{p'}(X^*). \tag{7.A.1}$$

Proof. First we suppose that $\mathbb{E}\nu(\mathbb{R}_+ \times J) < \infty$. By approximation we can assume that $\nu(\mathbb{R}_+ \times J) \leq N$ a.s., for some $N \in \mathbb{N}$. In this case we can proceed with a standard argument using the Radon-Nikodym property of X^*. Let $F \in (\mathscr{D}_q^p(X))^*$. On $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$ we can define and X^*-valued measure θ by setting

$$\langle\theta(A), x\rangle := F(\mathbf{1}_A \cdot x) \qquad (\mathscr{B}(\mathbb{R}_+)\otimes A \in \mathscr{F}\otimes\mathscr{J}, x \in X).$$

It is straightforward to verify that θ is σ-additive and absolutely continuous with respect to $\mathbb{P}\times\nu$. Moreover, θ is of finite variation. Indeed, if $A_1,\dots,A_n$ is a disjoint partition of $\mathbb{R}_+\times\Omega\times J$, then

$$\begin{aligned}\sum_{i=1}^n\|\theta(A_i)\| &= \sup_{(x_i)_{i=1}^n\subset B_X}\sum_{i=1}^n F(\mathbf{1}_{A_i}x_i)\\ &= \sup_{(x_i)_{i=1}^n\subset B_X} F\Big(\sum_{i=1}^n\mathbf{1}_{A_i}x_i\Big)\\ &\le \|F\|_{(\mathscr{D}_q^p(X))^*}\sup_{(x_i)_{i=1}^n\subset B_X}\Big(\mathbb{E}\Big(\int_{\mathbb{R}_+\times J}\Big\|\sum_{i=1}^n\mathbf{1}_{A_i}x_i\Big\|_X^q\,d\nu\Big)^{p/q}\Big)^{1/p}\\ &= \|F\|_{(\mathscr{D}_q^p(X))^*}\sup_{(x_i)_{i=1}^n\subset B_X}\Big(\mathbb{E}\Big(\int_{\mathbb{R}_+\times J}\sum_{i=1}^n\mathbf{1}_{A_i}\|x_i\|_X^q\,d\nu\Big)^{p/q}\Big)^{1/p}\\ &\le \|F\|_{(\mathscr{D}_q^p(X))^*}(\mathbb{E}\nu(\mathbb{R}_+\times J)^{p/q})^{1/p}. \end{aligned}\tag{7.A.2}$$

(Here B_X is a unit ball in X). By the Radon-Nikodym property of X^*, there exists an $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$-strongly measurable X^*-valued function f such that

$$F(g)=F_f(g)=\mathbb{E}\int_{\mathbb{R}_+\times J}\langle f,g\rangle\,\mathrm{d}\nu$$

for each $g\in\mathscr{D}_q^p(X)$. By Hölder's inequality, it is immediate that

$$\|F\|_{(\mathscr{D}_q^p(X))^*}\le\|f\|_{\mathscr{D}_{q'}^{p'}(X^*)}.$$

To show the reverse estimate, we may assume that $f\in\mathscr{D}_{q'}^{p'}(X^*)$ has norm 1 and show that $\|F_f\|_{(\mathscr{D}_q^p(X))^*}\ge 1$. By approximation, we may furthermore assume that f is simple, i.e.,

$$f=\sum_{m,n}\mathbf{1}_{A_n}\,\mathbf{1}_{B_{nm}}\,x^*_{nm},$$

for $A_n\in\mathscr{F}$ and $B_{nm}\in\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{J}$ disjoint and $x^*_{nm}\in X^*$. Define

$$g=\sum_{m,n}\mathbf{1}_{A_n}\Big(\sum_m\nu(B_{nm})\|x^*_{nm}\|_{X^*}^{q'}\Big)^{\frac{p'}{q'}-1}\,\mathbf{1}_{B_{nm}}\,x_{nm}\|x^*_{nm}\|_{X^*}^{q'-1},\tag{7.A.3}$$

where the $x_{nm}\in X$ satisfy the condition in Lemma 7.4.2, i.e. for some $0<\varepsilon<1$

$$(1-\varepsilon)\|x^*_{nm}\|\le\langle x_{nm},x_{nm^*}\rangle,\qquad\|x_{nm}\|_X=1.$$

By assumption,

$$\|f\|^{p'}_{D_{q'}^{p'}(X^*)}=\sum_n\mathbb{P}(A_n)\Big(\sum_m\nu(B_{nm})\|x^*_{nm}\|_{X^*}^{q'}\Big)^{p'/q'}=1.$$

Therefore, also

$$\begin{aligned}\|g\|_{D_q^p(X)}^q &= \sum_n \mathbb{P}(A_n)\Big(\sum_m \nu(B_{nm})\|x_{nm}^*\|_{X^*}^{q'}\Big)^{\left(\frac{p'}{q'}-1\right)p}\Big(\sum_m \nu(B_{nm})\|x_{nm}^*\|_{X^*}^{(q'-1)q}\Big)^{\frac{p}{q}}\\ &= \sum_n \mathbb{P}(A_n)\Big(\sum_m \nu(B_{nm})\|x_{nm}^*\|^{q'}\Big)^{p'/q'} = 1,\end{aligned}$$

as

$$qq' = q+q' \qquad \frac{pp'}{q'} - p + \frac{p}{q} = \frac{p'}{q'}. \tag{7.A.4}$$

Moreover,

$$\begin{aligned}F_f(g) &= \mathbb{E}\int \langle f, g\rangle \, d\nu\\ &= \sum_n \mathbb{P}(A_n)\sum_m \nu(B_{nm})\langle x_{nm}, x_{nm^*}\rangle \|x_{nm}^*\|_{X^*}^{q'-1}\Big(\sum_m \nu(B_{nm})\|x_{nm}^*\|_{X^*}^{q'}\Big)^{\frac{p'}{q'}-1}\\ &\geq \sum_n \mathbb{P}(A_n)\sum_m \nu(B_{nm})(1-\varepsilon)\|x_{nm}^*\|_{X^*}^{q'}\Big(\sum_m \nu(B_{nm})\|x_{nm}^*\|_{X^*}^{q'}\Big)^{\frac{p'}{q'}-1}\\ &= (1-\varepsilon)\sum_n \mathbb{P}(A_n)\Big(\sum_m \nu(B_{nm})\|x_{nm}^*\|_{X^*}^{q'}\Big)^{\frac{p'}{q'}} = (1-\varepsilon).\end{aligned}$$

Since ε was arbitrary, $\|F_f\|_{(\mathscr{D}_q^p(X))^*} \geq 1$.

Let now $\mathbb{E}\nu(\mathbb{R}_+ \times J) = \infty$ and assume that $\mathbb{P}\times\nu$ is σ-finite. Then there exists a sequence $(A_n)_{n\geq 1} \subset \mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$ such that $A_n \subset A_{n+1}$ for each $n \geq 1$, $\cup_{n\geq 1} A_n = \mathbb{R}_+\times\Omega\times J$, and $\mathbb{P}\otimes\nu(A_n) < \infty$ for each $n\geq 1$. Let $\nu_n := \nu\cdot\mathbf{1}_{A_n}$. Then each $F \in (\mathscr{D}_q^p(X))^*$ can be considered as a linear functional on the closed subspace of $\mathscr{D}_q^p(X)$ consisting of all functions with support in A_n. By the previous part of the proof, for each $n \geq 1$ there exists $f_n \in \mathscr{D}_{q'}^{p'}(X^*)$ with support in A_n such that

$$F(g\cdot\mathbf{1}_{A_n}) = F_{f_n}(g\cdot\mathbf{1}_{A_n}) = \mathbb{E}\int_{\mathbb{R}_+\times J}\langle f_n, g\rangle \mathbf{1}_{A_n}\,\mathrm{d}\nu$$

and

$$\|f_n\|_{\mathscr{D}_{q'}^{p'}(X^*)} \leq \|F_{f_n}\|_{(\mathscr{D}_q^p(X))^*} \leq \|F\|_{(\mathscr{D}_q^p(X))^*}.$$

Obviously $f_{n+1}\mathbf{1}_{A_n} = f_n$ for each $n \geq 1$, hence there exists $f : \Omega\times\mathbb{R}_+\times J \to X^*$ such that $f\mathbf{1}_{A_n} = f_n$ for each $n\geq 1$. But then Fatou's lemma implies

$$\|f\|_{\mathscr{D}_{q'}^{p'}(X^*)} \leq \liminf_{n\to\infty}\|f_n\|_{\mathscr{D}_{q'}^{p'}(X^*)} \leq \|F\|_{(\mathscr{D}_q^p(X))^*},$$

so $f \in \mathscr{D}_{q'}^{p'}(X^*)$. On the other hand, by Hölder's inequality

$$\|F\|_{(\mathscr{D}_q^p(X))^*} \leq \|f\|_{\mathscr{D}_{q'}^{p'}(X^*)}.$$

Since the bounded linear functionals F and F_f agree on a dense subset of $\mathscr{D}_q^p(X)$, it follows that $F = F_f$ and (7.A.1) holds.

Finally, let ν be general. Let $F \in (\mathscr{D}_q^p(X))^*$ be of norm 1. Let $\varepsilon_n \downarrow 0$ and let $(g_n)_{n\geq 1}$ be a sequence in the unit sphere of $\mathscr{D}_q^p(X)$ satisfying $F(g_n) \geq (1-\varepsilon_n)$. By strong measurability of the g_n, there exists an $A \in \mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$ so that $\mathbb{P}\times\nu$ is σ-finite on A and $g_n = 0$ on A^c $\mathbb{P}\times\nu$-a.e. Let $\tilde{F} \in (\mathscr{D}_q^p(X))^*$ be defined by $\tilde{F}(g) = F(g\mathbf{1}_A)$. The previous part of the proof shows that there exists an $f \in \mathscr{D}_{q'}^{p'}(X^*)$ so that $\tilde{F} = F_f$ and $\|\tilde{F}\|_{(\mathscr{D}_q^p(X))^*} = \|f\|_{\mathscr{D}_{q'}^{p'}(X^*)}$. It remains to show that $F = \tilde{F}$. To prove this, suppose that there exists a $g_0 \in \mathscr{D}_q^p(X)$ of norm 1 with $\mathrm{supp}(g_0) \subset A^c$ and $F(g_0) = \delta > 0$. Let $0 < \lambda < 1$. Then, for any $n \geq 1$,

$$\|(1-\lambda^p)^{1/p}g_0 + \lambda g_n\|_{\mathscr{D}_q^p(X)}^p = (1-\lambda^p)\|g_0\|_{\mathscr{D}_q^p(X)}^p + \lambda^p\|g_n\|_{\mathscr{D}_q^p(X)}^p = 1$$

and

$$F((1-\lambda^p)^{1/p}g_0 + \lambda g_n) \geq (1-\lambda^p)^{1/p}\delta + \lambda(1-\varepsilon_n).$$

As a consequence,

$$\|F\| \geq \sup_{0<\lambda<1} (1-\lambda^p)^{1/p}\delta + \lambda.$$

One easily checks that the supremum is attained in

$$\lambda = \left(1 + \delta^{1/(1-\frac{1}{p})}\right)^{-1/p}$$

and so $\|F\| > 1$, a contradiction. □

We now turn to proving a similar duality statement for $\hat{\mathscr{D}}_q^p(X)$, the space of all $\widetilde{\mathscr{P}}$-measurable functions in $\mathscr{D}_q^p(X)$. In the proof we will use the following 'reverse' version of the dual Doob inequality [52, Lemma 2.10].

Lemma 7.A.2 (Reverse dual Doob inequality)**.** *Fix $0 < p \leq 1$. Let $\mathbb{F} = (\mathscr{F}_n)_{n\geq 0}$ be a filtration and let $(\mathbb{E}_n)_{n\geq 0}$ be the associated sequence of conditional expectations. If $(f_n)_{n\geq 0}$ is a sequence of non-negative random variables in $L^1(\mathbb{P})$, then*

$$\Big(\mathbb{E}\Big|\sum_{n\geq 0} f_n\Big|^p\Big)^{\frac{1}{p}} \leq p^{-1}\Big(\mathbb{E}\Big|\sum_{n\geq 0}\mathbb{E}_n f_n\Big|^p\Big)^{\frac{1}{p}}.$$

Theorem 7.A.3. *Let X be a reflexive space and let ν be a predictable, $\widetilde{\mathscr{P}}$-σ-finite random measure on $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{J}$ that is non-atomic in time. Then, for $1 < p, q < \infty$,*

$$(L^p_{\widetilde{\mathscr{P}}}(\mathbb{P}; L^q(\nu; X)))^* = L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P}; L^{q'}(\nu; X^*))$$

with isomorphism given by

$$g \mapsto F_g, \qquad F_g(h) = \mathbb{E}\int_{\mathbb{R}_+\times J}\langle g, h\rangle\, d\nu \qquad (g \in L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P}; L^{q'}(\nu)), h \in L^p_{\widetilde{\mathscr{P}}}(\mathbb{P}; L^q(\nu))).$$

Moreover,

$$\min\Big\{\Big(\frac{p}{q}\Big)^{1/q}\frac{q'}{p'},\Big(\frac{p'}{q'}\Big)^{1/q'}\frac{q}{p}\Big\}\|g\|_{L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))} \le \|F_g\| \le \|g\|_{L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))}. \tag{7.A.5}$$

Proof. Step 1: reduction. It suffices to prove the result for $p \le q$. Indeed, once this is known we can deduce the case $q \le p$ as follows. Observe that $L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))$ is a closed subspace of $\mathscr{D}^{p'}_{q'}(X^*) = L^{p'}(\mathbb{P};L^{q'}(\nu;X^*))$. By Theorem 7.A.1, $\mathscr{D}^{p'}_{q'}(X^*)$ is reflexive and therefore $L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))$ is reflexive as well. Therefore, as $p' \le q'$,

$$(L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X)))^* = L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))^{**} = L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*)).$$

Hence, if $F \in (L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X)))^*$, then there exists an $f \in L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))$ so that for any $g \in L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X))$

$$F(g) = F_g(f) = \mathbb{E}\int_{\mathbb{R}_+\times J}\langle f,g\rangle d\nu.$$

Moreover, the bounds (7.A.5) follow from Lemma 7.4.4. Thus, for the remainder of the proof, we can assume that $p \le q$.

Step 2: norm estimates. Let us now show that (7.A.5) holds. Since the upper bound is immediate from Hölder's inequality, we only need to show that for any $g \in L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))$,

$$\|F_g\| \ge \Big(\frac{p}{q}\Big)^{1/q}\frac{q'}{p'}\|g\|_{L^{p'}(\mathbb{P};L^{q'}(\nu;X^*))}. \tag{7.A.6}$$

It suffices to show this on a dense subset of $L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))$. Indeed, suppose that $g_n \to g$ in $L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))$ and that (7.A.6) holds for g_n, for all $n \ge 1$. Then,

$$\Big(\frac{p}{q}\Big)^{1/q}\frac{q'}{p'}\|g_n\|_{L^{p'}(\mathbb{P};L^{q'}(\nu;X^*))} \le \|F_{g_n}\| \le \|F_g\| + \|g-g_n\|_{L^{p'}(\mathbb{P};L^{q'}(\nu;X^*))},$$

and by taking limits on both sides we see that g also satisfies (7.A.6).

Let us first assume that

$$\nu((s,t]\times J) \le (t-s) \qquad \text{a.s.,} \qquad \text{for all } 0 \le s \le t \tag{7.A.7}$$

By the previous discussion, we may assume that $\|g\|_{L^{p'}(\mathbb{P};L^{q'}(\nu;X^*))} = 1$ and that g is of the form

$$g = \sum_{n=0}^{N_{m_*}}\sum_{\ell=0}^{L}\mathbf{1}_{(n/2^{m_*},(n+1)/2^{m_*}]}\mathbf{1}_{B_\ell}g_{n\ell},$$

where $N_{m_*} < \infty$, $g_{n\ell}$ is simple and $\mathscr{F}_{n/2^{m_*}}$-measurable for all n and ℓ, and the B_ℓ are disjoint sets in $\mathscr{J}$ of finite $\mathbb{P}\otimes\nu$-measure. For $m \ge m_*$ define

$$g^{(m)} = \sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\mathbf{1}_{(n/2^m,(n+1)/2^m]}\mathbf{1}_{B_\ell}g^{(m)}_{n\ell}$$

so that $g^{(m)} = g$. Then clearly, $g_{n\ell}^{(m)}$ is $\mathscr{F}_{n/2^m}$-measurable for all n and ℓ. Let us now fix an $m \ge m_*$. We define, for any $0 \le k \le N_m$,

$$\bar{s}_{q'}^k(g) = \Big(\sum_{n=0}^{k}\sum_{\ell=0}^{L} \|g_{n\ell}^{(m)}\|^{q'}\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell)\Big)^{1/q'}$$

and set

$$\alpha = (\mathbb{E}\bar{s}_{q'}^{N_m}(g^{(m)})^{p'})^{1/p'}.$$

Let P_ε be as in Lemma 7.4.2. We define a $\widetilde{\mathscr{P}}$-measurable function h by

$$h = \sum_{n=0}^{N_m}\sum_{\ell=0}^{L} \mathbf{1}_{(n/2^m,(n+1)/2^m]}\mathbf{1}_{B_\ell} h_{n\ell}$$

where, for $0 \le n \le N_m$ and $0 \le \ell \le L$, $h_{n\ell}$ is the $\mathscr{F}_{n/2^m}$-measurable function

$$h_{n\ell} = \frac{1}{\alpha^{p'-1}}(\bar{s}_{q'}^n(g^{(m)}))^{p'-q'}\|g_{n\ell}^{(m)}\|^{q'-1}P_\varepsilon g_{n\ell}^{(m)}.$$

Since $p/q \le 1$, Lemma 7.A.2 implies

$$\begin{aligned}
\|h\|_{L^p(\mathbb{P};L^q(\nu))} &= \Big(\mathbb{E}\Big(\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|h_{n\ell}\|^q\nu((n/2^m,(n+1)/2^m]\times B_\ell)\Big)^{p/q}\Big)^{1/p}\\
&\le \Big(\frac{q}{p}\Big)^{1/q}\Big(\mathbb{E}\Big(\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|h_{n\ell}\|^q\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell)\Big)^{p/q}\Big)^{1/p}\\
&= \Big(\frac{q}{p}\Big)^{1/q}(\mathbb{E}\bar{s}_q^{N_m}(h)^p)^{1/p}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
\bar{s}_q^{N_m}(h)^q &= \sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|h_{n\ell}\|^q\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell)\\
&\le \frac{1}{\alpha^{(p'-1)q}}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|g_{n\ell}^{(m)}\|^{(q'-1)q}\bar{s}_{q'}^n(g^{(m)})^{(p'-q')q}\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell)\\
&\le \frac{1}{\alpha^{(p'-1)q}}\bar{s}_{q'}^{N_m}(g^{(m)})^{(p'-q')q}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|g_{n\ell}^{(m)}\|^{q'}\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell)\\
&= \frac{1}{\alpha^{(p'-1)q}}\bar{s}_{q'}^{N_m}(g^{(m)})^{p'q-q'q+q'}.
\end{aligned}$$

Using (7.A.4) it follows that

$$\begin{aligned}
\|h\|_{L^p(\mathbb{P};L^q(\nu))}^p &\le \Big(\frac{q}{p}\Big)^{p/q}\frac{1}{\alpha^{(p'-1)p}}\bar{s}_{q'}^{N_m}(g^{(m)})^{(p'q-q'q+q')p/q}\\
&= \Big(\frac{q}{p}\Big)^{p/q}\frac{1}{\alpha^{p'}}\mathbb{E}\bar{s}_{q'}^{N_m}(g^{(m)})^{p'} = \Big(\frac{q}{p}\Big)^{p/q}.
\end{aligned}$$

Moreover, by Lemma 7.4.2,

$$\begin{aligned}
F_g(h) &= \mathbb{E}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\langle g_{n\ell}^{(m)}, h_{n\ell}\rangle \nu((n/2^m,(n+1)/2^m]\times B_\ell)\\
&= \mathbb{E}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\langle g_{n\ell}^{(m)}, h_{n\ell}\rangle \mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell)\\
&\geq (1-\varepsilon)\frac{1}{\alpha^{p'-1}}\mathbb{E}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|g_{n\ell}^{(m)}\|^{q'}\bar{s}_{q'}^{n}(g^{(m)})^{p'-q'}\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell)\\
&= (1-\varepsilon)\frac{1}{\alpha^{p'-1}}\mathbb{E}\sum_{n=0}^{N_m}\bar{s}_{q'}^{n}(g^{(m)})^{p'-q'}(\bar{s}_{q'}^{n}(g^{(m)})^{q'}-\bar{s}_{q'}^{n-1}(g^{(m)})^{q'}).
\end{aligned}$$

Now apply (7.4.10) for $\alpha = p'/q' \geq 1$ and $x = \bar{s}_{q'}^{n}(g^{(m)})^{q'}/\bar{s}_{q'}^{n-1}(g^{(m)})^{q'} \geq 1$ to obtain

$$\begin{aligned}
F_g(h) &\geq (1-\varepsilon)\frac{1}{\alpha^{p'-1}}\mathbb{E}\sum_{n=0}^{N_m}\frac{q'}{p'}\big(\bar{s}_{q'}^{n}(g^{(m)})^{p'}-\bar{s}_{q'}^{n-1}(g^{(m)})^{p'}\big)\\
&= (1-\varepsilon)\frac{q'}{p'}\frac{1}{\alpha^{p'-1}}\mathbb{E}\bar{s}_{q'}^{N_m}(g^{(m)})^{p'}\\
&= (1-\varepsilon)\frac{q'}{p'}\big(\mathbb{E}\bar{s}_{q'}^{N_m}(g^{(m)})^{p'}\big)^{1/p'}\\
&= (1-\varepsilon)\frac{q'}{p'}\Big(\mathbb{E}\Big(\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|g_{n\ell}^{(m)}\|^{q'}\mathbb{E}_{n/2^m}\nu\big((n/2^m,(n+1)/2^m]\times B_\ell\big)\Big)^{p'/q'}\Big)^{1/p'}\\
&= (1-\varepsilon)\frac{q'}{p'}\Big(\mathbb{E}\Big(\sum_{n=0}^{N_m}\mathbb{E}_{n/2^m}\big((\|g\|^{q'}\star\nu)_{(n+1)/2^m}-(\|g\|^{q'}\star\nu)_{n/2^m}\big)\Big)^{p'/q'}\Big)^{1/p'}
\end{aligned}$$

In conclusion, for any $m \geq m_*$ we find

$$\|F_g\| \geq \Big(\frac{p}{q}\Big)^{1/q}\frac{q'}{p'}\Big(\mathbb{E}\Big(\sum_{n=0}^{N_m}\mathbb{E}_{n/2^m}((\|g\|^{q'}\star\nu)_{(n+1)/2^m}-(\|g\|^{q'}\star\nu)_{n/2^m})\Big)^{p'/q'}\Big)^{1/p'}.$$

Taking $m \to \infty$, we find using Corollary 7.5.19 that

$$\|F_g\| \geq \Big(\frac{p}{q}\Big)^{1/q}\frac{q'}{p'}\|g\|_{L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))}.$$

Let us now remove the additional restriction (7.A.7) on ν. In this case, we define a strictly increasing, predictable, continuous process

$$A_t := \nu([0,t]\times J) + t, \qquad t \geq 0$$

and a random time change $\tau = (\tau_s)_{s\geq 0}$ by

$$\tau_s = \{t \,:\, A_t = s\}.$$

By Proposition 7.5.21, $A\circ\tau(t) = t$ a.s. for any $t \geq 0$, and hence by continuity of A and τ, a.s. $A\circ\tau(t) = t$ for all $t \geq 0$. As was noted in (7.5.27), we have $\nu_\tau((s,t]\times J) \leq t-s$ a.s. for all $s \leq t$. By Proposition 7.5.21, we can now write

$$\begin{aligned}\|F_g\| &= \sup_{\|h\|_{L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X))}\leq 1} \mathbb{E}\int_{\mathbb{R}_+\times J}\langle g,h\rangle\,d\nu\\ &\geq \sup_{\|\tilde{h}\circ A\|_{L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X))}\leq 1} \mathbb{E}\int_{\mathbb{R}_+\times J}\langle g,\tilde{h}\circ A\rangle\,d\nu\\ &= \sup_{\|\tilde{h}\|_{L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu_\tau;X))}\leq 1} \mathbb{E}\int_{\mathbb{R}_+\times J}\langle g\circ\tau,\tilde{h}\rangle\,d\nu_\tau.\end{aligned}$$

Applying the previous part of the proof for $\nu = \nu_\tau$, we find

$$\|F_g\| \geq \Big(\frac{p}{q}\Big)^{1/q}\frac{q'}{p'}\|g\circ\tau\|_{L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu_\tau;X^*))} = \|g\|_{L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))}.$$

This completes our proof of (7.A.5).

Step 3: representation of linear functionals. It now remains to show that every $F\in(L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X)))^*$ is of the form F_g for a suitable $\widetilde{\mathscr{P}}$-measurable function g. We will first assume that $\mathbb{E}\nu(\mathbb{R}_+\times J)<\infty$. On $\widetilde{\mathscr{P}}$ we can define an X^*-valued measure θ by setting

$$\langle\theta(A),x\rangle := F(\mathbf{1}_A\cdot x) \qquad (A\in\widetilde{\mathscr{P}}, x\in X).$$

Then θ is σ-additive, absolutely continuous with respect to $\mathbb{P}\times\nu$. Moreover, by the same calculation as in (7.A.2), for any disjoint partition $A_1,\dots,A_n\in\widetilde{\mathscr{P}}$ of $\mathbb{R}_+\times\Omega\times J$,

$$\begin{aligned}\sum_{i=1}^n\|\theta(A_i)\| &\leq \|F\|_{(L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X)))^*}(\mathbb{E}\nu(\mathbb{R}_+\times J)^{p/q})^{1/p}\\ &\leq \|F\|_{(L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X)))^*}(\mathbb{E}\nu(\mathbb{R}_+\times J))^{1/q},\end{aligned}$$

so θ is of finite variation. By the Radon-Nikodym property of X^*, there exists a $\widetilde{\mathscr{P}}$-measurable X^*-valued function g such that

$$F(h) = F_g(h) = \mathbb{E}\int_{\mathbb{R}_+\times J}\langle g,h\rangle\,\mathrm{d}\nu$$

for each $h\in L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X))$. The extension to the general case, where ν is $\widetilde{\mathscr{P}}$-σ-finite, can now be obtained in the same way as in the proof of Theorem 7.A.1. □

Remark 7.A.4. The reader may wonder whether the duality

$$(L^p_{\widetilde{\mathscr{P}}}(\mathbb{P};L^q(\nu;X)))^* = L^{p'}_{\widetilde{\mathscr{P}}}(\mathbb{P};L^{q'}(\nu;X^*))$$

remains valid if ν is any random measure and $\widetilde{\mathscr{P}}$ is replaced by an arbitrary sub-σ-algebra of $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$. It turns out that, surprisingly, one cannot expect such a general result. Indeed, it was pointed out by Pisier [151] that there exist two probability spaces $(\Omega_1,\mathscr{F}_1,\mathbb{P}_1)$, $(\Omega_2,\mathscr{F}_2,\mathbb{P}_2)$ and a sub-σ-algebra $\mathscr{G}$ of $\mathscr{F}_1\otimes\mathscr{F}_2$, so that the duality

$$(L^p_{\mathscr{G}}(\mathbb{P}_1;L^q(\mathbb{P}_2)))^* = L^{p'}_{\mathscr{G}}(\mathbb{P}_1;L^{q'}(\mathbb{P}_2))$$

does not even hold isomorphically. This counterexample in particular shows that the duality results claimed in [109] are not valid without imposing additional assumptions.

7.A.2. $\mathscr{S}_q^p$ and $\hat{\mathscr{S}}_q^p$ spaces

Let ν be any random measure on $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{J}$. Recall that $\mathscr{S}_q^p$ is the space of all $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}\otimes\mathscr{J}$-strongly measurable functions $f:\mathbb{R}_+\otimes\Omega\otimes J\to L^q(S)$ satisfying

$$\|f\|_{\mathscr{S}_q^p} = \Big(\mathbb{E}\Big\|\Big(\int_{\mathbb{R}_+\times J}|f|^2\,\mathrm{d}\nu\Big)^{\frac12}\Big\|^p_{L^q(S)}\Big)^{\frac1p}<\infty. \tag{7.A.8}$$

The proof of the following result is analogous to Theorem 7.A.1. We leave the details to the reader.

