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Stability theory for semigroups using (L^p, L^q) Fourier multipliers ☆

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ABSTRACT

We study polynomial and exponential stability for C_0-semigroups using the recently developed theory of operator-valued (L^p, L^q) Fourier multipliers. We characterize polynomial decay of orbits of a C_0-semigroup in terms of the (L^p, L^q) Fourier multiplier properties of its resolvent. Using this characterization we derive new polynomial decay rates which depend on the geometry of the underlying space. We do not assume that the semigroup is uniformly bounded, our results depend only on spectral properties of the generator.

As a corollary of our work on polynomial stability we reprove and unify various existing results on exponential stability, and we also obtain a new theorem on exponential stability for positive semigroups.

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1. Introduction

In this article we study the asymptotic behavior of solutions to the abstract Cauchy problem

$$u'(t) + Au(t) = 0, \quad t \geq 0,$$
$$u(0) = x. \tag{1.1}$$

Here $-A$ is the generator of a C_0-semigroup $(T(t))_{t \geq 0}$ on a Banach space X and $x \in X$. The unique solution of (1.1) with initial data x is given by $u(t) = T(t)x$ for $t \geq 0$. One of the key difficulties in the asymptotic theory for solutions of (1.1) is that the classical Lyapunov stability criterion is in general not valid if X is infinite dimensional. However, asymptotic behavior can be deduced from the associated resolvent operators $R(\lambda, A) = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. For example, on a Hilbert space X the Gearhart–Prüss theorem [3, Theorem 5.2.1] states that $(T(t))_{t \geq 0}$ is exponentially stable if and only if $\sigma(A) \subset \mathbb{C}_+$ and $\sup_{\text{Re}(\lambda)<0} \|R(\lambda, A)\| < \infty$. A uniform bound for the resolvent is not sufficient to ensure exponential stability on general Banach spaces, but it was shown in [26,37] (see also [16,32,36,65]) that exponential stability can be characterized in terms of L^p Fourier multiplier properties of the resolvent. Outside of Hilbert spaces this multiplier condition is a strictly stronger assumption than uniform boundedness, and in applications it can be difficult to verify. On the other hand, cf. [47,49,64,65], uniform bounds for the resolvent do imply exponential stability for orbits in fractional domains, with the fractional domain parameter depending on the geometry of the underlying space. At the moment it is not fully understood how the characterization of exponential stability using Fourier multipliers is related to such concrete decay results.

In a separate development, over the past decade much attention has been paid to polynomial decay of semigroup orbits. The work of Lebeau [39,40] and Burq [13] on energy decay for damped wave equations raised the question of what the precise relation is between growth rates for the resolvent and decay rates for the semigroup. More precisely, if one has $\sigma(A) \subset \mathbb{C}_+$ in (1.1) but $\|R(i\xi, A)\| \to \infty$ as $|\xi| \to \infty$, then $(T(t))_{t \geq 0}$ is not exponentially stable and one typically encounters other asymptotic behavior. Since a uniform rate of decay for all solutions to (1.1) implies exponential stability of the semigroup, one can expect uniform asymptotic behavior only for orbits in suitable subspaces such as fractional domains, and in general the smoothness parameter of the fractional domain influences the decay behavior. In [4] Bátkai, Engel, Prüss and Schnaubelt proved that for uniformly bounded semigroups a polynomial growth rate of the resolvent implies a specific polynomial decay rate for classical solutions of (1.1) and vice versa, and they showed that this correspondence is optimal up to an arbitrarily small polynomial loss. In [8] Batty and Duyckaerts extended this correspondence to the setting of arbitrary resolvent growth and they reduced the loss to a logarithmic scale. Then Borichev and Tomilov proved in [12] that this logarithmic loss is sharp on general Banach spaces, but that it can be removed on Hilbert spaces in the case of polynomial resolvent growth. In particular,

on Hilbert spaces this yields a characterization of polynomial stability in terms of the growth of the resolvent. This result has been applied extensively in the study of partial differential equations (see e.g. [1,2,9,14,24,38,42,57] and references therein) and has been extended in [7,15,43,54,60,62,63] to finer scales of resolvent growth and semigroup decay.

Although much work has gone into determining the relation between resolvent growth and polynomial rates of decay, it is not clear how such asymptotic behavior relates to the Fourier analytic properties of the resolvent which characterize exponential stability. Furthermore, the currently available literature on polynomial decay deals almost exclusively with uniformly bounded semigroups. To the best of our knowledge, the only previously known result concerning polynomial decay for general semigroups is [4, Proposition 3.4]. There are many natural classes of examples where the generator has spectral properties as above but the semigroup is not uniformly bounded, or where it is unknown whether the semigroup is bounded. Typical examples of this phenomenon can be found in Section 4.7 and include semigroups whose generator is an operator matrix or a multiplication operator on a Sobolev space. In turn, such operators can be found in disguise in concrete partial differential equations. One example is the standard wave equation with periodic boundary conditions; here uniform boundedness fails. Other examples can be found in [50] for certain classes of perturbed wave equations and in [61] for delay equations. For infinite systems of equations the uniform boundedness condition leads to additional assumptions on the coefficients in [51].

In this article we deal with the problems outlined above in three ways. First, we characterize polynomial stability on general Banach spaces in terms of Fourier multiplier properties of powers of the resolvent, in Theorem 4.6. In doing so we extend the Fourier analytic characterization of exponential stability to this more refined setting. Then, using the theory of operator-valued (L^p, L^q) Fourier multipliers which was developed in [55,56] with applications to stability theory in mind, we derive concrete polynomial decay rates from this characterization. These results involve only growth bounds for the resolvent and are new even on Hilbert spaces. In particular, the following theorem can be found in the main text as Corollary 4.11.

Theorem 1.1. *Let* $-A$ *be the generator of a* C_0*-semigroup* $(T(t))_{t\geq 0}$ *on a Hilbert space* X *such that* $\sigma(A) \subset \mathbb{C}_+$ *and* $\|R(\lambda, A)\| \leq C(1 + |\lambda|)^\beta$ *for some* $\beta > 0$, $C \geq 0$ *and all* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) \leq 0$. *Then for each* $\tau \geq \beta$ *there exists a* $C_\tau \geq 0$ *such that*

$$\|T(t)A^{-\tau}\| \leq C_\tau t^{1-\tau/\beta} \qquad (t \in [1, \infty)). \tag{1.2}$$

Note that we do not assume that the semigroup is uniformly bounded. In fact, we show that one can derive polynomial decay behavior for initial values in suitable fractional domains given only spectral properties of the generator. In particular, by setting $\tau = \beta$ in Theorem 1.1 one obtains uniform boundedness of sufficiently smooth solutions. For uniformly bounded semigroups the parameter $1 - \tau/\beta$ in (1.2) can be replaced by $-\tau/\beta$, as was shown in [12], but in Example 4.20 we prove that $1 - \tau/\beta$ is optimal for general

semigroups if $\tau = \beta$. Our main theorems allow for A to have a singularity at zero, or even singularities at both zero and infinity. We also obtain versions of Theorem 1.1 on other Banach spaces; the decay rate in (1.2) then depends on the geometry of the underlying space.

Finally, as a direct corollary of our results on polynomial stability we recover in a unified manner various results on exponential stability from [26,37,47,49,64,65]. We also obtain a new stability result for positive semigroups, Theorem 5.8.

To prove our main results we rely on the theory of operator-valued Fourier multipliers from $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; Y)$, for X and Y Banach spaces. A Fourier multiplier characterization of exponential stability for general $p \in [1, \infty)$ and $q \in [p, \infty]$ was known from [37], but so far only the case where $p = q$ has been used (see [5,25,26,36,37,65]). Although in this setting very powerful multiplier theorems are available, see for example Weis' version of the Mikhlin multiplier theorem in [66] and [17,29,33], the assumptions of these theorems are in general too restrictive for applications to stability theory. Indeed, multiplier theorems on $L^p(\mathbb{R}; X)$ typically require both a geometric assumption on X, namely the UMD condition which excludes spaces of interest such as $X = L^1$, as well as smoothness of the multiplier and comparatively fast decay at infinity of its derivative. The latter assumption in particular is not satisfied in most applications to stability theory.

In this article we argue that for the study of asymptotic behavior it is more natural to consider general $p \in [1, \infty)$ and $q \in [p, \infty]$. It was observed in [55,56] that one can derive boundedness of Fourier multipliers from $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; Y)$ for $p < q$ under different geometric assumptions on X and Y than in the case where $p = q$, and assuming decay of the multiplier at infinity but no smoothness. In fact, the parameters p and q depend on the geometry of X, and the amount of decay which is required at infinity is proportional to $\frac{1}{p} - \frac{1}{q}$. Moreover, in Section 3.2 we prove that growth of the resolvent on X corresponds to uniform boundedness, and in fact even decay, of the resolvent from suitable fractional domain and range spaces to X. Then one can determine for which fractional domain and range parameters the conditions of the (L^p, L^q) multiplier theorems are satisfied for (powers of) the resolvent, and the Fourier multiplier characterizations of stability in Theorems 4.6 and 5.3 yield the corresponding asymptotic behavior. We emphasize that, although we use Fourier multiplier techniques for the proofs, our main theorems on concrete decay rates involve only growth bounds on the resolvent.

This article is organized as follows. In Section 2 we present some basics on Banach space geometry, Fourier multipliers and sectorial operators. In Section 3 we deduce multiplier properties of the resolvent and we prove Proposition 3.4 and Corollary 3.5. These are fundamental in later sections for relating resolvent growth on X to boundedness and decay from fractional domain and range spaces to X. In Section 4 we study polynomial decay of semigroups. We characterize polynomial stability using Fourier multipliers, and from this characterization we deduce concrete polynomial decay rates which depend on the geometry of the underlying space. In Section 5 we derive from these results various corollaries on exponential decay. We also prove a characterization of exponential stability using multipliers on Besov spaces, which in turn is used to obtain a new stability

result for positive semigroups. An appendix contains estimates for contour integrals and exponential functions.

1.1. Notation

The set of natural numbers is $\mathbb{N} = \{1, 2, \ldots\}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote by $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > 0\}$ and $\mathbb{C}_- = -\mathbb{C}_+$ the open complex right and left half-planes.

Nonzero Banach spaces over the complex numbers are denoted by X and Y. The space of bounded linear operators from X to Y is $\mathcal{L}(X, Y)$, and $\mathcal{L}(X) := \mathcal{L}(X, X)$. The identity operator on X is denoted by I_X, and we usually write λ for λI_X when $\lambda \in \mathbb{C}$. The domain of a closed operator A on X is $D(A)$, a Banach space with the norm

$$\|x\|_{D(A)} := \|x\|_X + \|Ax\|_X \qquad (x \in D(A)).$$

For an injective closed operator A we identify the range $\mathrm{ran}(A)$ of A with the Banach space $D(A^{-1})$. The spectrum of A is $\sigma(A)$ and the resolvent set is $\rho(A) = \mathbb{C} \setminus \sigma(A)$. We write $R(\lambda, A) = (\lambda - A)^{-1}$ for the resolvent operator of A at $\lambda \in \rho(A)$.

For $p \in [1, \infty]$ and Ω a measure space, $L^p(\Omega; X)$ is the Bochner space of equivalence classes of strongly measurable, p-integrable, X-valued functions on Ω. The Hölder conjugate of $p \in [1, \infty]$ is denoted by p' and is defined by $1 = \frac{1}{p} + \frac{1}{p'}$.

The class of X-valued Schwartz functions on $\mathbb{R}$ is denoted by $\mathcal{S}(\mathbb{R}; X)$, and the space of X-valued tempered distributions by $\mathcal{S}'(\mathbb{R}; X)$. The Fourier transform of $f \in \mathcal{S}'(\mathbb{R}; X)$ is denoted by $\mathcal{F}f$ or $\widehat{f}$. If $f \in L^1(\mathbb{R}; X)$ then

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-i\xi t} f(t) \, \mathrm{d}t \qquad (\xi \in \mathbb{R}).$$

We use the convention that $\frac{1}{0} = \infty$ and $\frac{0}{0} = \infty$.

For sets S and Z we occasionally denote a function $f : S \to Z$ of a variable s simply by $f = f(s)$. We use the notation $f(s) \lesssim g(s)$ for functions $f, g : S \to \mathbb{R}$ to indicate that $f(s) \leq Cg(s)$ for all $s \in S$ and a constant $C \geq 0$ independent of s, and similarly for $f(s) \gtrsim g(s)$. We write $f(s) \eqsim g(s)$ if $g(s) \lesssim f(s) \lesssim g(s)$ holds.

2. Preliminaries

2.1. Banach space geometry

Here we collect some background on Banach space geometry which is used for our results on non-Hilbertian Banach spaces.

A Banach space X has *Fourier type* $p \in [1, 2]$ if the Fourier transform $\mathcal{F}$ is bounded from $L^p(\mathbb{R}; X)$ to $L^{p'}(\mathbb{R}; X)$. We then set $\mathcal{F}_{p,X} := \|\mathcal{F}\|_{\mathcal{L}(L^p(\mathbb{R};X), L^{p'}(\mathbb{R};X))}$. To make our multiplier theorems more transparent, we say that X has *Fourier cotype* $q \in [2, \infty]$ if X

has Fourier type q'. Each Banach space has Fourier type 1, and X has Fourier type 2 if and only if X is isomorphic to a Hilbert space. For $r \in [1, \infty]$ and Ω a measure space, $L^r(\Omega)$ has Fourier type $\min(r, r')$. For more on Fourier type see [29,52].

A (real) Rademacher variable is a random variable $r : \Omega \to \{-1, 1\}$ on a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(r = -1) = \mathbb{P}(r = 1) = \frac{1}{2}$. A *Rademacher sequence* is a sequence $(r_k)_{k \geq 1}$ of independent Rademacher variables on some probability space.

Let $(r_k)_{k \geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathbb{P})$. A Banach space X has *type $p \in [1, 2]$* if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$ one has

$$\left(\mathbb{E} \Big\| \sum_{k=1}^{n} r_k x_k \Big\|^2 \right)^{1/2} \leq C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}.$$

Also, X has *cotype $q \in [2, \infty]$* if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$ one has

$$\left(\sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \leq C \left(\mathbb{E} \Big\| \sum_{k=1}^{n} r_k x_k \Big\|^2 \right)^{1/2},$$

with the obvious modification for $q = \infty$. We say that X has *nontrivial type* if X has type $p \in (1, 2]$, and *finite cotype* if X has cotype $q \in [2, \infty)$. Each Banach space has type $p = 1$ and cotype $q = \infty$, and X has type $p = 2$ and cotype $q = 2$ if and only if X is isomorphic to a Hilbert space, by Kwapień's theorem [34]. For $r \in [1, \infty)$ and Ω a measure space, $L^r(\Omega)$ has type $\min(r, 2)$ and cotype $\max(r, 2)$. For more on type and cotype see [18,30].

Let X be a Banach lattice and $p, q \in [1, \infty]$. We say that X is *p-convex* if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$ one has

$$\left\| \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\|_X \leq C \left(\sum_{k=1}^{n} \|x_k\|_X^p \right)^{1/p},$$

with the obvious modification for $p = \infty$. We say that X is *q-concave* if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$ one has

$$\left(\sum_{k=1}^{n} \|x_k\|_X^q \right)^{1/q} \leq C \left\| \left(\sum_{k=1}^{n} |x_k|^q \right)^{1/q} \right\|_X,$$

with the obvious modification for $q = \infty$. Each Banach lattice X is 1-convex and ∞-concave. For $r \in [1, \infty]$ and Ω a measure space, $L^r(\Omega)$ is r-convex and r-concave. For more on p-convexity and q-concavity we refer the reader to [21,41].

Let X and Y be Banach spaces and $\mathcal{T} \subseteq \mathcal{L}(X, Y)$. We say that $\mathcal{T}$ is *R-bounded* if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \mathcal{T}$ and $x_1, \ldots, x_n \in X$ one has

$$\left(\mathbb{E}\Big\|\sum_{k=1}^n r_k T_k x_k\Big\|_Y^2\right)^{1/2} \leq C\left(\mathbb{E}\Big\|\sum_{k=1}^n r_k x_k\Big\|_X^2\right)^{1/2}. \tag{2.1}$$

The smallest such C is the *R-bound* of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$. If we want to specify the underlying spaces X and Y then we write $R_{X,Y}(\mathcal{T})$ for the R-bound of $\mathcal{T}$, and we write $R_X(\mathcal{T}) = R_{X,Y}(\mathcal{T})$ if $X = Y$. Every R-bounded collection is uniformly bounded with supremum bound less than or equal to its R-bound, and the converse holds if and only if X has cotype 2 and Y has type 2. For $\lambda \in \mathbb{C}$ and an R-bounded collection $\mathcal{T} \subseteq \mathcal{L}(X,Y)$, the closed absolutely convex hull $\overline{\mathrm{aco}}(\lambda\mathcal{T}) \subseteq \mathcal{L}(X,Y)$ of $\lambda\mathcal{T} = \{\lambda T \mid T \in \mathcal{T}\}$ is R-bounded, and

$$R_{X,Y}(\overline{\mathrm{aco}}(\lambda\mathcal{T})) \leq 2|\lambda|R_{X,Y}(\mathcal{T}). \tag{2.2}$$

In particular, L^1-averages of R-bounded collections are again R-bounded, a fact which will be used frequently. For more on R-boundedness see [30,33,48].

The following lemma is used in the proof of Corollary 5.5. It can also be deduced from a corresponding statement in [31, Theorem 5.1] for the Besov space $B_{r,1}^{1/r}(\mathbb{R}; \mathcal{L}(X,Y))$. Here we give a more direct proof. For $r \in [1,\infty]$ and E a Banach space we denote by $W^{1,r}(\mathbb{R}; E)$ the Sobolev space of weakly differentiable $f : \mathbb{R} \to E$ such that $f, f' \in L^r(\mathbb{R}; E)$, with $\|f\|_{W^{1,r}(E)} := \|f\|_{L^r(\mathbb{R};E)} + \|f'\|_{L^r(\mathbb{R};E)}$.

Lemma 2.1. *Let X be a Banach space with cotype $q \in [2,\infty)$ and Y a Banach space with type $p \in [1,2]$, and let $r \in [1,\infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then there exists a constant $C \in [0,\infty)$ such that for all $f \in W^{1,r}(\mathbb{R}; \mathcal{L}(X,Y))$ the set $\{f(t) \mid t \in \mathbb{R}\} \subseteq \mathcal{L}(X,Y)$ is R-bounded, with*

$$R(\{f(t) \mid t \in \mathbb{R}\}) \leq C\|f\|_{W^{1,r}(\mathbb{R};\mathcal{L}(X,Y))}.$$

Proof. Let $f \in W^{1,r}(\mathbb{R}; \mathcal{L}(X,Y))$ and for $j \in \mathbb{Z}$ set $I_j := [j, j+1)$ and $\mathcal{T}_j := \{f(t) \mid t \in I_j\}$. Then [33, Example 2.18] and Hölder's inequality imply

$$R(\mathcal{T}_j) \lesssim \|f\|_{W^{1,1}(I_j;\mathcal{L}(X,Y))} \lesssim \|f\|_{W^{1,r}(I_j;\mathcal{L}(X,Y))}$$

for all $j \in \mathbb{Z}$. Now [20, Theorem 3.1] (see also [30, Proposition 9.1.10]) shows that $\{f(t) \mid t \in \mathbb{R}\} = \bigcup_{j\in\mathbb{Z}} \mathcal{T}_j$ is R-bounded, with

$$R(\{f(t) \mid t \in \mathbb{R}\}) \lesssim \|(R(\mathcal{T}_j))_j\|_{\ell^r(\mathbb{Z})} \lesssim \|(\|f\|_{W^{1,r}(I_j;\mathcal{L}(X,Y))})_j\|_{\ell^r(\mathbb{Z})}$$
$$\lesssim \|f\|_{W^{1,r}(\mathbb{R};\mathcal{L}(X,Y))}. \quad \square$$

By replacing the Rademacher random variables in (2.1) by Gaussian variables, one obtains the definition of a *γ-bounded* collection $\mathcal{T} \subseteq \mathcal{L}(X,Y)$. Each R-bounded collection is γ-bounded, and the converse holds if and only if X has finite cotype (see [35, Theorem 1.1]). We choose to work with R-boundedness in this article, both because the

notion of R-boundedness is more established and because those stability theorems in this article which use R-boundedness are only of interest on spaces with finite cotype.

2.2. Fourier multiplier theorems

To properly define Fourier multipliers for symbols with a singularity at zero, we briefly introduce the class of vector-valued homogeneous distributions. For more on these distributions see [56]. For X a Banach space let

$$\dot{\mathcal{S}}(\mathbb{R}; X) := \{f \in \mathcal{S}(\mathbb{R}; X) \mid \widehat{f}^{(k)}(0) = 0 \text{ for all } k \in \mathbb{N}_0\},$$

endowed with the subspace topology, and let $\dot{\mathcal{S}}'(\mathbb{R}; X)$ be the space of continuous linear mappings from $\dot{\mathcal{S}}(\mathbb{R}; \mathbb{C})$ to X. Then $\dot{\mathcal{S}}(\mathbb{R}; X)$ is dense in $L^p(\mathbb{R}; X)$ for all $p \in [1, \infty)$, and $L^p(\mathbb{R}; X)$ can be naturally identified with a subspace of $\dot{\mathcal{S}}'(\mathbb{R}; X)$ for all $p \in [1, \infty]$.

