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Self-duality and hydrodynamics in dynamic random environment

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SYMMETRIC INTERACTING PARTICLE SYSTEMS: SELF-DUALITY & HYDRODYNAMICS IN DYNAMIC RANDOM ENVIRONMENT

Federico SAU

SYMMETRIC INTERACTING PARTICLE SYSTEMS: SELF-DUALITY & HYDRODYNAMICS IN DYNAMIC RANDOM ENVIRONMENT

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A Giù

Preface

The general aim of this thesis is the study of detailed and scaling features of a class of conservative interacting particle systems.

In mathematical statistical physics, a challenging task is the rigorous derivation of the macroscopic laws arising from the underlying microscopic reality of interacting components. The target laws are typically described in terms of autonomous partial differential systems of first order in time, such as Fourier's law of heat conduction, Fick's law of diffusion, as well as Euler's conservation equations and Navier-Stokes' equations.

The mathematical procedure which connects the dynamics at micro and macro scales is known as hydrodynamic limit (see the surveys [31], [32], [85], [132]). In words, it consists of scaling down the size of the individual interacting units, rescaling time accordingly, averaging over many of these units and studying the evolution over macroscopic times of this average as governed by macroscopic autonomous differential equations. Key feature here – compared to other scaling procedures such as the thermodynamic limit – is that both space and time undergo a rescaling to obtain a sensible limiting behavior.

Stochastic interacting particle systems (IPS) [98] are suitable microscopic models for this mathematical investigation. Although the addition of stochasticity to deterministic Hamiltonian models takes away from microscopic systems some of their adherence to reality, a wide portion of literature devoted to the rigorous transition from micro to macro employs IPS, the gain being two-fold: whilst, on the one side, IPS are more manageable than Hamiltonian models as they do not require as many assumptions on their dynamical behavior as deterministic models do (see e.g. [32]), on the other side, they still display – on appropriate scales – some of the key features of macroscopic systems, such as phase transitions, metastable behaviors or formation of shocks (see e.g. [31]).

The number of microscopic models from which a given macroscopic law emerges is typically large, i.e. the map which associates a microscopic to a macroscopic description is, in general, far from being one-to-one. As a consequence, even substantially different particle systems exhibit resembling behaviors at macroscopic scales. The study of this connection stands at the core of the program of mathematical statistical physics.

Within this realm, an active line of research studies how the scaling behavior of stochastic systems is affected by the introduction of disorder at a microscopic scale (see [111], [129], [130] for some of the first works in this direction, e.g. [2], [10], [11], [127] and references therein for more recent developments). The analysis of the hydrodynamic limit of an interacting particle system in dynamic random environment is the content of the first part of this thesis.

The second part of the thesis focuses on a detailed property of conservative interacting particle systems, and, more generally, of Markov processes, called duality. This property refers to the possibility of studying a class of observables of the original system in terms of quantities of a – possibly simpler – dual system. In the context of conservative particle systems, practically speaking, this typically means that one may reduce the study of an observable of many particles to an observable of a system consisting of just one or two particles. The observables which enable this connection are indicated as duality functions.

As all exact methods, duality has the disadvantage of being a model dependent property. In general, duality may hold for a pair of processes but fail for small perturbations of either one or both of them. For instance, duality is a key tool for some symmetric particle systems (see e.g. [62], [98], [126]), but holds only for special asymmetric perturbations of these systems (see e.g. [21], [22], [59], [123], [124]).

For this reason, part of the research about duality consists of systematically finding, within a class of Markov processes, all pairs of dual processes and all possible duality functions. In words, this is the plan of what is presented in the second part of this thesis: within the context of conservative symmetric particle systems, by means of generating function and spectral methods, we characterize all dual processes and duality functions in a given factorized form.

Contents

Chapter 1

Introduction

This section is devoted to a mathematical presentation of the main notions and results to be found in the thesis.

Section 1.2 comes in form of an "expository route" throughout the chapters of the thesis, whilst Section 1.3 sketches schematically the content of each chapter. Section 1.1 is a panoramic introduction to the subject and its context. For expert readers, this part may be skipped at a first reading.

1.1 An overview on interacting particle systems

The "reading of a wave"[1] may turn from a contemplative and fulfilling activity into a frantic and stressful experience if taken over by the impatience of reaching a complete and definitive description of one's observation.

Either to isolate a single wave from the ones that follow and precede it, or to discern within a fixed 10×10 m square the propagation of all wave fronts which collapse and fragment one against the other? Or, maybe, to get closer to the shoreline, pull out of the pocket a microscope and trace a single water molecule which rapidly clashes erratically against its companions until it may, all of a sudden – and a bit fortuitously – jump right in front of its fellow molecules? Or to keep track of all these features together, all at the same time, at all scales, until, eventually, a new and unexpected occurrence enters and dissolves the complex picture which one has, punctiliously, just put together?

What a scientist would be expected to do in this situation is to tame complexity by reducing the observed reality to its simpler mechanisms.

[1]Free literal translation from the Italian *lettura di un'onda [16].*

Thermodynamic systems. In this reduction, the first step to take would be the identification of a precise and limited frame of observation. In the language of physics, we may think of this operation as corresponding to the setup of a *thermodynamic system*.

Examples that one may want to consider include the so-called open systems. If one is still observing a wave, there an open system may be anything confined within a limited portion of sea, with water and energy flowing in and out that region. Other examples of thermodynamic systems, depicted, for instance, by the content enclosed in a bottle stirred by the sea waves, are referred to as closed systems, for which the walls of the bottle forbid mass exchange with the surroundings, although external forces may still act on it. Idealized systems which are both closed and insensitive to any influence of the surroundings are called isolated.

Separation of scales. The second step would consist of choosing a specific *scale* – in space and time – at which to observe the system. Indeed, even though physical laws are meant to be universal, at different scales the system is studied by means of different sets of relevant observables, which, in turn, undergo different physical descriptions. Hence, depending on the phenomena that the physicist may want to catch, one scale may turn more suitable than another.

A classical separation is between *macroscopic* and *microscopic* scales.

On a macroscopic scale, the state of the system is described by a few continuous state variables, e.g. density, temperature, pressure etc., all following a deterministic dynamics encoded in a set of partial differential equations. For instance, the sea flow patterns are governed by hydrodynamic equations, such as Euler, reaction-diffusion, Navier-Stokes, etc., equations. These differential systems are versatile enough to depict some of the experimentally observed phenomena of macroscopic physical systems. Typical examples may be the formation and propagation of shock waves, the existence of metastable states and their transition to stationarity.

On a microscopic scale, things look quite different. Indeed, by increasing the degree of spatial resolution, one would see at some point water molecules moving extremely fast and colliding against each other in a random like fashion. There – up to ignoring quantum corrections [66] – the Hamiltonian discrete world consists of a myriad of, for instance, molecules, atoms, particles, etc., undergoing Newton's equations of motions.

However, as long as these two pictures describe the same physical system, it seems reasonable that they should "overlap", if properly rescaled.

To establish this connection is the program, commonly ascribed to the works of Ludwig E. Boltzmann by the end of the 19th century (for an exhaustive historical reconstruction of this scientific trajectory, see e.g. [23]), of statistical mechanics.

Statistical mechanics. Instead of opposing two views of reality – one regarded as a continuum whilst the other as made of discrete particles – statistical mechanics aims at deriving the macroscopic laws of thermodynamics in terms of the chaotic dynamics of its microscopic components. If successful, this procedure may elucidate on how macroscopic phenomena emerge as result of the concurrence of many microscopic effects as well as explain, for instance, the origin of irreversible macroscopic laws (e.g. the law of increasing entropy, i.e. the Second Law of thermodynamics) in the reversible laws of microscopic physics [66], [133].

Therefore, rather than guided by the study of specific phenomena, statistical mechanics may be regarded as a route linking two theories: thermodynamics for the macroscopic and kinetic theory for the microscopic picture of the same system.

Since the heuristics of Boltzmann, later attempts to legitimate this program were based mostly on physical rather than mathematical grounds, leaving unsolved the problems of controlling the validity of the approximations made and quantify the convergence of the schemes used [32]. This is the point where mathematicians embarked on the task of developing mathematical structures for statistical mechanics.

Equilibrium. In *equilibrium* mathematical statistical physics – when the stationary distributions are given by the so-called Boltzmann weights "$\exp(-\beta H)$", where H, the Hamiltonian, is the energy of a microstate – the rigorous scaling procedure which connects the micro to the macro description is known as *thermodynamic limit*. Gibbs measures are the basic tools within the so-called thermodynamic formalism [122], developed by Dobrushin, Lanford and Ruelle in the '60s of the last century. Within this framework, many macroscopic equilibrium phenomena, such as phase transition and symmetry breaking, may be derived as an outcome of the combination of Hamiltonian descriptions at microscopic scales and thermodynamic limits.

Non-equilibrium. As opposed to equilibrium, *non-equilibrium* mathematical statistical physics deals with systems for which a Gibbsian description of stationary distributions is not possible. Systems undergoing a net current of mass or energy, current which may be induced, for instance, by a coupling of the system with external baths at different temperatures or by the action of an external field, are typical examples of non-equilibrium systems. A wide range of complex phenomenologies, such as turbulence, dissipation, shocks, uphill diffusions, etc., arise from non-equilibrium systems, which, in turn, may go through transient as well as stationary non-equilibrium states.

At the current stage, whilst for thermostatics there is a well-established formalism, for non-equilibrium thermodynamics all attempted formulations so far all look much alike, but none of them has yet received a universal recognition – and, actually, the existence itself of a “unifying theory” of non-equilibrium is subject of debates.

For instance, a general framework to characterize stationary states – a non-equilibrium counterpart of the thermodynamic formalism at equilibrium – is missing in the context of non-equilibrium. Likewise, a crucial concept such as that of entropy, which in equilibrium counts at least five different equivalent formulations (Clausius’s entropy as the variation of heat over temperature; Boltzmann’s entropy as a combinatorial entropy; Onsager’s entropy as a statistical force; entropy as a Lyapunov function in the context of gradient flows; Kubo’s entropy in response theory), in non-equilibrium lacks of a solid interpretation and all these equivalent formulations in the context of equilibrium split up when lifting up to non-equilibrium (see e.g. [101], [102] and references therein for further details).

For these reasons, for non-equilibrium systems the derivation of macroscopic laws from a microscopic description is much less understood than in the context of equilibrium. Nonetheless, the production of a large inventory of examples for which the transition micro-to-macro may be rigorously established fits in the broader program of defining a definitive formalism of non-equilibrium (see e.g. [58], [103] as recent works on some aspects of non-equilibrium thermodynamics).

Ergodicity and mixing. In and out of equilibrium, partial differential equations governing macroscopic state variables should arise from the study of suitable averages of the microscopic activity. For general realistic systems, on one side, to recover autonomous equations from this averaging procedure is, in general, an infeasible task – one should integrate approximately 10^{23} “wildly coupled” (and non-linear) differential equations. On the other hand, the chaotic – more precisely, ergodic and mixing – behavior, though deterministic, of microscopic systems, is commonly believed to be a key ingredient in the foundations of statistical mechanics (see e.g. [32], [66], [132]) or, at least, if not strictly necessary (see e.g. [15]), is expected to be of help in this direction.

However, the rigorous derivation of even one among the most classical equations of thermodynamics – the heat equation – from a deterministic Hamiltonian many-particle system is beyond the reach of the present techniques. Likewise, the understanding of deterministic chaos in classical Hamiltonian physical systems with a large number of degrees of freedom and its role in the derivation of macroscopic autonomous equations remains mostly an unsolved (and fascinating) problem [100] (for instance, see [5] for an overview of old and new results on the notorious Fermi-Pasta-Ulam problem).

Stochastic interacting particle systems. One among other pragmatic modeling strategies to overcome this steep obstacle – the ergodicity assumption – has consisted, since the '70s of the 20th century (see e.g. [32] for one of the first reviews on this subject), of rendering the interaction among the microscopic components itself stochastic. In particular, if the whole configuration is required to evolve in a Markovian way, i.e. the stochastic law that governs the evolution depends solely on the present configuration, systems of this sort are known under the name of *(stochastic) interacting particle systems* (IPS) (see e.g. [98]).

The assignment of Markovian stochastic rules in place of a deterministic mechanical modeling of the microscopic reality, indeed, diverts towards a more idealized description of real systems. However, these models are constructed by following a sense of physical reality. For instance, conservation laws, such as the local conservation of mass, may be imposed, as well as the choice of the interactions among particles (e.g. repulsive, attractive, etc.) so to maintain the canonical Gibbs measures as equilibrium measures of the stochastic system. Furthermore, the time reversal invariance of Hamiltonian equations may be mimicked by reversible stochastic IPS satisfying the so-called detailed balance condition.

Initially developed in the context of probability theory by Spitzer [131] and Dobrushin [37] in the late '60s, IPS offer a wide assortment of modeling options in mathematical statistical physics. On one side, the simplification in the derivation of macroscopic equations arises because randomness at a microscopic scale introduces an intrinsic mechanism of relaxation and mixing in the system. On the other side, despite this simplification, the behavior of the rescaled systems grasp some of the phenomenologies present in real systems.

Moreover, this stochastic counterpart of Hamiltonian systems grafts a fruitful exchange of concepts and ideas from and to a more abstract framework of dynamical systems, as it has happened with statistical mechanics and ergodic theory in the 20s century.

Hydrodynamic limit. The scaling procedure to relate the dynamics of IPS at a microscale with the evolution of macroscopic quantities can be defined in mathematically precise terms and is called *hydrodynamic limit* [31], [32], [85], [132].

Once identified the macroscopic quantity of interest, e.g. the density of mass, and the microscopic description of the system in terms of IPS, one constructs a suitable empirical average of convenient microscopic quantities, e.g. an empirical average of the number of particles. The crucial idea lying behind this scale separation is the introduction of a scaling parameter, say $N \in \mathbb{N}$, to be sent to infinity. Here, N adopts the interpretation of a coarse-graining parameter: the smaller N is, the finer is the precision of the measurement tools we are employing to observe the system and the microscopic motion is registered, approximately, at its own internal time scale. As N grows, the measurement precision of the system gets coarser and the microscopic time moves enormously faster.

The task, then, is that of studying the convergence of the trajectory of the rescaled empirical averages defined in terms of details of the microscopic IPS to a macroscopic quantity. As one is dealing with stochastic processes, the convergence result will have to be stated in probabilistic terms, and the program will be successful if the microscopic contributions average out yielding closed equations, referred to as hydrodynamic equation, for the limit macroscopic quantity.

Local equilibrium. A system at a macroscale is a continuum described by assigning at each element values of macroscopic state variables, e.g. density, temperature, entropy etc. If the macroscopic system evolves according to a hydrodynamic equation, then these values change accordingly. In a microscopic description, the macroscopic element corresponds to an ensemble of microscopic units, while these element's values correspond to statistical local averages over the states of the ensemble.

Hydrodynamics looks at the correct space-time scaling at which, first, particles, due to local conservation laws, locally approach a state close to equilibrium, i.e. the invariant measure at a certain density, temperature, entropy, etc.

This condition is referred to in literature as *local equilibrium*. Then, particles evolve in such a way to *propagate* this local equilibrium – parametrized by the macroscopic observables' values – according to the hydrodynamic equation (for more details and precise mathematical definitions, see e.g. [31], [85]).

Donsker's invariance principle and "invariance principles". The good stochastic ergodic and mixing properties of IPS are those that ensure propagation of local equilibrium, feature which turns hard to verify for microscopic realistic models [32], [132]. This point should already justify the introduction of stochastic models – usually stochastically time-reversible – to derive deterministic – irreversible – macroscopic equations.

A further reason may be found in the robustness of averaging with respect to randomness. *Donsker's invariance principle* [9] is probably the most exemplary result of this fact.

Indeed, to recover the most universal stochastic process of all – the Brownian motion $\{B_t,\ t \geq 0\}$ – it suffices to suitably rescale an average of independent and identically distributed random contributions $\{X_n,\ n \in \mathbb{N}\}$ of zero mean and unit variance, *regardless of the precise form of their probability law*. The flexibility in the modeling choices of these contributions awarded this beautiful theorem with the name of "invariance principle".

This is, of course, not the only result in probability theory that shows this general feature of invariance when averaging with respect to randomness. The law of large numbers and the central limit theorem are two other key results that ground themselves in what may be referred to as *invariance principles*, i.e. the property that the same phenomenon may be the effect of underlying very diverse random activities.

Within this realm, the proof of a "new" Donsker's invariance principle, i.e. the convergence in law over finite time intervals of the trajectories of suitably rescaled random walks to Brownian motion trajectories, corresponds, roughly speaking, to the addition of a new "item" in the Brownian motion class – being the random walks constructed out of the i.i.d. contributions $\{X_n,\ n \in \mathbb{N}\}$ as described above only one among others.

By means of hydrodynamic limits, invariance principles for the solution of the heat equation, whose fundamental solutions may be represented in terms of Brownian motion probability densities, is another traditional domain of investigation. There, the question of convergence of rescaled random walks to Brownian motion is replaced by that of convergence of rescaled empirical density fields of interacting particle systems to macroscopic profiles solving the heat equation.

Random environments. The unveiling of all random objects which behave alike in a certain limit, is, essentially, one of the ultimate goals of probability theory and statistical physics. In particular, when studying hydrodynamics of IPS, this question translates into the inquiry of those key features of IPS which guarantee a prescribed limiting behavior.

The introduction of random impurities in microscopic systems, besides adding some sense of physical reality to the modeling, fulfills this desire of robustness in the mathematical modeling.

An extensive field of research has developed since the seminal works on homogenization theory in the '70s [6], [89], [111], which later prompted probabilistic investigations around *random walks in random environment* (RWRE) [127], [129], [130], invariance principles and related questions around them. A natural sequel has been the study of IPS in random environments [48], [57], [68] and, for the first time, in the first chapter of this thesis we study hydrodynamics of a classical IPS – the simple symmetric exclusion process – in a *dynamic* random environment (see also [116]).

Recent developments in non-equilibrium hydrodynamics. We have already mentioned above some reviews about the early accomplishments of hydrodynamics for IPS. Nowadays, even though a general theory of hydrodynamics is still lacking, the field is extremely active and has advanced considerably in the understanding of non-equilibrium phenomena via the study of non-equilibrium IPS, i.e. particle systems either in contact with infinite-capacity "conflicting" reservoirs or subject to space-time asymmetries in their dynamics.

Within the remarkable recent developments, we mention the so-called Macroscopic Fluctuation Theory (MFT) [7] as probably the only general theory so far for the study of large deviations around the hydrodynamic limit of open non-equilibrium IPS. Another milestone in the theory has proved to be the so-called relative entropy method introduced by Yau (see e.g. [85, Chapter 6]), recently adopted in [80] in the analysis of non-equilibrium fluctuations around the hydrodynamic limit of weakly asymmetric IPS for which no *a priori* knowledge of the stationary measures is required. Regarding non-equilibrium fluctuations of asymmetric IPS and convergence to solutions of the Kardar-Parisi-Zhang (KPZ) equation [82] (see also [71]), the seminal work of Bertini and Giacomin [8] opened an extensive field of research, see e.g. [29],

[67], [115] and references therein.

We conclude by mentioning the interplay between IPS and econophysics [24], which, in turn, is closely related to heat conduction and mass transport problems in statistical physics [64], [109]. The study of wealth distribution models, in which agents are modeled as spatial variables – and, typically, as nodes of random graphs – and money as particles/energy moving from agent to agent, has proved to be a challenging frontier of application and inspiration.

Duality and interacting particle systems. IPS are not interesting only when observed at a macroscopic scale. Indeed, the detailed study of exact properties of IPS acquired its own physical and mathematical interest.

For instance, in the understanding of microscopic structures in stationary non-equilibrium systems, the matrix ansatz method [34] (see also [99]) for the simple symmetric exclusion process in contact with reservoirs at different densities has been one of the most remarkable. In this direction, for analogous microscopic models, duality (see e.g. [98]) plays nowadays a significant role, see e.g. [20], [21], [22], [86].

This has lead to a whole line of investigation around duality and characterization of duality for IPS, see e.g. [62]. In particular, the third and forth chapters of this thesis go in this direction: while the third chapter focuses on the characterization problem for symmetric IPS (see also the second part of this introduction or [117]), the forth chapter deals with duality from a spectral point of view, providing, among other results, a characterization of finite state space Siegmund duality (see also [119]).

In the last chapter of this thesis, we present an application to econophysics, generalizing a wealth distribution model previously studied in [73], [83] based on a combination of splitting and exchange of wealth among agents. There, guided by the algebraic structure of infinitesimal Markov generators linked to well-known IPS (see e.g. [19]), we obtain full information about self-duality for these wealth distribution models.

1.2 Self-duality for symmetric interacting particle systems

Duality for Markov processes is the *ritournelle* throughout the chapters of this thesis. In particular, we focus on the related notion of *self-duality* and its connotation in the context of *conservative particle systems*. Within this framework, we derive hydrodynamic limits in dynamic random environment as an application of self-duality and discern which systems are – and are not – self-dual in this sense.

We start by introducing the notion of duality for two generic Markov processes $\{\xi_t,\ t \geq 0\}$ and $\{\eta_t,\ t \geq 0\}$ with corresponding state spaces $\widehat{\mathcal{X}}$ and $\mathcal{X}$ and infinitesimal generators $\widehat{L}$ and L. To avoid technicalities in this exposition, e.g. to specify function spaces, crucial subtleties about domains of generators, etc., we restrict now to the case of finite state spaces.

Duality with respect to a function. We say that the two processes are *dual* with *duality function* $D : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ defined on the product of the two state spaces if the action of $\widehat{L}$ on D w.r.t. the left variables equals the action of L on D w.r.t. the right variables, i.e.

$$\widehat{L}_{\text{left}} D(\xi,\eta) \ := \ \widehat{L} D(\cdot,\eta)(\xi) \ = \ L D(\xi,\cdot)(\eta) \ =: \ L_{\text{right}} D(\xi,\eta)\,, \qquad (1.1)$$

for all $\xi \in \widehat{\mathcal{X}}$ and $\eta \in \mathcal{X}$, where the subscript "left", resp. "right", refers to the action of an operator on the left, resp. right, variables.

In words, by integrating over time the relation in (1.1), duality w.r.t. the function D (to which we refer as simply *duality* when no emphasis on the specific form of the duality function D is strictly required) means that the expected outcome of the observable D, which depends on the state of both processes, is the same whether either we evolve one process while the other stays still, or viceversa. More precisely,

$$\widehat{\mathbb{E}}_{\xi}\left[D(\xi_t,\eta)\right] \ = \ \mathbb{E}_{\eta}\left[D(\xi,\eta_t)\right]$$

for all $\xi \in \widehat{\mathcal{X}}, \eta \in \mathcal{X}$ and $t \geq 0$. This connection may come as a consequence of a coupling of the two processes, but, in general, duality is a notion only concerning the distribution of the two processes w.r.t. special observables of the joint system. The "non-triviality" of these special observables measures the relevance of duality. For instance, the constant function is always a duality function for any pair of Markov processes, but carries no information about the underlying stochastic dynamics.

An evident advantage of the duality property is when one of the two processes is considerably simpler to study. A typical situation occurs when the process $\{\eta_t,\ t \geq 0\}$ is a system with many particles, while $\{\xi_t,\ t \geq 0\}$ consists of only a few particles. Another instance of duality occurs when information about stochastic models for genetic evolutions, such as Wright-Fisher and Moran-type of models, may be related to dual genealogical processes, as e.g. the so-called Kingman coalescent processes (see e.g. [38]).

Self-duality, I: self-duality w.r.t. a function. As a special case, we give an *ad hoc* name to duality when the two processes under consideration are equal in distribution, namely when $\widehat{L} = L$. In this situation, we speak about *self-duality w.r.t. a function*:

A Markov process $\{\eta_t,\ t \geq 0\}$ *is self-dual with self-duality function* $D : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *if* (1.1) *holds for all* $\xi, \eta \in \mathcal{X}$ *with* $\widehat{L} = L$.

This definition of self-duality holds for any Markov process and, in particular, for the following class of interacting particle systems (IPS), which we introduce below and which will be the main object of our study.

Conservative factorized symmetric IPS**.** The particles hop on a *discrete space*, say $(V, \sim)$, consisting of finitely many sites, which we typically denote by $x, y, z \in V$, and for which there exists a nearest-neighboring relation "$\sim$". Additionally, we assign to all unordered pairs, say $\{x, y\}$, of nearest-neighboring sites a positive weight $c(\{x, y\})$.

Next, we associate to each site a so-called *single-site state space*, say F. In this introductory exposition, for the sake of notational convenience, we stick to either $F = \{0, 1\}$ or $F = \mathbb{N}_0 = \{0, 1, \ldots\}$, but more general choices – such as $F = \{0, 1, \ldots, \alpha\} \subset \mathbb{N}_0$ and site-dependent spaces $\{F_x,\ x \in V\}$ – will be considered in e.g. Chapter 3. In all these cases, the integer number $\eta(x)$ has the interpretation of number of particles sitting at site $x \in V$ in the particle configuration $\eta \in F^V$. We call F^V the *configuration space* and $\eta(x)$ the *occupation variable at* $x \in V$.

At last, we describe the evolution of the particle system by specifying its infinitesimal generator L acting on functions $\varphi : F^V \to \mathbb{R}$ as follows:

$$L\varphi(\eta) = \sum_{x \sim y} c(\{x, y\})\, L_{\{x,y\}} \varphi(\eta)\,, \tag{1.2}$$

where the sum above runs over all unordered pairs of nearest-neighboring sites,

$L_{\{x,y\}}\varphi$ is given by

$$L_{\{x,y\}}\varphi(\eta) = g(\eta(x))\, h(\eta(y))\, (\varphi(\eta^{x,y}) - \varphi(\eta)) + g(\eta(y))\, h(\eta(x))\, (\varphi(\eta^{y,x}) - \varphi(\eta))\,, \quad \eta \in F^V\,, \quad (1.3)$$

and $\eta^{x,y}$ denotes the configuration obtained from η by removing a particle at x and placing it at $y \in V$ – provided that there is a particle at x. The rates of these particles' jumps are determined by the interaction functions $g, h : F \to \mathbb{R}_+$ which satisfy the following basic assumptions:

(i) $g(0) = 0$ and, for all $n \in F \setminus \{0\}$, $g(n) > 0$.

(ii) $h(0) > 0$ and, if $F = \{0, 1\}$, $h(1) = 0$.

We may refer to this class of interacting particle systems as *conservative*, *factorized* and *symmetric* IPS. Indeed, the dynamics preserves the total number of particles, the jump rates of each particle depend only on the number of particles in the departure and arrival sites in a factorized way and, in words, the interaction involving two neighboring sites is symmetric w.r.t. site interchange. More formally, besides $c(\{x,y\}) = c(\{y,x\})$ by definition, we have $\mathcal{L}_{\{x,y\}}\Psi_{\{x,y\}} = \Psi_{\{x,y\}}\mathcal{L}_{\{x,y\}}$, where

$$\Psi_{\{x,y\}}\varphi(\eta) = \varphi(\eta^{\{x,y\}}) \quad \text{and} \quad \eta^{\{x,y\}}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y \\ \eta(y) & \text{if } z = x \\ \eta(x) & \text{if } z = y\,. \end{cases} \quad (1.4)$$

We remark that, although this class of IPS are far from exhausting the list of conservative IPS being stationary w.r.t. product measures (see e.g. [90] for a more general discussion on this), from detailed balance computations it follows that these IPS admit a whole one-parameter family of stationary (actually reversible) homogeneous product measures $\{\otimes_{x\in V}\, \nu_\lambda,\ \lambda \in \Lambda \subset (0,\infty)\}$ with marginals given by

$$\nu_\lambda(n) = \frac{\lambda^n}{Z_\lambda} \prod_{m=1}^{n} \frac{h(m-1)}{g(m)}\,,$$

for all $n \in F$ (see Chapter 3 for more details).

Some notorious examples of interacting particle systems fall into this class.

For the following choice of interaction functions[2]

$$g(n) = n \quad \text{and} \quad h(n) = 1 + \sigma n\,, \quad n \in F\,,$$

one obtains:

(a) The *symmetric exclusion process* (SEP) for $\sigma = -1$ and $F = \{0, 1\}$.

(b) A system of *independent random walkers* (IRW) for $\sigma = 0$ and $F = \mathbb{N}_0$,

(c) The *symmetric inclusion process* (SIP) for $\sigma = 1$ and $F = \mathbb{N}_0$.

Given the interaction functions

$$g(n) \;\geq\; 0 \quad \text{and} \quad h(n) \;=\; 1\,, \quad n \in F\,,$$

one recovers, for the choice $F = \mathbb{N}_0$,

(d) The *symmetric zero-range process* with interaction function g.

A first application of duality: hydrodynamics. The first three examples of IPS presented above, namely SEP, IRW and SIP, posses a form of self-duality which we now derive and which proves to be immediately useful when deriving hydrodynamic limits.

In this setting, we start from a scaling parameter $N \in \mathbb{N}$ – to be sent to infinity – and a suitable sequence of (finite) discrete spaces $\{V_N,\ N \in \mathbb{N}\}$ which suitably "approximate" a macroscopic space $\mathcal{M}$ ($\frac{V_N}{N} \subset \mathcal{M}$ and "$\frac{V_N}{N} \to \mathcal{M}$ as $N \to \infty$", where typically either $\mathcal{M} = \mathbb{T}^d$ or $\mathbb{R}^d$). Moreover, we consider the sequence of empirical density fields $\{\mathsf{X}^N_\cdot,\ N \in \mathbb{N}\}$ associated to either one of these three particle systems on $\{V_N,\ N \in \mathbb{N}\}$:

$$\mathsf{X}^N_t \;=\; \frac{1}{|V_N|} \sum_{x \in V_N} \delta_{\frac{x}{N}}\, \eta_{t\vartheta_N}(x)\,, \quad t \geq 0\,,$$

where $\delta_{\frac{x}{N}}$ is the Dirac distribution on $\mathcal{M}$ concentrated in $\frac{x}{N} \in \frac{V_N}{N}$, $|V_N|$ is the cardinality of V_N and ϑ_N is a suitable time-scaling factor.

The study of the time evolution of these empirical averages requires an analysis of the evolution of the corresponding occupation variables. Hence, by

[2] As we will see later in this section and in Chapter 3, the value 1 in the expression of h does not play any crucial role for what we present and may be replaced by a generic $\alpha \in \mathbb{N}$ or $\mathbb{R}_+$, but we choose it here for agility in this exposition.

applying Dynkin's formula to the function $\varphi : F^{V_N} \to \mathbb{R}$ given by $\varphi(\eta) = \eta(x)$ for all $x \in V_N$, we obtain

$$d\eta_t(x) \;=\; L\eta_{t^-}(x)\,dt \;+\; dM_t(x)\,,$$

where $\{M_t(x),\ t \geq 0\}$ is a martingale for all $x \in V_N$. The action of the generator L on the observable $\eta(x)$ reads as follows:

$$L\eta(x) \;=\; \sum_{y:y\sim x} c(\{x,y\})\,L_{\{x,y\}}\eta(x)\,, \tag{1.5}$$

where, for $\sigma \in \{-1,0,1\}$,

$$L_{\{x,y\}}\eta(x) \;=\; \eta(x)\,(1+\sigma\eta(y))\,(-1) \;+\; \eta(y)\,(1+\sigma\eta(x))\,(+1)\,. \tag{1.6}$$

In words, $c(\{x,y\})\,\eta(x)\,(1+\sigma\eta(y))$ is the rate at which a particle leaves x and reaches y, whilst $c(\{x,y\})\,\eta(y)\,(1+\sigma\eta(x))$ is the rate at which a particle jumps from y to x. After cancellation of the terms $\sigma\eta(x)\eta(y)$ in (1.6), we read out from the expression in (1.5) the action of the generator A associated to a continuous-time symmetric (recall that $c(\{x,y\}) = c(\{y,x\})$) random walk on V_N, i.e.

$$L\eta(x) \;=\; \sum_{y:y\sim x} c(\{x,y\})\,(\eta(y)-\eta(x)) \;=\; A\eta(x)\,, \quad x \in V_N\,,$$

where A should be interpreted as acting on the function $\eta : V_N \to \mathbb{R}$ w.r.t. the x-variable. This is a first instance of duality between either one of the systems SEP, IRW or SIP ($\sigma = -1, 0$ or 1, respectively) and a symmetric random walk on the same discrete space, in which the duality function $D : V_N \times F^{V_N} \to \mathbb{R}$ is given by $D(x,\eta) = \eta(x)$:

$$A_{\mathrm{left}}D(x,\eta) \;=\; L_{\mathrm{right}}D(x,\eta)\,.$$

Then, we obtain a system of SDEs – linear in the drift part –

$$\begin{cases} d\eta_t(x) &=\; A\eta_{t^-}(x) \;+\; dM_t(x) \\ \eta_0(x) &=\; \eta(x)\,, \quad x \in V_N\,, \end{cases}$$

for which, in "column" notation,

$$\eta_t = e^{tA}\eta + \int_0^t e^{(t-s)A} dM_s\,, \quad t \geq 0\,,$$

is the unique solution, where $\{e^{tA},\ t \geq 0\}$ denotes the Feller semigroup associated to the continuous time symmetric random walk with generator A. As a consequence, the empirical density field $\{\mathsf{X}_t^N,\ t \geq 0\}$ decomposes in a first deterministic term (its mean) and a second term containing all stochasticity of the particle dynamics ("noise" around its mean).

As a consequence, up to choose the scaling factor ϑ_N such that the noise part of the empirical density fields vanishes in probability as $N \to \infty$, the program of deriving the hydrodynamic equation for either SEP, IRW or SIP in $\{V_N,\ N \in \mathbb{N}\}$ comes down to the following:

(i) *Consistency of the initial conditions.* At the starting time, the empirical density fields converge in probability to a macroscopic density profile $\rho_\bullet : \mathcal{Y} \to [0,1]$.

(ii) *Invariance principle.* All rescaled random walks with generator A and arbitrary starting positions converge in law to suitable diffusion processes on $\mathcal{M}$ with generator $\mathscr{A}$.

Here, duality proves to be a powerful tool as it boils down the study of the evolution of an interacting particle system to that of a single-particle, which moves in V_N without any interaction.

Duality and hydrodynamics: generalizations. This strategy to derive the hydrodynamic equation via duality applies to more general scenarios.

For instance, to the case of SEP, IRW and SIP in an *infinite discrete space*, e.g. $V_N = \mathbb{Z}^d$, in which the existence and non-explosion of the particle system is *a priori* not-guaranteed for all initial configurations.

A second instance is that of particle systems in *random environments*. In particular, for the particle systems with time-dependent infinitesimal generator

$$L_t\varphi(\eta) = \sum_{x\sim y} c_t(\{x,y\})\, L_{\{x,y\}}\varphi(\eta)\,, \tag{1.7}$$

with

$$L_{\{x,y\}}\varphi(\eta) = \eta(x)\,(\alpha_y + \sigma\eta(y))\,(\varphi(\eta^{x,y}) - \varphi(\eta)) + \eta(y)\,(\alpha_x + \sigma\eta(x))\,(\varphi(\eta^{y,x}) - \varphi(\eta))\,,$$

where

(i) $\boldsymbol{c} = \{c_t(\{x,y\}),\ t \geq 0,\ x \sim y\}$ represent dynamic (=time-dependent) *bond inhomogeneities*, also known as *conductances*.

(ii) $\boldsymbol{\alpha} = \{\alpha_x,\ x \in V\}$ represent static (=time-independent) *site inhomogeneities*.

Note that, in case of symmetric exclusion process (SEP, $\sigma = -1$), $\alpha_x \in \mathbb{N}$ has the interpretation of *maximal capacity* of the site $x \in V$, while, for IRW ($\sigma = 0$) and SIP ($\sigma = 1$), $\alpha_x \in (0,\infty)$ stands for *attraction parameter* of the site $x \in V$.

In presence of random environment $(\boldsymbol{c}, \boldsymbol{\alpha})$, firstly, detailed balance holds at all times w.r.t. a time-independent one-parameter family of product inhomogeneous (with $\boldsymbol{\alpha}$) measures (see Section 3.a for more details). Secondly, the duality to be established is between the particle system and the random walk evolving "backward" in the same environment and whose time-dependent infinitesimal generator is given by

$$A_t f(x) = \sum_{y:y\sim x} c_t(\{x,y\})\,\alpha_y\,(f(y) - f(x))\,,$$

with duality function $D(x,\eta) = \frac{\eta(x)}{\alpha_x}$ (see Section 3.a).

Part of this program is presented in full details in Chapter 2 of this thesis for the *symmetric simple exclusion process* (SSEP) in $\mathbb{Z}^d$ (which corresponds to SEP in $\mathbb{Z}^d$ with $\alpha_x \equiv 1$ and nearest-neighbor interactions) in presence of *dynamic* and *uniformly bounded conductances*. There, we prove existence and non-explosion of the process for all initial conditions via a graphical construction and derive the hydrodynamic equation from the invariance principle of the dual random walk – a random walk in dynamic random environment.

Jointly factorized duality. A simple computation showed that particle systems SEP, IRW and SIP are dual to a random walk – a one-particle system – with duality function $D(x,\eta) = \eta(x)$. With a bit more effort, one shows that

SEP, IRW and SIP are dual to a system of two indistinguishable particles with duality function

$$D(\{x,y\},\eta) \;=\; \begin{cases}\eta(x)\,\eta(y) & \text{if } x \neq y\\ \eta(x)\,(\eta(x)-1) & \text{if } x = y\,,\end{cases}$$

where the two particles evolve according to the interaction rules of SEP, IRW and SIP, respectively.

This fact continues to hold also when considering dual systems of three, four, $\ldots$, n particles evolving with the same interaction rules and, moreover, always with duality functions in the following factorized form:

$$D(\xi,\eta) \;=\; \prod_{x\in V} d(\xi(x),\eta(x))\,, \tag{1.8}$$

where ξ, η are configurations of particle systems of either SEP, IRW or SIP type. We give duality functions $D : F^V \times F^V \to \mathbb{R}$ in the form above the name of *jointly factorized duality functions* and to the function $d : F \times F \to \mathbb{R}$ the name of *single-site duality function*.

The explicit expression of some of these duality functions may be found in Section 3.1.4 of Chapter 3. Although, in general, the function $d = d(k,n)$ depends on the parameter $\sigma \in \{-1, 0, 1\}$, the single-site duality function $d = d(k,n)$ obtained is such that

$$d(1,n) \text{ is not a constant function of } n \in F \tag{1.9}$$

and

$$d(0,n) \;=\; 1 \text{ for all } n \in F\,. \tag{1.10}$$

Without the latter requirement, the duality function $D = \prod_{x\in V} d$ would degenerate in case of infinite systems in duality with finite systems (see Section 3.1.2 in Chapter 3 for further details on this condition). We refer to jointly factorized duality functions for which both conditions (1.9)–(1.10) hold as being "non-trivial" (see Sections 3.1.4–3.1.5, as well as Section 3.2 for another notion of "non-triviality" which we call *measure determining*).

Self-duality, II: jointly factorized self-duality for conservative IPS. Besides the explicit expression of the duality functions, we just showed that SEP, IRW and SIP are dual w.r.t. a non-trivial jointly factorized duality function to systems of n particles interacting with the same rules. This is the definition of *jointly factorized self-duality* for conservative IPS which we adopt:

> *A conservative interacting particle system is jointly factorized self-dual if systems of n and m particles are dual w.r.t. a non-trivial jointly factorized duality function for any choice of $n, m \in \mathbb{N}$, even $n \neq m$.*

Chapter 3 is devoted to the problem of determining which conservative factorized symmetric particle systems are self-dual in the above sense and finding systematically duality functions in a jointly factorized form.

A first partial answer to the characterization of self-dual (II) conservative IPS is the content of Theorem 3.3 in Chapter 3. There, we prove that, within the class of conservative factorized symmetric IPS, the only systems which are jointly factorized self-dual (see (1.8), (1.9) and (1.10)) must be one among SEP, IRW and SIP (and their generalizations as in (1.7), see also Theorem 3.27).

Theorem 3.3. *Let $\{\eta_t,\ t \geq 0\}$ be a conservative particle system with infinitesimal generator L as given in* (1.2)–(1.3). *Suppose that it is self-dual with a jointly factorized duality function*

$$D(\xi, \eta) \;=\; \prod_{x \in V} d(\xi(x), \eta(x))$$

as in (1.8) *and the single-site duality function $d = d(k, n)$ satisfies conditions* (1.9) *and* (1.10). *Then, depending whether $\sigma \in \{-1\}$ or $\{0, 1\}$, there exists a value $\alpha \in \mathbb{N}$, resp.* $(0, \infty)$, *such that*

$$\begin{aligned} g(n) \;&=\; n \\ h(n) \;&=\; \alpha + \sigma n\,, \end{aligned}$$

for all $n \in \{0, 1, \ldots, \alpha\}$, resp. $\mathbb{N}_0$.

As a consequence, the only self-dual symmetric zero-range process admitting this type of duality functions must be a system of independent random walkers.

The rest of Chapter 3 is dedicated to the characterization of all jointly factorized (self-)duality functions (among them the so-called orthogonal polynomial duality functions, previously obtained by explicit computations in [55]) for SEP, IRW and SIP. Main ingredients are a special relation of jointly factorized duality functions with stationary product measures (Section 3.2) and so-called *intertwining relations* between particle systems and their associated (possibly degenerate or improper) diffusion counterparts (Section 3.4).

Symmetries and intertwinings. Out of the framework of non-trivial jointly factorized dualities, yet the question whether other conservative particle systems than SEP, IRW and SIP are self-dual w.r.t. other duality functions remains unanswered.

In the problem of finding and characterizing duality for a Markov process, key ingredients are *symmetries* of the generator, i.e. operators S that commute with the generator:

$$S\,L \;=\; L\,S\,.$$

Various self-duality relations follow whenever symmetries S and self-duality functions D for L are available. Indeed,

$$L_{\text{left}} S_{\text{left}} D \;=\; S_{\text{left}} L_{\text{left}} D \;=\; S_{\text{left}} L_{\text{right}} D \;=\; L_{\text{right}} S_{\text{left}} D\,,$$

from which it follows that also $\widetilde{D} = S_{\text{left}} D$ is a self-duality function for L.

In the context of conservative particle systems, self-dualities for SEP, IRW and SIP have been thoroughly studied by means of Lie algebraic techniques in e.g. [19], [62], developing on earlier results in the pioneering work [126], where isomorphisms between stochastic particle systems and integrable quantum chains in one dimension shed light on self-dualities for symmetric as well as asymmetric systems (see also [123] and [135]). The advantage of these techniques is that, by viewing the infinitesimal generators of those particle systems as discrete representations of "special" (=central) elements of appropriate co-product Lie algebras, a full assortment of symmetries for L becomes at once accessible. If, in addition, a reversible measure μ for the particle system is known – and, for the systems there considered, this is the case – all these symmetries acting on the so-called "cheap" self-duality function $D_{\text{cheap}} = \text{diag}(\frac{1}{\mu})$ yield further self-duality relations (see e.g. [19], [21], [22] for an overview of the method).

Although we need not employ this Lie algebraic point of view until the end of Chapter 5, we, indeed, consider in various spots of the thesis generalizations of symmetries – referred to as *intertwiners* – as effective tools to produce duality relations. Roughly speaking, intertwiners of two generators L and $\widehat{L}$ are operators Λ for which the following intertwining relation holds:

$$\widehat{L}\,\Lambda \;=\; \Lambda\, L\,,$$

yielding, as a consequence, that the intertwiner is a symmetry for L as soon as $\widehat{L} = L$.

Intertwining relations, besides their utility in the generation of duality relations, acquire probabilistic interpretations – interesting on their own – anytime the intertwiner Λ is a stochastic operator. In fact, this will be the case in Appendix 3.b where we study "ladder" variants of symmetric exclusion processes, as well as in Section 3.4 of Chapter 3 and in Section 4.5 of Chapter 4 in which we establish intertwining relations between particle systems and their continuum counterparts and between particle systems with different number of particles, respectively.

We refer to Section 3.4 and Theorem 3.19 (see also Theorem 4.15) for a more extensive account on this notion and its relation to duality.

Duality and eigenfunctions. Abstracting the quest of duality relations from specific instances such as the explicit jointly factorized form of the duality functions as well as the knowledge of intertwiners, in Chapter 4, based on linear algebraic considerations, we adopt a point of view which turns to be rather powerful for the problem of existence and characterization of dualities for finite state space Markov processes and, in particular, for two specific situations: the study, on the one side, of Siegmund duality for monotone processes on finite totally ordered spaces (Section 4.4) and, on the other side, self-duality for conservative particle systems on finite spaces (Section 4.5). We present the main ideas of this approach by starting from the following observations.