Theorem 7.A.5. *Let $1<p,q<\infty$. Then $(\mathscr{S}_q^p)^* = \mathscr{S}_{q'}^{p'}$ and*

$$\|f\|_{\mathscr{S}_{q'}^{p'}} \eqsim_{p,q} \|f\|_{(\mathscr{S}_q^p)^*},\ \ f\in\mathscr{S}_{q'}^{p'}.$$

Let us now prove the desired duality for $\hat{\mathscr{S}}_q^p$, the subspace of all $\widetilde{\mathscr{P}}$-strongly measurable functions in $\mathscr{S}_q^p$.

Theorem 7.A.6. *Let $1<p,q<\infty$. Suppose that ν is a predictable, $\widetilde{\mathscr{P}}$-σ-finite random measure on $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{J}$ that is non-atomic in time. Then $(\hat{\mathscr{S}}_q^p)^* = \hat{\mathscr{S}}_{q'}^{p'}$ and*

$$\|f\|_{\hat{\mathscr{S}}_{q'}^{p'}} \eqsim_{p,q} \|f\|_{(\hat{\mathscr{S}}_q^p)^*},\ \ f\in\hat{\mathscr{S}}_{q'}^{p'}. \tag{7.A.9}$$

For the proof of Theorem 7.A.6 we will the following assertion. Given a filtration $\mathbb{F}=(\mathscr{F}_n)_{n\ge0}$ and $1<p,q<\infty$, we define Q_q^p to be the Banach space of all adapted $L^q(S)$-valued sequences $(f_n)_{n\ge0}$ satisfying

$$\|(f_n)_{n\ge0}\|_{Q_q^p} := \Big(\mathbb{E}\Big\|\Big(\sum_{n=0}^\infty|f_n|^2\Big)^{\frac12}\Big\|^p_{L^q(S)}\Big)^{\frac1p}<\infty. \tag{7.A.10}$$

Proposition 7.A.7. *Let $1<p,q<\infty$. Then $(Q_q^p)^* = Q_{q'}^{p'}$ isomorphically, with duality bracket given by*

$$\langle(f_n)_{n\ge0},(g_n)_{n\ge0}\rangle := \mathbb{E}\sum_{n=0}^\infty\langle f_n,g_n\rangle \qquad ((g_n)_{n\ge0}\in Q_{q'}^{p'},\ (f_n)_{n\ge0}\in Q_q^p).$$

Moreover,

$$\|(g_n)_{n\geq 0}\|_{Q^{p'}_{q'}} \eqsim_{p,q} \|(g_n)_{n\geq 0}\|_{(Q^p_q)^*}.$$

Proof. Consider the filtration $\mathbb{G} = (\mathscr{G}_n)_{n\geq 0} = (\mathscr{F}_{n+1})_{n\geq 0}$. Let S^p_q be the conditional sequence space defined in (7.1.4) for the filtration $\mathbb{G}$. First notice that Q^p_q is a closed subspace and

$$\|(f_n)_{n\geq 0}\|_{Q^p_q} = \|(f_n)_{n\geq 0}\|_{S^p_q}, \qquad \text{for all } (f_n)_{n\geq 0} \in Q^p_q.$$

Let F be in $(Q^p_q)^*$. Then by the Hahn-Banach theorem and [87] there exists $\tilde{g} = (\tilde{g}_n)_{n\geq 0} \in S^{p'}_{q'}$ such that $\|\tilde{g}\|_{S^{p'}_{q'}} \eqsim_{p,q} \|F\|_{(Q^p_q)^*}$ and

$$F(f) = \mathbb{E}\sum_{n=1}^{\infty}\langle f_n, \tilde{g}_n\rangle, \quad f = (f_n)_{n\geq 0} \in Q^p_q.$$

Now let $(g_n)_{n\geq 0}$ be the $\mathbb{F}$-adapted $L^q(S)$-valued sequence defined by $g_n = \mathbb{E}_n\tilde{g}_n$ for $n \geq 0$ (recall that $\mathbb{E}_n(\cdot) := \mathbb{E}(\cdot|\mathscr{F}_n)$). Then, on the one hand, the conditional Jensen inequality yields

$$\begin{aligned}\|(g_n)_{n\geq 0}\|^{p'}_{Q^{p'}_{q'}} = \|(g_n)_{n\geq 0}\|^{p'}_{S^{p'}_{q'}} &= \mathbb{E}\Big\|\Big(\sum_{n=1}^{\infty}|g_n|^2\Big)^{\frac{1}{2}}\Big\|^{p'}_{L^{q'}(S)} = \mathbb{E}\Big\|\Big(\sum_{n=1}^{\infty}|\mathbb{E}_n\tilde{g}_n|^2\Big)^{\frac{1}{2}}\Big\|^{p'}_{L^{q'}(S)}\\ &\leq \mathbb{E}\Big\|\Big(\sum_{n=1}^{\infty}\mathbb{E}_n|\tilde{g}_n|^2\Big)^{\frac{1}{2}}\Big\|^{p'}_{L^{q'}(S)} = \|(\tilde{g}_n)_{n\geq 0}\|^{p'}_{S^{p'}_{q'}},\end{aligned}$$

and, on the other hand, for each $f = (f_n)_{n\geq 0} \in Q^p_q$ the $\mathbb{F}$-adaptedness of $(f_n)_{n\geq 0}$ implies

$$F(f) = \mathbb{E}\sum_{n=1}^{\infty}\langle f_n, \tilde{g}_n\rangle = \mathbb{E}\sum_{n=1}^{\infty}\mathbb{E}_n\langle f_n, \tilde{g}_n\rangle = \mathbb{E}\sum_{n=1}^{\infty}\langle f_n, \mathbb{E}_n\tilde{g}_n\rangle = \mathbb{E}\sum_{n=1}^{\infty}\langle f_n, g_n\rangle.$$

Therefore, for each $F \in (Q^p_q)^*$ there exists a $(g_n)_{n\geq 0} \in Q^{q'}_{p'}$ such that

$$F(f) = \mathbb{E}\sum_{n\geq 0}\langle f_n, g_n\rangle, \quad f = (f_n)_{n\geq 0} \in Q^p_q,$$
$$\|(g_n)_{n\geq 0}\|_{Q^{q'}_{p'}} \lesssim_{p,q} \|F\|_{(Q^p_q)^*}.$$

The inequality $\|F\|_{(Q^p_q)^*} \leq \|(g_n)_{n\geq 0}\|_{Q^{q'}_{p'}}$ follows immediately from Hölder's inequality. □

Proof of Theorem 7.A.6. The proof contains two parts. In the first part, consisting of several steps, we will show that $\|f\|_{\hat{\mathscr{S}}^{p'}_{q'}} \eqsim_{p,q} \|f\|_{(\hat{\mathscr{S}}^p_q)^*}$. In the second part we show that $(\hat{\mathscr{S}}^p_q)^* = \hat{\mathscr{S}}^{p'}_{q'}$.

Step 1: J is finite, ν is Lebesgue. Let $J = \{j_1, \dots, j_K\}$, $\nu(\omega)$ be the product of Lebesgue measure and the counting measure on $\mathbb{R}_+ \times J$ for all $\omega \in \Omega$ (i.e. $\nu((s,t] \times j_k) = t - s$ for each $k = 1, \dots, K$ and $t \geq s \geq 0$). Fix $f \in \hat{\mathscr{S}}_{q'}^{p'}$. Without loss of generality we can assume that f is simple and that there exist $N, M \geq 1$ and a sequence of random variables $(f_{k,m})_{k=1,m=0}^{k=K,m=M}$ such that $f_{k,m}$ is $\mathscr{F}_{\frac{m}{N}}$-measurable and $f(t, j_k) = f_{k,m}$ for each $k = 1, \dots, K$, $m = 0, \dots, M$, and $t \in (\frac{m}{N}, \frac{m+1}{N}]$. Let $\mathbb{G} = (\mathscr{G}_{k,m})_{k=1,m=0}^{k=K,m=M} := (\mathscr{F}_{\frac{m}{N}})_{k=1,m=0}^{k=K,m=M}$. Then $\mathbb{G}$ forms a filtration with respect to the reverse lexicographic order on the pairs (k,m), $1 \leq k \leq K$ and $0 \leq m \leq M$. Let $Q_{q'}^{p'}$ be as defined in (7.A.10) for $\mathbb{G}$. Then

$$\|f\|_{\hat{\mathscr{S}}_{q'}^{p'}} = \frac{1}{\sqrt{N}} \big\|(f_{k,m})_{k=1,m=0}^{k=K,m=M}\big\|_{Q_{q'}^{p'}}. \tag{7.A.11}$$

By Proposition 7.A.7 there exists a $\mathbb{G}$-adapted $(g_{k,m})_{k=1,m=0}^{k=K,m=M} \in Q_q^p$ such that

$$\big\|(g_{k,m})_{k=1,m=0}^{k=K,m=M}\big\|_{Q_q^p} = 1$$

and

$$\big\langle (f_{k,m})_{k=1,m=0}^{k=K,m=M}, (g_{k,m})_{k=1,m=0}^{k=K,m=M}\big\rangle \eqsim_{p,q} \big\|(f_{k,m})_{k=1,m=0}^{k=K,m=M}\big\|_{Q_{q'}^{p'}}.$$

Let $g : \mathbb{R}_+ \times \Omega \times J \to L^q(S)$ be defined by setting $g(t, j_k) = \sqrt{N} g_{k,m}$ for each $k = 1, \dots, K$, $m = 1, \dots, M$, and $t \in (\frac{m}{N}, \frac{m+1}{N}]$. Then $g \in \hat{\mathscr{S}}_q^p$, and analogously to (7.A.11)

$$\|g\|_{\hat{\mathscr{S}}_q^p} = \big\|(g_{k,m})_{k=1,m=0}^{k=K,m=M}\big\|_{Q_q^p} = 1.$$

Moreover,

$$\begin{aligned}\langle f, g\rangle = \mathbb{E}\int_{\mathbb{R}_+ \times J} \langle f(t,j), g(t,j)\rangle \,\mathrm{d}t\,\mathrm{d}j &= \frac{1}{\sqrt{N}} \mathbb{E} \sum_{k=1,m=0}^{k=K,m=M} \langle f_{k,m}, g_{k,m}\rangle \\ &\eqsim_{p,q} \frac{1}{\sqrt{N}} \big\|(f_{k,m})_{k=1,m=0}^{k=K,m=M}\big\|_{Q_{q'}^{p'}} = \|f\|_{\hat{\mathscr{S}}_{q'}^{p'}},\end{aligned}$$

which finishes the proof.

Step 2: J is finite, $\nu((s,t] \times J) \leq t - s$ a.s. for each $t \geq s \geq 0$. Let ν_0 be the product of Lebesgue measure and the counting measure on $\mathbb{R}_+ \times J$ (see Step 1). Then clearly $\mathbb{P} \otimes \nu$ is absolutely continuous with respect to $\mathbb{P} \otimes \nu_0$ and by the Radon-Nikodym theorem there exists a $\widetilde{\mathscr{P}}$-measurable density $\phi : \mathbb{R}_+ \times \Omega \times J \to \mathbb{R}_+$ such that $d(\mathbb{P} \otimes \nu) = \phi\; d(\mathbb{P} \otimes \nu_0)$. Fix $f \in \hat{\mathscr{S}}_{q'}^{p'}$. Let $\hat{\mathscr{S}}_{q'}^{p',\nu_0}$ be as defined in (7.A.8) for the random measure ν_0. Then $f_0 := f \cdot \sqrt{\phi} \in \hat{\mathscr{S}}_{q'}^{p',\nu_0}$, and $\|f\|_{\hat{\mathscr{S}}_{q'}^{p'}} = \|f_0\|_{\hat{\mathscr{S}}_{q'}^{p',\nu_0}}$. By Step 1 there exists a $g_0 \in \hat{\mathscr{S}}_q^{p,\nu_0}$ such that $\|g_0\|_{\hat{\mathscr{S}}_q^{p,\nu_0}} = 1$ and $\langle f_0, g_0\rangle \eqsim_{p,q} \|f_0\|_{\hat{\mathscr{S}}_{q'}^{p',\nu_0}}$. Let $g = g_0 \mathbf{1}_{\phi \neq 0} \frac{1}{\sqrt{\phi}}$. Then

$$\langle f, g\rangle = \mathbb{E}\int_{\mathbb{R}_+ \times J} \langle f, g\rangle \,\mathrm{d}\nu = \mathbb{E}\int_{\mathbb{R}_+ \times J} \langle f, g\rangle \phi \,\mathrm{d}\nu_0 = \mathbb{E}\int_{\mathbb{R}_+ \times J} \langle f\sqrt{\phi}, g\sqrt{\phi}\rangle \,\mathrm{d}\nu_0$$

$$= \mathbb{E}\int_{\mathbb{R}_+\times J}\langle f_0, g_0\rangle \, \mathrm{d}\nu_0 = \langle f_0, g_0\rangle \eqsim_{p,q} \|f_0\|_{\hat{\mathscr{S}}_{q'}^{p',\nu_0}} = \|f\|_{\hat{\mathscr{S}}_{q'}^{p'}}$$

and

$$\begin{aligned}\|g\|_{\hat{\mathscr{S}}_q^p} &= \Big(\mathbb{E}\Big\|\Big(\int_{\mathbb{R}_+\times J}|g|^2\,\mathrm{d}\nu\Big)^{\frac12}\Big\|^p\Big)^{\frac1p} = \Big(\mathbb{E}\Big\|\Big(\int_{\mathbb{R}_+\times J}|g_0|^2\mathbf{1}_{\phi\neq 0}\frac{1}{\phi}\,\mathrm{d}\nu\Big)^{\frac12}\Big\|^p\Big)^{\frac1p}\\ &= \Big(\mathbb{E}\Big\|\Big(\int_{\mathbb{R}_+\times J}|g_0|^2\mathbf{1}_{\phi\neq 0}\,\mathrm{d}\nu_0\Big)^{\frac12}\Big\|^p\Big)^{\frac1p} \leq \Big(\mathbb{E}\Big\|\Big(\int_{\mathbb{R}_+\times J}|g_0|^2\,\mathrm{d}\nu_0\Big)^{\frac12}\Big\|^p\Big)^{\frac1p}\\ &= \|g_0\|_{\hat{\mathscr{S}}_q^{p,\nu_0}} = 1.\end{aligned}$$

Therefore $\|f\|_{\hat{\mathscr{S}}_{q'}^{p'}} \eqsim_{p,q} \|f\|_{(\hat{\mathscr{S}}_q^p)^*}$.

Step 3: J is finite, ν is general. Without loss of generality we can assume that $\mathbb{E}\nu(\mathbb{R}_+\times J) < \infty$. Then by a time-change argument as was used in the proof of Theorem 7.A.3, we can assume that $\nu((s,t]\times J) \leq t-s$ a.s. for each $t \geq s \geq 0$, and apply Step 2.

Step 4: J is general, ν is general. Without loss of generality assume that $\mathbb{E}\nu(\mathbb{R}_+\times J) < \infty$. Let f be simple $\widetilde{\mathscr{P}}$-measurable. Then there exists a $K \geq 1$ and a partition $J = J_1\cup\cdots\cup J_K$ of J into disjoint sets such that $f(i) = f(j)$ for all $i,j\in J_k$ and each $k = 1,\ldots,K$. Fix $j_k \in J_k$, $k = 1,\ldots,K$, and define $\widetilde{J} = \{j_1,\ldots,j_K\}$. Let $\widetilde{\nu}$ be a new random measure on $\mathbb{R}_+\times\Omega\times\widetilde{J}$ defined by

$$\widetilde{\nu}(A\times\{j_k\}) = \nu(A\times J_k), \qquad A\in\mathscr{P},\ k = 1,\ldots,K.$$

Let $\hat{\mathscr{S}}_{q'}^{p',\widetilde{\nu}}$ be as constructed in (7.A.8) for the measure $\widetilde{\nu}$. Let $\widetilde{f}\in\hat{\mathscr{S}}_{q'}^{p',\widetilde{\nu}}$ be such that $\widetilde{f}(j_k) = f(j_k)$ for each $k = 1,\ldots,K$. Then $\|\widetilde{f}\|_{\hat{\mathscr{S}}_{q'}^{p',\widetilde{\nu}}} = \|f\|_{\hat{\mathscr{S}}_{q'}^{p'}}$. By Step 3 there exists a $\widetilde{g}\in\hat{\mathscr{S}}_q^{p,\widetilde{\nu}}$ such that $\|\widetilde{g}\|_{\hat{\mathscr{S}}_q^{p,\widetilde{\nu}}} = 1$ and $\langle\widetilde{f},\widetilde{g}\rangle \eqsim_{p,q} \|\widetilde{f}\|_{\hat{\mathscr{S}}_{q'}^{p',\widetilde{\nu}}}$.

Define $g\in\hat{\mathscr{S}}_q^p$ by setting $g(j) = \widetilde{g}(j_k)$ for each $k = 1,\ldots,K$ and $j\in J_k$. Then $\|g\|_{\hat{\mathscr{S}}_q^p} = \|\widetilde{g}\|_{\hat{\mathscr{S}}_q^{p,\widetilde{\nu}}} = 1$. Moreover,

$$\begin{aligned}\langle f, g\rangle &= \mathbb{E}\int_{\mathbb{R}_+\times J}\langle f(t,j), g(t,j)\rangle\,\mathrm{d}\nu(t,j) = \mathbb{E}\sum_{k=1}^K\int_{\mathbb{R}_+\times J_k}\langle f(t,j), g(t,j)\rangle\,\mathrm{d}\nu(t,j)\\ &= \mathbb{E}\int_{\mathbb{R}_+\times\widetilde{J}}\langle\widetilde{f}(t,j),\widetilde{g}(t,j)\rangle\,\mathrm{d}\widetilde{\nu}(t,j) \eqsim_{p,q} \|\widetilde{f}\|_{\hat{\mathscr{S}}_{q'}^{p',\widetilde{\nu}}} = \|f\|_{\hat{\mathscr{S}}_{q'}^{p'}}.\end{aligned}$$

Hence, $\|f\|_{\hat{\mathscr{S}}_{q'}^{p'}} \eqsim_{p,q} \|f\|_{(\hat{\mathscr{S}}_q^p)^*}$.

Step 5: $(\hat{\mathscr{S}}_q^p)^* = \hat{\mathscr{S}}_{q'}^{p'}$. In Step 4 we proved that $\hat{\mathscr{S}}_{q'}^{p'} \hookrightarrow (\hat{\mathscr{S}}_q^p)^*$ isomorphically, so it remains to show that $(\hat{\mathscr{S}}_q^p)^* = \hat{\mathscr{S}}_{q'}^{p'}$. This identity follows from the same Radon-Nikodym argument that was presented in Step 3 of the proof of Theorem 7.A.3. □

Corollary 7.A.8. *Let $1 < p,q < \infty$. Then $\mathscr{I}_{p,q}^* = \mathscr{I}_{p',q'}$, where $\mathscr{I}_{p,q}$ is as defined in* (7.5.22)*, and*

$$\|f\|_{\mathscr{I}_{p',q'}} \eqsim_{p,q} \|f\|_{\mathscr{I}_{p,q}^*}, \quad f\in\mathscr{I}_{p',q'}. \tag{7.A.12}$$

Proof. The result follows by combining Theorem 7.A.3 (for $X = L^q(S)$), Theorem 7.A.6 and (7.2.2). $\square$

8

BURKHOLDER–DAVIS–GUNDY INEQUALITIES IN UMD BANACH FUNCTION SPACES

This chapter is based on the paper *Pointwise properties of martingales with values in Banach function spaces* by Mark Veraar and Ivan Yaroslavtsev, see [178].

In this chapter we consider local martingales with values in a UMD Banach function space. We prove that such martingales have a version which is a martingale field. Moreover, a new Burkholder–Davis–Gundy type inequality is obtained.

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8.1. INTRODUCTION

The discrete Burkholder–Davis–Gundy inequality (see [29, Theorem 3.2]) states that for any $p \in (1,\infty)$ and martingales difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega)$ one has

$$\Big\| \sum_{j=1}^n d_j \Big\|_{L^p(\Omega)} \eqsim_p \Big\| \Big(\sum_{j=1}^n |d_j|^2 \Big)^{1/2} \Big\|_{L^p(\Omega)}. \tag{8.1.1}$$

Moreover, there is the extension to continuous-time local martingales M (see [89, Theorem 26.12]) which states that for every $p \in [1,\infty)$,

$$\| \sup_{t\in[0,\infty)} |M_t| \|_{L^p(\Omega)} \eqsim_p \| [M]_\infty^{1/2} \|_{L^p(\Omega)}. \tag{8.1.2}$$

Here $t \mapsto [M]_t$ denotes the quadratic variation process of M.

In the case X is a UMD Banach function space the following variant of (8.1.1) holds (see [164, Theorem 3]): for any $p \in (1,\infty)$ and martingales difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega; X)$ one has

$$\Big\| \sum_{j=1}^n d_j \Big\|_{L^p(\Omega;X)} \eqsim_p \Big\| \Big(\sum_{j=1}^n |d_j|^2 \Big)^{1/2} \Big\|_{L^p(\Omega;X)}. \tag{8.1.3}$$

Moreover, the validity of the estimate also characterizes the UMD property.

It is a natural question whether (8.1.2) has a vector-valued analogue as well. The main result of this chapter states that this is indeed the case:

Theorem 8.1.1. *Let X be a UMD Banach function space over a σ-finite measure space (S,Σ,μ). Assume that $N : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R}$ is such that $N|_{[0,t]\times\Omega\times S}$ is $\mathscr{B}([0,t]) \otimes \mathscr{F}_t \otimes \Sigma$-measurable for all $t \geq 0$ and such that for almost all $s \in S$, $N(\cdot,\cdot,s)$ is a martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$ and $N(0,\cdot,s) = 0$. Then for all $p \in (1,\infty)$,*

$$\| \sup_{t\geq 0} |N(t,\cdot,\cdot)| \|_{L^p(\Omega;X)} \eqsim_{p,X} \sup_{t\geq 0} \| N(t,\cdot,\cdot) \|_{L^p(\Omega;X)} \eqsim_{p,X} \| [N]_\infty^{1/2} \|_{L^p(\Omega;X)}. \tag{8.1.4}$$

where $[N]$ denotes the quadratic variation process of N.

By standard methods we can extend Theorem 8.1.1 to spaces X which are isomorphic to a closed subspace of a Banach function space (e.g. Sobolev and Besov spaces, etc.)

The two-sided estimate (8.1.4) can for instance be used to obtain two-sided estimates for stochastic integrals for processes with values in infinite dimensions (see [126] and [177]). In particular, applying it with $N(t,\cdot,s) = \int_0^t \Phi(\cdot,s)\,\mathrm{d}W$ implies the following maximal estimate for the stochastic integral

$$\Big\| s \mapsto \sup_{t\geq 0} \Big| \int_0^t \Phi(\cdot,s)\,\mathrm{d}W \Big| \Big\|_{L^p(\Omega;X)} \eqsim_{p,X} \sup_{t\geq 0} \Big\| s \mapsto \int_0^t \Phi(\cdot,s)\,\mathrm{d}W \Big\|_{L^p(\Omega;X)}$$

$$\eqsim_{p,X} \Big\| s \mapsto \Big(\int_0^\infty \Phi^2(t,s)\,\mathrm{d}t\Big)^{1/2}\Big\|_{L^p(\Omega;X)}, \qquad (8.1.5)$$

where W is a Brownian motion and $\Phi : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R}$ is a progressively measurable process such that the right-hand side of (8.1.5) is finite. The second norm equivalence was obtained in [126]. The norm equivalence with the left-hand side is new in this generality. The case where X is an L^q-space was recently obtained in [4] using different methods.

It is worth noticing that the second equivalence of (8.1.4) in the case of $X = L^q$ was obtained by Marinelli in [110] for some range of $1 < p, q < \infty$ by using an interpolation method.

The UMD property is necessary in Theorem 8.1.1 by necessity of the UMD property in (8.1.3) and the fact that any discrete martingale can be transformed to a continuous-time one. Also in the case of continuous martingales, the UMD property is necessary in Theorem 8.1.1. Indeed, applying (8.1.5) with W replaced by an independent Brownian motion $\widetilde{W}$ we obtain

$$\Big\|\int_0^\infty \Phi\,\mathrm{d}W\Big\|_{L^p(\Omega;X)} \eqsim_{p,X} \Big\|\int_0^\infty \Phi\,\mathrm{d}\widetilde{W}\Big\|_{L^p(\Omega;X)},$$

for all predictable step processes Φ. The latter holds implies that X is a UMD Banach space (see [61, Theorem 1]).

In the special case that $X = \mathbb{R}$ the above reduces to (8.1.2). In the proof of Theorem 8.1.1 the UMD property is applied several times:

- The boundedness of the lattice maximal function (see [24, 60, 164]).
- The X-valued Meyer–Yoeurp decomposition of a martingale (see Theorem 4.3.1).
- The square-function estimate (8.1.3) (see [164]).

It remains open whether there exists a predictable expression for the right-hand side of (8.1.4). One would expect that one needs simply to replace $[N]$ by its predictable compensator, the *predictable quadratic variation* $\langle N\rangle$. Unfortunately, this does not hold true already in the scalar-valued case: if M is a real-valued martingale, then

$$\mathbb{E}|M|_t^p \lesssim_p \mathbb{E}\langle M\rangle_t^{\frac{p}{2}}, \quad t \geq 0, \; p < 2,$$

$$\mathbb{E}|M|_t^p \gtrsim_p \mathbb{E}\langle M\rangle_t^{\frac{p}{2}}, \quad t \geq 0, \; p > 2,$$

where both inequalities are known not to be sharp (see [29, p. 40], [114, p. 297], and [140]). The question of finding such a predictable right-hand side in (8.1.4) was answered only in the case $X = L^q$ for $1 < q < \infty$ by Dirsken and the author (see

[54]). The key tool exploited there was the so-called *Burkholder-Rosenthal inequalities*, which are of the following form:

$$\mathbb{E}\|M_N\|^p \eqsim_{p,X} |||(M_n)_{0\le n\le N}|||^p_{p,X},$$

where $(M_n)_{0\le n\le N}$ is an X-valued martingale, $|||\cdot|||_{p,X}$ is a certain norm defined on the space of X-valued L^p-martingales which depends only on *predictable moments* of the corresponding martingale. Therefore using approach of [54] one can reduce the problem of continuous-time martingales to discrete-time martingales. However, the Burkholder-Rosenthal inequalities are explored only in the case $X = L^q$.

Thanks to (8.1.2) the following natural question arises: can one generalize (8.1.4) to the case $p = 1$, i.e. whether

$$\Big\| \sup_{t\ge 0} |N(t,\cdot,\cdot)| \Big\|_{L^1(\Omega;X)} \eqsim_{p,X} \|[N]_\infty^{1/2}\|_{L^1(\Omega;X)} \tag{8.1.6}$$

holds true? Unfortunately the outlined earlier techniques cannot be applied in the case $p = 1$. Moreover, the obtained estimates cannot be simply extrapolated to the case $p = 1$ since those contain the *UMD$_p$ constant*, which is known to have infinite limit as $p \to 1$. Therefore (8.1.6) remains an open problem. Note that in the case of a continuous martingale M inequalities (8.1.4) can be extended to the case $p \in (0,1]$ due to the classical Lenglart approach (see Corollary 8.4.4).

8.2. PRELIMINARIES

Throughout the chapter any filtration satisfies the *usual conditions* (see [85, Definition 1.1.2 and 1.1.3]), unless the underlying martingale is continuous (then the corresponding filtration can be assumed general).

Recall that for a given measure space (S,Σ,μ), the linear space of all real-valued measurable functions is denoted by $L^0(S)$.

Definition 8.2.1. Let (S,Σ,μ) be a measure space. Let $n : L^0(S) \to [0,\infty]$ be a function which satisfies the following properties:

(i) $n(x) = 0$ if and only if $x = 0$,

(ii) for all $x, y \in L^0(S)$ and $\lambda \in \mathbb{R}$, $n(\lambda x) = |\lambda| n(x)$ and $n(x+y) \le n(x) + n(y)$,

(iii) if $x \in L^0(S)$, $y \in L^0(S)$, and $|x| \le |y|$, then $n(x) \le n(y)$,

(iv) if $0 \le x_n \uparrow x$ with $(x_n)_{n=1}^\infty$ a sequence in $L^0(S)$ and $x \in L^0(S)$, then $n(x) = \sup_{n\in\mathbb{N}} n(x_n)$.

Let X denote the space of all $x \in L^0(S)$ for which $\|x\| := n(x) < \infty$. Then X is called the *normed function space associated to n*. It is called a *Banach function space* when $(X, \|\cdot\|_X)$ is complete.

We refer the reader to [108, 124, 164, 178, 192] for details on Banach function spaces.