Let X and Y be Banach spaces. A function $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ is X-*strongly measurable* if $\xi \mapsto m(\xi)x$ is a strongly measurable Y-valued map for each $x \in X$. We say that m is *of moderate growth* if there exist $\alpha \in [0, \infty)$ and $g \in L^1(\mathbb{R})$ such that

$$|\xi|^{\alpha}(1 + |\xi|)^{-2\alpha}\|m(\xi)\|_{\mathcal{L}(X,Y)} \leq g(\xi) \qquad (\xi \in \mathbb{R}).$$

Let $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ be an X-strongly measurable map of moderate growth. Then $T_m : \dot{\mathcal{S}}(\mathbb{R}; X) \to \dot{\mathcal{S}}'(\mathbb{R}; Y)$,

$$T_m(f) := \mathcal{F}^{-1}(m \cdot \widehat{f}) \qquad (f \in \dot{\mathcal{S}}(\mathbb{R}; X)), \tag{2.3}$$

is the *Fourier multiplier operator* associated with m. One calls m the *symbol* of T_m, and we identify symbols which are equal almost everywhere. If $\|m(\cdot)\|_{\mathcal{L}(X,Y)} \in L^1_{\mathrm{loc}}(\mathbb{R})$ then (2.3) extends to all $f \in \mathcal{S}(\mathbb{R}; X)$ and defines an operator $T_m : \mathcal{S}(\mathbb{R}; X) \to \mathcal{S}'(\mathbb{R}; X)$.

For $p \in [1, \infty)$ and $q \in [1, \infty]$ we let $\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(X, Y))$ be the set of all X-strongly measurable $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ of moderate growth such that $T_m \in \mathcal{L}(L^p(\mathbb{R}; X), L^q(\mathbb{R}; Y))$, and

$$\|m\|_{\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(X,Y))} := \|T_m\|_{\mathcal{L}(L^p(\mathbb{R};X), L^q(\mathbb{R};Y))}.$$

We write $\| \cdot \|_{\mathcal{M}_{p,q}} = \| \cdot \|_{\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(X,Y))}$ when the spaces X and Y are clear from the context.

We now recall several (L^p, L^q) Fourier multiplier results from our earlier work. The first is [55, Proposition 3.9].

Proposition 2.2. *Let X be a Banach space with Fourier type $p \in [1, 2]$ and Y a Banach space with Fourier cotype $q \in [2, \infty]$, and let $r \in [1, \infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Let $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ be an X-strongly measurable map such that $\|m(\cdot)\|_{\mathcal{L}(X,Y)} \in L^r(\mathbb{R})$. Then $m \in \mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(X, Y))$ and*

$$\|m\|_{\mathcal{M}_{p,q}(\mathbb{R};\mathcal{L}(X,Y))} \leq \frac{1}{2\pi} \mathcal{F}_{p,X} \mathcal{F}_{q',Y} \left\| \|m(\cdot)\|_{\mathcal{L}(X,Y)} \right\|_{L^r(\mathbb{R})}. \tag{2.4}$$

Our next result follows from [56, Theorem 4.6 and Remark 4.8] and [55, Theorem 3.21 and Remark 3.22].

Proposition 2.3. *Let X be a Banach space with type $p \in [1,2]$ and Y a Banach space with cotype $q \in [2,\infty]$, and let $r \in [1,\infty]$ be such that $\frac{1}{r} > \frac{1}{p} - \frac{1}{q}$. Then there exists a constant $C \in [0,\infty)$ such that the following holds. Let $m : \mathbb{R} \to \mathcal{L}(X,Y)$ be an X-strongly measurable map such that $\{(1 + |\xi|)^r m(\xi) \mid \xi \in \mathbb{R}\} \subseteq \mathcal{L}(X,Y)$ is R-bounded. Then $m \in \mathcal{M}_{p,q}(\mathbb{R};\mathcal{L}(X,Y))$ and*

$$\|m\|_{\mathcal{M}_{p,q}(\mathbb{R};\mathcal{L}(X,Y))} \leq C R_{X,Y}(\{(1 + |\xi|)^r m(\xi) \mid \xi \in \mathbb{R}\}). \tag{2.5}$$

Moreover, if X is a complemented subspace of a p-convex Banach lattice with finite cotype and if Y is a Banach space continuously embedded in a q-concave Banach lattice for $q \in [1,\infty)$, then (2.5) also holds if $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$.

For $s \in \mathbb{R}$ and $p \in [1,\infty]$, the *inhomogeneous Bessel potential space* $H_p^s(\mathbb{R};X)$ consists of all $f \in \mathcal{S}'(\mathbb{R};X)$ such that $T_{m_s}(f) \in L^p(\mathbb{R};X)$, where $m_s(\xi) := (1+|\xi|^2)^{s/2}$ for $\xi \in \mathbb{R}$. It is a Banach space endowed with the norm

$$\|f\|_{H_p^s(\mathbb{R};X)} := \|T_{m_s}(f)\|_{L^p(\mathbb{R}^d;X)} \quad (f \in H_p^s(\mathbb{R};X)).$$

Moreover, $\dot{\mathcal{S}}(\mathbb{R}^d;X) \subseteq H_p^s(\mathbb{R}^d;X)$ is densely embedded for $p < \infty$.

The following proposition is proved in the same way as the corresponding homogeneous version in [55, Theorem 3.24]. We note that one can often avoid condition (2) by using approximation arguments.

Proposition 2.4. *Let $p \in [1,\infty)$ and $q \in [p,\infty)$. Let X be a p-convex Banach lattice with finite cotype and let Y be a q-concave Banach lattice, and let $r \in (1,\infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then there exists a constant $C \in [0,\infty)$ such that the following holds. Let $m : \mathbb{R} \to \mathcal{L}(X,Y)$ be such that there exists a $K : \mathbb{R} \to \mathcal{L}(X,Y)$ satisfying the following conditions:*

(1) $K(t) \in \mathcal{L}(X,Y)$ is a positive operator for all $t \in \mathbb{R}$;
(2) $K(\cdot)x \in L^1(\mathbb{R};Y)$ for all $x \in X$;
(3) $\mathcal{F}(K(\cdot)x)(\xi) = m(\xi)x$ for all $x \in X$ and $\xi \in \mathbb{R}$.

Then $T_m : H_p^{1/r}(\mathbb{R};X) \to L^q(\mathbb{R};Y)$ is bounded and

$$\|T_m\|_{\mathcal{L}(H_p^{1/r}(\mathbb{R};X),L^q(\mathbb{R};Y))} \leq C\|m(0)\|_{\mathcal{L}(X,Y)} \leq C \sup_{\xi \in \mathbb{R}} \|m(\xi)\|_{\mathcal{L}(X,Y)}.$$

2.3. Sectorial operators

For a C_0-semigroup $(T(t))_{t \geq 0} \subseteq \mathcal{L}(X)$ on a Banach space X we let

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} \mid \exists M \in [0, \infty) : \|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \text{ for all } t \in [0, \infty)\}.$$

For $\varphi \in (0, \pi)$ set

$$S_\varphi := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg(z)| < \varphi\},$$

and let $S_0 := (0, \infty)$. Recall that an operator A on a Banach space X is *sectorial of angle* $\varphi \in [0, \pi)$ if $\sigma(A) \subseteq \overline{S_\varphi}$ and if $\sup\{\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \mid \lambda \in \mathbb{C} \setminus \overline{S_\theta}\} < \infty$ for all $\theta \in (\varphi, \pi)$. Then we write $A \in \mathrm{Sect}(\varphi, X)$ and we let $\omega_A := \min\{\varphi \in [0, \pi) \mid A \in \mathrm{Sect}(\varphi, X)\}$. An operator A such that

$$M(A) := \sup\{\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \mid \lambda \in (0, \infty)\} < \infty \tag{2.6}$$

is sectorial of angle $\varphi = \pi - \arcsin(1/M(A))$.

For a sectorial operator A on a Banach space X one has $N(A) \cap \overline{\mathrm{Ran}(A)} = \{0\}$ and, if X is reflexive, $X = N(A) \oplus \overline{\mathrm{Ran}(A)}$. If $-A$ generates a C_0-semigroup $(T(t))_{t \geq 0} \subseteq \mathcal{L}(X)$ then $T(t)x = x$ for all $x \in N(A)$ and $t \geq 0$. Moreover, the restriction of $(T(t))_{t \geq 0}$ to $\overline{\mathrm{Ran}(A)}$ is generated by the part of A in $\overline{\mathrm{Ran}(A)}$, which is injective. Hence for the purposes of stability theory it is natural to assume that A is injective, and we will do so frequently.

For the definition and various properties of fractional powers of sectorial operators we refer to [23,44]. We shall use in particular that, for $\varphi \in [0, \pi)$, $A \in \mathrm{Sect}(\varphi, X)$ and $\alpha, \beta, \eta \in (0, \infty)$, one has

$$A^\alpha(\eta + A)^{-\alpha-\beta} = \frac{1}{2\pi i} \int_{\partial S_\theta} \frac{z^\alpha}{(\eta + z)^{\alpha+\beta}} R(z, A)\mathrm{d}z. \tag{2.7}$$

Here ∂S_θ is the positively oriented boundary of S_θ for $\theta \in (\varphi, \pi)$. Note that A^α is injective for A injective, and if A is invertible then one may let $\alpha = 0$ in (2.7).

For A a sectorial operator and $\alpha, \beta \in [0, \infty)$ we set $\Phi_\beta^\alpha(A) := A^\alpha(1 + A)^{-\alpha-\beta} \in \mathcal{L}(X)$. We will frequently use that $\Phi_0^\alpha(A) = (A(1 + A)^{-1})^\alpha$ and that

$$\Phi_{\beta_1}^{\alpha_1}(A)\Phi_{\beta_2}^{\alpha_2}(A) = \Phi_{\beta_1+\beta_2}^{\alpha_1+\alpha_2}(A) \tag{2.8}$$

for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, \infty)$, by [23, Proposition 3.1.1]. Let $X_\beta^\alpha := \mathrm{Ran}(\Phi_\beta^\alpha(A))$, $X^\alpha := X_0^\alpha$ and $X_\beta := X_\beta^0$. If A is injective then X_β^α is a Banach space with the norm

$$\|x\|_{X_\beta^\alpha} := \|x\|_X + \|\Phi_\beta^\alpha(A)^{-1}x\|_X = \|x\|_X + \|(1 + A)^{\alpha+\beta}A^{-\alpha}x\|_X \quad (x \in X_\beta^\alpha).$$

It follows from [7, Proposition 3.10(i)] (the restriction $\alpha, \beta \in [0,1]$ is not needed here) that $X_\beta^\alpha = \operatorname{ran}(A^\alpha) \cap D(A^\beta)$ with equivalence of norms. Finally, note that $\Phi_\beta^\alpha(A) : X \to X_\beta^\alpha$ is an isomorphism. More precisely, there exists a constant $C \geq 0$ such that

$$\|T\|_{\mathcal{L}(X_\beta^\alpha, X)} \leq \|T\Phi_\beta^\alpha(A)\|_{\mathcal{L}(X)} \leq C\|T\|_{\mathcal{L}(X_\beta^\alpha, X)} \qquad (T \in \mathcal{L}(X_\beta^\alpha, X)). \qquad (2.9)$$

3. Resolvent estimates and multipliers

In this section we prove some statements on Fourier multipliers and resolvents which will be used in later sections.

3.1. Resolvents and Fourier multipliers

Throughout this subsection $-A$ is the generator of a C_0-semigroup $(T(t))_{t \geq 0}$ on a Banach space X.

For the reader's convenience we include a proof of the following standard lemma.

Lemma 3.1. *Let* $n \in \mathbb{N}_0$, $x \in X$ *and* $\xi \in \mathbb{R}$. *Suppose that* $-i\xi \in \rho(A)$ *and that* $[t \mapsto t^n T(t)x] \in L^1([0,\infty); X)$. *Then*

$$\mathcal{F}[t \mapsto t^n T(t)x](\xi) = n!(i\xi + A)^{-n-1}x, \qquad (3.1)$$

$$\mathcal{F}\left(\int_0^\infty t^n T(t)g(\cdot - t)x \, dt\right)(\xi) = \widehat{g}(\xi)n!(i\xi + A)^{-n-1}x \qquad (g \in L^1(\mathbb{R})). \qquad (3.2)$$

Proof. It suffices to prove (3.1), as (3.2) follows from (3.1) by standard properties of convolutions. Since $\lambda(\lambda + A)^{-1}x \to x$ as $\lambda \to \infty$, by the dominated convergence theorem we may additionally assume that $x \in D(A)$ and that $[t \mapsto t^n T(t)Ax] \in L^1([0,\infty); X)$. Also, [45, Lemma 3.1.9] implies that $[t \mapsto T(t)x] \in C_0([0,\infty); X)$. Now the fundamental theorem of calculus yields

$$(i\xi + A) \int_0^\infty e^{-i\xi t}T(t)x \, dt = \left[-e^{-i\xi t}T(t)x \right]_0^\infty = x.$$

Hence $\int_0^\infty e^{-i\xi t}T(t)x \, dt = (i\xi + A)^{-1}x$ and

$$\int_0^\infty e^{-i\xi t}t^n T(t)x \, dt = \frac{1}{(-i)^n}\frac{d^n}{d\xi^n}\int_0^\infty e^{-i\xi t}T(t)x \, dt = n!(i\xi + A)^{-n-1}x. \quad \square$$

We will often use the following proposition, inspired by [37, Theorem 3.1].

Proposition 3.2. *Let Y be a Banach space that is continuously embedded in X and let $n \in \mathbb{N}$. Suppose that $i\mathbb{R} \setminus \{0\} \subseteq \rho(A)$ and that there exist $\psi \in L^\infty(\mathbb{R})$, $p \in [1, \infty)$ and $q \in [1, \infty]$ such that for $j \in \{n-1, n\} \cap \mathbb{N}$ one has*

$$m_1^j(\cdot) := \psi(\cdot)R(i\cdot, A)^j \in \mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(Y, X)),$$
$$m_2^j(\cdot) := (1 - \psi(\cdot))R(i\cdot, A)^j \in \mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X)).$$

Then $T_{R(i\cdot, A)^n} : L^p(\mathbb{R}; Y) \cap L^1(\mathbb{R}; Y) \to L^\infty(\mathbb{R}; X)$ is bounded and $\|T_{R(i\cdot, A)^n}\| \le 2MC_n$, where $M = \sup\{\|T(t)\|_{\mathcal{L}(X)} \mid t \in [0, 2]\}$,

$$C_n = \sum_{j=n-1}^{n} \|m_1^j\|_{\mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(Y, X))} + \|m_2^j\|_{\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X))}$$

for $n > 1$, and

$$C_1 = \|m_1^1\|_{\mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(Y, X))} + \|m_2^1\|_{\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X))} + \|I_Y\|_{\mathcal{L}(Y, X)}.$$

Proof. Let $K \in \mathbb{N}$, $f_1, \ldots, f_K \in \dot{\mathcal{S}}(\mathbb{R})$ and $x_1, \ldots, x_K \in Y$, and set $f := \sum_{k=1}^{K} f_k \otimes x_k$. Then $T_{m_1^n}(f) \in C_b(\mathbb{R}; X)$ and

$$\sup_{t \in \mathbb{R}} \|T_{m_1^n}(f)(t)\|_X \le \|m_1^n\|_{\mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(Y, X))} \|f\|_{L^1(\mathbb{R}; Y)}. \tag{3.3}$$

Also,

$$\|T_{m_2^n}(f)\|_{L^q(\mathbb{R}; X)} \le \|m_2^n\|_{\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X))} \|f\|_{L^p(\mathbb{R}; Y)}.$$

The latter inequality implies that for each $l \in \mathbb{Z}$ there exists a $t \in [l, l+1]$ such that

$$\|T_{m_2^n}(f)(t)\|_X \le 2\|m_2^n\|_{\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X))} \|f\|_{L^p(\mathbb{R}; Y)}. \tag{3.4}$$

Fix an $l \in \mathbb{Z}$ and let $t \in [l, l+1]$ be such that (3.4) holds. Then (3.3) implies

$$\|T_{R(i\cdot, A)^n}(f)(t)\|_X \le 2(\|m_1^n\|_{\mathcal{M}_{1,\infty}} + \|m_2^n\|_{\mathcal{M}_{p,q}}) \|f\|_{L^1(\mathbb{R}; Y) \cap L^p(\mathbb{R}; Y)}. \tag{3.5}$$

Let $\tau \in [0, 2]$ and note that

$$e^{i\xi\tau} T(\tau) R(i\xi, A)x = R(i\xi, A)x + \int_0^\tau e^{i\xi r} T(r)x \, dr$$

for all $\xi \in \mathbb{R} \setminus \{0\}$ and $x \in X$. Hence

$$T(\tau)T_{R(i\cdot,A)^n}(f)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(t-\tau)} e^{i\xi\tau} T(\tau) R(i\xi, A)^n \widehat{f}(\xi)\, d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(t-\tau)} R(i\xi, A)^n \widehat{f}(\xi)\, d\xi$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{\tau} e^{i\xi(t-\tau)} e^{i\xi r} T(r) R(i\xi, A)^{n-1} \widehat{f}(\xi)\, dr d\xi$$

$$= T_{R(i\cdot,A)^n}(f)(t-\tau) + \int_0^{\tau} T(r) T_{R(i\cdot,A)^{n-1}}(f)(t-\tau+r)\, dr.$$

Now (3.5) and Hölder's inequality yield

$$\|T_{R(i\cdot,A)^n}(f)(t-\tau)\|_X$$

$$\leq M \Big(\|T_{R(i\cdot,A)^n}(f)(t)\|_X + \int_0^{\tau} \|T_{R(i\cdot,A)^{n-1}}(f)(t-\tau+r)\|_X\, dr \Big)$$

$$\leq 2M(\|m_1^n\|_{\mathcal{M}_{1,\infty}} + \|m_2^n\|_{\mathcal{M}_{p,q}}) \|f\|_{L^1(\mathbb{R};Y) \cap L^p(\mathbb{R};Y)}$$

$$+ M(\tau \|T_{m_1^{n-1}}(f)\|_{L^\infty(\mathbb{R};X)} + \tau^{1/q'} \|T_{m_2^{n-1}}(f)\|_{L^q(\mathbb{R};X)})$$

$$\leq 2M \Big(\sum_{j=n-1}^{n} \|m_1^j\|_{\mathcal{M}_{1,\infty}} + \|m_2^j\|_{\mathcal{M}_{p,q}} \Big) \|f\|_{L^p(\mathbb{R};Y) \cap L^1(\mathbb{R};Y)}$$

for $n > 1$. For $n = 1$ the computation is similar, but one can directly estimate

$$\int_0^{\tau} \|f(t-\tau-r)\|_X\, dr \leq \|I_Y\|_{\mathcal{L}(Y,X)} \|f\|_{L^1(\mathbb{R};Y)}.$$

This concludes the proof, since $\tau \in [0,2]$ and $l \in \mathbb{Z}$ are arbitrary and since $\dot{\mathcal{S}}(\mathbb{R}) \otimes Y \subseteq L^p(\mathbb{R};Y) \cap L^1(\mathbb{R};Y)$ is dense. $\quad \square$

Remark 3.3. When applying Proposition 3.2 we will consider ψ with compact support. Then one may assume that $m_1^j \in \mathcal{M}_{u,v}(\mathbb{R}; \mathcal{L}(Y, X))$ for general $u \in [1, \infty)$ and $v \in [1, \infty]$. For $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $\text{supp}(\psi)$ one has $m_1^j = \chi m_1^j \in \mathcal{M}_{u,\infty}(\mathbb{R}; \mathcal{L}(Y, X))$ by Young's inequality. The same proof now shows that $T_{R(i\cdot,A)^n} : L^u(\mathbb{R};Y) \cap L^p(\mathbb{R};Y) \to L^\infty(\mathbb{R};X)$ is bounded, with

$$\|T_{R(i\cdot,A)^n}\| \leq 2M \Big(\sum_{j=n-1}^{n} \|m_1^j\|_{\mathcal{M}_{u,v}(\mathbb{R};\mathcal{L}(Y,X))} + \|m_2^j\|_{\mathcal{M}_{p,q}(\mathbb{R};\mathcal{L}(Y,X))} \Big)$$

for $n > 1$, and similarly for $n = 1$. However, Young's inequality also shows that $m_1^j = \chi m_1^j \chi \in \mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(Y, X))$, so that these assumptions are no more general than those in Proposition 3.2.

3.2. Resolvent estimates

We now present two propositions on resolvent growth. The assertions on uniform boundedness have for the most part been obtained by different methods in [65, Lemma 3.3], [28, Lemma 1.1], [37, Lemma 3.2] and [7, Theorem 5.5]. The proof below allows us to also deduce the corresponding statements on R-boundedness directly. Note that if A satisfies (3.6) with $\alpha \in (0, 1)$ then one may in fact let $\alpha = 0$, by elementary properties of resolvents.