Given two infinitesimal generators $\widehat{L}$ and L, if $\lambda \in \mathbb{C}$ is a *common* eigenvalue for the two generators with associated eigenfunctions $\widehat{\psi}$ and ψ, then $D(\xi,\eta) = \widehat{\psi}(\xi)\,\psi(\eta)$ is a duality function. Indeed,

$$\widehat{L}_{\text{left}}\,\widehat{\psi}(\xi)\,\psi(\eta) \;=\; \lambda\,\widehat{\psi}(\xi)\,\psi(\eta) \;=\; L_{\text{right}}\,\widehat{\psi}(\xi)\,\psi(\eta)\,.$$

Furthermore, if $\{\lambda_i\}$ is a collection of common eigenvalues with associated eigenfunctions $\{\widehat{\psi}_i\}$ and $\{\psi_i\}$, then

$$D(\xi,\eta) \;=\; \sum_k a_i\,\widehat{\psi}_i(\xi)\,\psi_i(\eta)$$

is a duality function for all $\{a_i\} \subset \mathbb{R}$.

Pushed by analogous considerations, all duality functions between finite state space generators can be expressed in terms of linear combinations of products of eigenfunctions associated to common eigenvalues (if the generators – viewed as matrices – are diagonalizable, then this statement is correct; if not, it becomes correct up to replace eigenfunctions with generalized eigenfunctions, see Theorem 4.10 in Section 4.3 for further details). As a consequence, for a pair of diagonalizable generators $\widehat{L}$ and L, the larger the number of eigenvalues they share is, the "richer" the family of duality functions between them is. Within this realm, $\widehat{L}$ and L must be in "maximal" duality whenever the number of eigenvalues (with multiplicities) in common is maximal, i.e. when the generators $\widehat{L}$ and L are *spectrally consistent*:

$$\Sigma(\widehat{L}) \subset \Sigma(L) \quad \text{or} \quad \Sigma(L) \subset \Sigma(\widehat{L})\,,$$

where $\Sigma(L)$ denotes the spectrum (with multiplicities) of L.

These considerations on maximal – in the sense of spectrum of generators – duality lead us to a third notion of self-duality for conservative IPS.

Self-duality, III: spectral self-duality for conservative IPS. For conservative interacting particle systems, "rich" dualities between systems with different number of particles may be expressed in terms of the notion of "spectral consistency" duality introduced above.

Given a conservative particle system in F^V with infinitesimal generator L, we denote by L_n the generator of the system confined to the invariant subset of configurations with $n \in \mathbb{N}$ particles. With this notation, we introduce below the notion of *spectral self-duality* for conservative IPS:

> *A conservative interacting particle system is spectrally self-dual if generators associated to systems of n and m particles are spectrally consistent, i.e. any of*
>
> $$\Sigma(L_n) \subset \Sigma(L_m) \quad \textit{and} \quad \Sigma(L_m) \subset \Sigma(L_n)$$
>
> *holds, for any choice of $n, m \in \mathbb{N}$, even $n \neq m$.*

We have seen above (and will formally prove in Chapter 3) that SEP, IRW and SIP are jointly factorized self-dual conservative IPS. In Theorem 4.23 in Chapter 4, we show that, actually, SEP, IRW and SIP are also spectrally self-dual. In particular, we obtain that jointly factorized self-duality (II) implies spectral self-duality (III) within the class of conservative factorized symmetric IPS which we consider.

Examples of (non-)spectrally self-dual conservative IPS. However, spectral self-duality (III) – which does not, in general, implies jointly factorized self-duality (II) – may provide an alternative and more abstract framework to explore self-duality for more general conservative IPS. Indeed, as shown in Theorem 3.3, jointly factorized self-duality (II) cannot be expected for conservative factorized symmetric particle systems other than SEP, IRW and SIP. Nonetheless, the quest of an informative self-duality relation – necessarily in a non-jointly factorized form – may be pursued e.g. for general zero-range processes via direct inspection of the spectrum of their generators.

To this purpose, in Section 4.5 we present a small inventory of well-known conservative IPS for which we prove – or disprove, at least on extremely simplified spatial structures $(V, \sim)$ – spectral self-duality.

Self-dualities I, II, III: a recap. We started from a rather weak definition of self-duality – self-duality w.r.t. a function (I) – if the self-duality function D is not further characterized – recall that the constant function $D \equiv c$ is always a duality function between any two Markov processes. Hence, guided by the idea of studying observables of a many-particle system in terms of those of a-few-particle system, we looked for notions of self-duality relating – via duality – particle systems with different number of particles.

Depending on the form of the duality functions involved in these duality relations between n-particle and m-particle systems, we identified two notions of self-duality for conservative IPS. On the one side, we speak about jointly factorized self-duality (II) if "non-trivial" jointly factorized duality functions – which play a central role in the study e.g. of hydrodynamic limits (see also the beginning of Chapter 3 for further applications) – are employed. On the other side, when we ask that the spectra of Markov generators are, roughly speaking, "nested" one into the other – which in case of reversible IPS corresponds to the existence of "full-rank" duality functions – we speak about spectral self-duality (III).

In the general context of conservative IPS, while it is clear that the first notion (I) always comes as a consequence of notions (II) and (III), the relation between (II) and (III) is somehow a bit more subtle because the definition of "non-triviality" of the jointly factorized duality functions alone does not seem to yield "full-rank" duality functions. However, from the characterization of all jointly factorized self-dual conservative, factorized and symmetric IPS (Theorem 3.3) and Proposition 4.24 – which proves that a class of jointly factorized duality functions are indeed "full-rank"– notion (II) yield (III) in this specific context.

1.3 Outline of the thesis

Here below we detail – to some extent – the main matter of all subsequent chapters. All chapters are based on publications – to be found either in journals, conference proceedings or currently submitted – reported therein.

Chapter 2. Symmetric simple exclusion process in dynamic environment: hydrodynamics. We consider the symmetric simple exclusion process in $\mathbb{Z}^d$ with quenched bounded dynamic random conductances and prove its hydrodynamic limit in path space. The main tool is the connection, due to the self-duality of the process, between the single particle invariance principle and the macroscopic behavior of the density field. While the hydrodynamic limit at fixed macroscopic times is obtained via a generalization to the time-inhomogeneous context of the strategy introduced in [108], in order to prove tightness for the sequence of empirical density fields we develop a criterion based on the notion of uniform conditional stochastic continuity, following [137].

Based on a joint work with Frank Redig (TU Delft) and Ellen Saada (Paris V):

[116] Redig, F., Saada, E. & Sau, F. Symmetric simple exclusion process in dynamic environment: hydrodynamics. *arXiv:1811.01366* (2018). In the revision process for *Electronic Journal of Probability*.

Chapter 3. Jointly factorized duality, stationary product measures and generating functions. We find all jointly factorized self-duality functions for a class of interacting particle systems, namely those that we call "conservative factorized symmetric" IPS. The functions we recover are self-duality functions for interacting particle systems such as symmetric exclusion processes, independent random walkers and symmetric inclusion processes, as well as duality and self-duality functions for their continuous counterparts.

The approach is based on, firstly, a general relation between jointly factorized duality functions and stationary product measures and, secondly, an intertwining relation provided by generating functions. For the interacting particle systems, these self-duality and duality functions turn out to be generalizations of those previously obtained in [62] and, more recently, in [18] and [55]. Thus, we discover that only these two families of jointly factorized dualities cover all possible cases. Moreover, the same method discloses all jointly factorized self-duality functions for interacting diffusion systems such as the Brownian energy process, where both the process and its dual are in continuous variables.

We further explore in one of the appendices jointly factorized duality and self-duality functions for conservative interacting particle as well as diffusion systems in presence of a (quenched) random environment obtained as combination of dynamic conductances and static site inhomogeneities. For these systems we obtain characterization of jointly factorized self-dual particle systems and recover – on the same footing as for homogeneous systems – all (self-)duality functions in a jointly factorized form.

Based on a joint work with Frank Redig (TU Delft):

[117] Redig, F. & Sau, F. Factorized Duality, Stationary Product Measures and Generating Functions. *Journal of Statistical Physics* **172**, 980–1008 (2018).

Chapter 4. Duality and eigenfunctions. We start from the observation that, anytime two Markov generators share an eigenvalue, the function constructed from the product of the two eigenfunctions associated to this common eigenvalue is a duality function. We push further this observation and provide a full characterization of duality relations in terms of spectral decompositions of the generators for finite state space Markov processes. Moreover, we study and revisit some well-known instances of duality, such as Siegmund duality, and extract spectral information from it. Next, we use the same formalism to construct all duality functions for some solvable examples, i.e. processes for which the eigenfunctions of the generator are explicitly known.

We conclude the chapter by reconsidering the problem of finding self-duality relations for conservative particle systems. In view of this spectral characterization of duality and by means of intertwining relations between systems with different number of particles, we first prove what we call "spectral self-duality" for SEP, IRW and SIP. Then, by going through the direct computation of spectra of Markov generators, we "disprove" the same property for other simple conservative particle systems of zero-range type.

Based on a joint work with Frank Redig (TU Delft):

[119] Redig, F. & Sau, F. Stochastic Duality and Eigenfunctions. in *Stochastic Dynamics Out of Equilibrium* (eds. Giacomin, G., Olla, S., Saada, E., Spohn, H. & Stoltz, G.) 621–649 (Springer International Publishing, 2019).

Chapter 5. Generalized immediate exchange models and their symmetries. We reconsider the immediate exchange model (IEM) with its discrete counterpart (IEM_d) and define a more general class of models where mass is split, exchanged and merged. By relating the splitting process of IEM_d to the symmetric inclusion process via "thermalization", we obtain symmetries and self-dualities for the generalized discrete immediate exchange model. We show that analogous properties hold for models where the splitting is related to the symmetric exclusion process, independent random walkers or interacting diffusions.

Based on a joint work with Frank Redig (TU Delft):

[118] Redig, F. & Sau, F. Generalized immediate exchange models and their symmetries. *Stochastic Processes and their Applications* **127**, 3251–3267 (2017).

Part I

Scaling limits in dynamic random environment

Chapter 2

Symmetric simple exclusion process in dynamic environment: hydrodynamics

Dynamic random environments are natural quantities to be inserted in probabilistic models in order to make them more realistic. But studying such models is challenging, and for a long time only models endowed with a static environment were considered. However, *random walks in dynamic random environment* (RWDRE) have been extensively studied in recent years (see e.g. [1], [2], [3], [10], [12], [35], [120] and references therein) and several results on the law of large numbers, invariance principles and heat kernel estimates have been obtained. A natural next step is to consider particle systems in such dynamic environments. There the first question concerns the derivation of hydrodynamic limits. In this article, we answer this question for the nearest-neighbor symmetric simple exclusion process.

For interacting particle systems with a form of *self-duality* and that evolve in a *static disorder*, the problem of deriving the macroscopic equation governing the hydrodynamic limit has been shown to be strongly connected to the asymptotic behavior of a single random walker in the same environment. Indeed, the feature that, if a rescaled test particle converges to a Brownian motion then the interacting particle system has a hydrodynamic limit, appears already in e.g. [31], [57], [94] and [114]. Our contribution is to carry out this connection between single particle behavior and diffusive hydrodynamic limit in the context of *dynamic environment* for a nearest-neighbor particle system, namely the *symmetric simple exclusion process* (SSEP) in a *quenched dynamic* *bond disorder*, for which we show that a suitable form of self-duality remains valid. Let us now first recall the definition of SSEP, then detail the known results on SSEP evolving in a static environment.

Symmetric simple exclusion process. In words, the *symmetric simple exclusion process without disorder* in $\mathbb{Z}^d$ with $d \geq 1$ [98], [131] is an interacting particle system consisting of indistinguishable particles which are forbidden to simultaneously occupy the same site, and which jump at a constant rate only to nearest-neighbor unoccupied sites. More precisely, let $\eta \in \{0,1\}^{\mathbb{Z}^d}$ be a configuration of particles, with $\eta(x)$ denoting the number of particles at site $x \in \mathbb{Z}^d$. The stochastic process $\{\eta_t,\ t \geq 0\}$ is Markovian and evolves on the state space $\{0,1\}^{\mathbb{Z}^d}$ according to the infinitesimal generator

$$L\,\varphi(\eta) = \sum_{|x-y|=1} \Big\{ \eta(x)\,(1-\eta(y))\,(\varphi(\eta^{x,y}) - \varphi(\eta)) + \eta(y)\,(1-\eta(x))\,(\varphi(\eta^{y,x}) - \varphi(\eta)) \Big\}\,, \tag{2.1}$$

where $|x-y| = \sum_{i=1}^d |x_i - y_i|$ and $\varphi : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$ is a bounded cylinder function, i.e. it depends only on a finite number of occupation variables $\{\eta(x),\ x \in \mathbb{Z}^d\}$. In (2.1) the finite summation is taken over all unordered pairs of nearest-neighboring sites – referred to as *bonds* – and $\eta^{x,y}$ is the configuration obtained from η by removing a particle from the occupied site x and placing it at the empty site y. The hydrodynamic limit [31], [60], [85] of the particle system described by (2.1) is known [31], [85] and, roughly speaking, prescribes that the trajectories of the particle density scales to the weak solution of the heat equation.

Static environment. For SSEP in a *quenched static bond disorder* in $\mathbb{Z}^d$, hydrodynamic limits – at a fixed macroscopic time – have been obtained by means of the *self-duality property* of the particle system, that is, by solving a homogenization problem (see e.g. [43, Theorem 2.1], [45, Theorem 2.4] and, more generally, [111]) or, alternatively, establishing an invariance principle (see e.g. [42], [108]) linked to the behavior of a single particle in the same environment. As examples, see [42], [108] for $d = 1$, [43] for $d \geq 1$ and [45] on the supercritical percolation cluster with $d \geq 2$. This method has been applied also to non-diffusive space-time rescalings, for which the hydrodynamic behavior is not described by a heat equation [42], [46], [47]. Nonetheless, all hydrodynamic limits obtained via this self-duality technique hold only at the level of finite-dimensional distributions and lack of a proof of relative compactness of the empirical density fields. Indeed, a direct application of the classical Aldous-Rebolledo criterion (see e.g. [85]) fails when following this approach.

Other techniques than self-duality – which apply to different particle systems – have also been studied in static environments. For instance, in quenched static bond disorder, the method based on the so-called *corrected empirical process* has been applied to prove hydrodynamics for the SSEP [79] and for zero-range processes [44], [68]. The non-gradient method [114], [136] (see also [85]) has found many applications to reversible lattice-gas models in a more general static environment, see e.g. [48].

Dynamic environment. In presence of *dynamic environment*, to the best of our knowledge, no hydrodynamic limit for interacting particle systems has been studied, yet.

On the one side, when looking at the hydrodynamic rescaling of a particle system in a quenched dynamic disorder, a space-time homogenization problem or, alternatively, an invariance principle for the associated RWDRE must be solved. On the other side, how this homogenization problem connects to the hydrodynamic behavior of the particle system depends on the interaction rules of the particles.

For the *symmetric simple exclusion process* in a *quenched dynamic bond disorder* in $\mathbb{Z}^d$, we show that a form of *self-duality* still holds and allows us to write the occupation variables of the particle system in terms of positions of suitable *time-inhomogeneous backward random walks* evolving in the same environment. The hydrodynamic limit is, thus, obtained by studying the diffusive behavior of forward and backward random walks evolving in this environment.

In absence of criteria for relative compactness of the empirical density fields that apply to our case, we develop a tightness criterion based on the notion of *uniform conditional stochastic continuity* introduced in [137]. We rely on two main assumptions for this tightness criterion to be effective: a quenched invariance principle for forward – as well as backward – random walks and a uniform bound on the maximal number of particles per site. Hence, under the invariance principle hypothesis, this tightness criterion applies in other contexts than the one considered in this chapter. Beyond the natural case of SSEP in a quenched *static* (rather than dynamic) bond disorder (cf. the models considered e.g. in [42], [43], [45], [46], [108]), tightness may be proved via the same strategy for generalizations – in quenched dynamic bond disorder – of SSEP in which up to $\alpha \in \mathbb{N}$ particles are allowed per site (see e.g. Section 1.2, Chapter 3 or [20], which differs from the so-called *generalized* or K-SSEP as e.g. in [85]), as well as for uniformly bounded site-inhomogeneous or "instantaneously thermalized" versions of it (see e.g. [20, Section 5] and references therein), i.e. all particle systems for which a self-duality property and a uniform bound on the maximal number of particles per site hold.

Besides the validity of the invariance principle for random walks in dynamic bond disorder with arbitrary starting positions (for recent results in this direction for "initially-anchored-at-the-origin" random walks, see e.g. [1], [2], [10], [12], [35]; concerning this assumption, see also the discussion at the end of Section 2.2), the other assumption on the environment that we require is the uniform boundedness (over the bonds and time) of the disorder. This assumption suffices to prove existence of the infinite particle system. Moreover, by means of this assumption alone and, in particular, without relying on uniform ellipticity of the environment (and consequent heat kernel estimates as in [1, Proposition 1.1]), we obtain an exponential upper bound for the transition probabilities of the random walks. This bound turns out to be useful in providing an explicit formula for some observables of the particle system. Finally, this uniform boundedness assumption yields the equivalence of the invariance principles for the forward and backward random walks in the same dynamic environment, which is a key fact in the proof of relative compactness of the empirical density fields.

Organization of the chapter. The rest of the chapter is organized as follows. In Section 2.1 we introduce the dynamic bond disorder and the model. In Section 2.2 we illustrate our approach in comparison with existing methods and state our main result, Theorem 2.3. In Section 2.3, from a graphical representation of the particle system (which we detail in Appendix 2.a), we deduce a representation of the occupation variables as *mild solutions* of an infinite system of linear stochastic differential equations (which is proved in Section 2.5). Section 2.4 is devoted to the proof of Theorem 2.3. We conclude the chapter with required observations on invariance principles for forward and backward random walks (Appendix 2.b) and the complete proof of the tightness criterion used (Appendix 2.c).

2.1 Setting

The space on which the particles move is the d-dimensional Euclidean lattice $(\mathbb{Z}^d, E_d)$ with $d \geq 1$. The set of bonds E_d consists in all unordered pairs of nearest-neighboring sites, i.e.

$$E_d \;=\; \{\{x,y\},\; x,y \in \mathbb{Z}^d \text{ with } |x-y| = 1\} \,.$$

Let us introduce our *dynamic environment* which is defined on the set of bonds E_d, so that we also refer to it as *(quenched) dynamic bond disorder* on $(\mathbb{Z}^d, E_d)$. Namely, we assign time-dependent non-negative weights to each bond $\{x,y\} \in E_d$ and we define as *environment* any càdlàg (w.r.t. the time variable t) function

$$\boldsymbol{c} \;=\; \{c_t(\{x,y\}),\, \{x,y\} \in E_d,\, t \geq 0\} \,, \tag{2.2}$$

where

$$c_t(\{x,y\}) \;=\; c_t(\{y,x\}) \;\geq\; 0 \tag{2.3}$$

is referred to as the *conductance* of the bond $\{x,y\} \in E_d$ at time $t \geq 0$. The environment $\boldsymbol{c}$ is said to be *static* if $c_t(\{x,y\}) = c_0(\{x,y\})$ for all $\{x,y\} \in E_d$ and $t \geq 0$.

We will need the following assumption on the environment.

Assumption 2.1 (BOUNDED CONDUCTANCES). *There exists a constant* $\mathfrak{a} > 0$ *such that* $c_t(\{x,y\}) \in [0, \mathfrak{a}]$, *for all bonds* $\{x,y\} \in E_d$ *and* $t \geq 0$.

Remark 2.1. *The boundedness of conductances guarantees, via a graphical construction (see Appendix 2.a), that all stochastic processes introduced in Sections 2.2 and 2.3 are well-defined.*

Given the environment $\boldsymbol{c}$ as defined in (2.2)–(2.3), we now introduce as a counterpart to the symmetric simple exclusion process without disorder (2.1) the time-evolution of the *symmetric simple exclusion process in the dynamic environment* $\boldsymbol{c}$ (SSEP($\boldsymbol{c}$)) by specifying its time-dependent infinitesimal generator L_t. For all $t \geq 0$ and every bounded cylinder function $\varphi : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$, we have

$$\begin{aligned} L_t\, \varphi(\eta) \;=\; \sum_{\{x,y\} \in E_d} c_t(\{x,y\}) \Big\{ &\eta(x)\,(1-\eta(y))\,(\varphi(\eta^{x,y}) - \varphi(\eta)) \\ &+\; \eta(y)\,(1-\eta(x))\,(\varphi(\eta^{y,x}) - \varphi(\eta)) \Big\} \,. \end{aligned} \tag{2.4}$$

Given any initial configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$, the time-dependent infinitesimal generators in (2.4) generate a time-inhomogeneous Markov (Feller) process $\{\eta_t,\ t \geq 0\}$ with sample paths in the Skorokhod space $\mathtt{D}([0,\infty), \{0,1\}^{\mathbb{Z}^d})$ such that $\eta_0 = \eta$. We postpone to Section 2.3 the construction of this infinite particle system via a graphical representation.

2.2 Main result: hydrodynamics

In the present section we discuss the hydrodynamic limit in path space of the particle system $\{\eta_t,\ t \geq 0\}$ evolving in the environment $\boldsymbol{c}$, described by (2.4), that is, roughly speaking, the convergence in law of empirical density fields' trajectories to (deterministic) measures whose density w.r.t. Lebesgue is solution of a Cauchy problem. Let us first detail what these density fields and the Cauchy problem with its solution are in our case.

Empirical density fields. We introduce for all $N \in \mathbb{N}$ the *empirical density field* $\{\mathsf{X}_t^N,\ t \geq 0\}$ as a process in $\mathtt{D}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$, the Skorokhod space of $\mathscr{S}'(\mathbb{R}^d)$-valued càdlàg trajectories (see e.g. [105]), where $\mathscr{S}(\mathbb{R}^d)$ denotes the Schwartz class of rapidly decreasing functions on $\mathbb{R}^d$ and $\mathscr{S}'(\mathbb{R}^d)$ its topological dual. Given the particle system $\{\eta_t,\ t \geq 0\}$ evolving in the environment $\boldsymbol{c}$, for any test function $G \in \mathscr{S}(\mathbb{R}^d)$, the empirical density evaluated at G reads as

$$\mathsf{X}_t^N(G) := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} G(\tfrac{x}{N})\, \eta_{tN^2}(x)\,, \qquad t \geq 0\,. \tag{2.5}$$

So we choose to view the empirical density field as taking values in the space of tempered distributions rather than in the space of Radon measures as e.g. in [42], [43]. Indeed, the space $\mathscr{S}'(\mathbb{R}^d)$ has the advantage that it is a good space for tightness criteria (see e.g. [105]) and we use the fact that $\mathscr{S}(\mathbb{R}^d)$ is closed under the action of the Brownian motion semigroup.

Heat equation. Let us denote by $\langle \cdot, \cdot \rangle$ the standard scalar product in $\mathbb{R}^d$. We denote by $\{\rho_t^\Sigma,\ t \geq 0\}$ the *unique* weak solution to the following Cauchy problem

$$\begin{cases} \partial_t \rho &= \frac{1}{2} \nabla \cdot (\Sigma\, \nabla \rho) \\ \rho_0 &= \rho_\bullet\,, \end{cases} \tag{2.6}$$

with $\rho_\bullet : \mathbb{R}^d \to [0,1]$ measurable and Σ being a d-dimensional real symmetric positive-definite matrix (see e.g. [41], [85]). We recall that, for $\{\rho_t^\Sigma,\ t \geq 0\}$, being a weak solution of (2.6) means that, for all $G \in \mathscr{S}(\mathbb{R}^d)$ and $t \geq 0$,

$$\langle G, \rho_t^\Sigma \rangle \ = \ \langle G, \rho_\bullet \rangle \ + \ \int_0^t \langle \tfrac{1}{2} \nabla \cdot (\Sigma\, \nabla G), \rho_s^\Sigma \rangle \, ds \ . \qquad (2.7)$$

In addition, due to the linearity of the heat equation in (2.6), $\{\rho_t^\Sigma,\ t \geq 0\}$ may be represented in terms of $\{\mathcal{S}_t^\Sigma,\ t \geq 0\}$, the transition semigroup associated to the d-dimensional Brownian motion $\{B_t^\Sigma,\ t \geq 0\}$, starting at the origin and with covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$ ([41]); namely, for all $G \in \mathscr{S}(\mathbb{R}^d)$,

$$\langle G, \rho_t^\Sigma \rangle \ = \ \langle \mathcal{S}_t^\Sigma G, \rho_\bullet \rangle \ . \qquad (2.8)$$

Hydrodynamics. The proof of hydrodynamic limits in path space may be divided into two steps. First, one proves that, for all $T > 0$, the sequence of distributions of the empirical density fields $\{\mathsf{X}_t^N,\ 0 \leq t \leq T\}$ is relatively compact in $\mathsf{D}([0,T], \mathscr{S}'(\mathbb{R}^d))$ by proving *tightness*. Then, one proves that all limiting probability measures in $D([0,T], \mathscr{S}'(\mathbb{R}^d))$ are supported on weak solutions of a Cauchy problem. By uniqueness of such a solution, the proof is concluded.

The "standard way" (e.g. [85]) to proceed is the following. To derive the convergence of the processes $\{\mathsf{X}_t^N,\ t \geq 0\}$ in $\mathsf{D}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$, we start from Dynkin's formula for the empirical density fields, i.e. for all $N \in \mathbb{N}$, $G \in \mathscr{S}(\mathbb{R}^d)$ and $t \geq 0$,

$$\mathsf{X}_t^N(G) \ = \ \mathsf{X}_0^N(G) + \int_0^t L_s\, \mathsf{X}_s^N(G)\, ds \ + \ \mathsf{M}_t^N(G)\, , \qquad (2.9)$$

with $\{\mathsf{M}_t^N(G),\ t \geq 0\}$ being a martingale. After obtaining tightness of the sequence $\{\mathsf{X}_t^N,\ 0 \leq t \leq T\}$ via an application of Aldous-Rebolledo criterion, the rest of the proof is carried out in two steps. First one shows that the martingale term $\mathsf{M}_t^N(G)$ vanishes in probability as $N \to \infty$. Secondly, all the remaining terms in (2.9) can be expressed in terms of the empirical density field only; i.e. one "closes" the equation, yielding as a unique limit the solution expressed as in (2.7).

Hydrodynamics & self-duality. In presence of (static or dynamic) disorder, the issue of "closing" equation (2.9) in terms of the empirical density field only cannot be directly achieved. To overpass this obstacle, in the static disorder case, the authors in [68], [79] solve this problem by introducing an auxiliary observable, called *corrected empirical density field*.

Here we follow the probabilistic approach initiated in [108] and further developed in e.g. [42], which is more natural in our context. Key ingredients of this method are the *self-duality property* of the particle system and an alternative to Dynkin's formula (2.9), namely representing the empirical density fields as *mild solutions* of an infinite system of semi-linear stochastic differential equations: for all $N \in \mathbb{N}$, $G \in \mathscr{S}(\mathbb{R}^d)$ and $t \geq 0$,

$$\mathsf{X}_t^N(G) \;=\; \mathsf{X}_0^N(\mathcal{S}_{0,t}^N G) + \int_0^t dM_s^N(\mathcal{S}_{s,t}^N G)\,, \tag{2.10}$$

where, for all $s \in [0,\infty)$, $\{\mathcal{S}_{s,t}^N,\ t \in [s,\infty)\}$ may be related to the semigroup of a *suitably rescaled random walk* (see also (2.37) below). Via this approach, the hydrodynamic limit is obtained in two steps: after proving tightness of the sequence of empirical density fields, first one ensures that the second term on the r.h.s. of (2.10) – which is not a martingale – vanishes in probability as $N \to \infty$; then, one checks whether the first term on the r.h.s. in (2.10) converges to $\langle \rho_t^\Sigma, G\rangle$ as given in (2.8), that is, $\langle \rho_t^\Sigma, G\rangle = \langle \mathcal{S}_t^\Sigma G, \rho_\bullet\rangle$. This latter convergence requires two ingredients: first, that the initial particle configurations rescale to a macroscopic density (assumption (a) in Theorem 2.3); secondly, that all random walks with arbitrary starting positions and evolving in the same dynamic environment rescale to Brownian motions with a given – space and time-independent – covariance matrix; namely, assumption (b) in Theorem 2.3).

In conclusion, while (2.7) is the representation of the solution $\{\rho_t^\Sigma,\ t \geq 0\}$ to the Cauchy problem (2.6) most commonly used when deriving hydrodynamic limits starting from Dynkin's formula (2.9), a method as the one we follow, based on the duality property of the particle system with suitable random walks, profits from this "mild solution" representation (2.8) of $\{\rho_t^\Sigma,\ t \geq 0\}$.

Let us now introduce the random walks alluded to above, used in our hydrodynamics result.

Definition 2.2 (FORWARD AND BACKWARD RANDOM WALKS). *For all $s \geq 0$, let $\{X_{s,t}^x,\ t \in [s,+\infty)\}$ be the* forward *random walk starting at $x \in \mathbb{Z}^d$ at time s and* *evolving in the environment* $\mathbf{c}$ *through the time-dependent infinitesimal generator*

$$A_t f(x) = \sum_{y:\{x,y\}\in E_d} c_t(\{x,y\})\,(f(y)-f(x))\,,$$

where $f:\mathbb{Z}^d\to\mathbb{R}$ *is a bounded function.*

Similarly, for all $t\geq 0$*, let* $\{\widehat{X}^y_{s,t},\ s\in[0,t]\}$ *be the* backward *random walk which starts at* $y\in\mathbb{Z}^d$ *at time* t *and "evolves backwards" in the environment* $\mathbf{c}$ *through the time-dependent infinitesimal generator*

$$A_{s^-} f(x) = \sum_{y:\{x,y\}\in E_d} c_{s^-}(\{x,y\})\,(f(y)-f(x))\,, \tag{2.11}$$

where $f:\mathbb{Z}^d\to\mathbb{R}$ *is as above and* $c_{s^-}(\{x,y\}) = \lim_{r\uparrow s} c_r(\{x,y\})$ *for all* $s\in[0,t]$.

We will give in Section 2.3.1 and Appendix 2.a.1 the construction of both those forward and backward random walks via a graphical representation.

We are now ready to state our main theorem, Theorem 2.3, followed by two remarks related to its proof.

Theorem 2.3 (PATH-SPACE HYDRODYNAMIC LIMIT). *For all* $N\in\mathbb{N}$*, we initialize the exclusion process* $\{\eta_t,\ t\geq 0\}$ *according to a probability measure* μ_N *on* $\{0,1\}^{\mathbb{Z}^d}$ *(Notation:* $\eta_0\sim\mu_N$*), and, consequently,* X^N_0 *is the random element in* $\mathscr{S}'(\mathbb{R}^d)$ *obtained from* $\eta_0\sim\mu_N$*. Besides Assumption 2.1, we further assume that*

(a) *The family of probability measures* $\{\mu_N,\ N\in\mathbb{N}\}$ *on* $\{0,1\}^{\mathbb{Z}^d}$ *is* associated to the density profile $\rho_\bullet:\mathbb{R}^d\to[0,1]$*; namely, for all* $G\in\mathscr{S}(\mathbb{R}^d)$ *and* $\delta>0$,

$$\mu_N\left(\left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\eta(x) - \int_{\mathbb{R}^d} G(u)\rho_\bullet(u)du\right| > \delta\right) \underset{N\to\infty}{\longrightarrow} 0\,. \tag{2.12}$$

(b) *The forward random walks* $\{X^x_{0,t},\ x\in\mathbb{Z}^d,\ t\in[0,+\infty)\}$ *with arbitrary starting positions satisfy an* invariance principle *with a non-degenerate covariance matrix* Σ*; namely, for all* $u\in\mathbb{R}^d$*, for all sequences* $\{x_N,\ N\in$

$\mathbb{N}\} \subset \mathbb{Z}^d$ *for which* $\frac{x_N}{N} \to u \in \mathbb{R}^d$ *as* $N \to \infty$ *and for all* $T > 0$,

$$\left\{\frac{X^{x_N}_{0,tN^2}}{N},\ t \in [0,T]\right\} \underset{N\to\infty}{\Longrightarrow} \left\{B^\Sigma_t + u,\ t \in [0,T]\right\}\ , \tag{2.13}$$

(Notation: $\Rightarrow$ *stands for convergence in law).*

Then, for all $T > 0$, *we have the following convergence*

$$\left\{\mathsf{X}^N_t,\ t \in [0,T]\right\} \underset{N\to\infty}{\Longrightarrow} \left\{\pi^\Sigma_t,\ t \in [0,T]\right\} \tag{2.14}$$

in $\mathsf{D}([0,T], \mathscr{S}'(\mathbb{R}^d))$, *with* $\pi^\Sigma_t(du) = \rho^\Sigma_t(u)du$ *and* $\{\rho^\Sigma_t,\ t \geq 0\}$ *being the unique weak solution to the Cauchy problem* (2.6).

Remark 2.4 (equivalence of forward & backward invariance principles). *In the proof of Theorem 2.3 and, in particular, in the proof of Proposition 2.12 below, we need, besides the invariance principle for the forward random walks (assumption* (b) *in Theorem 2.3), the invariance principle for the backward random walks in the same environment.*

In case of static *environment* $\mathbf{c}$, *the laws of forward and backward random walks coincide. In general, this is not true in presence of* dynamic *environment. However, under Assumption 2.1, the convergence in* (2.13) *is equivalent, by keeping the same notation and conditions, to*

$$\left\{\frac{\widehat{X}^{x_N}_{0,tN^2}}{N},\ t \in [0,T]\right\} \underset{N\to\infty}{\Longrightarrow} \left\{B^\Sigma_t + u,\ t \in [0,T]\right\}\ . \tag{2.15}$$

We will prove this claim in Appendix 2.b.

Remark 2.5 (uniform convergence over time). *If* $\pi^\Sigma_t(du) = \rho^\Sigma_t(u)du$ *for all* $t \geq 0$ *and* $\rho_0 = \rho_\bullet \in L^\infty(\mathbb{R}^d)$ *(which holds true in our setting), then* $\{\pi^\Sigma_t,\ t \geq 0\}$ *is a trajectory in* $\mathsf{C}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$, *the space of tempered distribution-valued continuous trajectories. Indeed, for all* $G \in \mathscr{S}(\mathbb{R}^d)$ *and* $t \geq 0$, *as long as* $\rho_\bullet \in L^\infty(\mathbb{R}^d)$, *by* (2.8), *we have*

$$\left|\langle \rho^\Sigma_t, G\rangle - \langle \rho^\Sigma_s, G\rangle\right| \leq \sup_{u\in\mathbb{R}^d} \rho_\bullet(u) \cdot \int_{\mathbb{R}^d} \left|\mathcal{S}^\Sigma_t G(u) - \mathcal{S}^\Sigma_s G(u)\right| du \underset{|t-s|\to 0}{\longrightarrow} 0\ . \tag{2.16}$$

Hence, because weakly continuous trajectories in the Montel space $\mathscr{S}'(\mathbb{R}^d)$ are strongly continuous (see e.g. [77, p. 145]), it follows that $\{\pi_t^\Sigma,\ t \geq 0\} \in$ $\mathtt{C}([0,\infty),\mathscr{S}'(\mathbb{R}^d))$. As a consequence, the convergence in $\mathtt{D}([0,T],\mathscr{S}'(\mathbb{R}^d))$ in (2.14) *becomes convergence w.r.t. the uniform topology in $\mathtt{C}([0,T],\mathscr{S}'(\mathbb{R}^d))$ (see e.g. [9, p. 124]), i.e. it can also be equivalently rewritten as follows: for all $G \in \mathscr{S}(\mathbb{R}^d)$,*

$$\sup_{0\le t\le T}\left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\,\eta_{tN^2}(x) - \int_{\mathbb{R}^d} G(u)\rho_t^\Sigma(u)du\right| \xrightarrow[N\to\infty]{\mathcal{P}} 0\,,$$

(Notation: $\xrightarrow{\mathcal{P}}$ stands for convergence in probability).

We will prove Theorem 2.3 in Section 2.4. As explained earlier, it relies on a mild solution representation for the particle system involving the forward and backward random walks of Definition 2.2 (see Section 2.3). This representation induces a mild solution representation of the empirical density fields, obtained in formula (2.37).

The proof of tightness for the empirical density fields – which cannot be achieved by means of more standard techniques (e.g. Censov or Aldous-Rebolledo criteria, resp. to be found e.g. in [31] and [85]) when representing the fields in this form – requires the elaboration of a new tightness criterion, presented in Appendix 2.c. The proof of relative compactness, done in Section 2.4.2, will use this criterion, formula (2.37), and the invariance principle for the forward as well as backward random walks. Indeed, in the proof of Proposition 2.12, we will need, besides the invariance principle for the forward random walks (that is, assumption (b) in Theorem 2.3), the invariance principle for the backward random walks in the same environment. In Appendix 2.b, under Assumption 2.1 on the dynamic environment, we obtain the invariance principle for the backward random walks as a consequence of the analogous invariance principle for the forward random walks.

The characterization of the limiting measures as concentrated on the unique weak solution of the Cauchy problem (2.6) boils down to prove convergence of finite-dimensional distributions, that is: for all $n \in \mathbb{N}$, for all $0 \le t_1 < \ldots < t_n \le T$ and for all $G_1,\ldots,G_n \in \mathscr{S}(\mathbb{R}^d)$,

$$\left(\mathsf{X}_{t_1}^N(G_1),\ldots,\mathsf{X}_{t_n}^N(G_n)\right) \xrightarrow[N\to\infty]{\mathcal{P}} \left(\langle G_1,\rho_{t_1}^\Sigma\rangle,\ldots,\langle G_n,\rho_{t_n}^\Sigma\rangle\right)\,. \tag{2.17}$$

As joint convergence in probability comes down to checking convergence in probability of the single marginal laws, it suffices to prove (2.17) for the choice $n = 1$ only, that is: for all $0 \leq t \leq T$ and $G \in \mathscr{S}(\mathbb{R}^d)$,

$$\left| \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} G(\tfrac{x}{N})\, \eta_{tN^2}(x) - \int_{\mathbb{R}^d} G(u) \rho_t^{\Sigma}(u) du \right| \xrightarrow[N\to\infty]{\mathcal{P}} 0\,. \tag{2.18}$$

In Section 2.4.1, we will then exploit the mild solution representation (2.37) to prove (2.18). For this, we will further generalize to the time-inhomogeneous context results originally developed in [108] and further extended in [42], [43], [45].

We end this section with a short discussion on assumption (b) in Theorem 2.3.

Assumption (b) **& examples of dynamic environments.** Our assumption (b) in Theorem 2.3 may be seen as the "dynamic" counterpart of the "static" arbitrary starting point quenched invariance principle in [108, Theorem 1] and [42, Proposition 4.3]. There, both authors derive this crucial result – rather than assuming it, as we do – from statistical properties of the conductances, namely strictly positive and uniformly bounded i.i.d. conductances with a forth-negative moment condition and a (strong) law of large numbers for the inverse of the conductances (= resistances), respectively. Via different techniques, the same authors show that those two assumptions suffice in dimension $d = 1$ and in presence of static conductances to derive [108, Theorem 1] and [42, Proposition 4.3], which, by Theorem 2.11, are both equivalent to our assumption (b) in presence of static environment.

In recent years (see e.g. [1], [2], [3], [10], [12], [35], [120]) there has been an intensive research in providing general examples of dynamical environments $\boldsymbol{c}$ leading to non-degenerate invariance principles for the forward random walk $\{X_{0,t}^0,\ t \geq 0\}$ starting at the origin $0 \in \mathbb{Z}^d$. In all these cases, $\boldsymbol{c}$ is obtained as a typical realization of a suitably constructed random environment process $(\boldsymbol{\Lambda}, \boldsymbol{F}, \boldsymbol{P})$, yielding, for $\boldsymbol{P}$-a.e. environment $\boldsymbol{c} \in \boldsymbol{\Lambda}$ and all $T > 0$,

$$\left\{ \frac{X_{0,tN^2}^0}{N},\ t \in [0,T] \right\} \underset{N\to\infty}{\Longrightarrow} \left\{ B_t^{\Sigma},\ t \in [0,T] \right\}\,. \tag{2.19}$$

Several examples of dynamic random environments which lead to invariance principles as those in (2.19) have been studied in the aforementioned references. In particular, it is worth mentioning that dynamic random environments driven by i.i.d. flipping and Markovian conductances taking values on a finite subset of $(0, \infty)$ fall in the setting studied in [1] for all dimensions $d \geq 1$; while in [12], [107], the authors consider – among other examples – the symmetric simple exclusion process in $\mathbb{Z}^d$ with $d \geq 2$ as an interacting particle system which induces the underlying dynamic random environment for the random walk.

In fact, more general random environments that fit our context have been studied. In particular, in the works [1], [2], [10], [12], [35], the authors obtain quenched invariance principles with deterministic and non-degenerate covariance matrices Σ for space-time ergodic random dynamical environments under conditions of either ellipticity or boundedness of p-moments of conductances and resistances.

However, all these quenched invariance principles in dynamic random environment are obtained for the random walk initially anchored at the origin, whilst our assumption (b) in Theorem 2.3 consists in a quenched invariance principle holding for *all* random walks centered around *all* macroscopic points $u \in \mathbb{R}^d$. The problem of deriving such "arbitrary starting point quenched invariance principles" (our assumption (b)) from (2.19) has been addressed in the case of static environment in [25, Appendix A.2], while – to the best of our knowledge – it remains unsolved in the dynamic setting.

2.3 Graphical constructions and mild solution

In Section 2.3.1 we construct the symmetric simple exclusion process in dynamic environment via a graphical representation. Relying on this construction, we express in Section 2.3.2 the occupation variables of the symmetric simple exclusion process (viewed as a stirring process) in dynamic environment as mild solution of a system of Poissonian stochastic differential equations.

2.3.1 Graphical construction of the particle system

The graphical construction employs, as a source of randomness, a collection of independent Poisson processes, each one attached to a bond of $\mathbb{Z}^d$. To take care of both space and time inhomogeneities, their intensities will depend both on the bond and time. As an intermediate step towards the graphical construction of the particle system, the same Poisson processes provide a graphical construction for all forward and backward random walks introduced in Definition 2.2. We explain this procedure below, leaving a detailed treatment to Appendix 2.a. Finally, we will relate the occupation variables of the particle system to the positions of backward random walks. This must be meant in a pathwise sense, expressing the pathwise duality of the symmetric simple exclusion process in the dynamic environment $\boldsymbol{c}$.

Poisson processes. We consider a family of independent inhomogeneous Poisson processes

$$\{\mathcal{N}_{\cdot}(\{x,y\}),\ \{x,y\} \in E_d\} \tag{2.20}$$

defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P})$, where $\{\mathcal{F}_t,\ t \geq 0\}$ is the natural filtration, $\mathcal{F} = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$ and such that $\mathcal{N}_{\cdot}(\{x,y\})$ has intensity measure $c_r(\{x,y\})dr$, that is

$$\mathbb{E}[\mathcal{N}_t(\{x,y\})] = \int_0^t c_r(\{x,y\})\, dr\,, \qquad t \geq 0\,,$$

where $\mathbb{E}$ denotes expectation w.r.t. $\mathbb{P}$ (for a constructive definition of the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P})$, we refer to Appendix 2.a.1). The associated compensated Poisson processes

$$\{\overline{\mathcal{N}}_{\cdot}(\{x,y\}),\ \{x,y\} \in E_d\}\,,$$

defined as

$$\overline{\mathcal{N}}_t(\{x,y\}) = \mathcal{N}_t(\{x,y\}) - \int_0^t c_r(\{x,y\})\, dr\,, \quad t \geq 0\,, \tag{2.21}$$

are a family of square integrable martingales w.r.t. $\{\mathcal{F}_t,\ t \geq 0\}$ of bounded variation, due to Assumption 2.1.

The associated picture is drawn as follows. On the space $\mathbb{Z}^d \times [0,\infty)$, where $\mathbb{Z}^d$ represents the sites and $[0,\infty)$ represents time which goes up, for each $z \in \mathbb{Z}^d$ draw a vertical line $\{z\} \times [0,\infty)$. Then for each $\{x,y\} \in E_d$, draw a horizontal two-sided arrow between x and y at each event time, i.e. jump time, of $\mathcal{N}_{\cdot}(\{x,y\})$.

Forward and backward random walks. We recover the walks defined in Definition 2.2 as follows. First, for all $\omega \in \Omega$, $x \in \mathbb{Z}^d$, $s \geq 0$ and $t \geq s$, $X^x_{s,t}[\omega]$ now denotes the position at time t of the random walk in $\mathbb{Z}^d$ that is at x at time s and that, between times s and t, crosses the bond $\{z,v\} \in E_d$

at an event time of $\mathcal{N}_{\cdot}(\{z,v\})[\omega]$ whenever at that time the walk is at location either z or v in $\mathbb{Z}^d$ (i.e. it follows the corresponding arrow in the graphical representation). We prove in Appendix 2.a, thanks to Assumption 2.1, that the trajectories of those walks are, for $\mathbb{P}$-a.e. realization $\omega \in \Omega$, well defined for all times and starting positions. In fact, they are all simultaneously defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P})$. In Appendix 2.a, we show that their associated generators are given by (2.2), so that, indeed, these walks are a version of the processes introduced in Definition 2.2.