Remark 8.2.2. Let X be a Banach function space over a measure space (S,Σ,μ). Then X is continuously embedded into $L^0(S)$ endowed with the topology of convergence in measure on sets of finite measure. Indeed, assume $x_n \to x$ in X and let $A \in \Sigma$ be of finite measure. We claim that $\mathbf{1}_A x_n \to \mathbf{1}_A x$ in measure. For this it suffices to show that every subsequence of $(x_n)_{n\geq 1}$ has a further subsequence which convergences a.e. to x. Let $(x_{n_k})_{k\geq 1}$ be a subsequence. Choose a subsubsequence $(\mathbf{1}_A x_{n_{k_\ell}})_{\ell\geq 1} =: (y_\ell)_{\ell\geq 1}$ such that $\sum_{\ell=1}^\infty \|y_\ell - x\| < \infty$. Then by [192, Exercise 64.1] $\sum_{\ell=1}^\infty |y_\ell - x|$ converges in X. In particular, $\sum_{\ell=1}^\infty |y_\ell - x| < \infty$ a.e. Therefore, $y_\ell \to x$ a.e. as desired.

Given a Banach function space X over a measure space S and Banach space E, let $X(E)$ denote the space of all strongly measurable functions $f : S \to E$ with $\|f\|_E \in X$. The space $X(E)$ becomes a Banach space when equipped with the norm $\|f\|_{X(E)} = \big\| s \mapsto \|f(s)\|_E \big\|_X$.

A Banach function space has the UMD property if and only if (8.1.3) holds for some (or equivalently, for all) $p \in (1,\infty)$ (see [164]). A broad class of Banach function spaces with UMD is given by the reflexive Lorentz–Zygmund spaces (see [43]) and the reflexive Musielak–Orlicz spaces (see [107]).

Definition 8.2.3. $N : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R}$ is called (continuous) (local) martingale field if $N|_{[0,t]\times\Omega\times S}$ is $\mathscr{B}([0,t]) \otimes \mathscr{F}_t \otimes \Sigma$-measurable for all $t \geq 0$ and $N(\cdot,\cdot,s)$ is a (continuous) (local) martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$ for almost all $s \in S$.

Let X be a Banach space, τ be a stopping time, $V : \mathbb{R}_+ \times \Omega \to X$ be a càdlàg process. Then we define $\Delta V_\tau : \Omega \to X$ as follows

$$\Delta V_\tau := V_\tau - \lim_{\varepsilon\to 0} V_{(\tau-\varepsilon)\vee 0}.$$

8.3. LATTICE DOOB'S MAXIMAL INEQUALITY

Doob's maximal L^p-inequality immediately implies that for martingale fields

$$\big\| \sup_{t\geq 0} \|N(t,\cdot)\|_X \big\|_{L^p(\Omega)} \leq \frac{p}{p-1} \sup_{t\geq 0} \|N(t)\|_{L^p(\Omega;X)}, \quad 1 < p < \infty.$$

In the next lemma we prove a stronger version of Doob's maximal L^p-inequality. As a consequence in Theorem 8.3.2 we will obtain the same result in a more general setting.

Lemma 8.3.1. *Let X be a UMD Banach function space and let $p \in (1,\infty)$. Let N be a càdlàg martingale field with values in a finite dimensional subspace of X. Then for all $T > 0$,*

$$\big\| \sup_{t\in[0,T]} |N(t,\cdot)| \big\|_{L^p(\Omega;X)} \eqsim_{p,X} \sup_{t\in[0,T]} \|N(t)\|_{L^p(\Omega;X)}$$

whenever one of the expression is finite.

Proof. Clearly, the left-hand side dominates the right-hand side. Therefore, we can assume the right-hand side is finite and in this case we have

$$\|N(T)\|_{L^p(\Omega;X)} = \sup_{t\in[0,T]} \|N(t)\|_{L^p(\Omega;X)} < \infty.$$

Since N takes values in a finite dimensional subspace it follows from Doob's L^p-inequality (applied coordinatewise) that the left-hand side is finite.

Since N is a càdlàg martingale field and by Definition 8.2.1(iv) we have that

$$\lim_{n\to\infty} \Big\| \sup_{0\le j\le n} |N(jT/n,\cdot)| \Big\|_{L^p(\Omega;X)} = \Big\| \sup_{t\in[0,T]} |N(t,\cdot)| \Big\|_{L^p(\Omega;X)}.$$

Set $M_j = N_{jT/n}$ for $j \in \{0,\dots,n\}$ and $M_j = M_n$ for $j > n$. It remains to prove

$$\Big\| \sup_{0\le j\le n} |M_j(\cdot)| \Big\|_{L^p(\Omega;X)} \le C_{p,X} \|M_n\|_{L^p(\Omega;X)}.$$

If $(M_j)_{j=0}^n$ is a Paley–Walsh martingale (see [79, Definition 3.1.8 and Proposition 3.1.10]), this estimate follows from the boundedness of the dyadic lattice maximal operator [164, pp. 199–200 and Theorem 3]. In the general case one can replace Ω by a divisible probability space and approximate (M_j) by Paley-Walsh martingales in a similar way as in [79, Corollary 3.6.7]. □

Theorem 8.3.2 (Doob's maximal L^p-inequality)**.** *Let X be a UMD Banach function space over a σ-finite measure space and let $p \in (1,\infty)$. Let $M : \mathbb{R}_+ \times \Omega \to X$ be a martingale such that*

1. *for all $t \ge 0$, $M(t) \in L^p(\Omega;X)$;*
2. *for a.a $\omega \in \Omega$, $M(\cdot,\omega)$ is in $\mathscr{D}([0,\infty);X)$.*

Then there exists a martingale field $N \in L^p(\Omega; X(\mathscr{D}_b([0,\infty))))$ such that for a.a. $\omega \in \Omega$, all $t \ge 0$ and a.a. $s \in S$, $N(t,\omega,\cdot) = M(t,\omega)(s)$ and

$$\Big\| \sup_{t\ge0} |N(t,\cdot)| \Big\|_{L^p(\Omega;X)} \eqsim_{p,X} \sup_{t\ge0} \|M(t,\cdot)\|_{L^p(\Omega;X)}. \tag{8.3.1}$$

Moreover, if M is continuous, then N can be chosen to be continuous as well.

Proof. We first consider the case where M becomes constant after some time $T > 0$. Then

$$\sup_{t\ge0} \|M(t,\cdot)\|_{L^p(\Omega;X)} = \|M(T)\|_{L^p(\Omega;X)}.$$

Let $(\xi_n)_{n\ge1}$ be simple random variables such that $\xi_n \to M(T)$ in $L^p(\Omega;X)$. Let $M_n(t) = \mathbb{E}(\xi_n|\mathscr{F}_t)$ for $t \ge 0$. Then by Lemma 8.3.1

$$\Big\| \sup_{t\ge0} |N_n(t,\cdot) - N_m(t,\cdot)| \Big\|_{L^p(\Omega;X)} \eqsim_{p,X} \Big\| |M_n(T,\cdot) - M_m(T,\cdot)| \Big\|_{L^p(\Omega;X)} \to 0$$

as $n, m \to \infty$. Therefore, $(N_n)_{n\geq 1}$ is a Cauchy sequence and hence converges to some N from the space $L^p(\Omega; X(\mathscr{D}_b([0,\infty))))$. Clearly, $N(t,\cdot) = M(t)$ and (8.3.1) holds in the special case that M becomes constant after $T > 0$.

In the case M is general, for each $T > 0$ we can set $M^T(t) = M(t \wedge T)$. Then for each $T > 0$ we obtain a martingale field N^T as required. Since $N^{T_1} = N^{T_2}$ on $[0, T_1 \wedge T_2]$, we can define a martingale field N by setting $N(t,\cdot) = N^T(t,\cdot)$ on $[0,T]$. Finally, we note that

$$\lim_{T\to\infty} \sup_{t\geq 0} \|M^T(t)\|_{L^p(\Omega;X)} = \sup_{t\geq 0} \|M(t)\|_{L^p(\Omega;X)}.$$

Moreover, by Definition 8.2.1(iv) we have

$$\lim_{T\to\infty} \big\| \sup_{t\geq 0} |N^T(t,\cdot)| \big\|_{L^p(\Omega;X)} = \big\| \sup_{t\geq 0} |N(t,\cdot)| \big\|_{L^p(\Omega;X)},$$

Therefore the general case of (8.3.1) follows by taking limits.

Now let M be continuous, and let $(M_n)_{n\geq 1}$ be as before. By the same argument as in the first part pf the proof we can assume that there exists $T > 0$ such that $M_t = M_{t\wedge T}$ for all $t \geq 0$. By Theorem 4.3.1 there exists a unique decomposition $M_n = M_n^c + M_n^d$ such that M_n^d is purely discontinuous and starts at zero and M_n^c has continuous paths a.s. Then by (4.3.1)

$$\|M(T) - M_n^c(T)\|_{L^p(\Omega;X)} \leq \beta_{p,X} \|M(T) - M_n(T)\|_{L^p(\Omega;X)} \to 0.$$

Since M_n^c takes values in a finite dimensional subspace of X we can define a martingale field N_n by $N_n(t,\omega,s) = M_n^c(t,\omega)(s)$. Now by Lemma 8.3.1

$$\big\| \sup_{0\leq t\leq T} |N_n(t,\cdot) - N_m(t,\cdot)| \big\|_{L^p(\Omega;X)} \eqsim_{p,X} \big\| |M_n^c(T,\cdot) - M_m^c(T,\cdot)| \big\|_{L^p(\Omega;X)} \to 0.$$

Therefore, $(N_n)_{n\geq 1}$ is a Cauchy sequence and hence converges to some N from the space $L^p(\Omega; X(C_b([0,\infty))))$. Analogously to the first part of the proof, $N(t,\cdot) = M(t)$ for all $t \geq 0$. ☐

Remark 8.3.3. Note that due to the construction of N we have that $\Delta M_\tau(s) = \Delta N(\cdot, s)_\tau$ for any stopping time τ and almost any $s \in S$. Indeed, let $(M_n)_{n\geq 1}$ and $(N_n)_{n\geq 1}$ be as in the proof of Theorem 8.3.2. Then on the one hand

$$\begin{aligned}\|\Delta M_\tau - \Delta(M_n)_\tau\|_{L^p(\Omega;X)} &\leq \big\| \sup_{0\leq t\leq T} \|M(t) - M_n(t)\|_X \big\|_{L^p(\Omega)} \\ &\eqsim_p \|M(T) - M_n(T)\|_{L^p(\Omega;X)} \to 0, \quad n \to \infty.\end{aligned}$$

On the other hand

$$\begin{aligned}\|\Delta N_\tau - \Delta(N_n)_\tau\|_{L^p(\Omega;X)} &\leq \big\| \sup_{0\leq t\leq T} |N(t) - N_n(t)| \big\|_{L^p(\Omega;X)} \\ &\eqsim_{p,X} \big\| |N(T) - N_n(T)| \big\|_{L^p(\Omega;X)} \to 0, \quad n \to \infty.\end{aligned}$$

Since $\|M_n(t) - N_n(t,\cdot)\|_{L^p(\Omega;X)} = 0$ for all $n \geq 0$, we have that by the limiting argument $\|\Delta M_\tau - \Delta N_\tau(\cdot)\|_{L^p(\Omega;X)} = 0$, so the desired follows from Definition 8.2.1(i).

One could hope there is a more elementary approach to derive continuity of N in the case M is continuous: if the filtration $\widetilde{\mathbb{F}} := (\widetilde{\mathscr{F}}_t)_{t\geq 0}$ is generated by M, then $M(s)$ is $\widetilde{\mathbb{F}}$-adapted for a.e. $s \in S$, and one might expect that M has a continuous version. Unfortunately, this is not true in general as follows from the next example.

Example 8.3.4. There exists a continuous martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, a filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathscr{F}}_t)_{t\geq 0}$ generated by M and all $\mathbb{P}$-null sets, and a purely discontinuous nonzero $\widetilde{\mathbb{F}}$-martingale $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$. Let $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a Brownian motion, $L : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a Poisson process such that W and L are independent. Let $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ be the filtration generated by W and L. Let σ be an $\mathbb{F}$-stopping time defined as follows

$$\sigma = \inf\{u \geq 0 : \Delta L_u \neq 0\}.$$

Let us define

$$M := \int \mathbf{1}_{[0,\sigma]} \, \mathrm{d}W = W^{\sigma}.$$

Then M is a martingale. Let $\widetilde{\mathbb{F}} := (\widetilde{\mathscr{F}}_t)_{t\geq 0}$ be generated by M. Note that $\widetilde{\mathscr{F}}_t \subset \mathscr{F}_t$ for any $t \geq 0$. Define a random variable

$$\tau = \inf\{t \geq 0 : \exists u \in [0,t) \, \text{such that} \, M \, \text{is a constant on} \, [u,t]\}.$$

Then $\tau = \sigma$ a.s. Moreover, τ is a $\widetilde{\mathbb{F}}$-stopping time since for each $u \geq 0$

$$\mathbb{P}\{\tau = u\} = \mathbb{P}\{\sigma = u\} = \mathbb{P}\{\Delta L^{\sigma}_u \neq 1\} \leq \mathbb{P}\{\Delta L_u \neq 1\} = 0,$$

and hence

$$\{\tau \leq u\} = \{\tau < u\} \cup \{\tau = u\} \subset \widetilde{\mathscr{F}}_u.$$

Therefore $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ defined by

$$N_t := \mathbf{1}_{[\tau,\infty)}(t) - t \wedge \tau \quad t \geq 0,$$

is an $\widetilde{\mathbb{F}}$-martingale since it is $\widetilde{\mathbb{F}}$-measurable and since $N_t = (L_t - t)^{\sigma}$ a.s. for each $t \geq 0$, hence for each $u \in [0,t]$

$$\mathbb{E}(N_t|\widetilde{\mathscr{F}}_u) = \mathbb{E}(\mathbb{E}(N_t|\mathscr{F}_u)|\widetilde{\mathscr{F}}_u) = \mathbb{E}(\mathbb{E}((L_t - t)^{\sigma}|\mathscr{F}_u)|\widetilde{\mathscr{F}}_u) = (L_u - u)^{\sigma} = N_u$$

due to the fact that $t \mapsto L_t - t$ is an $\widetilde{\mathbb{F}}$-measurable $\mathbb{F}$-martingale (see [93, Problem 1.3.4]). But $(N_t)_{t\geq 0}$ is not continuous since $(L_t)_{t\geq 0}$ is not continuous.

8.4. MAIN RESULT

Theorem 8.1.1 will be a consequence of the following more general result.

Theorem 8.4.1. *Let X be a UMD Banach function space over a σ-finite measure space (S, Σ, μ) and let $p \in (1, \infty)$. Let $M : \mathbb{R}_+ \times \Omega \to X$ be a local L^p-martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$ and assume $M(0,\cdot) = 0$. Then there exists a mapping $N : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R}$ such that*

(1) *for all* $t \geq 0$ *and a.a.* $\omega \in \Omega$, $N(t,\omega,\cdot) = M(t,\omega)$,

(2) N *is a local martingale field,*

(3) *the following estimate holds*

$$\big\| \sup_{t\geq 0} |N(t,\cdot,\cdot)| \big\|_{L^p(\Omega;X)} \eqsim_{p,X} \big\| \sup_{t\geq 0} \|M(t,\cdot)\|_X \big\|_{L^p(\Omega)} \eqsim_{p,X} \|[N]_\infty^{1/2}\|_{L^p(\Omega;X)}. \tag{8.4.1}$$

To prove Theorem 8.4.1 we first prove a completeness result.

Proposition 8.4.2. *Let* X *be a Banach function space over a* σ*-finite measure space* S, $1 \leq p < \infty$. *Let*

$$\begin{aligned} \mathrm{MQ}^p(X) := \{N : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R} : N \textit{ is a martingale field,}\\ N_0(s) = 0\ \forall s \in S, \textit{ and } \|N\|_{\mathrm{MQ}^p(X)} < \infty\}, \end{aligned}$$

where

$$\|N\|_{\mathrm{MQ}^p(X)} := \|[N]_\infty^{1/2}\|_{L^p(\Omega;X)}. \tag{8.4.2}$$

Then $(\mathrm{MQ}^p(X), \|\cdot\|_{\mathrm{MQ}^p(X)})$ *is a Banach space. Moreover, if* $N_n \to N$ *in* MQ^p, *then there exists a subsequence* $(N_{n_k})_{k\geq 1}$ *such that pointwise a.e. in* S, *we have* $N_{n_k} \to N$ *in* $L^1(\Omega;\mathscr{D}_b([0,\infty)))$.

Proof. Let us first check that $\mathrm{MQ}^p(X)$ is a normed vector space. For this only the triangle inequality requires some comments. By the well-known estimate for local martingales M,N (see [89, Theorem 26.6(iii)]) we have that a.s.

$$\begin{aligned} [M+N]_t &= [M]_t + 2[M,N]_t + [N]_t \\ &\leq [M]_t + 2[M]_t^{1/2}[N]_t^{1/2} + [N]_t = \big([M]_t^{1/2} + [N]_t^{1/2}\big)^2, \end{aligned} \tag{8.4.3}$$

Therefore, $[M+N]_t^{1/2} \leq [M]_t^{1/2} + [N]_t^{1/2}$ a.s. for all $t \in [0,\infty]$.

Let $(N_k)_{k\geq 1}$ be such that $\sum_{k\geq 1} \|N_k\|_{\mathrm{MQ}^p(X)} < \infty$. It suffices to show that $\sum_{k\geq 1} N_k$ converges in $\mathrm{MQ}^p(X)$. Observe that by monotone convergence in Ω and Jensen's inequality applied to $\|\cdot\|_X$ for any $n > m \geq 1$ we have

$$\begin{aligned} \Big\| \sum_{k=m+1}^n \mathbb{E}[N_k]_\infty^{1/2} \Big\|_X &= \Big\| \sum_{k=1}^n \mathbb{E}[N_k]_\infty^{1/2} - \sum_{k=1}^m \mathbb{E}[N_k]_\infty^{1/2} \Big\|_X \\ &= \Big\| \mathbb{E} \sum_{k=m+1}^n [N_k]_\infty^{1/2} \Big\|_X \leq \mathbb{E} \Big\| \sum_{k=m+1}^n [N_k]_\infty^{1/2} \Big\|_X \\ &= \Big\| \sum_{k=m+1}^n [N_k]_\infty^{1/2} \Big\|_{L^1(\Omega;X)} \leq \Big\| \sum_{k=m+1}^n [N_k]_\infty^{1/2} \Big\|_{L^p(\Omega;X)} \\ &\leq \sum_{k=m+1}^n \Big\| [N_k]_\infty^{1/2} \Big\|_{L^p(\Omega;X)} \to 0,\ \ m,n \to \infty, \end{aligned} \tag{8.4.4}$$

where the latter holds due to the fact that $\sum_{k\geq 1}\big\|[N_k]_\infty^{1/2}\big\|_{L^p(\Omega;X)}<\infty$. Thus $\sum_{k=1}^n \mathbb{E}[N_k]_\infty^{1/2}$ converges in X as $n\to\infty$, where the corresponding limit coincides with its pointwise limit $\sum_{k\geq 1}\mathbb{E}[N_k]_\infty^{1/2}$ by Remark 8.2.2. Therefore, since any element of X is finite a.s. by Definition 8.2.1, we can find $S_0\in\Sigma$ such that $\mu(S_0^c)=0$ and pointwise in S_0, we have $\sum_{k\geq 1}\mathbb{E}[N_k]_\infty^{1/2}<\infty$. Fix $s\in S_0$. In particular, we find that $\sum_{k\geq 1}[N_k]_\infty^{1/2}$ converges in $L^1(\Omega)$. Moreover, since by the scalar Burkholder-Davis-Gundy inequalities $\mathbb{E}\sup_{t\geq 0}|N_k(t,\cdot,s)|\eqsim \mathbb{E}[N_k(s)]_\infty^{1/2}$, we also obtain that

$$N(\cdot,s):=\sum_{k\geq 1}N_k(\cdot,s)\ \text{ converges in } L^1(\Omega;\mathscr{D}_b([0,\infty)). \tag{8.4.5}$$

Let $N(\cdot,s)=0$ for $s\notin S_0$. Then N defines a martingale field. Moreover, by the scalar Burkholder-Davis-Gundy inequalities

$$\lim_{m\to\infty}\Big[\sum_{k=n}^m N_k(\cdot,s)\Big]_\infty^{1/2}=\Big[\sum_{k=n}^\infty N_k(\cdot,s)\Big]_\infty^{1/2}$$

in $L^1(\Omega)$. Therefore, by considering an a.s. convergent subsequence and by (8.4.3) we obtain

$$\Big[\sum_{k=n}^\infty N_k(\cdot,s)\Big]_\infty^{1/2}\leq \sum_{k=n}^\infty [N_k(\cdot,s)]_\infty^{1/2}. \tag{8.4.6}$$

It remains to prove that $N\in \mathrm{MQ}^p(X)$ and $N=\sum_{k\geq 1}N_k$ with convergence in $\mathrm{MQ}^p(X)$. Let $\varepsilon>0$. Choose $n\in\mathbb{N}$ such that $\sum_{k\geq n+1}\|N_k\|_{\mathrm{MQ}^p(X)}<\varepsilon$. It follows from (8.4.4) that $\mathbb{E}\big\|\sum_{k\geq 1}[N_k]_\infty^{1/2}\big\|_X<\infty$, so $\sum_{k\geq 1}[N_k]_\infty^{1/2}$ a.s. converges in X. Now by (8.4.6), the triangle inequality and Fatou's lemma, we obtain

$$\begin{aligned}\Big\|\Big[\sum_{k\geq n+1}N_k\Big]_\infty^{1/2}\Big\|_{L^p(\Omega;X)}&\leq \Big\|\sum_{k=n+1}^\infty [N_k]_\infty^{1/2}\Big\|_{L^p(\Omega;X)}\\&\leq \sum_{k=n+1}^\infty \Big\|[N_k]_\infty^{1/2}\Big\|_{L^p(\Omega;X)}\\&\leq \liminf_{m\to\infty}\sum_{k=n+1}^m \Big\|[N_k]_\infty^{1/2}\Big\|_{L^p(\Omega;X)}<\varepsilon^p.\end{aligned}$$

Therefore, $N\in\mathrm{MQ}^p(X)$ and $\|N-\sum_{k=1}^n N_k\|_{\mathrm{MQ}^p(X)}<\varepsilon$.

For the proof of the final assertion assume that $N_n\to N$ in $\mathrm{MQ}^p(X)$. Choose a subsequence $(N_{n_k})_{k\geq 1}$ such that $\|N_{n_k}-N\|_{\mathrm{MQ}^p(X)}\leq 2^{-k}$. Then $\sum_{k\geq 1}\|N_{n_k}-N\|_{\mathrm{MQ}^p(X)}<\infty$ and hence by (8.4.5) we see that pointwise a.e. in S, the series $\sum_{k\geq 1}(N_{n_k}-N)$ converges in $L^1(\Omega;\mathscr{D}_b([0,\infty)))$. Therefore,

$$N_{n_k}\to N\ \text{ in } L^1(\Omega;\mathscr{D}_b([0,\infty);X))$$

as required. □

For the proof of Theorem 8.4.1 we will need the following lemma presented in [55, Théorème 2].

Lemma 8.4.3. *Let $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be an L^p-martingales. Let $T > 0$. For each $n \geq 1$ define*

$$R_n := \sum_{k=1}^{n} \big|M_{\frac{Tk}{n}} - M_{\frac{T(k-1)}{n}}\big|^2.$$

Then R_n converges to $[M]_T$ in $L^{p/2}$.

Proof of Theorem 8.4.1. The existence of the local martingale field N together with the first estimate in (8.4.1) follows from Theorem 8.3.2. It remains to prove

$$\Big\| \sup_{t\geq 0} \|M(t,\cdot)\|_X \Big\|_{L^p(\Omega)} \eqsim_{p,X} \|[N]_\infty^{1/2}\|_{L^p(\Omega;X)}. \tag{8.4.7}$$

Due to Definition 8.2.1(iv) it suffices to prove the above norm equivalence in the case M and N becomes constant after some fixed time T.

Step 1: The finite dimensional case. Assume that M takes values in a finite dimensional subspace Y of X and that the right hand side of (8.4.7) is finite. Then we can write $N(t,s) = M(t)(s) = \sum_{j=1}^n M_j(t)x_j(s)$, where each M_j is a scalar-valued martingale with $M_j(T) \in L^p(\Omega)$ and $x_1,\dots,x_n \in X$ form a basis of Y. Note that for any $c_1,\dots,c_n \in L^p(\Omega)$ we have that

$$\Big\| \sum_{j=1}^n c_j x_j \Big\|_{L^p(\Omega;X)} \eqsim_{p,Y} \sum_{j=1}^n \|c_j\|_{L^p(\Omega)}. \tag{8.4.8}$$

Fix $m \geq 1$. Then by (8.1.3) and Doob's maximal inequality

$$\begin{aligned}\Big\| \sup_{t\geq 0} \|M(t,\cdot)\|_X \Big\|_{L^p(\Omega)} &\eqsim_p \|M(T,\cdot)\|_{L^p(\Omega;X)} \\ &= \Big\| \sum_{i=1}^m M_{\frac{Ti}{m}} - M_{\frac{T(i-1)}{m}} \Big\|_{L^p(\Omega;X)} \\ &\eqsim_{p,X} \Big\| \Big(\sum_{i=1}^m \big|M_{\frac{Ti}{m}} - M_{\frac{T(i-1)}{m}}\big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;X)},\end{aligned} \tag{8.4.9}$$

and by (8.4.8) and Lemma 8.4.3 the right hand side of (8.4.9) converges to

$$\|[M]_\infty^{1/2}\|_{L^p(\Omega;X)} = \|[N]_\infty^{1/2}\|_{L^p(\Omega;X)}.$$

Step 2: Reduction to the case where M takes values in a finite dimensional subspace of X. Let $M(T) \in L^p(\Omega;X)$. Then we can find simple functions $(\xi_n)_{n\geq 1}$ in $L^p(\Omega;X)$ such that $\xi_n \to M(T)$. Let $M_n(t) = \mathbb{E}(\xi_n|\mathscr{F}_t)$ for all $t \geq 0$ and $n \geq 1$, $(N_n)_{n\geq 1}$ be the corresponding martingale fields. Then each M_n takes values in a finite dimensional subspace $X_n \subseteq X$, and hence by Step 1

$$\Big\| \sup_{t\geq 0} \|M_n(t,\cdot) - M_m(t,\cdot)\|_X \Big\|_{L^p(\Omega)} \eqsim_{p,X} \|[N_n - N_m]_\infty^{1/2}\|_{L^p(\Omega;X)}$$

for any $m,n \geq 1$. Therefore since $(\xi_n)_{n\geq 1}$ is Cauchy in $L^p(\Omega;X)$, $(N_n)_{n\geq 1}$ converges to some N in $\mathrm{MQ}^p(X)$ by the first part of Proposition 8.4.2.

Let us show that N is the desired local martingale field. Fix $t \geq 0$. We need to show that $N(\cdot,t,\cdot) = M_t$ a.s. on Ω. First notice that by the second part of Proposition 8.4.2 there exists a subsequence of $(N_n)_{n\geq 1}$ which we will denote by $(N_n)_{n\geq 1}$ as well such that $N_n(\cdot,t,\sigma) \to N(\cdot,t,\sigma)$ in $L^1(\Omega)$ for a.e. $\sigma \in S$. On the other hand by Jensen's inequality

$$\big\|\mathbb{E}|N_n(\cdot,t,\cdot) - M_t|\big\|_X = \big\|\mathbb{E}|M_n(t) - M(t)|\big\|_X \leq \mathbb{E}\|M_n(t) - M(t)\|_X \to 0, \quad n \to \infty.$$

Hence $N_n(\cdot,t,\cdot) \to M_t$ in $X(L^1(\Omega))$, and thus by Remark 8.2.2 in $L^0(S;L^1(\Omega))$. Therefore we can find a subsequence of $(N_n)_{n\geq 1}$ (which we will again denote by $(N_n)_{n\geq 1}$) such that $N_n(\cdot,t,\sigma) \to M_t(\sigma)$ in $L^1(\Omega)$ for a.e. $\sigma \in S$ (here we use that fact that μ is σ-finite), so $N(\cdot,t,\cdot) = M_t$ a.s. on $\Omega \times S$, and consequently by Definition 8.2.1(iii), $N(\omega,t,\cdot) = M_t(\omega)$ for a.a. $\omega \in \Omega$. Thus (8.4.7) follows by letting $n \to \infty$.