Proposition 3.4. *Let $\alpha \in \{0\} \cup [1, \infty)$, $\beta \in [0, \infty)$ and $\beta_0 \in [0, 1]$, and let A be an injective sectorial operator on a Banach space X. Let $\varphi \in (0, \frac{\pi}{2}]$ and $\Omega := \overline{\mathbb{C}_+} \setminus (S_\varphi \cup \{0\})$, and suppose that $-\Omega \subseteq \rho(A)$. Then the following statements hold:*

(1) *The collection*

$$\{\lambda^\alpha (\lambda + A)^{-1} \mid \lambda \in \Omega, |\lambda| \leq 1\} \subseteq \mathcal{L}(X) \tag{3.6}$$

is uniformly bounded if and only if

$$\{(\lambda + A)^{-1} \mid \lambda \in \Omega, |\lambda| \leq 1\} \subseteq \mathcal{L}(X^\alpha, X) \tag{3.7}$$

is uniformly bounded. Moreover, (3.6) is R-bounded if and only if (3.7) is R-bounded.
(2) *The collection*

$$\{\lambda^{-\beta} (\lambda + A)^{-1} \mid \lambda \in \Omega, |\lambda| \geq 1\} \subseteq \mathcal{L}(X) \tag{3.8}$$

is uniformly bounded if and only if

$$\{\lambda^{\beta_0} (\lambda + A)^{-1} \mid \lambda \in \Omega, |\lambda| \geq 1\} \subseteq \mathcal{L}(X_{\beta+\beta_0}, X) \tag{3.9}$$

is uniformly bounded. Moreover, (3.8) is R-bounded if and only if (3.9) is R-bounded.
(3) *The collection*

$$\left\{ (1 - \lambda)^{\beta_0} (\lambda + A)^{-1} A^\alpha (1 + A)^{-\alpha - \beta - \beta_0} - \frac{(-\lambda)^\alpha}{(1-\lambda)^{\alpha+\beta}} (\lambda + A)^{-1} \Big| \lambda \in \Omega \right\}$$

is R-bounded in $\mathcal{L}(X)$.

Proof. Fix $\theta \in (\max(\omega_A, \pi - \varphi), \pi)$ and let $\Gamma := \{re^{i\theta} \mid r \in [0, \infty)\} \cup \{re^{-i\theta} \mid r \in [0, \infty)\}$ be oriented from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$.

For (1) first note that, by the resolvent identity,

$$
\begin{aligned}
(\lambda + A)^{-1}A(1 + A)^{-1} &= (1 + A)^{-1} - \lambda(\lambda + A)^{-1}(1 + A)^{-1} \\
&= (1 + A)^{-1} - \frac{\lambda}{1 + \lambda}(\lambda + A)^{-1} - \frac{\lambda}{1 + \lambda}(1 + A)^{-1} \\
&= \frac{1}{1 + \lambda}(1 + A)^{-1} - \frac{\lambda}{1 + \lambda}(\lambda + A)^{-1}
\end{aligned}
$$

for all $\lambda \in \Omega$. Now (2.2) and (2.9) yield (1) for $\alpha = 1$.

Let $\alpha > 1$. Then

$$
\begin{aligned}
(\lambda + A)^{-1}A^\alpha(1 + A)^{-\alpha} &= (\lambda + A)^{-1}(1 + A)A^\alpha(1 + A)^{-\alpha-1} \\
&= A^\alpha(1 + A)^{-\alpha-1} + (1 - \lambda)(\lambda + A)^{-1}A^\alpha(1 + A)^{-\alpha-1}
\end{aligned}
\tag{3.10}
$$

for all $\lambda \in \Omega$. Since the singleton $\{A^\alpha(1 + A)^{-\alpha-1}\} \subseteq \mathcal{L}(X)$ is R-bounded, by (2.9) it suffices to show that (3.6) is uniformly bounded (or R-bounded) if and only if

$$
\{(1 - \lambda)(\lambda + A)^{-1}A^\alpha(1 + A)^{-\alpha-1} \mid \lambda \in \Omega, |\lambda| \leq 1\} \subseteq \mathcal{L}(X)
\tag{3.11}
$$

is uniformly bounded (or R-bounded). The resolvent identity and (2.7) yield

$$
\begin{aligned}
(\lambda + A)^{-1}A^\alpha(1 + A)^{-\alpha-1} &= \frac{1}{2\pi i}\int_\Gamma \frac{z^\alpha}{(1 + z)^{\alpha+1}}(\lambda + A)^{-1}R(z, A)\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{z^\alpha}{(1 + z)^{\alpha+1}(z + \lambda)}\,\mathrm{d}z(\lambda + A)^{-1} \\
&\quad + \frac{1}{2\pi i}\int_\Gamma \frac{z^\alpha}{(1 + z)^{\alpha+1}(z + \lambda)}R(z, A)\,\mathrm{d}z
\end{aligned}
$$

for $\lambda \in \Omega$. Hence, using (A.1) of Lemma 5.9,

$$
(1 - \lambda)(\lambda + A)^{-1}A^\alpha(1 + A)^{-\alpha-1} = \frac{(-\lambda)^\alpha}{(1 - \lambda)^\alpha}(\lambda + A)^{-1} + S_\lambda,
\tag{3.12}
$$

where

$$
S_\lambda := \frac{1}{2\pi i}\int_\Gamma \frac{z^\alpha}{(1 + z)^{\alpha+1}}\frac{1 - \lambda}{z + \lambda}R(z, A)\,\mathrm{d}z.
$$

Now fix $\varepsilon \in (0, \min(\alpha - 1, 1)]$. Then $z \mapsto \frac{z^\varepsilon}{(1+z)^{2\varepsilon}}R(z, A)$ is integrable on Γ, and

$$
\sup\left\{\frac{|z|^{\alpha-\varepsilon}}{|1 + z|^{\alpha+1-2\varepsilon}}\frac{|1 - \lambda|}{|z + \lambda|}\,\middle|\,\lambda \in \Omega, z \in \Gamma\right\} < \infty
$$

by (A.2) in Lemma 5.9. Hence [33, Corollary 2.17] implies that $\{S_\lambda \mid \lambda \in \Omega\} \subseteq \mathcal{L}(X)$ is R-bounded. Now (3.12) shows that the uniform boundedness (or R-boundedness) of (3.6) and (3.11) are equivalent, thereby proving (1).

For (2) we may suppose that $\beta + \beta_0 > 0$. Then (2.7), applied to the invertible sectorial operator $\frac{1}{2} + A$, and the resolvent identity imply that

$$
(\lambda + A)^{-1}(1 + A)^{-\beta - \beta_0} = \frac{1}{2\pi i} \int_\Gamma \frac{1}{(\frac{1}{2} + z)^{\beta + \beta_0}} (\lambda + A)^{-1} R(z, \tfrac{1}{2} + A) \, \mathrm{d}z
$$

$$
= \frac{1}{2\pi i} \int_\Gamma \frac{1}{(\frac{1}{2} + z)^{\beta + \beta_0}(z + \lambda - \frac{1}{2})} \, \mathrm{d}z (\lambda + A)^{-1}
$$

$$
+ \frac{1}{2\pi i} \int_\Gamma \frac{1}{(\frac{1}{2} + z)^{\beta + \beta_0}(z + \lambda - \frac{1}{2})} R(z, \tfrac{1}{2} + A) \, \mathrm{d}z
$$

for $\lambda \in \Omega$. Now (A.1) yields

$$
(1 - \lambda)^{\beta_0}(\lambda + A)^{-1}(1 + A)^{-\beta - \beta_0} = \frac{1}{(1 - \lambda)^\beta}(\lambda + A)^{-1} + (1 - \lambda)^{\beta_0} T_\lambda, \tag{3.13}
$$

where

$$
T_\lambda := \frac{1}{2\pi i} \int_\Gamma \frac{1}{(\frac{1}{2} + z)^{\beta + \beta_0}(z + \lambda - \frac{1}{2})} R(z, \tfrac{1}{2} + A) \, \mathrm{d}z.
$$

Fix $\varepsilon \in (0, \beta + \beta_0)$. Then $z \mapsto (z + \frac{1}{2})^{-\varepsilon} R(z, \frac{1}{2} + A)$ is integrable on Γ, and

$$
\sup \left\{ \frac{1 + |\lambda|}{|\frac{1}{2} + z|^{\beta + \beta_0 - \varepsilon}|z + \lambda - \frac{1}{2}|} \Big| \lambda \in \Omega, z \in \Gamma \right\} < \infty
$$

by (A.2). Hence [33, Corollary 2.17] implies that $\{(1 + |\lambda|)T_\lambda \mid \lambda \in \Omega\}$ is R-bounded. Since $|1 - \lambda|^{\beta_0} \leq 1 + |\lambda|$ for all $\lambda \in \Omega$, the proof of part (2) is completed using (2.2), (3.13) and (2.9).

Finally, for (3) we restrict to the case where $\alpha > 1$ and $\beta > 0$. The other cases follow in a similar manner from the proofs of (1) and (2). The operator family in (3) can be written as

$$
A^\alpha (1 + A)^{-\alpha} \left[(1 - \lambda)^{\beta_0}(\lambda + A)^{-1}(1 + A)^{-\beta - \beta_0} - (1 - \lambda)^{-\beta}(\lambda + A)^{-1} \right]
$$

$$
+ (1 - \lambda)^{-\beta} \left[(\lambda + A)^{-1} A^\alpha (1 + A)^{-\alpha} - \frac{(-\lambda)^\alpha}{(1 - \lambda)^\alpha}(\lambda + A)^{-1} \right]
$$

$$
=: A^\alpha (1 + A)^{-\alpha} V_\lambda^1 + (1 - \lambda)^{-\beta} V_\lambda^2.
$$

Using standard algebraic properties of R-boundedness (see [30, Proposition 8.1.19]), it suffices to prove that $\{V^i_\lambda \mid \lambda \in \Omega\} \subseteq \mathcal{L}(X)$ is R-bounded for $i \in \{1, 2\}$. The proof of (2), and in particular (3.13), shows that

$$R(\{V^1_\lambda \mid \lambda \in \Omega\}) = R(\{\{(1-\lambda)^{\beta_0} T_\lambda \mid \lambda \in \Omega\}) < \infty.$$

For the other term note that, by (3.10) and (3.12), $V^2_\lambda = A^\alpha (1 + A)^{-\alpha-1} + S_\lambda$. Hence the proof of (1) yields

$$R(\{V^2_\lambda \mid \lambda \in \Omega\}) \leq \|A^\alpha (1 + A)^{-\alpha-1}\|_{\mathcal{L}(X)} + R(\{S_\lambda \mid \lambda \in \Omega\}) < \infty. \quad \square$$

Corollary 3.5. *Let $\alpha \in [0, \infty)$ and $\alpha_0 \in [0, \alpha]$. Let A be an injective sectorial operator on a Banach space X such that $i\mathbb{R} \setminus \{0\} \subseteq \rho(A)$ and*

$$\sup\{\|\lambda^\alpha (\lambda + A)^{-1}\|_{\mathcal{L}(X)} \mid \lambda \in i\mathbb{R} \setminus \{0\}, |\lambda| \leq 1\} < \infty.$$

Then

$$\sup\{\|\lambda^{\alpha-\alpha_0} (\lambda + A)^{-1}\|_{\mathcal{L}(X^{\alpha_0}, X)} \mid \lambda \in i\mathbb{R} \setminus \{0\}, |\lambda| \leq 1\} < \infty.$$

Proof. First note that $0 \in \rho(A)$ for $\alpha < 1$, by elementary properties of resolvents. Hence, by Proposition 3.4 (1) it suffices to consider $\alpha \geq 1$ and $\alpha_0 \in (0, \alpha)$. By [23, Propositions 2.1.1.f and 3.1.9], $A(1 + A)^{-1}$ is a sectorial operator and $A^{\alpha_0} (1 + A)^{-\alpha_0} = (A(1 + A)^{-1})^{\alpha_0}$. Now the moment inequality [23, Proposition 6.6.4] and another application of [23, Proposition 3.1.9] yield

$$
\begin{aligned}
&\|\lambda^{\alpha-\alpha_0} (\lambda + A)^{-1} A^{\alpha_0} (1 + A)^{-\alpha_0} x\|_X \\
&= |\lambda|^{\alpha-\alpha_0} \|(A(1 + A)^{-1})^{\alpha_0} (\lambda + A)^{-1} x\|_X \\
&\lesssim |\lambda|^{\alpha-\alpha_0} \|(\lambda + A)^{-1} (A(1 + A)^{-1})^\alpha x\|_X^{\alpha_0/\alpha} \|(\lambda + A)^{-1} x\|_X^{(\alpha-\alpha_0)/\alpha} \\
&\leq \|(\lambda + A)^{-1} A^\alpha (1 + A)^{-\alpha}\|_{\mathcal{L}(X)}^{\alpha_0/\alpha} \|\lambda^\alpha (\lambda + A)^{-1}\|_{\mathcal{L}(X)}^{(\alpha-\alpha_0)/\alpha} \|x\|_X
\end{aligned}
$$

for all $\lambda \in i\mathbb{R} \setminus \{0\}$ and $x \in X$. Proposition 3.4 (1) and (2.9) conclude the proof. $\quad \square$

4. Polynomial stability

In this section we study polynomial stability for semigroups using Fourier multipliers. We first obtain some results valid on general Banach spaces. Then we establish the connection between polynomial stability and Fourier multipliers, and we use this link to deduce polynomial stability results under geometric assumptions on the underlying space. We also study the necessity of the spectral assumptions which we make, compare our theorems with the literature, and give examples to illustrate our results.

The following terminology will be used throughout this section.

Definition 4.1. Let $\alpha, \beta \in [0, \infty)$. An operator A on a Banach space X has *resolvent growth* (α, β) if the following conditions hold:

(i) $-A$ generates a C_0-semigroup $(T(t))_{t \geq 0}$ on X;
(ii) $\overline{\mathbb{C}_-} \setminus \{0\} \subseteq \rho(A)$, and

$$\left\{ \frac{\lambda^\alpha}{(1+\lambda)^{\alpha+\beta}} (\lambda + A)^{-1} \,\middle|\, \lambda \in \overline{\mathbb{C}_+} \setminus \{0\} \right\} \subseteq \mathcal{L}(X)$$

is uniformly bounded.

An operator A has *R-resolvent growth* (α, β) if A has resolvent growth (α, β) and

$$\left\{ \lambda^{-\beta} (\lambda + A)^{-1} \,\middle|\, \lambda \in \overline{\mathbb{C}_+}, |\lambda| \geq 1 \right\} \subseteq \mathcal{L}(X)$$

is *R*-bounded.

Note that we do not assume in (i) that the semigroup generated by $-A$ is uniformly bounded. We will implicitly use throughout that each operator A with resolvent growth (α, β), for $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$, is invertible and thus has resolvent growth $(0, \beta)$, as follows from the fact that $\|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \operatorname{dist}(\lambda, \sigma(A))^{-1}$ for all $\lambda \in \rho(A)$.

Recall that we use the convention that $\frac{0}{0} = \infty$, for simplicity of notation.

4.1. Results on general Banach spaces

The following lemma is used to interpolate between decay rates. Related results can be found in [4, Proposition 3.1] and [7, Lemma 4.2]. Recall the definition of the space X_β^α, for $\alpha, \beta \geq 0$, from Section 2.3.

Lemma 4.2. *Let A be an injective sectorial operator on a Banach space X such that $-A$ generates a C_0-semigroup $(T(t))_{t \geq 0}$ on X. For $j \in \{1, 2\}$ let $\alpha_j, \beta_j \in [0, \infty)$ be such that $\alpha_1 \geq \alpha_2$ and $\beta_1 \geq \beta_2$, and let $f_j : [0, \infty) \to [0, \infty)$ be such that $\|T(t)\|_{\mathcal{L}(X_{\beta_j}^{\alpha_j}, X)} \leq f_j(t)$ for all $t \in [0, \infty)$. Then for each $\theta \in [0, 1]$ there exists a $C_\theta \in [0, \infty)$ such that*

$$\|T(t)\|_{\mathcal{L}(X_{\theta\beta_1 + (1-\theta)\beta_2}^{\theta\alpha_1 + (1-\theta)\alpha_2}, X)} \leq C_\theta (f_1(t))^\theta (f_2(t))^{1-\theta} \qquad (t \in [0, \infty)). \tag{4.1}$$

Moreover, suppose that $f_1(t) = Ct^{-\mu}$ for some $C, \mu \in [0, \infty)$ and all $t \in [1, \infty)$. Then for each $\theta \in [1, \infty)$ there exists a $C_\theta \in [0, \infty)$ such that

$$\|T(t)\|_{\mathcal{L}(X_{\theta\beta_1}^{\theta\alpha_1}, X)} \leq C_\theta t^{-\mu\theta} \qquad (t \in [1, \infty)). \tag{4.2}$$

Proof. Let $t \in [0, \infty)$ and note that, by (2.9) and (2.8),

$$\|T(t)\|_{\mathcal{L}(X_{\theta\beta_1 + (1-\theta)\beta_2}^{\theta\alpha_1 + (1-\theta)\alpha_2}, X)} \leq \|T(t)\Phi_{\theta\beta_1 + (1-\theta)\beta_2}^{\theta\alpha_1 + (1-\theta)\alpha_2}(A)\|_{\mathcal{L}(X)}$$

$$= \|T(t)\Phi_{\theta(\beta_1-\beta_2)}^{\theta(\alpha_1-\alpha_2)}(A)\Phi_{\beta_2}^{\alpha_2}(A)\|_{\mathcal{L}(X)}.$$

Let $c := \alpha_1 - \alpha_2 + \beta_1 - \beta_2$. Then $\Phi_{(\beta_1-\beta_2)/c}^{(\alpha_1-\alpha_2)/c}(A) = A^{(\alpha_1-\alpha_2)/c}(1+A)^{-1}$ is sectorial, by [7, Proposition 3.10]. Hence [23, Theorem 2.4.2] yields

$$\Phi_{\theta(\beta_1-\beta_2)}^{\theta(\alpha_1-\alpha_2)}(A) = A^{\theta(\alpha_1-\alpha_2)}(1+A)^{-\theta(\alpha_1-\alpha_2+\beta_1-\beta_2)} = (\Phi_{(\beta_1-\beta_2)/c}^{(\alpha_1-\alpha_2)/c}(A))^{c\theta}.$$

The moment inequality [23, Proposition 6.6.4] and [23, Theorem 2.4.2] imply that

$$\|(\Phi_{(\beta_1-\beta_2)/c}^{(\alpha_1-\alpha_2)/c}(A))^{c\theta}x\|_X \lesssim \|(\Phi_{(\beta_1-\beta_2)/c}^{(\alpha_1-\alpha_2)/c}(A))^c x\|_X^\theta \|x\|_X^{1-\theta} = \|\Phi_{\beta_1-\beta_2}^{\alpha_1-\alpha_2}(A)x\|_X^\theta \|x\|_X^{1-\theta}$$

for $x \in D(\Phi_{\beta_1-\beta_2}^{\alpha_1-\alpha_2}(A))$. Combining all this with (2.8) and (2.9) shows that

$$\|T(t)\|_{\mathcal{L}(X_{\theta\beta_1 + (1-\theta)\beta_2}^{\theta\alpha_1 + (1-\theta)\alpha_2}, X)} \leq \|T(t)\Phi_{\theta(\beta_1-\beta_2)}^{\theta(\alpha_1-\alpha_2)}(A)\Phi_{\beta_2}^{\alpha_2}(A)\|_{\mathcal{L}(X)}$$

$$= \|(\Phi_{(\beta_1-\beta_2)/c}^{(\alpha_1-\alpha_2)/c}(A))^{c\theta}T(t)\Phi_{\beta_2}^{\alpha_2}(A)\|_{\mathcal{L}(X)}$$

$$\lesssim \|\Phi_{\beta_1-\beta_2}^{\alpha_1-\alpha_2}(A)T(t)\Phi_{\beta_2}^{\alpha_2}(A)\|_{\mathcal{L}(X)}^\theta \|T(t)\Phi_{\beta_2}^{\alpha_2}(A)\|_{\mathcal{L}(X)}^{1-\theta}$$

$$= \|T(t)\Phi_{\beta_1}^{\alpha_1}(A)\|_{\mathcal{L}(X)}^\theta \|T(t)\Phi_{\beta_2}^{\alpha_2}(A)\|_{\mathcal{L}(X)}^{1-\theta}$$

$$\lesssim \|T(t)\|_{\mathcal{L}(X_{\beta_1}^{\alpha_1}, X)}^\theta \|T(t)\|_{\mathcal{L}(X_{\beta_2}^{\alpha_2}, X)}^{1-\theta} \leq (f_1(t))^\theta (f_2(t))^{1-\theta},$$

thereby proving (4.1). As for (4.2), let $n \in \mathbb{N}$. Then

$$\|T(t)\|_{\mathcal{L}(X_{n\beta_1}^{n\alpha_1}, X)} \leq \|T(t)\Phi_{n\beta_1}^{n\alpha_1}(A)\|_{\mathcal{L}(X)} \leq \|T(\tfrac{t}{n})\Phi_{\beta_1}^{\alpha_1}(A)\|_{\mathcal{L}(X)}^n$$

$$\lesssim (f_1(\tfrac{t}{n}))^n = C^n n^{\mu n} t^{-\mu n},$$

which implies (4.2) for $\theta \in \mathbb{N}$. Finally, applying (4.1) to interpolate between $(n\alpha_1, n\beta_1)$ and $((n+1)\alpha_1, (n+1)\beta_1)$ yields (4.2) for all $\theta \in [1, \infty)$. $\quad\square$

The following result for C_0-semigroups on general Banach spaces extends [4, Proposition 3.4], where the case $\alpha = \rho = 0$ was considered.