We now provide a version of the backward random walks of Definition 2.2. For $\mathbb{P}$-a.e. $\omega \in \Omega$, $t \geq 0$ and $y \in \mathbb{Z}^d$, we implicitly define *backward* random walks' trajectories $\{\widehat{X}^y_{s,t}[\omega],\ s \in [0,t]\}$ by the following identity:

$$X_{s,t}^{\widehat{X}^y_{s,t}[\omega]}[\omega] \;=\; y\ . \tag{2.22}$$

In words, $\widehat{X}^y_{s,t}[\omega]$ denotes the position in $\mathbb{Z}^d$ at time s of the forward random walk that follows the Poissonian marks associated to $\omega \in \Omega$ and that is at $y \in \mathbb{Z}^d$ at time t with $t \geq s$. In particular, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $x, y \in \mathbb{Z}^d$, we have

$$X^x_{s,t}[\omega] \;=\; y \quad \text{if and only if} \quad \widehat{X}^y_{s,t}[\omega] \;=\; x\ . \tag{2.23}$$

Again, all these random walks are simultaneously $\mathbb{P}$-a.s. well-defined, and these backward random walks coincide in law with the ones in Definition 2.2 (see Appendix 2.a).

Transition probabilities. The Poissonian construction and the jump rules explained above ensure that each of the forward and backward random walks is Markovian.

For all $x, y \in \mathbb{Z}^d$, $s \geq 0$ and $t \geq s$, if we define

$$p_{s,t}(x,y) \;=\; \mathbb{P}\left(X^x_{s,t} = y\right) \quad \text{and} \quad \widehat{p}_{s,t}(y,x) \;=\; \mathbb{P}\left(\widehat{X}^y_{s,t} = x\right)\ , \tag{2.24}$$

we obtain families of transition probabilities respectively for the forward and backward random walks. In particular, for all $x, y \in \mathbb{Z}^d$ and $0 \leq s \leq r \leq t$, we have the Chapman-Kolmogorov equations

$$\sum_{z\in\mathbb{Z}^d} p_{s,r}(x,z)p_{r,t}(z,y) = p_{s,t}(x,y) \tag{2.25}$$

$$\sum_{z\in\mathbb{Z}^d} \widehat{p}_{r,t}(y,z)\widehat{p}_{s,r}(z,x) = \widehat{p}_{s,t}(y,x)\,. \tag{2.26}$$

From (2.23), we obtain that

$$p_{s,t}(x,y) = \widehat{p}_{s,t}(y,x)\,, \tag{2.27}$$

for all $x, y \in \mathbb{Z}^d$ and $t \geq s$. Then, the operators $\{S_{s,t},\ t \in [s,+\infty)\}$ and $\{\widehat{S}_{s,t},\ s \in [0,t]\}$, acting on bounded functions $f : \mathbb{Z}^d \to \mathbb{R}$ as, for $x \in \mathbb{Z}^d$,

$$S_{s,t}f(x) = \sum_{y\in\mathbb{Z}^d} p_{s,t}(x,y)f(y) \tag{2.28}$$

$$\widehat{S}_{s,t}f(x) = \sum_{y\in\mathbb{Z}^d} \widehat{p}_{s,t}(x,y)f(y)\,, \tag{2.29}$$

correspond to the transition semigroups (or, more properly, the "evolution systems" or "forward/backward propagators" as referred to in [14] and references therein) respectively associated to the forward and backward random walks. Then, as a consequence of (2.27), we obtain that

$$\sum_{x\in\mathbb{Z}^d} [S_{s,t}f(x)]\,g(x) = \sum_{x\in\mathbb{Z}^d} f(x)\,\widehat{S}_{s,t}g(x)\,, \tag{2.30}$$

for all $f, g : \mathbb{Z}^d \to \mathbb{R}$ for which the above summations are finite.

We refer to Appendix 2.a.2 for a more detailed treatment with further properties of the transition probabilities and associated time-inhomogeneous semigroups.

Stirring process. The stirring process relates the above introduced random walks with the occupation variables of the symmetric simple exclusion process in the environment $\boldsymbol{c}$ as follows. Due to the symmetry (2.3) of the environment and the one of the exclusion dynamics, we can rewrite the generator (2.4) as

$$L_t\,\varphi(\eta) = \sum_{\{x,y\}\in E_d} c_t(\{x,y\})\,(\varphi(\eta^{\{x,y\}}) - \varphi(\eta))\,,$$

where $\eta^{\{x,y\}}$ stands for the exchange of occupation numbers between sites x and y in configuration η, which takes place even if x, y are both occupied (due to the fact that particles are indistinguishable). This rewriting gives the stirring interpretation of the symmetric simple exclusion process in the environment $\boldsymbol{c}$ (similar to the stirring interpretation in the case (2.1) without disorder, as described in [31, p. 98] and [98, p. 399]), that we take from now on. This way, the stirring process can be constructed on the same graphical representation as before, and particles evolve as the forward random walks previously introduced.

Hence, similarly to [98, p. 399], we can write, for $\mathbb{P}$-a.e. $\omega \in \Omega$, for any initial configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$, for any $x \in \mathbb{Z}^d$ and $t \geq 0$, that

$$\eta_t(x)[\omega] = 1 \text{ if and only if there is a } y \in \mathbb{Z}^d \text{ so that } X^y_{0,t}[\omega] = x \text{ and } \eta(y) = 1$$

or, equivalently by using the associated backward random walks and (2.22),

$$\eta_t(x)[\omega] = 1 \text{ if and only if there is a } y \in \mathbb{Z}^d \text{ so that } \widehat{X}^x_{0,t}[\omega] = y \text{ and } \eta(y) = 1\,.$$

In other words,

$$\eta_t(x)[\omega] \;=\; \eta(\widehat{X}^x_{0,t}[\omega])\,, \qquad x \in \mathbb{Z}^d\,, \quad t \geq 0\,, \tag{2.31}$$

thus the stochastic process $\{\eta_t,\ t \geq 0\}$ (with $\eta_0 = \eta$) is defined for $\mathbb{P}$-a.e. $\omega \in \Omega$ on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P})$. Moreover, from the memoryless property of the inhomogeneous Poisson processes employed in the graphical construction of forward and backward random walks, given any initial configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$, we recover the Markov property of the process $\{\eta_t,\ t \geq 0\}$ w.r.t. $\{\mathcal{F}_t,\ t \geq 0\}$.

Remark 2.6 (PATHWISE SELF-DUALITY OF SSEP IN DYNAMIC ENVIRONMENT)**.** *What we obtained in* (2.31) *is the property of pathwise self-duality of the symmetric simple exclusion process with a single dual particle (=a one-particle system backward in time), which thus remains valid also in presence of the dynamic environment* $\boldsymbol{c}$.

Remark 2.7 (NOTATION)**.** *In Theorem 2.3, we have* $\eta_0 \sim \mu_N$. *We thus have to enlarge* Ω *and, accordingly, the filtration and the probability measure, to take into account possibly different initial particle configurations. Nevertheless, for the sake of simplicity, we will always write* $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\})$*, but we will write* $\mathbb{P}_{\mu_N}$ *(resp.* $\mathbb{P}_\eta$*) for the probability measure induced by the Poisson processes in* (2.20) *and the distribution* μ_N *(resp.* δ_η*) of the initial configuration* $\eta_0 \in \{0,1\}^{\mathbb{Z}^d}$ *of the* *exclusion process* $\{\eta_t,\ t \geq 0\}$ *(and* $\mathbb{E}_{\mu_N}$ *(resp.* $\mathbb{E}_\eta$*) for the corresponding expectation).*

2.3.2 Mild solution representation of the particle system

The above construction provides an alternative way of defining $\{\eta_t,\ t \geq 0\}$, the symmetric simple exclusion process in the environment $\boldsymbol{c}$ as strong solution of an infinite system of linear stochastic differential equations. This is the content of Proposition 2.8 below. For an analogous statement previously obtained in the time-homogeneous context, we refer to identity (13) in [108].

The motivation comes from an infinitesimal description of the stirring process, as explained through the following computation. For all $t > 0$ and $x \in \mathbb{Z}^d$, if we write $d\eta_t(x) = \eta_t(x) - \eta_{t^-}(x)$, we have

$$d\eta_t(x) \;=\; \sum_{y:\{x,y\}\in E_d} (\eta_{t^-}(y) - \eta_{t^-}(x))\, d\mathcal{N}_t(\{x,y\})\,. \tag{2.32}$$

By introducing the compensated Poisson process (2.21) in (2.32), we obtain

$$\begin{aligned} d\eta_t(x) \;=\; &\sum_{y:\{x,y\}\in E_d} (\eta_{t^-}(y) - \eta_{t^-}(x))\, c_t(\{x,y\})\, dt \\ &+ \sum_{y:\{x,y\}\in E_d} (\eta_{t^-}(y) - \eta_{t^-}(x))\, d\overline{\mathcal{N}}_t(\{x,y\})\,. \end{aligned} \tag{2.33}$$

Note that the terms in the second sum in the r.h.s. of (2.33) are increments of a martingale as products of bounded predictable terms and increments of the compensated Poisson processes. Moreover, like the latter, such martingales are square integrable and of bounded variation.

After observing that the first sum on the r.h.s. of (2.33) corresponds to the definition of the infinitesimal generator in (2.2) at time t of the forward random walk, we rewrite (2.33) as

$$d\eta_t(x) \;=\; A_t\, \eta_{t^-}(x)\, dt \;+\; dM_t(\eta_{t^-}, x)\,, \qquad x \in \mathbb{Z}^d\,, \quad t > 0\,, \tag{2.34}$$

where A_t acts on the x variable and where

$$dM_t(\eta, x) \;:=\; \sum_{y:\{x,y\}\in E_d} (\eta(y) - \eta(x))\, d\overline{\mathcal{N}}_t(\{x,y\})\,. \tag{2.35}$$

In the following proposition, whose proof is postponed to Section 2.5, we state that the so-called "mild solution" [113, Chapter 9] associated to the system of differential equations (2.34) equals $\mathbb{P}$-a.s. the process obtained via the stirring procedure in (2.31). The mild solution is defined as in (2.36) below, i.e. by formally applying the method of variation of constants to (2.34). Recall that $\{\widehat{S}_{s,t},\ s \in [0,t]\}$ and $\{\widehat{p}_{s,t}(y,x),\ x,y \in \mathbb{Z}^d,\ s \in [0,t]\}$ are, respectively, the semigroup and transition probabilities of the backward random walks of Definition 2.2.

Proposition 2.8. *Fix an initial configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$. Consider, for all $x \in \mathbb{Z}^d$ and $t \geq 0$,*

$$\begin{aligned} \zeta_t(x) &= \widehat{S}_{0,t}\eta(x) + \int_0^t \widehat{S}_{r,t}\, dM_r(\eta_{r^-}, x) \\ &= \sum_{y \in \mathbb{Z}^d} \widehat{p}_{0,t}(x,y)\eta(y) + \int_0^t \sum_{y \in \mathbb{Z}^d} \widehat{p}_{r,t}(x,y)\, dM_r(\eta_{r^-}, y)\,, \end{aligned} \tag{2.36}$$

where $\{\eta_t(x),\ x \in \mathbb{Z}^d,\ t \geq 0\}$ is defined in (2.31)*, $\eta_0 = \eta$ and dM_r is given in* (2.35)*. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\zeta_t(x)[\omega] = \eta_t(x)[\omega]\,, \qquad x \in \mathbb{Z}^d\,, \quad t \geq 0\,.$$

Remark 2.9. *Systems of equations of type* (2.34) *are studied in [113] in the context of Hilbert spaces. There it is proved that for a large class of semi-linear infinite-dimensional SDEs the so-called mild solutions coincide with weak solutions.*

2.4 Proof of Theorem 2.3

The key ingredient to prove Theorem 2.3 is the decomposition of the occupation variables of the process $\{\eta_t,\ t \geq 0\}$ provided in Proposition 2.8.

Let $G \in \mathscr{S}(\mathbb{R}^d)$ and consider the empirical density fields $\mathsf{X}_t^N(G)$ as in (2.5). By using first (2.5), then Proposition 2.8 and, finally, identity (2.30), we obtain, for any fixed initial configuration $\eta_0 = \eta \in \{0,1\}^{\mathbb{Z}^d}$,

$$\begin{aligned}
\mathsf{X}_t^N(G) &= \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\, \eta_{tN^2}(x) \\
&= \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\, \widehat{S}_{0,tN^2}\eta(x) + \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N}) \int_0^{tN^2} \widehat{S}_{r,tN^2}\, dM_r(\eta_{r^-},x) \\
&= \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} [S^N_{0,tN^2} G(\tfrac{x}{N})]\, \eta(x) + \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \int_0^{tN^2} [S^N_{r,tN^2} G(\tfrac{x}{N})]\, dM_r(\eta_{r^-},x) \\
&= \mathsf{X}_0^N(S^N_{0,tN^2}G) + \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \int_0^{tN^2} [S^N_{r,tN^2} G(\tfrac{x}{N})]\, dM_r(\eta_{r^-},x)\,, \qquad (2.37)
\end{aligned}$$

where

$$S^N_{s,t} G(\tfrac{x}{N}) := S_{s,t} G(\tfrac{\cdot}{N})(x)\,, \qquad x \in \mathbb{Z}^d\,. \qquad (2.38)$$

Note that the decomposition (2.37) (different from Dynkin's formula) is the one presented in (2.10).

We then proceed as announced after (2.10): in Section 2.4.2 we exploit the tightness criterion given in Appendix 2.c to prove relative compactness of the empirical density fields. In Section 2.4.1 we prove convergence of finite-dimensional distributions, that is (2.18), by showing that, for any $\delta > 0$,

$$\mathbb{P}_{\mu_N}\left(\left|\frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \int_0^{tN^2} S^N_{r,tN^2} G(\tfrac{x}{N})\, dM_r(\eta_{r^-},x)\right| > \frac{\delta}{2}\right) \underset{N\to\infty}{\longrightarrow} 0 \qquad (2.39)$$

and

$$\mu_N\left(\left|\frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} S^N_{0,tN^2} G(\tfrac{x}{N})\eta(x) - \int_{\mathbb{R}^d} G(u)\rho_t^\Sigma(u)du\right| > \frac{\delta}{2}\right) \underset{N\to\infty}{\longrightarrow} 0\,, \quad (2.40)$$

We do not prove tightness first because the computation done to prove (2.39) in Lemma 2.10 of Section 2.4.1 will be used again to prove tightness in Proposition 2.14 in Section 2.4.2.

Let us now shed more light on (2.39) and (2.40). Observe that the first term in the r.h.s. of (2.37) is deterministic – once $\eta_0 = \eta \in \{0,1\}^{\mathbb{Z}^d}$ is fixed – whereas the second term has mean zero and contains all stochasticity derived from the stirring construction. Indeed,

$$\begin{aligned}\mathbb{E}_\eta[\mathsf{X}_t^N(G)] &= \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\,\mathbb{E}_\eta[\eta_{tN^2}(x)] = \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\,\mathbb{E}_\eta[\eta(\widehat{X}^x_{0,tN^2})]\\ &= \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\,\widehat{S}_{0,tN^2}\eta(x) = \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}[S^N_{0,tN^2}G(\tfrac{x}{N})]\,\eta(x)\,.\end{aligned}$$

Thus, the decomposition (2.37) can be written as

$$\mathsf{X}_t^N(G) = \mathbb{E}_\eta[\mathsf{X}_t^N(G)] + \left(\mathsf{X}_t^N(G) - \mathbb{E}_\eta[\mathsf{X}_t^N(G)]\right)\,,$$

where the first term on the r.h.s. is the expectation of the empirical density field and the second one is "noise", i.e. the (stochastic) deviation from the mean; hence it satisfies

$$\mathbb{E}_\eta\left[\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\int_0^{tN^2}[S^N_{r,tN^2}G(\tfrac{x}{N})]\,dM_r(\eta_{r^-},x)\right] = 0\,.$$

Therefore, when deriving the hydrodynamic limit – basically a Weak Law of Large Numbers (WLLN) – the proof of (2.18) reduces to proving that, firstly, the "noise" vanishes in probability and, secondly, that the expectation – when initialized according to μ_N – converges to the correct deterministic limit corresponding to the macroscopic equation; that is (2.39) and (2.40), respectively.

2.4.1 Convergence of finite dimensional distributions

In the present section, we prove (2.18) by means of (2.39) and (2.40).

Proof of (2.39). The convergence (2.39) is a consequence of Chebyshev's inequality and the following lemma, derived through an adaptation of the proof of Lemma 12 in [108].

Lemma 2.10. *For all initial configurations $\eta \in \{0,1\}^{\mathbb{Z}^d}$ and $t \geq 0$, we have*

$$\mathbb{E}_\eta\left[\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\int_0^{tN^2} S^N_{r,tN^2}G(\tfrac{x}{N})\,dM_r(\eta_{r^-},x)\right)^2\right] \underset{N\to\infty}{\longrightarrow} 0\,.$$

Proof. By (2.35), we can rearrange

$$\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\int_0^{tN^2} S^N_{r,tN^2}G(\tfrac{x}{N})\,dM_r(\eta_{r^-},x)$$

as follows:

$$\frac{1}{N^d}\sum_{\{x,y\}\in E_d}\int_0^{tN^2}\left(S^N_{r,tN^2}G(\tfrac{x}{N})-S^N_{r,tN^2}G(\tfrac{y}{N})\right)(\eta_{r^-}(y)-\eta_{r^-}(x))\,d\overline{\mathcal{N}}_r(\{x,y\})\,.$$

Recall that the compensated Poisson processes $\{\overline{\mathcal{N}}_\cdot(\{x,y\}),\ \{x,y\}\in E_d\}$ are of bounded variation in view of Assumption 2.1 and, moreover, they are independent over bonds. Thus by Itō's isometry for jump processes and the independence over the bonds of the Poisson processes in (2.20), we obtain

$$\begin{aligned}\mathcal{V}^N_{t,\eta}(G) &:= \mathbb{E}_\eta\left[\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\int_0^{tN^2} S^N_{r,tN^2}G(\tfrac{x}{N})\,dM_r(\eta_{r^-},x)\right)^2\right]\\ &= \frac{1}{N^{2d}}\sum_{\{x,y\}\in E_d}\int_0^{tN^2}\left(S^N_{r,tN^2}G(\tfrac{x}{N})-S^N_{r,tN^2}G(\tfrac{y}{N})\right)^2\xi_{r,\eta}(\{x,y\})\,c_r(\{x,y\})\,dr\,,\end{aligned}$$

where $\xi_{r,\eta}(\{x,y\}) := \mathbb{E}_\eta\left[(\eta_{r^-}(y)-\eta_{r^-}(x))^2\right]$. Note that, for all $r\geq 0$ and $\eta\in\{0,1\}^{\mathbb{Z}^d}$, $\xi_{r,\eta}(\{x,y\})\in[0,1]$. Then recall definition (2.2), so that

$$\begin{aligned}\mathcal{V}^N_{t,\eta}(G) \leq\ & \frac{1}{N^{2d}}\sum_{\{x,y\}\in E_d}\int_0^{tN^2}\left(S^N_{r,tN^2}G(\tfrac{x}{N})-S^N_{r,tN^2}G(\tfrac{y}{N})\right)^2 c_r(\{x,y\})\,dr\\ =\ & \frac{1}{N^{2d}}\sum_{x\in\mathbb{Z}^d}\int_0^{tN^2} S^N_{r,tN^2}G(\tfrac{x}{N})\,(-A_rS_{r,tN^2}G)(\tfrac{\cdot}{N})(x)\,dr\,,\end{aligned}$$

which, by Kolmogorov backward equation (2.91) for the forward transition semigroup, equals

$$\begin{aligned}&\frac{1}{N^{2d}}\sum_{x\in\mathbb{Z}^d}\int_0^{tN^2} S^N_{r,tN^2}G(\tfrac{x}{N})\,\partial_r\,S^N_{r,tN^2}G(\tfrac{x}{N})\,dr\\ &= \frac{1}{N^{2d}}\sum_{x\in\mathbb{Z}^d}\int_0^{tN^2}\frac{1}{2}\,\partial_r\left(S^N_{r,tN^2}G(\tfrac{x}{N})\right)^2dr\,.\end{aligned}$$

After integration, we further write

$$\begin{aligned}
&\frac{1}{N^{2d}} \sum_{x\in\mathbb{Z}^d} \int_0^{tN^2} \frac{1}{2}\,\partial_r \left(S^N_{r,tN^2} G(\tfrac{x}{N})\right)^2 dr \\
&= \frac{1}{2N^d} \cdot \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left(G(\tfrac{x}{N})^2 - \left(S^N_{0,tN^2} G(\tfrac{x}{N})\right)^2 \right) \\
&\leq \frac{1}{2N^d} \cdot \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})^2 \,.
\end{aligned} \tag{2.41}$$

Because $\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\frac{x}{N})^2 \rightarrow \int_{\mathbb{R}^d} G(u)^2 du < \infty$ as $N \rightarrow \infty$, and since $\mathcal{V}^N_{t,\eta}(G) \geq 0$, the conclusion follows. □

Proof of (2.40). Note that for proving (2.39) neither assumptions (a) nor (b) of Theorem 2.3 have been invoked. In what follows, the invariance principle of forward and backward random walks, i.e. assumption (b) of Theorem 2.3, will play a crucial role. More precisely, we exploit conditions, given in terms of convergence of semigroups, that are equivalent to the invariance principle. In the time-homogeneous context, the correspondence between weak convergence of Feller processes and convergence of Feller semigroups is due to Trotter and Kurtz [40], [93]. For the sake of completeness, in the next theorem we point out how this correspondence translates in the time-inhomogeneous setting.

Theorem 2.11 (INVARIANCE PRINCIPLE). *The following statements are equivalent:*

(A) Weak convergence in path-space. *The forward random walks* $\{X^x_{0,t},\ x \in \mathbb{Z}^d,\ t \in [0,+\infty)\}$ *satisfy an* invariance principle with arbitrary starting positions *with covariance matrix* Σ *(assumption* (b) *in Theorem 2.3).*

(B) Uniform convergence of transition semigroups. *For all* $T > 0$ *and* $G \in \mathscr{S}(\mathbb{R}^d)$,

$$\sup_{0\leq s\leq t\leq T}\ \sup_{x\in\mathbb{Z}^d} \left| S^N_{sN^2,tN^2} G(\tfrac{x}{N}) - \mathcal{S}^\Sigma_{t-s} G(\tfrac{x}{N}) \right| \underset{N\to\infty}{\longrightarrow} 0\,, \tag{2.42}$$

where $\{S^N_{s,t},\ t \in [s,+\infty)\}$ *with* $s \in [0,\infty)$ *is the forward semigroup defined in* (2.38) *and* $\{\mathcal{S}^\Sigma_t,\ t \in [0,\infty)\}$ *is the Brownian motion semigroup, introduced before* (2.8).

An analogous equivalence holds for the backward random walks when replacing $\{X^{\cdot}_{0,t},\ t \in [0,+\infty)\}$ *and* $S^N_{sN^2,tN^2}$ *by* $\{\widehat{X}^{\cdot}_{0,t},\ s \in [0,t]\}$ *and* $\widehat{S}^N_{sN^2,tN^2}$*, respectively.*

We do not provide a detailed proof of Theorem 2.11, but we just mention the main lines. Firstly, by Assumption 2.1, the random walks under consideration are Feller processes (see Appendix 2.a.2). Secondly, by viewing $\{\mathcal{S}^{\Sigma}_t,\ t \geq 0\}$ as an operator semigroup on $C_0(\mathbb{R}^d)$, the Schwartz space $\mathscr{S}(\mathbb{R}^d)$, being a dense and $\mathcal{S}^{\Sigma}_t$-invariant (for all $t \geq 0$) subset of $C_0(\mathbb{R}^d)$, is a core for the associated infinitesimal generator $\mathcal{A}^{\Sigma}$. As a consequence, the idea is to conclude by means of [81, Theorem 19.25] (up to required adaptations as e.g. in [40], Theorem 6.1 in Chapter 1 and Corollary 8.7 in Chapter 4, because pre-limit and limit processes do not take values in the same state space), which applies to the time-homogeneous setting, only. Hence, we first consider the transition semigroup for the (time-homogeneous) space-time process $\{(X^x_{s,s+\cdot}, s+\cdot),\ x \in \mathbb{Z}^d,\ s \geq 0\}$ defined in Appendix 2.a.2, we apply [81, Theorem 19.25] in this time-homogeneous setting and, then, by considering only functions $\widetilde{G} \in \mathscr{S}(\mathbb{R}^d \times (-\infty,\infty))$ which do not depend on the time-variable within a compact interval of $(-\infty,\infty)$ and smoothly vanish outside of it, we obtain Theorem 2.11.

Having an invariance principle for both the forward and the backward random walks in the environment $\boldsymbol{c}$ allows to replace the uniform convergence (w.r.t. $x \in \mathbb{Z}^d$) in (2.42) with convergence in mean (w.r.t. the counting measure). The more precise statement is the content of the following proposition. We state only the forward case, the backward one being analogous.

Proposition 2.12. *Keep the same notation as in Theorem 2.11. Assume that assumption* (b) *in Theorem 2.3 holds true. Then, for all* $T > 0$ *and* $G \in \mathscr{S}(\mathbb{R}^d)$*, we have*

$$\sup_{0 \leq s \leq t \leq T} \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \left| S^N_{sN^2,tN^2} G(\tfrac{x}{N}) - \mathcal{S}^{\Sigma}_{t-s} G(\tfrac{x}{N}) \right| \underset{N\to\infty}{\longrightarrow} 0 \,. \tag{2.43}$$

Proof. The proof consists in proving a compact containment condition uniformly over time and space. More precisely, we want to show that, for all $\varepsilon > 0$, we can find a compact subset $\mathcal{K}_\varepsilon \subset \mathbb{R}^d$ for which we have

$$\limsup_{N\to\infty} \sup_{0 \leq t \leq T} \frac{1}{N^d} \sum_{\frac{x}{N} \notin \mathcal{K}_\varepsilon} |\mathcal{S}^{\Sigma}_t G(\tfrac{x}{N})| < \varepsilon \tag{2.44}$$

and

$$\limsup_{N\to\infty} \sup_{0\le s\le t\le T} \frac{1}{N^d} \sum_{\frac{x}{N}\notin\mathcal{K}_\varepsilon} |S^N_{sN^2,tN^2} G(\tfrac{x}{N})| \;<\; \varepsilon\,. \tag{2.45}$$

The bound (2.44) is a consequence of the uniform bound for the tails of $\{\mathcal{S}^\Sigma_t G,\ t\ge 0\}$ over finite time intervals. Indeed, there exist a compact subset $\mathcal{J}\subset\mathbb{R}^d$ and a constant $C>0$ such that

$$\sup_{0\le t\le T} |\mathcal{S}^\Sigma_t G(u)| \;\le\; \frac{C}{1+|u|^{2d}}\,, \qquad u\notin\mathcal{J}\,.$$

This follows from the fact that $G\in\mathscr{S}(\mathbb{R}^d)$, $\mathcal{S}^\Sigma_t$ acts as convolution with a non-degenerate Gaussian kernel and the use of Fourier transformation. Then, it suffices to choose $\mathcal{K}_\varepsilon\supset\mathcal{J}$ such that

$$\int_{(\mathcal{K}_\varepsilon)^c} \frac{C}{1+|u|^{2d}}\,du \;<\; \varepsilon\,.$$

We turn now to (2.45). Let $\mathcal{H}_\varepsilon\subset\mathbb{R}^d$ be a compact subset such that, for all $N\in\mathbb{N}$, it holds

$$\frac{1}{N^d}\sum_{\frac{x}{N}\notin\mathcal{H}_\varepsilon} |G(\tfrac{x}{N})| \;<\; \frac{\varepsilon}{2}\,. \tag{2.46}$$

As a consequence, for all $N\in\mathbb{N}$, we have the following upper bound:

$$\begin{aligned}
&\sup_{0\le s\le t\le T} \frac{1}{N^d} \sum_{\frac{x}{N}\notin\mathcal{K}_\varepsilon} |S^N_{sN^2,tN^2} G(\tfrac{x}{N})| \\
&\quad\le\; \sup_{0\le s\le t\le T} \frac{1}{N^d} \sum_{y\in\mathbb{Z}^d} |G(\tfrac{y}{N})| \sum_{\frac{x}{N}\notin\mathcal{K}_\varepsilon} p_{sN^2,tN^2}(x,y) \\
&\quad\le\; \frac{\varepsilon}{2} + C_G \sup_{0\le s\le t\le T}\ \sup_{\frac{y}{N}\in\mathcal{H}_\varepsilon} \sum_{\frac{x}{N}\notin\mathcal{K}_\varepsilon} \widehat{p}_{sN^2,tN^2}(y,x)\,,
\end{aligned} \tag{2.47}$$

where, in the second inequality,

$$C_G \;:=\; \sup_{N\in\mathbb{N}} \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left|G(\tfrac{x}{N})\right| \;<\; \infty\,, \tag{2.48}$$

and we used (2.27), (2.46), and that $\sum_{\frac{x}{N}\notin\mathcal{K}_\varepsilon} \widehat{p}_{sN^2,tN^2}(y,x)\le 1$. Note that, for all $y \in \mathbb{Z}^d$, $\mathcal{L}_\varepsilon \subset \mathcal{K}_\varepsilon \subset \mathbb{R}^d$ and functions F_ε in $\mathscr{S}(\mathbb{R}^d)$ satisfying the following conditions,

$$F_\varepsilon(u) \in \begin{cases} \{1\} & \text{if } u \in \mathcal{L}_\varepsilon \\ \{0\} & \text{if } u \notin \mathcal{K}_\varepsilon \\ [0,1] & \text{otherwise}, \end{cases}$$

we obtain:

$$\begin{aligned} \sum_{\frac{x}{N} \notin \mathcal{K}_\varepsilon} \widehat{p}_{sN^2,tN^2}(y,x) &= \mathbb{P}\left(\frac{\widehat{X}^y_{sN^2,tN^2}}{N} \notin \mathcal{K}_\varepsilon \right) \le 1 - \widehat{S}^N_{sN^2,tN^2} F_\varepsilon(\tfrac{y}{N}) \\ &\le \left(1 - \mathcal{S}^\Sigma_{t-s} F_\varepsilon(\tfrac{y}{N})\right) + \left| \widehat{S}^N_{sN^2,tN^2} F_\varepsilon(\tfrac{y}{N}) - \mathcal{S}^\Sigma_{t-s} F_\varepsilon(\tfrac{y}{N}) \right| . \end{aligned} \tag{2.49}$$

Regarding the first term in the last line of (2.49), given $\mathcal{H}_\varepsilon \subset \mathbb{R}^d$, we choose the subsets $\mathcal{L}_\varepsilon \subset \mathcal{K}_\varepsilon \subset \mathbb{R}^d$ large enough such that $\mathcal{H}_\varepsilon \subset \mathcal{L}_\varepsilon$ and

$$\begin{aligned} \sup_{0 \le s \le t \le T} \sup_{\frac{y}{N} \in \mathcal{H}_\varepsilon} \left(1 - \mathcal{S}^\Sigma_{t-s} F_\varepsilon(\tfrac{y}{N})\right) &= 1 - \inf_{0 \le s \le t \le T} \inf_{u \in \mathcal{H}_\varepsilon} \mathcal{S}^\Sigma_{t-s} F_\varepsilon(u) \\ &\le 1 - \inf_{0 \le s \le t \le T} \inf_{u \in \mathcal{H}_\varepsilon} P\left(B^\Sigma_{t-s} + u \in \mathcal{L}_\varepsilon\right) < \frac{\varepsilon}{4} \, (C_G)^{-1} , \end{aligned} \tag{2.50}$$

where P denotes the probability distribution of the Brownian motion $\{B^\Sigma_t,\ t \ge 0\}$. For the second term on the last line of (2.49), the invariance principle for the backward random walks allows us to conclude. Indeed, by the analogue of (2.42) for the backward random walk (cf. Remark 2.4), there exists $N_\varepsilon = N_\varepsilon(F_\varepsilon) \in \mathbb{N}$ for which, for all $N \ge N_\varepsilon$,

$$\sup_{0 \le s \le t \le T} \sup_{y \in \mathbb{Z}^d} \left| \widehat{S}^N_{sN^2,tN^2} F_\varepsilon(\tfrac{y}{N}) - \mathcal{S}^\Sigma_{t-s} F_\varepsilon(\tfrac{y}{N}) \right| \le \frac{\varepsilon}{4} \, (C_G)^{-1} . \tag{2.51}$$

As a consequence of (2.49), (2.50) and (2.51), we obtain (2.45). Together with (2.44) and (2.42), this concludes the proof of (2.43).

□

We apply Proposition 2.12 and assumption (a) of Theorem 2.3 to prove (2.40) and conclude the characterization of the finite-dimensional distributions of the limiting density field.

Let $\{\rho^\Sigma_t,\ t \ge 0\}$ be the unique weak solution of the Cauchy problem as given in (2.8). Moreover, note that $\mathcal{S}^\Sigma_t \mathscr{S}(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d)$ for all $t \ge 0$. Hence, for any family of probability measures $\{\mu_N, N \in \mathbb{N}\}$ associated to the density profile $\rho_\bullet$ (see (2.12) for the definition), we obtain

$$\mu_N\left(\left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\mathcal{S}_t^\Sigma G(\tfrac{x}{N})\,\eta(x)-\int_{\mathbb{R}^d}\mathcal{S}_t^\Sigma G(u)\,\rho_\bullet(u)du\right|>\delta\right)\underset{N\to\infty}{\longrightarrow}0\,,\quad(2.52)$$

for all $t \geq 0$ and all $\delta > 0$. In turn, (2.40) comes as a consequence of (2.52) and the following lemma.

Lemma 2.13. *For all $t \geq 0$, all $G \in \mathscr{S}(\mathbb{R}^d)$ and for any sequence of probability measures $\{\tilde{\mu}_N,\ N \in \mathbb{N}\}$ in $\{0,1\}^{\mathbb{Z}^d}$, we have, for all $\delta > 0$,*

$$\tilde{\mu}_N\left(\left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left(S^N_{0,tN^2}G(\tfrac{x}{N})-\mathcal{S}_t^\Sigma G(\tfrac{x}{N})\right)\eta(x)\right|>\delta\right)\underset{N\to\infty}{\longrightarrow}0\,.\qquad(2.53)$$

Proof. Because $\eta(x) \leq 1$, we obtain

$$\left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left(S^N_{0,tN^2}G(\tfrac{x}{N})-\mathcal{S}_t^\Sigma G(\tfrac{x}{N})\right)\eta(x)\right|\leq\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|S^N_{0,tN^2}G(\tfrac{x}{N})-\mathcal{S}_t^\Sigma G(\tfrac{x}{N})\right|\,.$$

Then we obtain (2.53) via Proposition 2.12 together with Markov's inequality. □

2.4.2 Tightness

In this section we prove tightness of the sequence of density fields $\{\mathsf{X}^N_\cdot,\ N \in \mathbb{N}\}$ in the Skorokhod space $\mathsf{D}([0,T],\mathscr{S}'(\mathbb{R}^d))$. Note that tightness of the distributions $\{\mathsf{X}^N_\cdot,\ N \in \mathbb{N}\}$ is implied by tightness of the single density fields evaluated at all functions $G \in \mathscr{S}(\mathbb{R}^d)$ (see [105]). Hence, it suffices to discuss tightness of the sequence $\{\mathsf{X}^N_\cdot(G),\ N \in \mathbb{N}\}$ in $\mathsf{D}([0,T],\mathbb{R})$, for all $G \in \mathscr{S}(\mathbb{R}^d)$.

The criterion we use is given in Appendix 2.c. Note that we cannot use Aldous-Rebolledo criterion (see e.g. [85]), which relies ultimately on Doob's maximal martingale inequality. Indeed, instead of decomposing the empirical density fields into a predictable term and a martingale term, we employed the mild solution representation (2.36) for which maximal inequalities for martingales do not apply. We postpone to Appendix 2.c any precise statements and anticipate that in our case the proof boils down to prove the following.

Proposition 2.14. *For any initial configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ and all $G \in \mathscr{S}(\mathbb{R}^d)$, the single density fields $\{\mathsf{X}^N_\cdot(G),\ N \in \mathbb{N}\}$ satisfy the following conditions:*

(a) Boundedness. *For all $t \in [0,T]$,*

$$\lim_{m\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(|\mathsf{X}^N_t(G)| > m\right) = 0\,.$$

(b) Equicontinuity. *For all $\varepsilon > 0$, there exist values $h_\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ such that for all $N \geq N_\varepsilon$ we find deterministic functions $\psi^N_\varepsilon, \psi_\varepsilon : [0,h_\epsilon] \to [0,1)$ and non-negative values ϕ^N_ε satisfying the following properties:*

(i) *$\psi^N_\varepsilon, \psi_\varepsilon$ are non-decreasing and $\psi^N_\varepsilon(h), \psi_\varepsilon(h) \to 0$ as $h \to 0$.*

(ii) *For all $h \in [0,h_\varepsilon]$ and $t \in [0,T]$, it holds*

$$\mathbb{P}\left(|\mathsf{X}^N_{t+h}(G) - \mathsf{X}^N_t(G)| > \varepsilon \,|\, \mathsf{F}^N_t\right) \leq \psi^N_\varepsilon(h)\,, \quad \mathbb{P}\text{-a.s.}\,,$$

where, for all $N \in \mathbb{N}$, $\{\mathsf{F}^N_t,\ t \geq 0\}$ denotes the natural filtration associated to $\{\mathsf{X}^N_t,\ t \geq 0\}$.

(iii) *$\phi^N_\varepsilon \to 0$ as $N \to \infty$.*

(iv) *For all $h \in [0,h_\varepsilon]$ and $N \geq N_\varepsilon$, it holds $\psi^N_\varepsilon(h) \leq \psi_\varepsilon(h) + \phi^N_\varepsilon < 1$.*

As a consequence, $\{\mathsf{X}^N_\cdot,\ N \in \mathbb{N}\}$ is a tight sequence in $\mathsf{D}([0,T], \mathscr{S}'(\mathbb{R}^d))$.

Proof. Statement (a) is a direct consequence of (2.18) proved in Section 2.4.1. We prove equicontinuity.

For all $N \in \mathbb{N}$ and $t, t+h \in [0,T]$, writing $\mathsf{X}^N_{t+h}(G)$ via (2.37), then using for these terms (2.30), Chapman-Kolmogorov equation for $\{S_{s,t},\ t \in [s,+\infty)\}$ (see Proposition 2.18(f)), and (2.36), we get the decomposition

$$\begin{aligned}\mathsf{X}^N_{t+h}(G) - \mathsf{X}^N_t(G) &= \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} (S^N_{tN^2,(t+h)N^2} G(\tfrac{x}{N}) - G(\tfrac{x}{N}))\, \eta_{tN^2}(x) \\ &\quad + \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \int_{tN^2}^{(t+h)N^2} S^N_{r,(t+h)N^2} G(\tfrac{x}{N})\, dM_r(\eta_{r^-}, x)\,.\end{aligned}$$

Thus, we obtain

$$\mathbb{P}\left(|\mathsf{X}^N_{t+h}(G) - \mathsf{X}^N_t(G)| > \varepsilon \,\middle|\, \mathsf{F}^N_t\right) \tag{2.54}$$

$$\leq \mathbb{P}\left(\left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}(S^N_{tN^2,(t+h)N^2}G(\tfrac{x}{N}) - G(\tfrac{x}{N}))\,\eta_{tN^2}(x)\right| > \frac{\varepsilon}{2}\,\middle|\,\mathsf{F}^N_t\right) \tag{2.55}$$

$$+ \mathbb{P}\left(\left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\int_{tN^2}^{(t+h)N^2} S^N_{r,(t+h)N^2}G(\tfrac{x}{N})\,dM_r(\eta_{r^-},x)\right| > \frac{\varepsilon}{2}\,\middle|\,\mathsf{F}^N_t\right) \tag{2.56}$$

and we estimate separately the two terms in (2.55) and (2.56). We start with the term in (2.55), that we call $\mathcal{X}^N_{t,t+h}(\varepsilon)$. The bound $\eta_t(x) \leq 1$ yields

$$\mathcal{X}^N_{t,t+h}(\varepsilon) \leq \mathbb{P}\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|S^N_{tN^2,(t+h)N^2}G(\tfrac{x}{N}) - G(\tfrac{x}{N})\right| > \frac{\varepsilon}{2}\,\middle|\,\mathsf{F}^N_t\right)$$

and the probability on the r.h.s. vanishes as $N \to \infty$. This can be seen as follows:

(α) by Proposition 2.12 we can deduce that there exists a sufficiently large $N_{\varepsilon,1} \in \mathbb{N}$ such that, for all $N \geq N_{\varepsilon,1}$, we have

$$\sup_{0\leq t\leq t+h\leq T}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|S^N_{tN^2,(t+h)N^2}G(\tfrac{x}{N}) - \mathcal{S}^\Sigma_h G(\tfrac{x}{N})\right| \leq \frac{\varepsilon}{4}\,; \tag{2.57}$$

(β) by the strong continuity of $\{\mathcal{S}^\Sigma_h,\ h \geq 0\}$ and the uniform integrability of $\{\mathcal{S}^\Sigma_h G,\ h \in [0,T]\}$ also used in the proof of Proposition 2.12, one can show that there exists $h_\varepsilon > 0$ – independent of $N \in \mathbb{N}$ – such that, for all $h \in [0, h_\varepsilon]$ and $N \geq N_{\varepsilon,1}$, we have

$$\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|\mathcal{S}^\Sigma_h G(\tfrac{x}{N}) - G(\tfrac{x}{N})\right| \leq \frac{\varepsilon}{4}\,. \tag{2.58}$$

We then obtain, for all $h \in [0, h_\varepsilon]$ and $N \geq N_{\varepsilon,1}$,

$$\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|S^N_{tN^2,(t+h)N^2}G(\tfrac{x}{N}) - G(\tfrac{x}{N})\right| \leq \frac{\varepsilon}{2}\,,$$

from which the conclusion follows. More precisely, for all $N \geq N_{\varepsilon,1}$ and

$h \in [0, h_\varepsilon]$,

$$\mathcal{X}^N_{t,t+h}(\varepsilon) \;=\; 0\,. \tag{2.59}$$

To bound the term in (2.56), that we call $\mathcal{Y}^N_{t,t+h}(\varepsilon)$, we combine Chebyshev's inequality and the argument in the proof of Lemma 2.10 (which gave (2.41) – note that we applied Itō's isometry for the conditional expectation) to get

$$\mathcal{Y}^N_{t,t+h}(\varepsilon) \;\leq\; \frac{4}{\varepsilon^2}\cdot\frac{1}{2N^d}\cdot\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left(G^2(\tfrac{x}{N}) - (S^N_{tN^2,(t+h)N^2}G)^2(\tfrac{x}{N})\right)\,. \tag{2.60}$$

Recall the values $N_{\varepsilon,1} \in \mathbb{N}$ and $h_\varepsilon > 0$ obtained from conditions (2.57) and (2.58). For all $N \geq N_{\varepsilon,1}$ and $h \in [0, h_\varepsilon]$, define the function $\psi^N_\varepsilon : [0, h_\varepsilon] \to [0,\infty)$ as

$$\psi^N_\varepsilon(h) \;=\; \left(\frac{4\,C_G}{\varepsilon^2 N^d}\right)\mathcal{Z}^N_h\,, \tag{2.61}$$

where C_G was defined in (2.48), and

$$\mathcal{Z}^N_h \;:=\; \sup_{t\in[0,T]}\,\sup_{h'\in[0,h]}\,\sup_{x\in\mathbb{Z}^d}\left|G(\tfrac{x}{N}) - S^N_{tN^2,(t+h')N^2}G(\tfrac{x}{N})\right|\,, \tag{2.62}$$

Since we want the functions ψ^N_ε to take values in $[0,1)$ rather than $[0,\infty)$, we note that there exists $N_{\varepsilon,2} \in \mathbb{N}$ such that

$$M_\varepsilon \;:=\; 2\sup_{u\in\mathbb{R}^d}|G(u)|\left(\frac{4\,C_G}{\varepsilon^2\,(N_{\varepsilon,2})^d}\right) \;<\; 1\,. \tag{2.63}$$

Hence, for all $h \in [0, h_\varepsilon]$ and $N \geq N_{\varepsilon,2}$, we have $\psi^N_\varepsilon(h) \in [0,1)$ because

$$\sup_{N\in\mathbb{N}}\,\sup_{h\in[0,h_\varepsilon]}\mathcal{Z}^N_h \;\leq\; \sup_{x\in\mathbb{Z}^d}\left|G(\tfrac{x}{N})\right| + \sup_{s\leq t}\,\sup_{x\in\mathbb{Z}^d}\left|S^N_{s,t}G(\tfrac{x}{N})\right| \;\leq\; 2\sup_{u\in\mathbb{R}^d}|G(u)|\,.$$

If we call

$$N_{\varepsilon,1,2} := \max\{N_{\varepsilon,1}, N_{\varepsilon,2}\} \tag{2.64}$$

we observe that, for all $N \geq N_{\varepsilon,1,2}$, ψ^N_ε is non-decreasing: indeed $\mathcal{Z}^N_{h'} \leq \mathcal{Z}^N_{h''}$ if $h' \leq h''$, given that $h', h'' \in [0, h_\varepsilon]$. By the strong continuity of the transition semigroup $\{S_{s,t},\ t \in [s,+\infty)\}$ (see Proposition 2.18(g)), for all $N \geq N_{\varepsilon,1,2}$ we have $\psi^N_\varepsilon(h) \to 0$ as $h \to 0$. This yields (i) for the functions $\{\psi^N_\varepsilon,\ N \geq N_{\varepsilon,1,2}\}$.