Step 3: Reduction to the case where the left-hand side of (8.4.7) *is finite.* Assume that the left-hand side of (8.4.7) is infinite, but the right-hand side is finite. Since M is a local L^p-martingale we can find a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n \uparrow \infty$ and $\|M_T^{\tau_n}\|_{L^p(\Omega;X)} < \infty$ for each $n \geq 1$. By the monotone convergence theorem and Definition 8.2.1(iv)

$$\begin{aligned}
\|[N]_\infty^{1/2}\|_{L^p(\Omega;X)} &= \lim_{n\to\infty} \|[N^{\tau_n}]_\infty^{1/2}\|_{L^p(\Omega;X)} \eqsim_{p,X} \limsup_{n\to\infty} \|M_T^{\tau_n}\|_{L^p(\Omega;X)}\\
&= \lim_{n\to\infty} \|M_T^{\tau_n}\|_{L^p(\Omega;X)} = \lim_{n\to\infty} \Big\|\sup_{0\leq t\leq T} \|M_t^{\tau_n}\|_X\Big\|_{L^p(\Omega)}\\
&= \Big\|\sup_{0\leq t\leq T} \|M_t\|_X\Big\|_{L^p(\Omega)} = \infty
\end{aligned}$$

and hence the right-hand side of (8.4.7) is infinite as well. □

We use an extrapolation argument to extend part of Theorem 8.4.1 to $p \in (0,1]$ in the continuous-path case.

Corollary 8.4.4. *Let X be a UMD Banach function space over a σ-finite measure space and let $p \in (0,\infty)$. Let M be a continuous local martingale $M : \mathbb{R}_+ \times \Omega \to X$ with $M(0,\cdot) = 0$. Then there exists a continuous local martingale field $N : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R}$ such that for a.a. $\omega \in \Omega$, all $t \geq 0$, and a.a. $s \in S$, $N(t,\omega,\cdot) = M(t,\omega)(s)$ and*

$$\Big\|\sup_{t\geq 0} \|M(t,\cdot)\|_X\Big\|_{L^p(\Omega)} \eqsim_{p,X} \big\|[N]_\infty^{1/2}\big\|_{L^p(\Omega;X)}. \tag{8.4.10}$$

Proof. By a stopping time argument we can reduce to the case where $\|M(t,\omega)\|_X$ is uniformly bounded in $t \in \mathbb{R}_+$ and $\omega \in \Omega$ and M becomes constant after a fixed time T. Now the existence of N follows from Theorem 8.4.1 and it remains to prove (8.4.10) for $p \in (0,1]$. For this we can use a classical argument due to Lenglart. Indeed, for both estimates we can apply [106] or [156, Proposition IV.4.7] to the continuous increasing processes $Y, Z : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ given by

$$Y_u = \mathbb{E} \sup_{t\in[0,u]} \|M(t,\cdot)\|_X,$$

$$Z_u = \| s \mapsto [N(\cdot,\cdot,s)]_u^{1/2} \|_X,$$

where $q \in (1,\infty)$ is a fixed number. Then by (8.4.1) for any bounded stopping time τ, we have

$$\begin{aligned}\mathbb{E}Y_\tau^q = \sup_{t\geq 0} \|M(t\wedge\tau,\cdot)\|_X^q &\eqsim_{q,X} \mathbb{E}\|s\mapsto [N(\cdot\wedge\tau,\cdot,s)]_\infty^{1/2}\|_X^q\\ &\overset{(*)}{=} \mathbb{E}\|s\mapsto [N(\cdot,\cdot,s)]_\tau^{1/2}\|_X^q = \mathbb{E}Z_\tau^q,\end{aligned}$$

where we used [89, Theorem 17.5] in $(*)$. Now (8.4.10) for $p \in (0,q)$ follows from [106] or [156, Proposition IV.4.7]. $\square$

As we saw in Theorem 8.3.2, continuity of M implies pointwise continuity of the corresponding martingale filed N. The following corollaries of Theorem 8.4.1 are devoted to proving the same type of assertions concerning pure discontinuity, quasi-left continuity, and having accessible jumps.

Let τ be a stopping time. Then τ is called *predictable* if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n < \tau$ a.s. on $\{\tau > 0\}$ for each $n \geq 1$ and $\tau_n \nearrow \tau$ a.s. A càdlàg process $V : \mathbb{R}_+ \times \Omega \to X$ is called to have *accessible jumps* if there exists a sequence of predictable stopping times $(\tau_n)_{n\geq 1}$ such that $\{t \in \mathbb{R}_+ : \Delta V \neq 0\} \subset \{\tau_1,\ldots,\tau_n,\ldots\}$ a.s.

Corollary 8.4.5. *Let X be a UMD function space over a measure space (S,Σ,μ), $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous L^p-martingale with accessible jumps. Let N be the corresponding martingale field. Then $N(\cdot,s)$ is a purely discontinuous martingale with accessible jumps for a.e. $s \in S$.*

Proof of Corollary 8.4.5. Without loss of generality we can assume that there exists $T \geq 0$ such that $M_t = M_T$ for all $t \geq T$, and that $M_0 = 0$. Since M has accessible jumps, there exists a sequence of predictable stopping times $(\tau_n)_{n\geq 1}$ such that a.s.

$$\{t \in \mathbb{R}_+ : \Delta M \neq 0\} \subset \{\tau_1,\ldots,\tau_n,\ldots\}.$$

For each $m \geq 1$ define a process $M^m : \mathbb{R}_+ \times \Omega \to X$ in the following way:

$$M^m(t) := \sum_{n=1}^m \Delta M_{\tau_n} \mathbf{1}_{[0,t]}(\tau_n), \quad t \geq 0.$$

Note that M^m is a purely discontinuous L^p-martingale with accessible jumps by Lemma 2.4.5. Let N^m be the corresponding martingale field. Then $N^m(\cdot,s)$ is a purely discontinuous martingale with accessible jumps for almost any $s \in S$ due to Remark 8.3.3. Moreover, for any $m \geq \ell \geq 1$ and any $t \geq 0$ we have that a.s. $[N^m(\cdot,s)]_t \geq [N^\ell(\cdot,s)]_t$. Define $F : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R}_+ \cup \{+\infty\}$ in the following way:

$$F(t,\cdot,s) := \lim_{m\to\infty} [N^m(\cdot,s)]_t, \quad s \in S, t \geq 0.$$

Note that $F(\cdot,\cdot,s)$ is a.s. finite for almost any $s\in S$. Indeed, by Theorem 8.4.1 and 3.3.17 we have that for any $m\geq 1$

$$\big\|[N^m]_\infty^{1/2}\big\|_{L^p(\Omega;X)}\eqsim_{p,X}\|M^m(T,\cdot)\|_{L^p(\Omega;X)}\leq\beta_{p,X}\|M(T,\cdot)\|_{L^p(\Omega;X)},$$

so by Definition 8.2.1(iv), $F(\cdot,\cdot,s)$ is a.s. finite for almost any $s\in S$ and

$$\begin{aligned}\big\|F_\infty^{1/2}\big\|_{L^p(\Omega;X)}=\big\|F_T^{1/2}\big\|_{L^p(\Omega;X)}&=\lim_{m\to\infty}\big\|[N^m]_T^{1/2}\big\|_{L^p(\Omega;X)}\\&\lesssim_{p,X}\limsup_{m\to\infty}\|M^m(T,\cdot)\|_{L^p(\Omega;X)}\lesssim_{p,X}\|M(T,\cdot)\|_{L^p(\Omega;X)}.\end{aligned}$$

Moreover, for almost any $s\in S$ we have that $F(\cdot,\cdot,s)$ is pure jump and

$$\{t\in\mathbb{R}_+:\Delta F\neq 0\}\subset\{\tau_1,\ldots,\tau_n,\ldots\}.$$

Therefore to this end it suffices to show that $F(s)=[N(s)]$ a.s. on Ω for a.e. $s\in S$. Note that by Definition 8.2.1(iv),

$$\big\|(F-[N^m])^{1/2}(\infty)\big\|_{L^p(\Omega;X)}\to 0,\quad m\to\infty \tag{8.4.11}$$

so by Theorem 8.4.1 $(M^m(T))_{m\geq1}$ is a Cauchy sequence in $L^p(\Omega;X)$. Let ξ be its limit, $M^0:\mathbb{R}_+\times\Omega\to X$ be a martingale such that $M^0(t)=\mathbb{E}(\xi|\mathscr{F}_t)$ for all $t\geq 0$. Then by Proposition 2.2.16 M^0 is purely discontinuous. Moreover, for any stopping time τ a.s.

$$\Delta M_\tau^0=\lim_{m\to\infty}\Delta M_\tau^m=\lim_{m\to\infty}\Delta M_\tau\mathbf{1}_{\{\tau_1,\ldots,\tau_m\}}(\tau)=\Delta M_\tau,$$

where the latter holds since the set $\{\tau_1,\ldots,\tau_n,\ldots\}$ exhausts the jump times of M. Therefore $M=M^0$ since both M and M^0 are purely discontinuous with the same jumps, and hence $[N]=F$ (where $F(s)=[M^0(s)]$ by (8.4.11)). Consequently $N(\cdot,\cdot,s)$ is purely discontinuous with accessible jumps for almost all $s\in S$. □

Remark 8.4.6. Note that the proof of Corollary 8.4.5 also implies that $M_t^m\to M_t$ in $L^p(\Omega;X)$ for each $t\geq 0$.

A càdlàg process $V:\mathbb{R}_+\times\Omega\to X$ is called *quasi-left continuous* if $\Delta V_\tau=0$ a.s. for any predictable stopping time τ.

Corollary 8.4.7. *Let X be a UMD function space over a measure space (S,Σ,μ), $1<p<\infty$, $M:\mathbb{R}_+\times\Omega\to X$ be a purely discontinuous quasi-left continuous L^p-martingale. Let N be the corresponding martingale field. Then $N(\cdot,s)$ is a purely discontinuous quasi-left continuous martingale for a.e. $s\in S$.*

Proof. Without loss of generality we can assume that there exists $T\geq 0$ such that $M_t=M_T$ for all $t\geq T$, and that $M_0=0$. Let μ be a random measure defined on $\mathbb{R}_+\times X$ in the following way

$$\mu(A\times B)=\sum_{t\geq 0}\mathbf{1}_A(t)\mathbf{1}_{B\setminus\{0\}}(\Delta M_t),$$

where $A \subset \mathbb{R}_+$ is a Borel set, and $B \subset X$ is a ball. For each $k, \ell \geq 1$ we define a stopping time $\tau_{k,\ell}$ as follows

$$\tau_{k,\ell} = \inf\{t \in \mathbb{R}_+ : \#\{u \in [0,t] : \|\Delta M_u\|_X \in [1/k, k]\} = \ell\}.$$

Since M has càdlàg trajectories, $\tau_{k,\ell}$ is a.s. well-defined and takes its values in $[0,\infty]$. Moreover, $\tau_{k,\ell} \to \infty$ for each $k \geq 1$ a.s. as $\ell \to \infty$, so we can find a subsequence $(\tau_{k_n,\ell_n})_{n\geq 1}$ such that $k_n \geq n$ for each $n \geq 1$ and $\inf_{m\geq n} \tau_{k_m,\ell_m} \to \infty$ a.s. as $n \to \infty$. Define $\tau_n = \inf_{m\geq n} \tau_{k_m,\ell_m}$ and define $M^n := (\mathbf{1}_{[0,\tau_n]}\mathbf{1}_{B_n}) \star \bar{\mu}$, where $\bar{\mu} = \mu - \nu$ is such that ν is a compensator of μ and $B_n = \{x \in X : \|x\| \in [1/n, n]\}$. Then M^n is a purely discontinuous quasi-left continuous martingale by Lemma 2.8.1. Moreover, a.s.

$$\Delta M_t^n = \Delta M_t \mathbf{1}_{[0,\tau_n]}(t)\mathbf{1}_{[1/n,n]}(\|\Delta M_t\|), \quad t \geq 0.$$

so by Theorem 3.3.17 M^n is an L^p-bounded martingale.

The rest of the proof is analogous to the proof of Corollary 8.4.5 and uses the fact that $\tau_n \to \infty$ monotonically a.s. □

Corollary 8.4.8. *Let X be a UMD Banach function space, $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be an L^p-martingale. Let N be the corresponding martingale field. Let $M = M^c + M^q + M^a$ be the canonical decomposition, N^c, N^q, and N^a be the corresponding martingale fields. Then $N(s) = N^c(s) + N^q(s) + N^a(s)$ is the canonical decomposition of $N(s)$ for a.e. $s \in S$. In particular, if $M_0 = 0$ a.s., then M is continuous, purely discontinuous quasi-left continuous, or purely discontinuous with accessible jumps if and only if $N(s)$ is so for a.e. $s \in S$.*

Proof. The first part follows from Theorem 8.3.2, Corollary 8.4.5, and Corollary 8.4.7 and the fact that $N(s) = N^c(s)+N^q(s)+N^a(s)$ is then a canonical decomposition of a local martingale $N(s)$ which is unique due to Remark 2.4.24. Let us show the second part. One direction follows from Theorem 8.3.2, Corollary 8.4.5, and Corollary 8.4.7. For the other direction assume that $N(s)$ is continuous for a.e. $s \in S$. Let $M = M^c + M^q + M^a$ be the canonical decomposition, N^c, N^q, and N^a be the corresponding martingale fields of M^c, M^q, and M^a. Then by the first part of the theorem and the uniqueness of the canonical decomposition (see Remark 2.4.24) we have that for a.e. $s \in S$, $N^q(s) = N^a(s) = 0$, so $M^q = M^a = 0$, and hence M is continuous. The proof for the case of pointwise purely discontinuous quasi-left continuous N or pointwise purely discontinuous N with accessible jumps is similar. □

9

Burkholder–Davis–Gundy inequalities and stochastic integration in general UMD Banach spaces

This chapter is based on the paper *Burkholder–Davis–Gundy inequalities in UMD Banach spaces* by Ivan Yaroslavtsev, see [187].

In this chapter we prove Burkholder–Davis–Gundy inequalities for a general martingale M with values in a UMD Banach space X. Assuming that $M_0 = 0$, we show that the following two-sided inequality holds for all $1 \leq p < \infty$:

$$\mathbb{E} \sup_{0 \leq s \leq t} \|M_s\|^p \eqsim_{p,X} \mathbb{E}\gamma([\![M]\!]_t)^p, \quad t \geq 0. \tag{$\star$}$$

Here $\gamma([\![M]\!]_t)$ is the L^2-norm of the unique Gaussian measure on X having

$$[\![M]\!]_t(x^*, y^*) := [\langle M, x^* \rangle, \langle M, y^* \rangle]_t$$

as its covariance bilinear form. This extends to general UMD spaces Theorem 8.1.1, where a pointwise version of $(\star)$ was proved for UMD Banach functions spaces X.

We show that for continuous martingales, $(\star)$ holds for all $0 < p < \infty$, and that for purely discontinuous martingales the right-hand side of $(\star)$ can be expressed more explicitly in terms of the jumps of M. For martingales with independent increments, $(\star)$ is shown to hold more generally in reflexive Banach spaces X with finite cotype. In the converse direction, we show that the validity of $(\star)$ for arbitrary martingales implies the UMD property for X.

As an application we prove various Itô isomorphisms for vector-valued stochastic integrals with respect to general martingales, which extends earlier results by van Neerven, Veraar, and Weis for vector-valued stochastic integrals with respect to a Brownian motion. We also provide Itô isomorphisms for vector-valued stochastic integrals with respect to compensated Poisson and general random measures.

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9.1. INTRODUCTION

In the celebrated paper [40], Burkholder, Davis, and Gundy proved that if $M = (M_t)_{t\geq 0}$ is a real-valued martingale satisfying $M_0 = 0$, then for all $1 \leq p < \infty$ and $t \geq 0$ one has the two-sided inequality

$$\mathbb{E}\sup_{0\leq s\leq t} |M_s|^p \eqsim_p \mathbb{E}[M]_t^{\frac{p}{2}}, \tag{9.1.1}$$

where $[M]$ is the quadratic variation of M, i.e.,

$$[M]_t := \mathbb{P}-\lim_{\mathrm{mesh}(\pi)\to 0} \sum_{n=1}^N |M(t_n) - M(t_{n-1})|^2, \tag{9.1.2}$$

where the limit in probability is taken over partitions $\pi = \{0 = t_0 < \ldots < t_N = t\}$ whose mesh approaches 0. Later, Burkholder [36, 38] and Kallenberg and Sztencel [90] extended (9.1.1) to Hilbert space-valued martingales (see also [115]). They showed that if M is a martingale with values in a Hilbert space H satisfying $M_0 = 0$, then for all $1 \leq p < \infty$ and $t \geq 0$ one has

$$\mathbb{E}\sup_{0\leq s\leq t} \|M_s\|^p \eqsim_p \mathbb{E}[M]_t^{\frac{p}{2}}, \tag{9.1.3}$$

where the quadratic variation $[M]$ is defined as in (9.1.2) with absolute values replaced by norms in H. A further result along these lines was obtained in Chapter 8. There it is shown that if M is an L^p-bounded martingale, $1 < p < \infty$, with $M_0 = 0$, that takes values in a UMD Banach function space X over a measure space (S, Σ, μ) (see Section 8.2 for the definition), then for all $t \geq 0$:

$$\mathbb{E}\sup_{0\leq s\leq t} \|M_s(\sigma)\|^p \eqsim_{p,X} \mathbb{E}\big\|[M(\sigma)]_t^{\frac{1}{2}}\big\|^p, \tag{9.1.4}$$

where the quadratic variation $[M(\sigma)]_t$ is considered pointwise in $\sigma \in S$. Although this inequality seems to be particularly useful from a practical point of view, it does not give any hint how to work with a general Banach space since not every (UMD) Banach space has a Banach function space structure (e.g. noncommutative L^q-spaces).

Therefore the following natural question is rising up. *Given a Banach space X. Is there an analogue of* (9.1.3) *for a general X-valued local martingale M and how then should the right-hand side of* (9.1.3) *look like?* In the current article we present the following complete solution to this problem for local martingales M with values in a UMD Banach space X.

Theorem 9.1.1. *Let X be a UMD Banach space. Then for any local martingale $M : \mathbb{R}_+ \times \Omega \to X$ with $M_0 = 0$ and any $t \geq 0$ the covariation bilinear form $[[M]]_t$ is well-defined and bounded almost surely, and for all $1 \leq p < \infty$ we have*

$$\mathbb{E}\sup_{0\leq s\leq t} \|M_s\|^p \eqsim_{p,X} \mathbb{E}\gamma([[M]]_t)^p. \tag{9.1.5}$$

Here $\gamma(V)$, where $V : X^* \times X^* \to \mathbb{R}$ is a given nonnegative symmetric bilinear form, is the L^2-norm of an X-valued Gaussian random variable ξ with

$$\mathbb{E}\langle \xi, x^*\rangle^2 = V(x^*, x^*), \quad x^* \in X^*.$$

We call $\gamma(V)$ the *Gaussian characteristic* of V (see Section 9.3).

Let us explain briefly the main steps of the proof of Theorem 9.1.1. This discussion will also clarify the meaning of the term on the right-hand side, which is equivalent to the right-hand side of (9.1.3) if X is a Hilbert space, and of (9.1.4) (up to a multiplicative constant) if X is a UMD Banach function space.

In Section 9.2 we start by proving the discrete-time version of Theorem 9.1.1, which takes the following simple form

$$\mathbb{E} \sup_{1\le m\le N} \Big\| \sum_{n=1}^m d_n \Big\|^p \eqsim_{p,X} \mathbb{E}\Big(\mathbb{E}_\gamma \Big\| \sum_{n=1}^N \gamma_n d_n \Big\|^2\Big)^{\frac{p}{2}}, \tag{9.1.6}$$

where $(d_n)_{n=1}^N$ is an X-valued martingale difference sequence and $(\gamma_n)_{n=1}^N$ is a sequence of independent standard Gaussian random variables defined on a probability space $(\Omega_\gamma, \mathbb{P}_\gamma)$. (9.1.6) follows from a decoupling inequality due to Garling [61] and a martingale transform inequality due to Burkholder [35] (each of which holds if and only if X has the UMD property) together with the equivalence of Rademacher and Gaussian random sums with values in spaces with finite cotype due to Maurey and Pisier (see [117]).

Theorem 9.1.1 is derived from (9.1.6) by finite-dimensional approximation and discretization. This is a rather intricate procedure and depends on some elementary, but nevertheless important properties of a Gaussian characteristic $\gamma(\cdot)$. In particular in Section 9.3 we show that for a finite dimensional Banach space X there exists a proper continuous extension of the Gaussian characteristic to all (not necessarily nonnegative) symmetric bilinear forms $V : X^* \times X^* \to \mathbb{R}$, with the bound

$$(\gamma(V))^2 \lesssim_X \sup_{\|x^*\|\le 1} V(x^*, x^*).$$

Next, in Section 9.5, under the assumptions of Theorem 9.1.1 we show that M has a well-defined *covariation bilinear form*, i.e. for each $t \ge 0$ and for almost all $\omega \in \Omega$ there exists a symmetric bilinear form $[[M]]_t(\omega) : X^* \times X^* \to \mathbb{R}$ such that for all $x^*, y^* \in X^*$ one has

$$[[M]]_t(x^*, y^*) = [\langle M, x^*\rangle, \langle M, y^*\rangle]_t \ \text{ a.s.}$$

Next we prove that the bilinear form $[[M]]_t(\omega)$ has a finite Gaussian characteristic $\gamma([[M]]_t)$ for almost all $\omega \in \Omega$. After these preparations we prove Theorem 9.1.1. We also show that the UMD property is necessary for the conclusion of the theorem to hold true (see Subsection 9.7.3).

In Section 9.6 we develop three ramifications of our main result:

- if M is continuous, the conclusion of Theorem 9.1.1 holds for all $0 < p < \infty$.
- if M is purely discontinuous, the theorem can be reformulated in terms of the jumps of M.
- if M has independent increments, the UMD assumption on X can be weakened to reflexivity and finite cotype.

The first two cases are particularly important in view of the fact that any UMD space-valued local martingale has a unique *Meyer-Yoeurp decomposition* into a sum of a continuous local martingale and a purely discontinuous local martingale (see [184, 185]).

A reasonable part of the chapter, namely Section 9.7, is devoted to applications of Theorem 9.1.1 and results related to Theorem 9.1.1. Let us outline some of them. In Subsection 9.7.1 we develop a theory of vector-valued stochastic integration. Our starting point is a result of van Neerven, Veraar, and Weis [126]. They proved that if W_H is a cylindrical Brownian motion in a Hilbert space H and $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H, X)$ is an elementary predictable process, then for all $0 < p < \infty$ and $t \geq 0$ one has the two-sided inequality

$$\mathbb{E} \sup_{0\leq s\leq t} \Big\| \int_0^s \Phi \, \mathrm{d}W_H \Big\|^p \eqsim_{p,X} \mathbb{E} \|\Phi\|^p_{\gamma(L^2([0,t];H),X)}. \tag{9.1.7}$$

Here $\|\Phi\|_{\gamma(L^2([0,t];H),X)}$ is the *γ-radonifying norm* of Φ as an operator from a Hilbert space $L^2([0,t];H)$ into X (see (2.9.1) for the definition); this norm coincides with the Hilbert-Schmidt norm given X is a Hilbert space. This result was extended to continuous local martingales in [175, 177].

Theorem 9.1.1 directly implies (9.1.7). More generally, if $M = \int \Phi \, \mathrm{d}\widetilde{M}$ for some H-valued martingale $\widetilde{M}$ and elementary predictable process $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H, X)$, then it follows from Theorem 9.1.1 that for all $1 \leq p < \infty$ and $t \geq 0$ one has

$$\mathbb{E} \sup_{0\leq s\leq t} \Big\| \int_0^s \Phi \, \mathrm{d}\widetilde{M} \Big\|^p \eqsim_{p,X} \mathbb{E} \|\Phi q_{\widetilde{M}}^{1/2}\|^p_{\gamma(L^2(0,t;[\widetilde{M}]),X)}. \tag{9.1.8}$$

Here $q_{\widetilde{M}}$ is the quadratic variation derivative of $\widetilde{M}$ and $\gamma(L^2(0,t;[\widetilde{M}]),X)$ is a suitable space of γ-radonifying operator associated with $\widetilde{M}$ (see Subsection 9.7.1 for details). This represents a significant improvement of (9.1.7).

In Subsection 9.7.2 we apply our results to vector-valued stochastic integrals with respect to a compensated Poisson random measure $\widetilde{N}$. We show that if N is a Poisson random measure on $\mathbb{R}_+ \times J$ for some measurable space $(J, \mathscr{J})$, ν is its compensator, $\widetilde{N} := N - \nu$ is the corresponding compensated Poisson random measure, then for any UMD Banach space X, for any elementary predictable $F : J \times \mathbb{R}_+ \times \Omega \to X$, and for any $1 \leq p < \infty$ one has that

$$\mathbb{E} \sup_{0\leq s\leq t} \Big\| \int_{J\times[0,s]} F \, \mathrm{d}\widetilde{N} \Big\|^p \eqsim_{p,X} \mathbb{E} \|F\|^p_{\gamma(L^2(J\times[0,t];N),X)}, \quad t \geq 0. \tag{9.1.9}$$

We also show that (9.1.9) holds if one considers a general quasi-left continuous random measure μ instead of N.

In Subsection 9.7.4 we prove the following *martingale domination inequality*: for all local martingales M and N with values in a UMD Banach space X such that

$$\|N_0\| \leq \|M_0\| \text{ a.s.},$$

and

$$[\langle N, x^*\rangle]_\infty \leq [\langle M, x^*\rangle]_\infty \text{ almost surely, for all } x^* \in X^*,$$

for all $1 \leq p < \infty$ we have that

$$\mathbb{E}\sup_{t\geq 0}\|N_t\|^p \lesssim_{p,X} \mathbb{E}\sup_{t\geq 0}\|M_t\|^p.$$

This extends *weak differential subordination* L^p-estimates obtained in [184, 189] (which used to be known to hold only for $1 < p < \infty$, see [146, 184, 189]).

Finally, in Section 9.8, we prove that for any UMD Banach function space X over a measure space (S, Σ, μ), that any X-valued local martingale M has a pointwise local martingale version $M(\sigma)$, $\sigma \in S$, such that if $1 \leq p < \infty$, then for μ-almost all $\sigma \in S$ one has

$$\mathbb{E}\sup_{0\leq s\leq t}\|M_s(\sigma)\|^p \eqsim_{p,X} \mathbb{E}\big\|[M(\sigma)]_t^{\frac{1}{2}}\big\|^p$$

for all $t \geq 0$, which extends (9.1.4) to the case $p = 1$ and general local martingales.

In conclusion we wish to notice that it remains open whether one can find a *predictable* right-hand side in (9.1.5): so far such a predictable right-hand side was explored only in the real-valued case and in the case $X = L^q(S)$, $1 < q < \infty$, see *Burkholder–Novikov–Rosenthal inequalities* in the forthcoming paper [53].

9.2. BURKHOLDER–DAVIS–GUNDY INEQUALITIES: THE DISCRETE TIME CASE

Let us show discrete Burkholder–Davis–Gundy inequalities.