Proposition 4.3. *Let $\alpha, \beta \in [0, \infty)$ and let A be an injective sectorial operator with resolvent growth (α, β) on a Banach space X. Let $\sigma, \tau \in [0, \infty)$ be such that $\sigma > \alpha - 1$ and $\tau > \beta + 1$. Then for each $\rho \in [0, \min(\frac{\sigma+1}{\alpha} - 1, \frac{\tau-1}{\beta} - 1))$ there exists a $C_\rho \in [0, \infty)$ such that*

$$\|T(t)\|_{\mathcal{L}(X_\tau^\sigma, X)} \leq C_\rho t^{-\rho} \qquad (t \in [1, \infty)). \tag{4.3}$$

Proof. By elementary calculations the proposition is equivalent to the following statement: for all $s \geq 0$ and $\delta, \varepsilon > 0$ there exists a $C_{s,\delta,\varepsilon} \geq 0$ such that

$$\|T(t)\|_{\mathcal{L}(X_{\nu}^{\mu}, X)} \leq C_{s,\delta,\varepsilon} t^{-s} \qquad (t \in [1, \infty)), \tag{4.4}$$

where $\mu = \max((s+1)\alpha - 1 + \delta, 0)$ and $\nu = (s+1)\beta + 1 + \varepsilon$. Furthermore, by Lemma 4.2 it suffices to prove (4.4) for $n := s \in \mathbb{N}_0$.

Let $x \in X_{\nu+1}^{\mu}$ and set $y := \Phi_{\nu}^{\mu}(A)x = A^{-\mu}(1 + A)^{\mu+\nu}x \in D(A)$. Then

$$g(t) := \frac{1}{2\pi i} \int\limits_{i\infty}^{-i\infty} e^{-\lambda t} \frac{\lambda^{\mu}}{(1 + \lambda)^{\mu+\nu}} R(\lambda, A) y \, \mathrm{d}\lambda$$

is a well defined element of X for all $t \geq 0$. One can check that g is continuously differentiable with $g'(t) = -Ag(t)$. Also,

$$g(0) = \frac{1}{2\pi i} \int\limits_{i\infty}^{-i\infty} \frac{\lambda^{\mu}}{(1 + \lambda)^{\mu+\nu}} R(\lambda, A) y \, \mathrm{d}\lambda = A^{\mu}(1 + A)^{-\mu-\nu} y = x.$$

Here we have deformed the path of integration to the curve $\Gamma = \{re^{i\theta} \mid r \in [0, \infty)\} \cup \{re^{-i\theta} \mid r \in [0, \infty)\}$ in (2.7), for $\theta \in (\omega_A, \pi)$, which we may do by the assumptions on A. Now $g(t) = T(t)x$, by uniqueness of the Cauchy problem associated with $-A$. Integration by parts yields

$$t^n T(t)x = \frac{t^n}{2\pi i} \int\limits_{i\mathbb{R}} e^{-\lambda t} \frac{\lambda^{\mu}}{(1 + \lambda)^{\mu+\nu}} R(\lambda, A) y \, \mathrm{d}\lambda$$

$$= \frac{(-1)^n}{2\pi i} \int\limits_{i\mathbb{R}} \left(\frac{d^n}{d\lambda^n} e^{-\lambda t} \right) \frac{\lambda^{\mu}}{(1 + \lambda)^{\mu+\nu}} R(\lambda, A) y \, \mathrm{d}\lambda$$

$$= \frac{1}{2\pi i} \int\limits_{i\mathbb{R}} e^{-\lambda t} p(\lambda, A) y \, \mathrm{d}\lambda.$$

Here $p(\lambda, A)$ is a finite linear combination of terms of the form

$$\frac{\lambda^{\mu-j}}{(1 + \lambda)^{\mu+\nu+(k-j)}} R(\lambda, A)^{n-k+1}$$

for $0 \leq j \leq k \leq n$, where we let $j = 0$ if $\mu = 0$. Then

$$\|t^n T(t)x\|_X \leq \frac{1}{2\pi} \int\limits_{i\mathbb{R}} \|p(\lambda, A)\|_{\mathcal{L}(X)} \|y\|_X \mathrm{d}|\lambda| \lesssim \|(-A)^{-\mu}(1 + A)^{\mu+\nu} x\|_X \leq \|x\|_{X_{\nu}^{\mu}}$$

with implicit constants independent of t and x. Since $X_{\nu+1}^\mu$ is dense in X_ν^μ, the proof is concluded. □

The following corollary of Proposition 4.3 and Lemma 4.2 takes into account the growth behavior of $(T(t))_{t\geq 0}$ on X. It also extends Proposition 4.3 by providing stability rates on X_τ^σ for $\sigma \in [0, \alpha - 1]$ and $\tau \in [0, \beta + 1]$. The same approach was used in [4, Theorem 3.5] for uniformly bounded semigroups and $\alpha = 0$.

Corollary 4.4. *Let $\alpha, \beta \in [0, \infty)$ and let A be an injective sectorial operator with resolvent growth (α, β) on a Banach space X. Let $\sigma, \tau \in [0, \infty)$. Then for each $\rho \in [0, \min(\frac{\sigma}{\alpha}, \frac{\tau}{\beta}))$ there exists a $C_\rho \in [0, \infty)$ such that*

$$\|T(t)\|_{\mathcal{L}(X_\tau^\sigma, X)} \leq C_\rho \max(1, \|T(t)\|_{\mathcal{L}(X)})t^{-\rho} \qquad (t \in [1, \infty)). \tag{4.5}$$

Proof. By elementary calculations it suffices to prove the following: for all $s \geq 0$ and $\delta, \varepsilon > 0$ there exists a constant $C_{s,\delta,\varepsilon} \geq 0$ such that

$$\|T(t)\|_{\mathcal{L}(X_\nu^\mu, X)} \leq C_{s,\delta,\varepsilon} \max(1, \|T(t)\|_{\mathcal{L}(X)})t^{-s} \qquad (t \in [1, \infty)), \tag{4.6}$$

where $\mu = s\alpha + \delta$ and $\nu = s\beta + \varepsilon$. Let $\widetilde{\varepsilon} > 0$ and for $\theta \in (0, 1)$ set $\widetilde{s} := s/\theta$, $\widetilde{\mu} := \max((\widetilde{s}+1)\alpha - 1 + \widetilde{\varepsilon}, 0)$ and $\widetilde{\nu} := (\widetilde{s}+1)\beta + 1 + \widetilde{\varepsilon}$. Then, by Lemma 4.2 and (4.4),

$$\|T(t)\|_{\mathcal{L}(X_{\widetilde{\nu}\theta}^{\widetilde{\mu}\theta}, X)} \lesssim \|T(t)\|_{\mathcal{L}(X)}^{1-\theta}\|T(t)\|_{\mathcal{L}(X_{\widetilde{\nu}}^{\widetilde{\mu}}, X)}^{\theta} \lesssim \max(1, \|T(t)\|_{\mathcal{L}(X)})t^{-s}$$

for all $t \geq 1$. Next, note that $\widetilde{\mu}\theta = \max(s\alpha + \theta(\alpha - 1 + \widetilde{\varepsilon}), 0)$ and $\widetilde{\nu}\theta = s\beta + \theta(\beta + 1 + \widetilde{\varepsilon})$. Now the proof is concluded by letting $\theta \in (0, 1)$ be such that $\widetilde{\mu}\theta \leq s\alpha + \varepsilon$ and $\widetilde{\nu}\theta \leq s\beta + \varepsilon$. □

4.2. Polynomial stability and Fourier multipliers

In this subsection we relate polynomial stability of a semigroup to Fourier multiplier properties of the resolvent of its generator.

In order to state our abstract result on polynomial stability we introduce a class of admissible spaces.

Definition 4.5. Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and let $n \in \mathbb{N}_0$. A Banach space Y which is continuously embedded in X is (A, n)-*admissible* if the following conditions hold:

(i) there exists a constant $C_T \in [0, \infty)$ such that $T(t)Y \subseteq Y$ and

$$\|T(t)\restriction_Y\|_{\mathcal{L}(Y)} \leq C_T\|T(t)\|_{\mathcal{L}(X)} \qquad (t \in [0, \infty));$$

(ii) there exists a dense subspace $Y_0 \subseteq Y$ such that $[t \mapsto t^n T(t)y] \in L^1([0, \infty); X)$ for all $y \in Y_0$.

Let $\alpha, \beta \in [0, \infty)$ and let A be an injective sectorial operator with resolvent growth (α, β). Then $Y = X_\tau^\sigma$ is (A, n)-admissible for all $\sigma, \tau \in [0, \infty)$ and $n \in \mathbb{N}_0$, by Proposition 4.3.

The following theorem is our main result relating polynomial stability and Fourier multipliers. It follows from (4.10) and (4.12) below that one can obtain quantitative bounds in each of the implications between (1) and (2).

Theorem 4.6 *(Characterization of polynomial stability). Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and assume that A has resolvent growth (α, β) for some $\alpha, \beta \in [0, \infty)$. Let $n \in \mathbb{N}_0$ and let Y be an (A, n)-admissible space. Then the following statements are equivalent:*

(1) $\sup_{t \geq 0} \| t^n T(t) \|_{\mathcal{L}(Y, X)} < \infty;$
(2) *there exist $\psi \in C_c^\infty(\mathbb{R})$, $p \in [1, \infty)$ and $q \in [p, \infty]$ such that*

$$\psi(\cdot) R(i\cdot, A)^k \in \mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(Y, X)),$$
$$(1 - \psi(\cdot)) R(i\cdot, A)^k \in \mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X))$$

for all $k \in \{n - 1, n, n + 1\} \cap \mathbb{N}$.

Moreover, if (1) or (2) holds then $R(i\cdot, A)^k \in \mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X))$ for:

(i) *$n \geq 2$, $k \in \{1, \ldots, n - 1\}$ and $1 \leq p \leq q \leq \infty;$*
(ii) *$k = n \geq 1$ and $1 \leq p < q \leq \infty;$*
(iii) *$k = n + 1$, $p = 1$ and $q = \infty$.*

Proof. $(2) \Rightarrow (1)$: Let $\omega, M_\omega \geq 1$ be such that $\|T(t)\|_{\mathcal{L}(X)} \leq M_\omega e^{t(\omega-1)}$ for all $t \geq 0$, and set

$$m(\xi) := n!(i\xi + A)^{-n}(I_X + \omega(i\xi + A)^{-1}) \in \mathcal{L}(Y, X) \qquad (\xi \in \mathbb{R} \setminus \{0\}).$$

Since $(i \cdot + A)^{-1} = -R(-i\cdot, A)$, it follows from Proposition 3.2 that

$$T_m : L^p(\mathbb{R}; Y) \cap L^1(\mathbb{R}; Y) \to L^\infty(\mathbb{R}; X)$$

is bounded with

$$\|T_m\| \leq 2Mn!(C_n + \omega C_{n+1}). \tag{4.7}$$

Here $M := \sup_{t \in [0,2]} \|T(t)\|_{\mathcal{L}(X)}$, C_k is as in Proposition 3.2 for $k \in \mathbb{N}$, and $C_0 := \|I_Y\|_{\mathcal{L}(Y, X)}$. Now let $Y_0 \subseteq Y$ be as in Definition 4.5 and fix $x \in Y_0$. Lemma 3.1 yields

$$\mathcal{F}[t \mapsto t^n T(t)x](\cdot) = n!(i \cdot + A)^{-n-1}x. \tag{4.8}$$

Set $f(t) := e^{-\omega t} T(t) x$ for $t \geq 0$, and $f \equiv 0$ on $(-\infty, 0)$. Then

$$\|f(t)\|_Y \leq \|e^{-\omega t} T(t)\|_{\mathcal{L}(Y)} \|x\|_Y \leq C_T \|e^{-\omega t} T(t)\|_{\mathcal{L}(X)} \|x\|_Y \qquad (t \in [0, \infty)). \qquad (4.9)$$

Hence $f \in L^1(\mathbb{R}; Y) \cap L^\infty(\mathbb{R}; Y)$ and $\|f\|_{L^r(\mathbb{R}; Y)} \leq C_T M_\omega \|x\|_Y$ for all $r \in [1, \infty]$. By Lemma 3.1, $\widehat{f}(\cdot) = (w + i \cdot + A)^{-1} x$. Therefore, by the resolvent identity,

$$m(\xi) \widehat{f}(\xi) = n! (i\xi + A)^{-n-1} x \qquad (\xi \in \mathbb{R} \setminus \{0\}).$$

Combining (4.8) and (4.9) with (4.7) yields

$$\sup_{t \geq 0} \|t^n T(t) x\|_X \leq \|T_m\| \big(\|f\|_{L^p(\mathbb{R}; Y)} + |f|_{L^1(\mathbb{R}; Y)} \big) \leq C \|x\|_Y, \qquad (4.10)$$

where $C = 4 M n! C_T M_\omega (C_n + \omega C_{n+1})$. The required result now follows since $Y_0 \subseteq Y$ is dense.

(1) $\Rightarrow$ (2): Set $K_n := \sup_{t \geq 0} \|t^n T(t) x\|_X$ and let $Y_0 \subseteq Y$ be as in Definition 4.5. Let $f \in \dot{\mathcal{S}}(\mathbb{R}) \otimes Y_0$ and set $S_k(f)(s) := \int_0^\infty t^k T(t) f(s - t) dt$ for $s \in \mathbb{R}$ and $k \in \{0, 1, \ldots, n\}$. Lemma 3.1 yields

$$S_k(f) = k! \mathcal{F}^{-1}((i \cdot + A)^{-k-1} \widehat{f}(\cdot)) = k! T_{(i \cdot + A)^{-k-1}}(f). \qquad (4.11)$$

Now, for $n \geq 2$, $k \in \{0, \ldots, n-2\}$ and $r \in [1, \infty]$,

$$\big\| [t \mapsto \|t^k T(t)\|] \big\|_{L^r([0, \infty); \mathcal{L}(Y, X))} dt \leq M + K_n \|[t \mapsto t^{-2}]\|_{L^r(1, \infty)} \leq M + K_n.$$

Similarly, for $n \geq 1$ and $r \in (1, \infty]$,

$$\big\| [t \mapsto \|t^{n-1} T(t)\|] \big\|_{L^r([0, \infty); \mathcal{L}(Y, X))} \leq M + \frac{K_n}{(r-1)^{1/r}}.$$

By combining these estimates with (4.11) and with Young's inequality for operator-valued kernels in [3, Proposition 1.3.5] one obtains, for $p \in [1, \infty)$ and $q \in [p, \infty]$,

$$\|R(i\cdot, A)^k\|_{\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X))} \leq \frac{M + K^n}{(k-1)!} \qquad (n \geq 2, k \in \{1, \ldots, n-1\}),$$

$$\|R(i\cdot, A)^n\|_{\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X))} \leq \frac{M + (r-1)^{-1/r} K_n}{(n-1)!} \qquad (n \geq 1, p < q), \qquad (4.12)$$

$$\|R(i\cdot, A)^{n+1}\|_{\mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(Y, X))} \leq \frac{K_n}{n!}.$$

Now (4.11) and (4.12) yield statements (i)-(iii) for $(i\cdot + A)^{-1}$, and by reflection these statements hold for $R(i\cdot, A)$ as well. Finally, for (2) let $\psi \in C_c^\infty(\mathbb{R})$. Then Young's inequality and (4.12) yield $\psi(\cdot) R(i\cdot, A)^k \in \mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(Y, X))$ for all $k \in \{1, \ldots, n+1\}$, and one obtains (4.12) for $\psi(\cdot) R(i\cdot, A)$ with an additional multiplicative factor $\|\mathcal{F}^{-1}(\psi)\|_{L^1(\mathbb{R})}$. Similarly, (4.12) holds with an additional multiplicative factor $\|\mathcal{F}^{-1}(1 - \psi)\|_{L^1(\mathbb{R})}$ upon replacing $R(i\cdot, A)$ by $(1 - \psi(\cdot)) R(i\cdot, A)$. $\square$

The assumption in Theorem 4.6 that A has resolvent growth (α, β) for some $\alpha, \beta \in [0, \infty)$ is only made to ensure that $T_{R(i\cdot, A)}$ is well-defined, and the specific choice of α and β is irrelevant here. Inspection of the proof of Theorem 4.6 also shows that one could assume in (2) that for each $k \in \{n-1, n, n+1\} \cap \mathbb{N}$ there exist $p_k, q_k \in [1, \infty]$ such that

$$(1 - \psi(\cdot))R(i\cdot, A)^k \in \mathcal{M}_{p_k, q_k}(\mathbb{R}; \mathcal{L}(Y, X)).$$

However, we will not need this generality in the remainder. As was already mentioned in Remark 3.3, the assumption

$$\psi(\cdot)R(i\cdot, A)^k \in \mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(Y, X))$$

in (2) is the most general (L^p, L^q) Fourier multiplier condition for $\psi(\cdot)R(i\cdot, A)^k$.

Remark 4.7. The theory of (L^p, L^p) Fourier multipliers alone cannot yield a characterization of polynomial stability as in Theorem 4.6, and in general it is necessary to also consider the case where $p < q$ in condition (2). To see this, consider a uniformly bounded C_0-semigroup $(T(t))_{t \geq 0} \subseteq \mathcal{L}(X)$ with generator $-A$ such that $\overline{\mathbb{C}_-} \subseteq \rho(A)$ but A is not of type $(0,0)$. Let $n = 0$ and $Y = X$. Then $R(i\cdot, A) \notin \mathcal{M}_{p,p}(\mathbb{R}; \mathcal{L}(X))$ for each $p \in [1, \infty)$ since $\sup\{\|R(i\xi, A)\|_{\mathcal{L}(X)} \mid \xi \in \mathbb{R} \setminus \{0\}\} = \infty$. Nonetheless, (1) holds since $(T(t))_{t \geq 0}$ is uniformly bounded, and $R(i\cdot, A) \in \mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(X))$. Indeed,

$$\mathcal{F}^{-1}(R(i\cdot, A)\widehat{f}(\cdot))(t) = \int\limits_0^\infty T(t-s)f(s)\mathrm{d}s \qquad (t \in \mathbb{R})$$

defines an element of $L^\infty(\mathbb{R}; X)$ for each $f \in \mathcal{S}(\mathbb{R}; X)$.

A variation of the proof of Theorem 4.6 yields the following result, which will also be used in Section 5. In particular, it provides a simple condition for powers of the resolvent to be Fourier multipliers.

Proposition 4.8. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t \geq 0}$ on a Banach space X, and suppose that $\overline{\mathbb{C}_-} \setminus \{0\} \subseteq \rho(A)$. Let $q \in [1, \infty)$, $n \in \mathbb{N}_0$ and let Y be an (A, n)-admissible space. Then the following statements are equivalent:*

(1) *there exists a constant $C \in [0, \infty)$ such that $[t \mapsto t^n T(t)x] \in L^q([0, \infty); X)$ for all $x \in Y$, and*

$$\|[t \mapsto t^n T(t)x]\|_{L^q([0,\infty);X)} \leq C\|x\|_Y \qquad (x \in Y);$$

(2) *for each $k \in \{n, n+1\} \cap \mathbb{N}$ one has $R(i\cdot, A)^k \in \mathcal{M}_{1,q}(\mathbb{R}; \mathcal{L}(Y, X))$;*

(3) *there exist $\psi \in C_c^\infty(\mathbb{R})$ and $p \in [1, q]$ such that*

$$\psi(\cdot)R(i\cdot, A)^k \in \mathcal{M}_{1,q}(\mathbb{R}; \mathcal{L}(Y, X)) \text{ and } (1 - \psi(\cdot))R(i\cdot, A)^k \in \mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(Y, X))$$

for $k \in \{n, n+1\} \cap \mathbb{N}$.

Proof. $(2) \Rightarrow (3)$ is trivial. For $(3) \Rightarrow (1)$ one proceeds in an almost identical manner as in the proof of Theorem 4.6, except that now it is not necessary to appeal to Proposition 3.2.