Now we prove (ii). We claim that $\mathcal{Y}^N_{t,t+h}(\varepsilon) \le \psi^N_\varepsilon(h)$. Indeed, by (2.60),

$$\begin{aligned}
&\frac{4}{\varepsilon^2}\cdot\frac{1}{2N^d}\cdot\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\Big(G^2(\tfrac{x}{N}) - (S^N_{tN^2,(t+h)N^2}G)^2(\tfrac{x}{N})\Big)\\
&\quad\le \frac{4}{\varepsilon^2}\cdot\frac{1}{2N^d}\cdot\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}|G(\tfrac{x}{N})|\cdot|G(\tfrac{x}{N}) - S^N_{tN^2,(t+h)N^2}G(\tfrac{x}{N})|\\
&\quad+ \frac{4}{\varepsilon^2}\cdot\frac{1}{2N^d}\cdot\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}|S^N_{tN^2,(t+h)N^2}G(\tfrac{x}{N})|\cdot|G(\tfrac{x}{N}) - S^N_{tN^2,(t+h)N^2}G(\tfrac{x}{N})|\\
&\quad\le \left(\frac{4}{\varepsilon^2}\cdot\frac{1}{2N^d}\cdot\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}|G(\tfrac{x}{N})| + |S^N_{tN^2,(t+h)N^2}G(\tfrac{x}{N})|\right)\cdot\mathcal{Z}^N_h \;\le\; \psi^N_\varepsilon(h)\,, \qquad (2.65)
\end{aligned}$$

where in the last inequality we used that, by (2.28) and (2.27), we have

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d}|S^N_{tN^2,(t+h)N^2}G(\tfrac{x}{N})|\\
&\qquad\le \sum_{x\in\mathbb{Z}^d}S^N_{tN^2,(t+h)N^2}|G|(\tfrac{x}{N}) \;=\; \sum_{x\in\mathbb{Z}^d}|G(\tfrac{x}{N})| \;\le\; N^d\,C_G\,.
\end{aligned}$$

As a consequence, for our choices of $N_{\varepsilon,1,2}$ and h_ε (see also (2.54) and its bounds, (2.59) and (2.65)), we have

$$\mathbb{P}\left(|\mathsf{X}^N_{t+h}(G) - \mathsf{X}^N_t(G)| > \varepsilon \,\middle|\, \mathsf{F}^N_t\right) \;\le\; \mathcal{X}^N_{t,t+h}(\varepsilon) \;+\; \mathcal{Y}^N_{t,t+h}(\varepsilon) \;\le\; \psi^N_\varepsilon(h)\,,$$

and, in turn, (ii).

Now we prove the last two items, namely (iii) and (iv). By the triangle inequality, we obtain

$$\mathcal{Z}^N_h \;\le\; \sup_{h'\in[0,h]}\sup_{u\in\mathbb{R}^d}|G(u) - \mathcal{S}^\Sigma_{h'}G(u)| + \sup_{0\le s\le t\le T}\sup_{x\in\mathbb{Z}^d}|S^N_{sN^2,tN^2}G(\tfrac{x}{N}) - \mathcal{S}^\Sigma_{t-s}G(\tfrac{x}{N})|\,. \qquad (2.66)$$

This leads us to the following definitions: for all $h\in[0,h_\varepsilon]$ and $N \ge N_{\varepsilon,1,2}$,

$$\psi_\varepsilon(h) \;:=\; \left(\frac{4\,C_G}{\varepsilon^2(N_{\varepsilon,1,2})^d}\right)\sup_{h'\in[0,h]}\sup_{u\in\mathbb{R}^d}|G(u) - \mathcal{S}^\Sigma_{h'}G(u)| \qquad (2.67)$$

$$\phi^N_\varepsilon \;:=\; \left(\frac{4\,C_G}{\varepsilon^2N^d}\right)\sup_{0\le s\le t\le T}\sup_{x\in\mathbb{Z}^d}|S^N_{sN^2,tN^2}G(\tfrac{x}{N}) - \mathcal{S}^\Sigma_{t-s}G(\tfrac{x}{N})|\,. \qquad (2.68)$$

We obtain (iii), i.e. $\phi_\varepsilon^N \to 0$ as $N \to \infty$ from (2.42), i.e. forward semigroup uniform convergence. Alternatively, the contraction property of the semigroups $\{S_{s,t},\ t \in [s,+\infty)\}$ and $\{\mathcal{S}_t^\Sigma,\ t \geq 0\}$ (cf. Proposition 2.18(d)), yields

$$\sup_{0\leq s\leq t\leq T}\ \sup_{x\in\mathbb{Z}^d} |S^N_{sN^2,tN^2}G(\tfrac{x}{N}) - \mathcal{S}^\Sigma_{t-s}G(\tfrac{x}{N})| \leq 2 \sup_{u\in\mathbb{R}^d} |G(u)| \,. \tag{2.69}$$

By combining (2.69) with $\phi_\varepsilon^N \geq 0$ and $\phi_\varepsilon^N \leq (\sup_u |G(u)| \frac{8}{\varepsilon^2} C_G)\frac{1}{N^d}$ leads to (iii).

In particular, there exists $N_{\varepsilon,3} \in \mathbb{N}$ such that, for all $N \geq N_{\varepsilon,3}$, $\phi_\varepsilon^N \in [0, 1 - M_\varepsilon)$.

From the definitions of M_ε in (2.63), $N_{\varepsilon,1,2}$ in (2.64) and $\psi_\varepsilon(h)$ in (2.67) above, we obtain that $\psi_\varepsilon(h) \in [0, M_\varepsilon]$ for all $h \in [0, h_\varepsilon]$ and that ψ_ε is non-decreasing. The property $\psi_\varepsilon(h) \to 0$ as $h \to 0$ is a consequence of the strong continuity of the contraction semigroup $\{\mathcal{S}_t^\Sigma,\ t \geq 0\}$. This yields (i) for the function $\psi_\varepsilon : [0, h_\varepsilon] \to [0, 1)$ (recall (2.63)).

In conclusion, (2.66) and $N \geq N_\varepsilon := \max\{N_{\varepsilon,1,2}, N_{\varepsilon,3}\}$ yield (iv) and, in particular, $\psi_\varepsilon(h) + \phi_\varepsilon^N \in [0, 1)$ for all $h \in [0, h_\varepsilon]$. This concludes the proof. □

2.5 Proof of Proposition 2.8

We have to show in this section that the infinite summation on the r.h.s. of (2.36) is absolutely convergent and that it equals $\eta_t(x)[\omega]$. The proof relies on the construction of *active islands* (introduced in Appendix 2.a.1) and on a finer control on their radius, which allows to obtain exponential bounds on the transition probabilities of the random walks. As a consequence, we prove identity (2.36) for all initial conditions $\eta \in \{0, 1\}^{\mathbb{Z}^d}$ and all times $t \geq 0$.

The plan is the following. First we show that, when restricting to a finite summation, formula (2.36) indeed holds for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then, based only on a percolation result on the radius of active islands in sufficiently small time intervals [69] and the uniform boundedness assumption of the conductances (Assumption 2.1), we obtain an exponential upper bound for the heat kernel. In conclusion, we prove that, for all initial conditions $\eta \in \{0, 1\}^{\mathbb{Z}^d}$, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the infinite summation in (2.36) is absolutely convergent, hence a rearrangement of the order of the summation, which does not change its value, gives us the result.

Finite summations. Among all *active islands in* $[s,t]$, namely the connected subgraphs of $(\mathbb{Z}^d, E_d)$ consisting of sites of $\mathbb{Z}^d$ and bonds $\{y,z\} \in E_d$ for which Poissonian events occurred in the time window $[s,t]$, i.e. for which

$$\mathcal{N}_t(\{y,z\}) - \mathcal{N}_{s^-}(\{y,z\}) \;\geq\; 1\,,$$

we denote by $\mathcal{G}_{[s,t]}[x]$ the unique active island in $[s,t]$ containing $x \in \mathbb{Z}^d$. Due to Assumption 2.1 (see Appendix 2.a.1 for the detailed argument), there exists $h_c(d,\mathfrak{a}) > 0$ such that, for $\mathbb{P}$-a.e. realization $\omega \in \Omega$, for all $x \in \mathbb{Z}^d$ and all $s,t \in [0,\infty)$ with $0 < t-s < h_c(d,\mathfrak{a})$, the active island $\mathcal{G}_{[s,t]}(x)[\omega]$ is finite. As a consequence, both trajectories $\{X^x_{s,r}[\omega],\ s \leq r \leq t\}$ and $\{\widehat{X}^x_{r,t}[\omega],\ s \leq r \leq t\}$ are well-defined (see Section 2.3.1 and Appendix 2.a.1). For the same reason, given $\{\eta_s(x)[\omega],\ x \in \mathbb{Z}^d\}$, the definition (2.31), i.e. $\eta_t(x)[\omega] = \eta_s(\widehat{X}^x_{s,t}[\omega])[\omega]$ in terms of the stirring process, poses no problem.

In the following lemma, due to the finiteness of active islands, we can give a precise meaning to (2.36) when restricting the summation only to particle positions within the same active island.

Lemma 2.15. *Fix* $x,z \in \mathbb{Z}^d$ *and* $s,t \in [0,\infty)$ *with* $0 < t-s < h_c(d,\mathfrak{a})$. *Then, for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *and any configuration* $\eta_s \in \{0,1\}^{\mathbb{Z}^d}$, *we have*

$$\sum_{y \in \mathcal{G}_{[s,t]}(z)[\omega]} \left(\widehat{p}_{s,t}(x,y)\eta_s(y) + \int_s^t \widehat{p}_{r,t}(x,y)\, dM_r(\eta_{r^-}[\omega],y)[\omega] \right) = \begin{cases} \eta_t(x)[\omega] & \text{if } x \in \mathcal{G}_{[s,t]}(z)[\omega] \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.70}$$

where $\eta_t(x)[\omega] = \eta_s(\widehat{X}^x_{s,t}[\omega])$.

Proof. For notational convenience, let us set $s = 0$ and $t < h_c(d,\mathfrak{a})$. By recalling the definition of dM_r in (2.35) and the following backward master equation (obtained by using (2.91))

$$\partial_r \widehat{p}_{r,t}(x,\cdot)(y) \;=\; -\sum_{v:\{y,v\}\in E_d} c_r(\{y,v\})\,(\widehat{p}_{r,t}(x,v) - \widehat{p}_{r,t}(x,y))\,, \tag{2.71}$$

we rearrange the l.h.s. in (2.70) to obtain:

$$\sum_{y\in\mathcal{G}_{[0,t]}(z)[\omega]}\left(\widehat{p}_{0,t}(x,y)\eta(y)+\int_0^t \eta_{r^-}(y)[\omega]\,\partial_r\widehat{p}_{r,t}(x,\cdot)(y)\,dr\right.$$
$$\left.+\int_0^t \eta_{r^-}(y)[\omega]\sum_{v:\{y,v\}\in E_d}\left(\widehat{p}_{r,t}(x,v)-\widehat{p}_{r,t}(x,y)\right)d\mathcal{N}_r(\{y,v\})[\omega]\right)\,. \quad (2.72)$$

Now, for all $y\in\mathcal{G}_{[0,t]}(z)[\omega]$, we denote by

$$0\ \le\ s_1(y)[\omega]\ <\ \ldots\ <s_{n(y)[\omega]}(y)[\omega]\ \le\ t$$

the $n(y)[\omega]$ jump times occurred in a bond incident to y in the time interval $[0,t]$, with the convention

$$s_0\ =\ 0\quad\text{and}\quad s_{n(y)[\omega]+1}\ =\ t\ .$$

Note that Assumption 2.1 assures that $n(y)[\omega]<\infty$. By recalling definition (2.31), each y-term in (2.72) admits the following decomposition (we write s_k instead of $s_k(y)[\omega]$ for readability):

$$\widehat{p}_{0,t}(x,y)\eta(y)+\sum_{k=0}^{n(y)[\omega]}\eta(\widehat{X}^y_{0,s_k}[\omega])\left(\widehat{p}_{s_{k+1},t}(x,y)-\widehat{p}_{s_k,t}(x,y)\right)$$
$$+\sum_{k=0}^{n(y)[\omega]}\eta(\widehat{X}^y_{0,s_k}[\omega])\left(\widehat{p}_{s_{k+1},t}(x,X^y_{s_k,s_{k+1}}[\omega])-\widehat{p}_{s_{k+1},t}(x,y)\right)\,,$$

which further simplifies as

$$\widehat{p}_{t,t}(x,y)\eta(\widehat{X}^y_{0,t}[\omega])+\sum_{k=0}^{n(y)[\omega]-1}\left(\eta(\widehat{X}^y_{0,s_k}[\omega])\widehat{p}_{s_{k+1},t}(x,X^y_{s_k,s_{k+1}}[\omega])\right.$$
$$\left.-\eta(\widehat{X}^y_{0,s_{k+1}}[\omega])\widehat{p}_{s_{k+1},t}(x,y)\right)\,. \quad (2.73)$$

Now, for all $k=0,\ldots,n(y)[\omega]-1$, there exists a unique neighbor of y, here denoted by $v\in\mathbb{Z}^d$, for which $d\mathcal{N}_{s_{k+1}(y)}(\{y,v\})[\omega]=1$. Note that $v\in\mathcal{G}_{[0,t]}(z)[\omega]$. As a consequence of the construction of the forward and backward random walks, we have

$$X^y_{s_k,s_{k+1}}[\omega] = v\,, \quad \widehat{X}^y_{0,s_k}[\omega] = \widehat{X}^v_{0,s_{k+1}}[\omega] \quad \text{and} \quad \widehat{X}^v_{0,s_k}[\omega] = \widehat{X}^y_{0,s_{k+1}}[\omega]\,.$$

In turn, there will be exactly one term in the following sum

$$\sum_{\ell=0}^{n(v)[\omega]-1} \left(\eta(\widehat{X}^v_{0,s_\ell}[\omega])\widehat{p}_{s_{\ell+1},t}(x, X^v_{s_\ell,s_{\ell+1}}[\omega]) - \eta(\widehat{X}^v_{0,s_{\ell+1}}[\omega])\widehat{p}_{s_{\ell+1},t}(x,v)\right)$$

which cancels the corresponding k-th term in (2.73). Hence, after reordering these finite summations, (2.72) reduces to the following

$$\sum_{y\in\mathcal{G}_{[0,t]}(z)[\omega]} \widehat{p}_{t,t}(x,y)\eta(\widehat{X}^y_{0,t}[\omega])\,.$$

The observation that $\widehat{p}_{t,t}(x,y) = \mathbf{1}_{\{x=y\}}$ concludes the proof. □

Radius of active islands and absolute convergence. We start by presenting a key estimate, direct consequence of [69, Theorem 3.4] and Assumption 2.1, on the radius of active islands:

Fact 2.1 ([69, THEOREM 3.4])**.** *For all $s,t \in [0,\infty)$, with $0 < t-s < h_c(d,\mathfrak{a})$, there exists $\chi(t-s) > 0$, such that*

$$\mathbb{P}\left(\exists\, y \in \mathbb{Z}^d : |y-x| = n \text{ and } y \in \mathcal{G}_{[s,t]}(x)\right) \le e^{-\chi(t-s)n}\,, \tag{2.74}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$. In words, the probability that the active island in $(s,t]$ containing $x \in \mathbb{Z}^d$ contains at least one site at distance n from $x \in \mathbb{Z}^d$ is smaller than $e^{-\chi(t-s)n}$, for all $n \in \mathbb{N}$. The function $\chi : (0, h_c(d,\mathfrak{a})) \to (0,\infty)$ can be chosen to be non-increasing.

For all $x \in \mathbb{Z}^d$, $t \ge 0$ and $\eta \in \{0,1\}^{\mathbb{Z}^d}$, we need to give a precise meaning to the infinite sum in (2.36) for $\mathbb{P}$-a.e. realization $\omega \in \Omega$. More precisely, we need to ensure that this infinite sum is absolutely convergent, allowing us to reorder the summation so as to sum over finite active islands (over space and time) first, and then, to apply Lemma 2.15. This is the content of the following lemma.

Lemma 2.16. *Fix $x \in \mathbb{Z}^d$, $t > 0$ and a partition $\{0 = t_0, t_1, \ldots, t_n, t_{n+1} = t\}$ of $[0,t]$ finer than $h_c(d,\mathfrak{a})$, i.e. $t_{k+1} > t_k$ and $t_{k+1} - t_k < h_c(d,\mathfrak{a})$, for $k = 0,\ldots,n$.*

Then:

(a) There exist two constants $C, \overline{\chi} > 0$ (depending only on $t > 0$ and the partition $\{t_0, \dots, t_{n+1}\}$ of $[0, t]$) such that, for all $m \in \mathbb{N}$,

$$\sum_{y:|y-x|=m} \sum_{z_1 \in \mathbb{Z}^d} \cdots \sum_{z_n \in \mathbb{Z}^d} p_{0,t_1}(x, z_1) \cdots p_{t_n,t}(z_n, y) \leq C\, e^{-\overline{\chi} m} . \quad (2.75)$$

An analogous result (with the same constants) holds for the backward transition probabilities:

$$\sum_{y:|y-x|=m} \sum_{z_n \in \mathbb{Z}^d} \cdots \sum_{z_1 \in \mathbb{Z}^d} \widehat{p}_{t_n,t}(x, z_n) \cdots \widehat{p}_{0,t_1}(z_1, y) \leq C\, e^{-\overline{\chi} m} . \quad (2.76)$$

(b) For $\mathbb{P}$-a.e. $\omega \in \Omega$ and all initial configurations $\eta \in \{0,1\}^{\mathbb{Z}^d}$,

$$\sum_{z_n \in \mathbb{Z}^d} \cdots \sum_{z_1 \in \mathbb{Z}^d} \widehat{p}_{t_n,t}(x, z_n) \cdots \widehat{p}_{t_1,t_2}(z_2, z_1) \times$$
$$\times \int_0^{t_1} \sum_{y \in \mathbb{Z}^d} \widehat{p}_{r,t_1}(z_1, y)\, d\left|M_r(\eta_{r^-}[\omega], y)[\omega]\right| < \infty . \quad (2.77)$$

(c) For $\mathbb{P}$-a.e. $\omega \in \Omega$, $k = 0, \dots, n$ and all initial configurations $\eta \in \{0,1\}^{\mathbb{Z}^d}$, the infinite summation

$$\sum_{y \in \mathbb{Z}^d} \widehat{p}_{t_k,t_{k+1}}(x, y) \eta_{t_k}(y)[\omega] + \int_{t_k}^{t_{k+1}} \sum_{y \in \mathbb{Z}^d} \widehat{p}_{r,t_{k+1}}(x, y)\, dM_r(\eta_{r^-}[\omega], y)[\omega] \quad (2.78)$$

is absolutely convergent and equals $\eta_{t_{k+1}}(x)[\omega]$.

Proof. For item (a), all terms being non-negative, we can reorder the summation on the l.h.s. in (2.75) to obtain

$$\sum_{z_1 \in \mathbb{Z}^d} \cdots \sum_{z_{n-1} \in \mathbb{Z}^d} p_{0,t_1}(x, z_1) \cdots p_{t_{n-2},t_{n-1}}(z_{n-2}, z_{n-1}) \times$$
$$\times \left(\sum_{m_n=0}^{\infty} \sum_{z_n:|z_n-x|=m_n} p_{t_{n-1},t_n}(z_{n-1}, z_n) \sum_{y:|y-x|=m} p_{t_n,t}(z_n, y) \right) . \quad (2.79)$$

As a consequence of the graphical construction of forward and backward random walks (see Appendix 2.a), triangle inequality and (2.74), we have, for all $z_n \in \mathbb{Z}^d$ such that $|z_n - x| = m_n$, first when $m_n \neq m$,

$$\begin{aligned}\sum_{y:|y-x|=m} p_{t_n,t}(z_n, y) \;&=\; \mathbb{P}\left(|X^{z_n}_{t_n,t} - x| = m\right)\\ &\le\; \mathbb{P}\left(\exists\, y \in \mathbb{Z}^d : |y - x| = m \text{ and } y \in \mathcal{G}_{[t_n,t]}(z_n)\right)\\ &\le\; \mathbb{P}\left(\exists\, w \in \mathbb{Z}^d : |w - z_n| = |m - m_n| \text{ and } w \in \mathcal{G}_{[t_n,t]}(z_n)\right)\\ &\le\; e^{-\chi(t-t_n)|m-m_n|}\;. \end{aligned} \tag{2.80}$$

Then when $m_n = m$, we simply bound

$$\sum_{y:|y-x|=m} p_{t_n,t}(z_n, y) \;\le\; 1\;. \tag{2.81}$$

Hence, by (2.80) and (2.81), (2.79) is bounded above by

$$\sum_{m_n=0}^{\infty} e^{-\chi(t-t_n)|m-m_n|}\left(\sum_{z_n:|z_n-x|=m_n}\sum_{z_1\in\mathbb{Z}^d}\cdots\sum_{z_{n-1}\in\mathbb{Z}^d} p_{0,t_1}(x, z_1)\cdots p_{t_{n-1},t_n}(z_{n-1}, z_n)\right)\;.$$

By iterating this procedure for a finite number of steps, we obtain the following upper bound for (2.79):

$$\sum_{m_n=0}^{\infty}\cdots\sum_{m_2=0}^{\infty} e^{-\chi(t-t_n)|m-m_n|}\cdots e^{-\chi(t_2-t_1)|m_2-m_1|}\left(\sum_{m_1=0}^{\infty}\sum_{z_1:|z_1-x|=m_1} p_{0,t_1}(x, z_1)\right)\;.$$

If we bound the last summation in parenthesis as follows (see also (2.80))

$$\begin{aligned}&\sum_{z_1:|z_1-x|=m_1} p_{0,t_1}(x, z_1)\\ &\quad\le\; \mathbb{P}\left(\exists\, v \in \mathbb{Z}^d : |v - x| = m_1 \text{ and } v \in \mathcal{G}_{[0,t_1]}(x)\right) \;\le\; e^{-\chi(t_1)m_1}\;, \end{aligned} \tag{2.82}$$

we obtain

$$\sum_{y:|y-x|=m}\sum_{z_1\in\mathbb{Z}^d}\cdots\sum_{z_n\in\mathbb{Z}^d} p_{0,t_1}(x,z_1)\cdots p_{t_n,t}(z_n,y)$$
$$\le \sum_{m_n=0}^{\infty}\cdots\sum_{m_2=0}^{\infty}\sum_{m_1=0}^{\infty} e^{-\chi(t-t_n)|m-m_n|}\ldots e^{-\chi(t_2-t_1)|m_2-m_1|}e^{-\chi(t_1)m_1},$$

hence the bound (2.75). An analogous argument yields (2.76).

We now prove item (b). For all initial conditions $\eta\in\{0,1\}^{\mathbb{Z}^d}$, for all realizations $\omega\in\Omega$, by the definition of dM_r in (2.35) and the bound $\eta_{r^-}(z)\le 1$ for all $z\in\mathbb{Z}^d$, we have that

$$\sum_{z_n\in\mathbb{Z}^d}\cdots\sum_{z_1\in\mathbb{Z}^d}\widehat{p}_{t_n,t}(x,z_n)\cdots\widehat{p}_{t_1,t_2}(z_2,z_1)\int_0^{t_1}\sum_{y\in\mathbb{Z}^d}\widehat{p}_{r,t_1}(z_1,y)\,d\left|M_r(\eta_{r^-}[\omega],y)[\omega]\right|$$

is bounded above by

$$\sum_{z_n\in\mathbb{Z}^d}\cdots\sum_{z_1\in\mathbb{Z}^d}\widehat{p}_{t_n,t}(x,z_n)\cdots\widehat{p}_{t_1,t_2}(z_2,z_1)\int_0^{t_1}\sum_{y\in\mathbb{Z}^d}\widehat{p}_{r,t_1}(z_1,y)\sum_{z:\{y,z\}\in E_d}c_r(\{y,z\})\,dr$$
$$+\sum_{z_n\in\mathbb{Z}^d}\cdots\sum_{z_1\in\mathbb{Z}^d}\widehat{p}_{t_n,t}(x,z_n)\cdots\widehat{p}_{t_1,t_2}(z_2,z_1)\int_0^{t_1}\sum_{y\in\mathbb{Z}^d}\widehat{p}_{r,t_1}(z_1,y)\sum_{z:\{y,z\}\in E_d}d\mathcal{N}_r(\{y,z\})[\omega]\,.$$

We estimate the two terms on the above r.h.s. separately. First, by Assumption 2.1 and (2.76), we obtain

$$\sum_{z_n\in\mathbb{Z}^d}\cdots\sum_{z_1\in\mathbb{Z}^d}\widehat{p}_{t_n,t}(x,z_n)\cdots\widehat{p}_{t_1,t_2}(z_2,z_1)\times$$
$$\times\int_0^{t_1}\sum_{y\in\mathbb{Z}^d}\widehat{p}_{r,t_1}(z_1,y)\sum_{z:\{y,z\}\in E_d}c_r(\{y,z\})\,dr\ \le\ 2Cd\mathfrak{a}t_1\sum_{m=0}^{\infty}e^{-\overline{\chi}m}\ <\ \infty\,.$$

For the second term, we observe that, for all $m\in\mathbb{N}$ and $y\in\mathbb{Z}^d$ with $|y-x|=m$, by independence of the Poisson processes over the bonds and Assumption 2.1, we have

$$\mathbb{P}\left(\sum_{z:\{y,z\}\in E_d}\mathcal{N}_{t_1}(\{y,z\})>m\right)\ \le\ \sum_{k=m+1}^{\infty}\frac{(2d\mathfrak{a}t_1)^k e^{-2d\mathfrak{a}t_1}}{k!}\ \le\ c\,\frac{(2d\mathfrak{a}t_1)^m}{m!}$$

for some constant $c > 0$ independent of m. As a consequence, we obtain

$$\sum_{m=0}^{\infty} \sum_{y:|y-x|=m} \mathbb{P}\left(\sum_{z:\{y,z\}\in E_d} \mathcal{N}_{t_1}(\{y,z\}) > m \right) \leq \sum_{m=0}^{\infty} \sum_{y:|y-x|=m} c\, \frac{(2d\mathfrak{a}t_1)^m}{m!} \leq \sum_{m=0}^{\infty} c\, \frac{(4d^2\mathfrak{a}t_1)^m}{m!} < \infty .$$

Hence, by a Borel-Cantelli argument, we can conclude that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists a constant $c[\omega] > 0$ for which

$$\sum_{z:\{y,z\}\in E_d} \mathcal{N}_{t_1}(\{y,z\})[\omega] \leq c[\omega]\, m \tag{2.83}$$

holds for all $m \in \mathbb{N}$ and $y \in \mathbb{Z}^d$ with $|y-x| = m$. Therefore, for $\mathbb{P}$-a.e. $\omega \in \Omega$, by (2.83), we get

$$\begin{aligned} &\sum_{z_n\in\mathbb{Z}^d} \cdots \sum_{z_1\in\mathbb{Z}^d} \widehat{p}_{t_n,t}(x,z_n) \cdots \widehat{p}_{t_1,t_2}(z_2,z_1) \times \\ &\qquad \times \int_0^{t_1} \sum_{y\in\mathbb{Z}^d} \widehat{p}_{r,t_1}(z_1,y) \sum_{z:\{y,z\}\in E_d} d\mathcal{N}_r(\{y,z\})[\omega] \\ &\leq \sum_{m=0}^{\infty} \left(\sup_{0\leq r\leq t_1} \sum_{y:|y-x|=m} \sum_{z_n\in\mathbb{Z}^d} \cdots \sum_{z_1\in\mathbb{Z}^d} \widehat{p}_{t_n,t}(x,z_n) \cdots \widehat{p}_{r,t_1}(z_1,y) \right) 2d\mathfrak{a}t_1 c[\omega] m . \end{aligned}$$

The term in parenthesis, by using (2.76), is exponentially small in $m \in \mathbb{N}$, yielding (2.77).

For item (c), for all initial conditions $\eta \in \{0,1\}^{\mathbb{Z}^d}$, we observe that, in view of (2.76), the bound $\eta_0(y) \leq 1$ for all $y \in \mathbb{Z}^d$ and (2.77), for $\mathbb{P}$-a.e. $\omega \in \Omega$, the infinite summation in (2.78) is absolutely convergent. More precisely, for all

$\varepsilon > 0$, there exists an integer $n_k = n_{[t_k,t_{k+1}],\varepsilon}(x)[\omega] > 0$ such that

$$\left| \sum_{y:|y-x|\geq n_k} \widehat{p}_{t_k,t_{k+1}}(x,y)\eta_{t_k}(y)[\omega] + \right.$$
$$\left. + \int_{t_k}^{t_{k+1}} \sum_{y:|y-x|\geq n_k} \widehat{p}_{r,t_{k+1}}(x,y)\, dM_r(\eta_{r^-}[\omega],y)[\omega] \right| < \varepsilon \,. \quad (2.84)$$

Once we have determined $n_k = n_{[t_k,t_{k+1}],\varepsilon}(x)[\omega]$ for which (2.84) is in force, let us define the finite subset $\mathcal{U}_k = \mathcal{U}_{[t_k,t_{k+1}],\varepsilon}(x)[\omega]$ of $\mathbb{Z}^d$ obtained as union of all active islands in $[t_k,t_{k+1}]$ which contain at least a site at a distance n_k (or less) from $x \in \mathbb{Z}^d$. Therefore, for all finite $\mathcal{V} \subset \mathbb{Z}^d$ containing $\mathcal{U}_k$, we have that the absolute value of

$$\left(\sum_{y\in\mathcal{V}} \widehat{p}_{t_k,t_{k+1}}(x,y)\eta_{t_k}(y)[\omega] + \int_{t_k}^{t_{k+1}} \sum_{y\in\mathcal{V}} \widehat{p}_{r,t_{k+1}}(x,y)\, dM_r(\eta_{r^-}[\omega],y)[\omega] \right) - \eta_{t_{k+1}}(x)[\omega]$$
$$= \sum_{y\in\mathcal{V}\setminus\mathcal{U}_k} \widehat{p}_{t_k,t_{k+1}}(x,y)\eta_{t_k}(y)[\omega] + \int_{t_k}^{t_{k+1}} \sum_{y\in\mathcal{V}\setminus\mathcal{U}_k} \widehat{p}_{r,t_{k+1}}(x,y)\, dM_r(\eta_{r^-}[\omega],y)[\omega]$$

(in this identity we used Lemma 2.15) is bounded above by

$$\sum_{y\in\mathcal{V}\setminus\mathcal{U}_k} \left(\widehat{p}_{t_k,t_{k+1}}(x,y)\eta_{t_k}(y)[\omega] + \int_{t_k}^{t_{k+1}} \widehat{p}_{r,t_{k+1}}(x,y)\, d\left|M_r(\eta_{r^-}[\omega],y)[\omega]\right| \right)$$
$$\leq \sum_{y:|y-x|\geq n_k} \left(\widehat{p}_{t_k,t_{k+1}}(x,y)\eta_{t_k}(y)[\omega] + \int_{t_k}^{t_{k+1}} \widehat{p}_{r,t_{k+1}}(x,y)\, d\left|M_r(\eta_{r^-}[\omega],y)[\omega]\right| \right),$$

where this last inequality follows from $\mathcal{V} \setminus \mathcal{U}_k \subset \{y \in \mathbb{Z}^d : |y-x| \geq n_k\}$. By (2.84) the proof is concluded. □

We are now ready to conclude the proof of Proposition 2.8.

Proof of Proposition 2.8. For all initial conditions $\eta \in \{0,1\}^{\mathbb{Z}^d}$, $\mathbb{P}$-a.e. realization $\omega \in \Omega$, $x \in \mathbb{Z}^d$ and $t > 0$, by applying Lemma 2.16(c) for all $k = 0,\ldots,n$, and reordering the summations thanks to Lemma 2.16(a)–(b),

$$\eta_t(x)[\omega] = \sum_{z_n\in\mathbb{Z}^d} \widehat{p}_{t_n,t}(x,z_n)\eta_{t_n}(z_n)[\omega] + \int_{t_n}^{t} \sum_{z_n\in\mathbb{Z}^d} \widehat{p}_{r,t}(x,z_n)\, dM_r(\eta_{r^-}[\omega],z_n)[\omega]$$

$$
\begin{aligned}
&= \sum_{z_n \in \mathbb{Z}^d} \cdots \sum_{z_1 \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \widehat{p}_{t_n,t}(x, z_n) \cdots \widehat{p}_{0,t_1}(z_1, y)\eta_0(y) \\
&+ \sum_{z_n \in \mathbb{Z}^d} \cdots \sum_{z_1 \in \mathbb{Z}^d} \widehat{p}_{t_n,t}(x, z_n) \cdots \widehat{p}_{t_1,t_2}(z_2, z_1) \int_0^{t_1} \sum_{y \in \mathbb{Z}^d} \widehat{p}_{r,t_1}(z_1, y)\, dM_r(\eta_{r^-}[\omega], y)[\omega] \\
&+ \ldots + \int_{t_n}^{t} \sum_{z_n \in \mathbb{Z}^d} \widehat{p}_{r,t}(x, z_n)\, dM_r(\eta_{r^-}[\omega], z_n)[\omega]\,,
\end{aligned}
$$

where the hidden terms are of the following form:

$$
\begin{aligned}
&\sum_{z_n \in \mathbb{Z}^d} \cdots \sum_{z_k \in \mathbb{Z}^d} \widehat{p}_{t_n,t}(x, z_n) \cdots \widehat{p}_{t_k,t_{k+1}}(z_{k+1}, z_k) \times \\
&\qquad \times \int_{t_{k-1}}^{t_k} \sum_{z_{k-1} \in \mathbb{Z}^d} \widehat{p}_{r,t_k}(z_k, z_{k-1})\, dM_r(\eta_{r^-}[\omega], z_{k-1})[\omega]\,, \qquad (2.85)
\end{aligned}
$$

with $k = 1, \ldots, n-1$. Hence, by using Chapman-Kolmogorov equation (2.25) for the backward transition probabilities, each term in (2.85) equals

$$
\int_{t_{k-1}}^{t_k} \sum_{z_{k-1} \in \mathbb{Z}^d} \widehat{p}_{r,t}(x, z_{k-1})\, dM_r(\eta_{r^-}[\omega], z_{k-1})[\omega]\,.
$$

Thus, by piecing together the above integrals for all $k = 0, \ldots, n$, we finally obtain (2.36).

□

2.a Time-inhomogeneous random walks: graphical construction and properties

In this appendix we collect some basic facts about time-inhomogeneous random walks. In particular, first we detail a dynamic version of Harris graphical construction [72] based on a percolation argument, which was summarized in Section 2.3.1. Then, we show that the random walks obtained are indeed Feller processes. We rely on the notation in Section 2.3.1.

2.a.1 Graphical construction of random walks

In this section we explain in detail the graphical construction which defines the percolation structure on which we build the families of forward and backward random walks given in Section 2.3.1. The main difficulty comes from the loss of space-time translation invariance due to the dynamic environment c. We deal with this difficulty by using Assumption 2.1 about the uniform boundedness of the conductances by $\mathfrak{a} > 0$. It will enable us to relate the percolation structure built from the inhomogeneous Poisson processes to a *bond percolation model in* $\mathbb{Z}^d$ [69]. Using the latter, we can construct the families of random walks by piecing together paths defined on sufficiently small time intervals which cover the whole positive real line.

Remark 2.17. *The uniform boundedness assumption could in principle be relaxed as long as results from bond percolation models transfer to our inhomogeneous setting. For examples of weaker assumptions on the conductances when* $d = 1$, *see e.g. [42, Lemma 2.1] or [10].*

Stochastic domination. Let $\{\mathfrak{N}.(\{x,y\}),\ \{x,y\} \in E_d\}$ be a family of i.i.d. Poisson processes of intensity $\mathfrak{a}$ defined on the probability space $(\Xi, \mathfrak{F}, \{\mathfrak{F}_t,\ t \geq 0\}, \mathfrak{P})$. By a thinning procedure (see e.g. [97]), we construct the family of inhomogeneous Poisson processes $\{\mathcal{N}.(\{x,y\}),\ \{x,y\} \in E_d\}$ given in (2.20) as follows: for all $n \in \mathbb{N}$ and $\{x,y\} \in E_d$, if we denote by $T_n(\{x,y\})$ the random time at which the n-th event of $\mathfrak{N}.(\{x,y\})$ has occurred, we erase this random time with probability

$$1 - \mathfrak{a}^{-1} \cdot c_{T_n(\{x,y\})}(\{x,y\}) \,.$$

We proceed analogously and independently for all random times $\{T_m(\{x,y\}),\ m \in \mathbb{N},\ \{x,y\} \in E_d\}$. We denote the probability space induced by $\{\mathfrak{N}.(\{x,y\}),\ \{x,y\} \in E_d\}$ and this thinning procedure by $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P})$. Then the remaining random points form the family of inhomogeneous Poisson process $\{\mathcal{N}.(\{x,y\}),\ \{x,y\} \in E_d\}$ introduced in (2.20), see also [97].

Given this construction, for all $\{x,y\} \in E_d$ and $t \geq s$, the number of Poissonian events of $\mathfrak{N}.(\{x,y\})$ in the time interval $[s,t]$ $\mathbb{P}$-a.s. dominates the number of events of $\mathcal{N}.(\{x,y\})$ in the same time interval.

Percolation and active islands. Let us first consider the family of i.i.d. Poisson processes $\{\mathfrak{N}.(\{x,y\}),\ \{x,y\} \in E_d\}$. For all $t \geq s$, we say that the bond

$\{x,y\} \in E_d$ is *open in* $[s,t]$ if

$$\mathfrak{N}_t(\{x,y\}) - \mathfrak{N}_{s^-}(\{x,y\}) \geq 1 .$$

We call the connected components of the subgraph consisting of sites of $\mathbb{Z}^d$ and bonds that are open in $[s,t]$ *open clusters in* $[s,t]$.

Bonds $\{x,y\} \in E_d$ are open in $[s,t]$ independently of each other with probability $\mathfrak{p}_{s,t}(\mathfrak{a}) = 1 - e^{\mathfrak{a}(t-s)}$. Hence, for all $t \geq s$, this induces a *bond percolation model in* $\mathbb{Z}^d$ *with density* $\mathfrak{p}_{s,t}(\mathfrak{a})$. As a consequence of the existence of a critical probability $\mathfrak{p}(d) \in (0,1]$ for bond percolation in $\mathbb{Z}^d$ (see [69, p. 13] for the case $d = 1$ and [69, Theorems 1.10–11] for the case $d \geq 2$), for any $d \geq 1$ there exists a value $h_c(d,\mathfrak{a}) > 0$ such that for all $s,t \in [0,\infty)$ with $0 < t-s < h_c(d,\mathfrak{a})$, the open clusters in $[s,t]$ are all finite $\mathfrak{P}$-almost surely. In particular, if we fix an interval size $\bar{h} < h_c(d,\mathfrak{a})$ and consider, for all $k \geq 0$, the interval $I_k = [k\bar{h}, (k+1)\bar{h}]$, we have

$$\mathfrak{P}(\text{for all } k \in \mathbb{N} \text{ all open clusters in } I_k \text{ are finite}) = 1 . \tag{2.86}$$

We turn now to the inhomogeneous Poisson processes $\{\mathcal{N}_\cdot(\{x,y\}),\ \{x,y\}\}$. A bond $\{x,y\} \in E_d$ such that

$$\mathcal{N}_t(\{x,y\}) - \mathcal{N}_{s^-}(\{x,y\}) \geq 1$$

is said to be *active in* $[s,t]$. Note that, due to the thinning procedure, active bonds are open, but not necessarily viceversa. Hence, the connected components of the subgraph consisting of sites of $\mathbb{Z}^d$ and bonds that are active in $[s,t]$, denoted as *active island in* $[s,t]$, are $\mathbb{P}$-a.s. finite if $0 < t-s < h_c(d,\mathfrak{a})$. Combining this with (2.86), we get

$$\mathbb{P}(\text{for all } k \in \mathbb{N} \text{ all active islands in } I_k \text{ are finite}) = 1 . \tag{2.87}$$

Random walks. Let us denote by $\mathcal{G}_{[s,t]}(x)$ the active island in $[s,t]$ containing the site $x \in \mathbb{Z}^d$. Then, by rephrasing (2.87), we have

$$\mathbb{P}(\text{for all } k \in \mathbb{N} \text{ and } x \in \mathbb{Z}^d,\ \mathcal{G}_{I_k}(x) \text{ is finite}) = 1 . \tag{2.88}$$

Moreover, $\mathbb{P}$-a.s. each $\mathcal{G}_{I_k}(x)$ contains at most finitely-many Poissonian marks (and no marks at the times $k\bar{h}$, for all $k \in \mathbb{N}$).

As a consequence, for $\mathbb{P}$-a.e. $\omega \in \Omega$, if we choose $s,t \in I_k$ for some $k \in \mathbb{N}$

with $0 < t - s < h_c(d, \mathfrak{a})$, the random walks' paths $\{X^x_{s,r}[\omega],\ x \in \mathbb{Z}^d,\ s \le r \le t\}$ and $\{\widehat{X}^y_{r,t}[\omega],\ y \in \mathbb{Z}^d,\ s \le r \le t\}$ are all simultaneously well-defined. Indeed, for all x and $y \in \mathbb{Z}^d$, it suffices to consider only finitely many Poissonian marks within $\mathcal{G}_{I_k}(x)[\omega]$ and $\mathcal{G}_{I_k}(y)[\omega]$, respectively, when performing the jumps (right-continuous for the forward random walks and left-continuous for the backward random walks). This procedure uniquely defines $X^x_{s,t}[\omega]$ and $\widehat{X}^y_{s,t}[\omega]$ simultaneously for all x and $y \in \mathbb{Z}^d$ (see also Section 2.3).

If $s, t \in [0, \infty)$ with $t - s > 0$ belong to different intervals $I_{k(s)}$ and $I_{k(t)}$, respectively, with $k(s) < k(t)$, then, by piecing together the well-defined paths

$$\{X^x_{s,r}[\omega],\ r \ge s \text{ and } r \in I_{k(s)}\}\ ,\ \ldots\ ,\ \{X^x_{r,t}[\omega],\ r \le t \text{ and } r \in I_{k(t)}\}$$

in ascending order w.r.t. $k \in \{k(s), \ldots, k(t)\}$ and

$$\{\widehat{X}^y_{r,t}[\omega],\ r \le t \text{ and } r \in I_{k(t)}\}\ ,\ \ldots\ ,\ \{\widehat{X}^y_{s,r}[\omega],\ r \ge s \text{ and } r \in I_{k(s)}\}$$

in descending order w.r.t. $k \in \{k(s), \ldots, k(t)\}$, we obtain $X^x_{s,t}[\omega]$ and $\widehat{X}^y_{s,t}[\omega]$ for all x and $y \in \mathbb{Z}^d$.

The property of the inhomogeneous Poisson processes $\{\mathcal{N}_\cdot(\{z,v\}),\ \{z,v\} \in E_d\}$ for which past and future are independent conditioned on the present state and our construction rules of the random walks imply that the processes $\{X^x_{s,t},\ t \ge s\}$ and $\{\widehat{X}^y_{s,t},\ s \le t\}$ for all x and $y \in \mathbb{Z}^d$ are Markovian w.r.t. the induced natural filtrations. This justifies the introduction in Section 2.3 of the transition probabilities $\{p_{s,t}(x,y),\ x, y \in \mathbb{Z}^d\}$ and $\{\widehat{p}_{s,t}(y,x),\ x, y \in \mathbb{Z}^d\}$, as well as of the semigroups $\{S_{s,t},\ t \ge s\}$ and $\{\widehat{S}_{s,t},\ s \le t\}$.

2.a.2 Feller transition semigroups and generators

We study properties of the transition semigroups $\{S_{s,t},\ t \ge s\}$ and $\{\widehat{S}_{s,t},\ s \le t\}$ introduced in (2.28) and (2.29) and their associated infinitesimal generators solving the associated Kolmogorov forward and backward equations as in (2.90) and (2.91), which turn out to be $\{A_t,\ t \ge 0\}$ and $\{A_{t^-},\ t \ge 0\}$ defined in (2.2) and (2.11). Indeed, for all $x, y \in \mathbb{Z}^d$ with the convention $c_t(\{x,y\}) = 0$ if $\{x,y\} \notin E_d$, we have

$$\lim_{t \downarrow s} \frac{p_{s,t}(x,y)}{t-s} = c_{s^+}(\{x,y\}) \quad \text{and} \quad \lim_{s \uparrow t} \frac{\widehat{p}_{s,t}(x,y)}{t-s} = c_{t^-}(\{x,y\})\,, \tag{2.89}$$

where the following limits $c_{s^\pm}(\{x,y\}) = \lim_{h\downarrow 0} c_{s\pm h}(\{x,y\})$ exist and, as the conductances are assumed to be càdlàg, $c_{s^+}(\{x,y\}) = c_s(\{x,y\})$.