Theorem 9.2.1. *Let X be a UMD Banach space, $(d_n)_{n\geq 1}$ be an X-valued martingale difference sequence. Then for any $1 \leq p < \infty$*

$$\mathbb{E}\sup_{m\geq 1}\Big\|\sum_{n=1}^m d_n\Big\|^p \eqsim_{p,X} \mathbb{E}\|(d_n)_{n=1}^\infty\|_{\gamma(\ell^2,X)}^p. \tag{9.2.1}$$

Proof. Without loss of generality we may assume that there exists $N \geq 1$ such that $d_n = 0$ for all $n > N$. Let $(r_n)_{n\geq 1}$ be a sequence of independent Rademacher random variables, $(\gamma_n)_{n\geq 1}$ be a sequence of independent standard Gaussian random variables. Then

$$\mathbb{E}\sup_{m\geq 1}\Big\|\sum_{n=1}^m d_n\Big\|^p \overset{(i)}{\eqsim}_p \mathbb{E}\mathbb{E}_r\sup_{m\geq 1}\Big\|\sum_{n=1}^N r_n d_n\Big\|^p \overset{(ii)}{\eqsim}_{p,X} \mathbb{E}\mathbb{E}_r\Big\|\sum_{n=1}^N r_n d_n\Big\|^p$$

$$\overset{(iii)}{\eqsim}_{p,X} \mathbb{E}\mathbb{E}_\gamma\Big\|\sum_{n=1}^N \gamma_n d_n\Big\|^p \overset{(iv)}{\eqsim}_p \mathbb{E}\Big(\mathbb{E}_\gamma\Big\|\sum_{n=1}^N \gamma_n d_n\Big\|^2\Big)^{\frac{p}{2}} \tag{9.2.2}$$
$$= \mathbb{E}\|(d_n)_{n=1}^\infty\|_{\gamma(\ell^2,X)}^p,$$

where (i) follows from [35, (8.22)], (ii) holds by [80, Proposition 6.1.12], (iii) follows from [80, Corollary 7.2.10 and Proposition 7.3.15], and (iv) follows from [80, Proposition 6.3.1]. □

Remark 9.2.2. If we collect all the constants in (9.2.2) then one can see that those constants behave well as $p \to 1$, i.e. for any $1 < r < \infty$ there exist positive $C_{r,X}$ and $c_{r,X}$ such that for any $1 \le p \le r$

$$c_{r,X}\mathbb{E}\|(d_n)_{n=1}^\infty\|_{\gamma(\ell^2,X)}^p \le \mathbb{E}\sup_{m\ge 1}\Big\|\sum_{n=1}^m d_n\Big\|^p \le C_{r,X}\mathbb{E}\|(d_n)_{n=1}^\infty\|_{\gamma(\ell^2,X)}^p.$$

Remark 9.2.3. Fix $1 < p < \infty$ and a UMD Banach space X. By Doob's maximal inequality (2.2.1) and Theorem 9.2.1 we have that

$$\mathbb{E}\Big\|\sum_{n=1}^\infty d_n\Big\|^p \eqsim_p \mathbb{E}\sup_{m\ge 1}\Big\|\sum_{n=1}^m d_n\Big\|^p \eqsim_{p,X} \mathbb{E}\|(d_n)_{n=1}^\infty\|_{\gamma(\ell^2,X)}^p.$$

Let us find the constants in the equivalence

$$\mathbb{E}\Big\|\sum_{n=1}^\infty d_n\Big\|^p \eqsim_{p,X} \mathbb{E}\|(d_n)_{n=1}^\infty\|_{\gamma(\ell^2,X)}^p.$$

Since X is UMD, it has a finite cotype q (see [80, Definition 7.1.1. and Proposition 7.3.15]), and therefore by modifying (9.2.2) (using decoupling inequalities [79, p. 282] instead of [35, (8.22)] and [80, Proposition 6.1.12]) one can show that

$$\frac{1}{\beta_{p,X}c_{p,X}}\Big(\mathbb{E}\|(d_n)_{n=1}^\infty\|_{\gamma(\ell^2,X)}^p\Big)^{\frac{1}{p}} \le \Big(\mathbb{E}\Big\|\sum_{n=1}^m d_n\Big\|^p\Big)^{\frac{1}{p}}$$
$$\le 2\beta_{p,X}\kappa_{p,2}\Big(\mathbb{E}\|(d_n)_{n=1}^\infty\|_{\gamma(\ell^2,X)}^p\Big)^{\frac{1}{p}},$$

where $c_{p,X}$ depends on p, the cotype of X, and the Gaussian cotype constant of X (see [80, Proposition 7.3.15]), while $\kappa_{p,q}$ is the Kahane-Khinchin constant (see [80, Section 6.2]).

In the following theorem we show that X having the UMD property is necessary for Theorem 9.2.1 to hold.

Theorem 9.2.4. *Let X be a Banach space and $1 \le p < \infty$ be such that* (9.2.1) *holds for any martingale difference sequence $(d_n)_{n\ge 1}$. Then X is UMD.*

Proof. Note that for any set $(x_n)_{n=1}^N$ of elements of X and for any $[-1,1]$-valued sequence $(\varepsilon_n)_{n=1}^N$ we have that $\|(\varepsilon_n x_n)_{n=1}^N\|_{\gamma(\ell^2_N,X)} \leq \|(x_n)_{n=1}^N\|_{\gamma(\ell^2_N,X)}$ by the ideal property (see [80, Theorem 9.1.10]). Therefore if (9.2.1) holds for any X-valued martingale difference sequence $(d_n)_{n\geq 1}$, then we have that for any $[-1,1]$-valued sequence $(\varepsilon_n)_{n\geq 1}$

$$\mathbb{E}\sup_{m\geq 1}\Big\|\sum_{n=1}^m \varepsilon_n d_n\Big\|^p \lesssim_{p,X} \mathbb{E}\sup_{m\geq 1}\Big\|\sum_{n=1}^m d_n\Big\|^p. \tag{9.2.3}$$

If $p>1$, then (9.2.3) together with (2.2.1) implies the UMD property. If $p=1$, then (9.2.3) for $p=1$ implies (9.2.3) for any $p>1$ (see [79, Theorem 3.5.4]), and hence it again implies UMD. □

Now we turn to the continuous-time case. It turns out that in this case the right-hand side of (9.2.1) transforms to a so-called *Gaussian characteristic* of a certain bilinear form generated by a quadratic variation of the corresponding martingale. Therefore before proving our main result (Theorem 9.5.1) we will need to outline some basic properties of a Gaussian characteristic (see Section 9.3). We will also need some preliminaries concerning continuous-time Banach space-valued martingales (see Section 9.4).

9.3. GAUSSIAN CHARACTERISTICS

The current section is devoted to the definition and some basic properties of one of the main object of the chapter – a Gaussian characteristic of a bilinear form. Many of the statements here might seem to be obvious for the reader. Nevertheless we need to show them before reaching our main Theorem 9.5.1.

9.3.1. Basic definitions

Let us first recall some basic facts on Gaussian measures. Let X be a Banach space. An X-valued random variable ξ is called *Gaussian* if $\langle \xi, x^*\rangle$ has a Gaussian distribution for all $x^* \in X^*$. Gaussian random variables enjoy a number of useful properties (see [21, 100]). We will need the following *Gaussian covariance domination inequality* (see [21, Corollary 3.3.7] and [80, Theorem 6.1.25] for the case $\Phi = \|\cdot\|^p$).

Lemma 9.3.1. *Let X be a Banach space, ξ, η be centered X-valued Gaussian random variables. Assume that $\mathbb{E}\langle \eta, x^*\rangle^2 \leq \mathbb{E}\langle \xi, x^*\rangle^2$ for all $x^* \in X^*$. Then $\mathbb{E}\Phi(\eta) \leq \mathbb{E}\Phi(\xi)$ for any convex symmetric continuous function $\Phi : X \to \mathbb{R}_+$.*

Let X be a Banach space. We denote the linear space of all continuous $\mathbb{R}$-valued bilinear forms on $X \times X$ by $X^* \otimes X^*$. Note that this linear space can be endowed with the following natural norm:

$$\|V\| := \sup_{x\in X, \|x\|\leq 1} |V(x,x)|, \tag{9.3.1}$$

where the latter expression is finite due to bilinearity and continuity of V. A bilinear form V is called *nonnegative* if $V(x,x) \geq 0$ for all $x \in X$, and V is called *symmetric* if $V(x,y) = V(y,x)$ for all $x,y \in X$.

Let X be a Banach space, ξ be a centered X-valued Gaussian random variable. Then ξ has a *covariance bilinear form* $V : X^* \times X^* \to \mathbb{R}$ such that

$$V(x^*, y^*) = \mathbb{E}\langle \xi, x^*\rangle\langle \xi, y^*\rangle, \quad x^*, y^* \in X.$$

Notice that a covariance bilinear form is always continuous, symmetric, and nonnegative. It is worth noticing that one usually considers a *covariance operator* $Q : X^* \to X^{**}$ defined by

$$\langle Qx^*, y^*\rangle = \mathbb{E}\langle \xi, x^*\rangle\langle \xi, y^*\rangle, \quad x^*, y^* \in X.$$

But since there exists a simple one-to-one correspondence between bilinear forms and $\mathcal{L}(X^*, X^{**})$, we will work with covariance bilinear forms instead. We refer the reader to [21, 48, 68, 173] for details.

Let $V : X^* \times X^* \to \mathbb{R}$ be a symmetric continuous nonnegative bilinear form. Then V is said to have a finite *Gaussian characteristic* $\gamma(V)$ if there exists a centered X-valued Gaussian random variable ξ such that V is the covariance bilinear form of ξ. Then we set $\gamma(V) := (\mathbb{E}\|\xi\|^2)^{\frac{1}{2}}$ (this value is finite due to the Fernique theorem, see [21, Theorem 2.8.5]). Otherwise we set $\gamma(V) = \infty$. Note that then for all $x^*, y^* \in X^*$ one has the following control of continuity of V:

$$\begin{aligned} |V(x^*, x^*)^{\frac{1}{2}} - V(y^*, y^*)^{\frac{1}{2}}| &= (\mathbb{E}|\langle \xi, x^*\rangle|^2)^{\frac{1}{2}} - (\mathbb{E}|\langle \xi, y^*\rangle|^2)^{\frac{1}{2}} \\ &\leq (\mathbb{E}|\langle \xi, x^* - y^*\rangle|^2)^{\frac{1}{2}} \leq (\mathbb{E}\|\xi\|^2)^{\frac{1}{2}}\|x^* - y^*\| = \|x^* - y^*\|\gamma(V). \end{aligned} \tag{9.3.2}$$

Remark 9.3.2. Note that for any V with $\gamma(V) < \infty$ the distribution of the corresponding centered X-valued Gaussian random variable ξ is *uniquely determined* (see [21, Chapter 2]).

Remark 9.3.3. Note that if X is finite dimensional, then $\gamma(V) < \infty$ for any nonnegative symmetric bilinear form V. Indeed, in this case X is isomorphic to a finite dimensional Hilbert space H, so there exists an eigenbasis $(h_n)_{n=1}^d$ making V diagonal, and then the corresponding Gaussian random variable will be equal to $\xi := \sum_{n=1}^d V(h_n, h_n)\gamma_n h_n$, where $(\gamma_n)_{n=1}^d$ are independent standard Gaussian.

9.3.2. Basic properties of $\gamma(\cdot)$

Later we will need the following technical lemmas.

Lemma 9.3.4. *Let X be a reflexive (separable) Banach space, $V : X^* \times X^* \to \mathbb{R}$ be a symmetric continuous nonnegative bilinear form. Then there exist a (separable) Hilbert space H and $T \in \mathcal{L}(H, X)$ such that*

$$V(x^*, y^*) = \langle T^* x^*, T^* y^*\rangle, \quad x^*, y^* \in X^*.$$

Proof. See [27, pp. 57-58] or [100, p. 154]. □

The following lemma connects Gaussian characteristics and γ-norms (see (2.9.1)) and it can be found e.g. in [125, Theorem 7.4] or in [27, 129].

Lemma 9.3.5. *Let X be a separable Banach space, H be a separable Hilbert space, $T \in \mathscr{L}(H,X)$, $V : X^* \times X^* \to \mathbb{R}$ be a symmetric continuous nonnegative bilinear form such that $V(x^*, y^*) = \langle T^*x^*, T^*y^*\rangle$ for all $x^*, y^* \in X^*$. Then $\gamma(V) = \|T\|_{\gamma(H,X)}$.*

Remark 9.3.6. Fix a Hilbert space H and a Banach space X. Note that even though by the lemma above there exists a natural embedding of γ-radonifying operators from $\mathscr{L}(H,X)$ to the space of symmetric nonnegative bilinear forms on $X^* \times X^*$, this embedding is neither *injective* nor *linear*. This also explains why we need to use bilinear forms with finite Gaussian characteristics instead of γ-radonifying operators: in the proof of our main result – Theorem 9.5.1 – we will need various statements (like triangular inequalities and convergence theorems) for *bilinear forms*, not *operators*.

Now we will prove some statements about approximation of nonnegative symmetric bilinear forms by finite dimensional ones in $\gamma(\cdot)$.

Lemma 9.3.7. *Let X be a reflexive Banach space, $Y \subset X^*$ be a finite dimensional subspace. Let $P : Y \hookrightarrow X^*$ be an inclusion operator. Let $V : X^* \times X^* \to \mathbb{R}$ and $V_0 : Y \times Y \to \mathbb{R}$ be symmetric continuous nonnegative bilinear forms such that $V_0(x_0^*, y_0^*) = V(Px_0^*, Py_0^*)$ for all $x_0^*, y_0^* \in Y$. Then $\gamma(V_0)$ is well-defined and $\gamma(V_0) \le \gamma(V)$.*

Proof. First of all notice that $\gamma(V_0)$ is well-defined since Y is finite dimensional, hence reflexive, and thus has a predual space coinciding with its dual. Without loss of generality assume that $\|V\|_\gamma < \infty$. Let ξ_V be a centered X-valued Gaussian random variable with V as the covariance bilinear form. Define $\xi_{V_0} := P^*\xi_V$ (note that $Y^* \hookrightarrow X$ due to the Hahn-Banach theorem). Then for all $x_0^*, y_0^* \in X_0^*$

$$\mathbb{E}\langle \xi_{V_0}, x_0^*\rangle\langle \xi_{V_0}, y_0^*\rangle = \mathbb{E}\langle \xi_V, Px_0^*\rangle\langle \xi_V, Py_0^*\rangle = V(Px_0^*, Py_0^*) = V_0(x_0^*, y_0^*),$$

so V_0 is the covariance bilinear form of ξ_{V_0} and since $\|P^*\| = \|P\| = 1$

$$\gamma(V_0) = (\mathbb{E}\|\xi_{V_0}\|^2)^{\frac{1}{2}} = (\mathbb{E}\|P^*\xi_V\|^2)^{\frac{1}{2}} \le (\mathbb{E}\|\xi_V\|^2)^{\frac{1}{2}} = \gamma(V). \tag{9.3.3}$$

□

Proposition 9.3.8. *Let X be a separable reflexive Banach space, $V : X^* \times X^* \to \mathbb{R}$ be a symmetric continuous nonnegative bilinear form. Let $Y_1 \subset Y_2 \subset \ldots \subset Y_m \subset \ldots$ be a sequence of finite dimensional subspaces of X^* with $\overline{\cup_m Y_m} = X^*$. Then for each $m \ge 1$ a symmetric continuous nonnegative bilinear form $V_m = V|_{Y_m \times Y_m}$ is well-defined and $\gamma(V_m) \to \gamma(V)$ as $m \to \infty$.*

Proof. First of all notice that V_m's are well-defined since each of the Y_m is finite dimensional, hence reflexive, and thus has a predual space coinciding with its dual (which we will call X_m and which can even be embedded into X due to the Hahn-Banach theorem). Let $P_m : Y_m \hookrightarrow X^*$ be the inclusion operator (thus is particular $\|P_m\| \le 1$). Let a Hilbert space H and an operator $T \in \mathscr{L}(H, X)$ be as constructed in Lemma 9.3.4. Let $(h_n)_{n\ge1}$ be an orthonormal basis of H, and $(\gamma_n)_{n\ge1}$ be a sequence of standard Gaussian random variables. For each $N \ge 1$ define a centered Gaussian random variable $\xi_N := \sum_{n=1}^N \gamma_n Th_n$. Then for each $m \ge 1$ the centered Gaussian random variable $\sum_{n=1}^\infty \gamma_n P_m^* Th_n$ is well-defined (since $P_m^* T$ has a finite rank, and every finite rank operator has a finite γ-norm, see [80, Section 9.2]), and for any $x^* \in Y_m$ we have that

$$V_m(x^*, x^*) = V(x^*, x^*) = \|T^* x^*\| = \|T^* P_m x^*\| = \mathbb{E}\Big\langle \sum_{n=1}^\infty \gamma_n P_m^* Th_n, x^* \Big\rangle^2,$$

so V_m is the covariance bilinear form of $\sum_{n=1}^\infty \gamma_n P_m^* Th_n$, and

$$\gamma(V_m) = \Big(\mathbb{E}\Big\| \sum_{n=1}^\infty \gamma_n P_m^* Th_n \Big\|^2\Big)^{\frac12} = \Big(\mathbb{E}\Big\| P_m^* \sum_{n=1}^\infty \gamma_n Th_n \Big\|^2\Big)^{\frac12}.$$

The latter expression converges to $\gamma(V)$ by Lemma 9.3.5 and due to the fact that $\|P_m^* x\| \to \|x\|$ monotonically for each $x \in X$ as $m \to \infty$. □

The next lemma provides the Gaussian characteristic with the triangular inequality.

Lemma 9.3.9. *Let X be a reflexive Banach space, $V, W : X^* \times X^*$ be symmetric continuous nonnegative bilinear forms. Then $\gamma(V + W) \le \gamma(V) + \gamma(W)$.*

Proof. If $\max\{\gamma(V), \gamma(W)\} = \infty$ then the lemma is obvious. Let $\gamma(V), \gamma(W) < \infty$. Let ξ_V and ξ_W be X-valued centered Gaussian random variables corresponding to V and W respectively. Without loss of generality we can set ξ_V and ξ_W independent. Let $\xi_{V+W} = \xi_V + \xi_W$. Then ξ_{V+W} is an X-valued centered Gaussian random variable (see [21]) and for any $x^* \in X^*$ due to the independence of ξ_V and ξ_W

$$\mathbb{E}\langle \xi_{V+W}, x^*\rangle^2 = \mathbb{E}\langle \xi_V + \xi_W, x^*\rangle^2 = \mathbb{E}\langle \xi_V, x^*\rangle^2 + \mathbb{E}\langle \xi_W, x^*\rangle^2 = (V + W)(x^*, x^*).$$

So ξ_{V+W} has $V + W$ as the covariation bilinear form, and therefore

$$\gamma(V + W) = (\mathbb{E}\|\xi_{V+W}\|^2)^{\frac12} \le (\mathbb{E}\|\xi_V\|^2)^{\frac12} + (\mathbb{E}\|\xi_W\|^2)^{\frac12} = \gamma(V) + \gamma(W).$$

□

Now we discuss such important properties of $\gamma(\cdot)$ as monotonicity and monotone continuity.

Lemma 9.3.10. *Let X be a separable Banach space, $V, W : X^* \times X^* \to \mathbb{R}$ be symmetric continuous nonnegative bilinear forms such that $W(x^*, x^*) \le V(x^*, x^*)$ for all $x^* \in X^*$. Then $\gamma(W) \le \gamma(V)$.*

Proof. The lemma follows from Lemma 9.3.5 and [80, Theorem 9.4.1]. □

Lemma 9.3.11. *Let X be a separable reflexive Banach space, $(V_n)_{n\ge1}$ be symmetric continuous nonnegative bilinear forms on $X^* \times X^*$ such that $V_n(x^*, x^*) \to 0$ for any $x^* \in X^*$ monotonically as $n \to \infty$. Assume additionally that $\gamma(V_n) < \infty$ for some $n \ge 1$. Then $\gamma(V_n) \to 0$ monotonically as $n \to \infty$.*

Proof. Without loss of generality assume that $\gamma(V_1) < \infty$. First notice that by Lemma 9.3.10 the sequence $(\gamma(V_n))_{n\ge1}$ is monotone and bounded by $\gamma(V_1)$.

By Lemma 9.3.4 we may assume that there exists a separable Hilbert space H and a sequence of operators $(T_n)_{n\ge1}$ from H to X such that $V_n(x^*, x^*) = \|T_n^* x^*\|^2$ for all $x^* \in X^*$ (note that we are working with one Hilbert space since all the separable Hilbert spaces are isometrically isomorphic). Let $T \in \mathscr{L}(H, X)$ be the zero operator. Then $T_n^* x^* \to T^* x^* = 0$ as $n \to \infty$ for all $x^* \in X^*$, and hence by [80, Theorem 9.4.2], Lemma 9.3.5, and the fact that $\|T_n x^*\| \le \|T_1 x^*\|$ for all $x^* \in X^*$

$$\lim_{n\to\infty} \gamma(V_n) = \lim_{n\to\infty} \|T_n\|_{\gamma(H,X)} = \|T\|_{\gamma(H,X)} = 0.$$

□

The following lemma follows for Lemma 9.3.9 and 9.3.11.

Lemma 9.3.12. *Let X be a separable reflexive Banach space, V, $(V_n)_{n\ge1}$ be symmetric continuous nonnegative bilinear forms on $X^* \times X^*$ such that $V_n(x^*, x^*) \nearrow V(x^*, x^*)$ for any $x^* \in X^*$ monotonically as $n \to \infty$. Then $\gamma(V_n) \nearrow \gamma(V)$ monotonically as $n \to \infty$.*

Remark 9.3.13. Notice that $\gamma(\cdot)$ is not a norm. Indeed, it is easy to see that $\gamma(\alpha V) = \sqrt{\alpha}\gamma(V)$ for any $\alpha \ge 0$ and any nonnegative symmetric bilinear form V: if we fix any X-valued Gaussian random variable ξ having V as its covariance bilinear form, then $\sqrt{\alpha}\xi$ has $\alpha\gamma(V)$ as its covariance bilinear form. Therefore it is a natural question whether $\gamma(\cdot)^2$ satisfies the triangle inequality and hence has the norm properties. It is easy to check the triangle inequality if X is Hilbert: indeed, for any V and W

$$\gamma(V+W)^2 = \mathbb{E}\|\xi_{V+W}\|^2 = \mathbb{E}\|\xi_V\|^2 + \mathbb{E}\|\xi_W\|^2 + 2\mathbb{E}\langle \xi_V, \xi_W\rangle = \gamma(V)^2 + \gamma(W)^2,$$

where ξ_V, ξ_W, and ξ_{V+W} are as in the latter proof.

It turns out that if such a triangular inequality holds for some Banach space X, then this Banach space must have a *Gaussian type* 2 (see [80, Subsection 7.1.d]). Indeed, let X be such that for all nonnegative symmetric bilinear forms V and W on $X^* \times X^*$

$$\gamma(V+W)^2 \le \gamma(V)^2 + \gamma(W)^2. \tag{9.3.4}$$

Fix $(x_i)_{i=1}^n \subset X$ and a sequence of independent standard Gaussian random variables $(\xi_i)_{i=1}^n$. For each $i = 1,\dots,n$ define a symmetric bilinear form $V_i : X^* \times X^* \to \mathbb{R}$ as $V_i(x^*,y^*) := \langle x_i, x^*\rangle \cdot \langle x_i, y^*\rangle$. Let $V = V_1 + \cdots + V_n$. Then by (9.3.4) and the induction argument

$$\mathbb{E}\Big\|\sum_{i=1}^n \xi_i x_i\Big\|^2 \overset{(*)}{=} \gamma(V)^2 \leq \sum_{i=1}^n \gamma(V_i)^2 \overset{(**)}{=} \sum_{i=1}^n \mathbb{E}\|\xi_i x_i\|^2 = \sum_{i=1}^n \|x_i\|^2,$$

where $(*)$ follows from the fact that $\sum_{i=1}^n \xi_i x_i$ is a centered Gaussian random variable the fact that for all $x^*, y^* \in X^*$

$$\mathbb{E}\Big\langle \sum_{i=1}^n \xi_i x_i, x^*\Big\rangle \cdot \Big\langle \sum_{i=1}^n \xi_i x_i, y^*\Big\rangle = \sum_{i=1}^n \langle x_i, x^*\rangle \cdot \langle x_i, y^*\rangle = V(x^*,y^*),$$

while $(**)$ follows analogously by exploiting the fact that $\xi_i x_i$ is a centered Gaussian random variable with the covariance bilinear form V_i. Therefore by [80, Definition 7.1.17], X has a Gaussian type 2 with the corresponding Gaussian type constant $\tau_{2,X}^{\gamma} = 1$.

9.3.3. Finite dimensional case

Even though a Gaussian characteristic is well-defined only for some nonnegative symmetric forms, it can be extended in a proper continuous way to all the symmetric forms given X is finite dimensional. Let X be a finite dimensional Banach space. Notice that in this case $\gamma(V) < \infty$ for any nonnegative symmetric bilinear form V (see Remark 9.3.3). Let us define $\gamma(V)$ for a general symmetric $V \in X^{**}\otimes X^{**} = X \otimes X$ in the following way:

$$\gamma(V) := \inf\{\gamma(V^+) + \gamma(V^-) : V^+, V^- \text{ are nonnegative and } V = V^+ - V^-\}. \tag{9.3.5}$$

Notice that $\gamma(V)$ is well-defined and finite for any symmetric V. Indeed, by a well known linear algebra fact (see e.g. [166, Theorem 6.6 and 6.10]) any symmetric bilinear form V has an eigenbasis $(x_n^*)_{n=1}^d$ of X^* that *diagonalizes* V, i.e. there exists $(\lambda_n)_{n=1}^d \in \mathbb{R}$ such that for all $(a_n)_{n=1}^d, (b_n)_{n=1}^d \in \mathbb{R}$ we have that for $x^* = \sum_{n=1}^d a_n x_n^*$ and $y^* = \sum_{n=1}^d b_n x_n^*$

$$V(x^*,y^*) = \sum_{n=1}^d \sum_{m=1}^d a_n b_m V(x_n^*, x_m^*) = \sum_{n=1}^d \lambda_n a_n b_n.$$

Therefore it is sufficient to define

$$V^+(x^*,y^*) := \sum_{n=1}^d \mathbf{1}_{\lambda_n \geq 0} \lambda_n a_n b_n, \quad V^-(x^*,y^*) := \sum_{n=1}^d \mathbf{1}_{\lambda_n < 0} (-\lambda_n) a_n b_n$$

and then $\gamma(V) \leq \gamma(V^+) + \gamma(V^-) < \infty$ due to the fact that V^+ and V^- are nonnegative and by Remark 9.3.3. (In fact, one can check that $\gamma(V) = \gamma(V^+) + \gamma(V^-)$, but we will not need this later, so we leave this fact without a proof).

Now we will develop some basic and *elementary* (but nonetheless important) properties of such a general $\gamma(\cdot)$.

Lemma 9.3.14. *Let $V : X^* \times X^* \to \mathbb{R}$ be a* nonnegative *symmetric bilinear form. Then $\gamma(V)$ defined by* (9.3.5) *coincides with $\gamma(V)$ defined in Subsection 9.3.1. In other words, these definitions* agree *given V is nonnegative.*

Proof. Fix nonnegative V^+ and V^- such that $V = V^+ - V^-$. Then $\gamma(V^+) + \gamma(V^-) = \gamma(V + V^-) + \gamma(V^-) \geq \gamma(V) + \gamma(V^-) \geq \gamma(V)$ by Lemma 9.3.10, so $\gamma(V)$ does not change. □

Lemma 9.3.15. *Let $V, W : X^* \times X^* \to \mathbb{R}$ be symmetric bilinear forms. Then $\gamma(V) - \gamma(W) \leq \gamma(V - W)$.*

Proof. Denote $V - W$ by U. Fix $\varepsilon > 0$. Then there exist symmetric nonnegative bilinear forms W^+, W^-, U^+, U^- such that $W = W^+ - W^-$, $U = U^+ - U^-$, and

$$\gamma(W) \geq \gamma(W^+) + \gamma(W^-) - \varepsilon,$$

$$\gamma(U) \geq \gamma(U^+) + \gamma(U^-) - \varepsilon.$$

Then since $V = U + W$ by (9.3.5) and Lemma 9.3.9

$$\begin{aligned}\gamma(V) - \gamma(W) &= \gamma((W^+ + U^+) - (W^- + U^-)) - \gamma(W^+ - W^-)\\ &\leq \gamma(W^+ + U^+) + \gamma(W^- + U^-) - \gamma(W^+) - \gamma(W^-) + \varepsilon\\ &\leq \gamma(U^+) + \gamma(U^-) + \varepsilon \leq \gamma(U) + 2\varepsilon,\end{aligned}$$

and by sending $\varepsilon \to 0$ we conclude the desired. □

Lemma 9.3.16. *Let $V : X^* \times X^* \to \mathbb{R}$ be a symmetric bilinear form. Then $\gamma(V) = \gamma(-V)$ and $\gamma(\alpha V) = \sqrt{\alpha}\gamma(V)$ for any $\alpha \geq 0$.*

Proof. The first part follows directly from (9.3.5). For the second part we have that due to (9.3.5) it is enough to justify $\gamma(\alpha V) = \sqrt{\alpha}\gamma(V)$ only for nonnegative V, which was done in Remark 9.3.13. □

Proposition 9.3.17. *The function $\gamma(\cdot)$ defined by* (9.3.5) *is continuous on the linear space of all symmetric bilinear forms endowed with $\|\cdot\|$ defined by* (9.3.1)*. Moreover, $\gamma(V)^2 \lesssim_X \|V\|$ for any symmetric bilinear form $V : X^* \times X^* \to \mathbb{R}$.*

Proof. Due to Lemma 9.3.15 and 9.3.16 it is sufficient to show that $\gamma(\cdot)$ is bounded on the unit ball with respect to the norm $\|\cdot\|$ in order to prove the first part of the proposition. Let us show this boundedness. Let U be a fixed symmetric nonnegative element of $X \otimes X$ such that $U + V$ is nonnegative and such that $U(x^*, x^*) \geq V(x^*, x^*)$ for any symmetric V with $\|V\| \leq 1$ (since X is finite dimensional, one can take $U(x^*) := c|||x^*|||^2$ for some Euclidean norm $|||\cdot|||$ on X^* and some big enough constant $c>0$). Fix a symmetric $V: X^*\times X^* \to \mathbb{R}$ with $\|V\|\leq 1$. Then $V=(U+V)-U$, and by (9.3.5)

$$\gamma(V)\leq \gamma(U+V)+\gamma(U)=\gamma(2U)+\gamma(U),$$

which does not depend on V.