$(1) \Rightarrow (2)$: Let $Y_0 \subseteq Y$ be as in Definition 4.5. Then $[t \mapsto t^k T(t)x] \in L^q(\mathbb{R}_+; X)$ for all $k \in \{0, \ldots, n\}$ and $x \in Y_0$. Hence, for $f \in L^1(\mathbb{R}) \otimes Y_0$, Minkowski's inequality yields

$$\left(\int_{\mathbb{R}} \Big\| \int_{\mathbb{R}} (t-s)^n T(t-s)f(s)\mathrm{d}s \Big\|^q \mathrm{d}t \right)^{1/q}$$

$$\leq \int_{\mathbb{R}} \left(\int_s^\infty \|(t-s)^n T(t-s)f(s)\|^q \mathrm{d}t \right)^{1/q} \mathrm{d}s \leq C \int_{\mathbb{R}} \|f(s)\| \mathrm{d}s.$$

Now the proof is concluded using Lemma 3.1. $\quad\square$

4.3. Results under Fourier type assumptions

Here we apply Theorem 4.6 to obtain polynomial stability results under assumptions on the Fourier type of the underlying space. The following theorem coincides with Proposition 4.3 for $p = 1$. In the case where $\alpha = 0$ it was already stated in [4] that an improvement of Proposition 4.3 might be possible using ideas from [46, §4.2], but no details are given there.

Theorem 4.9. *Let $\alpha, \beta \in [0, \infty)$ and let A be an injective sectorial operator with resolvent growth (α, β) on a Banach space X with Fourier type $p \in [1, 2]$. Let $r \in [1, \infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$, and let $\sigma, \tau \in [0, \infty)$ be such that $\sigma > \alpha - 1$ and $\tau > \beta + \frac{1}{r}$. Then for each $\rho \in [0, \min(\frac{\sigma+1}{\alpha} - 1, \frac{\tau - r^{-1}}{\beta} - 1))$ there exists a $C_\rho \in [0, \infty)$ such that*

$$\|T(t)\|_{\mathcal{L}(X_\tau^\sigma, X)} \leq C_\rho t^{-\rho} \qquad (t \in [1, \infty)). \tag{4.13}$$

If $p = 2$ then (4.13) also holds for $\tau \geq \beta$ and $\rho \in [0, \infty)$ with $\rho < \frac{\sigma+1}{\alpha} - 1$ and $\rho \leq \frac{\tau}{\beta} - 1$.

Proof. We prove the following equivalent statement: for all $s \geq 0$ and $\delta, \varepsilon > 0$ there exists a constant $C_{s,\delta,\varepsilon} \geq 0$ such that

$$\|T(t)\|_{\mathcal{L}(X_\nu^\mu, X)} \leq C_{s,\delta,\varepsilon} t^{-s} \qquad (t \in [1, \infty)), \tag{4.14}$$

where $\mu = \max((s+1)\alpha - 1 + \delta, 0)$, $\nu = (s+1)\beta + \frac{1}{r} + \varepsilon$ for $p \in [1, 2)$, and $\nu = (s+1)\beta$ for $p = 2$. By Lemma 4.2 it suffices to consider $n := s \in \mathbb{N}_0$, and the case where $p = 1$

follows from Proposition 4.3. For $p \in (1,2)$ set $\beta_0 := \frac{1}{r} + \varepsilon$, and for $p = 2$ we let $\beta_0 = 0$. We may assume that $\beta_0 \in [0,1)$.

By Proposition 3.4 and because $R(i\xi, A)$ commutes with $A^\alpha(1 + A)^{-\alpha-\beta}$ for all $\xi \in \mathbb{R} \setminus \{0\}$, one has

$$\sup\{\|R(i\xi, A)^k\|_{\mathcal{L}(X^{n\alpha}_{n\beta}, X)} \mid \xi \in \mathbb{R} \setminus \{0\}\} < \infty \qquad (k \in \{1, \ldots, n\}). \tag{4.15}$$

Now, the part $\widetilde{A}$ of A in $X^{n\alpha}_{n\beta}$ satisfies the conditions of Proposition 3.4 and Corollary 3.5, and $R(i\xi, \widetilde{A}) = R(i\xi, A)\restriction_{X^{n\alpha}_{n\beta}}$ for all $\xi \in \mathbb{R} \setminus \{0\}$. Hence

$$\left\{ \frac{|\xi|^{1-\delta}}{(1 + |\xi|)^{1-\delta-\beta_0}} R(i\xi, A) \Big| \xi \in \mathbb{R} \setminus \{0\} \right\} \subseteq \mathcal{L}(X^\mu_\nu, X^{n\alpha}_{n\beta}) \tag{4.16}$$

is uniformly bounded. Let $k \in \{1, \ldots, n+1\}$. Then (4.15) and (4.16) show that

$$\left\{ \frac{|\xi|^{1-\delta}}{(1 + |\xi|)^{1-\delta-\beta_0}} R(i\xi, A)^k \Big| \xi \in \mathbb{R} \setminus \{0\} \right\} \subseteq \mathcal{L}(X^\mu_\nu, X) \tag{4.17}$$

is uniformly bounded. Let $\psi \in C^\infty_c(\mathbb{R})$ be such that $\psi \equiv 1$ on $[-1, 1]$. Since $\delta > 0$, it follows from (4.17) and Proposition 2.2 that

$$\psi(\cdot)R(i\cdot, A)^k \in L^1(\mathbb{R}; \mathcal{L}(X^\mu_\nu, X)) \subseteq \mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(X^\mu_\nu, X)).$$

Another application of (4.17) yields $\|(1 - \psi(\cdot))R(i\cdot, A)^k\|_{\mathcal{L}(X^\mu_\nu, X)} \in L^r(\mathbb{R})$. Note that X^μ_ν has Fourier type p, since X^μ_ν is isomorphic to X. Hence Proposition 2.2 yields

$$(1 - \psi(\cdot))R(i\cdot, A)^k \in \mathcal{M}_{p,p'}(\mathbb{R}; \mathcal{L}(X^\mu_\nu, X)).$$

Now Theorem 4.6 concludes the proof. $\quad\square$

Remark 4.10. One can show that the constant C_ρ in (4.13) depends only on the following variables: α, β, σ, τ, ρ, $\mathcal{F}_{p,X}$, the sectoriality constant $M(A)$ from (2.6),

$$M_{\alpha,\beta} := \sup\left\{ \frac{|\xi|^\alpha}{|1 + \xi|^{\alpha+\beta}} \|R(i\xi, A)\|_{\mathcal{L}(X)} \Big| \xi \in i\mathbb{R} \setminus \{0\} \right\}$$

and the semigroup growth constants M, ω and M_ω which appear in (4.10).

It is an open question whether (4.13) also holds for $\rho = \min(\frac{\sigma+1}{\alpha} - 1, \frac{\tau - r^{-1}}{\beta} - 1)$ if $\alpha + \beta > 0$.

A Hilbert space has Fourier type 2 by Plancherel's identity. Hence we may distill from Theorem 4.9 the following important corollary, which in particular implies Theorem 1.1. It follows from Example 4.20 and Remark 4.17 that, up to ε loss, the polynomial rate of decay in Corollary 4.11 is optimal for $\alpha = 0$ and $\tau = \beta \in [0, \infty)$, and for $\alpha = 1$ and

$\beta = 0$. We do not know whether the rate of decay is also optimal for other values of α, β, σ and τ.

Corollary 4.11. *Let $\alpha, \beta \in [0, \infty)$ and let A be an injective sectorial operator with resolvent growth (α, β) on a Hilbert space. Let $\sigma, \tau \in [0, \infty)$ be such that $\sigma > \alpha - 1$ and $\tau \geq \beta$. Then for each $\rho \in [0, \infty)$ such that $\rho < \frac{\sigma+1}{\alpha} - 1$ and $\rho \leq \frac{\tau}{\beta} - 1$ there exists a $C_\rho \in [0, \infty)$ such that*

$$\|T(t)\|_{\mathcal{L}(X_\tau^\sigma, X)} \leq C_\rho t^{-\rho} \qquad (t \in [1, \infty)).$$

Remark 4.12. Corollary 4.4 yields a faster decay rate than Theorem 4.9 when $\|T(t)\|_{\mathcal{L}(X)}$ grows slowly as $t \to \infty$. More precisely, with notation as in Theorem 4.9, let $\mu_0 \in [0, \infty)$ be such that

$$\min\left(\frac{\sigma}{\alpha}, \frac{\tau}{\beta}\right) - \mu_0 = \min\left(\frac{\sigma+1}{\alpha} - 1, \frac{\tau - r^{-1}}{\beta} - 1\right).$$

If there exists a $\mu < \mu_0$ such that $\limsup_{t \to \infty} t^{-\mu} \|T(t)\|_{\mathcal{L}(X)} < \infty$ then Corollary 4.4 yields a sharper decay rate than Theorem 4.9, namely

$$\|T(t)\|_{\mathcal{L}(X_\tau^\sigma, X)} \lesssim t^{-\rho} \qquad (t \in [1, \infty))$$

for each $\rho < \min(\frac{\sigma}{\alpha}, \frac{\tau}{\beta}) - \mu$. Otherwise Theorem 4.9 yields at least as sharp a decay rate as Corollary 4.4. In particular, on Hilbert spaces Corollary 4.11 yields faster decay than Corollary 4.4 if $\alpha = 0$ and $\|T(\cdot)\|_{\mathcal{L}(X)}$ grows at least linearly. Note also that in many cases (4.27) below yields a faster decay rate than Corollary 4.4.

4.4. Results under type and cotype assumptions

Here we consider polynomial decay rates under type and cotype assumptions on the underlying space.

The following result also holds for $q = \infty$. However, in this case Proposition 4.3 yields a more general statement, since each Banach space has type $p = 1$ and cotype $q = \infty$ and because a Banach space with nontrivial type also has finite cotype.

Theorem 4.13. *Let $\alpha, \beta \in [0, \infty)$ and let A be an injective sectorial operator with R-resolvent growth (α, β) on a Banach space X with type $p \in [1, 2]$ and cotype $q \in [2, \infty)$. Let $r \in [1, \infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, and let $\sigma, \tau \in [0, \infty)$ be such that $\sigma > \alpha - 1$ and $\tau > \beta + \frac{1}{r}$. Then for each $\rho < \min(\frac{\sigma+1}{\alpha} - 1, \frac{\tau - r^{-1}}{\beta} - 1)$ there exists a $C_\rho \in [0, \infty)$ such that*

$$\|T(t)\|_{\mathcal{L}(X_\tau^\sigma, X)} \leq C_\rho t^{-\rho} \qquad (t \in [1, \infty)).$$

If $p = q = 2$ then (4.13) also holds for $\tau \geq \beta$ and $\rho \in [0, \infty)$ with $\rho < \frac{\sigma+1}{\alpha} - 1$ and $\rho \leq \frac{\tau}{\beta} - 1$.

Proof. The proof is similar to that of Theorem 4.9. The case where $p = q = 2$ is already contained in Corollary 4.11, since each Banach space with type 2 and cotype 2 is isomorphic to a Hilbert space, and because every uniformly bounded collection on a Hilbert space is R-bounded. So we may assume that $r \in (1, \infty)$ and derive (4.14) for $n := s \in \mathbb{N}_0$. Set $\beta_0 := \frac{1}{r} + \varepsilon$ and let $k \in \{1, \dots, n+1\}$. We may suppose that $\beta_0 \in (0, 1)$. As in the proof of Theorem 4.9, using Proposition 3.4 and Corollary 3.5, one sees that

$$\{|\xi|^{1-\delta} R(i\xi, A)^k \mid \xi \in \mathbb{R} \setminus \{0\}, |\xi| \leq 1\} \subseteq \mathcal{L}(X^\mu, X)$$

is uniformly bounded and that

$$\{|\xi|^{\beta_0} R(i\xi, A)^k \mid \xi \in \mathbb{R} \setminus \{0\}, |\xi| \geq 1\} \subseteq \mathcal{L}(X_\nu, X) \tag{4.18}$$

is R-bounded. Now let $\psi \in C_c^\infty(\mathbb{R})$ be such that $\psi \equiv 1$ on $[-1/2, 1/2]$ and such that $\operatorname{supp}(\psi) \subseteq [-1, 1]$. Then Proposition 2.2 shows that $\psi(\cdot) R(i\cdot, A)^k \in \mathcal{M}_{1,\infty}(\mathcal{L}(X^\mu, X))$, and $(1 - \psi(\cdot)) R(i\cdot, A)^k \in \mathcal{M}_{p,q}(\mathcal{L}(X_\nu, X))$ by the first statement in Proposition 2.3. Theorem 4.6 concludes the proof. □

A similar dependence on the underlying parameters as in Remark 4.10 holds for the constant C_ρ in Theorem 4.13.

Using the second statement in Proposition 2.3 we obtain the following improvement of Theorem 4.13 on Banach lattices, which allows one to deal with the limit case in the fractional domain exponent.

Theorem 4.14. *Let $\alpha, \beta \in [0, \infty)$ and let A be an injective sectorial operator with R-resolvent growth (α, β) on a Banach lattice X which is p-convex and q-concave for $p \in [1, 2]$ and $q \in [2, \infty)$. Let $r \in (1, \infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, and let $\sigma, \tau \in [0, \infty)$ be such that $\sigma > \alpha - 1$ and $\tau \geq \beta + \frac{1}{r}$. Then for each $\rho \in [0, \infty)$ such that $\rho < \frac{\sigma+1}{\alpha} - 1$ and $\rho \leq \frac{\tau - r^{-1}}{\beta} - 1$ there exists a $C_\rho \in [0, \infty)$ such that*

$$\|T(t)\|_{\mathcal{L}(X_\tau^\sigma, X)} \leq C_\rho t^{-\rho} \qquad (t \in [1, \infty)).$$

We do not know whether the R-boundedness assumption in Theorems 4.13 and 4.14 is necessary. This question is relevant even in the case where $\alpha = \beta = 0$, cf. the remark following Corollary 5.5.

Remark 4.15. Each Banach space X with Fourier type $p \in [1, 2]$ has type p and cotype p', but the converse does not hold in general. In particular, if $X = L^u(\Omega)$ for $u \in [1, \infty)$ and for some measure space Ω, then X has Fourier type $\widetilde{p} = \min(u, u')$, type $p = \min(u, 2)$ and cotype $q = \max(u, 2)$. In this case the parameter $\frac{1}{r}$ in Theorems 4.13 and 4.14 is

strictly smaller than in Theorem 4.9 for $u \in [1, \infty) \setminus \{2\}$. However, the R-boundedness assumption on the resolvent of A is in general stronger than the assumption in Theorem 4.9.

We suspect that the R-boundedness condition in Theorems 4.13 and 4.14 can be removed at the cost of a larger parameter $\frac{1}{r}$. For $\alpha = \beta = 0$ this is indeed the case, with $\frac{1}{r} = 2(\frac{1}{p} - \frac{1}{q})$, as is shown in Corollary 5.5.

4.5. Results for asymptotically analytic semigroups

Here we consider polynomial stability for the asymptotically analytic semigroups from [11]. Define the *non-analytic growth bound* $\zeta(T)$ of a C_0-semigroup $(T(t))_{t \geq 0}$ on a Banach space X as

$$\zeta(T) := \inf \left\{ \omega \in \mathbb{R} \,\middle|\, \sup_{t > 0} e^{-\omega t} \|T(t) - S(t)\| < \infty \text{ for some } S \in \mathcal{H}(\mathcal{L}(X)) \right\},$$

where $\mathcal{H}(\mathcal{L}(X))$ is the set of $S \colon (0, \infty) \to \mathcal{L}(X)$ having an exponentially bounded analytic extension to some sector containing $(0, \infty)$. One says that $(T(t))_{t \geq 0}$ is *asymptotically analytic* if $\zeta(T) < 0$. In this case $s_0^\infty(-A) < 0$, where $s_0^\infty(-A)$ is the infimum over all $\omega \in \mathbb{R}$ for which there exists an $R > 0$ such that

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) \geq \omega, |\operatorname{Im}(\lambda)| \geq R\} \subseteq \rho(-A)$$

and

$$\sup\{\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \mid \operatorname{Re}(\lambda) \geq \omega, |\operatorname{Im}(\lambda)| \geq R\} < \infty.$$

The converse implication holds if X is a Hilbert space. More generally, it was shown in [10, Theorem 3.6] that $\zeta(T) < 0$ if and only if $s_0^\infty(-A) < 0$ and there exist $R > 0$ and $\psi \in C_c^\infty(\mathbb{R})$ such that $i(\mathbb{R} \setminus [-R, R]) \subseteq \rho(A)$, $\psi \equiv 1$ on $[-R, R]$ and

$$(1 - \psi(\cdot))R(i\cdot, A) \in \mathcal{M}_{p,p}(\mathbb{R}; \mathcal{L}(X))$$

for some (in which case it holds for all) $p \in [1, \infty)$.

Note that if $(T(t))_{t \geq 0}$ is analytic, and in particular if A is bounded, then trivially $\zeta(T) = -\infty$. More generally, if $(T(t))_{t \geq 0}$ is eventually differentiable then $\zeta(T) = -\infty$. For these facts and for more on the non-analytic growth bound see [6,10,11].

Theorem 4.16. *Let $\alpha \in [0, \infty)$ and let A be an injective sectorial operator with resolvent growth $(\alpha, 0)$ on a Banach space X. Suppose that $(T(t))_{t \geq 0}$ is asymptotically analytic, and let $\sigma \in [0, \infty)$ be such that $\sigma > \alpha - 1$. Then for each $\rho \in [0, \frac{\sigma+1}{\alpha} - 1)$ there exists a $C_\rho \in [0, \infty)$ such that*

$$\|T(t)\|_{\mathcal{L}(X^\sigma, X)} \leq C_\rho t^{-\rho} \qquad (t \in [1, \infty)).$$

Proof. It suffices to obtain (4.14) with $\mu = \max((n+1)\alpha - 1 + \delta, 0)$ and $\nu = 0$ for $n \in \mathbb{N}_0$. There exist $R \in (0, \infty)$, $\psi \in C_c^\infty(\mathbb{R})$ and $p \in [1, \infty)$ such that

$$(1 - \psi(\cdot))R(i\cdot, A)^k \in \mathcal{M}_{p,p}(\mathbb{R}; \mathcal{L}(X)) \qquad (k \in \{0, \ldots, n+1\}). \tag{4.19}$$

Since the inclusion $X^\mu \subseteq X$ is continuous, (4.19) also holds with $\mathcal{L}(X)$ replaced by $\mathcal{L}(X^\mu, X)$. It follows as in the proof of Theorem 4.9 that

$$\psi(\cdot)R(i\cdot, A)^k \in L^1(\mathbb{R}; \mathcal{L}(X^\mu, X) \subseteq \mathcal{M}_{1,\infty}(\mathbb{R}; \mathcal{L}(X^\mu, X)) \qquad (k \in \{0, \ldots, n+1\}).$$

Now Theorem 4.6 yields the required estimate. □

Remark 4.17. An injective sectorial operator A of angle $\varphi \in (0, \pi/2)$ has resolvent growth $(1, 0)$. The semigroup $(T(t))_{t \geq 0}$ generated by $-A$ is analytic and for any $\sigma \geq 0$ one has

$$\|T(t)\|_{\mathcal{L}(X^\sigma, X)} \lesssim t^{-\sigma} \qquad (t \in [1, \infty)).$$

This follows from [23, Proposition 2.6.11]. This decay rate is optimal for the multiplication semigroup $(T(t))_{t \geq 0}$ on $L^p[0, \infty)$, $p \in [1, \infty)$, given by $T(t)f(s) = e^{-ts}f(s)$ for $f \in L^p[0, \infty)$ and $t, s \geq 0$.

4.6. Necessary conditions

In this subsection we study the necessity of the assumptions in our results.

4.6.1. Spectral conditions

The following lemma, an extension of [7, Proposition 6.4], shows that one can deduce spectral properties of an operator A given uniform decay on suitable subspaces of the associated semigroup. The proof follows that of [7, Proposition 6.4] and uses the Hille–Phillips functional calculus for semigroup generators. For more on this calculus see [23, Section 3.3] or [27, Chapter XV].

Lemma 4.18. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t \geq 0}$ on a Banach space X. Suppose that there exist $\alpha, \beta \in \mathbb{N}_0$, $\eta \in \rho(-A)$, and a sequence $(t_n)_{n \in \mathbb{N}} \subseteq [0, \infty)$ such that*

$$\lim_{n \to \infty} \|T(t_n)A^\alpha(\eta + A)^{-\alpha - \beta}\|_{\mathcal{L}(X)} = 0. \tag{4.20}$$

Then $\overline{\mathbb{C}_-} \setminus \{0\} \subseteq \rho(A)$.