In what follows, for a differentiable function $\phi : (-\infty,\infty) \to (\mathcal{X}, d)$, with $(\mathcal{X}, d)$ a metric space, we define

$$\partial_\tau \phi(\tau) = \lim_{h\downarrow 0} \tfrac{1}{h}(\phi(\tau+h) - \phi(\tau)) \quad \text{and} \quad \partial_{\tau^-}\phi(\tau) = \lim_{h\downarrow 0} \tfrac{1}{h}(\phi(\tau) - \phi(\tau - h)) \,.$$

Moreover, $C_0(\mathbb{R}^d)$ denotes the Banach space of real-valued continuous functions on $\mathbb{R}^d$ vanishing at infinity endowed with the sup norm $\|\cdot\|_\infty$. By $C_0(\mathbb{Z}^d)$ we denote the space of functions obtained as restrictions to $\mathbb{Z}^d$ of functions in $C_0(\mathbb{R}^d)$.

The proofs of the next two propositions, which follow from Assumption 2.1, are left to the reader. For notational convenience, we extend the definitions of conductances, transition semigroups and generators to negative times.

Proposition 2.18 (TRANSITION SEMIGROUPS)**.** *For all $f \in C_0(\mathbb{Z}^d)$ and $s \le r \le t$, the following hold true:*

(a) Operators on $C_0(\mathbb{Z}^d)$. $S_{s,t} f \in C_0(\mathbb{Z}^d)$ *and* $\widehat{S}_{s,t} f \in C_0(\mathbb{Z}^d)$.

(b) Identity. $S_{t,t} f = \widehat{S}_{t,t} f = f$.

(c) Positivity. *If* $f \ge 0$, *then* $S_{s,t} f \ge 0$ *and* $\widehat{S}_{s,t} f \ge 0$.

(d) Contraction. $\|S_{s,t} f\|_\infty \le \|f\|_\infty$ *and* $\|\widehat{S}_{s,t} f\|_\infty \le \|f\|_\infty$.

(e) Conservativity. $S_{s,t} 1 = 1$ *and* $\widehat{S}_{s,t} 1 = 1$.

(f) Chapman-Kolmogorov equation.

$$S_{s,r} S_{r,t} f = S_{s,t} f \qquad\qquad \widehat{S}_{r,t} \widehat{S}_{s,r} f = \widehat{S}_{s,t} f \ .$$

(g) Strong continuity. *For all* $T > 0$, $\lim_{h\downarrow 0} \sup_{0\le s\le T} \|S_{s,s+h} f - f\|_\infty = 0$.

Proposition 2.19 (INFINITESIMAL GENERATORS)**.** *For all $f \in C_0(\mathbb{Z}^d)$ and $t \in (-\infty,\infty)$, the following hold true:*

(a) Domain. $A_t f$ *and* $A_{t^-} f \in C_0(\mathbb{Z}^d)$.

(b) Kolmogorov forward and backward equations.

$$\begin{aligned} \partial_t S_{s,t} f &= S_{s,t} A_t f & \qquad \partial_t \widehat{S}_{s,t} f &= A_t \widehat{S}_{s,t} f \\ \partial_{t^-} S_{s,t} f &= S_{s,t} A_{t^-} f & \qquad \partial_{t^-} \widehat{S}_{s,t} f &= A_{t^-} \widehat{S}_{s,t} f \end{aligned} \tag{2.90}$$

and

$$\begin{aligned} \partial_s S_{s,t} f &= -A_s S_{s,t} f & \qquad \partial_s \widehat{S}_{s,t} f &= -\widehat{S}_{s,t} A_s f \\ \partial_{s^-} S_{s,t} f &= -A_{s^-} S_{s,t} f & \qquad \partial_{s^-} \widehat{S}_{s,t} f &= -\widehat{S}_{s,t} A_{s^-} f \,, \end{aligned} \tag{2.91}$$

where derivatives are meant w.r.t. $\| \cdot \|_\infty$.

Feller property. We now consider the space-time processes [139, Section 8.5.5]

$$\{(X^x_{s,s+\cdot}, s+\cdot),\ x \in \mathbb{Z}^d,\ s \in (-\infty,\infty)\} \tag{2.92}$$

$$\{(\widehat{X}^y_{t-\cdot,t}, t-\cdot),\ y \in \mathbb{Z}^d,\ t \in (-\infty,\infty)\} \tag{2.93}$$

associated to forward and backward random walks, respectively. These processes are time-homogeneous Markov processes on the state space $\mathbb{Z}^d \times (-\infty,\infty)$ with infinitesimal generators $\mathcal{B}$ and $\widehat{\mathcal{B}}$ given by

$$\mathcal{B} f(x,s) = A_s f(x,s) + \partial_s f(x,s) \tag{2.94}$$

$$\widehat{\mathcal{B}} f(x,s) = A_{s^-} f(x,s) + \partial_s f(x,s) \,, \tag{2.95}$$

for all $f \in C_0(\mathbb{Z}^d \times (-\infty,\infty))$ [14], [121, Chapter III.2], [139, Section 8.5.5]. Hence, by passing to this formulation, Propositions 2.18 and 2.19 guarantee that the forward and backward random walks are Feller processes, i.e. in the sense that their associated space-time processes are Feller processes as in [81, Chapter 19. Conditions (F1)–(F3)].

2.b Forward and backward invariance principle

As announced in Remark 2.4, we prove that an invariance principle for the forward random walks (2.13) holds if and only if an analogous result holds for the backward random walks (2.15). For this, next to the two equivalent formulations (A) and (B) of the invariance principle for the forward random walks in Theorem 2.11, we add a third one below:

(C) Uniform convergence of infinitesimal generators. *For all* $T > 0$ *and* $G \in \mathscr{S}(\mathbb{R}^d)$, *there exists a sequence* $\{G_N,\ N \in \mathbb{N}\}$ *with* $G_N \in C_0(\frac{\mathbb{Z}^d}{N})$ *such that*

$$\sup_{x\in\mathbb{Z}^d} \left|G_N(\tfrac{x}{N}) - G(\tfrac{x}{N})\right| \underset{N\to\infty}{\longrightarrow} 0$$

and

$$\sup_{0\le t\le T}\ \sup_{x\in\mathbb{Z}^d} \left|A^N_{tN^2} G_N(\tfrac{x}{N}) - \mathcal{A}^{\Sigma} G(\tfrac{x}{N})\right| \underset{N\to\infty}{\longrightarrow} 0\,,$$

where

$$A^N_t G(\tfrac{x}{N}) \;=\; N^2 \cdot A_t G(\tfrac{\cdot}{N})(x)\,, \qquad x \in \mathbb{Z}^d\,,$$

and $\mathcal{A}^{\Sigma} = \frac{1}{2}\nabla(\Sigma\cdot\nabla)$ *is the infinitesimal generator of the Brownian motion* $\{B^{\Sigma}_t,\ t \ge 0\}$ *with covariance matrix* Σ.

The proof of the equivalence of (A), (B) and (C) can be found in [81, Theorem 19.25], [40, Chapter 1. Theorem 6.1] after considering the generator (2.94) of the associated space-time process. The analogous condition for the backward random walks reads as follows:

($\widehat{\mathrm{C}}$) Uniform convergence of infinitesimal generators. *For all* $T > 0$ *and* $G \in \mathscr{S}(\mathbb{R}^d)$, *there exists a sequence* $\{G_N,\ N \in \mathbb{N}\}$ *with* $G_N \in C_0(\frac{\mathbb{Z}^d}{N})$ *such that*

$$\sup_{x\in\mathbb{Z}^d} \left|G_N(\tfrac{x}{N}) - G(\tfrac{x}{N})\right| \underset{N\to\infty}{\longrightarrow} 0$$

and

$$\sup_{0\le t\le T}\ \sup_{x\in\mathbb{Z}^d} \left|A^N_{(tN^2)^-} G_N(\tfrac{x}{N}) - \mathcal{A}^{\Sigma} G(\tfrac{x}{N})\right| \underset{N\to\infty}{\longrightarrow} 0\,,$$

where the notation is as in (C).

As a consequence, if

$$\sup_{0\le t\le T}\ \sup_{x\in\mathbb{Z}^d} \left|A^N_{tN^2} G(\tfrac{x}{N}) - A^N_{(tN^2)^-} G(\tfrac{x}{N})\right| \underset{N\to\infty}{\longrightarrow} 0 \qquad (2.96)$$

holds for all $G \in \mathscr{S}(\mathbb{R}^d)$, then, by triangle inequality, (C) and ($\widehat{\mathrm{C}}$) are equivalent. In turn, the invariance principles in Theorem 2.3(b) would also be equivalent.

We end this section by showing that in our context (C) and ($\widehat{\mathrm{C}}$) are always equivalent, even without relying on (2.96).

Proposition 2.20. *Under Assumption 2.1, for all* $f \in C_0(\mathbb{Z}^d)$ *and* $t \in (-\infty, \infty)$, *we have that*

$$\lim_{r \uparrow t} \sup_{x \in \mathbb{Z}^d} \left| A_{t^-} f(x) - A_r f(x) \right| = 0 . \tag{2.97}$$

As a consequence, (C) *holds if and only if* $(\widehat{\mathrm{C}})$ *holds.*

Proof. We start with the proof of (2.97). Let $\ell \in \mathbb{N}$. By Assumption 2.1, we obtain

$$\begin{aligned}\sup_{x \in \mathbb{Z}^d} \left| A_{t^-} f(x) - A_r f(x) \right| \le\ & C_f \cdot \sup_{|x| \le \ell} \max_{y : |y-x|=1} |c_{t^-}(\{x,y\}) - c_r(\{x,y\})| \\ & + 4\mathfrak{a} d \cdot \sup_{|x| > \ell} \max_{y : |y-x|=1} |f(y) - f(x)| ,\end{aligned}$$

where C_f is a constant depending only on $f \in C_0(\mathbb{Z}^d)$. For any $\varepsilon > 0$, we choose $\ell > 0$ large enough and, consequently, $\delta > 0$ small enough so that

$$\sup_{|x| > \ell} \max_{y : |y-x|=1} |f(y) - f(x)| \le \frac{\varepsilon}{8\mathfrak{a} d}$$

and

$$\sup_{0 \le r \le t : |r-t| < \delta} \sup_{|x| \le \ell} \max_{y : |y-x|=1} |c_{t^-}(\{x,y\}) - c_r(\{x,y\})| \le \frac{\varepsilon}{2C_f} .$$

This proves (2.97). Now assume (C). Then, for all $N \in \mathbb{N}$, we have

$$\begin{aligned}&\sup_{0 \le t \le T} \sup_{x \in \mathbb{Z}^d} \left| A^N_{(tN^2)^-} G_N(\tfrac{x}{N}) - \mathcal{A}^\Sigma G(\tfrac{x}{N}) \right| \\ &\quad \le \sup_{0 \le t \le T} \sup_{x \in \mathbb{Z}^d} \left| A^N_{(tN^2)^-} G_N(\tfrac{x}{N}) - A^N_{rN^2} G_N(\tfrac{x}{N}) \right| + \sup_{x \in \mathbb{Z}^d} \left| A^N_{rN^2} G_N(\tfrac{x}{N}) - \mathcal{A}^\Sigma G(\tfrac{x}{N}) \right|\end{aligned}$$

for all $0 \le r \le T$. By (2.97), for all $\varepsilon > 0$, $N \in \mathbb{N}$ and $0 \le t \le T$, we choose $0 \le r^N_{\varepsilon,t} \le T$ such that

$$\sup_{x \in \mathbb{Z}^d} \left| A^N_{(tN^2)^-} G_N(\tfrac{x}{N}) - A^N_{r^N_{\varepsilon,t} N^2} G_N(\tfrac{x}{N}) \right| < \varepsilon . \tag{2.98}$$

The uniform bound (2.98) and (C) give $(\widehat{\mathrm{C}})$. The converse implication is obtained analogously. □

2.c Tightness criterion

We present a tightness criterion for processes in the Skorokhod space $\mathtt{D}([0,T],\mathbb{R})$ of real-valued càdlàg functions on $[0,T]$ (see e.g. [9]). This criterion relies on the notion of *uniform conditional stochastic continuity* of a process [137, Appendix A]. The study of this property allows to extract information on the modulus of continuity of the trajectories.

By following closely the argument in [137, Appendix A], we get a quantitative estimate for the modulus of continuity leading to a sufficient condition for tightness. To the best of our knowledge, this strategy has not been remarked before with this purpose, therefore we provide below a detailed proof.

As a first step, we specify the topological setting as in [9].

Definition 2.21 (MODULUS OF CONTINUITY). *Given $z:[0,T]\to\mathbb{R}$ a bounded function, for all $\delta>0$, the δ-*modulus of continuity *$w_z'''(\delta)$ ([9, Problem 12.4, p. 137]) for the function z is given by*

$$\begin{aligned} &w_z'''(\delta)\\ &=\max\left\{\sup_{\substack{0\le s\le t\le T\\ t-s<\delta}}\inf_{r\in(s,t)}\max\left\{w_z(s,r),w_z(r,t)\right\},\ |z_\delta-z_0|,\ |z_{T^-}-z_{T-\delta}|\right\}, \end{aligned}$$

where

$$w_z(s,t)=\sup_{s\le s'\le t'\le t}|z_{t'}-z_{s'}|\ .$$

Roughly speaking, given $\delta>0$, the δ-modulus of continuity $w_z'''(\delta)$ (referred to as $\overline{w}_z(\delta)$ in [137, Appendix A.2]) "allows" for one jump in intervals of size at most δ. We refer to [9, Chapter 3. Section 12] for further details on w''' and its relation to the space $\mathtt{D}([0,T],\mathbb{R})$. Note that our definition of $w_z'''(\delta)$ slightly differs from the one given in [9, Problem 12.4, p. 137] as we include also information about z near 0 and T, i.e. $|z_\delta-z_0|$ and $|z_{T^-}-z_{T-\delta}|$.

In what follows, we state a general tightness criterion, namely Theorem 2.22, in $\mathtt{D}([0,T],\mathbb{R})$ in terms of the modulus of continuity w''' introduced above. We remark that Theorem 2.22 below is a rewriting of Theorem 13.2, the corresponding Corollary and (13.8) to be found at pp. 139–141 of [9]. There the author refers to moduli of continuity (w' and w'' defined in (12.6), p. 122, and (12.27), p. 131, respectively) which are different, though "equivalent" (cf. [9, (12.31)–(12.32), p. 132] and [9, Problem 12.4, p. 137], respectively), to the one we employ.

Theorem 2.22 ([9])**.** *A family of probability measures* $\{\mathtt{P}^N,\ N \in \mathbb{N}\}$ *on* $\mathtt{D}([0,T],\mathbb{R})$*, whose canonical coordinate processes are denoted by* $\{\mathtt{Z}^N_\cdot,\ N \in \mathbb{N}\}$*, is* tight *if the following conditions hold:*

(T1) *For all* t *in a dense subset of* $[0,T]$ *containing* T*,*

$$\lim_{m\to\infty} \limsup_{N\to\infty} \mathtt{P}_N \left(|\mathtt{Z}^N_t| > m \right) = 0 .$$

(T2) *For all* $\varepsilon > 0$*,*

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathtt{P}_N \left(w'''_{\mathtt{Z}^N}(\delta) > \varepsilon \right) = 0 .$$

In Theorem 2.24, we will present a condition alternative to (T2) on the uniform control of the modulus of continuity w'''. First we need Theorem 2.23, which is a slight modification of [137, Theorem A.6]. Indeed, the proof of Theorem 2.23 follows closely the one of [137, Theorem A.6]. Only in the last part, the two proofs differ yielding a different upper bound (2.100). For the sake of completeness, though, we include the whole proof at the end of this section.

Theorem 2.23 ([137, THEOREM A.6])**.** *Let* $\{\mathtt{Z}_t,\ t \geq 0\}$ *be a continuous-time real-valued stochastic process, whose associated distribution and filtration are denoted by* $\mathtt{P}$ *and* $\{\mathtt{F}_t,\ t \geq 0\}$*, respectively.*

Fix $T > 0$ *and* $\varepsilon > 0$ *and suppose that there exist a positive value* $h_\varepsilon > 0$ *and a deterministic function* $\psi_\varepsilon : [0, h_\varepsilon] \to [0, 1)$ *such that:*

(i) ψ_ε *is non-decreasing and* $\psi_\varepsilon(h) \to 0$ *as* $h \to 0$ *.*

(ii) *For all* $h \in [0, h_\varepsilon]$ *and* $t \in [0,T]$*, we have*

$$\mathtt{P}\left(|\mathtt{Z}_{t+h} - \mathtt{Z}_t| > \varepsilon \,|\, \mathtt{F}_t \right) \leq \psi_\varepsilon(h) , \qquad \mathtt{P}\text{-a.s.} . \tag{2.99}$$

Then the following bound on the modulus of continuity $w'''_{\mathtt{Z}}$

$$\mathtt{P}\left(w'''_{\mathtt{Z}}(h) > 4\varepsilon \right) \leq (k+1) \cdot \frac{\psi_\varepsilon(h)}{1-\psi_\varepsilon(h)} + 2 \cdot \frac{\psi_\varepsilon(\frac{2T}{k})}{1-\psi_\varepsilon(\frac{2T}{k})} \tag{2.100}$$

holds for all $h \in [0, h_\varepsilon]$ *and* $k > k_\varepsilon = 2T/h_\varepsilon$*.*

We remark that, if (2.100) holds for all $\varepsilon > 0$, then the process $\{\mathtt{Z}_t,\ t \in [0,T]\}$ ca be realized in the Skorokhod space $\mathtt{D}([0,T],\mathbb{R})$, [137, Theorem A.6].

Theorem 2.24. *Let* $\{\mathtt{P}^N,\ N \in \mathbb{N}\}$ *and* $\{\mathtt{Z}^N_\cdot, N \in \mathbb{N}\}$ *be as in Theorem 2.22. Let, for all* $N \in \mathbb{N}$, $\{\mathtt{Z}^N_t,\ 0 \le t \le T\}$ *be adapted to the filtration* $\{\mathtt{F}^N_t,\ 0 \le t \le T\}$.

Fix $\varepsilon > 0$ *and suppose that there exist values* $h_\varepsilon > 0$ *and* $N_\varepsilon \in \mathbb{N}$ *such that for all* $N \ge N_\varepsilon$ *there exist deterministic functions* $\psi^N_\varepsilon, \psi_\varepsilon : [0, h_\varepsilon] \to [0, 1)$ *and non-negative values* ϕ^N_ε *satisfying the following properties:*

(i) $\psi^N_\varepsilon, \psi_\varepsilon$ *are non-decreasing and* $\psi^N_\varepsilon(h), \psi_\varepsilon(h) \to 0$ *as* $h \to 0$.

(ii) *For all* $h \in [0, h_\varepsilon]$, $t \in [0, T]$ *and* $N \ge N_\varepsilon$, *we have*

$$\mathtt{P}^N\left(|\mathtt{Z}^N_{t+h} - \mathtt{Z}^N_t| > \varepsilon \mid \mathtt{F}^N_t\right) \le \psi^N_\varepsilon(h), \quad \mathtt{P}^N\text{-a.s.}\ .$$

(iii) $\phi^N_\varepsilon \to 0$ *as* $N \to \infty$.

(iv) *For all* $h \in [0, h_\varepsilon]$ *and* $N \ge N_\varepsilon$, *we have* $\psi^N_\varepsilon(h) \le \psi_\varepsilon(h) + \phi^N_\varepsilon < 1$.

Then we obtain

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathtt{P}^N\left(w'''_{\mathtt{Z}^N}(\delta) > 4\varepsilon\right) = 0\ . \tag{2.101}$$

If this is true for all $\varepsilon > 0$, *then condition* (T2) *in Theorem 2.22 holds for* $\{\mathtt{Z}^N_\cdot,\ N \in \mathbb{N}\}$.

Proof. Fix $\varepsilon > 0$. Due to (i) and (ii) we can apply Theorem 2.23 to get an estimate for $\mathtt{P}^N(w'''_{\mathtt{Z}^N}(h) > 4\varepsilon)$ of the form (2.100) with ψ^N_ε. By using, in addition, item (iv), we obtain the bound

$$\mathtt{P}^N\left(w'''_{\mathtt{Z}^N}(h) > 4\varepsilon\right) \le (k+1) \cdot \frac{\psi_\varepsilon(h) + \phi^N_\varepsilon}{1 - \psi_\varepsilon(h) - \phi^N_\varepsilon} + 2 \cdot \frac{\psi_\varepsilon(\frac{2T}{k}) + \phi^N_\varepsilon}{1 - \psi_\varepsilon(\frac{2T}{k}) - \phi^N_\varepsilon},$$

which is valid for all $h \in [0, h_\varepsilon]$, $N \ge N_\varepsilon$ and $k > 2T/h_\varepsilon$. Now observe that, by (iii), we have

$$\limsup_{N \to \infty} \mathtt{P}^N\left(w'''_{\mathtt{Z}^N}(h) > 4\varepsilon\right) \le (k+1) \cdot \frac{\psi_\varepsilon(h)}{1 - \psi_\varepsilon(h)} + 2 \cdot \frac{\psi_\varepsilon(\frac{2T}{k})}{1 - \psi_\varepsilon(\frac{2T}{k})}\ . \tag{2.102}$$

We are left to show that the r.h.s. in (2.102) vanishes as $h \to 0$. We use the fact that $\psi_\varepsilon(h) \to 0$ as $h \to 0$. First observe that, for any arbitrary small $\sigma > 0$, due to (i) there exists $k_{\varepsilon,\sigma} > k_\varepsilon = 2T/h_\varepsilon$ such that

$$2 \cdot \frac{\psi_\varepsilon(\frac{2T}{k_{\varepsilon,\sigma}})}{1-\psi_\varepsilon(\frac{2T}{k_{\varepsilon,\sigma}})} \;\le\; \frac{\sigma}{2}\,.$$

We can then choose $h_{\varepsilon,\sigma} \in (0, h_\varepsilon)$ so that to control also the first term on the r.h.s. of (2.102). Namely, we can pick a value $h_{\varepsilon,\sigma}$ such that we have

$$(k_{\varepsilon,\sigma}+1) \cdot \frac{\psi_\varepsilon(h)}{1-\psi_\varepsilon(h)} \;\le\; \frac{\sigma}{2}$$

for all $h \in [0, h_{\varepsilon,\sigma}]$. Choosing σ sufficiently small yields (2.101). □

Proof of Theorem 2.23. We follow here [137, Theorem A.6]. We fix $\varepsilon > 0$, $\tau_{\varepsilon,0} = 0$ and define $\tau_{\varepsilon,1}$ as the first time $|Z_t - Z_0|$ exceeds 2ε, $\tau_{\varepsilon,1}+\tau_{\varepsilon,2}$ as the first time $|Z_t - Z_{\tau_{\varepsilon,1}}|$ does and so on, up to reach time T and with the convention that, if $\tau_{\varepsilon,1}+\ldots+\tau_{\varepsilon,n} > T$, we set $\tau_{\varepsilon,1}+\ldots+\tau_{\varepsilon,n}$ equal to $T+1$. As a consequence of these definitions, if we define $\sigma_{\varepsilon,n} = \tau_{\varepsilon,0}+\tau_{\varepsilon,1}+\ldots+\tau_{\varepsilon,n}$, we have: for all $n \in \mathbb{N}_0$,

$$\mathrm{P}\left(\sup_{\sigma_{\varepsilon,n}\le s\le t<\sigma_{\varepsilon,n+1}} |Z_t - Z_s| \le 4\varepsilon\right) \;=\; 1\,,$$

and, for all $h \in [0, h_\varepsilon]$,

$$\mathrm{P}\left(\sup_{0\le h'\le h} |Z_{\sigma_{\varepsilon,n}+h'} - Z_{\sigma_{\varepsilon,n}}| > 2\varepsilon \,\middle|\, \mathrm{F}_{\sigma_{\varepsilon,n}}\right) \;=\; \mathrm{P}\left(\tau_{\varepsilon,n+1} \le h \,|\, \mathrm{F}_{\sigma_{\varepsilon,n}}\right)\,. \tag{2.103}$$

We rewrite the probability in (2.103) as follows:

$$\begin{aligned}\mathrm{P}\left(\tau_{\varepsilon,n+1} \le h \,|\, \mathrm{F}_{\sigma_{\varepsilon,n}}\right) \;=\;& \mathrm{P}\left(\tau_{\varepsilon,n+1} \le h,\, |Z_{\sigma_{\varepsilon,n}+h} - Z_{\sigma_{\varepsilon,n}}| \le \varepsilon \,|\, \mathrm{F}_{\sigma_{\varepsilon,n}}\right)\\ &+ \mathrm{P}\left(\tau_{\varepsilon,n+1} \le h,\, |Z_{\sigma_{\varepsilon,n}+h} - Z_{\sigma_{\varepsilon,n}}| > \varepsilon \,|\, \mathrm{F}_{\sigma_{\varepsilon,n}}\right)\,.\end{aligned} \tag{2.104}$$

Concerning the first term on the r.h.s. in (2.104), we have the following upper bound (recall that $\sigma_{\varepsilon,n+1} = \sigma_{\varepsilon,n} + \tau_{\varepsilon,n+1}$):

$$\mathrm{P}\left(\tau_{\varepsilon,n+1} \le h,\, |Z_{\sigma_{\varepsilon,n}+h} - Z_{\sigma_{\varepsilon,n+1}}| > \varepsilon \,|\, \mathrm{F}_{\sigma_{\varepsilon,n}}\right)\,,$$

which, in turn, rewrites as follows:

$$\mathtt{E}\left[\mathtt{E}\left[\mathbf{1}_{\{\tau_{\sigma_{\varepsilon,n+1}}\le h\}}\mathbf{1}_{\{|\mathtt{Z}_{\sigma_{\varepsilon,n}+h}-\mathtt{Z}_{\sigma_{\varepsilon,n+1}}|>\varepsilon\}}\,\middle|\,\mathtt{F}_{\sigma_{\varepsilon,n+1}}\right]\,\middle|\,\mathtt{F}_{\sigma_{\varepsilon,n}}\right]$$
$$= \mathtt{E}\left[\mathbf{1}_{\{\tau_{\sigma_{\varepsilon,n+1}}\le h\}}\,\mathtt{E}\left[\mathbf{1}_{\{|\mathtt{Z}_{\sigma_{\varepsilon,n}+h}-\mathtt{Z}_{\sigma_{\varepsilon,n+1}}|>\varepsilon\}}\,\middle|\,\mathtt{F}_{\sigma_{\varepsilon,n+1}}\right]\,\middle|\,\mathtt{F}_{\sigma_{\varepsilon,n}}\right].$$

By (2.99) - which holds true also when considering σ-fields associated to stopping times being the bound (2.99) uniform in time - we obtain, $\mathtt{P}$-a.s.,

$$\begin{aligned}\mathtt{P}\left(\tau_{\varepsilon,n+1}\le h,\ |\mathtt{Z}_{\sigma_{\varepsilon,n}+h}-\mathtt{Z}_{\sigma_{\varepsilon,n}}|\le\varepsilon\,|\,\mathtt{F}_{\sigma_{\varepsilon,n}}\right)&\\ \le \mathtt{E}\left[\mathbf{1}_{\{\tau_{\sigma_{\varepsilon,n+1}}\le h\}}\,\psi_\varepsilon(h-\tau_{\varepsilon,n+1})\,|\,\mathtt{F}_{\sigma_{\varepsilon,n}}\right]&\\ \le \psi_\varepsilon(h)\,\mathtt{P}\left(\tau_{\sigma_{\varepsilon,n+1}}\le h\,|\,\mathtt{F}_{\sigma_{\varepsilon,n}}\right)\,,&\end{aligned}\tag{2.105}$$

where in the last inequality we used the monotonicity of ψ_ε ($\psi_\varepsilon(h')\le\psi_\varepsilon(h'')$ if $h'\le h''$). Analogously by (2.99) and uniformity in time of this bound, for the second term on the r.h.s. in (2.104), we have, for all $h\in[0,h_\varepsilon]$ and $\mathtt{P}$-a.s. as a consequence of (2.99),

$$\begin{aligned}\mathtt{P}\left(\tau_{\varepsilon,n+1}\le h,\ |\mathtt{Z}_{\sigma_{\varepsilon,n}+h}-\mathtt{Z}_{\sigma_{\varepsilon,n}}|>\varepsilon\,|\,\mathtt{F}_{\sigma_{\varepsilon,n}}\right)&\\ \le \mathtt{P}\left(|\mathtt{Z}_{\sigma_{\varepsilon,n}+h}-\mathtt{Z}_{\sigma_{\varepsilon,n}}|>\varepsilon\,|\,\mathtt{F}_{\sigma_{\varepsilon,n}}\right)\le\psi_\varepsilon(h)\,.&\end{aligned}\tag{2.106}$$

As a consequence of (2.103), (2.104), (2.105) and (2.106), we obtain, for all $h\in[0,h_\varepsilon]$ and $n\in\mathbb{N}_0$,

$$\mathtt{P}\left(\sup_{0\le h'\le h}|\mathtt{Z}_{\sigma_{\varepsilon,n}+h'}-\mathtt{Z}_{\sigma_{\varepsilon,n}}|>2\varepsilon\,\middle|\,\mathtt{F}_{\sigma_{\varepsilon,n}}\right)\le\frac{\psi_\varepsilon(h)}{1-\psi_\varepsilon(h)}\,,\quad \mathtt{P}\text{-a.s.}\,.\tag{2.107}$$

Recall Definition 2.21 of the modulus of continuity w'''. For any choice of $k\in\mathbb{N}$, the probability $\mathtt{P}\left(w'''_{\mathtt{Z}}(h)>4\varepsilon\right)$ can be bounded above by

$$\mathtt{P}\left(w'''_{\mathtt{Z}}(h)>4\varepsilon,\ \sigma_{\varepsilon,k}>T,\ \min\{\tau_{\varepsilon,1},\dots,\tau_{\varepsilon,k}\}>h,\ |\mathtt{Z}_{T^-}-\mathtt{Z}_{T-h}|\le4\varepsilon\right)\tag{2.108}$$
$$+\ \mathtt{P}\left(\sigma_{\varepsilon,k}\le T\right)\ +\ \mathtt{P}\left(\min\{\tau_{\varepsilon,1},\dots,\tau_{\varepsilon,k}\}\le h\right)\ +\ \mathtt{P}\left(|\mathtt{Z}_{T^-}-\mathtt{Z}_{T-h}|>4\varepsilon\right)\,.\tag{2.109}$$

The probability in (2.108) vanishes. Indeed, if the events

$$\{\sigma_{\varepsilon,k}>T\}\ \text{ and }\ \{\min\{\tau_{\varepsilon,1},\dots,\tau_{\varepsilon,k}\}>h\}$$

occur, then necessarily in any subinterval of size h of $[0,T]$ there can be at most one $\sigma_{\varepsilon,\ell}$, for some $0 \leq \ell \leq k$, making, together with

$$\{|Z_{T^-} - Z_{T-h}| \leq 4\varepsilon\} \ ,$$

the event $\{w''_{Z}(h) > 4\varepsilon\}$ impossible.

Now we estimate each term in (2.109) and consider $h \in [0, h_\varepsilon]$. For the second one, by (2.103) and (2.107), we get

$$\begin{aligned} &\mathrm{P}\left(\min\{\tau_{\varepsilon,1},\dots,\tau_{\varepsilon,k}\} \leq h\right) \\ &\qquad \leq \sum_{\ell=1}^{k} \mathrm{E}\left[\mathrm{P}\left(\tau_{\varepsilon,\ell} \leq h \,|\, \mathrm{F}_{\sigma_{\varepsilon,\ell-1}}\right)\right] \ \leq \ k \cdot \frac{\psi_\varepsilon(h)}{1-\psi_\varepsilon(h)} \ . \end{aligned} \tag{2.110}$$

For the third term, we obtain

$$\begin{aligned} \mathrm{P}\left(|Z_{T^-} - Z_{T-h}| > 4\varepsilon\right) \ &\leq \ \mathrm{P}\left(\sup_{T-h\leq t<T} |Z_t - Z_{T-h}| > 2\varepsilon\right) \\ &= \ \mathrm{E}\left[\mathrm{P}\left(\sup_{T-h\leq t<T} |Z_t - Z_{T-h}| > 2\varepsilon \,\middle|\, \mathcal{F}_{T-h}\right)\right] \ \leq \ \frac{\psi_\varepsilon(h)}{1-\psi_\varepsilon(h)} \ , \end{aligned} \tag{2.111}$$

where in the last inequality we argued as to obtain (2.107) and used (2.99). It is slightly more involved to control the first term in (2.109). We have, for all $\delta \in [0, h_\varepsilon]$,

$$\begin{aligned} T \cdot \mathrm{P}\left(\sigma_{\varepsilon,k} \leq T\right) \ &\geq \ \mathrm{E}\left[\sigma_{\varepsilon,k} \cdot \mathbf{1}_{\{\sigma_{\varepsilon,k} \leq T\}}\right] \ = \ \sum_{\ell=1}^{k} \mathrm{E}\left[\tau_{\varepsilon,\ell} \cdot \mathbf{1}_{\{\sigma_{\varepsilon,k} \leq T\}}\right] \\ &\geq \ \sum_{\ell=1}^{k} \mathrm{E}\left[\tau_{\varepsilon,\ell} \cdot \mathbf{1}_{\{\sigma_{\varepsilon,k} \leq T\}} \cdot \mathbf{1}_{\{\tau_{\varepsilon,\ell} > \delta\}}\right] \ \geq \ \delta \cdot \sum_{\ell=1}^{k} \mathrm{P}\left(\sigma_{\varepsilon,k} \leq T,\ \tau_{\varepsilon,\ell} > \delta\right) \\ &\geq \ \delta \cdot \sum_{\ell=1}^{k} \mathrm{P}\left(\sigma_{\varepsilon,k} \leq T\right) - \delta \cdot \sum_{\ell=1}^{k} \mathrm{P}\left(\tau_{\varepsilon,\ell} \leq \delta\right) \\ &\geq \ \delta \cdot k \cdot \mathrm{P}\left(\sigma_{\varepsilon,k} \leq T\right) - \delta \cdot k \cdot \frac{\psi_\varepsilon(\delta)}{1-\psi_\varepsilon(\delta)} \ . \end{aligned}$$

where this last inequality follows from (2.103) and (2.107) as in (2.110). Hence, whenever $\delta \cdot k > T$, we obtain

$$\mathrm{P}\left(\sigma_{\varepsilon,k} \leq T\right) \;\leq\; \frac{\delta \cdot k}{\delta \cdot k - T} \cdot \frac{\psi_\varepsilon(\delta)}{1 - \psi_\varepsilon(\delta)} \,. \tag{2.112}$$

To conclude, the bounds (2.110), (2.112) with the choice $\delta = \frac{2T}{k}$ (and, as a consequence, $k > 2T/h_\varepsilon$) and (2.111) lead to the final result (2.100).

□

Part II

Duality

Jointly factorized duality, stationary product measures and generating functions

Duality and self-duality are very useful and powerful tools that allow to analyze properties of a complicated system in terms of a simpler one. In case of self-duality for particle systems, the dual system is the same and the simplification arises because in the dual one considers only a finite number of particles (see e.g. [31]). The connection between the evolution of the empirical density field of a many-particle system in Chapter 2 and that of a single random walker is another typical instance of self-duality for interacting particle systems.

Further applications of duality in the context of interacting particle systems range from the study of hydrodynamic limits and fluctuations, see e.g. [29], [31], [85], to the characterization of extremal measures, see e.g. [98]; from the derivation of the Fourier law of transport, as in e.g. [20, 86], to the explicit form of correlation inequalities, see e.g. [61], [63]. Other fields rich of applications are population genetics, where the coalescent process arises as a natural dual process (see e.g. [33] and references therein) and branching-coalescing processes (see e.g. [39]). Duality and related notions such as, for instance, intertwinings have already been used in the study of spectral gaps and convergence to stationarity by several authors (see e.g. [26], [36], [52], [104], [112]).

Several methods are available to construct dual processes and duality relations. In the context of population dynamics, the starting point to find dualities is to consider the coalescent, the simplest example here being the duality between Kingman's coalescent block-counting process and the Wright-Fisher diffusion (for an overview of this kind of dualities in more general contexts, see e.g. [30]). Another method is provided by the pathwise dualities based on graphical constructions and time reversals, see e.g. [98], [134].

The search of dualities for conservative IPS. In the context of conservative interacting particle systems such as the exclusion process and its generalizations, zero range processes, etc. [31], [98], the algebraic method first initiated in [126] and further developed in [19], [21], [22], [62] offers a general framework to construct self-duality functions starting from a reversible product measure by using symmetries of the generator, i.e. operators commuting with the generator. Additionally, if these symmetries are in product form, i.e. of the form $\prod_x S_x$, where the action of S_x depends only on the variables associated to site x, then the self-duality functions produced by these reversible product measures and product-like symmetries also *jointly factorize over the sites*, i.e. they are a product over the sites of functions that depend only on the variables associated to that site. In this thesis, we call such duality functions "*jointly factorized duality functions*".

The search of jointly factorized duality. A complete picture of how to obtain *all* jointly factorized self-duality functions for such particle systems is missing (except in the simplest case of symmetric exclusion with at most one particle per site, see e.g. [125] and references therein). One of the useful applications of disposing of all jointly factorized self-duality functions is that, depending on the target, one can choose appropriate ones: e.g. in the hydrodynamic limit and the study of the structure of the stationary measures, the "classical" duality functions (Sections 3.1.4 and 3.3.1) are the appropriate ones (see e.g. [31]), whereas in the study of (stationary and non-stationary) fluctuation fields and associated Boltzmann-Gibbs principles [85, Chapter 11], [4], as well as in the study of speed of relaxation to stationarity in L^2 or in the study of perturbation theory around models with duality [31], "orthogonal" duality functions (Section 3.3.1) turn out to be very useful.

In this search of jointly factorized dualities, natural associated questions are: which of these conservative particle systems allow this form of self-duality? And then, is it possible to obtain all jointly factorized self-duality functions for these systems?

In this chapter, we develop an approach to answer the above questions and systematically determine all jointly factorized self-duality functions for a specific class of conservative interacting particle systems – particle systems which we call *conservative, factorized* and *symmetric* (cf. Sections 1.2 and 3.1.3). As a consequence, by considering many-particle limits, we also obtain jointly factorized duality and self-dualities for a class of conservative diffusion processes, such as the *Brownian energy process* (BEP), see e.g. Section 3.1.6.

Jointly factorized duality functions and stationary product measures. In this route, starting from examples, we first investigate a general connection between stationary product measures and jointly factorized duality functions. This shows, in particular, that for infinite systems with factorized self-duality functions, the only stationary measures which are ergodic (w.r.t. either space-translation or time) are, in fact, product measures. Then we use this connection between stationary product measures and jointly factorized duality functions to recover all possible candidate jointly factorized duality functions from the stationary product measures. More precisely, we show that, given the "first" duality function, i.e. the duality function with a single dual particle, all remaining "higher order" jointly factorized duality functions are determined. This provides a machinery to obtain all jointly factorized self-duality functions in processes such as symmetric exclusion processes (SEP), systems of independent random walkers (IRW) and symmetric inclusion processes (SIP). In particular, we recover via this method all "orthogonal polynomial" duality functions obtained in [55].

Moreover, we prove that in the context of conservative symmetric particle systems where the rates for particle hopping depend only on the number of particles in the departure and arrival sites in a product way, the processes SEP, IRW and SIP are the only systems which have self-duality with "non-trivial" jointly factorized self-duality functions and that the first duality function is necessarily an affine function of the number of particles, see Theorem 3.3.

Generating functions as intertwiners. Next, in order to prove that the only candidate jointly factorized self-duality functions derived via the method described above are actual self-duality functions, we develop a method based on generating functions. This method, via an intertwining relation, allows to go from discrete systems (particle hopping dynamics) to continuous systems (such as diffusion processes or deterministic solutions of differential systems) and back, and also allows to pass from self-duality to duality and back. In fact, we show equivalence between self-duality of SIP, duality between SIP and BEP and self-duality of BEP, which intertwines with SIP via a product-like generating function. The proof of a self-duality relation in a discrete system then reduces to the same property in a continuous system, which is much easier to check directly.

As a consequence of this generating function method, we provide new examples of self-duality for processes in the continuum – such as for the Brownian energy process (BEP). Likewise, this method based on intertwinings may be viewed as a generalization of the procedure of producing dualities from symmetries of the generator, being a symmetry an intertwining of the generator with itself.

Organization of the rest of the chapter. In Section 3.1 we introduce the basic definitions of duality and systems considered. Additionally, in Theorem 3.3 we prove which particle systems out of those considered admit jointly factorized self-duality. In Section 3.2 we investigate a general relation between jointly factorized duality functions and stationary product measures. We treat separately the finite and infinite contexts in which this relation arises; in the latter case, we exploit this connection to draw some conclusions on the product structure of ergodic measures. Section 3.3 is devoted to the derivation of all possible jointly factorized self-duality and duality functions. Here Theorem 3.3 and the general relation studied in the previous section are the two key ingredients. In Section 3.4, after an introductory example and a brief introduction on the general connection between duality and intertwining relations, we establish an intertwining between the discrete and the continuum processes. This intertwining relation is then used to produce all self-duality functions for the Brownian energy process. We conclude this chapter with two appendices. In Appendix 3.a we show how all these results extend to the case of particle systems in a (quenched) random environment, while in Appendix 3.b we study an intertwiner relation of independent interest between symmetric exclusion processes and labeled variations of it, known as "ladder" symmetric exclusion processes [62].

3.1 Setting

We start defining what we mean by *duality (w.r.t. a function)* for Markov processes. Then, we introduce a general class of Markov interacting particle systems with associated interacting diffusion systems arising as many-particle limits.

Theorem 3.3 states that the only conservative particle systems described by the infinitesimal generator (3.7) below which admit a "non-trivial" factorized self-duality are necessarily SEP, IRW and SIP-type of processes. Moreover, in the same statement, we find the general form of the "first" self-duality function for such systems.

3.1.1 Duality with respect to a function

Given two (Polish) state spaces $\mathcal{X}$ and $\widehat{\mathcal{X}}$ and two *Markov processes* $\{\eta_t,\ t \geq 0\}$ and $\{\xi_t,\ t \geq 0\}$ evolving on them, we say that they are *dual* with *duality function* $D : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ (where D is a measurable function) if, for all $t > 0$, $\xi \in \widehat{\mathcal{X}}$ and $\eta \in \mathcal{X}$, we have the so-called duality relation

$$\widehat{\mathbb{E}}_\xi \left[D(\xi_t, \eta) \right] \;=\; \mathbb{E}_\eta \left[D(\xi, \eta_t) \right] \,. \tag{3.1}$$

If the laws of the two processes coincide, we speak about *self-duality*.

More generally, we say that two *semigroups* $\{S_t,\ t \geq 0\}$ and $\{\widehat{S}_t,\ t \geq 0\}$ are dual with duality function D if, for all $t \geq 0$,

$$(\widehat{S}_t)_{\text{left}} D \;=\; (S_t)_{\text{right}} D \,, \tag{3.2}$$

where "left" (resp. "right") refers to action on the left (resp. right) variable. In the case that these semigroups are Markov semigroups, (3.1) is exactly the same as (3.2). Even more generally, we say that two *operators* L and $\widehat{L}$ are dual to each other with duality function D if

$$\widehat{L}_{\text{left}} D \;=\; L_{\text{right}} D \,. \tag{3.3}$$

In the context of Markov processes, the operators L and $\widehat{L}$ which we have in mind here are Markov infinitesimal generators. Moreover, we refer to [78] for more technical aspects of duality relations, e.g. when generator duality implies semigroup duality or which are the exact restrictions on the state spaces needed. In order not to overload notation, we use the expression $A_{\text{left}} D(\xi, \eta)$ for $(AD(\cdot, \eta))(\xi)$ and, similarly, $B_{\text{right}} D(\xi, \eta) = (BD(\xi, \cdot))(\eta)$. We will often write $D(\xi, \eta)$ in place of $(\xi, \eta) \mapsto D(\xi, \eta)$.

3.1.2 Lattice, factorization over sites and jointly factorized duality

The underlying geometry of all systems that we will look at consists of a set of sites V, either finite or $V = \mathbb{Z}^d$. Moreover, we are given a family of bond weights $\boldsymbol{c} = \{c(\{x, y\}),\ x, y \in V\}$, satisfying the following conditions: for all $x, y \in V$,

(1) *Symmetry*: $c(\{x,y\}) = c(\{y,x\})$,

(2) *Vanishing diagonal*: $c(\{x,x\}) = 0$,

(3) *Irreducibility*: there exist $x_1 = x$, x_2, …, $x_{m+1} = y$ such that

$$\prod_{i=1}^{m} c(\{x_i, x_{i+1}\}) > 0 \,.$$

In case of $V = \mathbb{Z}^d$, we further require the following:

(4) *Finite-range*: there exists $R > 0$ such that, for all $x, y \in V$, $c(\{x,y\}) = 0$ if $|x-y| > R$,

(5) *Uniform bound on the total jump rate*: $\sup_{x\in V} \sum_{y\in V} c(\{x,y\}) < \infty$.