Let us show the second part. Due to the latter consideration there exists a constant C_X depending only on X such that $\gamma(V)\leq C_X$ if $\|V\|\leq 1$. Therefore by Lemma 9.3.16 we have that for a general symmetric V

$$\gamma(V)^2 = \|V\|\gamma(V/\|V\|)^2 \leq C_X^2\|V\|.$$

□

Later we will also need the following elementary lemma.

Lemma 9.3.18. *There exists vectors $(x_i^*)_{i=1}^n$ in X^* such that*

$$|||V||| := \sum_{i=1}^n |V(x_i^*, x_i^*)| \tag{9.3.6}$$

defines a norm on the space of all symmetric bilinear forms on $X^*\times X^*$. In particular we have that $\|V\|\eqsim_X |||V|||$ for any symmetric bilinear form $V: X^*\times X^*\to\mathbb{R}$.

We will demonstrate here the proof for the convenience of the reader.

Proof. First notice that $|||\cdot|||$ clearly satisfies the triangular inequality. Let us show that there exists a set $(x_i^*)_{i=1}^n$ such that $|||V|||=0$ implies $V=0$. Let $(y_i^*)_{i=1}^d$ be a basis of X^*. Then there exist $i,j\in\{1,\dots,d\}$ such that

$$0\neq V(y_i^*, y_j^*) = (V(y_i^*+y_j^*, y_i^*+y_j^*) - V(y_i^*-y_j^*, y_i^*-y_j^*))/4$$

(otherwise $V=0$). This means that for these i and j

$$|V(y_i^*+y_j^*, y_i^*+y_j^*)| + |V(y_i^*-y_j^*, y_i^*-y_j^*)|\neq 0,$$

so in particular

$$\sum_{i=1}^d\sum_{j=1}^d |V(y_i^*+y_j^*, y_i^*+y_j^*)| + |V(y_i^*-y_j^*, y_i^*-y_j^*)|\neq 0.$$

It remains to notice that the latter sum has the form (9.3.6) for a proper choice of $(x_i^*)_{i=1}^n$ independent of V.

In order to show the last part of the lemma we need to notice that the space of symmetric bilinear forms is finite dimensional if X is so, so all the norms on the linear space of symmetric bilinear forms are equivalent, and therefore $\|V\|\eqsim_X |||V|||$ for any symmetric bilinear form $V: X^*\times X^*\to\mathbb{R}$. □

9.4. COVARIATION BILINEAR FORMS

We continue with the definition of a covariation bilinear form and its basic properties.

Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a local martingale. Fix $t \geq 0$. Then M is said to have a *covariation bilinear from* $[[M]]_t$ at $t \geq 0$ if there exists a continuous bilinear form-valued random variable $[[M]]_t : X^* \times X^* \times \Omega \to \mathbb{R}$ such that for any fixed $x^*, y^* \in X^*$ a.s. $[[M]]_t(x^*, y^*) = [\langle M, x^*\rangle, \langle M, y^*\rangle]_t$.

Remark 9.4.1. Let us outline some basic properties of the covariation bilinear forms, which follow directly from [89, Theorem 26.6] (here we presume the existence of $[[M]]_t$ and $[[N]]_t$ for all $t \geq 0$)

(i) $t \mapsto [[M]]_t$ is nondecreasing, i.e. $[[M]]_t(x^*, x^*) \geq [[M]]_s(x^*, x^*)$ a.s. for all $0 \leq s \leq t$ and $x^* \in X^*$,

(ii) $[[M]]^\tau = [[M^\tau]]$ a.s. for any stopping time τ,

(iii) $\Delta[[M]]_\tau(x^*, x^*) = |\langle \Delta M_\tau, x^*\rangle|^2$ a.s. for any stopping time τ.

Remark 9.4.2. If X is finite dimensional, then it is isomorphic to a Hilbert space, and hence existence of $[[M]]_t$ follows from existence of $[M]_t$ with the following estimate a.s.

$$\|[[M]]_t\| = \sup_{x^* \in X^*, \|x^*\| \leq 1} [[M]]_t(x^*, x^*) = \sup_{x^* \in X^*, \|x^*\| \leq 1} [\langle M, x^*\rangle, \langle M, x^*\rangle]_t \lesssim_X [M]_t.$$

For a general infinite dimensional Banach space the existence of $[[M]]_t$ remains an open problem. In Theorem 9.5.1 we show that if X has the UMD property, then existence of $[[M]]_t$ follows automatically; moreover, in this case $\gamma([[M]]_t) < \infty$ a.s. (see Section 9.3 and Theorem 9.5.1, which is way stronger than continuity.

9.5. BURKHOLDER–DAVIS–GUNDY INEQUALITIES: THE CONTINUOUS-TIME CASE

The following theorem is the main theorem of the chapter.

Theorem 9.5.1. *Let X be a UMD Banach space. Then for any local martingale M : $\mathbb{R}_+ \times \Omega \to X$ with $M_0 = 0$ and any $t \geq 0$ the covariation bilinear form $[[M]]_t$ is well-defined and bounded almost surely, and for all $1 \leq p < \infty$*

$$\mathbb{E} \sup_{0 \leq s \leq t} \|M_s\|^p \eqsim_{p,X} \mathbb{E}\gamma([[M]]_t)^p. \tag{9.5.1}$$

For the proof we will need the following technical lemma which follows from [180, Theorem 6].

Lemma 9.5.2. *Let X be a finite dimensional Banach space, $M:\mathbb{R}_+\times\Omega\to X$ be a martingale such that $\mathbb{E}\sup_{t\geq 0}\|M_t\|<\infty$. Then there exists a sequence $(M^n)_{n\geq 1}$ of X-valued uniformly bounded martingales such that $\mathbb{E}\sup_{t\geq 0}\|M_t-M_t^n\|\to 0$ as $n\to\infty$.*

Proof of Theorem 9.5.1. Step 1: finite dimensional case. First note that in this case $[\![M]\!]_t$ exists and bounded a.s. due to Remark 9.4.2. Fix $1\leq p<\infty$. We will prove separately the cases $p>1$ and $p=1$.

Case $p>1$. For each $N\geq 1$ fix a partition $0=t_1^N<\ldots<t_{n_N}^N=t$ with the mesh not exceeding $1/N$. For each $\omega\in\Omega$ and $N\geq 1$ define a bilinear form $V_N:X^*\times X^*\to\mathbb{R}$ as follows:

$$V_N(x^*,x^*):=\sum_{i=1}^{n_N}\langle M_{t_i^N}-M_{t_{i-1}^N},x^*\rangle^2,\quad x^*\in X^*. \tag{9.5.2}$$

Note that $(M_{t_i^N}-M_{t_{i-1}^N})_{i=1}^{n_N}$ is a martingale difference sequence with respect to the filtration $(\mathscr{F}_{t_i^N})_{i=1}^{n_N}$, so by Theorem 9.2.1 and (2.2.1)

$$\begin{aligned}\mathbb{E}\|M_t\|^p=\mathbb{E}\Big\|\sum_{i=1}^{n_N}M_{t_i^N}-M_{t_{i-1}^N}\Big\|^p&\eqsim_{p,X}\mathbb{E}\Big(\mathbb{E}_\gamma\Big\|\sum_{i=1}^{n_N}\gamma_i(M_{t_i^N}-M_{t_{i-1}^N})\Big\|^2\Big)^{\frac{p}{2}}\\&=\mathbb{E}\gamma(V_N)^p,\end{aligned}\tag{9.5.3}$$

where $(\gamma_i)_{i=1}^{n_N}$ is a sequence of independent Gaussian standard random variables, and the latter equality holds due to the fact that for any fixed $\omega\in\Omega$ the random variable $\sum_{i=1}^{n_N}\gamma_i(M_{t_i^N}-M_{t_{i-1}^N})(\omega)$ is Gaussian and by (9.5.2)

$$V_N(x^*,x^*)=\mathbb{E}_\gamma\Big\langle\sum_{i=1}^{n_N}\gamma_i(M_{t_i^N}-M_{t_{i-1}^N})(\omega),x^*\Big\rangle^2,\quad x^*\in X^*.$$

Therefore it is sufficient to show that $\gamma(V_N-[\![M]\!]_t)\to 0$ in $L^p(\Omega)$ as $N\to\infty$. Indeed, if this is the case, then by (9.5.3) and by Lemma 9.3.15

$$\mathbb{E}\gamma([\![M]\!]_t)^p=\lim_{N\to\infty}\mathbb{E}\gamma(V_N)^p\eqsim_{p,X}\mathbb{E}\|M_t\|^p.$$

Let us show this convergence. Note that by Proposition 9.3.17 and Lemma 9.3.18 a.s.

$$\gamma(V_N-[\![M]\!]_t)^2\lesssim_X\|V_N-[\![M]\!]_t\|\lesssim_X|\!|\!|V_N-[\![M]\!]_t|\!|\!|$$

(where $|\!|\!|\cdot|\!|\!|$ is as in (9.3.6)) Therefore we need to show that $|\!|\!|V_N-[\![M]\!]_t|\!|\!|\to 0$ in $L^{\frac{p}{2}}(\Omega)$, which follows from the fact that for any x_i^* from Lemma 9.3.18, $i=1,\ldots,n$, we have that

$$V_N(x_i^*,x_i^*)=\sum_{i=1}^{n_N}\langle M_{t_i^N}-M_{t_{i-1}^N},x_i^*\rangle^2\to[\langle M,x_i^*\rangle]_t$$

in $L^{\frac{p}{2}}$-sense by [55, Théorème 2].

Case $p=1$. First assume that M is an L^2-bounded martingale. Then due to Remark 9.2.2 and Case $p>1$ we can show that

$$c_X\mathbb{E}\gamma([\![M]\!]_t)^p\leq\mathbb{E}\sup_{0\leq s\leq t}\|M_s\|^p\leq C_X\mathbb{E}\gamma([\![M]\!]_t)^p \tag{9.5.4}$$

for any $1 < p < 2$ for some universal positive constants c_X and C_X. (9.5.1) then follows as p in (9.5.4) approaches to 1 by the dominated convergence theorem.

Now let M be a general martingale. Then by Lemma 9.5.2 there exists a sequence of X-valued L^2-bounded martingales $(M^m)_{m\geq 1}$ such that $\mathbb{E}\sup_{0\leq s\leq t}\|M_s - M_s^m\| \to 0$ as $m\to\infty$. In particular then we have that

$$\mathbb{E}\sup_{0\leq s\leq t}\|M_s^m\| \to \mathbb{E}\sup_{0\leq s\leq t}\|M_s\|, \quad m\to\infty, \tag{9.5.5}$$

since $M\mapsto \mathbb{E}\sup_{0\leq s\leq t}\|M_s\|$ defines a norm. Notice that due to the real-valued Burkholder–Davis–Gundy inequality for any fixed $x^*\in X^*$

$$\mathbb{E}[\langle M_t - M_t^m, x^*\rangle]^{\frac12} \eqsim \mathbb{E}\sup_{0\leq s\leq t}|\langle M_s - M_s^m, x^*\rangle| \to 0, \quad m\to\infty,$$

so $[\langle M_t^m - M_t, x^*\rangle]^{\frac12} \to 0$ in L^1-sense. Therefore by Lemma 9.3.15, 9.3.18, and Proposition 9.3.17

$$|\gamma([[M^m]]_t) - \gamma([[M]]_t)| \leq \gamma([[M^m]]_t - [[M]]_t) \lesssim_X \sum_{i=1}^n [\langle M^m - M, x_i^*\rangle]_t \to 0$$

in $L^{\frac12}$, where $(x_i^*)_{i=1}^n \subset X^*$ is as in Lemma 9.3.18. Hence we have that

$$\mathbb{E}\gamma[[M]]_t^{\frac12} = \lim_{n\to\infty}\mathbb{E}\gamma[[M^m]]_t^{\frac12} \overset{(*)}{\eqsim}_X \lim_{n\to\infty}\mathbb{E}\sup_{0\leq s\leq t}\|M_s^m\| \overset{(**)}{=} \mathbb{E}\sup_{0\leq s\leq t}\|M_s\|,$$

where $(*)$ follows from the first part of this case, and $(**)$ holds by (9.5.5).

Step 2: infinite dimensional case. First assume that M is an L^p-bounded martingale. Without loss of generality we can assume X to be separable. Since X is UMD, X is reflexive, so X^* is separable as well. Let $Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset \ldots$ be a family of finite dimensional subspaces of X^* such that $\overline{\cup_n Y_n} = X^*$. For each $n\geq 1$ let $P_n : Y_n \to X^*$ be the inclusion operator. Then $\|P_n^*\| \leq 1$ and $P_n^* M$ is a well-defined Y_n^*-valued L^p-bounded martingale. By Step 1 this martingale a.s. has a covariation bilinear form $[[P_n^* M]]_t$ acting on $Y_n \times Y_n$ and

$$\mathbb{E}\gamma([[P_n^* M]]_t)^p \overset{(*)}{\eqsim}_{p,X} \mathbb{E}\sup_{0\leq s\leq t}\|P_n^* M_s\|^p \leq \mathbb{E}\sup_{0\leq s\leq t}\|M_s\|^p, \tag{9.5.6}$$

where $(*)$ is independent of n due to [79, Proposition 4.2.17]. Note that a.s. $[[P_n^* M]]_t$ and $[[P_m^* M]]_t$ agree for all $m\geq n\geq 1$, i.e. a.s.

$$[[P_m^* M]]_t(x^*, y^*) = [[P_n^* M]]_t(x^*, y^*) = [\langle M, x^*\rangle, \langle M, y^*\rangle]_t, \quad x^*, y^* \in Y_n. \tag{9.5.7}$$

Let $\Omega_0 \subset \Omega$ be a subset of measure 1 such that (9.5.7) holds for all $m\geq n\geq 1$. Fix $\omega\in\Omega_0$. Then by (9.5.7) we can define a bilinear form (not necessarily continuous!) V on $Y\times Y$ (where $Y := \cup_n Y_n \subset X^*$) such that $V(x^*, y^*) = [[P_n^* M]]_t(x^*, y^*)$ for all $x^*, y^* \in Y_n$ and $n\geq 1$.

Let us show that V is continuous (and hence has a continuous extension to $X^* \times X^*$) and $\gamma(V) < \infty$ a.s. on Ω_0. Notice that by Lemma 9.3.7 the sequence $(\gamma([[P_n^* M]]_t))_{n\geq 1}$ is increasing a.s. on Ω_0. Moreover, by the monotone convergence theorem and (9.5.6) $(\gamma([[P_n^* M]]_t))_{n\geq 1}$ has a limit a.s. on Ω_0. Let $\Omega_1 \subset \Omega_0$ be a subset of full measure such that $(\gamma([[P_n^* M]]_t))_{n\geq 1}$ has a limit on Ω_1. Then by (9.3.2) V is continuous on Ω_1 and hence has a continuous extension to $X^* \times X^*$ (which we will denote by V as well for simplicity). Then by Proposition 9.3.8 $\gamma(V) = \lim_{n\to\infty} \gamma([[P_n^* M]]_t)$ monotonically on Ω_1 and hence by monotone convergence theorem ans the fact that $\|P_n^* x\| \to \|x\|$ as $n \to \infty$ monotonically for all $x \in X$

$$\mathbb{E} \sup_{0\leq s\leq t} \|M_s\|^p = \lim_{n\to\infty} \mathbb{E} \sup_{0\leq s\leq t} \|P_n^* M_s\|^p \eqsim_{p,X} \lim_{n\to\infty} \mathbb{E}\gamma([[P_n^* M]]_t)^p = \mathbb{E}(\gamma(V))^p.$$

It remains to show that $V = [[M]]_t$ a.s., i.e. $V(x^*, x^*) = [\langle M, x^*\rangle]_t$ a.s. for any $x^* \in X^*$. If $x^* \in Y$, then the desired follows from the construction of V. Fix $x^* \in X^* \setminus Y$. Since Y is dense in X^*, there exists a Cauchy sequence $(x_n^*)_{n\geq 1}$ in Y converging to x^*. Then since $V(x_n^*, x_n^*) = [\langle M, x_n^*\rangle]_t$ a.s. for all $n \geq 1$,

$$\begin{aligned}\lim_{n\to\infty} |V(x_n^*, x_n^*) - [\langle M, x^*\rangle]_t|^{\frac{p}{2}} &\lesssim_p \lim_{n\to\infty} [\langle M, x^* - x_n^*\rangle]_t^{\frac{p}{2}} \eqsim_p \lim_{n\to\infty} \mathbb{E}|\langle M, x^* - x_n^*\rangle|^p \\ &\leq \lim_{n\to\infty} \mathbb{E}\|M\|^p \|x^* - x_n^*\|^p = 0,\end{aligned}$$

so due to a.s. continuity of V, $V(x^*, x^*)$ and $[\langle M, x^*\rangle]_t$ coincide a.s.

Now let M be a general local martingale. By a stopping time argument we can assume that M is an L^1-bounded martingale, and then the existence of $[[M]]_t$ follows from the case $p = 1$.

Let us now show (9.5.1). If the left-hand side is finite then M is an L^p-bounded martingale and the desired follows from the previous part of the proof. Let the left-hand side be infinite. Then it is sufficient to notice that by Step 1

$$\mathbb{E} \sup_{0\leq s\leq t} \|P_n^* M_s\|^p \eqsim_{p,X} \mathbb{E}\gamma([[P_n^* M]]_t)^p,$$

for any (finite or infinite) left-hand side, and the desired will follow as $n \to \infty$ by the fact that $\|P_n^* M_s\| \to \|M_s\|$ and $\gamma([[P_n^* M]]_t) \to \gamma([[M]]_t)$ monotonically a.s. as $n \to \infty$, and the monotone convergence theorem. □

Remark 9.5.3. Note that X being a UMD Banach space is necessary in Theorem 9.5.1 (see Theorem 9.2.4 and [126]).

Remark 9.5.4. Because of Lemma 9.3.5 the reader may suggest that if X is a UMD Banach space, then for any X-valued local martingale M, for any $t \geq 0$, and for a.a. $\omega \in \Omega$ there exist a natural choice of a Hilbert space $H(\omega)$ and a natural choice of an operator $T(\omega) \in \mathcal{L}(H(\omega), X)$ such that for all $x^*, y^* \in X^*$ a.s.

$$[[M]]_t(x^*, y^*) = \langle T^* x^*, T^* y^*\rangle.$$

If this is the case, then by Lemma 9.3.5 and Theorem 9.5.1

$$\mathbb{E} \sup_{0\le s\le t} \|M_s\|^p \eqsim_{p,X} \mathbb{E}\|T\|^p_{\gamma(H,X)}.$$

Such a natural pair of $H(\omega)$ and $T(\omega)$, $\omega \in \Omega$, is known for purely discontinuous local martingales (see Theorem 9.6.5) and for stochastic integrals (see Subsection 9.7.1 and 9.7.2). Unfortunately, it remains open how such H and T should look like for a general local martingale M.

9.6. RAMIFICATIONS OF THEOREM 9.5.1

Let us outline some ramifications of Theorem 9.5.1.

9.6.1. Continuous and purely discontinuous martingales

In the following theorems we will consider separately the cases of continuous and purely discontinuous martingales. First we show that if M is continuous, then Theorem 9.5.1 holds for the whole range $0 < p < \infty$.

Theorem 9.6.1. *Let X be a UMD Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a continuous local martingale. Then we have that for any $0 < p < \infty$*

$$\mathbb{E} \sup_{0\le s\le t} \|M_s\|^p \eqsim_{p,X} \mathbb{E}\gamma([[M]]_t)^p, \quad t \ge 0. \tag{9.6.1}$$

For the proof we will need the following technical lemma.

Lemma 9.6.2. *Let X be a UMD Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a continuous local martingale. Then the function $t \mapsto \gamma([[M]]_t)$, $t \ge 0$, is continuous a.s.*

Proof. By Pettis measurability theorem [79, Theorem 1.1.20] X can be assumed to be separable, and since X is UMD, it is reflexive, so X^* is separable as well. Therefore there exists a linearly independent set $(x_n^*)_{n\ge1} \subset X^*$ such that $\overline{\operatorname{span}(x_n^*)_{n\ge1}} = X^*$. By [89, Theorem 26.6(iv)] there exists a set $\Omega_0 \in \Omega$ of full measure such that for any $m, n \ge 1$ the function $t \mapsto [\langle M, x_n^*\rangle, \langle M, x_m^*\rangle]_t$, $t \ge 0$, is continuous on Ω_0, and such that $[\langle M, x_n^*\rangle, \langle M, x_m^*\rangle]_t = [[M]]_t(x_n^*, x_m^*)$ on Ω_0 for all $t \ge 0$ (see the construction of $[[M]]$ in the proof of Theorem 9.5.1).

Fix $y^* \in X^*$. Let us show that $t \mapsto [[M]]_t(y^*, y^*)$, $t \ge 0$, is continuous on Ω_0. Let $(y_k^*)_{k\ge1} \in \operatorname{span}(x_n^*)_{n\ge1}$ be such that $y_k^* \to y^*$ as $k \to \infty$. Fix $\omega \in \Omega_0$ and $t \ge 0$. We know that $[[M]]_s$, $0 \le s \le t$, defines a bounded symmetric bilinear form such that $[[M]]_s(x^*, x^*) \ge 0$ for all $x^* \in X^*$. Therefore $x^* \mapsto \sqrt{[[M]]_s(x^*, x^*)}$ defines a Euclidean norm, and hence since $[[M]]_t(x^*, x^*) \ge [[M]]_s(x^*, x^*)$ for all $x^* \in X^*$ and $0 \le s \le t$

$$\begin{aligned}\sup_{0\le s\le t} |[[M]]_s(y^*, y^*) - [[M]]_s(y_k^*, y_k^*)| &\lesssim \sup_{0\le s\le t} |[[M]]_s(y^* - y_k^*, y^* - y_k^*)| \\ &= [[M]]_t(y^* - y_k^*, y^* - y_k^*),\end{aligned} \tag{9.6.2}$$

where the latter vanishes as $k \to \infty$. Notice that $s \mapsto [[M]]_s(y_k^*, y_k^*)$, $0 \le s \le t$, is continuous since $[[M]]_s(y_k^*, y_k^*)$ is a finite linear combination of $[[M]]_s(x_n^*, x_m^*)$ by bilinearity and because $(y_k^*)_{k\ge1} \in \operatorname{span}(x_n^*)_{n\ge1}$. Consequently $s \mapsto [[M]]_s(y^*, y^*)$, $0 \le s \le t$, is continuous as well by (9.6.2).

Therefore we have that $t \mapsto [[M]]_t(y^*, y^*)$, $t \ge 0$, is continuous for all $y^* \in X^*$ on Ω_0, and then continuity of $t \mapsto \gamma([[M]]_t)$ on Ω_0 follows from Lemma 9.3.12. □

Proof of Theorem 9.6.1. The case $p \ge 1$ follows from Theorem 9.5.1. Let us treat the case $0 < p < 1$. First we show that $(\gamma([[M]]_t))_{t\ge0}$ is a predictable process: $(\gamma([[M]]_t))_{t\ge0}$ is a monotone limit of processes $(\gamma([[P_n^* M]]_t))_{t\ge0}$ (where P_n's are as in the proof of Theorem 9.5.1), which are predictable due to the fact that $([[P_n^* M]]_t)_{t\ge0}$ is a $Y_n^* \otimes Y_n^*$-valued predictable process and $\gamma : Y_n^* \otimes Y_n^* \to \mathbb{R}_+$ is a fixed measurable function. Moreover, by Lemma 9.6.2 $(\gamma([[M]]_t))_{t\ge0}$ is continuous a.s., and by Remark 9.4.1 and Lemma 9.3.10 $(\gamma([[M]]_t))_{t\ge0}$ is increasing a.s.

Now since $(\gamma([[M]]_t))_{t\ge0}$ is continuous predictable increasing, (9.6.1) follows from the case $p \ge 1$ and Lenglart's inequality (see [106] and [156, Proposition IV.4.7]). □

Theorem 9.6.3. *Let X be a UMD Banach space, $(M^n)_{n\ge1}$ be a sequence of X-valued continuous local martingales such that $M_0^n = 0$ for all $n \ge 1$. Then $\sup_{t\ge0} \|M_t^n\| \to 0$ in probability as $n \to \infty$ if and only if $\gamma([[M^n]]_\infty) \to 0$ in probability as $n \to \infty$.*

Proof. The proof follows from the classical argument due do Lenglart (see [106]), but we will recall this argument for the convenience of the reader. We will show only one direction, the other direction follows analogously. Fix $\varepsilon, \delta > 0$. For each $n \ge 1$ define a stopping time τ_n in the following way:

$$\tau_n := \inf\{t \ge 0 : M_t^n > \varepsilon\}.$$

Then by (9.5.1) and Chebyshev's inequality

$$\begin{aligned}
\mathbb{P}(\gamma([[M^n]]_\infty) > \delta) &\le \mathbb{P}(\tau_n < \infty) + \mathbb{P}(\gamma([[M^n]]_{\tau_n}) > \delta)\\
&\le \mathbb{P}(\sup_{t\ge0} \|M_t^n\| > \varepsilon) + \delta^{-\frac12} \mathbb{E}\gamma([[M^n]]_{\tau_n})^{\frac12}\\
&\lesssim_X \mathbb{P}(\sup_{t\ge0} \|M_t^n\| > \varepsilon) + \delta^{-\frac12} \mathbb{E}\|M_{\tau_n}^n\|\\
&\le \mathbb{P}(\sup_{t\ge0} \|M_t^n\| > \varepsilon) + \delta^{-\frac12} \varepsilon,
\end{aligned}$$

and the latter vanishes for any fixed $\delta > 0$ as $\varepsilon \to 0$ and $n \to \infty$. □

Remark 9.6.4. Note that Theorem 9.6.3 does not hold for general martingales even in the real-valued case, see [89, Exercise 26.5].

For the next theorem recall that $\ell^2([0,t])$ is the *nonseparable* Hilbert space consisting of all functions $f : [0,t] \to \mathbb{R}$ which support $\{s \in [0,t] : f(s) \neq 0\}$ is countable and $\|f\|_{\ell^2([0,t])} := \sum_{0\le s\le t} |f(s)|^2 < \infty$.

Theorem 9.6.5. *Let X be a UMD Banach space, $1 \le p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous martingale. Then for any $t \ge 0$*

$$\mathbb{E} \sup_{0\le s\le t} \|M_s\|^p \eqsim_{p,X} \mathbb{E}\|(\Delta M_s)_{0\le s\le t}\|^p_{\gamma(\ell^2([0,t]),X)}.$$

Proof. It is sufficient to notice that for any $x^* \in X^*$ a.s.

$$[\langle M, x^*\rangle]_t = \sum_{0\le s\le t} |\langle \Delta M_s, x^*\rangle|^2,$$

and apply Theorem 9.5.1 and Lemma 9.3.5. □

Remark 9.6.6. Note that martingales in Theorem 9.6.1 and 9.6.5 cover all the martingales if X is UMD. More specifically, if X has the UMD property, then any X-valued local martingale M has a unique decomposition $M = M^c + M^d$ into a sum of a continuous local martingale M^c and a purely discontinuous local martingale M^d (see Chapter 4 and 5).

9.6.2. Martingales with independent increments

Here we show that both Theorem 9.2.1 and 9.5.1 hold in much more general Banach spaces given the corresponding martingale has independent increments.

Proposition 9.6.7. *Let X be a Banach space, $(d_n)_{n\ge1}$ be an X-valued martingale difference sequence with independent increments. Then for any $1 < p < \infty$*

$$\mathbb{E}\sup_{m\ge1}\Big\|\sum_{n=1}^m d_n\Big\|^p \lesssim_p \mathbb{E}\|(d_n)_{n=1}^\infty\|^p_{\gamma(\ell^2,X)}.$$

Moreover, if X has a finite cotype, then

$$\mathbb{E}\sup_{m\ge1}\Big\|\sum_{n=1}^m d_n\Big\|^p \eqsim_{p,X} \mathbb{E}\|(d_n)_{n=1}^\infty\|^p_{\gamma(\ell^2,X)}.$$

Proof. Let $(r_n)_{n\ge1}$ be a sequence of independent Rademacher random variables, $(\gamma_n)_{n\ge1}$ be a sequence of independent standard Gaussian random variables. Then

$$\begin{aligned}\mathbb{E}\sup_{m\ge1}\Big\|\sum_{n=1}^m d_n\Big\|^p &\overset{(i)}{\eqsim}_p \mathbb{E}\Big\|\sum_{n=1}^\infty d_n\Big\|^p \overset{(ii)}{\eqsim}_p \mathbb{E}\mathbb{E}_r\Big\|\sum_{n=1}^N r_n d_n\Big\|^p\\ &\overset{(iii)}{\lesssim}_p \mathbb{E}\mathbb{E}_\gamma\Big\|\sum_{n=1}^N \gamma_n d_n\Big\|^p \overset{(iv)}{\eqsim}_p \mathbb{E}\Big(\mathbb{E}_\gamma\Big\|\sum_{n=1}^N \gamma_n d_n\Big\|^2\Big)^{\frac{p}{2}}\\ &= \mathbb{E}\|(d_n)_{n=1}^\infty\|^p_{\gamma(\ell^2,X)},\end{aligned}$$

where (i) follows from (2.2.1), (ii) follows from [104, Lemma 6.3], (iii) holds by [80, Proposition 6.3.2], and finally (iv) follows from [80, Proposition 6.3.1].