Proof. Without loss of generality we may consider $\beta \in \mathbb{N}$ and $\eta > \omega_0(T)$. Let $t \geq 0$ and set $f_t(\lambda) := e^{-t\lambda}\lambda^\alpha(\eta + \lambda)^{-\alpha - \beta}$ for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\eta$. Let

$$k(s) := \begin{cases} \frac{1}{(\alpha+\beta-1)!} \frac{d^\alpha}{ds^\alpha}(s^{\alpha+\beta-1}e^{-\eta s}), & s \in (0, \infty), \\ 0, & s \in (-\infty, 0]. \end{cases}$$

Then f_t is the Laplace transform of $\delta_t * k$, where δ_t is the Dirac point mass at t. Moreover, $f_t(A)$ is defined through the Hille–Phillips functional calculus for A and

$$f_t(A) = T(t)A^\alpha(\eta + A)^{-\alpha-\beta}.$$

By the spectral inclusion theorem for the Hille–Phillips functional calculus in [27, Theorem 16.3.5] one obtains $f_t(\sigma(A)) \subseteq \sigma(f_t(A))$. Let $\lambda \in \sigma(A) \setminus \{0\}$ and $n \in \mathbb{N}$. Then $f_{t_n}(\lambda) \in \sigma(f_{t_n}(A))$, so [19, Corollary IV.1.4] shows that

$$e^{-\operatorname{Re}(\lambda)t_n} \frac{|\lambda|^\alpha}{|\eta + \lambda|^{\alpha+\beta}} = |f_{t_n}(\lambda)| \le \|f_{t_n}(A)\| = \|T(t_n)A^\alpha(\eta + A)^{-\alpha-\beta}\|.$$

This concludes the proof since the right-hand side tends to zero as $n \to \infty$. □

If $\eta + A$ is a sectorial operator in Lemma 4.18, then one may consider $\beta \in [0, \infty)$ in (4.20). Similarly, if A is a sectorial operator then one may let $\alpha \in [0, \infty)$.

A similar statement as in the following proposition can be obtained for more general subspaces. It follows from Example 4.22 that the conclusion is sharp.

Proposition 4.19. *Let A be an injective sectorial operator such that $-A$ generates a C_0-semigroup $(T(t))_{t \ge 0}$ on X. Suppose that there exist $\alpha \in \{0\} \cup [1, \infty)$, $\beta \in [0, \infty)$ and a $g \in L^1(\mathbb{R}_+)$ such that $\|T(t)\|_{\mathcal{L}(X_\beta^\alpha, X)} \le g(t)$ for $t \ge 0$. Then $\overline{\mathbb{C}_-} \setminus \{0\} \subseteq \rho(A)$ and*

$$\{\lambda^\alpha(1 + \lambda)^{-\alpha-\beta}(\lambda + A)^{-1} \mid \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}\} \subseteq \mathcal{L}(X) \tag{4.21}$$

is R-bounded. In particular, A has R-resolvent growth (α, β). Furthermore, if $\alpha = 0$ then also $0 \in \rho(A)$.

Proof. Lemma 4.18 and the remark following it show that $\overline{\mathbb{C}_-} \setminus \{0\} \subseteq \rho(A)$. By assumption, $T(\cdot)A^\alpha(1 + A)^{-\alpha-\beta}x \in L^1(\mathbb{R}_+; X)$ for all $x \in X$, with

$$\int_0^\infty \|T(t)A^\alpha(1 + A)^{-\alpha-\beta}x\|_X dt \le \|g\|_{L^1(\mathbb{R}_+)}\|x\|_X.$$

Moreover, for each $\lambda \in \overline{\mathbb{C}_+}$ one has $[t \mapsto e^{-\lambda t}] \in L^\infty(\mathbb{R}_+)$. Set

$$F(\lambda)x := \int_0^\infty e^{-\lambda t}T(t)A^\alpha(1 + A)^{-\alpha-\beta}x \, dt$$

for $\lambda \in \overline{\mathbb{C}_+}$ and $x \in X$. By [33, Corollary 2.17], $\{F(\lambda) \mid \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}\} \subseteq \mathcal{L}(X)$ is R-bounded. Lemma 3.1, applied to the semigroup $(e^{-\lambda t} T(t))_{t \geq 0}$ generated by $-(\lambda + A)$, shows that $F(\lambda) = (\lambda + A)^{-1} A^{\alpha} (1 + A)^{-\alpha - \beta}$ for $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$. Now Proposition 3.4 (3) implies that

$$\{\lambda^{\alpha}(1 + \lambda)^{-\alpha - \beta}(\lambda + A)^{-1} \mid \lambda \in \mathbb{C} \setminus \{0\}, |\arg(\lambda)| \in [\varphi, \pi/2]\} \subseteq \mathcal{L}(X) \qquad (4.22)$$

is R-bounded for each $\varphi \in (0, \pi/2)$. In particular, since A is a sectorial operator, the collection in (4.21) is uniformly bounded. Now a standard argument, considering a convolution with the Poisson kernel (see e.g. [30, Proposition 8.5.8]), shows that (4.21) is R-bounded.

For the second statement suppose that $\alpha = 0$. To show that $0 \in \rho(A)$ we may consider $\beta \in \mathbb{N}$, since $(1 + A)^{-(\lceil \beta \rceil - \beta)} \in \mathcal{L}(X)$. Note that $F(0) \in \mathcal{L}(X, D(A))$, with

$$AF(0)x = -\lim_{h \downarrow 0} \frac{T(h) - I_X}{h} F(0) = \lim_{h \downarrow 0} \frac{1}{h} \int_0^h T(t)(1 + A)^{-\beta} x \, dt = (1 + A)^{-\beta} x$$

for all $x \in X$. Similarly, $F(0)Ay = (1 + A)^{-\beta} y$ for $y \in D(A)$. By iteration one obtains that $F(0) \in \mathcal{L}(X, X_\beta)$. This shows that the part of A in X_β is invertible, with inverse $F(0)(1 + A)^{\beta}|_{X_\beta}$. Using the similarity transform $(1 + A)^{-\beta} : X \to X_\beta$ one obtains $0 \in \rho(A)$, which concludes the proof. $\square$

4.6.2. Operators which are not sectorial

In several of the results up to this point we have considered operators A with resolvent growth (α, β), for $\alpha, \beta \in [0, \infty)$, which are in addition assumed to be sectorial. Here we discuss which results are still valid when one drops the sectoriality assumption. A complicating factor is then that A^{α} is not well defined through the sectorial functional calculus, and we only consider $\alpha \in \mathbb{N}_0$.

Let A be an injective operator, not necessarily sectorial, with resolvent growth (α, β) on a Banach space X. First note that $\varepsilon + A$ is a sectorial operator for each $\varepsilon > 0$, since $-A$ generates a C_0-semigroup and $\overline{\mathbb{C}_-} \subseteq \rho(\varepsilon + A)$. Hence the fractional domains

$$X_\beta = D((1 + A)^{-\beta}) = D((1 - \varepsilon + \varepsilon + A)^{-\beta})$$

are well defined via the sectorial functional calculus for $\varepsilon + A$, $\varepsilon \in (0, 1)$, and up to norm equivalence they do not depend on the choice of ε.

If $\alpha_1 = \alpha_2 \in \mathbb{N}_0$ in Lemma 4.2, then (4.1) still holds and (4.2) is replaced by

$$\|T(t)\|_{\mathcal{L}(X_{\nu\beta_1}^{\lceil \nu\alpha_1 \rceil}, X)} \leq C_\nu t^{-\mu\nu} \qquad (t \in [1, \infty)). \qquad (4.23)$$

The proofs are identical except that one obtains (4.23) for $\nu \notin \mathbb{N}$ by applying (4.1) to the pairs $(\lceil \nu \rceil \alpha_1, \lfloor \nu \rfloor \beta_1)$ and $(\lceil \nu \rceil \alpha_1, \lceil \nu \rceil \beta_1)$. One can also show that for each $\tau \in [0, \infty)$ there exists a $\sigma \in \mathbb{N}$ such that (4.5) holds for all $\rho \in [0, \min(\frac{\sigma}{\alpha}, \frac{\tau}{\beta}))$.

Suppose that X has Fourier type $p \in [1, 2]$ and let $r \in [1, \infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$. Then for all $s \geq 0$ and $\varepsilon > 0$ there exists a $C_{s,\varepsilon} \geq 0$ such that

$$\|T(t)\|_{\mathcal{L}(X_\nu^\mu, X)} \leq C_{s,\varepsilon} t^{-s} \qquad (t \in [1, \infty)), \tag{4.24}$$

where $\mu = \lfloor (\lceil s \rceil + 1)\alpha \rfloor \in \mathbb{N}_0$, $\nu = (s + 1)\beta + \frac{1}{r} + \varepsilon$ for $p \in [1, 2)$, and $\nu = (s + 1)\beta$ for $p = 2$. Versions of (4.24) in the settings of Theorems 4.13, 4.14 and 4.16 also hold. In particular, if $(T(t))_{t \geq 0}$ is asymptotically analytic then (4.24) holds with $\mu := \lfloor (s+1)\alpha \rfloor$ and $\nu = 0$ for each $s \in \mathbb{N}_0$.

4.7. Comparison and examples

In this subsection we compare the decay rates which we have obtained to what can be found in the literature, and we present examples to illustrate our results.

4.7.1. Comparison

Let $\alpha, \beta \geq 0$ and let A be an injective sectorial operator with resolvent growth (α, β) on a Banach space X. The decay rates which we have obtained so far are in general not optimal when $(T(t))_{t \geq 0} \subseteq \mathcal{L}(X)$ is uniformly bounded. Indeed, for $\sigma, \tau \geq 0$ and $N := \sup\{\|T(t)\|_{\mathcal{L}(X)} \mid t \in [0, \infty)\} < \infty$ it follows from [8,15] that there exists a $C_\rho \geq 0$ such that

$$\|T(t)\|_{\mathcal{L}(X_\tau^\sigma, X)} \leq C_\rho N (1 + \log(t))^\rho t^{-\rho} \qquad (t \in [1, \infty)), \tag{4.25}$$

where $\rho = \frac{\sigma}{\alpha}$ if $\beta = 0$, $\rho = \frac{\tau}{\beta}$ if $\alpha = 0$, and $\rho = \min(\sigma, \tau) \cdot \min(\frac{1}{\alpha}, \frac{1}{\beta})$ if $\alpha\beta > 0$. It was shown in [12] that (4.25) is optimal on general Banach spaces if $\alpha = 0$, but on Hilbert spaces (4.25) can be improved to

$$\|T(t)\|_{\mathcal{L}(X_\tau^\sigma, X)} \leq C_\rho N^2 t^{-\rho} \qquad (t \in [1, \infty)), \tag{4.26}$$

cf. [7,12]. Moreover, (4.26) is optimal, in the sense that for $\sigma, \tau \in \{0, 1\}$ (4.26) implies that A has resolvent growth (α, β) (see [7,8]).

For unbounded semigroups (4.25) and (4.26) do not hold in general. Indeed, [45, Example 4.2.9] gives an example of an operator A with R-resolvent growth $(0, 0)$ on $X := L^p(1, \infty) \cap L^{p'}(1, \infty)$, $p \in [1, 2)$, such that $\|T(\cdot)\|_{\mathcal{L}(X)}$ grows exponentially. Moreover, Example 4.20 shows that on Hilbert spaces (4.26) can fail for $\alpha = 0$ and $\beta > 0$, and Corollary 4.11 is optimal for this example when $\tau = \beta$.

Note that (4.25) need not be optimal for uniformly bounded semigroups when $\alpha\beta > 0$, and that Corollary 4.4 yields a sharper decay rate if e.g. $\alpha = \sigma = 1/\varepsilon$ and $\beta = \tau = \varepsilon$ for $\varepsilon \in (0, 1)$. On Hilbert spaces one can use [7, Theorem 4.7], Proposition 3.4 and Lemma 4.2 to let $\rho = \min(\frac{\sigma}{\alpha}, \frac{\tau}{\beta})$ in (4.26), but a similar improvement of (4.25) on Banach spaces using the methods of [8,15] is not immediate.

The characterization of polynomial stability in Theorem 4.6 is new even for uniformly bounded semigroups.

A scaling argument can be used to apply (4.25) to polynomially growing semigroups, leading to sharper decay rates than those in Corollary 4.4. Suppose $\alpha = 0$, $\beta > 0$ and that $\|T(t)\|_{\mathcal{L}(X)} \lesssim t^{\mu}$ for all $t \geq 1$ and some $\mu \geq 0$. For $a > 0$ one has

$$\sup\{\|e^{-at}T(t)\|_{\mathcal{L}(X)} \mid t \in [0, \infty)\} \lesssim a^{-\mu}.$$

Now (4.25) yields

$$\|e^{-at}T(t)\|_{\mathcal{L}(X_\tau, X)} \lesssim a^{-\mu}(1 + \log(t))^{\tau/\beta} t^{-\tau/\beta} \qquad (t \in [1, \infty)).$$

For $t \geq 1$ set $a := 1/t$. Then

$$\|T(t)\|_{\mathcal{L}(X_\sigma, X)} \lesssim (1 + \log(t))^{\tau/\beta} t^{\mu - \tau/\beta}, \tag{4.27}$$

which improves the rates from Corollary 4.4. However, other results in this section yield faster decay rates than (4.27) for large μ, such as Corollary 4.11 for $\mu \geq 1$.

In this article we make polynomial growth assumptions on the resolvent, whereas in [7,8,15,54] more general resolvent growth is allowed. The scaling argument from above can be used in certain cases to obtain decay estimates corresponding to more general resolvent growth, but this depends on the growth behavior of the semigroup on X. We do not know whether the techniques from this article can be used to obtain nontrivial decay estimates for unbounded semigroups under, for example, exponential or logarithmic growth conditions on the resolvent.

4.7.2. An exponentially unstable semigroup with polynomial resolvent

We now apply our theorems to an operator from [68, Example 4.1], which in turn is a variation of a classical example in stability theory from [69] (see also [45, Example 1.2.4]). This example shows that Corollary 4.11 is optimal in the case where $\alpha = 0$ and $\tau = \beta$.

Example 4.20. We show that for all $\beta \in (0, \infty)$ and $\varepsilon \in (0, 1)$ there exists an operator A with resolvent growth $(0, \beta)$ on a Hilbert space X such that $\|T(\cdot)\|_{\mathcal{L}(X_\tau, X)}$ is unbounded for $\tau \in [0, (1-\varepsilon)\beta)$. In fact, $\|T(t)\|_{\mathcal{L}(X_\tau, X)}$ grows exponentially in t for $\tau \in [0, (1-\varepsilon)\beta)$. By Corollary 4.11 $\|T(\cdot)\|_{\mathcal{L}(X_\beta, X)}$ is uniformly bounded, and therefore the exponent τ in Corollary 4.11 is optimal.

It suffices to show that for all $\gamma, \delta \in (0, 1)$ there exists an operator A with resolvent growth $\left(0, \frac{\log(1/\gamma)}{\log(1/\delta)}\right)$ on a Hilbert space X such that $\|T(\cdot)\|_{\mathcal{L}(X_\tau, X)}$ is unbounded for all $\tau \in [0, \frac{1-\gamma}{\log(1/\delta)})$, as follows from the fact that $1 - \gamma$ is the first order Taylor approximation of $\log(1/\gamma)$ near $\gamma = 1$. Set $\beta_0 := \frac{\log(1/\gamma)}{\log(1/\delta)}$, and for $n \in \mathbb{N}$ let the $n \times n$ matrix B_n be given by

$$B_n(k,l) := \begin{cases} 1 & \text{for } l = k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $m(n) := \left\lfloor \frac{\log(n)}{\log(1/\delta)} \right\rfloor \in \mathbb{N}_0$ and let $n_0 \in \mathbb{N}$ be such that $m(n_0) \geq 2$. Next, let $X = \bigoplus_{n \geq n_0} \ell^2_{m(n)}$ be the ℓ^2 direct sum of the $m(n)$-dimensional $\ell^2_{m(n)}$ spaces for $n \geq n_0$, and consider the operator $A := (-in + \gamma - B_{m(n)})_{n \geq n_0}$ on X. As shown in [68, Example 4.1], $-A$ generates a C_0-semigroup $(T(t))_{t \geq 0} \subseteq \mathcal{L}(X)$ such that $\omega_0(T) = 1 - \gamma$. We claim that $\overline{\mathbb{C}_-} \subseteq \rho(A)$ and that there exists a $C \geq 0$ such that

$$\|(\eta + i\xi + A)^{-1}\|_{\mathcal{L}(X)} \leq C(|\xi|^{\beta_0} + 1) \qquad (\eta \in [0, \infty), \ \xi \in \mathbb{R}), \qquad (4.28)$$

which implies that A has resolvent growth $(0, \beta_0)$.

To prove the claim let $z := \eta + i\xi$ and note that $B^{m(n)}_{m(n)} = 0 \in \mathcal{L}(\ell^2_{m(n)})$ and $\|B_{m(n)}\|_{\mathcal{L}(\ell^2_{m(n)})} = 1$ for all $n \geq n_0$. Hence

$$\|(z - in + \gamma - B_{m(n)})^{-1}\|_{\mathcal{L}(\ell^2_{m(n)})} \leq \sum_{k=0}^{m(n)-1} \frac{\|B^k_{m(n)}\|_{\mathcal{L}(\ell^2_{m(n)})}}{|z - in + \gamma|^{k+1}} \leq \sum_{k=0}^{m(n)-1} \frac{1}{|z - in + \gamma|^{k+1}}.$$

Fix $\xi \in \mathbb{R}$, and let $n_1 \in \mathbb{N}$ be such that $n_1 \geq n_0$ and $|n_1 - \xi| = \min\{|n - \xi| \mid n \in \mathbb{N}, n \geq n_0\}$. Note that $|z - in + \gamma| \geq \gamma$ for all $n \in \mathbb{N}$. Hence for $\xi \geq 0$ and $n \in \{n_0, \ldots, n_1 + 1\}$ one has

$$\|(z - in + \gamma - B_{m(n)})^{-1}\|_{\mathcal{L}(\ell^2_{m(n)})} \leq \sum_{k=0}^{m(n)-1} \frac{1}{\gamma^{k+1}} = \frac{\gamma^{-m(n)} - 1}{1 - \gamma}$$

$$\leq (1 - \gamma)^{-1} \gamma^{-m(n_1+1)} \leq (1 - \gamma)^{-1} (n_1 + 1)^{\beta_0} \lesssim \xi^{\beta_0} + 1,$$

where we used that $n_1 \leq \xi + 2$. If $\xi < 0$ or $n \geq n_1 + 2$ then $|z - in + \gamma| \geq c_\gamma := \sqrt{1 + \gamma^2} > 1$. Therefore

$$\|(z - in + \gamma - B_{m(n)})^{-1}\|_{\mathcal{L}(\ell^2_{m(n)})} \leq \sum_{k=0}^{\infty} \frac{1}{c_\gamma^{k+1}} < \infty,$$

and now (4.28) follows. In fact, (4.28) is optimal for $\eta = 0$ (see [68, Example 4.1]).

We now show that $\|T(\cdot)\|_{\mathcal{L}(X_\tau, X)}$ is unbounded for $\tau \in [0, \frac{1-\gamma}{\log(1/\delta)})$. First note that $\|T(t)\|_{\mathcal{L}(X_\tau, X)} \geq \frac{\|T(t)x\|_X}{\|(1+A)^\tau x\|_X}$ for each $x \in X_\tau$ with $1 = \|x\|_{X_\tau} \gtrsim \|(1 + A)^\tau x\|_X$. Let $n \geq n_0$ and let $x = (x^{(k)})_{k \geq n_0} \in X$ be such that $x^{(k)} = 0$ for all $k \neq m(n)$ and $x^{(m(n))} = (0, \ldots, 0, 1)$. Then, for $\tau \in \mathbb{N}_0$, Newton's binomial formula yields

$$\|(1 + A)^\tau x\|_X = \|(-in + 1 + \gamma - B_{m(n)})^\tau x^{(m(n))}\|_{\ell^2_{m(n)}} \lesssim n^\tau. \qquad (4.29)$$

The moment inequality [23, Proposition 6.6.4] extends (4.29) to all $\tau \in [0, \infty)$. Now set $t := m(n) - 1 \in [1, \infty)$. Lemma 5.10 yields

$$\|T(t)x\|_X = e^{-\gamma t}\|e^{tB_{m(n)}}x^{(n)}\|_{\ell^2_{m(n)}} = e^{-\gamma t}\left(\sum_{k=0}^{m(n)-1}\left(\frac{t^k}{k!}\right)^2\right)^{1/2}$$

$$\gtrsim \frac{e^{(1-\gamma)m(n)}}{(m(n))^{1/4}} \gtrsim \frac{n^{\frac{1-\gamma}{\log(1/\delta)}}}{\log(n)^{1/4}}.$$

Combining this with (4.29) shows that, with $v := \frac{1-\gamma}{\log(1/\delta)} - \tau$,

$$\|T(t)\|_{\mathcal{L}(X_\tau,X)} \gtrsim \frac{n^v}{\log(n)^{1/4}} \approx \frac{e^{tv}}{t^{1/4}}$$

for an implicit constant independent of $n \geq n_0$ and $t \geq 1$. The latter is bounded as $n \to \infty$ if and only if $\tau \geq \frac{1-\gamma}{\log(1/\delta)}$ holds, and otherwise it grows exponentially.

4.7.3. Operator matrices

We now give an example of an operator $\mathcal{A}$ with resolvent growth $(n, 0)$, for $n \in \mathbb{N} \setminus \{1\}$, such that $\|T(\cdot)\|_{\mathcal{L}(X^m,X)}$ is unbounded for all $m \in \{0, \ldots, n-2\}$. Moreover, $\|T(t)\|_{\mathcal{L}(X^{n-1},X)}$ does not tend to zero as $t \to \infty$. Hence the example would show that the exponent $\frac{\sigma+1}{\alpha} - 1$ in Theorem 4.16 is sharp, if A were a sectorial operator. However, it turns out that this is not the case. As noted in Section 4.6.2, our theory also applies to operators which are not sectorial.