Notice that when $\boldsymbol{c}$ is finite-range and translation invariant, i.e.

$$c(\{x+z, y+z\}) \;=\; c(\{x,y\})$$

for all $x, y, z \in V$, then the uniform bound on the total jump rate follows automatically. Note also that the finite-range assumption is not necessary and can be relaxed; however, we assume it here for simplicity in order to avoid existence problems of the processes in case $V = \mathbb{Z}^d$. Furthermore, we employ $\boldsymbol{c}$ – which, in perfect analogy with the terminology in Chapter 2, we call *conductances* (for the set V) – to equip V with a nearest-neighboring relation: $x \sim y$ if and only if $c(\{x,y\}) > 0$.

To each site $x \in V$ we associate a variable $\eta(x) \in F = \{0, \ldots, \alpha\}$ or $\mathbb{N}_0$, for $\alpha \in \mathbb{N}$ or $(0,\infty)$, with the interpretation of either the number of particles or the amount of energy associated to the site $x \in V$. Configurations are denoted by $\eta \in \mathcal{X} = F^V$.

As discussed in the introduction, we look at duality functions which *jointly factorize over sites*, i.e. of the form

$$D(\xi,\eta) \;=\; \prod_{x\in V} d(\xi(x),\eta(x))\,, \quad \eta \in F^V,\ \xi \in \widehat{F}^V\,. \tag{3.4}$$

We then call the functions $d(\cdot,\cdot) : \widehat{F} \times F \to \mathbb{R}$ the *single-site duality functions* and further assume

$$d(0,\cdot) \;\equiv\; 1\,. \tag{3.5}$$

The above condition (3.5) is related to the fact that we want to have duality functions which make sense for infinite systems when the dual configuration $\xi \in \widehat{F}^V$ has a finite total mass. A typical example is when $\eta, \xi \in \mathbb{N}_0^{\mathbb{Z}^d}$, where η is an infinite configuration while ξ is a finite configuration, so that in the product (3.4) there are only a finite number of factors different from $d(0, \cdot)$. In this sense, the choice $d(0, \cdot) \equiv 1$ is the only sensible one for infinite systems.

When V is finite and $F = \{0, \ldots, \alpha\}$ or $\mathbb{N}_0$, this condition is not necessary and e.g. if a (positive) reversible product measure $\mu = \otimes_{x \in V} \nu$ exists, then the so-called "cheap" self-duality function

$$D_{\text{cheap}}(\xi, \eta) \;=\; \tfrac{1}{\mu(\xi)} \mathbf{1}_{\{\xi = \eta\}} \;=\; \prod_{x \in V} \frac{1}{\nu(\xi(x))} \mathbf{1}_{\{\xi(x) = \eta(x)\}}$$

is a self-duality function (see also Section 4.1) in the form (3.4), but does *not* satisfy (3.5).

3.1.3 Conservative factorized symmetric interacting particle systems

The class of interacting particle systems we consider is described by the (formal) infinitesimal generator acting on cylindrical functions $\varphi : \mathcal{X} \to \mathbb{R}$ as follows:

$$L\varphi(\eta) \;=\; \sum_{x \sim y} c(\{x, y\}) \, L_{\{x,y\}} \varphi(\eta) \,, \quad \eta \in \mathcal{X} \,, \tag{3.6}$$

where the summation above runs over all nearest-neighboring sites and $L_{\{x,y\}}$, the *single-bond generator*, is defined as

$$\begin{aligned} L_{\{x,y\}} \varphi(\eta) \;=\; & g(\eta(x))\, h(\eta(y))\, (\varphi(\eta^{x,y}) - \varphi(\eta)) \\ & +\; g(\eta(y))\, h(\eta(x))\, (\varphi(\eta^{y,x}) - \varphi(\eta)) \,, \quad \eta \in \mathcal{X} \,, \end{aligned} \tag{3.7}$$

where $\eta^{x,y}$ denotes the configuration arising from η by removing one particle at x and putting it at y, i.e. $\eta^{x,y}(x) = \eta(x) - 1$, $\eta^{x,y}(y) = \eta(y) + 1$, while $\eta^{x,y}(z) = \eta(z)$ if $z \neq x, y \in V$. Note the conservative nature of the system and the form of the particle jump rates in (3.7) which depend on the number of particles in the departure and arrival sites in a factorized form. As already anticipated in Section 1.2, we refer to these particle systems as *conservative, factorized* and *symmetric* particle systems.

Minimal requirements on the so-called interaction functions g and h :

$\mathbb{N}_0 \to [0, \infty)$, namely

(i) $g(0) = 0$, $g(1) = 1$ and $g(n) > 0$ for all $n > 0$,

(ii) $h(0) \neq 0$ and $h(\alpha) = 0$ if $F = \{0, \ldots, \alpha\}$ and in all other cases $h(n) > 0$,

guarantee the existence of a one-parameter family of *stationary* (actually *reversible*) *product measures* $\{\mu_\lambda = \otimes \nu_\lambda,\ \lambda > 0\}$ with marginals ν_λ given by

$$\nu_\lambda(n) \ = \ \phi(n) \frac{\lambda^n}{n!} \frac{1}{Z_\lambda} \, , \quad n \in \{0, \ldots, \alpha\} \text{ or } \mathbb{N}_0 \, , \tag{3.8}$$

for all $\lambda > 0$ for which the normalizing constant $Z_\lambda < \infty$ and with $\phi(n) = n! \prod_{m=1}^{n} \frac{h(m-1)}{g(m)}$.

Remark 3.1 (SINGLE-BOND DUALITY). *In this context, we speak about* single-bond duality *(w.r.t. a function) if all single-bond generators in* (3.7) *are dual with a common duality function* $D : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$, *namely, for all* $x, y \in V$,

$$(\widehat{L}_{\{x,y\}})_{\text{left}} D(\xi, \eta) \ = \ (L_{\{x,y\}})_{\text{right}} D(\xi, \eta) \tag{3.9}$$

for all $\xi \in \widehat{\mathcal{X}}, \eta \in \mathcal{X}$. *By linearity, single-bond duality yields duality w.r.t. the same duality function for the generators in* (3.6), *i.e.*

$$\begin{aligned} \widehat{L}_{\text{left}} D(\xi, \eta) \ &= \ \sum_{x \sim y} c(\{x, y\}) \, (\widehat{L}_{\{x,y\}})_{\text{left}} D(\xi, \eta) \\ &= \ \sum_{x \sim y} c(\{x, y\}) \, (L_{\{x,y\}})_{\text{right}} D(\xi, \eta) \ = \ L_{\text{right}} D(\xi, \eta) \, , \end{aligned} \tag{3.10}$$

for all $\xi \in \widehat{\mathcal{X}}, \eta \in \mathcal{X}$. *The converse statement, in general, is false. However, if we have* (3.10) *for all conductances* $\mathbf{c} = \{c(\{x, y\}),\ x, y \in V\}$, *then duality as in* (3.9) *for all single-bond generators follows.*

Additionally, if the duality function is in a jointly factorized form $D = \prod_x d$ *as in* (3.4), *then, for each bond* $\{x, y\}$, (3.9) *reduces to the following duality relation involving only the single-site duality functions associated to the sites* x *and* $y \in V$, *namely*

$$(\widehat{L}_{\{x,y\}})_{\text{left}} D_{\{x,y\}}(\xi, \eta) \ = \ (L_{\{x,y\}})_{\text{right}} D_{\{x,y\}}(\xi, \eta) \, , \tag{3.11}$$

for all $\xi \in \widehat{\mathcal{X}}, \eta \in \mathcal{X}$, *where*

$$D_{\{x,y\}}(\xi, \eta) \ := \ d(\xi(x), \eta(x)) \cdot d(\xi(y), \eta(y)) \, .$$

In particular, all (self-)duality relations in this chapter will be in the form (3.11)*, yielding automatically (self-)duality relations for the generators* (3.6)*.*

Remark 3.2 (EXISTENCE OF THE PARTICLE SYSTEMS). *The existence of the processes with formal generator* (3.6) *poses no problem if V is finite. If V is infinite, further growth conditions on the functions $g(n)$ and $h(n)$ are required in order to ensure non-explosion. For the processes that we will be considering in the next section, which have the self-duality property, existence can be proved via self-duality in the spirit of [31, §2.2].*

3.1.4 Examples of conservative particle systems with jointly factorized self-duality

We recall here the basic examples of self-dual (conservative factorized symmetric) interacting particle systems and corresponding jointly factorized self-duality functions known in literature (see e.g. [62]). In the next section – Theorem 3.3 below – we prove that these are the only particle systems (within our defined class) that are self-dual with jointly factorized self-duality functions.

(a) *Symmetric exclusion process* (SEP(α), $\alpha \in \mathbb{N}$).

- $F = \{0, \ldots, \alpha\}$,
- $g(n) = n$, $h(n) = \alpha - n$,
- $\nu_\lambda \sim \text{Binomial}(\alpha, \frac{\lambda}{1+\lambda})$, $\nu_\lambda(n) = \frac{\alpha!}{(\alpha-n)!}\frac{\lambda^n}{n!}\left(\frac{1}{1+\lambda}\right)^\alpha$, $\lambda > 0$,
- $d(k,n) = \frac{(\alpha-k)!}{\alpha!}\frac{n!}{(n-k)!}\mathbf{1}_{\{k \le n\}}$.

(b) *Independent random walkers* (IRW(α), $\alpha > 0$).

- $F = \mathbb{N}_0$,
- $g(n) = n$, $h(n) = \alpha$,
- $\nu_\lambda \sim \text{Poisson}(\alpha\lambda)$, $\nu_\lambda(n) = \frac{(\alpha\lambda)^n}{n!}e^{-\alpha\lambda}$, $\lambda > 0$,
- $d(k,n) = \frac{1}{\alpha^k}\frac{n!}{(n-k)!}\mathbf{1}_{\{k \le n\}}$.

(c) *Symmetric inclusion process* (SIP(α), $\alpha > 0$).

- $F = \mathbb{N}_0$,
- $g(n) = n$, $h(n) = \alpha + n$,
- $\nu_\lambda \sim \text{Gamma}_\text{d}(\alpha, \lambda)$, $\nu_\lambda(n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\frac{\lambda^n}{n!}(1-\lambda)^\alpha$, $\lambda \in (0,1)$,

- $d(k,n) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}\frac{n!}{(n-k)!}\mathbf{1}_{\{k\le n\}}$.

The jointly factorized self-duality functions $D(\xi,\eta) = \prod d(\xi(x),\eta(x))$ constructed out of the single-site self-duality functions above may be considered to be fully informative in the following sense.

Besides satisfying (3.5) ($d(0,\cdot) = 1$) and being $\{d(k,\cdot),\ k \in F \setminus \{0\}\}$ non-constant as functions on F, the integrals of the single-site self-duality functions determine uniquely the marginals ν_λ of the corresponding stationary product measures μ_λ: for all $k \in F$, the expression

$$\theta(\lambda) \;=\; \sum_{n\in F} d(k,n)\, \nu_\lambda(n)$$

reads

$$\sum_{n=k}^{\alpha} \frac{(\alpha-k)!}{\alpha!}\frac{n!}{(n-k)!}\frac{\alpha!}{(\alpha-n)!}\frac{\lambda^n}{n!}\left(\frac{1}{1+\theta}\right)^{\alpha} \;=\; \left(\frac{\lambda}{1+\lambda}\right)^{k}$$

for SEP(α),

$$\sum_{n=k}^{\infty} \frac{1}{\alpha^k}\frac{n!}{(n-k)!}\frac{(\alpha\lambda)^n}{n!}e^{-\alpha\theta} \;=\; \lambda^k$$

for IRW(α), and

$$\sum_{n=k}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}\frac{n!}{(n-k)!}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\frac{\lambda^n}{n!}(1-\lambda)^{\alpha} \;=\; \left(\frac{\lambda}{1-\lambda}\right)^{k}$$

for SIP(α). Moreover, in all three cases, the marginals ν_λ of μ_λ are the unique probability measures w.r.t. which the integrals of the functions $\{d(k,\cdot),\ k \in F\}$ – these functions corresponding to weighted factorial moments – equal $\{\theta(\lambda),\ k \in F\}$: for all probability measures $\nu_\star$ on F such that

$$\sum_{n\in F} d(k,n)\, \nu_\star(n) \;=\; \theta(\lambda)$$

for all $k \in F$, we have $\nu_\star = \nu_\lambda$.

This property, which will be reconsidered under the name of "measure determining at stationarity" in Section 3.2, will play a crucial role in the characterization of ergodic measures for processes in duality w.r.t. jointly factorized duality functions.

3.1.5 Characterization of particle systems with jointly factorized self-duality

In the following theorem we show that the only processes with generator of the type (3.7) which have "non-trivial" jointly factorized self-duality functions are of one of the types described in the examples above, i.e. SEP, IRW or SIP. Here by "non-trivial" we mean that $d(0,\cdot) \equiv 1$ (condition (3.5)) and that the first single-site self-duality function $d(1,n)$ is not a constant (as a function of n).

Theorem 3.3. *Assume that the process with generator given by the single-bond generator $L_{\{x,y\}}$ as in (3.7) is self-dual with jointly factorized self-duality function $D = \prod_x d$ in the form (3.4) with $d(0,\cdot) \equiv 1$ as in (3.5). If $d(1,n)$ is not constant as a function of n, then*

$$\begin{aligned} g(n) &= n \\ h(n) &= h(0) + (h(1) - h(0))\, n\,, \end{aligned} \tag{3.12}$$

and the first single-site self-duality function is of the form

$$d(1,n) = a + \tfrac{b}{h(0)} n\,, \tag{3.13}$$

for some $a \in \mathbb{R}$ and $b \neq 0$.

Proof. Using the self-duality relation for $\xi = \delta_x$, i.e. with one particle at $x \in V$ and no particles elsewhere, together with $g(0) = 0$, we obtain the identity

$$\begin{aligned} &g(1)\, h(0)\, [d(1,\eta(y)) - d(1,\eta(x))] \\ &\quad = g(\eta(x))\, h(\eta(y))\, [d(1,\eta(x)-1) - d(1,\eta(x))] \\ &\quad + g(\eta(y))\, h(\eta(x))\, [d(1,\eta(x)+1) - d(1,\eta(x))]\,. \end{aligned} \tag{3.14}$$

Setting $\eta(x) = \eta(y) = n \geq 1$, this yields, anytime $g(n)\, h(n) \neq 0$,

$$d(1,n+1) + d(1,n-1) - 2d(1,n) = 0\,, \tag{3.15}$$

from which we derive $d(1,n) = a + \frac{b}{h(0)} n$ for some $a, b \in \mathbb{R}$. Because $d(1,n)$ is not constant as a function of n, we must have $b \neq 0$. Inserting $d(1,n) =$

$a + \frac{b}{h(0)}n$ in (3.14) we obtain

$$g(\eta(x))\, h(\eta(y)) - g(\eta(y))\, h(\eta(x)) \;=\; -g(1)\, h(0)\, (\eta(y) - \eta(x))\,,$$

from which, by setting $\eta(x) = n$ and $\eta(y) = 0$ we obtain the first in (3.12), while via $\eta(x) = n$ and $\eta(y) = 1$ we get the second condition. □

Remark 3.4. *More generally, if we replace* (3.5) *with* $d(0, n) \neq 0$ *in the above statement, we analogously obtain* (3.12) *and*

$$\begin{aligned} d(0, n) \;&=\; c^n \\ d(1, n) \;&=\; (a + \tfrac{b}{h(0)}n) \cdot c^n\,, \end{aligned}$$

for some constants $a, b, c \in \mathbb{R}$, $b, c \neq 0$.

3.1.6 Interacting diffusion systems as many-particle limits

Conservative interacting diffusion processes arise as many-particle limits of the particle systems in Section 3.1.4 above (see e.g. [62]). More in details, by "many-particle limit" we refer to the limit process of the particle systems $\{\frac{1}{N}\eta^N_\cdot,\ N \in \mathbb{N}\}$, where the initial conditions

$$\tfrac{1}{N}\eta^N_0 \;=\; \left\{\tfrac{1}{N}(\lfloor Nz(x)\rfloor),\ x \in V\right\}$$

converge to some $z \in F^V$, with $F = [0, \infty)$, as $N \to \infty$.

In case of IRW(α), one obtains a deterministic (hence degenerate) diffusion process $\{z_t,\ t \geq 0\}$ whose evolution is described by a first-order differential operator. In case of SIP(α), the many-particle limit is a proper Markov process of interacting diffusions known as *Brownian energy process* (BEP(α)) [62]. For the SEP(α), this limit cannot be taken in the sense of Markov processes, but we can extend the SEP(α) generator to a larger class of functions defined on a larger configuration space and take the many-particle limit. The limiting operator is then not a Markov generator, but still a second-order differential operator. We will explain this more in detail below.

Many-particle limits of IRW **and** SIP**.** The limiting differential operators in the case of IRW(α) and SIP(α) can be described as acting on smooth functions

$\varphi : [0,\infty)^V \to \mathbb{R}$ as follows:

$$\mathscr{L}\varphi(\zeta) \;=\; \sum_{x\sim y} c(\{x,y\})\,\mathscr{L}_{\{x,y\}}\varphi(\zeta)\,, \quad \zeta \in [0,\infty)^V\,, \tag{3.16}$$

with single-bond generators $\mathscr{L}_{\{x,y\}}$ given, respectively, by

$$\mathscr{L}_{\{x,y\}}\varphi(\zeta) \;=\; \big(-\alpha(\zeta(x)-\zeta(y))\,(\partial_x-\partial_y)\big)\,\varphi(\zeta)\,, \tag{3.17}$$

and

$$\mathscr{L}_{\{x,y\}}\varphi(\zeta) \;=\; \Big(-\alpha(\zeta(x)-\zeta(y))\,(\partial_x-\partial_y) \;+\; \zeta(x)\,\zeta(y)\,(\partial_x-\partial_y)^2\Big)\,\varphi(\zeta)\,, \tag{3.18}$$

where $\zeta \in [0,\infty)^V$ and ∂_x denotes partial derivative w.r.t. the variable $\zeta(x) \in [0,\infty)$. The operator $\mathscr{L}$ given in (3.16) with single-bond generators $\mathscr{L}_{\{x,y\}}$ as in (3.18) corresponds to the generator of the Brownian energy process with parameter $\alpha > 0$ (BEP(α)).

Many-particle limit of SEP. For the SEP(α) we proceed as follows. For each $N \in \mathbb{N}$, consider the operator L^N working on functions $\varphi : (\mathbb{N}_0/N)^V \to \mathbb{R}$ as

$$L^N\varphi(\tfrac{1}{N}\eta) \;=\; \sum_{x\sim y} c(\{x,y\})\,L^N_{\{x,y\}}\varphi(\tfrac{1}{N}\eta)\,, \quad \eta \in \mathbb{N}_0^V\,, \tag{3.19}$$

where

$$\begin{aligned} L^N_{\{x,y\}}\varphi(\tfrac{1}{N}\eta) \;=\; &\eta(x)\,(\alpha-\eta(y))\,(\varphi(\tfrac{1}{N}\eta^{x,y})-\varphi(\tfrac{1}{N}\eta)) \\ &+\; \eta(y)\,(\alpha-\eta(x))\,(\varphi(\tfrac{1}{N}\eta^{y,x})-\varphi(\tfrac{1}{N}\eta))\,, \quad \eta \in \mathbb{N}_0^V\,. \end{aligned}$$

This operator is not a Markov generator anymore, because the factors $\eta(x)\,(\alpha-\eta(y))$ may become negative. With this operator, we consider the limit

$$\lim_{N\to\infty} L^N\varphi(\tfrac{1}{N}\eta^N)\,,$$

where $\eta^N = \{\lfloor N\zeta(x)\rfloor,\ x \in V\}$ and $\varphi : [0,\infty)^V \to \mathbb{R}$ is a smooth function. This then gives the differential operator $\mathscr{L}$ which is the analogue of (3.16) in the context of SEP(α). This differential operator $\mathscr{L}$, with single-bond operators

$$\mathscr{L}_{\{x,y\}}\varphi(\zeta) = \Big(-\alpha\,(\zeta(x)-\zeta(y))\,(\partial_x-\partial_y) \\ - \zeta(x)\,\zeta(y)\,(\partial_x-\partial_y)^2\Big)\,\varphi(\zeta)\,, \quad \zeta\in[0,\infty)^V\,, \quad (3.20)$$

does not generate a Markov process but it is still useful because, as we will see in Section 3.4 below, via generating functions, it is intertwined with the operator (3.19) for the choice $N = 1$.

Remark 3.5 (EXISTENCE OF THE DIFFUSION PROCESSES). *The existence and ergodic properties of diffusion processes with generator of type* (3.16) *in the context of infinite volume $V = \mathbb{Z}^d$ has been addressed in [76]. In the finite-volume case,* BEP *is a multi-type Wright-Fisher diffusion with mutation, whose existence, i.e. the well-posedness of the martingale problem for the operator* (3.16), *is well-known (see e.g. [40, Theorem 2.8 of Chapter 8]).*

Stationary product measures and jointly factorized duality. Naturally, as we can see for the case of SIP(α) and BEP(α), when going to the many-particle limit, some properties concerning stationary measures and duality pass to the limit [62]. Indeed, BEP(α) admits a one-parameter family of stationary product measures $\{\mu_\lambda = \otimes\, \nu_\lambda,\ \lambda > 0\}$, where $\nu_\lambda \sim \text{Gamma}(\alpha,\lambda)$, namely

$$\nu_\lambda(dz) = z^{\alpha-1}e^{-\lambda z}\,\frac{\lambda^\alpha}{\Gamma(\alpha)}\,dz\,, \qquad (3.21)$$

and is dual to SIP(α) with jointly factorized duality function given by

$$D(\xi,\zeta) = \prod_{x\in V} d(\xi(x),\zeta(x))$$

with single-site duality functions

$$d(k,z) = z^k\,\frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}\,, \quad k\in\mathbb{N}_0\,,\ z\in[0,\infty)\,.$$

After noting that property (3.5) holds also in this situation, we show that the first single-site duality functions $d(1,z)$ between SIP(α) and BEP(α) are affine functions of $z \in [0,\infty)$, as we found earlier for single-site self-dualities in Theorem 3.3.

Proposition 3.6. *Assume that* SIP(α) *and* BEP(α)*'s single-bond generators are* *dual with jointly factorized duality function*

$$D(\xi,\zeta) \;=\; \prod_{x\in V} d(\xi(x),\zeta(x))$$

with $d(0,\cdot)\equiv 1$ *as in* (3.4)–(3.5). *Then*

$$d(1,z) \;=\; a+\tfrac{b}{\alpha}z\,, \tag{3.22}$$

for some $a,b\in\mathbb{R}$.

Proof. The duality relation for $\xi=\delta_x$, by using (3.5), reads

$$\begin{aligned}&\alpha\,[d(1,\zeta(y))-d(1,\zeta(x))]\\&\qquad=\;-\alpha\,(\zeta(x)-\zeta(y))\,\partial_x d(1,\zeta(x))+\zeta(x)\,\zeta(y)\,\partial_x^2 d(1,\zeta(x))\,.\end{aligned}$$

If we set $\zeta(x)=\zeta(y)=z$, then $z^2\frac{d^2}{dz^2}d(1,z)=0$ leads to (3.22) as unique solution. □

3.2 Jointly factorized functions and stationary product measures: a general relation

In the examples of duality that we have encountered in the previous section, we have a general relation between the stationary product measures and the jointly factorized duality functions. Given $\{\eta_t,\ t\geq 0\}$, if there is a dual process $\{\xi_t,\ t\geq 0\}$ with jointly factorized duality functions $D(\xi,\eta)=\prod d(\xi(x),\eta(x))$ as in (3.4)–(3.5) and stationary product measures $\{\mu_\lambda=\otimes\,\nu_\lambda\}$, then there is a relation between these measures and these functions; namely, there exists a function $\theta(\lambda)$ such that

$$\int D(\xi,\eta)\,\mu_\lambda(d\eta) \;=\; \prod_{x\in V}\int d(\xi(x),\eta(x))\,\nu_\lambda(d\eta(x)) \;=\; \theta(\lambda)^{|\xi|}\,. \tag{3.23}$$

This function $\theta(\lambda)$ is then the expectation of the first single-site duality function, i.e.

$$\theta(\lambda) \;=\; \int d(1,z)\,\nu_\lambda(dz)\,.$$

In the examples of Section 3.1.4 of self-dual particle systems, we have $\theta(\lambda)=\frac{\lambda}{1+\lambda}$ for SEP(α), $\theta(\lambda)=\lambda$ for IRW(α) and $\theta(\lambda)=\frac{\lambda}{1-\lambda}$ for SIP(α).

In this section we first investigate under which general conditions this relation holds, and further use it in Section 3.2.3 as a criterion of characterization of all extremal measures. We refer to Sections 3.1.1–3.1.2 for the general setting in which these results hold. Later on, we will see that this relation (3.23) is actually a characterizing property of jointly factorized duality functions, meaning that all jointly factorized duality functions are determined once the first single-site duality function is fixed.

Main results of the section. Theorem 3.9 establishes the equivalence between existence of a stationary product measure and (3.23) in the finite-volume context, while Theorem 3.12 establishes the same equivalence in the infinite-volume context ($V = \mathbb{Z}^d$) under the condition [BHT] defined below. As a consequence, in the same infinite-volume context, we obtain Theorems 3.13–3.15 stating that, under [BHT] and existence of jointly factorized duality, the only ergodic invariant measures are product measures.

3.2.1 Finite case

We start with the simplest situation in which V is a finite set. First, we assume that the total number of particles/the total energy of the dual process is the only conserved quantity. More precisely, we assume the following property, which we refer to as *harmonic triviality* of the dual system $\{\xi_t,\ t \geq 0\}$:

[HT] If $H : \widehat{F}^V \to \mathbb{R}$ is harmonic, i.e. such that, for all $t > 0$,

$$\widehat{\mathbb{E}}_\xi \left[H(\xi_t) \right] = H(\xi) ,$$

then $H(\xi)$ is only a function of $|\xi| := \sum_{x \in V} \xi(x)$.

Remark 3.7 (IRREDUCIBLE CONSERVATIVE PARTICLE SYSTEMS AND [HT]). *Under the assumption of irreducibility of the conductances* **c** *(cf.* (3) *in Section 3.1.2), if the dual system* $\{\xi_t,\ t \geq 0\}$ *is a conservative particle system on* $(V, \sim)$*, then property* [HT] *is verified. This is the case of duality with, for instance, either one of the particle systems considered in Sections 3.1.3–3.1.4.*

Secondly, we may require that the duality functions under consideration are "full rank" in some sense. To this purpose, we need the following definition, which, roughly speaking, may be seen as related to the notion of determinate moment problem.

Definition 3.8 (MEASURE DETERMINING FOR A MEASURE). *Let $D : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ be a measurable function and μ a probability measure on $\mathcal{X}$. We say that $D(\xi, \eta)$ is* measure determining for μ *if, for all probability measures $\mu_\star$ on $\mathcal{X}$ for which*

$$\int_{\mathcal{X}} D(\xi, \eta)\, \mu_\star(d\eta) \;=\; \int_{\mathcal{X}} D(\xi, \eta)\, \mu(d\eta) \;<\; \infty$$

holds for all $\xi \in \widehat{\mathcal{X}}$, we have $\mu_\star = \mu$.

In particular, in case $D(\xi, \eta)$ is a duality function and $\{\eta_t,\ t \geq 0\}$ admits a one-parameter family of stationary measures $\{\mu_\lambda,\ \lambda \in \Delta\}$, we may require the following additional property for $D(\xi, \eta)$:

[MDS] The function $D(\xi, \eta)$ is *measure determining at stationarity*, i.e. measure determining for μ_λ for all $\lambda \in \Delta$.

Then we have the following.

Theorem 3.9. *Assume that $\{\eta_t,\ t \geq 0\}$ and $\{\xi_t,\ t \geq 0\}$ are dual as in* (3.1) *with jointly factorized duality function* (3.4) *satisfying condition* (3.5)*. Moreover, assume that* [HT] *holds and that μ is a probability measure on $\mathcal{X}$. We distinguish two cases:*

(1) Interacting particle system case. *If $\widehat{F}$ is a subset of $\mathbb{N}_0$, then we assume that the duality functions $D(\xi, \cdot)$ are μ-integrable for all $\xi \in \widehat{\mathcal{X}}$.*

(2) Interacting diffusion case. *If $\widehat{F} = [0, \infty)$, then we assume the following integrability condition: for each $\varepsilon > 0$, there exists a μ-integrable function $f_\varepsilon : \mathcal{X} \to \mathbb{R}$ such that*

$$\sup_{\xi \in \widehat{\mathcal{X}},\, |\xi| = \varepsilon} |D(\xi, \eta)| \;\leq\; f_\varepsilon(\eta)\,, \qquad \eta \in \mathcal{X}\,. \tag{3.24}$$

Then

(a) μ is a stationary product measure for the process $\{\eta_t,\ t \geq 0\}$

implies

(b) For all $\xi \in \mathcal{X}$ and for all $x \in V$, we have

$$\int D(\xi, \eta)\, \mu(d\eta) \;=\; \left(\int D(\delta_x, \eta)\, \mu(d\eta) \right)^{|\xi|}\,, \tag{3.25}$$

where δ_x denotes the configuration with a single particle at $x \in V$ and no particles elsewhere. Moreover, if condition [MDS] *holds, the two statements (a) and (b) are equivalent.*

Proof. First assume that μ is a stationary product measure and define $H(\xi) = \int D(\xi,\eta)\,\mu(d\eta)$. By μ-integrability in the interacting particle system case, resp. (3.24) in the interacting diffusion case, self-duality and invariance of μ, we have

$$\begin{aligned}\mathbb{E}_\xi\left[H(\xi(t))\right] &= \int \mathbb{E}_\xi\left[D(\xi(t),\eta)\right]\,\mu(d\eta)\\ &= \int \mathbb{E}_\eta\left[D(\xi,\eta(t))\right]\,\mu(d\eta) = \int D(\xi,\eta)\,\mu(d\eta) = H(\xi)\,.\end{aligned}$$

Therefore by [HT] we conclude that $H(\xi) = \psi(|\xi|)$. By using $d(0,\cdot) \equiv 1$ and the joint factorization of the duality functions, we have that $\psi(0) = 1$. For the particle case, we obtain that

$$\int D(\delta_x,\eta)\,\mu(d\eta) = \psi(1)\,.$$

In particular, we obtain that the l.h.s. does not depend on x. Next, for $n \geq 2$, put

$$\int D(n\delta_x,\eta)\,\mu(d\eta) = \psi(n)\,,$$

then we have for $x \neq y \in V$, using the joint factorized form of the duality function and the product form of the measure,

$$\begin{aligned}\psi(n) &= \int D(n\delta_x,\eta)\,\mu(d\eta)\\ &= \int D(\delta_y,\eta)\,D((n-1)\delta_x,\eta)\,\mu(d\eta) = \psi(1)\,\psi(n-1)\,,\end{aligned}$$

from which it follows that $\psi(n) = \psi(1)^n$. Via an analogous reasoning that uses the factorization of $D(\xi,\eta)$ and the product form of μ, for the diffusion case we obtain, for all $\varepsilon, \varepsilon' \geq 0$,

$$\psi(\varepsilon+\varepsilon') = \psi(\varepsilon)\,\psi(\varepsilon')\,,$$

and hence, by measurability of $\psi(\varepsilon)$, we get $\psi(\varepsilon) = \psi(1)^\varepsilon$.

To prove the other implication, put

$$\int D(\delta_x, \eta)\, \mu(d\eta) \;=\; \kappa\,.$$

We then have by assumption

$$\int D(\xi, \eta)\, \mu(d\eta) \;=\; \kappa^{|\xi|}\,,$$

and so it follows that μ is stationary by self-duality, μ-integrability, the conservation of the number of particles and the measure determining property at stationarity. Indeed,

$$\begin{aligned}\int \mathbb{E}_\eta\left[D(\xi, \eta_t)\right]\, \mu(d\eta) \;&=\; \int \mathbb{E}_\xi\left[D(\xi_t, \eta)\right]\, \mu(d\eta)\\ &=\; \mathbb{E}_\xi\left[(\kappa^{|\xi_t|})\right] \;=\; \kappa^{|\xi|} \;=\; \int D(\xi, \eta)\, \mu(d\eta)\,.\end{aligned}$$

From the factorized form of $D(\xi, \eta)$, (3.25) implies that for all $x \in V$ and $\xi(x) \in \widehat{F}$

$$\int d(\xi(x), \eta(x))\, \mu(d\eta) \;=\; \kappa^{\xi(x)}\,,$$

and also

$$\int D(\xi, \eta)\, \mu(d\eta) \;=\; \kappa^{|\xi|} \;=\; \prod_{x \in V} \kappa^{\xi(x)} \;=\; \prod_{x \in V} \int d(\xi(x), \eta(x))\, \mu(d\eta)\,,$$

therefore μ is a product measure by the fact that there exists $\mu_\star$ stationary product measure for $\{\eta_t,\ t \geq 0\}$ such that

$$\int D(\xi, \eta)\, \mu_\star(d\eta) \;=\; \kappa^{|\xi|}$$

for all $\xi \in \widehat{F}^V$ and that D is measure determining at stationarity [MDS]. □

3.2.2 Infinite case

If $V = \mathbb{Z}^d$, then one needs essentially two extra conditions to state an analogous result in which a general relation between duality functions and corresponding stationary measures can be derived.

In this section we will assume that the dual process is a discrete particle system, i.e. $\widehat{F}$ is a subset of $\mathbb{N}_0$, in which the number of particles is conserved. In this case we need an additional property ensuring that for the dynamics of a finite number of particles there are no bounded harmonic functions other than those depending on the total number of particles. Therefore, we introduce the condition of existence of a successful coupling for the discrete dual process with a finite number of particles. This is defined below.

Definition 3.10 (SUCCESSFUL COUPLING PROPERTY). *We say that the discrete dual process* $\{\xi_t,\ t \geq 0\}$ *has the* successful coupling property *when the following holds: if we start with n particles then there exists a labeling such that for the corresponding labeled process* $\{X_1(t), \ldots, X_n(t),\ t \geq 0\}$ *there exists a successful coupling. This means that for every two initial positions* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$, *there exists a coupling with path space measure* $\mathbb{P}_{\mathbf{x},\mathbf{y}}$ *such that the coupling time*

$$\tau = \inf\{s > 0 : \mathbf{X}(t) = \mathbf{Y}(t),\ \forall t \geq s\}$$

is finite $\mathbb{P}_{\mathbf{x},\mathbf{y}}$ *almost surely.*

Notice that the successful coupling property is the most common way to prove the following equivalent property (see e.g. [98]) of *bounded harmonic triviality* of the dual process – analogue of [HT]:

[BHT] If H is a bounded harmonic function for the process $\{\xi_t,\ t \geq 0\}$, then $H(\xi) = \psi(|\xi|)$ for some bounded $\psi : \widehat{F} \to \mathbb{R}$.

Remark 3.11 (SUCCESSFUL COUPLING FOR SEP, IRW AND SIP). *The condition of the existence of a successful coupling (and the consequent bounded harmonic triviality) is quite natural in the context of conservative interacting particle systems, where we have that a finite number of walkers behave as independent walkers, except when they are close and interact. Therefore, the successful coupling needed is a variation of the Ornstein coupling of independent walkers. For the specific instances of coupling for symmetric exclusion, resp. inclusion, particles with* homogeneous *nearest-neighbor interactions, i.e.* $c(\{x, y\}) = c > 0$ *for all* $x, y \in \mathbb{Z}^d$ *with* $|x - y| = 1$ *and zero otherwise, cf. e.g. [31, §6.6] and [49], resp. [92].*

Furthermore, we need a form of uniform μ-integrability of the duality functions which we introduce below and call *uniform domination property* of D w.r.t. μ (note the analogy with condition (3.24)):

[UD] Given μ a probability measure on $\mathcal{X}$, the functions $\{D(\xi,\cdot),\ |\xi| = n\}$ are uniformly μ-integrable, i.e. for all $n \in \mathbb{N}$ there exists a function f_n such that f_n is μ-integrable and such that for all $\eta \in \mathcal{X}$

$$\sup_{\xi \in \widehat{\mathcal{X}},\ |\xi|=n} |D(\xi,\eta)| \leq f_n(\eta)\ .$$

Under these conditions, the following result holds, whose proof resembles that of Theorem 3.9.

Theorem 3.12. *Assume as in* (3.1) *that* $\{\eta_t,\ t \geq 0\}$ *is dual to the discrete process* $\{\xi_t,\ t \geq 0\}$ *with jointly factorized duality function as in* (3.4)–(3.5)*. Moreover, assume* [BHT] *in place of* [HT] *for the dual process and that μ is a probability measure on $\mathcal{X}$ for which* [UD] *holds. Then the same conclusions as in Theorem 3.9 follow, where* (3.25) *holds for all finite configurations* $\xi \in \widehat{\mathcal{X}}$.

3.2.3 Characterization of ergodic measures for infinite systems

In this section, we show that in the infinite context and in presence of jointly factorized duality functions, minimal ergodicity assumptions on a stationary probability measure μ on $\mathcal{X}$ are needed to ensure (3.25) and, as a consequence, that μ is product measure.

Translation invariance. Here we restrict to the case $V = \mathbb{Z}^d$ as we will use ergodicity w.r.t. space-translation.

Theorem 3.13 (ergodicity & translation invariance)**.** *In the setting of Theorem 3.12 with $D(\xi,\eta)$ jointly factorized duality function and μ probability measure on $\mathcal{X}$, if μ is a translation invariant and ergodic (under translations) stationary measure for* $\{\eta_t,\ t \geq 0\}$*, then we have* (3.25) *for all finite configurations ξ; as a consequence, μ is a product measure.*

Proof. To start, let us consider a configuration $\xi = \sum_{i=1}^n \delta_{x_i}$. By bounded harmonic triviality [BHT], combined with the bound (3.2.2) for all such configurations, $\int D(\xi,\eta)\,\mu(d\eta)$ is only depending on n and, therefore, we can replace ξ by $\sum_{i=1}^{n-1} \delta_{x_i} + \delta_y$, where y is arbitrary in $\mathbb{Z}^d$. Let us call $B_N = [-N,N]^d \cap \mathbb{Z}^d$ and fix N_0 such that B_{N_0} contains all the points $x_1, \ldots x_{n-1}$. For y outside B_{N_0}, by the factorization property,

$$D(\delta_{x_1} + \ldots \delta_{x_{n-1}} + \delta_y, \eta) \;=\; D(\delta_{x_1} + \ldots + \delta_{x_{n-1}}, \eta)\, D(\delta_y, \eta)\ .$$

By the Birkhoff ergodic theorem, we have that

$$\frac{1}{(2N+1)^d} \sum_{y \in B_N} D(\delta_y, \eta) \underset{N\to\infty}{\longrightarrow} \int D(\delta_0, \eta)\, \mu(d\eta)$$

for μ-a.e. $\eta \in \mathcal{X}$. Using this, together with (3.2.2), we have

$$\begin{aligned}
&\int D(\xi, \eta)\, \mu(d\eta) \\
&= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{y \in B_N \setminus B_{N_0}} \int D(\delta_{x_1} + \ldots \delta_{x_{n-1}}, \eta)\, D(\delta_y, \eta)\, \mu(d\eta) \\
&= \int D(\delta_{x_1} + \ldots + \delta_{x_{n-1}}, \eta)\, \mu(d\eta) \int D(\delta_0, \eta)\, \mu(d\eta)\,.
\end{aligned}$$

Iterating this argument gives (3.25). □

Remark 3.14 (APPROXIMATE JOINTLY FACTORIZED DUALITY). *As it follows clearly from the proof, the condition of joint factorization of the duality function can be replaced by the weaker condition of*

$$\lim_{|y|\to\infty} D(\delta_{x_1} + \ldots + \delta_{x_{n-1}} + \delta_y, \eta) - D(\delta_{x_1} + \ldots + \delta_{x_{n-1}}, \eta) D(\delta_y, \eta) \;=\; 0\,,$$

for all $x_1, \ldots, x_{n-1}$ *and* μ*-a.e.* $\eta \in \mathcal{X}$. *We note that this approximate joint factorization of the duality function leads to* (3.25), *though* μ *is not necessarily a product measure.*

No translation invariance. We continue here with $V = \mathbb{Z}^d$ but drop the assumption of translation invariance. Indeed, equality (3.25) is also valid in contexts where one cannot rely on translation invariance. Examples include spatially inhomogeneous SEP($\boldsymbol{\alpha}$), IRW($\boldsymbol{\alpha}$) and SIP($\boldsymbol{\alpha}$), where the parameters $\boldsymbol{\alpha} = \{\alpha_x, x \in V\}$ in Section 3.1.3 may depend on the site accordingly. In this inhomogeneous setting, self-duality functions jointly factorize over sites and the stationary measures are in product form, with single-site duality functions and marginals both site-dependent (see Section 3.a for further details).

Going back to the general setting of Theorem 3.12, in what follows we show that relation (3.25) between jointly factorized self-duality functions and ergodic stationary measures still holds and, as a consequence, these ergodic stationary measures are, in fact, product measures. The idea is that the averaging over space w.r.t. μ, used in the proof of Theorem 3.13 above, can be replaced by a time average.

If we start with a single dual particle, the dual process is a continuous-time random walk on V, for which we denote by $p_t(x, y)$ the transition probability to go from x to y in time $t > 0$. A basic assumption will then be: for all $x, y \in V$,

$$\lim_{t\to\infty} p_t(x, y) = 0 \tag{3.26}$$

Theorem 3.15 (ERGODICITY & NO TRANSLATION INVARIANCE). *In the setting of Theorem 3.12 with $D(\xi, \eta)$ jointly factorized duality function and μ probability measure on $\mathcal{X}$, if μ is an ergodic stationary measure for the process $\{\eta_t,\ t \ge 0\}$ and* (3.26) *holds for the dual particle, then we have* (3.25) *for all finite configurations ξ; as a consequence, μ is a product measure.*

Proof. The idea is to replace the spatial average in the proof of Theorem 3.13 by a Cesaro average over time, which we can deal with by combining assumption (3.26) with the assumed temporal ergodicity.

Fix $x_1, \ldots, x_n \in V$, $y \in V$. Define

$$\begin{aligned} H_{n+1}(x_1, \ldots, x_n, y) &= \int D(\delta_{x_1} + \ldots + \delta_{x_n} + \delta_y, \eta)\, \mu(d\eta) \\ H_1(y) &= \int D(\delta_y, \eta)\, \mu(d\eta) \\ H_n(x_1, \ldots, x_n) &= \int D(\delta_{x_1} + \ldots + \delta_{x_n}, \eta)\, \mu(d\eta)\, . \end{aligned}$$

It is sufficient to obtain

$$H_{n+1}(x_1, \ldots, x_n, y) = H_1(y)\, H_n(x_1, \ldots, x_n)\, .$$

We already know by the bounded harmonic triviality that H_n only depends on n and not on the given locations $x_1, \ldots, x_n$. Therefore, we have for all $t \ge 0$,

$$H_{n+1}(x_1, \ldots, x_n, y) = \sum_{z\in V} p_t(y, z)\, H_{n+1}(x_1, \ldots, x_n, z)\, .$$

As a consequence, we have

$$H_{n+1}(x_1, \ldots, x_n, y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{z\in V} p_t(y, z)\, H_{n+1}(x_1, \ldots, x_n, z)\, dt\ ,$$

while assumption (3.26) implies

$$H_{n+1}(x_1,\dots,x_n,y) \;=\; \lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{z\notin\{x_1,\dots,x_n\}} p_t(y,z)\, H_{n+1}(x_1,\dots,x_n,z)\, dt\ .$$

By the factorization property of D, we obtain

$$\begin{aligned} &H_{n+1}(x_1,\dots,x_n,y)\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{z\notin\{x_1,\dots,x_n\}} p_t(y,z) \int D(\delta_{x_1}+\dots+\delta_{x_n},\eta)\, D(\delta_y,\eta)\, \mu(d\eta)\, dt\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{z\in V} p_t(y,z) \int D(\delta_{x_1}+\dots+\delta_{x_n},\eta)\, D(\delta_y,\eta)\, \mu(d\eta)\, dt\ , \end{aligned}$$

In conclusion, we have

$$\begin{aligned} \lim_{T\to\infty}\frac{1}{T}\int_0^T \int D(\delta_{x_1}+\dots+\delta_{x_n},\eta)\, \mathbb{E}_\eta\left[D(\delta_y,\eta_t)\right]\, \mu(d\eta)\, dt&\\ = H_1(y)\, H_n(x_1,\dots,x_n)\,&, \end{aligned}$$

where in this last step we used the assumed temporal ergodicity of μ and Birkhoff ergodic theorem. □

3.3 From stationary product measures to jointly factorized duality functions

As we have just illustrated in Section 3.2.3, on the one side relations (3.23)-(3.25) turn out to be useful in deriving information about the product structure of stationary ergodic measures from the knowledge of jointly factorized duality functions. On the other side, granted some information on the stationary product measures – which follows usually from a simple detailed balance computation – up to which extent do relations (3.23)-(3.25) say something about the possible factorized duality functions?