If X has a finite cotype, then one has $\eqsim_{p,X}$ instead of $\lesssim_p$ in (iii) (see [80, Corollary 7.2.10]), and the second part of the proposition follows. □

Based on Proposition 9.6.7 and the proof of Theorem 9.5.1 one can show the following assertion. Notice that we presume the reflexivity of X since it was assumed in the whole Section 9.3.

Proposition 9.6.8. *Let X be a reflexive Banach space, $1 \le p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be an L^p-bounded martingale with independent increments such that $M_0 = 0$. Let $t \ge 0$. If M has a covariation bilinear form $[[M]]_t$ at t, then*

$$\mathbb{E} \sup_{0\le s\le t} \|M_s\|^p \lesssim_{p,X} \mathbb{E}\gamma([[M]]_t)^p.$$

Moreover, if X has a finite cotype, then the existence of $[[M]]_t$ is guaranteed, and

$$\mathbb{E} \sup_{0\le s\le t} \|M_s\|^p \eqsim_{p,X} \mathbb{E}\gamma([[M]]_t)^p.$$

9.7. APPLICATIONS AND MISCELLANEA

Here we provide further applications of Theorem 9.5.1.

9.7.1. Itô isomorphism: general martingales

Let H be a Hilbert space, X be a Banach space. For each $x \in X$ and $h \in H$ we denote the linear operator $g \mapsto \langle g, h\rangle x$, $g \in H$, by $h \otimes x$. The process $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H, X)$ is called *elementary predictable* with respect to the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\ge 0}$ if it is of the form

$$\Phi(t,\omega) = \sum_{k=1}^{K}\sum_{m=1}^{M} \mathbf{1}_{(t_{k-1},t_k]\times B_{mk}}(t,\omega) \sum_{n=1}^{N} h_n \otimes x_{kmn}, \quad t \ge 0,\ \omega \in \Omega,$$

where $0 = t_0 < \ldots < t_K < \infty$, for each $k = 1, \ldots, K$ the sets $B_{1k}, \ldots, B_{Mk}$ are in $\mathscr{F}_{t_{k-1}}$ and the vectors $h_1, \ldots, h_N$ are in H. Let $\widetilde{M} : \mathbb{R}_+ \times \Omega \to H$ be a local martingale. Then we define the *stochastic integral* $\Phi \cdot \widetilde{M} : \mathbb{R}_+ \times \Omega \to X$ of Φ with respect to $\widetilde{M}$ as follows:

$$(\Phi\cdot\widetilde{M})_t := \sum_{k=1}^{K}\sum_{m=1}^{M} \mathbf{1}_{B_{mk}} \sum_{n=1}^{N} \langle(\widetilde{M}(t_k\wedge t) - \widetilde{M}(t_{k-1}\wedge t)), h_n\rangle x_{kmn}, \quad t \ge 0. \tag{9.7.1}$$

Notice that for any $t \ge 0$ the stochastic integral $\Phi\cdot\widetilde{M}$ obtains a covariation bilinear form $[[\Phi\cdot\widetilde{M}]]_t$ which is a.s. continuous on $X^* \times X^*$ and which has the following form due to (2.2.6) and (9.7.1)

$$\begin{aligned}[[\Phi\cdot\widetilde{M}]]_t(x^*, x^*) &= \Big[\Big\langle \int_0^{\cdot} \Phi \,\mathrm{d}\widetilde{M}, x^*\Big\rangle\Big]_t = \Big[\int_0^{\cdot} (\Phi^* x^*)^* \,\mathrm{d}\widetilde{M}\Big]_t \\ &= \int_0^t \|q_{\widetilde{M}}^{1/2}(s)\Phi^*(s)x^*\|^2 \,\mathrm{d}[\widetilde{M}]_s, \quad t \ge 0.\end{aligned} \tag{9.7.2}$$

Remark 9.7.1. If $X = \mathbb{R}$, then by the real-valued Burkholder–Davis–Gundy inequality and the fact that for any elementary predictable Φ

$$\Big[\int_0^{\cdot} \Phi \,\mathrm{d}\widetilde{M}\Big]_t = \int_0^t \|q_{\widetilde{M}}^{1/2}(s)\Phi^*(s)\|^2 \,\mathrm{d}[\widetilde{M}]_s, \quad t \ge 0,$$

one has an isomorphism

$$\mathbb{E}\sup_{t\geq 0}|(\Phi\cdot\widetilde{M})_t| \eqsim \mathbb{E}\Big(\int_0^\infty \|q_{\widetilde{M}}^{1/2}(s)\Phi(s)\|^2\,\mathrm{d}[\widetilde{M}]_s\Big)^{\frac{1}{2}},$$

so one can extend the definition of a stochastic integral to *all* predictable $\Phi:\mathbb{R}_+\times\Omega\to H$ with

$$\mathbb{E}\Big(\int_0^\infty \|q_{\widetilde{M}}^{1/2}(s)\Phi(s)\|^2\,\mathrm{d}[\widetilde{M}]_s\Big)^{\frac{1}{2}} < \infty, \tag{9.7.3}$$

by extending the stochastic integral operator from a dense subspace of all elementary predictable processes satisfying (9.7.3). We refer the reader to [89, 121, 123] for details.

Remark 9.7.2. Let $X=\mathbb{R}^d$ for some $d\geq 1$. Then analogously to Remark 9.7.1 one can extend the definition of a stochastic integral to all predictable processes $\Phi:\mathbb{R}_+\times\Omega\to\mathscr{L}(H,\mathbb{R}^d)$ with

$$\begin{aligned}\mathbb{E}\Big(\sum_{n=1}^d\int_0^\infty \|q_{\widetilde{M}}^{1/2}(s)\Phi^*(s)e_n\|^2\,\mathrm{d}[\widetilde{M}]_s\Big)^{\frac{1}{2}} &= \mathbb{E}\|q_{\widetilde{M}}^{1/2}\Phi^*\|_{HS(\mathbb{R}^d,L^2(\mathbb{R}_+;[\widetilde{M}]))}\\ &= \mathbb{E}\|\Phi q_{\widetilde{M}}^{1/2}\|_{HS(L^2(\mathbb{R}_+;[\widetilde{M}]),\mathbb{R}^d)} < \infty,\end{aligned}$$

where $(e_n)_{n=1}^d$ is a basis of $\mathbb{R}^d$, $\|T\|_{HS(H_1,H_2)}$ is the Hilbert-Schmidt norm of an operator T acting form a Hilbert space H_1 to a Hilbert space H_2, and $L^2(\mathbb{R}_+;A)$ for a given increasing $A:\mathbb{R}_+\to\mathbb{R}$ is a Hilbert space of all functions $f:\mathbb{R}_+\to\mathbb{R}$ such that $\int_{\mathbb{R}_+}\|f(s)\|^2\,\mathrm{d}A(s)<\infty$.

Now we present the Itô isomorphism for vector-valued stochastic integrals with respect to general martingales, which extends [126, 175, 177].

Theorem 9.7.3. *Let H be a Hilbert space, X be a UMD Banach space, $\widetilde{M}:\mathbb{R}_+\times\Omega\to H$ be a local martingale, $\Phi:\mathbb{R}_+\times\Omega\to\mathscr{L}(H,X)$ be elementary predictable. Then for all $1\leq p<\infty$*

$$\mathbb{E}\sup_{0\leq s\leq t}\Big\|\int_0^s\Phi\,\mathrm{d}\widetilde{M}\Big\|^p \eqsim_{p,X} \mathbb{E}\|\Phi q_{\widetilde{M}}^{1/2}\|^p_{\gamma(L^2([0,t],[\widetilde{M}];H),X)},\quad t\geq 0,$$

where $[\widetilde{M}]$ is the quadratic variation of $\widetilde{M}$, $q_{\widetilde{M}}$ is the quadratic variation derivative (see Section 2.2.1), and $\|\Phi q_{\widetilde{M}}^{1/2}\|^p_{\gamma(L^2([0,t],[\widetilde{M}];H),X)}$ is the γ-norm (see (2.9.1)*).*

Proof. Fix $t\geq 0$. Then the theorem holds by Theorem 9.5.1, Lemma 9.3.5, and the fact that by (9.7.2) for any fixed $x^*\in X^*$ a.s.

$$\begin{aligned}\Big[\Big\langle\int_0^\cdot\Phi\,\mathrm{d}\widetilde{M},x^*\Big\rangle\Big]_t = \Big[\int_0^\cdot\langle\Phi,x^*\rangle\,\mathrm{d}\widetilde{M}\Big]_t &= \int_0^t\|q_M^{\frac{1}{2}}\Phi^*x^*\|^2\,\mathrm{d}[\widetilde{M}]_s\\ &= \|q_M^{\frac{1}{2}}\Phi^*x^*\|^2_{L^2([0,t],[\widetilde{M}];H)}.\end{aligned}$$

□

Theorem 9.7.3 allows us to provide the following general stochastic integration result. Recall that a predictable process $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H,X)$ is called *strongly predictable* if there exists a sequence $(\Phi_n)_{n\geq 1}$ of elementary predictable $\mathscr{L}(H,X)$-valued processes such that Φ is a pointwise limit of $(\Phi_n)_{n\geq 1}$.

Corollary 9.7.4. *Let H be a Hilbert space, X be a UMD Banach space, $\widetilde{M} : \mathbb{R}_+ \times \Omega \to H$ be a local martingale, $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(H,X)$ be strongly predictable such that $\mathbb{E}\|\Phi q_{\widetilde{M}}^{1/2}\|_{\gamma(L^2(\mathbb{R}_+,[\widetilde{M}];H),X)} < \infty$. Then there exists a martingale $\Phi \cdot \widetilde{M}$ which coincides with the stochastic integral given Φ is elementary predictable such that*

$$\langle \Phi \cdot \widetilde{M}, x^* \rangle = (\Phi^* x^*) \cdot \widetilde{M}, \quad x^* \in X^*, \tag{9.7.4}$$

where the latter integral is defined as in Remark 9.7.1. Moreover, then we have that for any $1 \leq p < \infty$

$$\mathbb{E} \sup_{t\geq 0} \|(\Phi \cdot \widetilde{M})_t\|^p \eqsim_{p,X} \mathbb{E}\|\Phi q_{\widetilde{M}}^{1/2}\|^p_{\gamma(L^2(\mathbb{R}_+,[\widetilde{M}];H),X)}. \tag{9.7.5}$$

For the proof we will need the following technical lemma.

Lemma 9.7.5. *Let X be a reflexive separable Banach space, $Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset \ldots \subset X^*$ be finite dimensional subspaces such that $\overline{\cup_n Y_n} = X^*$. Let $P_n : Y_n \hookrightarrow X^*$, $n \geq 1$, and $P_{n,m} : Y_n \hookrightarrow Y_m$, $m \geq n \geq 1$, be the inclusion operators. For each $n \geq 1$ let $x_n \in Y_n^*$ be such that $P_{n,m}^* x_m = x_n$ for all $m \geq n \geq 1$. Assume also that $\sup_n \|x_n\| < \infty$. Then there exists $x \in X$ such that $P_n^* x = x_n$ for all $n \geq 1$ and $\|x\| = \lim_{n\to\infty} \|x_n\|$ monotonically.*

Proof. Set $C = \sup_n \|x_n\|$. First notice that $(x_n)_{n\geq 1}$ defines a bounded linear functional on $Y = \cup_n Y_n$. Indeed, fix $y \in Y_n$ for some fixed $n \geq 1$ (then automatically $y \in Y_m$ for any $m \geq n$). Define $\ell(y) = \langle x_n, y\rangle$. Then this definition of ℓ agrees for different n's since for any $m \geq n$ we have that

$$\langle x_m, y_n\rangle = \langle x_m, P_{n,m} y_n\rangle = \langle P_{n,m}^* x_m, y_n\rangle = \langle x_n, y_n\rangle.$$

Moreover, this linear functional is bounded since $|\langle x_n, y_n\rangle| \leq \|x_n\|\|y_n\| \leq C\|y_n\|$. So, it can be continuously extended to the whole space X^*. Since X is reflexive, there exists $x \in X$ such that $\ell(x^*) = \langle x^*, x\rangle$ for any $x^* \in X^*$. Then for any fixed $n \geq 1$ and for any $y \in Y_n$ we have that

$$\langle x_n, y\rangle = \ell(y) = \langle x, y\rangle = \langle x, P_n y\rangle = \langle P_n^* x, y\rangle,$$

so $P_n^* x = x_n$. The latter follows from the fact that $\|P_n^* x\| \to \|x\|$ monotonically as $n \to \infty$ for any $x \in X$. □

Proof of Corollary 9.7.4. We will first consider the finite dimensional case and then deduce the infinite dimensional case.

Finite dimensional case. Since X is finite dimensional, it is isomorphic to a finite dimensional Euclidean space, and so the γ-norm is equivalent to the Hilbert-Schmidt norm (see e.g. [80, Proposition 9.1.9]). Then Φ is stochastically integrable with respect to $\widetilde{M}$ due to Remark 9.7.2, so (9.7.4) clearly holds and we have that for any $x^* \in X^*$ a.s.

$$[\langle \Phi \cdot \widetilde{M}, x^* \rangle]_t = [(\Phi^* x^*) \cdot \widetilde{M}]_t = \int_0^t \| q_{\widetilde{M}}^{1/2}(s) \Phi^*(s) x^* \|^2 \, \mathrm{d}[\widetilde{M}]_s, \quad t \geq 0,$$

thus (9.7.5) follows from Theorem 9.5.1 and Lemma 9.3.5.

Infinite dimensional case. Let now X be general. Since Φ is strongly predictable, it takes values in a separable subspace of X, so we may assume that X is separable. Since X is UMD, it is reflexive, so X^* is separable as well, and there exists a sequence $Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset \ldots \subset X^*$ of finite dimensional subsets of X^* such that $\overline{\cup_n Y_n} = X^*$. For each $m \geq n \geq 1$ define inclusion operators $P_n : Y_n \hookrightarrow X^*$ and $P_{n,m} : Y_n \hookrightarrow Y_m$. Notice that by the ideal property [80, Theorem 9.1.10] $\mathbb{E}\|P_n^* \Phi q_{\widetilde{M}}^{1/2}\|_{\gamma(L^2(\mathbb{R}_+, [\widetilde{M}]; H), Y_n^*)} < \infty$ for any $n \geq 1$, so since Y_n^* is finite dimensional, the stochastic integral $(P_n^* \Phi) \cdot \widetilde{M}$ is well-defined by the case above and

$$\mathbb{E} \sup_{t \geq 0} \big\| \big((P_n^* \Phi) \cdot \widetilde{M} \big)_t \big\| \eqsim_X \mathbb{E} \|P_n^* \Phi q_{\widetilde{M}}^{1/2}\|_{\gamma(L^2(\mathbb{R}_+, [\widetilde{M}]; H), Y_n^*)}, \tag{9.7.6}$$

where the equivalence is independent of n since $Y_n \subset X^*$ for all $n \geq 1$ and due to [79, Proposition 4.2.17] and Theorem 9.5.1. Denote the stochastic integral $(P_n^* \Phi) \cdot \widetilde{M}$ by Z^n. Note that Z^n is Y_n^*-valued, and since $P_{n,m}^* P_m^* \Phi = P_n^* \Phi$ for all $m \geq n \geq 1$, $P_{m,n}^* Z_t^m = Z_t^n$ a.s. for any $t \geq 0$. Therefore by Lemma 9.7.5 there exists a process $Z : \mathbb{R}_+ \times \Omega \to X$ such that $P_n^* Z = Z^n$ for all $n \geq 1$. Let us show that Z is integrable. Fix $t \geq 1$. Notice that by Lemma 9.7.5 the limit $\|Z_t\| = \lim_{n \to \infty} \|P_n^* Z_t\| = \lim_{n \to \infty} \|Z_t^n\|$ is monotone, so by the monotone convergence theorem, (9.7.6), and the ideal property [80, Theorem 9.1.10]

$$\begin{aligned} \mathbb{E}\|Z_t\| = \lim_{n \to \infty} \mathbb{E}\|Z_t^n\| &\lesssim_X \limsup_{n \to \infty} \mathbb{E}\|P_n^* \Phi q_{\widetilde{M}}^{1/2}\|_{\gamma(L^2(\mathbb{R}_+, [\widetilde{M}]; H), Y_n^*)} \\ &\leq \mathbb{E}\|\Phi q_{\widetilde{M}}^{1/2}\|_{\gamma(L^2(\mathbb{R}_+, [\widetilde{M}]; H), X)}. \end{aligned}$$

Now let us show that Z is a martingale. Since Z is integrable, due to [79, Section 2.6] it is sufficient to show that $\mathbb{E}(\langle Z_t, x^* \rangle | \mathscr{F}_s) = \langle Z_s, x^* \rangle$ for all $0 \leq s \leq t$ for all x^* from some dense subspace Y of X^*. Set $Y = \cup_n Y_n$ and $x^* \in Y_n$ for some $n \geq 1$. Then for all $0 \leq s \leq t$

$$\begin{aligned} \mathbb{E}(\langle Z_t, x^* \rangle | \mathscr{F}_s) &= \mathbb{E}(\langle Z_t, P_n x^* \rangle | \mathscr{F}_s) = \mathbb{E}(\langle P_n^* Z_t, x^* \rangle | \mathscr{F}_s) \\ &= \mathbb{E}(\langle Z_t^n, x^* \rangle | \mathscr{F}_s) = \langle Z_s^n, x^* \rangle = \langle Z_s, x^* \rangle, \end{aligned}$$

so Z is a martingale. Finally, let us show (9.7.5). First notice that for any $n \geq 1$ and $x^* \in Y_n \subset X^*$ a.s.

$$[\langle Z, x^* \rangle]_t = [\langle Z^n, x^* \rangle]_t = \int_0^t \| q_{\widetilde{M}}^{1/2}(s) \Phi^*(s) x^* \|^2 \, \mathrm{d}[\widetilde{M}]_s, \quad t \geq 0;$$

the same holds for a general $x^* \in X^*$ by a density argument. Then (9.7.5) follows from Theorem 9.5.1 and Lemma 9.3.5. □

Remark 9.7.6. As the reader can judge, the basic assumptions on Φ in Corollary 9.7.4 can be weakened by a stopping time argument. Namely, one can assume that $\Phi q_{\widetilde{M}}^{1/2}$ is *locally* in $L^1(\Omega, \gamma(L^2(\mathbb{R}_+, [\widetilde{M}]; H), X))$ (i.e. there exists an increasing sequence $(\tau_n)_{n\geq 1}$ of stopping times such that $\tau_n \to \infty$ a.s. as $n \to \infty$ and $\Phi q_{\widetilde{M}}^{1/2}\mathbf{1}_{[0,\tau_n]}$ is in $L^1(\Omega, \gamma(L^2(\mathbb{R}_+, [\widetilde{M}]; H), X))$ for all $n \geq 1$). Notice that such an assumption is a natural generalization of classical assumptions for stochastic integration in the real-valued case (see e.g. [89, p. 526]).

9.7.2. Itô isomorphism: Poisson and general random measures

Let $(J, \mathscr{J})$ be a measurable space, N be a Poisson random measure on $J \times \mathbb{R}_+$, $\widetilde{N}$ be the corresponding compensated Poisson random measure (see e.g. [51, 89, 95, 165] and Section 2.8 for details). Then by Theorem 9.6.5 for any UMD Banach space X, for any $1 \leq p < \infty$, and for any elementary predictable $F : J \times R_+ \times \Omega \to X$ we have that

$$\mathbb{E} \sup_{0\leq s\leq t} \Big\| \int_{J\times[0,s]} F \, d\widetilde{N} \Big\|^p \eqsim_{p,X} \mathbb{E}\|F\|^p_{\gamma(L^2(J\times[0,t];N),X)}, \quad t \geq 0. \tag{9.7.7}$$

The same holds for a general quasi-left continuous random measure (see Section 2.8 for the definition and the details): if μ is a general quasi-left continuous random measure on $J \times \mathbb{R}_+$, ν is its compensator, and $\bar{\mu} := \mu - \nu$, then for any $1 \leq p < \infty$

$$\mathbb{E} \sup_{0\leq s\leq t} \Big\| \int_{J\times[0,t]} F \, d\bar{\mu} \Big\|^p \eqsim_{p,X} \mathbb{E}\|F\|^p_{\gamma(L^2(J\times[0,t];\mu),X)}, \quad t \geq 0. \tag{9.7.8}$$

The disadvantage of right-hand sides of (9.7.7) and (9.7.8) is that both of them are not predictable and do not depend continuously on time a.s. on Ω (therefore they seem not to be useful from the SPDE's point of view since one may not produce a fixed point argument). For example, if $X = L^q$ for some $1 < q < \infty$, then such predictable a.s. continuous in time right-hand sides do exist (see [51] and Chapter 7).

9.7.3. Necessity of the UMD property

As it follows from Remark 9.5.3, Theorem 9.5.1 holds only in the UMD setting. The natural question is whether there exists an appropriate right-hand side of (9.5.1) in terms of $([\langle M, x^*\rangle, \langle M, y^*\rangle])_{x^*,y^*\in X^*}$ for some non-UMD Banach space X and some $1 \leq p < \infty$. Here we show that this is impossible.

Assume that for some Banach space X and some $1 \leq p < \infty$ there exists a function G acting on families of stochastic processes parametrized by $X^* \times X^*$ (i.e. each family has the form $V = (V_{x^*,y^*})_{x^*,y^*\in X^*}$) taking values in $\mathbb{R}$ such that for any X-valued local martingale M starting in zero we have that

$$\mathbb{E} \sup_{t\geq 0} \|M_t\|^p \eqsim_{p,X} G([[M]]), \tag{9.7.9}$$

where we denote $[[M]] = ([\langle M,x^*\rangle,\langle M,y^*\rangle])_{x^*,y^*\in X^*}$ for simplicity (note that the latter might not have a proper bilinear structure). Let us show that then X must have the UMD property.

Fix any X-valued L^p-bounded martingale difference sequence $(d_n)_{n=1}^N$ and any $\{-1,1\}$-valued sequence $(\varepsilon_n)_{n=1}^N$. Let $e_n := \varepsilon_n d_n$ for all $n = 1,\dots,N$. For every $x^*, y^* \in X^*$ define a stochastic process $V_{x^*,y^*} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ as

$$V_{x^*}(t) = \sum_{n=1}^{N\wedge[t]} \langle d_n, x^*\rangle \cdot \langle d_n, y^*\rangle = \sum_{n=1}^{N\wedge[t]} \langle e_n, x^*\rangle \cdot \langle e_n, y^*\rangle, \quad t \geq 0$$

(recall that $[t]$ is the integer part of t). Let $V := (V_{x^*,y^*})_{x^*,y^*\in X^*}$. Then by (9.7.9)

$$\mathbb{E}\sup_{k\geq 0}\Big\|\sum_{n=1}^{k} d_n\Big\|^p \eqsim_{p,X} G(V) \eqsim_{p,X} \mathbb{E}\sup_{k\geq 0}\Big\|\sum_{n=1}^{k} e_n\Big\|^p. \tag{9.7.10}$$

Since N, $(d_n)_{n=1}^N$, and $(\varepsilon_n)_{n=1}^N$ are general, (9.7.10) implies that X is a UMD Banach space (see the proof of Theorem 9.2.4).

9.7.4. Martingale domination

The next theorem shows that under some natural domination assumptions on martingales one gets L^p-estimates.

Theorem 9.7.7. *Let X be a UMD Banach space, $M, N : \mathbb{R}_+ \times \Omega \to X$ be local martingales such that $\|N_0\| \leq \|M_0\|$ a.s. and $[\langle N,x^*\rangle]_\infty \leq [\langle M,x^*\rangle]_\infty$ a.s. for all $x^* \in X^*$. Then for all $1 \leq p < \infty$*

$$\mathbb{E}\sup_{t\geq 0}\|N_t\|^p \lesssim_{p,X} \mathbb{E}\sup_{t\geq 0}\|M_t\|^p. \tag{9.7.11}$$

Note that the assumptions in Theorem 9.7.7 are a way more general than the weak differential subordination assumptions, so Theorem 9.7.7 significantly improves Theorem 4.4.1 and 6.4.26, and extends these results to the case $p = 1$ as well.

Proof of Theorem 9.7.7. First notice that by a triangular inequality

$$\mathbb{E}\sup_{t\geq 0}\|M_t\|^p \eqsim_p \mathbb{E}\|M_0\|^p + \mathbb{E}\sup_{t\geq 0}\|M_t - M_0\|^p,$$

$$\mathbb{E}\sup_{t\geq 0}\|N_t\|^p \eqsim_p \mathbb{E}\|N_0\|^p + \mathbb{E}\sup_{t\geq 0}\|N_t - N_0\|^p.$$

Consequently we can reduce the statement to the case $M_0 = N_0 = 0$ a.s. (by setting $M := M - M_0$, $N := N - N_0$), and then the proof follows directly from Theorem 9.5.1 and Lemma 9.3.10. □

Remark 9.7.8. It is not known what the sharp constant is in (9.7.11). Nevertheless, sharp inequalities of such type have been discovered in the scalar case by

Osękowski in [137]. It was shown there that if M and N are real-valued L^p-bounded martingales such that a.s.

$$[N]_t \leq [M]_t, \quad t \geq 0, \quad \text{if } 1 < p \leq 2,$$

$$[N]_\infty - [N]_{t-} \leq [M]_\infty - [M]_{t-}, \quad t \geq 0, \quad \text{if } 2 \leq p < \infty,$$

then

$$(\mathbb{E}|N_\infty|^p)^{\frac{1}{p}} \leq (p^* - 1)(\mathbb{E}|M_\infty|^p)^{\frac{1}{p}}, \quad 1 < p < \infty,$$

where $p^* := \max\{p, \frac{p}{p-1}\}$.

9.7.5. Martingale approximations

The current subsection is devoted to approximation of martingales. Namely, we will extend Lemma 9.5.2 by Weisz (see [180, Theorem 6]) to general UMD Banach space-valued martingales. Here is the main theorem of the current subsection.

Theorem 9.7.9. *Let X be a UMD Banach space, $1 \leq p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be a martingale such that $\mathbb{E}\sup_{t\geq 0} \|M_t\|^p < \infty$. Then there exists a sequence $(M^n)_{n\geq 1}$ of X-valued L^∞-bounded martingales such that $\mathbb{E}\sup_{t\geq 0} \|M_t - M_t^n\|^p \to 0$ as $n \to \infty$.*

In order to prove Theorem 9.7.9 we will need to show similar approximation results for quasi-left continuous purely discontinuous martingales and purely discontinuous martingales with accessible jumps. Both cases will be considered separately.

QUASI-LEFT CONTINUOUS PURELY DISCONTINUOUS MARTINGALES

Before stating the corresponding approximation theorem let us show the following proposition.