Example 4.21. Fix $n \in \mathbb{N} \setminus \{1\}$. We give an example of an injective bounded operator $\mathcal{A}$ with dense range on a Hilbert space X such that $\sigma(\mathcal{A}) = [0, 1]$,

$$\sup\left\{\frac{|\lambda|^n}{(1+|\lambda|)^n}\|(\lambda+A)^{-1}\|_{\mathcal{L}(X)}\,\Big|\,\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}\right\} < \infty, \tag{4.30}$$

and

$$\|T(t)\mathcal{A}^m\|_{\mathcal{L}(X)} \approx t^{n-1-m} \qquad (t \in [1, \infty)) \tag{4.31}$$

for all $m \in \{0, \ldots, n-1\}$, where $(T(t))_{t\geq 0}$ is the C_0-semigroup generated by $-\mathcal{A}$. Moreover, $\mathcal{A}$ is not sectorial.

We construct $\mathcal{A}$ using operator matrices. Let $A \in \mathcal{L}(L^2(0,1))$ be the multiplication operator given by $Af(x) := xf(x)$ for $f \in L^2(0,1)$ and $x \in (0,1)$. Set $X := (L^2(0,1))^n$ and let $N \in \mathcal{L}(X)$ be the nilpotent operator matrix with $N_{k,k+1} = I_{L^2(0,1)}$ for $k \in \{1, \ldots, n-1\}$, and $N_{k,l} = 0 \in \mathcal{L}(L^2(0,1))$ for $k, l \in \{1, \ldots, n\}$ with $l \neq k+1$. Set $\mathcal{A} := AI_X - N$. Then $\mathcal{A}$ is bounded and has dense range. Let $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ and $(S(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be the C_0-semigroups generated by $-\mathcal{A}$ and $-AI_X$. Then $T(t) =$

$S(t)e^{tN}$ for all $t \in [0, \infty)$, where we use that AI_X and N commute. Since $N^k \neq 0$ if and only if $k \leq n - 1$, one has $\|T(t)\|_{\mathcal{L}(X)} \eqsim t^{n-1}$ for $t \geq 0$. Also, $\sigma(\mathcal{A}) = [0, 1]$ and, using the Neumann series for the resolvent,

$$R(\lambda, \mathcal{A}) = R(\lambda, A)(I_X + R(\lambda, A)N)^{-1} = \sum_{k=0}^{n-1} R(\lambda, A)^{k+1}(-N)^k$$

for $\lambda \in \mathbb{C} \setminus [0, 1]$. This implies (4.30).

Fix $m \in \{0, \ldots, n-1\}$. Then $\mathcal{A}^m = \sum_{k=0}^m \binom{m}{k}(-1)^k A^{m-k} N^k$ and

$$T(t)\mathcal{A}^m = \sum_{k=0}^m \binom{m}{k}(-1)^k S(t) A^{m-k} e^{tN} N^k \qquad (t \in [0, \infty)). \tag{4.32}$$

Let $k \in \{0, \ldots, m\}$ and $t \geq m$. Then

$$\|S(t)\upharpoonright_{L^2(0,1)}(-A)^{m-k}\|_{\mathcal{L}(L^2(0,1))} = \sup_{s \in (0,1)} e^{-ts} s^{m-k} \eqsim t^{k-m}.$$

The dominating matrix element of $e^{tN} N^k$ is $\frac{t^{n-k-1}}{(n-k-1)!} I_{L^2(0,1)}$. Hence

$$\|S(t) A^{n-1-k} e^{tN} N^k\|_{\mathcal{L}(X)} \eqsim t^{k-m} t^{n-k-1}$$

for an implicit constant independent of t. Now (4.31) follows from (4.32).

4.7.4. Multiplication operators on Sobolev spaces

We now consider another typical setting where one encounters generators of unbounded semigroups with polynomial growth of the resolvent. It is included to show that even straightforward multiplication operators can generate unbounded C_0-semigroups when the underlying space is a Sobolev space. The example also shows that Proposition 4.19 is sharp.

Example 4.22. Fix $a \in (0, \infty)$ and $b \in (0, 1)$ with $a + b \geq 1$. Set $\varphi(s) := s^{-a} + is^b$ for $s \in (1, \infty)$. Let $X := W^{1,2}(1, \infty)$ and let A be the multiplication operator on X associated with φ. Then $\sigma(A) \subseteq \mathbb{C}_+$ and $-A$ generates the C_0-semigroup $(T(t))_{t \geq 0} \subseteq \mathcal{L}(X)$ given by $T(t)f(s) = e^{-t\varphi(s)} f(s)$ for $t \in [0, \infty)$, $f \in X$ and $s \in (1, \infty)$. We prove that A has resolvent growth $(0, \frac{b-1+2a}{b})$, by showing that $\|(\eta - i\xi + A)^{-1}\|_{\mathcal{L}(X)} \lesssim g(\xi) := |\xi|^{\frac{b-1+2a}{b}}$ for each $\eta \in [0, \infty)$ and $\xi \in \mathbb{R}$.

First note that the operator $(\eta - i\xi + A)^{-1}$ is the multiplication operator on X associated with $s \mapsto -(\eta + s^{-a} + i(s^b - \xi))^{-1}$. Furthermore,

$$\sup\{\|(\eta - i\xi + A)^{-1}\|_{\mathcal{L}(X)} \mid \eta \in [0, \infty), \xi \in [-(a/b)^{b/(a+b)}, (a/b)^{b/(a+b)}]\} < \infty,$$

where we use that $-A$ is a semigroup generator and that $\sigma(A) \subseteq \mathbb{C}_+$. For $\xi \in \mathbb{R}$ with $|\xi| > (a/b)^{b/(a+b)}$ we bound $\|(\eta - i\xi + A)^{-1}\|_{\mathcal{L}(X)}$, using the supremum norm of $s \mapsto -(\eta + s^{-a} + i(s^b - \xi))^{-1}$ and its derivative, by

$$
\begin{aligned}
&\sup_{s \in (1,\infty)} \frac{1}{|\eta + s^{-a} + i(s^b - \xi)|} + \sup_{s \in (1,\infty)} \frac{|-as^{-a-1} + ibs^{b-1}|}{|\eta + s^{-a} + i(s^b - \xi)|^2} \\
&\leq \sqrt{2} \sup_{s \in (1,\infty)} \frac{1}{s^{-a} + |s^b - \xi|} + \sup_{s \in (1,\infty)} \frac{as^{-a-1} + bs^{b-1}}{s^{-2a} + (s^b - \xi)^2}.
\end{aligned}
\tag{4.33}
$$

For the first term in (4.33) note that

$$
\sup\left\{ \frac{1}{s^{-a} + |s^b - \xi|} \,\Big|\, x \in [1, |\xi|^{1/b}] \right\} \leq \xi^{a/b} \leq g(\xi)
$$

and that $s \mapsto (s^{-a} + |s^b - \xi|)^{-1}$ is decreasing for $s > |\xi|^{1/b} > (a/b)^{1/(a+b)}$. For the second term and for $|\xi| > (a/b)^{b/(a+b)} > 1$ and $s \in (1, (|\xi| + \frac{1}{2})^{1/b})$, write

$$
\frac{as^{-a-1} + bs^{b-1}}{s^{-2a} + (s^b - \xi)^2} \leq as^{a-1} + \frac{bs^{b-1}}{s^{-2a}} \lesssim g(\xi).
$$

We conclude that A indeed has resolvent growth $(0, \frac{b-1+2a}{b})$.

Let $t \in [1, \infty)$ and write

$$
\begin{aligned}
\|T(t)\|_{\mathcal{L}(X)} &\eqsim \sup_{k \in \{0,1\}} \sup_{s \in (1,\infty)} \left| \frac{d^k}{ds^k}[e^{-t\phi(s)}] \right| \\
&\eqsim \sup_{s \in (1,\infty)} |ibs^{b-1}te^{-ts^{-a}} - as^{-a-1}te^{-ts^{-a}}| \eqsim \sup_{s \in (1,\infty)} s^{b-1}te^{-ts^{-a}} \eqsim t^{1-\frac{1-b}{a}}
\end{aligned}
$$

for implicit constants independent of t. It follows from Corollary 4.4 that

$$
\|T(t)\|_{\mathcal{L}(X_\tau, X)} \lesssim t^{1 - \frac{1-b}{a} - \rho} \qquad (t \in [1,\infty))
$$

for each $\tau \in [0, \infty)$ and $\rho \in [0, \tau b/(b-1+2a))$. On the other hand, explicit computations yield

$$
\begin{aligned}
\|T(t)\|_{\mathcal{L}(X_\tau, X)} &\eqsim \sup_{k \in \{0,1\}} \sup_{s \in (1,\infty)} \left| \frac{d^k}{ds^k}[e^{-t\phi(s)}\phi(s)^{-\tau}] \right| \\
&\eqsim \sup_{s \in (1,\infty)} s^{b-1-b\tau}te^{-ts^{-a}} \eqsim t^{1 - \frac{1-b+b\tau}{a}}.
\end{aligned}
\tag{4.34}
$$

Thus $\|T(\cdot)\|_{\mathcal{L}(X_\tau, X)}$ decays faster than Corollary 4.4 would imply. We also obtain from (4.34) that $\|T(t)\|_{\mathcal{L}(X_\tau, X)} \in L^1[0, \infty)$ if and only if $\tau > \frac{b-1+2a}{b}$. Therefore Proposition 4.19 yields that A has resolvent growth $(0, \beta)$ for each $\beta > \frac{b-1+2a}{b}$. Since the

notions of uniform boundedness and R-boundedness coincide on the Hilbert space X, this shows that the parameters in Proposition 4.19 cannot be improved.

5. Exponential stability

In this section we use the theory from the previous sections to derive in a unified manner various corollaries on exponential stability.

Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq0}$ on a Banach space X. Set $s(-A) := \sup\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(-A)\}$, and for $\beta \in [0, \infty)$ let

$$s_\beta(-A) := \inf\{\omega > s(-A) \mid \sup\{(1 + |\lambda|)^{-\beta}\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \mid \mathrm{Re}(\lambda) \geq \omega\} < \infty\},$$
$$s_R(-A) := \inf\{\omega > s(-A) \mid R_X(\{(\lambda + A)^{-1} \mid \mathrm{Re}(\lambda) \geq \omega\}) < \infty\}.$$

Then Proposition 4.19 yields

$$s_0(-A) \leq s_R(-A) \leq \omega_0(T).$$

In particular, for each $\eta \in (\omega_0(T), \infty)$ the operator $A+\eta$ is sectorial. Hence for $\beta \in [0, \infty)$ the fractional domain $X_\beta = D((\eta + A)^\beta)$ is defined as in Section 2.3, and up to norm equivalence X_β does not depend on the choice of $\eta \in (\omega_0(T), \infty)$. Throughout this section we fix a choice of $\eta \in (\omega_0(T), \infty)$ and the associated spaces X_β for $\beta \in [0, \infty)$. For $x \in X$ let

$$\omega(x) := \inf\{\omega \in \mathbb{R} \mid \lim_{t\to\infty} \|e^{-\omega t}T(t)x\|_X = 0\},$$

and for a Banach space Y continuously embedded in X set

$$\omega_Y(T) := \sup\{\omega(x) \mid x \in Y\}.$$

For $\beta \in (0, \infty)$ we write $\omega_\beta(T) := \omega_{X_\beta}(T)$. The uniform boundedness principle implies that for all $\omega > \omega_Y(T)$ there exists an $M \in (0, \infty)$ such that

$$\|T(t)\|_{\mathcal{L}(Y,X)} \leq Me^{\omega t} \qquad (t \in [0, \infty)). \tag{5.1}$$

We need two preparatory lemmas. The first is [65, Lemma 3.5], and it follows directly from Lemma 4.2 and from basic properties of convex functions.

Lemma 5.1. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq0}$ on a Banach space X. Then the function $(0, \infty) \to [-\infty, \infty)$, $\beta \mapsto \omega_\beta(T)$, is continuous on open subintervals of $\{\beta \in \mathbb{R} \mid \omega_\beta(T) \in (-\infty, \infty)\}$.*

The following lemma is [65, Theorem 3.1] (see also [67, Theorem 3.2]). We show that it follows directly from Lemma 5.1 and Proposition 4.3.

Lemma 5.2. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Then $\omega_{\beta+1}(T) \leq s_\beta(-A)$ for all $\beta \in [0,\infty)$.*

Proof. First note that by Lemma 5.1 it suffices to show that $\omega_{\beta+1+\varepsilon}(T) \leq s_\beta(-A)$ for all $\beta \geq 0$ and $\varepsilon \in (0,1)$. Also, by a scaling argument we may suppose that $s_\beta(-A) < 0$ and prove that $\omega_{\beta+1+\varepsilon}(T) \leq 0$. But in this case A has resolvent growth $(0,\beta)$, and Proposition 4.3 then shows that $\sup_{t\geq 0} \|T(t)\|_{\mathcal{L}(X_{\beta+1+\varepsilon},X)} < \infty$. $\quad\square$

5.1. The resolvent as an (L^p, L^q) Fourier multiplier

The following theorem is the main link between exponential stability and (L^p, L^q)-Fourier multipliers. This result appeared in [26] and in full generality in [37, Theorem 3.6]. Here we give a proof using Theorem 4.6.

Theorem 5.3. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and let $\beta \in [0,\infty)$. Then, for all $p \in [1,\infty)$ and $q \in [p,\infty]$,*

$$\omega_\beta(T) = \inf\{\omega > s_\beta(-A) \mid (\omega + i \cdot + A)^{-1} \in \mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(X_\beta, X))\}. \tag{5.2}$$

In fact, $(\omega + i \cdot + A)^{-1} \in \mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(X_\beta, X))$ for all $\omega > \omega_\beta(T)$.

Proof. Fix $p \in [1,\infty)$ and $q \in [p,\infty]$, and denote the right-hand side of (5.2) by $\mu_{p,q,\beta}(A)$. We first show that $\omega_\beta(T) \geq \mu_{p,q,\beta}(A)$. Let $\omega > \omega_\beta(T)$, and apply Proposition 4.19 to $(e^{-\omega t} T(t))_{t\geq 0}$ to obtain $\omega > s_\beta(-A)$. Now $(\omega + i \cdot + A)^{-1} \in \mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(X_\beta, X))$ follows from Lemma 3.1 and Young's inequality for convolutions. Hence $\omega \geq \mu_{p,q,\beta}(A)$, and the statement follows by letting $\omega \downarrow \omega_\beta(T)$.

To prove the reverse inequality it suffices to assume that $\mu_{p,q,\beta}(A) < 0$ and show that $\omega_\beta(A) \leq 0$. Note that $R(i\cdot, A) \in \mathcal{M}_{p,q,\beta}(\mathbb{R}; \mathcal{L}(X_\beta, X))$. Indeed, this follows by using Proposition 3.4 and [53, Theorem 5.18] to express $(i \cdot + A)^{-1} \in L^\infty(\mathbb{R}; \mathcal{L}(X_\beta, X))$ as a convolution of the Poisson kernel with $(\omega + i \cdot + A)^{-1}$ for $s_\beta(-A) < \omega < 0$, and by applying Young's inequality. From Theorem 4.6 with $\psi \equiv 0$ one now obtains $\sup_{t\geq 0} \|T(t)\|_{\mathcal{L}(X_\beta, X)} < \infty$ and $\omega_\beta(T) \leq 0$. Here one may use Lemma 5.2 to see that X_β satisfies the assumptions of Theorem 4.6. $\quad\square$

The first part of the following theorem is [65, Theorem 3.2] (see also [49, Theorem 4.4] and [67, Remark 3.3]). The proof avoids the use of Mikhlin's multiplier theorem on Besov spaces (see [65, Theorem 2.1]) and instead relies on the elementary Proposition 2.2. Part (2) is the main result of [47].

Theorem 5.4. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Then the following assertions hold:*

(1) *If X has Fourier type $p \in [1,2]$ then $\omega_{\frac{1}{p}-\frac{1}{p'}}(T) \leq s_0(-A)$.*
(2) *If X has type $p \in [1,2]$ and cotype $q \in [2,\infty]$ then $\omega_{\frac{1}{p}-\frac{1}{q}}(T) \leq s_R(-A)$.*

Proof. By Lemma 5.1 and a scaling argument, for (1) we may assume that $s_0(-A) < 0$ and show that $\omega_{\frac{1}{p}-\frac{1}{p'}+\varepsilon}(T) \leq 0$ for any $\varepsilon > 0$. The latter follows directly from Theorem 4.9. In the same way (2) follows from Theorem 4.13. Alternatively, one can give direct proofs by combining Theorem 5.3 with Proposition 3.4 and the multiplier results in Propositions 2.2 and 2.3. $\square$

The geometry of X and regularity of the resolvent can be used to obtain R-bounds from uniform bounds, leading to the following corollary.

Corollary 5.5. *Let $-A$ be the generator of a C_0-semigroup on a Banach space X with type $p \in [1, 2]$ and cotype $q \in [2, \infty]$. Then $\omega_{\frac{2}{p}-\frac{2}{q}}(T) \leq s_0(-A)$.*

Proof. Let $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and $\beta > \frac{1}{p} - \frac{1}{q}$. By Lemma 5.2 we may suppose that $\beta \in (0, 1/2)$. By Lemma 5.1, Theorem 5.3 and Proposition 2.3 it suffices to show that for each $\omega > s_0(A)$ the set $\{(1+|\xi|)^\beta(\omega+i\xi+A)^{-1} \mid \xi \in \mathbb{R}\} \subseteq \mathcal{L}(X_{2\beta}, X)$ is R-bounded. Let $E := \mathcal{L}(X_{2\beta}, X)$ and define $f : \mathbb{R} \to E$ by $f(\xi) := (1+|\xi|)^\beta(\omega+i\xi+A)^{-1}$ for $\xi \in \mathbb{R}$. Then $\|f(\xi)\|_E \leq C(1+|\xi|)^{-\beta}$ by Proposition 3.4, so that $f \in L^r(\mathbb{R}; E)$. Moreover,

$$\|f'(\xi)\|_{\mathcal{L}(E)} \leq \beta(1+|\xi|)^{\beta-1}\|(\omega+i\xi+A)^{-1}\|_E + (1+|\xi|)^\beta\|(\omega+i\xi+A)^{-2}\|_E$$
$$\lesssim (1+|\xi|)^{-\beta-1} + (1+|\xi|)^{-\beta}.$$

So $f \in W^{1,r}(\mathbb{R}; E)$, and Lemma 2.1 shows that the range of f is R-bounded. $\square$

For $X = L^r(\Omega)$ with $r \in [1, \infty)$, Corollary 5.5 and part (1) of Theorem 5.4 yield the same conclusion. It is an open question whether in this case the index $|\frac{2}{r} - 1|$ can be improved.

Remark 5.6. In Theorem 5.4 and Corollary 5.5 one can add a parameter $\beta \in [0, \infty)$, as in Lemma 5.2. Then Theorem 5.4 (1) says that $\omega_{\beta+\frac{1}{p}-\frac{1}{p'}}(T) \leq s_\beta(-A)$, and (2) that $\omega_{\beta+\frac{1}{p}-\frac{1}{q}}(T) \leq s_{R,\beta}(-A)$. Here

$$s_{R,\beta}(-A) := \inf\{\omega > s(-A) \mid R_X(\{(1+|\lambda|)^{-\beta}(\lambda+A)^{-1} \mid \operatorname{Re}(\lambda) \geq \omega\}) < \infty\}.$$

In Corollary 5.5 the more general inequality is $\omega_{2\beta+\frac{2}{p}-\frac{2}{p'}}(T) \leq s_\beta(-A)$. The proofs are the same, using Proposition 3.4.

5.2. The resolvent as a Fourier multiplier on Besov spaces

In this subsection we give an alternative characterization of $\omega_\beta(T)$, $\beta > 0$, using Fourier multipliers on Besov spaces. We then use this characterization to obtain a new stability result for positive semigroups.

For the definition and basic properties of vector-valued Sobolev and Besov spaces which are used below we refer to [58,59].

Theorem 5.7. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and let $\beta \in (0, \infty)$. Then for all $p \in [1, \infty)$ and $q \in [p, \infty]$,*

$$\omega_\beta(T) = \inf\{\omega > s_\beta(-A) \mid T_{m_\omega} \in \mathcal{L}(B^\beta_{p,1}(\mathbb{R}; X), L^q(\mathbb{R}; X))\},$$

where $m_\omega(\cdot) := (\omega + i \cdot + A)^{-1}$ for $\omega > s_\beta(-A)$.