In the context of conditions (3.4)–(3.5) and in presence of a one-parameter family of stationary product measures $\{\mu_\lambda = \otimes\, \nu_\lambda,\ \lambda > 0\}$, relation (3.23) for

$\xi = k\delta_x \in \widehat{\mathcal{X}}$ for some $k \in \mathbb{N}_0$ reads

$$\int_F d(k,z)\, \nu_\lambda(dz) \;=\; \left(\int_F d(1,z)\, \nu_\lambda(dz)\right)^k \;=\; \theta(\lambda)^k\,. \qquad (3.27)$$

As a consequence, knowing the first single-site duality function $d(1,\cdot)$ and the explicit expression of the marginal ν_λ is enough to recover the l.h.s. in (3.27). However, rather than obtaining $d(k,\cdot)$, at this stage the l.h.s. has still the form of an "integral transform"-type of expression for $d(k,\cdot)$.

In the next two subsections, we show how to recover $d(k,z)$ from (3.27) and the knowledge of $\theta(\lambda)$. This then leads to the characterization of all possible jointly factorized (self-)duality functions.

Main results of the section. Relation (3.23), together with the knowledge of the first single-site self-duality function, determines all candidate jointly factorized (self)-duality functions. This is shown in Section 3.3.1 for particle systems (self-duality for SEP, IRW and SIP) and in Section 3.3.2 for diffusion processes (duality between SIP and BEP).

3.3.1 Particle systems: classical and orthogonal self-duality

Going back to the interacting particle systems introduced in Section 3.1.3 with infinitesimal generator (3.6) and stationary product measures with marginals given in (3.8), the integral relation (3.27) rewrites, for each $k \in \mathbb{N}_0$ and $\lambda > 0$ for which $Z_\lambda < \infty$, as

$$\sum_{n\in\mathbb{N}_0} d(k,n)\, \nu_\lambda(n) \;=\; \sum_{n\in\mathbb{N}_0} d(k,n)\, \phi(n)\, \frac{\lambda^n}{n!}\, \frac{1}{Z_\lambda} \;=\; \theta(\lambda)^k\,,$$

where $\phi(n) = n! \prod_{m=1}^n \frac{h(m-1)}{g(m)}$. Now, if we rewrite the above relation as

$$\sum_{n\in\mathbb{N}_0} d(k,n)\, \phi(n)\, \frac{\lambda^n}{n!} \;=\; \theta(\lambda)^k Z_\lambda$$

and interpret

$$\sum_{n\in\mathbb{N}_0} d(k,n)\, \phi(n)\, \frac{\lambda^n}{n!}$$

as the Taylor series expansion around $\lambda = 0$ of the function $\theta(\lambda)^k Z_\lambda$, we can re-obtain the explicit formula of $d(k,n)\,\phi(n)$ as its n-th order derivative evaluated at $\lambda = 0$, namely

$$d(k,n)\,\phi(n) \;=\; \left(\left[\frac{d^n}{d\lambda^n}\right]_{\lambda=0} \theta(\lambda)^k Z_\lambda\right) ,$$

and hence, anytime $\phi(n) > 0$,

$$d(k,n) \;=\; \frac{1}{\phi(n)}\left(\left[\frac{d^n}{d\lambda^n}\right]_{\lambda=0} \theta(\lambda)^k Z_\lambda\right) . \qquad (3.28)$$

Together with the full characterization obtained in Theorem 3.3 of the first single-site self-duality functions $d(1,\cdot)$ – and $\theta(\lambda)$ in turn – we obtain via this procedure a full characterization of all remaining single-site self-duality functions $\{d(k,\cdot),\ k > 1\}$. Besides recovering the "classical" single-site self-duality functions illustrated in Section 3.1.3, we also obtain single-site self-duality functions in terms of orthogonal polynomials $\{\mathfrak{p}_k(n),\ k \in \mathbb{N}_0\}$ of a discrete variable (see e.g. [110]) recently discovered via a different approach in [55]. Remarkable property of these "orthogonal" single-site self-duality functions is that they can be obtained from the classical ones via a Gram-Schmidt orthogonalization procedure w.r.t. the correct probability measures on $\{0,\ldots,\alpha\}$ or $\mathbb{N}_0$, namely the marginals of the associated stationary product measures (see [55] for further details).

For notational purposes, as we will represent these orthogonal polynomials in terms of hypergeometric functions, we recall here the explicit form of those generalized hypergeometric functions which will be of interest for us in this chapter (cf. e.g. [110, Section 2.7]):

$$\begin{aligned} &{}_mF_0\left[\begin{matrix}-n_1 & \ldots & -n_m \\ & - & \end{matrix} ; u\right] \\ &\quad= \sum_{r=0}^{\min\{n_1,\ldots,n_m\}} \frac{n_1!}{(n_1-r)!}\cdots\frac{n_m!}{(n_m-r)!}\,\frac{((-1)^m u)^r}{r!} , \end{aligned} \qquad (3.29)$$

$$\begin{aligned} &{}_mF_1\left[\begin{matrix}-n_1 & \ldots & -n_m \\ & \chi & \end{matrix} ; u\right] \\ &\quad= \sum_{r=0}^{\min\{n_1,\ldots,n_m\}} \frac{n_1!}{(n_1-r)!}\cdots\frac{n_m!}{(n_m-r)!}\,\frac{\Gamma(\chi)}{\Gamma(\chi+r)}\,\frac{((-1)^m u)^r}{r!} , \end{aligned} \qquad (3.30)$$

and

$$ {}_0F_1\left[\begin{matrix} - \\ \chi \end{matrix}; u\right] \;=\; \sum_{r=0}^{\infty} \frac{\Gamma(\chi)}{\Gamma(\chi+r)} \frac{u^r}{r!} \,, \tag{3.31} $$

for all $m \in \mathbb{N}$, $n_1, \ldots, n_m \in \mathbb{N}_0$, $\chi \in \mathbb{R}$ and $u \in \mathbb{R}$.

We divide the discussion in three parts: one dealing with processes of IRW-type, the second one with SEP and SIP and the last one covering all remaining conservative factorized symmetric particle systems introduced in Section 3.1.3 for which only "trivial" (cf. Section 3.1.5) jointly factorized self-duality functions may be found.

Independent random walkers. We recall that the IRW(α)-case corresponds to the choice of values in (3.12) satisfying the relations

$$ g(n) \;=\; n \,, \quad h(0) \;=\; \alpha \;>\; 0 \quad \text{and} \quad h(n) \;=\; h(0) \,. $$

As a consequence,

$$ \phi(n) \;=\; \alpha^n \,, \quad Z_\lambda \;=\; e^{\alpha\lambda} \,, $$

and, if we compute $\theta(\lambda)$ for the general first single-site self-duality function $d(1, n) = a + \frac{b}{\alpha} n$ obtained in (3.13), we get

$$ \theta(\lambda) \;=\; \sum_{n \in \mathbb{N}_0} \left(a + \tfrac{b}{\alpha} n\right) \phi(n) \frac{\lambda^n}{n!} \frac{1}{Z_\lambda} \;=\; a + b\lambda \,. $$

and, in turn via relation (3.28), we recover all functions $d(k, \cdot)$ for $k > 1$:

$$ \begin{aligned} d(k,n) \;&=\; \frac{1}{\alpha^n} \left(\left[\frac{d^n}{d\lambda^n}\right]_{\lambda=0} (a+b\lambda)^k \, e^{\alpha\lambda} \right) \\ &=\; \sum_{r=0}^{n} \binom{n}{r} k(k-1)\cdots(k-r+1) \left(\frac{b}{\alpha}\right)^r a^{k-r} \,. \end{aligned} $$

In case $a = 0$, $d(k, n) = 0$ for $n < k$, while for $n \geq k$, in the summation all terms but the one corresponding to $r = k$ vanish, thus we obtain

$$ d(k,n) \;=\; \left(\frac{b}{\alpha}\right)^k \frac{n!}{(n-k)!} \, \mathbf{1}_{\{k \leq n\}} \,. \tag{3.32} $$

In case $a \neq 0$ (recall (3.29) with $m = 2$),

$$d(k,n) \;=\; a^k \sum_{r=0}^{\min(k,n)} \binom{n}{r}\binom{k}{r} r! \left(\frac{b}{a\alpha}\right)^r \;=\; a^k {}_2F_0\left[\begin{matrix}-k \quad -n \\ - \end{matrix}\,;\, \frac{b}{a\alpha}\right]. \qquad (3.33)$$

In conclusion, for the choice $a \cdot b < 0$,

$$d(k,n) \;=\; a^k C_k^{(\beta)}(n)\,,$$

where $\beta = -\frac{a\alpha}{b}$ in this case and $\{C_k^{(\beta)}(n),\ k \in \mathbb{N}_0\}$ are the Poisson-Charlier polynomials – orthogonal polynomials w.r.t. the Poisson distribution of parameter $\beta > 0$ (cf. [110]).

Exclusion and inclusion processes. For SEP(α) and SIP(α) we are in the case $h(0) = \alpha$ and $h(1) \neq h(0)$, and hence we abbreviate

$$\sigma \;=\; h(1) - h(0)\,.$$

We recall from Section 3.1.4 that for SEP(α) we have $\sigma = -1$ with $F = \{0, \ldots, \alpha\}$, while for SIP($\alpha$) we have $\sigma = 1$ with $F = \mathbb{N}_0$ and, in both cases,

$$\phi(n) \;=\; \sigma^n \frac{\Gamma(\sigma\alpha + n)}{\Gamma(\sigma\alpha)}\,, \quad Z_\lambda \;=\; (1 - \sigma\lambda)^{-\sigma\alpha}\,.$$

Hence, if we compute $\theta(\lambda)$ for $d(1,n) = a + \frac{b}{\alpha}n$ in (3.13), we get

$$\theta(\lambda) \;=\; a + b\lambda\,(1 - \sigma\lambda)^{-1} \;=\; \frac{a + (b - a\sigma)\,\lambda}{1 - \sigma\lambda}\,.$$

By applying formula (3.28), we obtain all functions $d(k,\cdot)$ for $k > 1$ as follows:

$$\begin{aligned} d(k,n) \;&=\; \frac{1}{\phi(n)} \left(\left[\frac{d^n}{d\lambda^n}\right]_{\lambda=0} (a + (b - a\sigma)\,\lambda)^k\,(1 - \sigma\lambda)^{-k-\sigma\alpha} \right) \\ &=\; \frac{\Gamma(\sigma\alpha)}{\Gamma(\sigma\alpha + n)} \sum_{r=0}^{n} \binom{n}{r}\binom{k}{r} r!\, a^{k-r} \left(\frac{b}{\sigma} - a\right)^r \frac{\Gamma(\sigma\alpha + n + k - r)}{\Gamma(\sigma\alpha + k)}\,. \qquad (3.34) \end{aligned}$$

In case $a = 0$, clearly $d(k,n) = 0$ for $n < k$, while for $n \geq k$ only the term for $r = k$ is nonzero in the summation:

$$d(k,n) = \left(\frac{b}{\sigma}\right)^k \frac{\Gamma(\sigma\alpha)}{\Gamma(\sigma\alpha + k)} \frac{n!}{(n-k)!} \mathbf{1}_{\{k \leq n\}} \,. \tag{3.35}$$

Note that the functions $d(k,\cdot)$ in (3.35) further simplify into

$$d(k,n) = b^k \frac{(\alpha - k)!}{\alpha!} \frac{n!}{(n-k)!} \mathbf{1}_{\{k \leq n\}}$$

for the choice $\sigma = -1$ (SEP(α), $\alpha \in \mathbb{N}$), and into

$$d(k,n) = b^k \frac{\Gamma(\alpha)}{\Gamma(\alpha + k)} \frac{n!}{(n-k)!} \mathbf{1}_{\{k \leq n\}}$$

for $\sigma = 1$ (SIP(α), $\alpha > 0$). In case $a \neq 0$, if we go back to (3.34) and use there the "known relation" ([110], p. 51) for the hypergeometric functions in (3.30), we have (see also (3.30) with $m = 2$)

$$\begin{aligned} d(k,n) &= a^k \frac{\Gamma(\sigma\alpha)\Gamma(\sigma\alpha + n + k)}{\Gamma(\sigma\alpha + n)\Gamma(\sigma\alpha + k)} {}_2F_1\left[\begin{matrix} -n \quad -k \\ -n - k - \sigma\alpha + 1 \end{matrix}; 1 - \frac{b}{a\sigma}\right] \\ &= a^k {}_2F_1\left[\begin{matrix} -n \quad -k \\ \sigma\alpha \end{matrix}; \frac{b}{a\sigma}\right] . \end{aligned} \tag{3.36}$$

If $\sigma = -1$, $\alpha \in \mathbb{N}$ and given the additional requirements $0 < -\frac{a}{b} < 1$, from the expression in (3.36) we have a representation of the single-site duality functions in terms of the Kravchuk polynomials as defined in [110], i.e.

$$d(k,n) = b^k K_k^{(\alpha,p)}(n) \frac{1}{\binom{\alpha}{k}} ,$$

where $p = -\frac{a}{b}$ in our case and $\{K_k^{(\alpha,p)}(n),\ k \in \mathbb{N}_0\}$ are the Kravchuk polynomials – orthogonal polynomials w.r.t. the Binomial distribution Binomial(α, p) (see Section 3.1.4).

If $\sigma = 1$, $\alpha > 0$ and if $a \cdot b < 0$, we recognize in (3.36) the Meixner polynomials as defined in [110], i.e.

$$d(k,n) = a^k \frac{\Gamma(\alpha)}{\Gamma(\alpha + k)} M_k^{(\alpha,\beta)}(n) ,$$

where in our case $\beta = \frac{a}{a-b} \in (0,1)$ and $\{M_k^{(\alpha,\beta)}(n),\ k \in \mathbb{N}_0\}$ are the Meixner polynomials – orthogonal polynomials w.r.t. the discrete Gamma distribution $\text{Gamma}_{\text{d}}(\alpha,\beta)$ (see Section 3.1.4).

Remark 3.16 (CLASSICAL & ORTHOGONAL SELF-DUALITIES). *We emphasize that all jointly factorized self-duality functions (up to irrelevant factors depending on the total number of particles) for independent random walkers, exclusion and inclusion processes satisfying* (3.5) *are necessarily in either one of the following two forms: in the "classical" form of Section 3.1.4 (case $a = 0$) or in the form of products of rescaled versions of orthogonal polynomials (case $a \neq 0$). Other jointly factorized self-duality functions for* $\text{SEP}(\alpha)$, $\text{IRW}(\alpha)$ *and* $\text{SIP}(\alpha)$ *than those just mentioned do not exist.*

Remark 3.17 (TRIANGULAR VS. SYMMETRIC SELF-DUALITY FUNCTIONS). *Apart from the leading factor a^k, the remaining polynomials in the expressions of $d(k,n)$ for $a \neq 0$ are* "self-dual" *in the sense of orthogonal polynomials literature, i.e. $\mathfrak{p}_n(k) = \mathfrak{p}_k(n)$ in our context (see e.g. [87, Definition 3.1]). Henceforth, if $d(k,n)$ is interpreted as a countable matrix with elements indexed by $k, n \in \mathbb{N}_0$, the value $a \in \mathbb{R}$ is the only responsible for the asymmetry of $d(k,n)$: upper-triangular for $a = 0$ while symmetric for $a = 1$.*

"Trivial" jointly factorized self-duality. For the sake of completeness, we can implement the same machinery to cover all jointly factorized self-dualities with property (3.5) for all particle systems of type (3.6).

Indeed, from the proof of Theorem 3.3, if the process is neither of the types IRW, SIP and SEP, then the only possible choice is $d(1,n) = a$ for some $a \in \mathbb{R}$, i.e. $d(1,n)$ is not depending on n. From this we get $\theta(\lambda) = a$ and $d(k,n) = a^k$ from formula (3.28). Hence, the self-duality functions must be of the form

$$D(\xi,\eta) \;=\; \prod_{x \in V} d(\xi(x),\eta(x)) \;=\; a^{|\xi|}\,,$$

i.e. depending only on the total number of dual particles (and not on the configuration η). Hence, the duality relation in that case reduces to the trivial relation, for all $t \geq 0$ and $\xi \in \widehat{\mathcal{X}}$,

$$\mathbb{E}_\xi\left[a^{|\xi_t|}\right] \;=\; a^{|\xi|}\,, \quad a \in \mathbb{R}\,,$$

which is just conservation of the number of particles in the dual process. No other self-duality relation with jointly factorized self-duality functions can exist for a conservative factorized symmetric IPS different from SEP, IRW or SIP.

3.3.2 Interacting diffusions and particle systems: classical and orthogonal duality

As shown in Theorem 3.9, relation (3.27) still holds whenever the discrete right-variables $n \in \mathbb{N}_0$ are replaced by continuous variables $z \in [0,\infty)$ and sums by integrals. With this observation in mind, we provide a second method to characterize all jointly factorized duality functions between the continuous process BEP(α) and its discrete dual SIP(α).

More precisely, if $d(k,z)$ is a single-site duality function with property (3.5) between BEP(α) and SIP(α), and ν_λ is the stationary product measure marginal for BEP(α) as in (3.21), then, from the analogue of relation (3.27) for $k = 1$, namely

$$\int_{[0,\infty)} d(1,z)\, z^{\alpha-1} e^{-\lambda z} \frac{\lambda^\alpha}{\Gamma(\alpha)}\, dz \;=\; \theta(\lambda)\,, \tag{3.37}$$

we necessarily have by Theorem 3.9 that

$$\int_{[0,\infty)} d(k,z)\, \frac{z^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda z}\, dz \;=\; \theta(\lambda)^k \lambda^{-\alpha}\,. \tag{3.38}$$

As a consequence, the function $d(k,z)\frac{z^{\alpha-1}}{\Gamma(\alpha)}$ is the inverse Laplace transform of $\theta(\lambda)^k \lambda^{-\alpha}$. Given the first single-site duality function $d(1,z)$ in (3.22), from (3.37) we obtain

$$\theta(\lambda) \;=\; \int_{[0,\infty)} \left(a + \tfrac{b}{\alpha} z\right) z^{\alpha-1} e^{-\lambda z} \frac{\lambda^\alpha}{\Gamma(\alpha)}\, dz \;=\; \left(a\lambda + b\right) \lambda^{-1}\,. \tag{3.39}$$

As a consequence, the r.h.s. in (3.38) becomes

$$\theta(\lambda)^k \lambda^{-\alpha} \;=\; \left(a\lambda + b\right)^k \lambda^{--\alpha-k}\,, \tag{3.40}$$

and there exist explicit expressions for the inverse Laplace transform of this function. We split the computation in two cases. In case $a = 0$, since the inverse Laplace transform of $\lambda^{-\alpha-k}$ is $\frac{z^{\alpha+k-1}}{\Gamma(\alpha+k)}$, we obtain

$$d(k,z) \; = \; b^k \, z^k \, \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)} \, ,$$

i.e. the "classical" single-site duality function as in Section 3.1.6. In case $a \neq 0$, the inverse Laplace transform of (3.40) is more elaborated (recall (3.30) with $m = 1$):

$$a^k \, \frac{z^{\alpha-1}}{\Gamma(\alpha)} \, {}_1F_1\!\left[\begin{matrix} -k \\ \alpha \end{matrix} ; -\frac{b}{a} z\right] .$$

As the above expression must equal $d(k,z) \, \frac{z^{\alpha-1}}{\Gamma(\alpha)}$, it follows that

$$d(k,z) \; = \; a^k \, {}_1F_1\!\left[\begin{matrix} -k \\ \alpha \end{matrix} ; -\frac{b}{a} z\right] .$$

As a final consideration, we note that for the choice $a \cdot b < 0$,

$$d(k,z) \; = \; a^k \, k! \, \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)} \, L_k^{(\alpha-1,\beta)}(z) \, ,$$

where $\beta = -\frac{b}{a}$ here and $\{L_k^{(\alpha-1,\beta)}(z), \; k \in \mathbb{N}_0\}$ are the generalized Laguerre polynomials – orthogonal polynomials w.r.t. to the Gamma distribution of shape parameter α and rate parameter β as defined in [110].

3.4 Intertwining and generating functions

In this section, we introduce the generating function method, which allows to go from a self-duality of a discrete process towards duality between a discrete and continuous process, and further towards a self-duality of a continuous process, and back. This then allows e.g. to simplify the proof of a discrete self-duality by lifting it to a continuous self-duality, which is usually easier to verify. The key ingredient of this method are *intertwining* relations. In particular, we find intertwining relations between discrete multiplication and derivation operators by means of appropriate generating functions.

After an introduction to intertwining in Section 3.4.2 and its relation with duality in Theorem 3.19, we find in Propositions 3.20–3.22 (product) intertwiners between particle system generators introduced in Section 3.1.4 and their diffusion counterparts of Section 3.1.6. As a corollary of Propositions

3.20–3.22, in Section 3.4.4 we prove that all the candidate jointly factorized self-duality functions produced in Section 3.3.1 (from the stationary product measures via (3.23)) are actual self-duality functions. Seemingly, we also produce several (self-)duality functions for the diffusion counterparts of the particle systems.

We start with the introductory example of IRW(1) on two sites ($V = \{x, y\}$) in Section 3.4.1, showing that the single-bond generator intertwines with the first order differential operator given in (3.17) with $\alpha = 1$, and from that recover in an easy way self-duality of IRW(1) with "classical" self-duality functions as in Section 3.1.4.

3.4.1 Introductory example: independent random walkers

To make the method clear, let us start with a simple example of independent random walkers on a single bond $\{x, y\}$. The generator is

$$\begin{aligned} L\,\varphi(n_x, n_y) \;&= n_x\,(\varphi(n_x - 1, n_y + 1) - \varphi(n_x, n_y)) \\ &\quad + n_y\,(\varphi(n_x + 1, n_y - 1) - \varphi(n_x, n_y))\,, \end{aligned}$$

with $n_x, n_y \in \mathbb{N}_0$. Define now the (exponential) generating function

$$\mathscr{G}\varphi(z_x, z_y) \;=\; \sum_{n_x, n_y = 0}^{\infty} \varphi(n_x, n_y) \frac{z_x^{n_x}\, z_y^{n_y}}{n_x!\, n_y!}\,, \quad z_x, z_y \in [0, \infty)\,.$$

Then it is easy to see that $\mathscr{G}$ produces an intertwining relation

$$\mathscr{L}\mathscr{G} \;=\; \mathscr{G}L$$

involving L and $\mathscr{L}$, where

$$\mathscr{L} \;=\; -\left(z_x - z_y\right)\left(\partial_x - \partial_y\right)$$

is the operator given in (3.17) with $\alpha = 1$ and with ∂_x denoting partial derivation w.r.t. $z_x \in [0, \infty)$. Now assume that we have a self-duality function for the particle system, i.e.

$$L_{\text{left}} D \;=\; L_{\text{right}} D\,.$$

Then, combining the self-duality with the intertwining relations above, we obtain

$$L_{\text{left}} \mathscr{D} \;=\; \mathscr{L}_{\text{right}} \mathscr{D}\,,$$

where

$$\begin{aligned}\mathscr{D}((k_x,k_y),(z_x,z_y)) &= \mathscr{G}_{\text{right}} D((k_x,k_y),(z_x,z_y)) \\ &= \sum_{n_x,n_y=0}^{\infty} D((k_x,k_y),(n_x,n_y))\,\frac{z_x^{n_x}\,z_y^{n_y}}{n_x!\,n_y!}\ . \end{aligned} \tag{3.41}$$

In words, a self-duality function for L is "lifted" to a duality function between the IRW generator L and its continuous counterpart $\mathscr{L}$ by applying the generating function $\mathscr{G}$ to the n-variables. However, if we read out of the last expression in (3.41) the Taylor series expansion of $\mathscr{D}$, we obtain an inverse statement: given a duality function between the independent random walk generator L and its continuous counterpart $\mathscr{L}$, its Taylor coefficients provide a self-duality function of L.

We can then also take the generating function w.r.t. the k-variables in the function $\mathscr{D}$ to produce a self-duality function for $\mathscr{L}$, i.e. defining

$$\begin{aligned}\mathcal{D}((v_x,v_y),(z_x,z_y)) &= \mathscr{G}_{\text{left}} \mathscr{D}((v_x,v_y),(z_x,z_y)) \\ &= \sum_{k_x,k_y=0}^{\infty} \mathscr{D}((k_x,k_y),(z_x,z_y))\,\frac{v_x^{k_x}\,v_y^{k_y}}{k_x!\,k_y!}\ ,\end{aligned}$$

we have

$$\mathscr{L}_{\text{left}}\mathcal{D} = \mathscr{L}_{\text{right}}\mathcal{D}\ .$$

For the classical self-duality function

$$D((k_x,k_y),(n_x,n_y)) = \frac{n_x!}{(n_x-k_x)!}\,\frac{n_y!}{(n_y-k_y)!}\mathbf{1}_{\{k_x\le n_x\}}\mathbf{1}_{\{k_y\le n_y\}}\ ,$$

we find that

$$\mathcal{D}((v_x,v_y),(z_x,z_y)) = e^{z_x+z_y}\,e^{v_x z_x+v_y z_y}\ .$$

Beside the factor $e^{z_x+z_y}$ which depends only on the conserved quantity z_x+z_y, to check the self-duality relation for $\mathscr{L}$ w.r.t. the function $e^{v_x z_x+v_y z_y}$ is rather straightforward, the computation involving only derivatives of exponentials. By looking at the Taylor coefficients w.r.t. both v and z-variables of this self-duality relation, we obtain the self-duality relation for L w.r.t. D where we started from.

In conclusion, all these duality relations turn out to be equivalent, and the proof of self-duality for particle systems requiring rather intricate combinatorial arguments (see e.g. [31]) is superfluous once the more direct self-duality for diffusion systems is checked.

3.4.2 Intertwining and duality

The notion of *intertwining* between stochastic processes was originally introduced by Yor in [140] in the context of Markov chains and later pursued in [36] and [53] as an abstract framework, in discrete-time and continuous-time respectively, for the problem of Markov functionals, i.e. finding sufficient and necessary conditions under which a random function of a Markov chain is again Markovian.

For later purposes, we adopt a rather general definition of intertwining, in which $\{\eta_t,\ t \geq 0\}$ and $\{\zeta_t,\ t \geq 0\}$ are continuous-time stochastic processes on the Polish spaces $\mathcal{X}$ and $\mathcal{X}'$, respectively, whose expectations read $\mathbb{E}$, $\mathbb{E}'$ resp., and $\mathcal{M}(\mathcal{X})$ denotes the space of signed measures on $\mathcal{X}$. We say that $\{\zeta_t,\ t \geq 0\}$ is *intertwined on top* of $\{\eta_t,\ t \geq 0\}$ if there exists a mapping $\Lambda : \mathcal{X}' \to \mathcal{M}(\mathcal{X})$ such that, for all $t \geq 0$, $\zeta \in \mathcal{X}'$ and smooth $\varphi : \mathcal{X} \to \mathbb{R}$,

$$\mathbb{E}'_\zeta \left[\int_{\mathcal{X}'} \varphi(\eta)\, \Lambda(\zeta_t)(d\eta) \right] = \int_{\mathcal{X}} \mathbb{E}_\eta \left[\varphi(\eta(t)) \right] \Lambda(\zeta)(d\eta)\,. \tag{3.42}$$

Working at the abstract level of *semigroups*, we say that $\{\mathscr{S}_t,\ t \geq 0\}$ on a space of functions $f : \mathcal{X}' \to \mathbb{R}$ denoted by $\mathcal{F}(\mathcal{X}')$, is *intertwined on top* of $\{S_t,\ t \geq 0\}$, a semigroup on a space of functions $f : \mathcal{X} \to \mathbb{R}$ denoted by $\mathcal{F}(\mathcal{X})$, with *intertwiner* Λ if Λ is a linear operator from $\mathcal{F}(\mathcal{X})$ into $\mathcal{F}(\mathcal{X}')$ and if, for all $t \geq 0$ and $\varphi : \mathcal{X} \to \mathbb{R}$,

$$\mathscr{S}_t \Lambda \varphi = \Lambda S_t \varphi\,.$$

Similarly, *operators* $\mathscr{L}$ with domain $\mathcal{D}(\mathscr{L})$ and L with domain $\mathcal{D}(L)$ are *intertwined* with intertwiner Λ if, for all $\varphi \in \mathcal{D}(L)$, $\Lambda\varphi \in \mathcal{D}(\mathscr{L})$ and

$$\mathscr{L}\Lambda = \Lambda L \varphi\,. \tag{3.43}$$

Notice that with a slight abuse of notation we used the same symbol Λ for an abstract intertwining operator as for the intertwining mapping. In other words, in case the intertwining mapping as in (3.42) is given by $\tilde{\Lambda}$, then the corresponding operator is

$$\Lambda\varphi(\zeta) = \int \varphi(\eta)\,\tilde{\Lambda}(\zeta)(d\eta)\,.$$

An intertwining mapping Λ has a probabilistic interpretation if it takes values in the subset of probability measures on $\mathcal{X}$. Indeed, in (3.42) the process $\{\zeta_t,\ t \geq 0\}$ may be viewed as an added structure on top of $\{\eta_t,\ t \geq 0\}$ or, alternatively, the process $\{\eta_t,\ t \geq 0\}$ as a random functional of $\{\zeta_t,\ t \geq 0\}$, in which Λ provides this link.

Remark 3.18 (FINITE STATE SPACE). *The connection with duality introduced in Section 3.1.1 becomes transparent when $\mathcal{X}$, $\widehat{\mathcal{X}}$ and $\mathcal{X}'$ are finite sets and the operators and functions L, $\widehat{L}$ $\mathscr{L}$, D and Λ in* (3.3) *and* (3.43) *are represented in terms of matrices. There, relations* (3.3) *and* (3.43), *once rewritten in matrix notation respectively as*

$$\widehat{L}D = DL^{\mathsf{T}}\,, \tag{3.44}$$

where L^{T} denotes the transpose of L, and

$$\mathscr{L}\Lambda = \Lambda L\,, \tag{3.45}$$

differ essentially only in the terms L^{T} versus L in the r.h.s. of both identities. The presence or absence of transposition can be interpreted as a forward-*versus*-backward *evolution against a* forward-*versus*-forward *evolution. More precisely, if L, $\widehat{L}$ and $\mathscr{L}$ are generators of Markov processes $\{\eta_t, t \geq 0\}$, $\{\xi_t,\ t \geq 0\}$ and $\{\zeta_t,\ t \geq 0\}$, respectively, then* (3.44) *and* (3.45) *relate the evolution of $\{\eta_t,\ t \geq 0\}$ to that of $\{\xi_t,\ t \geq 0\}$, resp. $\{\zeta_t,\ t \geq 0\}$; however, while in* (3.45) *the processes run both along the same direction in time, in* (3.44) *the processes run along opposite time directions.*

Intertwiners as Λ in (3.45) may be also interpreted as natural generalizations of *symmetries of generators*, indeed (3.45) with $\mathscr{L} = L$ just means that Λ commutes with L, which is the definition of a symmetry of L. As outlined in [62, Theorem 2.6], the knowledge of symmetries of a generator and dualities of this generator leads to the construction of new dualities. The following theorem presents the analogue procedure in presence of intertwiners: a duality and an intertwining lead to a new duality.

Theorem 3.19. *Let L, $\widehat{L}$ and $\mathscr{L}$ be operators on real-valued functions on $\mathcal{X}$, $\widehat{\mathcal{X}}$ and $\mathcal{X}'$, respectively. Suppose that there exists an intertwiner Λ such that for all*

$\varphi \in \mathcal{D}(L)$, $\Lambda\varphi \in \mathcal{D}(\mathscr{L})$,

$$\mathscr{L}\Lambda\varphi \;=\; \Lambda L\varphi\,, \tag{3.46}$$

and a duality function $D : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ for $\widehat{L}$ and L, namely $D(\xi, \cdot) \in \mathcal{D}(L)$ for all $\xi \in \widehat{\mathcal{X}}$, $D(\cdot, \eta) \in \mathcal{D}(\widehat{L})$ for all $\eta \in \mathcal{X}$ and

$$\widehat{L}_{\text{left}} D \;=\; L_{\text{right}} D\,.$$

Then, if $\Lambda_{\text{right}} D(\xi, \cdot) \in \mathcal{D}(\mathscr{L})$ for all $\xi \in \widehat{\mathcal{X}}$ and $\Lambda_{\text{right}} D(\cdot, \zeta) \in \mathcal{D}(\widehat{L})$ for all $\zeta \in \mathcal{X}'$, $\Lambda_{\text{right}} D$ is a duality function for $\widehat{L}$ and $\mathscr{L}$, i.e.

$$\widehat{L}_{\text{left}} \Lambda_{\text{right}} D \;=\; \mathscr{L}_{\text{right}} \Lambda_{\text{right}} D\,.$$

Proof.

$$\widehat{L}_{\text{left}} \Lambda_{\text{right}} D \;=\; \Lambda_{\text{right}} \widehat{L}_{\text{left}} D \;=\; \Lambda_{\text{right}} L_{\text{right}} D \;=\; \mathscr{L}_{\text{right}} \Lambda_{\text{right}} D\,.$$

Here in the first equality we used that left and right actions commute, in the second equality we used the assumed duality of $\widehat{L}$ and L, and in the third equality we used the assumed intertwining. □

3.4.3 Intertwining between continuum and discrete processes

In this section we prove the existence of an intertwining relation between the interacting diffusion processes presented in Section 3.1.6 and the particle systems of Section 3.1.4. This intertwining relation provides a second connection, besides the many-particle limit procedure (see Section 3.1.6), between continuum and discrete processes, which proves to be better suited for the goal of establishing duality relations among these processes. Indeed, the characterization of all possible factorized self-dualities for particle systems obtained in Section 3.3 and the intertwining relation below, via the application of Theorem 3.19, produces a characterization of all possible dualities, resp. self-dualities, between the discrete and the continuum processes, resp. of the continuum process.

In the following proposition, we prove the intertwining relation for operators L^σ and $\mathscr{L}^\sigma$ defined, respectively, on functions $\varphi : F^V \to \mathbb{R}$, $F = \mathbb{N}_0$, as

$$L^\sigma \varphi(\eta) \;=\; \sum_{x \sim y} c(\{x,y\})\, L^\sigma_{\{x,y\}} \varphi(\eta)\,, \quad \eta \in F^V\,, \tag{3.47}$$

where

$$L^\sigma_{\{x,y\}}\varphi(\eta) = \eta(x)\,(\alpha+\sigma\eta(y))\,(\varphi(\eta^{x,y})-\varphi(\eta)) + \eta(y)\,(\alpha+\sigma\eta(x))\,(\varphi(\eta^{y,x})-\varphi(\eta))\,,$$

and, on real analytic functions $\varphi : \mathbb{R}^V \to \mathbb{R}$, as

$$\mathscr{L}^\sigma\varphi(\zeta) = \sum_{x\sim y} c(\{x,y\})\,\mathscr{L}^\sigma_{\{x,y\}}\varphi(\zeta)\,, \quad \zeta\in\mathbb{R}^V\,, \tag{3.48}$$

where

$$\mathscr{L}^\sigma_{\{x,y\}}\varphi(\zeta) = \left(-\alpha\,(\zeta(x)-\zeta(y))\,(\partial_x-\partial_y)+\sigma\,\zeta(x)\,\zeta(y)\,(\partial_x-\partial_y)^2\right)\varphi(\zeta)\,.$$

Note that L^σ in (3.47) is a special instance of the generator L in (3.6) with conditions (3.12), while $\mathscr{L}^\sigma$ above matches – on a common sub-domain and for particular choices of the parameters σ and α – those in (3.16).

Proposition 3.20 (INTERTWINING "$\mathscr{L}^\sigma G^\otimes = G^\otimes L^\sigma$"). *Let G be the Poisson probability kernel defined as the operator that maps functions $f : \mathbb{N}_0 \to \mathbb{R}$ into functions $Gf : \mathbb{R}\to\mathbb{R}$ as*

$$Gf(z) = \sum_{n=0}^\infty f(n)\,\frac{z^n}{n!}\,e^{-z}\,, \quad z\in\mathbb{R}\,. \tag{3.49}$$

Then, whenever $Gf : \mathbb{R}\to\mathbb{R}$ is a real analytic function, if $G^\otimes = \otimes_{x\in V} G_x$ denotes the tensorized operator mapping functions $\varphi : \mathbb{N}_0^V \to \mathbb{R}$ into functions $G^\otimes\varphi : \mathbb{R}^V\to\mathbb{R}$ accordingly, $\mathscr{L}^\sigma$ and L^σ are intertwined with intertwiner $G^\otimes$, namely

$$\mathscr{L}^\sigma G^\otimes\varphi(\zeta) = G^\otimes L^\sigma\varphi(\zeta)\,, \quad \zeta\in\mathbb{R}^V\,. \tag{3.50}$$

Proof. Let us introduce the non-normalized operator

$$\bar{G}f(z) = \sum_{n=0}^\infty f(n)\,\frac{z^n}{n!}\,, \quad z\in\mathbb{R}\,,$$

and the associated tensorized operator $\bar{G}^\otimes = \otimes_{x\in V}\bar{G}_x$. Due to the factorized structure of L^σ, $\mathscr{L}^\sigma$ and $\bar{G}^\otimes$, the proof of the intertwining relation (3.50) with

$\bar{G}^{\otimes}$ as an intertwiner reduces to consider and combine the following relations:

$$\sum_{n=0}^{\infty} n\, f(n-1)\,\frac{z^n}{n!} \;=\; z\,\bar{G}f(z) \qquad \sum_{n=0}^{\infty} n\, f(n)\,\frac{z^n}{n!} \;=\; z\,\frac{d}{dz}\bar{G}f(z)$$
$$\sum_{n=0}^{\infty} f(n+1)\,\frac{z^n}{n!} \;=\; \frac{d}{dz}\bar{G}f(z) \qquad \sum_{n=0}^{\infty} n\, f(n+1)\,\frac{z^n}{n!} \;=\; z\,\frac{d^2}{dz^2}\bar{G}f(z)\,.$$

As a first consequence, we have

$$\mathscr{L}^{\sigma}\bar{G}^{\otimes}\varphi(\zeta) \;=\; \bar{G}^{\otimes}L^{\sigma}\varphi(\zeta)\,, \quad \zeta \in \mathbb{R}^V\,.$$

We obtain (3.50) by observing that ($|\zeta| = \sum_{x\in V}\zeta(x)$)

$$G^{\otimes}\varphi(\zeta) \;=\; e^{-|\zeta|}\,\bar{G}^{\otimes}\varphi(\zeta)\,, \quad \zeta \in \mathbb{R}^V\,,$$

and that, for $\psi(\zeta) = \bar{\psi}(\zeta)\cdot e^{-|\zeta|}$,

$$\mathscr{L}^{\sigma}\psi(\zeta) \;=\; e^{-|\zeta|}\,\mathscr{L}^{\sigma}\bar{\psi}(\zeta)\,, \quad \zeta \in \mathbb{R}^V\,.$$

□

Remark 3.21 (PROBABILISTIC INTERPRETATION). *The intertwiner $G^{\otimes}$ has a nice probabilistic interpretation: from an "energy" configuration $\zeta \in [0,\infty)^V$, the associated particle configurations are generated by placing – independently over the sites – a number of particles on site $x \in V$ distributed according to a Poisson random variable with intensity $\zeta(x)$.*

In the remaining part of this section, under some reasonable regularity assumptions, we are able to invert the intertwining relation (3.50), namely to find an operator $H^{\otimes} = \otimes_{x\in V}H_x$ that intertwines L^{σ} and $\mathscr{L}^{\sigma}$, in this order. The natural candidate for H is the "inverse operator" of G, whenever this is well-defined. In general, this "inverse intertwiner" lacks any probabilistic interpretation, but indeed establishes a second intertwining relation useful in the subsequent section.

Proposition 3.22 (INTERTWINING "$L^{\sigma}H^{\otimes} = H^{\otimes}\mathscr{L}^{\sigma}$"). *Let H be the differential operator mapping real analytic functions $g : \mathbb{R} \to \mathbb{R}$ into functions $Hg : \mathbb{N}_0 \to \mathbb{R}$ as*

$$Hg(n) \;=\; \left(\left[\frac{d^n}{dz^n}\right]_{z=0} e^z g(z)\right)\,, \quad n \in \mathbb{N}_0\,. \tag{3.51}$$

Then H is the inverse operator of G, namely, for all $f : \mathbb{N}_0 \to \mathbb{R}$ such that $Gf : \mathbb{R} \to \mathbb{R}$ is real analytic, we have

$$GHg(z) = g(z), \quad z \in \mathbb{R}, \quad HGf(n) = f(n), \quad n \in \mathbb{N}_0 .$$

Moreover, the tensorized operator $H^{\otimes} = \otimes_{x \in V} H_x$ is an intertwiner for L^{σ} and $\mathscr{L}^{\sigma}$, i.e. for all real analytic $\psi : \mathbb{R}^V \to \mathbb{R}$,

$$L^{\sigma} H^{\otimes} \psi(\eta) = H^{\otimes} \mathscr{L}^{\sigma} \psi(\eta), \quad \eta \in \mathbb{N}_0^V . \tag{3.52}$$

Before giving the proof, we need the following lemma.

Lemma 3.23 (SYMMETRY FOR L^{σ})**.** *Let A be the operator acting on functions $f : \mathbb{N}_0 \to \mathbb{R}$ defined as*

$$Af(n) = \sum_{k=0}^{n} \binom{n}{k} f(k), \quad n \in \mathbb{N}_0 .$$

Then, the tensorized operator $A^{\otimes} = \otimes_{x \in V} A_x$ is a symmetry for the generator L^{σ}, i.e. for all $\varphi : \mathbb{N}_0^V \to \mathbb{R}$

$$A^{\otimes} L^{\sigma} \varphi(\eta) = L^{\sigma} A^{\otimes} \varphi(\eta), \quad \eta \in \mathbb{N}_0^V . \tag{3.53}$$

Proof. Instead of going through tedious computations, we exploit the fact that the operator $A^{\otimes}$ has the form

$$A^{\otimes} = \otimes_{x \in V} A_x = \otimes_{x \in V} e^{\mathcal{K}_x^-} = \otimes_{x \in V} \sum_{\ell=0}^{\infty} \frac{\left(\mathcal{K}_x^-\right)^{\ell}}{\ell!} ,$$

where $\mathcal{K}_x^-$ is an operator defined for functions $\varphi : \mathbb{N}_0^V \to \mathbb{R}$ which acts only on the x-th coordinates as

$$\mathcal{K}_x^- \varphi(\eta) = \eta(x) f(\eta - \delta_x), \quad \eta \in \mathbb{N}_0^V .$$

Since all these operators $\{\mathcal{K}_x^-, \ x \in V\}$ commute over the sites, we have

$$A^{\otimes} = \otimes_{x \in V} e^{\mathcal{K}_x^-} = e^{\sum_{x \in V} \mathcal{K}_x^-} .$$

We conclude the proof by noting that the operator

$$K^- := \sum_{x\in V} \mathcal{K}_x^-$$

is a symmetry for the generator L^σ, see e.g. [20], [62] or Section 4.5.1. □

Proof of Proposition 3.22. First we compute the following key relations:

$$\begin{aligned}
\left(\left[\frac{d^n}{dz^n}\right]_{z=0} g'(z)\right) &= \left(\left[\frac{d^{n+1}}{dz^{n+1}}\right]_{z=0} g(z)\right)\\
\left(\left[\frac{d^n}{dz^n}\right]_{z=0} zg(z)\right) &= n\left(\left[\frac{d^{n-1}}{dz^{n-1}}\right]_{z=0} g(z)\right)\\
\left(\left[\frac{d^n}{dz^n}\right]_{z=0} zg'(z)\right) &= n\left(\left[\frac{d^{n}}{dz^{n}}\right]_{z=0} g(z)\right)\\
\left(\left[\frac{d^n}{dz^n}\right]_{z=0} zg''(z)\right) &= n\left(\left[\frac{d^{n+1}}{dz^{n+1}}\right]_{z=0} g(z)\right) .
\end{aligned}$$

Hence, if we introduce the operator

$$\bar{H}g(n) = \left(\left[\frac{d^n}{dz^n}\right]_{z=0} g(z)\right), \quad n \in \mathbb{N}_0 ,$$

and the associated tensorized operator $\bar{H}^\otimes = \otimes_{x\in V}\bar{H}_x$, we obtain

$$L^\sigma \bar{H}^\otimes \psi(\eta) = \bar{H}^\otimes \mathscr{L}^\sigma \psi(\eta), \quad \eta \in \mathbb{N}_0^V . \tag{3.54}$$

Now, by using Lemma 3.23 and noting that

$$Hg(n) = \sum_{k=0}^{n} \binom{n}{k} \bar{H}g(k) = A\bar{H}g(n), \quad n \in \mathbb{N}_0 ,$$

and, by the mixed property of the tensor product,

$$H^\otimes \psi(\eta) = (A\bar{H})^\otimes \psi(\eta) = A^\otimes \bar{H}^\otimes \psi(\eta), \quad \eta \in \mathbb{N}_0^V , \tag{3.55}$$

we get (3.52) by applying first (3.55), then (3.53) and finally (3.54):

$$L^\sigma H^\otimes \psi = L^\sigma A^\otimes \bar{H}^\otimes \psi = A^\otimes L^\sigma \bar{H}^\otimes \psi = A^\otimes \bar{H}^\otimes \mathscr{L}^\sigma \psi = H^\otimes \mathscr{L}^\sigma \psi .$$

□

3.4.4 Generating functions and duality

As anticipated in the previous section, from the intertwining relation (3.50) and the functions obtained in Section 3.3.1, we obtain new duality relations.