Proposition 9.7.10. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous quasi-left continuous martingale. Then there exist sequences of positive numbers $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, and a sequence of X-valued purely discontinuous quasi-left continuous martingales $(M^n)_{n\geq 1}$ such that*

$$\sup_t \|\Delta M_t^n\| \leq a_n, \quad \#\{t \geq 0 : \Delta M_t^n \neq 0\} \leq b_n \text{ a.s. } \forall n \geq 1,$$

$$\{t \geq 0 : \Delta M_t^n \neq 0\} \subset \{t \geq 0 : \Delta M_t^m \neq 0\} \text{ a.s. } \forall m \geq n \geq 1, \tag{9.7.12}$$

$$\Delta M_t^n = \Delta M_t \quad \forall t \geq 0 \text{ s.t. } \Delta M_t^n \neq 0 \text{ a.s. } \forall n \geq 1, \tag{9.7.13}$$

and

$$\cup_{n\geq 1} \{t \geq 0 : \Delta M_t^n \neq 0\} = \{t \geq 0 : \Delta M_t \neq 0\} \text{ a.s.} \tag{9.7.14}$$

Sketch of the proof. Let μ^M be a random measure defined on $(\mathbb{R}_+ \times X, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(X))$ by

$$\mu^M(A \times B) = \sum_{t \in A} \mathbf{1}_{\Delta M_t \in B \setminus \{0\}}, \quad A \in \mathscr{B}(\mathbb{R}_+), B \in \mathscr{B}(X).$$

Let ν^M be the corresponding compensator, $\bar{\mu}^M := \mu^M - \nu^M$. Due to the proof of Lemma 7.5.11 there exists an a.s. increasing sequence $(\tau_n)_{n\geq 1}$ of stopping times such $\tau_n \to \infty$ a.s. as $n \to \infty$, and such = that there exist positive sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ with $(a_n)_{n\geq 1}$ being increasing natural and with

$$\#\{t \geq 0 : \|\Delta M_t^{\tau_n}\| \in [1/a_n, a_n]\} \leq b_n.$$

Define a predictable set $A_n := [0, \tau_n] \times B_n \subset \mathbb{R}_+ \times X$, where $B_n := \{x \in X : \|x\| \in [1/a_n, a_n]\}$. Then the desired M^n equals the stochastic integral

$$M_t^n := \int_{[0,t]\times X} \mathbf{1}_{A_n}(s,x) x \, \mathrm{d}\bar{\mu}^M(\,\mathrm{d}s,\,\mathrm{d}x), \quad t \geq 0,$$

where the latter is a well-defined martingale since by Section 2.8 it is sufficient to check that for any $t \geq 0$

$$\begin{aligned}\int_{[0,t]\times X} \|\mathbf{1}_{A_n}(s,x)x\| \,\mathrm{d}\mu^M(\,\mathrm{d}s,\,\mathrm{d}x) &= \int_{A_n \cap [0,t]\times X} \|x\| \,\mathrm{d}\mu^M(\,\mathrm{d}s,\,\mathrm{d}x) \\ &= \sum_{t \in [0, \tau_n \wedge t]} \|\Delta M_t^{\tau_n}\| \mathbf{1}_{\Delta M_t^{\tau_n} \in [1/a_n, a_n]} \leq a_n b_n < \infty.\end{aligned}$$

All the properties of the sequence $(M^n)_{n\geq 1}$ then follow from the construction, namely from the fact that A_n are a.s. increasing with $\cup_n A_n = \mathbb{R}_+ \times X \setminus \{0\}$ a.s., and the fact that ν^M is non-atomic in time since M is quasi-left continuous (see Section 2.8 and Subsection 7.5.4). □

In the next theorem we show that the martingales obtained in Proposition 9.7.10 approximate M in the strong L^p-sense.

Theorem 9.7.11. *Let X be a UMD Banach space, M be an X-valued martingale, $(M^n)_{n\geq 1}$ be a sequence of X-valued martingales constructed in Proposition 9.7.10. Assume that for some fixed $1 \leq p < \infty$, $\mathbb{E}\sup_{t\geq 0} \|M_t\|^p < \infty$. Then $\mathbb{E}\sup_{t\geq 0} \|M_t^n\|^p < \infty$ for all $n \geq 1$ and*

$$\mathbb{E}\sup_{t\geq 0} \|M_t - M_t^n\|^p \to 0, \quad n \to \infty.$$

Proof. First of all notice that by Theorem 9.6.5, (9.7.13), and [80, Proposition 6.1.5] for any $n \geq 1$

$$\begin{aligned}\mathbb{E}\sup_{t\geq 0} \|M_t^n\|^p &\eqsim_{p,X} \mathbb{E}\Big(\mathbb{E}_\gamma \Big\| \sum_{t\geq 0} \gamma_s \Delta M_s^n \Big\|^2\Big)^{\frac{p}{2}} \\ &\leq \mathbb{E}\Big(\mathbb{E}_\gamma \Big\| \sum_{t\geq 0} \gamma_s \Delta M_s \Big\|^2\Big)^{\frac{p}{2}} \eqsim_{p,X} \mathbb{E}\sup_{t\geq 0} \|M_t\|^p.\end{aligned}$$

Let us show the second part of the theorem. Note that by (9.7.13) a.s. for all $x^* \in X^*$

$$[[M - M^n]]_\infty(x^*, x^*) = \sum_{t\geq 0} \langle \Delta M_t, x^*\rangle^2 \mathbf{1}_{\Delta M_t \neq \Delta M_t^n},$$

which monotonically vanishes as $n\to\infty$ by (9.7.12) and (9.7.14). Consequently, the desired follows form Theorem 9.5.1, Lemma 9.3.11, and the monotone convenience theorem. □

PURELY DISCONTINUOUS MARTINGALES WITH ACCESSIBLE JUMPS

Now let us turn to purely discontinuous martingales with accessible jumps.

Let X be a Banach space $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous L^p-bounded martingale with accessible jumps, $(\tau_n)_{n\geq 0}$ be a set of predictable stopping times with disjoint graphs such that (2.4.4) holds. Thanks to Lemma 2.4.5 for each $n \geq 1$ we can define a martingale

$$M_t^n = \sum_{i=1}^n \Delta M_{\tau_i} \mathbf{1}_{[0,t]}(\tau_i), \quad t\geq 0. \tag{9.7.15}$$

Does $(M^n)_{n\geq 1}$ converge to M in strong L^p-sense? The following theorem answers this question in the UMD case.

Theorem 9.7.12. *Let X be a UMD Banach space, $M : \mathbb{R}_+ \times \Omega \to X$ be a martingale with accessible jumps, $(M^n)_{n\geq 1}$ be as in* (9.7.15)*. Assume that $\mathbb{E}\sup_{t\geq 0} \|M_t\|^p < \infty$ for some fixed $1 \leq p < \infty$. Then $\mathbb{E}\sup_{t\geq 0} \|M_t^n\|^p < \infty$ for all $n\geq 1$ and*

$$\mathbb{E}\sup_{t\geq 0} \|M_t - M_t^n\|^p \to 0, \quad n\to\infty.$$

Proof. The proof is fully analogous to the proof of Theorem 9.7.11. □

PROOF OF THEOREM 9.7.9

Let us now prove Theorem 9.7.9. Since X is a UMD Banach space, M has the canonical decomposition, i.e. there exist an X-valued continuous local martingale M^c, an X-valued purely discontinuous quasi-left continuous local martingale M^q, and an X-valued purely discontinuous local martingale M^a with accessible jumps such that $M_0^c = M_0^q = 0$ and $M = M^c + M^q + M^a$ (see Chapter 4 and 5 for details). Moreover, by (9.7.17) and a triangle inequality

$$\mathbb{E}\sup_{t\geq 0}\Big(\|M_t^c\|^p + \|M_t^q\|^p + \|M_t^a\|^p\Big) \eqsim_{p,X} \mathbb{E}\sup_{t\geq 0}\|M_t\|^p,$$

so it is sufficient to show Theorem 9.7.9 for each of these three cases separately. By [79, Theorem 1.3.2 and 3.3.16] M converges a.s., so we can assume that there exists $T > 0$ such that $M_t = M_T$ a.s. for all $t \geq T$.

Case 1: M is continuous. The theorem follows from the fact that every continuous martingale is locally bounded and the fact that $M_t = M_T$ for all $t \geq T$.

Case 2: M is purely discontinuous quasi-left continuous. By Theorem 9.7.11 one can assume that M has uniformly bounded jumps. Then the theorem follows from the fact that any adapted càdlàg process with uniformly bounded jumps is local uniformly bounded and the fact that $M_t = M_T$ for all $t \geq T$.

Case 3: M is purely discontinuous with accessible jumps. By Theorem 9.7.12 we can assume that there exist predictable stopping times $(\tau_n)_{n=1}^N$ with disjoint graphs such that

$$M_t = \sum_{n=1}^N \Delta M_{\tau_n} \mathbf{1}_{[0,t]}(\tau_n), \quad t \geq 0.$$

Fix $\varepsilon > 0$. Without loss of generality we may assume that the stopping times $(\tau_n)_{n=1}^N$ are bounded a.s. Due to the proof of Theorem 7.5.5 we may additionally assume that $(\tau_n)_{n=1}^N$ is a.s. increasing. Then by the proof of Theorem 7.5.5 (or [89, Lemma 26.18] in the real-valued case) the sequence $(0, \Delta M_{\tau_1}, 0, \Delta M_{\tau_2}, \ldots, 0, \Delta M_{\tau_N})$ is a martingale difference sequence with respect to the filtration

$$\mathbb{G} := (\mathscr{F}_{\tau_1 -}, \mathscr{F}_{\tau_1}, \mathscr{F}_{\tau_2 -}, \mathscr{F}_{\tau_2}, \ldots, \mathscr{F}_{\tau_N -}, \mathscr{F}_{\tau_N})$$

(see [89, Lemma 25.2] for the definition of $\mathscr{F}_{\tau -}$). As any discrete L^p-bounded martingale difference sequence, $(0, \Delta M_{\tau_1}, 0, \Delta M_{\tau_2}, \ldots, 0, \Delta M_{\tau_N})$ can be approximated in a strong L^p-sense by a uniformly bounded X-valued $\mathbb{G}$-martingale difference sequence $(0, d_1^\varepsilon, 0, d_2^\varepsilon, \ldots, 0, d_N^\varepsilon)$ such that

$$\mathbb{E} \sup_{n=1}^N \Big\| \sum_{i=1}^n \Delta M_{\tau_i} - d_i^\varepsilon \Big\|^p < \varepsilon.$$

The martingale difference sequence $(0, d_1^\varepsilon, 0, d_2^\varepsilon, \ldots, 0, d_N^\varepsilon)$ can be translated back to a martingale on $\mathbb{R}_+$ in the same way as it was shown in the proof of Theorem 7.5.5, i.e. one can define a process $N^\varepsilon : \mathbb{R}_+ \times \Omega \to X$ such that

$$N_t^\varepsilon := \sum_{n=1}^N d_n \mathbf{1}_{[0,t]}(\tau_n), \quad t \geq 0,$$

which is a martingale by Lemma 2.4.5 (or see [89, Lemma 26.18] for the real valued version) with

$$\mathbb{E} \sup_{t\geq 0} \|M_t - N_t^\varepsilon\|^p = \mathbb{E} \sup_{t \geq 0} \Big\| \sum_{0 \leq s \leq t} \Delta M_s - \Delta N_s^\varepsilon \Big\|^p = \mathbb{E} \sup_{n=1}^N \Big\| \sum_{i=1}^n \Delta M_{\tau_i} - d_i \Big\|^p < \varepsilon,$$

which terminates the proof.

Remark 9.7.13. Clearly Theorem 9.7.9 holds true if X has a Schauder basis. Therefore it remain open for whether Theorem 9.7.9 holds true for a general Banach space.

9.7.6. *The canonical decomposition*

As it was shown in Chapter 4 and 5, the canonical decomposition of a UMD Banach space-valued martingale is unique, and by Section 4.3 together with (2.2.1) we have that for any $1 < p < \infty$ and for any $i = c, q, a$

$$\mathbb{E}\sup_{t\geq 0}\|M_t^i\|^p \lesssim_{p,X} \mathbb{E}\sup_{t\geq 0}\|M_t\|^p. \tag{9.7.16}$$

Theorem 9.7.7 allows us to extend (9.7.16) to the case $p = 1$. Indeed, due to Subsection 2.4.3 we have that for any $x^* \in X^*$ a.s.

$$[\langle M, x^*\rangle]_t = [\langle M^c, x^*\rangle]_t + [\langle M^q, x^*\rangle]_t + [\langle M^a, x^*\rangle]_t, \quad t \geq 0,$$

so by Theorem 9.7.7

$$\mathbb{E}\sup_{t\geq 0}\|M_t^i\|^p \lesssim_{p,X} \mathbb{E}\sup_{t\geq 0}\|M_t\|^p, \tag{9.7.17}$$

for all $1 \leq p < \infty$ and any $i = c, q, a$.

9.7.7. *Covariation bilinear forms for pairs of martingales*

Let X be a UMD Banach space, $M, N : \mathbb{R}_+ \times \Omega \to X$ be local martingales. Then for any fixed $t \geq 0$ and any $x^*, y^* \in X^*$ we have that by [89, Theorem 26.6(iii)] a.s.

$$[\langle M, x^*\rangle, \langle N, y^*\rangle]_t \leq [[M]]_t(x^*, x^*)[[N]]_t(y^*, y^*).$$

Thus analogously the proof of Theorem 9.5.1 (by exploiting a subspace Y of X^* that is a linear span of a countable subset of X^*) there exists a bounded bilinear form-valued random variable $[[M,N]]_t : \Omega \to X \otimes X$ such that $[\langle M, x^*\rangle, \langle N, y^*\rangle]_t = [[M,N]]_t(x^*, y^*)$ for any $x^*, y^* \in X^*$ a.s.

Now let X and Y be UMD Banach spaces (perhaps different), $M : \mathbb{R}_+ \times \Omega \to X$, $N : \mathbb{R}_+ \times \Omega \to Y$ be local martingales. Then we can show that for any $t \geq 0$ there exists a bilinear form-valued process $[[M,N]]_t : \Omega \to X \otimes Y$ such that $[[M,N]]_t = [\langle M, x^*\rangle, \langle N, y^*\rangle]_t$ a.s. for any $x^* \in X^*$ and $y^* \in Y^*$. Indeed, one can presume the Banach space to be $X \times Y$ and extend both M and N to take values in this Banach space. Then by the first part of the present subsection there exists a bilinear form $[[M,N]]_t$ acting on $(X \times Y)^* \times (X \times Y)^*$ such that for any $x^* \in X^*$ and $y^* \in Y^*$ a.s.

$$\begin{aligned} [[M,N]]_t\big((x^*, y^*), (x^*, y^*)\big) &= [\langle M, (x^*, y^*)\rangle, \langle N, (x^*, y^*)\rangle]_t \\ &= [\langle M, x^*\rangle, \langle N, y^*\rangle]_t. \end{aligned} \tag{9.7.18}$$

It remains to restrict $[[M,N]]_t$ back to $X \otimes Y$ from $(X \times Y) \otimes (X \times Y)$ which is possible by (9.7.18).

Interesting things happen given $Y = \mathbb{R}$. In this case $[[M,N]]_t$ takes values in $X \otimes \mathbb{R} \simeq X$, so $[[M,N]]_t$ is simply X-valued, and it is easy to see that

$$[[M,N]]_t = \mathbb{P} - \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} (M(t_n) - M(t_{n-1}))(N(t_n) - N(t_{n-1})), \tag{9.7.19}$$

where the limit in probability is taken over partitions $0 = t_0 < \ldots < t_n = t$, and it is taken in a *weak* sense (i.e. (9.7.19) holds under action of any linear functional $x^* \in X^*$). It remains open whether (9.7.19) holds in a strong sense.

9.8. UMD BANACH FUNCTION SPACES

The goal of the present section is to show that a weaker version of (8.4.1) holds for $p = 1$.

Theorem 9.8.1. *Let X be a UMD Banach function space over a σ-finite measure space (S,Σ,μ), $M : \mathbb{R}_+ \times \Omega \to X$ be a local martingale. Then there exists a local martingale field $N : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R}$ such that $N(\omega, t, \cdot) = M_t(\omega)$ for all $t \geq 0$ for a.a. $\omega \in \Omega$, and for all $1 \leq p < \infty$*

$$\mathbb{E}\sup_{t\geq 0} \|M_t\|^p \eqsim_{p,X} \mathbb{E}\|[N]_\infty^{1/2}\|^p, \tag{9.8.1}$$

Let us first show the discrete version of Theorem 9.8.1, which was shown in [164, Theorem 3] for the case $p \in (1,\infty)$.

Proposition 9.8.2. *Let X be a UMD Banach function space over a measure space (S,Σ,μ), $(d_n)_{n\geq 1}$ be an X-valued martingale difference sequence. Then for all $1 \leq p < \infty$*

$$\mathbb{E}\sup_{N\geq 1}\Big\|\sum_{n=1}^N d_n\Big\|^p \eqsim_{p,X} \mathbb{E}\Big\|\Big(\sum_{n=1}^\infty |d_n|^2\Big)^{\frac12}\Big\|^p.$$

Proof. The proof follows from Theorem 9.2.1 and the equivalence [80, (9.26)] between the γ-norm and the square function. □

Remark 9.8.3. By Remark 9.2.2 and [80, (9.26)] one has that for any $r \in (1,\infty)$ there exist positive $C_{r,X}$ and $c_{r,X}$ such that for any $1 \leq p \leq r$

$$c_{r,X}\mathbb{E}\Big\|\Big(\sum_{n=1}^\infty |d_n|^2\Big)^{\frac12}\Big\|^p \leq \mathbb{E}\sup_{N\geq 1}\Big\|\sum_{n=1}^N d_n\Big\|^p \leq C_{r,X}\mathbb{E}\Big\|\Big(\sum_{n=1}^\infty |d_n|^2\Big)^{\frac12}\Big\|^p.$$

Proof of Theorem 9.8.1. We will consider separately the cases $p > 1$ and $p = 1$.

Case $p > 1$. This case was covered in Theorem 8.4.1. Nevertheless, we wish to notice that by modifying the proof from Theorem 8.4.1 by using Proposition 9.8.2 one can obtain better behavior of the equivalence constants in (9.8.1). Namely, by exploiting the same proof together with Proposition 9.8.2 and Remark 9.8.3 one obtains that for any $p' \in (1,\infty)$ there exist positive $C_{p',X}$ and $c_{p',X}$ (the same as in Remark 9.8.3) such that for any $1 < p \leq p'$

$$c_{p',X}\mathbb{E}\|[N]_\infty^{1/2}\|^p \leq \mathbb{E}\sup_{t\geq 0}\|M_t\|^p \leq C_{p',X}\mathbb{E}\|[N]_\infty^{1/2}\|^p. \tag{9.8.2}$$

Case $p = 1$. By Theorem 9.7.9 there exists a sequence $(M^n)_{n\geq 1}$ of uniformly bounded X-valued martingales such that

$$\mathbb{E}\sup_{t\geq 0}\|M_t - M_t^n\| \to 0, \quad n \to \infty. \tag{9.8.3}$$

Since M^n is uniformly bounded for any $n \geq 1$, $\mathbb{E}\sup_{t\geq 0}\|M^n_t\|^2 < \infty$, so by Case $p>1$ there exists a local martingale field N^n such that $N^n(\omega, t, \cdot) = M^n_t(\omega)$ for all $t \geq 0$ for a.a. $\omega \in \Omega$. By (9.8.2) one has that there exist positive constants C_X and c_X such that for all $m, n \geq 1$

$$c_X\mathbb{E}\|[N^n - N^m]^{1/2}_\infty\| \leq \mathbb{E}\sup_{t\geq 0}\|M^n_t - M^m_t\| \leq C_X\mathbb{E}\|[N^n - N^m]^{1/2}_\infty\|,$$

hence due to (9.8.3) $(N^n)_{n\geq 1}$ is a Cauchy sequence in $\mathrm{MQ}^1(X)$. Since by Proposition 8.4.2 the linear space $\mathrm{MQ}^1(X)$ endowed with the norm (8.4.2) is Banach, there exists a limit N of $(N^n)_{n\geq 1}$ in $\mathrm{MQ}^1(X)$.

Let us show that N is the desired local martingale field. Fix $t \geq 0$. We need to who that $N(\cdot, t, \cdot) = M_t$ a.s. on Ω. First notice that by the last part of Proposition 8.4.2 there exists a subsequence of $(N^n)_{n\geq 1}$ which we will denote by $(N^n)_{n\geq 1}$ as well such that $N^n(\cdot, t, \sigma) \to N(\cdot, t, \sigma)$ in $L^1(\Omega)$ for a.e. $\sigma \in S$. On the other hand by Jensen's inequality

$$\big\|\mathbb{E}|N^n(\cdot, t, \cdot) - M_t|\big\| = \big\|\mathbb{E}|M^n_t - M_t|\big\| \leq \mathbb{E}\|M^n_t - M_t\| \to 0, \quad n \to \infty.$$

Hence $N^n(\cdot, t, \cdot) \to M_t$ in $X(L^1(\Omega))$, and thus by Remark 8.2.2 in $L^0(S; L^1(\Omega))$. Therefore we can find a subsequence of $(N^n)_{n\geq 1}$ (which we will again denote by $(N^n)_{n\geq 1}$) such that $N^n(\cdot, t, \sigma) \to M_t(\sigma)$ in $L^1(\Omega)$ for a.e. $\sigma \in S$ (here we use that fact that μ is σ-finite), so $N(\cdot, t, \cdot) = M_t$ a.s. on $\Omega \times S$, and consequently by Definition 8.2.1(iii), $N(\omega, t, \cdot) = M_t(\omega)$ for a.a. $\omega \in \Omega$.

Let us finally show (9.8.1). Since $N^n \to N$ in $\mathrm{MQ}^1(X)$ and by (9.8.3)

$$\mathbb{E}\|[N]^{1/2}_\infty\| = \lim_{n\to\infty}\mathbb{E}\|[N^n]^{1/2}_\infty\| \eqsim_X \lim_{n\to\infty}\mathbb{E}\sup_{t\geq 0}\|M^n_t\| = \mathbb{E}\sup_{t\geq 0}\|M_t\|,$$

which terminates the proof. □

Remark 9.8.4. It was shown in Theorem 8.4.1 that in the case $p>1$ the equivalence (9.8.1) can be strengthen. Namely, in this case one can show that

$$\mathbb{E}\Big\|\sup_{t\geq 0}|M_t|\Big\|^p \eqsim_{p,X} \mathbb{E}\|[N]^{1/2}_\infty\|^p, \tag{9.8.4}$$

i.e. one has the same equivalence with a pointwise supremum in S. The techniques that provide such an improvement were discovered by Rubio de Francia in [164]. Unfortunately, it remains open whether (9.8.4) holds for $p = 1$. Surprisingly, (9.8.4) holds for $p = 1$ and for $X = L^1(S)$ by a simple Fubini-type argument, so it might be that (9.8.4) holds for $p = 1$ even for other nonreflexive Banach spaces.

INDEX

$F \star \bar{\mu}$, 34
M^*, 19
M^a, 28
M^c, 21, 28
M^d, 21
M^q, 28
$M^{\tau-}$, 25
$N \ll M$, 5, 53
$N \overset{w}{\ll} M$, 6, 53
$X + Y$, 176
$X \cap Y$, 176
$[M,N]$, 20, 21
$[M]$, 20
$[M]^a$, 28
$[M]^c$, 21, 28
$[M]^d$, 21
$[M]^q$, 28
ΔM_τ, 24
ΔV_τ, 96
$\gamma(\cdot)$, 249
$\hbar_{p,X}$, 131
$\mathbb{R}_+$, 17
$\mathbb{T}$, 126
$\mathscr{F}_{\tau-}$, 24
$\mathscr{F}_\tau$, 24
$\mathscr{H}_X^{\mathbb{T}}$, 126
$\mathscr{H}_X^{\mathbb{R}}$, 143
$\mathscr{H}_X^{\mathrm{dis}}$, 143
$\mathscr{M}_X^p$, 19
$\mathscr{M}_X^p(\mathbb{F})$, 19
$\mathscr{M}_X^{1,\infty}$, 96
$\mathscr{M}_X^{1,\infty}(\mathbb{F})$, 96
$\mathscr{M}_X^{p,a}$, 29
$\mathscr{M}_X^{p,q}$, 29
$\mathscr{M}_X^{p,c}$, 21
$\mathscr{M}_X^{p,c}(\mathbb{F})$, 21
$\mathscr{M}_X^{p,d}$, 22
$\mathscr{M}_X^{p,d}(\mathbb{F})$, 22
μ^M, 98, 197
$\overline{\mathbb{R}}_+$, 17
$\widetilde{\Omega}$, 33
$\widetilde{\mathscr{O}}$, 33
$\widetilde{\mathscr{P}}$, 33
$a \eqsim_A b$, 17
$a \vee b$, 17
$a \wedge b$, 17
$a \gtrsim_A b$, 17
$a \lesssim_A b$, 17
df_n, 20
$g \ll f$, 45
$g \overset{w}{\ll} f$, 45
$h \otimes x$, 32
p', 17
p^*, 4, 17, 23
q_M, 20

Beurling-Ahlfors transform, 60
Brownian representation, 33

canonical decomposition of martingales, 7, 28, 30, 83, 101, 241, 274
compensator, 34, 97
corresponding dual basis, 36
covariation, 20, 21
covariation bilinear form, 257

decomposition
  canonical, 7, 28, 30, 83, 101, 241, 274
  Gundy's, 97
  Meyer-Yoeurp, 21, 23, 73, 108, 130
  Yoeurp, 8, 28, 30, 81, 110
decoupling constant, 151

differential subordination, 4, 5, 45, 53
double Riesz transform, 61

filtration
usual conditions, 18
Fourier multiplier, 56
Fourier transform, 56
function
biconcave, 35
Burkholder, 48
concave, 35
convex, 35
plurisubharmonic, 128
step, 143
subharmonic, 128
zigzag-concave, 35

γ-radonifying operator, 35
Gaussian characteristic, 249
Gundy's decomposition, 97

Hilbert transform, 61, 126, 143

induced filtration, 31
inequality
Burkholder–Davis–Gundy, 21, 234, 243, 275
Burkholder-Rosenthal, 178
decoupling, 151
Doob's, 19
lattice Doob's, 231
Novikov's, 199, 200
Itô's formula, 37

Lebesgue measure, 49, 72

martingale, 18
L^p-bounded, 19
approximation, 270
continuous, 21, 261
domination, 269
local, 18
orthogonal, 127
Paley-Walsh, 19
purely discontinuous, 21, 22, 261
quasi-left continuous, 27
τ-continuous, 31
with accessible jumps, 27
with independent increments, 263
measure
Gaussian, 249
Lebesgue, 49, 72
random, 33, 199
Meyer-Yoeurp decomposition, 21, 73, 108, 130

operator
Beurling-Ahlfors transform, 60
double Riesz transform, 61
Fourier multiplier, 56
Fourier transform, 56
γ-radonifying, 35
Hilbert, 166
Hilbert transform, 61, 126, 143
Riesz transform, 166
singular integral, 165
orthogonal martingales, 127

process
adapted, 18
càdlàg, càglàd, càd, càg, làd, làg, 17
elementary predictable, 32, 264
optional, 33
predictable, 18
pure jump, 21
quasi-left continuous, 27
Wiener, 33
Wiener multidimensional, 33
with accessible jumps, 27

quadratic variation, 5, 20
quadratic variation derivative, 20

Rademacher distribution, 19
random measure, 33, 199
compensator, 34

integer-valued, 33
non-atomic in time, 33
optional, 33
predictable, 33

σ-algebra
optional, 18
predictable, 18
Skorohod space, 18
space
Banach function, 275
Skorohod, 18
UMD, 23
step function, 143
stochastic integral, 32, 189, 209, 264
w.r.t. random measure, 33, 199, 268
stopping time, 24
graph of, 27
optional, 24
predictable, 25
totally inaccessible, 25

time-change, 31
totally inaccessible, 25

ucp topology, 96
UMD Banach space, 23
usual conditions, 18

variation, 97

weak differential subordination, 4, 6, 45, 53, 84, 102, 157
of harmonic functions, 161
transform, 102
Wiener process
multidimensional, 33

Yoeurp decomposition, 8, 28, 30, 81, 110

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I wish everybody the best. Have fun, party like a Russian, and do not be afraid of something new!

Delft, February 2019

Ivan S. Yaroslavtsev

Curriculum Vitæ

Ivan Yaroslavtsev

Ivan Sergeevich Yaroslavtsev was born on January 1, 1991, in Neftekamsk, the Soviet Union. He completed Kolmogorov school in Moscow in 2009. In the same year he began his studies in Mathematics at Lomonosov Moscow State University. He obtained his Specialist degree (which is equivalent to MSc) in 2014. His diploma thesis "Polylinear functions on infinite-dimensional spaces with measures" was done under the supervision of Professor Vradimir Bogachev and was devoted to solving an open problem concerning polynomials and multilinear forms on linear spaces with Gaussian measures posed by Heinrich von Weizsäcker in the late 1980's (see [21, p. 275] for the problem and [6] for its solution).

In March 2015 he started his PhD research under the supervision of Professor Mark Veraar and Professor Jan van Neerven at the Delft University of Technology. Part of this research was carried out during his stays at University College London, RWTH Aachen University, University of Warsaw, Australian National University, and a three month stay at University of Jyväskylä, Finland.

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List of Publications

(1) L.M. Arutyunyan and I.S. Yaroslavtsev. On measurable polynomials on infinite-dimensional spaces. *Dokl. Ross. Akad. Nauk* 449, no. 6, 627–631 (in Russian); English transl.: *Dokl. Math.* 87, no. 2, 214–217, 2013.

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(14) I.S. Yaroslavtsev. Burkholder–Davis–Gundy inequalities in UMD Banach spaces. *arXiv:1807.05573*, 2018.

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(17) S. Dirksen, C. Marinelli, and I.S. Yaroslavtsev. Stochastic evolution equations in L^p-spaces driven by jump noise. *In preparation*.

(18) S. Geiss and I.S. Yaroslavtsev. Dyadic and stochastic shifts and Volterra-type operators. *In preparation*.

For notes