Proof. Denote the right-hand side of (5.2) by $\nu_{p,q,\beta}(A)$. We first show that $\omega_\beta(T) \leq \nu_{p,q,\beta}(A)$. By shifting A and using Lemma 5.1 we may assume that $\nu_{p,q,\beta}(A) < 0$ and prove that $\omega_{\beta+\varepsilon}(T) \leq 0$ for any $\varepsilon > 0$. Without loss of generality we may suppose that $(\beta, \beta + \varepsilon) \cap \mathbb{N} = \varnothing$. Let $n := \lceil \beta + \varepsilon \rceil$, $\alpha \in (\beta, \beta + \varepsilon)$ and let $D_A(\alpha, 1) = (X, D(A^n))_{\alpha/n,1}$ be the appropriate real interpolation space. Set $m(\xi) := (i\xi + A)^{-1}$ for $\xi \in \mathbb{R}$. As in Theorem 5.3 one sees that $T_m \in \mathcal{L}(B^\alpha_{p,1}(\mathbb{R}; X), L^q(\mathbb{R}; X))$, where we also use that $B^\alpha_{p,1}(\mathbb{R}; X) \subseteq B^\beta_{p,1}(\mathbb{R}; X)$ continuously.

Let $\omega > \omega_0(T)$ and let $F \in \mathcal{L}(X, L^p(\mathbb{R}; X))$ be given by $Fx(t) := t^n e^{-\omega t} T(t)x$ for $t \in \mathbb{R}$ and $x \in X$, where we extend the semigroup by zero to all of $\mathbb{R}$. Then $F : X_n \to W^{n,p}(\mathbb{R}; X)$ is bounded and, by real interpolation,

$$F : D_A(\alpha, 1) \to (L^p(\mathbb{R}; X), W^{n,p}(\mathbb{R}; X))_{\alpha/n,1} = B^\alpha_{p,1}(\mathbb{R}; X)$$

is bounded. Now fix $x \in X_{\alpha+1}$ and let $f := Fx$. Then

$$\|T_m(f)\|_{L^q(\mathbb{R};X)} \lesssim \|f\|_{B^\alpha_{p,1}(\mathbb{R};X)} \lesssim \|x\|_{D_A(\alpha,1)} \lesssim \|x\|_{X_{\beta+\varepsilon}}, \tag{5.3}$$

where we have also used that $X_{\beta+\varepsilon} \hookrightarrow X_{D_A(\alpha,1)}$. By Lemmas 3.1 and 5.2 one has

$$T * f(t) := \int_0^t T(t-s)f(s)\mathrm{d}s = T_m(f)(t) \qquad (t \in [0, \infty)) \tag{5.4}$$

and $T_m(f)(t) = 0$ for $t \in (-\infty, 0)$. On the other hand, $T(t-s)f(s) = s^n e^{-\omega s} T(t)x$ for $s \in [0, t]$. Hence, for $t \geq 1$,

$$\|T(t)x\|_X \lesssim \|T(t)x\|_X \int_0^t s^n e^{-\omega s}\mathrm{d}s = \|T * f(t)\|_X.$$

Now, using that $\sup_{t\in[0,1]} \|T(t)\|_{\mathcal{L}(X)} < \infty$, it follows from (5.3) and (5.4) that

$$\left(\int_0^\infty \|T(t)x\|_X^q\mathrm{d}t\right)^{\frac{1}{q}} \lesssim \|x\|_X + \|T * f\|_{L^q([1,\infty);X)} \lesssim \|x\|_{X_{\beta+\varepsilon}}.$$

Therefore Proposition 4.8 implies that $R(i\cdot, A) \in \mathcal{M}_{1,q}(\mathbb{R}; \mathcal{L}(X_{\beta+\varepsilon}, X))$. Here one may again use Lemma 5.2 to see that $X_{\beta+\varepsilon}$ satisfies the conditions of Proposition 4.8. Finally, Theorem 5.3 shows that $\omega_{\beta+\varepsilon}(A) \le 0$.

Next, we prove that $\omega_\beta(T) \ge \nu_{p,q,\beta}(A)$. To do so it suffices by Lemma 5.1 to show that for all $\alpha \in (0, \beta)$ and $\omega > \omega_\alpha(T)$ one has $T_{m_\omega} \in \mathcal{L}(B_{p,1}^\beta(\mathbb{R}; X), L^q(\mathbb{R}; X))$. Moreover, we may suppose that $\alpha \notin \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $\alpha, \beta \in (n-1, n]$. By (5.1) there exist $M, \varepsilon \in (0, \infty)$ such that $e^{-\omega t}\|T(t)\|_{\mathcal{L}(X_\alpha, X)} \le Me^{-\varepsilon t}$ for all $t \ge 0$. For $f \in \mathcal{S}(\mathbb{R}; X)$ set $S_\alpha f := (\omega + A)^{-\alpha} T_{m_\omega} f$. Then $S_\alpha \in \mathcal{L}(L^p(\mathbb{R}; X), L^q(\mathbb{R}; X))$ by Theorem 5.3. We claim that $S_\alpha \in \mathcal{L}(W^{k,p}(\mathbb{R}; X), L^q(\mathbb{R}; X_k))$ for each $k \in [1, \ldots, n]$. Indeed, let $f \in C_c^k(\mathbb{R}) \otimes X$ and $t \in [0, \infty)$. Lemma 3.1 and integration by parts yield

$$(\omega + A)^k S_\alpha f(t) = \int_{-\infty}^{t} (\omega + A)^k e^{-\omega(t-s)} T(t-s)(\omega + A)^{-\alpha} f(s)\mathrm{d}s$$

$$= -\int_{-\infty}^{t} \frac{d}{ds}[(\omega + A)^{k-1} e^{-\omega(t-s)} T(t-s)](\omega + A)^{-\alpha} f(s)\mathrm{d}s$$

$$= -(\omega + A)^{k-1-\alpha} f(t) + \int_{-\infty}^{t} (\omega + A)^{k-1} e^{\omega(t-s)} T(t-s)(\omega + A)^{-\alpha} f'(s)\mathrm{d}s$$

$$= -(\omega + A)^{k-1-\alpha} f(t) + (\omega + A)^{k-1} S_\alpha f'(t).$$

By iterating this procedure one obtains

$$(\omega + A)^k S_\alpha f(t) = -\sum_{j=1}^{k} (\omega + A)^{k-j-\alpha} f^{(j-1)}(t) + S_\alpha f^{(k)}(t).$$

Since $k - 1 - \alpha < 0$ this yields

$$\|(\omega + A)^k S_\alpha f\|_{L^q(\mathbb{R};X)} \le \sum_{j=1}^{k} \|(\omega + A)^{k-j-\alpha} f^{(j-1)}\|_{L^q(\mathbb{R};X)} + \|S_\alpha f^{(k)}\|_{L^q(\mathbb{R};X)}$$

$$\lesssim \|f\|_{W^{k-1,q}(\mathbb{R};X)} + \|f^{(k)}\|_{L^p(\mathbb{R};X)} \lesssim \|f\|_{W^{k,p}(\mathbb{R};X)},$$

and the claim follows since $(\omega + A)^{-k} : X \to X_k$ is an isomorphism.

Now, if $\beta = n$ then $B_{p,1}^\beta(\mathbb{R}; X) \subseteq W_{n,p}(\mathbb{R}; X)$ continuously so $S_\alpha : B_{p,1}^\beta(\mathbb{R}; X) \to L^q(\mathbb{R}; X_\beta)$ is bounded. On the other hand, if $\beta < n$ then real interpolation for the exponents $k = n - 1$ and $k = n$ shows that $S_\alpha \in \mathcal{L}(B_{p,1}^\beta(\mathbb{R}; X), L^q(\mathbb{R}; D_A(\beta, 1)))$. Since $\alpha < \beta$, in both cases we obtain that $T_{m_\omega} : B_{p,1}^\beta(\mathbb{R}; X) \to L^q(\mathbb{R}; X)$ is bounded, which concludes the proof. $\square$

The following theorem unifies [64, Theorem 1] and [47, Corollary 1.3] and is new for $1 \leq p < q < 2$ and $2 < p < q < \infty$. Here there is no use in adding an additional parameter β as in Lemma 5.2 and Remark 5.6, since $s_0(-A) = s(-A)$.

Theorem 5.8. *Let $-A$ be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X which is p-convex and q-concave for $p \in [1,\infty)$ and $q \in [p,\infty)$. Then $\omega_{\frac{1}{p}-\frac{1}{q}}(T) \leq s(-A)$.*

Proof. First note that $s_0(-A) = s(-A)$ (see [3, Theorem 5.3.1]). Let $r \in (1,\infty]$ satisfy $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and let $\omega > s_0(-A)$. By Theorem 5.7 it suffices to show that $T_m \in \mathcal{L}(B_{p,1}^{1/r}(\mathbb{R};X), L^q(\mathbb{R};X))$ for $m(\xi) := (\omega + i\xi + A)^{-1}$, $\xi \in \mathbb{R}$. For $n \in \mathbb{N}$ with $n > \omega_0(T)$ set $K_n(t) := e^{-\omega t}T(t)n(n+A)^{-1}$, $t \geq 0$, and let $K_n \equiv 0$ on $(-\infty,0)$. Then $K_n(t) \in \mathcal{L}(X)$ is positive for all $t \in \mathbb{R}$, and $K_n(\cdot)x \in L^1(\mathbb{R};X)$ for all $x \in X$ by Lemma 5.2. Furthermore, $\mathcal{F}(K_n x)(\xi) = m_n(\xi)x$ for $\xi \in \mathbb{R}$, where $m_n(\xi) := (\omega + i\xi + A)^{-1}n(n+A)^{-1}$. Note that $\sup_{\xi \in \mathbb{R}} \|m_n(\xi)\|_{\mathcal{L}(X)} < \infty$. Now, since X has cotype $q < \infty$ (see [18, p. 332]), it follows from Proposition 2.4 and the continuous embedding $B_{p,1}^{1/r}(\mathbb{R};X) \subseteq H_p^{1/r}(\mathbb{R};X)$ that

$$C := \sup_n \|T_{m_n}\|_{\mathcal{L}(B_{p,1}^{1/r}(\mathbb{R};X), L^q(\mathbb{R};X))} < \infty.$$

Now fix $f \in \mathcal{S}(\mathbb{R};X)$. Then $\|T_{m_n}(f)\|_{L^q(\mathbb{R};Y)} \leq C\|f\|_{B_{p,1}^{1/r}(\mathbb{R};X)}$ for all n. Moreover, for $\xi \in \mathbb{R}$ and $x \in X$ one has $\lim_{n\to\infty} m_n(\xi)x = m(\xi)x$ by [19, Lemma 3.4]. Now [55, Lemma 3.1] implies that $T_m \in \mathcal{L}(B_{p,1}^{1/r}(\mathbb{R};X), L^q(\mathbb{R};X))$, as required. $\quad\square$

Using Theorem 5.8 and [22, Example 5.5b] one can modify an example due to Arendt (see [3, Example 5.1.11], [67, Section 4] and [47, Example 1.4]) to construct for all $1 \leq p \leq q < \infty$ a positive C_0-semigroup $(T(t))_{t\geq 0}$ on $X = L^p(1,\infty) \cap L^q(1,\infty)$ with generator $-A$ such that $\omega_0(T) = -\frac{1}{p}$,

$$\omega_{\frac{1}{p}-\frac{1}{q}}(T) = s_R(-A) = s_0(-A) = -\frac{1}{q},$$

and such that $\alpha \mapsto \omega_\alpha(T)$ is linear on $[0, \frac{1}{p} - \frac{1}{q}]$. This shows that the index $\frac{1}{p} - \frac{1}{q}$ in part (2) of Theorem 5.4 and in Theorem 5.8 is optimal, which shows in turn that [55, Theorem 3.24] is optimal. Moreover, it follows from [65, Example 4.4] that the positivity assumption in Theorem 5.8 cannot be omitted.

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Appendix A. Technical estimates

In this section we provide the proofs of a few technical results which are used in the main text.

A.1. Contour integrals

We start with a lemma which is needed when dealing with certain contour integrals in Proposition 3.4.

Lemma 5.9. *Let $\varphi \in (0, \frac{\pi}{2}]$ and $\theta \in (\pi - \varphi, \pi)$. Set $\Omega := \overline{\mathbb{C}_+} \setminus (S_\varphi \cup \{0\})$ and let $\Gamma := \{re^{i\theta} \mid r \in [0, \infty)\} \cup \{re^{-i\theta} \mid r \in [0, \infty)\}$ be oriented from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$. Then for all $\alpha \in [0, \infty)$, $\beta \in (0, \infty)$, $\eta \in (0, 1]$ and $\lambda \in \Omega$ one has*

$$\frac{1}{2\pi i} \int_\Gamma \frac{z^\alpha}{(\eta + z)^{\alpha+\beta}(z + \lambda + \eta - 1)} dz = \frac{(1 - \eta - \lambda)^\alpha}{(1 - \lambda)^{\alpha+\beta}}. \tag{A.1}$$

Furthermore, for all $\gamma \in [1, \infty)$ and $\delta \in [0, \infty)$ there exists a constant $C \in [0, \infty)$ such that

$$\frac{|z|^\gamma}{|1 + z|^{\gamma+\delta}} \frac{|1 - \lambda|}{|z + \lambda|} \leq C \quad and \quad \frac{1 + |\lambda|}{|\frac{1}{2} + z|^\delta |z + \lambda - \frac{1}{2}|} \leq C \tag{A.2}$$

for all $z \in \Gamma$ and $\lambda \in \Omega$.

Proof. Let $\alpha \in [0, \infty)$, $\beta \in (0, \infty)$, $\eta \in (0, 1]$ and $\lambda \in \Omega$. For $r \in (0, \text{Im}(\lambda)/2)$ and $R \geq 2|\lambda| + 2$ set $\Gamma_+ := \{se^{i\theta} \mid s \in [r, R]\}$, $\Gamma_- := \{se^{-i\theta} \mid s \in [r, R]\}$, $\Gamma_r := \{re^{i\nu} \mid \nu \in [-\theta, \theta]\}$ and $\Gamma_R := \{Re^{i\nu} \mid \nu \in [-\theta, \theta]\}$, and let $\Gamma_{r,R} := \Gamma_+ \cup \Gamma_r \cup \Gamma_- \cup \Gamma_R$ be oriented counterclockwise. Then

$$\int_{\Gamma_R} \frac{|z|^\alpha}{|\eta + z|^{\alpha+\beta}|z + \lambda + \eta - 1|} d|z| = \int_{-\theta}^{\theta} \frac{R^{1+\alpha}}{|\eta + Re^{i\nu}|^{\alpha+\beta}|Re^{i\nu} + \lambda + \eta - 1|} d\nu$$

$$= R^{-\beta} \int_{-\theta}^{\theta} \frac{1}{|\frac{\eta}{R} + e^{i\nu}|^{\alpha+\beta}|e^{i\nu} + \frac{\lambda + \eta - 1}{R}|} d\nu$$

$$\leq 2^{2+\alpha+\beta} \theta R^{-\beta},$$

and the latter tends to zero as $R \to \infty$. Similarly, one sees that

$$\int_{\Gamma_r} \frac{|z|^\alpha}{|\eta + z|^{\alpha+\beta}|z + \lambda + \eta - 1|} \mathrm{d}|z|$$

tends to zero as $r \to 0$. Now Cauchy's integral theorem yields

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{z^\alpha}{(\eta + z)^{\alpha+\beta}(z + \lambda + \eta - 1)} \mathrm{d}z$$

$$= \lim_{r \to 0, R \to \infty} \frac{1}{2\pi i} \int_{\Gamma_{r,R}} \frac{z^\alpha}{(\eta + z)^{\alpha+\beta}(z + \lambda + \eta - 1)} \mathrm{d}z = \frac{(1 - \eta - \lambda)^\alpha}{(1 - \lambda)^{\alpha+\beta}},$$

which proves (A.1).

Next, let $\gamma \in [1, \infty)$, $\delta \in (0, \infty)$, $z \in \Gamma$ and $\lambda \in \Omega$. Note that $|z + \lambda| = |z| \, |e^{\pm i\theta} + \lambda'|$ for some $\lambda' \in \Omega$. Since the distance from $e^{\pm i\theta}$ to $-\Omega$ is nonzero, there exists a constant $C_1 \in (0, \infty)$ such that $|z + \lambda| \geq C_1 |z|$. Hence

$$\frac{|z|^\gamma}{|1 + z|^{\gamma+\delta}} \frac{|1 - \lambda|}{|z + \lambda|} \leq \frac{|z|^\gamma}{|1 + z|^{\gamma+\delta}} \Big(\frac{|1 + z|}{|z + \lambda|} + 1 \Big) \leq \frac{|z|^\gamma}{|1 + z|^{\gamma+\delta}} \Big(\frac{C_1^{-1}}{|z|} + C_1^{-1} + 1 \Big),$$

and the latter is uniformly bounded in $z \in \Gamma$.

For the second term in (A.2) first note that the distances from $z - \frac{1}{2}$ to Γ, and hence to $-\Omega$, and from $z + \frac{1}{2}$ to 0 are bounded uniformly from below by a constant $C_2 > 0$. Hence $|z + \lambda - \frac{1}{2}| \geq C_2$ and $|\frac{1}{2} + z| \geq C_2$ for all $z \in \Gamma$ and $\lambda \in \Omega$. Therefore, for the second term in (A.2) it suffices to bound $\frac{|\lambda|}{|z + \lambda - \frac{1}{2}|}$ uniformly. Let $\nu \in [\frac{\pi}{2}, \frac{\pi}{2}]$ be such that $\lambda = |\lambda| e^{i\nu}$, and set $w := \frac{z}{|\lambda|} \in \Gamma$. Then

$$\frac{|\lambda|}{|z + \lambda - \frac{1}{2}|} = \frac{1}{|w + e^{i\nu} - \frac{1}{2|\lambda|}|}.$$

Now the required results follows, since by geometric inspection one sees that

$$|(w - \tfrac{1}{2|\lambda|}) - (-e^{i\nu})| \geq \mathrm{dist}(\Gamma, -e^{i\nu}) \geq \mathrm{dist}(\Gamma, -e^{i\varphi}). \quad \Box$$

A.2. Estimates for exponential functions

The following lemma provides a two-sided exponential estimate, one part of which is used in Example 4.20.

Lemma 5.10. *Let $m \in \mathbb{N}$. Then*

$$\frac{e^m}{m^{1/4}e^2} \leq \Big(\sum_{j=0}^{m} \Big(\frac{m^j}{j!} \Big)^2 \Big)^{1/2} \leq \frac{e^m}{m^{1/4}}. \tag{A.3}$$

Proof. Both estimates are clear for $m = 1$, so we may consider $m \geq 2$ throughout. Let $k \in \mathbb{N} \cap [1, \lfloor \sqrt{m} \rfloor]$ and note that

$$\log \left(\frac{m^{m-k}}{(m-k)!} \right) = (m-k)\log(m) - \log((m-k)!)$$

$$\geq (m-k)\log(m) - (m-k+\tfrac{1}{2})\log(m-k) + m - k - 1,$$

where we used Stirling's formula. Moreover,

$$\log(m-k) = \log(m) + \log(1 - \tfrac{k}{m}) \leq \log(m) - \tfrac{k}{m},$$

where we used that $\log(1-s) \leq -s$ for $s \in (0,1)$. Hence

$$\log \left(\frac{m^{m-k}}{(m-k)!} \right) \geq (m-k)\log(m) - (m-k+\tfrac{1}{2})(\log(m) - \tfrac{k}{m}) + m - k - 1$$

$$\geq -\frac{1}{2}\log(m) - \frac{k^2}{m} + m - 1 \geq -\frac{1}{2}\log(m) + m - 2,$$

where in the last step we used that $k^2 \leq m$ holds. We now see that $\frac{m^{m-k}}{(m-k)!} \geq m^{-1/2}e^{-2}e^m$, from which we deduce the first inequality in (A.3):

$$\sum_{j=0}^{m} \left(\frac{m^j}{j!} \right)^2 \geq \sum_{k=0}^{\lfloor \sqrt{m} \rfloor} \left(\frac{m^{m-k}}{(m-k)!} \right)^2 \geq m^{\frac{1}{2}} m^{-1} e^{-4} e^{2m} = m^{-1/2} e^{-4} e^{2m}.$$

For the second inequality let $a_j := \frac{m^j}{j!}$ for $j \in \{0, \ldots, m\}$. Then another application of Stirling's formula yields

$$a_{m-1} \leq \frac{m^{m-1}}{\sqrt{2\pi}(m-1)^{m-\frac{1}{2}}e^{-(m-1)}} = \frac{e^m}{e\sqrt{2\pi}\sqrt{m}} \left(1 + \frac{1}{m-1} \right)^{m-\frac{1}{2}} \leq \frac{e^m}{\sqrt{m}}.$$

Also, $a_j \leq a_{m-1}$ for each $j \in \{0, \ldots, m-1\}$, where we use that $a_{j+1}/a_j \geq 1$ for each $j \in \{0, \ldots, m-1\}$ and that $\frac{a_m}{a_{m-1}} = 1$. Since $\sum_{j=0}^{m} a_j \leq e^m$, the upper estimate in (A.3) follows from

$$\sum_{j=0}^{m} a_j^2 \leq a_{m-1} \sum_{j=0}^{m} a_j \leq \frac{e^{2m}}{\sqrt{m}}. \quad \square$$

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