Due to the jointly factorized form (3.4) of the self-duality functions with single-site functions (3.32), (3.33), (3.35) and (3.36) and the tensor form of the intertwiner $G^{\otimes}$ in (3.50), the new duality functions inherit the same joint factorized form. Moreover, from the definition of G in (3.49), the whole computation reduces to determine *(exponential) generating functions* [54] of (3.32), (3.33), (3.35) and (3.36). To this purpose and, in particular, for the functions (3.33) and (3.36), some identities for hypergeometric functions are available, see . e.g. the tables in [87, Chapter 9]. Moreover, all generating functions obtained satisfy the requirements of analyticity for suitable choices of the parameters σ, α, a and b (see e.g. [87]), hence all operations below make sense.

However, just as the functions found in Section 3.3.1, the functions here obtained will only be "candidate" (self-)duality functions, since no duality relation as in (3.1) has been proved, yet. By using the "inverse" intertwining (3.52), all these "possible" dualities turn out to be equivalent, i.e. one implies all the others. Thus, in Proposition 3.24 below, we choose to prove directly the self-duality relation for the continuum – possibly improper – processes, more immediate to verify due to the simpler form of the self-duality functions. Indeed, while the single-site self-duality functions for the SIP(α) process, for instance, have the generic form of an hypergeometric function

$$ {}_2F_1\left[\begin{matrix}-k \quad -n \\ \alpha\end{matrix}; \frac{b}{a}\right], \quad k,\ n \in \mathbb{N}_0\,, $$

(cf. (3.30)–(3.31)), the single-site duality functions between discrete and continuum processes involve in their expressions hypergeometric functions with an argument less

$$ {}_1F_1\left[\begin{matrix}-k \\ \alpha\end{matrix}; -\frac{b}{a}z\right], \quad k \in \mathbb{N}_0\,,\ z \in [0,\infty)\,, $$

while those for the self-duality of continuum processes are even simpler, namely

$$ {}_0F_1\left[\begin{matrix}- \\ \alpha\end{matrix}; \frac{b}{a}vz\right], \quad v,\ z \in [0,\infty)\,, $$

as the number of arguments of the hypergeometric functions drops.

Tables of single-site (self-)duality functions. The tables below schematically report all single-site (self-)duality functions for the operators L^σ in (3.47) and $\mathscr{L}^\sigma$ in (3.48). Regarding the parameters a, $b \in \mathbb{R}$ in (3.12), we impose $b = 1$ and, consequently, we choose $a \in \mathbb{R}$ to be either equal to 0 – yielding "classical" (self-)duality functions – or $a < 0$ in case of "orthogonal" (self-)duality functions. We remark that for SEP(α), we further require $a \in (-1, 0)$. We refer to Section 3.3.1 for more general sensible choices of these parameters.

As a further instance of jointly factorized (self-)duality function – although not satisfying condition (3.5) – we add to the tables below the so-called "cheap" (self-)duality functions, parametrized here by a constant $c \in \mathbb{R}$.

(Notation: "Cl." = Classical polynomial single-site duality functions. "Or." = Orthogonal polynomial single-site duality functions. "Ch." = "Cheap" single-site duality functions.)

(a) *Symmetric exclusion process* (SEP(α), $\alpha \in \mathbb{N}$): $\sigma = -1$.

Cl.	$\frac{(\alpha-k)!}{\alpha!}\frac{n!}{(n-k)!}\mathbf{1}_{\{k\leq n\}}$	$z^k\frac{(\alpha-k)!}{\alpha!}$	$e^{-v}{}_0F_1\left[{- \atop -\alpha}; -vz\right]$
Or.	$a^k{}_2F_1\left[{-n \; -k \atop -\alpha}; -\frac{1}{a}\right]$	$a^k{}_1F_1\left[{-k \atop -\alpha}; \frac{z}{a}\right]$	$e^{(a-1)v}{}_0F_1\left[{- \atop -\alpha}; -vz\right]$
Ch.	$c^k\frac{k!(\alpha-k)!}{\alpha!}\mathbf{1}_{\{k=n\}}$	$e^{-z}(cz)^k\frac{(\alpha-k)!}{\alpha!}$	$e^{-(v+z)}{}_0F_1\left[{- \atop -\alpha}; -cvz\right]$

(b) *Independent random walkers* (IRW(α), $\alpha > 0$): $\sigma = 0$.

Cl.	$\frac{1}{\alpha^k}\frac{n!}{(n-k)!}\mathbf{1}_{\{k\le n\}}$	$\left(\frac{z}{\alpha}\right)^k$	$e^{-v}e^{\frac{vz}{\alpha}}$
Or.	$a^k{}_2F_0\left[\begin{smallmatrix}-k\ -n\\ -\end{smallmatrix};\frac{1}{a\alpha}\right]$	$\left(a+\frac{z}{\alpha}\right)^k$	$e^{(a-1)v}e^{\frac{vz}{\alpha}}$
Ch.	$c^k\frac{k!}{\alpha^k}\mathbf{1}_{\{k=n\}}$	$e^{-z}\left(\frac{cz}{\alpha}\right)^k$	$e^{-(v+z)}e^{c\frac{vz}{\alpha}}$

(c) *Symmetric inclusion process* (SIP(α), $\alpha > 0$): $\sigma = 1$.

Cl.	$\frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}\frac{n!}{(n-k)!}\mathbf{1}_{\{k\le n\}}$	$z^k\frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}$	$e^{-v}{}_0F_1\left[\begin{smallmatrix}-\\ \alpha\end{smallmatrix};vz\right]$
Or.	$a^k{}_2F_1\left[\begin{smallmatrix}-n\ -k\\ \alpha\end{smallmatrix};\frac{1}{a}\right]$	$a^k{}_1F_1\left[\begin{smallmatrix}-k\\ \alpha\end{smallmatrix};-\frac{z}{a}\right]$	$e^{(a-1)v}{}_0F_1\left[\begin{smallmatrix}-\\ \alpha\end{smallmatrix};vz\right]$
Ch.	$c^kk!\frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}\mathbf{1}_{\{k=n\}}$	$e^{-z}(cz)^k\frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}$	$e^{-(v+z)}{}_0F_1\left[\begin{smallmatrix}-\\ \alpha\end{smallmatrix};cvz\right]$

More in detail, on the left-most column we place the single-site self-duality functions $d(k,n)$ for the particle systems of Section 3.1.3: while the top-left functions are those already appearing in e.g. [19], [62], see also Section 3.1.4 and (3.32), (3.35) – and, thus, for this reason denoted here as the "classical" ones – the second-to-the-top functions are those derived in Section 3.3.1 in (3.33), (3.36) and being related to suitable families of orthogonal polynomials. While these two classes of single-site self-duality functions satisfy condition (3.5) (they are the only ones doing so by Theorem 3.3), the bottom-left single-site self-duality functions correspond to the "cheap" self-duality (cf. end of Section 3.1.2), namely the detailed-balance condition w.r.t. the measures $\{\otimes_{x\in V}\nu_\lambda,\ \lambda > 0\}$ with marginals (3.8).

On the mid-column, we find the single-site duality functions between the difference operators L^σ and the differential operators $\mathscr{L}^\sigma$, obtained from their left-neighbors by a direct application of the operator G in (3.49) on the n-variables. The new functions will depend hence on the two variables $k \in \mathbb{N}_0$ and $z \in [0, \infty)$.

A second application w.r.t. the k-variables of the same operator G on the functions just obtained gives us back the right-most column, functions depending now on variables $v, z \in [0, \infty)$. These functions represent the single-site self-duality functions for the differential operator $\mathscr{L}^\sigma$ acting on functions φ depending on variables $v, \zeta \in [0, \infty)^V$. As an immediate consequence of Proposition 3.22, we could also proceed from right to left by applying the inverse intertwiner H in (3.51).

Proof of jointly factorized (self-)duality. Note that the single-site self-duality functions for $\mathscr{L}^\sigma$ on the right-most columns, though they have been derived from different discrete analogues, i.e. classical, orthogonal and cheap single-site functions, within the same table they differ only of a factor which depends solely on the conserved quantities $|\zeta| = \sum_{x \in V} \zeta(x)$ and $|v| = \sum_{x \in V} v(x)$. Henceforth, when proving the self-duality relation, this extra-factor does not play any role and it is enough to check that the functions

$$\begin{aligned} d(v, z) &= {}_0F_1\left[\begin{matrix} - \\ -\alpha \end{matrix}; cvz\right] \\ d(v, z) &= e^{cvz} \\ d(v, z) &= {}_0F_1\left[\begin{matrix} - \\ \alpha \end{matrix}; cvz\right], \quad v, z \in [0, \infty), \end{aligned} \tag{3.56}$$

for arbitrary constants $c \in \mathbb{R}$, are single-site self-duality functions for the operators $\mathscr{L}^{-1}$, $\mathscr{L}^0$ and $\mathscr{L}^1$, respectively. This final computation is the content of the next proposition.

Proposition 3.24. *Fix a constant $c \in \mathbb{R}$. Then the functions $d(v, z)$ in (3.56) are single-site self-duality functions for the differential operators $\mathscr{L}^\sigma$ with $\sigma = -1$, $\sigma = 0$ and $\sigma = 1$, respectively.*

Proof. We first start with the case $\sigma = 0$. To prove that

$$d(v, z) = e^{cvz}, \quad v, z \in [0, \infty),$$

is a single-site self-duality function for the differential operator $\mathscr{L}^0$, we first observe that

$$\partial_z \, d(v, z) = c \, v \, d(v, z).$$

Hence, the self-duality relation for the single-bond generator $\mathscr{L}^0_{\{x,y\}}$ rewrites

$$\begin{aligned}&-\alpha\,(v(x)-v(y))\,(c\zeta(x)-c\zeta(y))\,d(v(x),\zeta(x))\,d(v(y),\zeta(y))\\ &\qquad = \ -\alpha\,(\zeta(x)-\zeta(y))\,(cv(x)-cv(y))\,d(v(x),\zeta(x))\,d(v(y),\zeta(y))\,,\end{aligned}$$

which indeed holds.

Then we consider $\sigma = 1$. For the proof of self-duality for the single-site function

$$d(v,z) \ = \ {}_0F_1\left[{- \atop \alpha};cvz\right], \quad v,z \in [0,\infty)\,,$$

we use the following shortcut: for $x \in V$,

$$W_x(\alpha) \ = \ {}_0F_1\left[{- \atop \alpha};c\,v(x)\,\zeta(x)\right].$$

Additionally, we recall a formula for the $\zeta(x)$-derivative of W_x, namely

$$\left(\frac{\partial}{\partial\zeta(x)}\right)W_x(\alpha) \ = \ \frac{c\,v(x)}{\alpha}\,W_x(\alpha+1)\,, \tag{3.57}$$

and a recurrence identity

$$W_x(\alpha+1) \ = \ W_x(\alpha) - \frac{c\,v(x)\,\zeta(x)}{\alpha(\alpha+1)}\,W_x(\alpha+2)\,. \tag{3.58}$$

Hence, the l.h.s. of the self-duality relation for $\mathscr{L}^1_{\{x,y\}}$ w.r.t. the function $W_x(\alpha)W_y(\alpha)$ rewrites by using (3.57) as

$$\begin{aligned}&c\,\zeta(x)\,v(y)\,W_x(\alpha+1)\,W_y(\alpha) \ + \ c\,v(x)\,\zeta(y)\,W_x(\alpha)\,W_y(\alpha+1)\\ &\qquad + \ \frac{c^2(v(x)\,\zeta(x))\,(v(y)\,\zeta(x))}{\alpha(\alpha+1)}\,W_x(\alpha+2)\,W_y(\alpha)\\ &\qquad + \ \frac{c^2(v(x)\,\zeta(y))\,(v(y)\,\zeta(y))}{\alpha(\alpha+1)}\,W_x(\alpha)\,W_y(\alpha+2)\end{aligned}$$

while the r.h.s. equals

$$c\,\upsilon(x)\,\zeta(y)\,W_x(\alpha+1)\,W_y(\alpha)\;+\;c\,\upsilon(y)\,\zeta(x)\,W_x(\alpha)\,W_y(\alpha+1)$$
$$+\;\frac{c^2(\upsilon(x)\,\zeta(x))\,(\upsilon(x)\,\zeta(y))}{\alpha(\alpha+1)}\,W_x(\alpha+2)\,W_y(\alpha)$$
$$+\;\frac{c^2(\zeta(x)\,\upsilon(y))\,(\zeta(y)\,\upsilon(y))}{\alpha(\alpha+1)}\,W_x(\alpha)\,W_y(\alpha+2)\,.$$

By substituting (3.58), the duality relation holds.

The proof for the case $\sigma = -1$ follows the same lines of the previous proof for $\sigma = 1$ and we omit it. □

3.a Inhomogeneous systems

In this section we further investigate jointly factorized self-duality for conservative factorized symmetric particle systems in the inhomogeneous context, i.e. in which inhomogeneities may be interpreted as realizations of a random environment. In what follows, we show to which extent the procedure to determine jointly factorized self-duality functions extends to IPS and their associated interacting diffusion processes in presence of a quenched combined bond-site disorder. As a final outcome, we recover inhomogeneous versions of the jointly factorized self-duality functions listed in Section 3.4.4 for SEP, IRW and SIP in a quenched random environment.

Inhomogeneous conservative factorized symmetric IPS. We start by introducing the (formal) generator associated to inhomogeneous variations of the particle systems introduced in Section 3.1.3. As a natural inhomogeneous counterpart of the generator in (3.6), we consider the following time-dependent operator acting on cylindrical functions $\varphi : \mathbb{N}_0^V \to \mathbb{R}$ and given by

$$L_t\,\varphi(\eta) \;=\; \sum_{x\sim y} c_t(\{x,y\})\,L_{\{x,y\}}\varphi(\eta)\,,\qquad \eta\in\mathbb{N}_0^V\,. \tag{3.59}$$

with

$$L_{\{x,y\}}\varphi(\eta) \;=\; g_x(\eta(x))\,h_y(\eta(y))\,(\varphi(\eta^{x,y})-\varphi(\eta)) + \; g_y(\eta(y))\,h_x(\eta(x))\,(\varphi(\eta^{y,x})-\varphi(\eta))\,,\qquad \eta\in\mathbb{N}_0^V\,. \tag{3.60}$$

We note that the above summation runs over all unordered pairs of nearest-neighboring sites, $\boldsymbol{c} = \{c_t(\{x,y\}),\ t \geq 0,\ x,y \in V\}$ stands for the time-dependent conductances (for the set V) as in Chapter 2 (see also Section 3.1.2) and the interaction functions $\{g_x,\ h_x : \mathbb{N}_0 \to [0,\infty),\ x \in V\}$ are chosen to depend on the locations $x \in V$. We impose the following assumptions on the interaction functions as those appearing in Section 3.1.3: for all $x \in V$,

(i) $g_x(0) = 0$ and $g_x(n) > 0$ for all $n > 0$,

(ii) $h_x(0) \neq 0$ and if $h_x(m) = 0$, then $h_x(n) = 0$ for all $n > m$.

In particular, if $\alpha_x := \min\{m \in \mathbb{N} : h_x(m) = 0\} \neq \emptyset$, then we set $F_x = \{0,\ldots,\alpha_x\}$, $F_x = \mathbb{N}_0$ otherwise. In view of this definition, the product space

$$\mathcal{X} = \prod_{x \in V} F_x$$

is the configuration space of the time-inhomogeneous particle system associated to the generators $\{L_t,\ t \geq 0\}$ above. In accordance with the notation of Chapter 2, we denote by $\{S_{s,t},\ t \geq s\}$, resp. $\{\widehat{S}_{s,t},\ s \leq t\}$, the forward, resp. backward, semigroups for the particle systems associated to the infinitesimal generators $\{L_t,\ t \geq 0\}$.

Self-duality in the time-inhomogeneous context. In the time-inhomogeneous context, the most natural self-duality relations which may be established are those involving a time-independent duality function $D : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ for which

$$(L_t)_{\text{left}} D(\xi,\eta) = (L_t)_{\text{right}} D(\xi,\eta)\,, \tag{3.61}$$

holds for all $\xi,\eta \in \mathcal{X}$ and $t \geq 0$. In the following proposition, we show that relations as in (3.61) yield a duality relation between forward and backward semigroups with the same D as duality function. Namely, if $\{\eta^\eta_{s,t}, t \geq s\}$ and $\{\xi^\xi_{s,t},\ s \leq t\}$ represent forward and backward, respectively, Markov processes on $\mathcal{X}$ starting from η and $\xi \in \mathcal{X}$, respectively, associated to the generators $\{L_t,\ t \geq 0\}$, we obtain

$$\widehat{\mathbb{E}}\left[D(\xi^\xi_{s,t},\eta)\right] = \mathbb{E}\left[D(\xi,\eta^\eta_{s,t})\right] .$$

for all $\xi,\eta \in \mathcal{X}$ and $s \leq t$. We first deal with the case of Markov processes on a finite state space to provide a sharper proof and refer to Remark 3.26 below for its analogue in the infinite setting.

Proposition 3.25 (FORWARD-BACKWARD SELF-DUALITY: FINITE STATE SPACE). *Let the state space $\mathcal{X}$ be finite. Then, generator self-duality in* (3.61) *implies the following backward-forward semigroup self-duality: for all $s \leq t$ and $\xi, \eta \in \mathcal{X}$,*

$$(\widehat{S}_{s,t})_{\text{left}} D(\xi,\eta) \;=\; (S_{s,t})_{\text{right}} D(\xi,\eta)\,.$$

Proof. By Kolmogorov backward equations (cf. e.g. (2.91) in Chapter 2), for all $s < t$ and $\varphi : \mathcal{X} \to \mathbb{R}$, we have in $L^2(\mathcal{X})$

$$\partial_s S_{s,t}\varphi \;=\; -L_s S_{s,t}\varphi \tag{3.62}$$

$$\partial_s \widehat{S}_{s,t}\varphi \;=\; -\widehat{S}_{s,t} L_s \varphi\,. \tag{3.63}$$

Furthermore, for all $t \geq 0$ and $\varphi : \mathcal{X} \to \mathbb{R}$, $\{S_{s,t}\varphi,\ s \leq t\}$ is the unique solution to the following backward Cauchy problem in $L^2(\mathcal{X})$:

$$\begin{cases} \frac{d}{ds}\varphi_s &=& -L_s\varphi_s\,, \quad s < t\,, \\ \varphi_t &=& \varphi\,. \end{cases} \tag{3.64}$$

As a consequence of equation (3.63), generator self-duality (3.61) and commutation of operators acting on different variables, we obtain in $L^2(\mathcal{X}) \otimes L^2(\mathcal{X})$

$$\begin{aligned} \partial_s (\widehat{S}_{s,t})_{\text{left}} D &= -(\widehat{S}_{s,t})_{\text{left}} (L_s)_{\text{left}} D \\ &= -(\widehat{S}_{s,t})_{\text{left}} (L_s)_{\text{right}} D \\ &= -(L_s)_{\text{right}} (\widehat{S}_{s,t})_{\text{left}} D\,. \end{aligned} \tag{3.65}$$

If we add the observation that $(\widehat{S}_{t,t})_{\text{left}} D = D = (S_{t,t})_{\text{right}} D$ for all $\xi, \eta \in \mathcal{X}$ and $t \geq 0$, then $\{(S_{s,t})_{\text{right}} D,\ s \leq t\}$ solves (3.65), namely (3.64) with $\varphi = D(\xi, \cdot)$ for all $\xi \in \mathcal{X}$. By uniqueness of the solution of the Cauchy problem (3.64), we conclude the proof. □

Remark 3.26 (INFINITE STATE SPACE). *The analogue of Proposition 3.25 in the context of an infinite (Polish) state space $\mathcal{X}$ requires – although the proof may be carried out along the same lines as in the finite context – additional assumptions. In case the semigroups $\{S_{s,t},\ t \geq s\}$ and $\{\widehat{S}_{s,t},\ s \leq t\}$ act on a Banach space $(\mathcal{F}(\mathcal{X}), \|\cdot\|_{\mathcal{X}})$, we further assume:*

(i) *The self-duality function $D : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is measurable and such that, for* *all $\xi, \eta \in \mathcal{X}$ and $t \geq 0$, $D(\xi, \cdot)$ and $D(\cdot, \eta) \in \mathcal{D}(L_t)$, where $\mathcal{D}(L_t)$ stands for the domain of the generator L_t.*

(ii) *For all $s < t$, the following holds:*

$$\lim_{h \to 0} \left\| \frac{(\widehat{S}_{s+h,t})_{\text{left}} D(\cdot, \cdot) - (\widehat{S}_{s,t})_{\text{left}} D(\cdot, \cdot)}{h} + (\widehat{S}_{s,t})_{\text{left}} (L_s)_{\text{left}} D(\cdot, \cdot) \right\|_{\mathcal{X} \times \mathcal{X}} = 0 .$$

In the rest of this section, we will consider only generator self-dualities as in (3.61).

Stationary product measures. Recall that in the time-inhomogeneous context by stationarity we mean that there exists a probability measure μ on $\mathcal{X}$ for which, for all $\varphi : \mathcal{X} \to \mathbb{R}$ cylinder functions and for all $s \leq t$, the following identity holds:

$$\int_{\mathcal{X}} S_{s,t} \varphi(\eta) \, \mu(d\eta) = \int_{\mathcal{X}} \varphi(\eta) \, \mu(d\eta) .$$

For the inhomogeneous particle systems constructed above, via a detailed balance computation, one may show that there exists a one-parameter family of (time-independent) stationary product measures

$$\{\mu_\lambda = \otimes_{x \in V} \nu_{x,\lambda} \, , \; \lambda \in \Lambda\} \, ,$$

where – note the analogy with (3.8) – the site-dependent dependent marginals $\nu_{x,\lambda}$ are given by

$$\nu_{x,\lambda}(n) = \phi_x(n) \frac{\lambda^n}{n!} \frac{1}{Z_{x,\lambda}} \, , \quad n \in F_x \, , \tag{3.66}$$

with

$$\phi_x(n) = n! \prod_{m=1}^{n} \frac{h_x(m-1)}{g_x(m)}$$

and $\Lambda \in (0, \infty)$ defined in such a way that all normalizing constants $Z_{x,\lambda}$ are finite, for all $x \in V$ and $\lambda \in \Lambda$.

Characterization of inhomogeneous particle systems with jointly factorized self-duality. In analogy with what we have found for the stationary product measures, jointly factorized self-duality functions in quenched random environment are expected to be products of site-dependent single-site self-duality functions as follows:

$$D(\xi,\eta) \;=\; \prod_{x\in V} d_x(\xi(x),\eta(x))\;. \tag{3.67}$$

Moreover, the assumptions of "non-triviality" for the single-site duality functions as, for instance, in (1.9)–(1.10) (see also (3.5) and Section 3.1.5) apply to this setting as well: for all $x \in V$,

$$d_x(0,\cdot) \;=\; 1 \tag{3.68}$$

and

$$d_x(1,n) \text{ is not a constant function of } n \in F_x\;. \tag{3.69}$$

Going back to the homogeneous case in Section 3.1.1, we recall that a single-bond generator as in (3.6) and a "non-trivial" jointly factorized self-duality function as in (3.4) already suffice to characterize all possible self-dual particle systems as well as the explicit form of the first single-site self-duality functions. This is the content of Theorem 3.3.

In the inhomogeneous setting, an analogous result holds for particle systems associated to the single-bond generator in (3.60). In the following theorem, we show that all inhomogeneous self-dual particle systems with "non-trivial" jointly factorized self-duality must be inhomogeneous variants of SEP, IRW and SIP. Furthermore, we find the general form of the candidate first single-site self-duality functions.

Theorem 3.27. *Consider, for all $x \sim y$, the single-bond generator $L_{\{x,y\}}$ given in* (3.60). *Assume that $L_{\{x,y\}}$ is self-dual with jointly factorized self-duality function given by*

$$D(\xi,\eta) \;=\; d_x(\xi(x),\eta(x))\cdot d_y(\xi(y),\eta(y))\,,$$

and whose single-site self-duality functions $d_x(\cdot,\cdot)$ and $d_y(\cdot,\cdot)$ satisfy conditions (3.68)–(3.69). *Then, the interaction functions appearing in* (3.60) *are of the following form*

$$g_x(n) \;=\; g_x(1)\,n \tag{3.70}$$

$$h_x(n) \;=\; h_x(0) \,+\, (h_x(1)-h_x(0))\,n\,, \qquad n \in F_x\,, \tag{3.71}$$

and, analogously, for $g_y, h_y : F_y \to [0,\infty)$, satisfying the following constraint:

$$\frac{h_x(1) - h_x(0)}{g_x(1)} \;=\; \frac{h_y(1) - h_y(0)}{g_y(1)}\,. \tag{3.72}$$

Moreover, there exist constants $a, b \in \mathbb{R}$, $b \neq 0$ – independent of the location $x, y \in V$ – for which

$$d_x(1,n) \;=\; a + \tfrac{b}{h_x(0)}\, g_x(n)\,, \quad n \in F_x \tag{3.73}$$

and, analogously, for $d_y(1,n)$.

Proof. The self-duality relation (3.1) for $L_{\{x,y\}}$ w.r.t. the self-duality function $d_x \cdot d_y$ with $\xi = \delta_x$ rewrites as follows:

$$\begin{aligned} g_x(1)\, h_y(0)\, &[d_y(1,\eta(y)) - d_x(1,\eta(x))] \\ &= g_x(\eta(x))\, h_y(\eta(y))\, [d_x(1,\eta(x)-1) - d_x(1,\eta(x))] \\ &+ g_y(\eta(y))\, h_x(\eta(x))\, [d_x(1,\eta(x)+1) - d_x(1,\eta(x))]\,. \end{aligned} \tag{3.74}$$

First, we observe that

$$d_x(1,0) \;=\; d_y(1,0) \;=:\; a \in \mathbb{R}\,. \tag{3.75}$$

Indeed, with the choice $\eta(x) = \eta(y) = 0$, because $g_x(0) = g_y(0) = 0$, the r.h.s. of (3.74) vanishes yielding (3.75). Let us now consider (3.74) with $\eta(x) = 0$ and $\eta(y) = n$ for some $n \in F_y$:

$$g_x(1)\, h_y(0)\, [d_y(1,n) - a] \;=\; g_y(n)\, h_x(0)\, [d_x(1,1) - a]\,,$$

which we rewrite as follows

$$d_y(1,n) \;=\; a + \tfrac{h_x(0)}{h_y(0)} \tfrac{g_y(n)}{g_x(1)}\, [d_x(1,1) - a]\,. \tag{3.76}$$

An analogous relation holds for $d_x(1,n)$ by considering the self-duality relation with $\xi = \delta_y$:

$$d_x(1,n) \;=\; a + \tfrac{h_y(0)}{h_x(0)} \tfrac{g_x(n)}{g_y(1)}\, [d_y(1,1) - a]\,,$$

which, for the choice $n = 1$, we can substitute into (3.76) to obtain:

$$d_y(1,n) \;=\; a + \tfrac{g_y(n)}{g_y(1)}\, [d_y(1,1) - a]\,,$$

We obtain an analogous expression for $d_x(1,n)$, namely

$$d_x(1,n) \;=\; a + \tfrac{g_x(n)}{g_x(1)}\,[d_x(1,1) - a]\,,$$

with the condition that

$$d_x(1,1) \;=\; a + \tfrac{h_y(0)}{h_x(0)}\tfrac{g_x(1)}{g_y(1)}\,[d_y(1,1) - a]\,.$$

By further specifying, for some $b \in \mathbb{R} \setminus \{0\}$, $d_y(1,1) = a + \frac{b}{h_y(0)} g_y(1)$ and $d_x(1,1)$ accordingly, we get (3.73).

By substituting the explicit expression obtained for $d_x(1,\cdot)$ and $d_y(1,\cdot)$ into the self-duality relation with $\xi = \delta_x$, we have

$$\begin{aligned} g_x(1)\, h_y(0) &\left[\tfrac{1}{h_y(0)} g_y(\eta(y)) - \tfrac{1}{h_x(0)} g_x(\eta(x))\right] \\ &= g_x(\eta(x))\, h_y(\eta(y))\, \tfrac{1}{h_x(0)}\, \left[g_x(\eta(x)-1) - g_x(\eta(x))\right] \\ &+ g_y(\eta(y))\, h_x(\eta(x))\, \tfrac{1}{h_x(0)}\, \left[g_x(\eta(x)+1) - g_x(\eta(x))\right]\,. \end{aligned}$$

With the choice $\eta(x) = n \in F_x$ and $\eta(y) = 0$, we get

$$g_x(n) \;=\; g_x(n-1) + g_x(1)\,,$$

and, hence, (3.70). With the choice $\eta(x) = n$ and $\eta(y) = 1$, we obtain

$$h_x(n) \;=\; h_x(0) + \tfrac{g_x(1)}{g_y(1)}\,(h_y(1) - h_y(0))\, n\,, \quad n \in F_x\,.$$

As a consequence, this leads to condition (3.72) and, then, (3.71). Analogously, by choosing $\xi = \delta_y$, we find g_y and h_y. □

Inhomogeneous SEP, IRW **and** SIP. As we have already mentioned above, Theorem 3.27 asserts that the interaction functions of a non-trivially jointly factorized self-dual particle system yield inhomogeneous counterparts of SEP, IRW and SIP. Indeed, by introducing the parameters

$$\begin{aligned} \Sigma &= \frac{h_x(1) - h_x(0)}{g_x(1)} = \frac{h_y(1) - h_y(0)}{g_y(1)} \in \mathbb{R}\,, \\ \sigma &= \operatorname{sign}(\Sigma) \in \{-1, 0, 1\}\,, \end{aligned}$$

and

$$\alpha_x \;=\; \begin{cases} \frac{h_x(0)}{g_x(1)} \cdot \frac{1}{|\Sigma|} & \text{if } \Sigma \neq 0 \\ \frac{h_x(0)}{g_x(1)} & \text{otherwise}\,, \end{cases}$$

we rewrite the single-bond generator $L_{\{x,y\}}$ with interaction functions given by (3.70), (3.71) and (3.72) as follows:

$$L_{\{x,y\}}\varphi(\eta) \;=\; g_x(1)\, g_y(1)\, \widetilde{L}_{\{x,y\}}\varphi(\eta)\,, \quad \eta \in \mathcal{X}\,,$$

with

$$\begin{aligned} \widetilde{L}_{\{x,y\}}\varphi(\eta) \;=\; & \eta(x)\,(\alpha_y + \sigma\eta(y))\,(\varphi(\eta^{x,y}) - \varphi(\eta)) \\ & + \;\eta(y)\,(\alpha_x + \sigma\eta(x))\,(\varphi(\eta^{y,x}) - \varphi(\eta))\,, \quad \eta \in \mathcal{X}\,. \end{aligned} \tag{3.77}$$

Hence, the term $g_x(1)g_y(1)$ may be considered as an additional multiplicative factor of the bond conductance $c(\{x,y\})$ in (3.59), while out of (3.77) we derive the expression of the single-bond generators of SEP, IRW and SIP with quenched site disorder $\{\alpha_x,\ x \in V\} \subset (0,\infty)$ (see also Section 1.2 for further details).

More specifically, we refer to $(\boldsymbol{c},\boldsymbol{\alpha})$-inhomogeneous SEP, IRW and SIP – shortly denoted by SEP($\boldsymbol{\alpha}$), IRW($\boldsymbol{\alpha}$) and SIP($\boldsymbol{\alpha}$), respectively, where we specify only site-inhomogeneities $\boldsymbol{\alpha}$ if there is no confusion on the choice of the conductances $\boldsymbol{c}$ – as those particle systems with time-dependent generator L_t as in (3.59) and single-bond generators given by (3.77) with the following choices:

(a) SEP($\boldsymbol{\alpha}$) with $\sigma = -1$ and $\{\alpha_x,\ x \in V\} \subset \mathbb{N}$.

(b) IRW($\boldsymbol{\alpha}$) with $\sigma = 0$ and $\{\alpha_x,\ x \in V\} \subset (0,\infty)$.

(c) SIP($\boldsymbol{\alpha}$) with $\sigma = 1$ and $\{\alpha_x,\ x \in V\} \subset (0,\infty)$.

We further note that, as a consequence of Theorem 3.27, we obtain that

$$d_x(1,n) \;=\; a + \tfrac{b}{\alpha_x}\, n\,, \quad n \in F_x\,, \tag{3.78}$$

is the most general form of the first single-site self-duality functions for the three particle systems above, where the constants $a, b \in \mathbb{R}$ are independent of the location $x \in V$.

From the first to the k-th single-site self-duality function. In Section 3.3, in the derivation of all possible single-site self-duality functions $\{d(k,\cdot),\ k \in$

$F\}$ from the knowledge of $d(0,\cdot)$ and $d(1,\cdot)$, the general relation between jointly factorized self-duality functions and stationary product measures studied in Section 3.2 played a crucial role. Minor adaptations to the proofs of Theorems 3.9 and 3.12 yield a natural inhomogeneous counterpart of this relation as well as characterizations of ergodic measures for infinite systems, with

$$D(\xi,\eta) \;=\; \prod_{x\in V} d_x(\xi(x),\eta(x))$$

as jointly factorized self-duality function with $d_x(0,\cdot) \equiv 1$ for all $x \in V$, $\mu = \otimes_{x\in V} \nu_x$ as stationary product measure for the forward process and the backward process as the dual process satisfying the condition of (bounded) harmonic triviality [HT] (or [BHT]).

In the specific instance of SEP($\boldsymbol{\alpha}$), IRW($\boldsymbol{\alpha}$) and SIP($\boldsymbol{\alpha}$), by combining the general form of the first single-site self-duality functions given in (3.78), the explicit knowledge of the marginals $\nu_{x,\lambda}$ of the stationary product measures μ_λ as given in (3.66) and the following relation

$$\sum_{n\in F_x} d_x(k,n)\, \nu_{x,\lambda}(n) \;=\; \theta(\lambda)^k\,, \quad k \in F_x\,,$$

we may proceed as in Section 3.3.1 and recover all possible single-site self-duality functions $\{d_x(k,\cdot),\ k \in F_x\}$, for all $x \in V$. We remark that the same computations can be carried all thorough with the only addition of the dependence of α on the location $x \in V$. As a consequence, given $a, b \in \mathbb{R}$, we recover

$$d_x(k,n) \;=\; \begin{cases} \left(\dfrac{b}{\alpha_x}\right)^k \dfrac{n!}{(n-k)!}\,\mathbf{1}_{\{k\le n\}} & a = 0 \\[2ex] a^k\,{}_2F_0\!\left[\begin{matrix} -k \quad -n \\ - \end{matrix}\,;\, \dfrac{b}{a\alpha_x}\right] & a \neq 0 \end{cases} \tag{3.79}$$

for IRW($\boldsymbol{\alpha}$), while for SEP($\boldsymbol{\alpha}$) and SIP($\boldsymbol{\alpha}$) – corresponding to $\sigma = -1$ and $\sigma = 1$, respectively – we get

$$d_x(k,n) \;=\; \begin{cases} b^k \dfrac{\Gamma(\sigma\alpha_x)}{\Gamma(\sigma\alpha_x + k)} \dfrac{n!}{(n-k)!}\mathbf{1}_{\{k\le n\}} & a = 0 \\[2ex] a^k\,{}_2F_1\!\left[\begin{matrix} -k \quad -n \\ \sigma\alpha_x \end{matrix}\,;\, \dfrac{b}{a\sigma}\right] & a \neq 0\,. \end{cases} \tag{3.80}$$

Generating functions and intertwining. As in the homogeneous case, the functions $D = \prod_{x\in V} d_x$ obtained from the functions d_x in (3.79)–(3.80) are only *candidate* jointly factorized self-duality functions. However, the intertwining relations between discrete and continuum processes derived from (exponential) generating functions and established in Propositions 3.20–3.22 carry through even in the inhomogeneous setting with the same intertwining operators $G^{\otimes}$ and $H^{\otimes}$ (cf. Propositions 3.20–3.22) and inhomogeneous counterparts of the differential operators $\mathscr{L}^{\sigma}$ given in (3.48).

The following proposition – whose proof resembles those of Propositions 3.20–3.22 and, hence, is left to the reader – collects all intertwining relations in this inhomogeneous setting. We refer to Section 3.4.2 for the precise conditions on the domains of the operators.

Proposition 3.28. *Recall the definitions of the operators $G^{\otimes}$ and $H^{\otimes}$ in Propositions 3.20–3.22. For all $\mathbf{c} = \{c_t(\{x,y\}),\ t \geq 0,\ x \sim y\}$, $\boldsymbol{\alpha} = \{\alpha_x,\ x \in V\}$, $\sigma \in \{-1,0,1\}$ and $t \geq 0$, we denote by L_t^{σ} and $\mathscr{L}_t^{\sigma}$ the following difference and differential operators:*

$$L_t^{\sigma}\varphi(\eta) = \sum_{x\sim y} c_t(\{x,y\})\, L_{\{x,y\}}^{\sigma}\varphi(\eta), \quad \eta \in \mathbb{N}_0^V, \tag{3.81}$$

with

$$\begin{aligned} L_{\{x,y\}}^{\sigma}\varphi(\eta) = {} & \eta(x)\,(\alpha_y + \sigma\eta(y))\,(\varphi(\eta^{x,y}) - \varphi(\eta)) \\ & + \eta(y)\,(\alpha_x + \sigma\eta(x))\,(\varphi(\eta^{y,x}) - \varphi(\eta)), \end{aligned}$$

and

$$\mathscr{L}_t^{\sigma}\varphi(\zeta) = \sum_{x\sim y} c_t(\{x,y\})\, \mathscr{L}_{\{x,y\}}^{\sigma}\varphi(\zeta), \quad \zeta \in [0,\infty)^{\{x,y\}}. \tag{3.82}$$

$$\begin{aligned} \mathscr{L}_{\{x,y\}}^{\sigma}\varphi(\zeta) = {} & -(\alpha_y\zeta(x) - \alpha_x\zeta(y))\,(\partial_x - \partial_y)\,\varphi(\zeta) \\ & + \sigma\zeta(x)\zeta(y)\,(\partial_x - \partial_y)^2\varphi(\zeta). \end{aligned}$$

Then, for all pairs $x \sim y$, the following intertwining relations hold,

$$L_{\{x,y\}}^{\sigma} G^{\otimes} = G^{\otimes}\mathscr{L}_{\{x,y\}}^{\sigma}, \quad \mathscr{L}_{\{x,y\}}^{\sigma} H^{\otimes} = H^{\otimes} L_{\{x,y\}}^{\sigma},$$

and, as a consequence, similarly for L_t^{σ} and $\mathscr{L}_t^{\sigma}$.

The above intertwining relations are, also in this case, the key ingredient to produce all possible jointly factorized duality and self-dualities for the inhomogeneous difference operators – corresponding to the particle systems SEP($\boldsymbol{\alpha}$), IRW($\boldsymbol{\alpha}$) and SIP($\boldsymbol{\alpha}$) – and the associated differential operators. In particular, we obtain tables containing all possible single-site (self-)duality functions as those in Section 3.4.4 with, this time, the parameter $\alpha \in (0,\infty)$ depending on $x \in V$.

Proof of jointly factorized self-duality. In conclusion, by the equivalence of all these duality relations, self-duality for the difference operators follows from self-duality for the associated differential operators, the latter one being more direct to check. For the proof of the following statement, we refer to the proof of Proposition 3.24.

Proposition 3.29. *For all* $\sigma \in \{-1,0,1\}$ *and* $t \geq 0$, *the differential operators* $\mathscr{L}_t^\sigma$ *in* (3.82) *is self-dual with jointly factorized self-duality function given by*

$$D(v,\zeta) \;=\; \prod_{x\in V} d_x(v(x),\zeta(x))\,,$$

where, for the case $\sigma = 0$,

$$d_x(v,z) \;=\; e^{\frac{cvz}{\alpha_x}}\,, \quad v,z \in [0,\infty)\,, \tag{3.83}$$

while, for the case $\sigma \in \{-1,1\}$,

$$d_x(v,z) \;=\; {}_0F_1\left[\begin{matrix} -\;- \\ \sigma\alpha_x \end{matrix};cvz\right], \quad v,z \in [0,\infty)\,, \tag{3.84}$$

for some constant $c \in \mathbb{R}$. *As a consequence, the difference operators* L_t^σ *are self-dual with jointly factorized self-duality functions given by*

$$D(\xi,\eta) \;=\; \prod_{x\in V} d_x(\xi(x),\eta(x))\,,$$

where the functions $d_x(\cdot,\cdot)$ *are given in* (3.79)–(3.80).

3.b Intertwining and ladder symmetric exclusion processes

In this section we provide an application of Theorem 3.19. First we establish intertwining relations between symmetric exclusion processes and some "ladder" variants of them. Afterwards, we obtain, as in Theorem 3.19, a large class of jointly factorized self-duality functions for both processes. In particular, we recover all single-site self-duality functions for SEP found in Section 3.3.1. All throughout the section, a finite set of sites $(V, \sim)$, $\boldsymbol{c}$ and $\boldsymbol{\alpha} \in \mathbb{N}^V$ are fixed.

Ladder symmetric exclusion process. The *ladder symmetric exclusion process* with parameter $\boldsymbol{\alpha} \in \mathbb{N}^V$ (LSEP($\boldsymbol{\alpha}$)) is the interacting particle system with (finite) configuration space

$$\tilde{\mathcal{X}} \;=\; \{\tilde{\eta} : \tilde{\eta}(x,i) \in \{0,1\},\; x \in V,\; i \in \{1,\ldots,\alpha_x\}\} \tag{3.85}$$

and with infinitesimal generator $\tilde{L}$ acting on functions $\tilde{\varphi} : \tilde{\mathcal{X}} \to \mathbb{R}$ as

$$\tilde{L}\,\tilde{\varphi}(\tilde{\eta}) \;=\; \sum_{x\sim y} c(\{x,y\})\,\tilde{L}_{\{x,y\}}\tilde{\varphi}(\tilde{\eta})$$

where

$$\begin{aligned}\tilde{L}_{\{x,y\}}\tilde{\varphi}(\tilde{\eta}) \;=\; \sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y} \Big\{ & \tilde{\eta}(x,i)\,(1-\tilde{\eta}(y,j))\,(\tilde{\varphi}(\tilde{\eta}^{(x,i),(y,j)}) - \tilde{\varphi}(\tilde{\eta})) \\ & + \tilde{\eta}(y,j)\,(1-\tilde{\eta}(x,i))\,(\tilde{\varphi}(\tilde{\eta}^{(y,j),(x,i)}) - \tilde{\varphi}(\tilde{\eta}))\Big\}\;, \quad \tilde{\eta} \in \tilde{\mathcal{X}}\;,\end{aligned}$$

where $\tilde{\eta}^{(x,i),(y,j)}$ denotes, also in this context, the configuration obtained from $\tilde{\eta}$ by removing a particle at position (x,i) and placing it at (y,j). Indeed, this process may be considered as a special case of a symmetric exclusion process SEP(1) on the set $\tilde{V} = \{(x,i),\; x \in V,\; i \in \{1,\ldots,\alpha_x\}\}$ with conductances

$$\tilde{c}(\{(x,i),(y,j)\}) \;=\; c(\{x,y\})\;, \quad (x,i)\,,\,(y,j) \in \tilde{V}\;.$$

Deterministic intertwiner with SEP. If we denote by L the infinitesimal generator of SEP($\boldsymbol{\alpha}$) (see Section 3.a), L and $\tilde{L}$ are intertwined via a deterministic intertwining operator Λ mapping functions $\varphi : \mathcal{X} \to \mathbb{R}$ into functions

$\tilde{\varphi} : \tilde{\mathcal{X}} \to \mathbb{R}$, where $\mathcal{X} = \{\eta : \eta(x) \in \{0, \ldots, \alpha_x\}\}$. Given the mapping $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$,

$$\pi(\tilde{\eta}) = \{|\tilde{\eta}(x,\cdot)|,\ x \in V\} \in \mathcal{X}\,, \quad |\tilde{\eta}(x,\cdot)| := \sum_{i=1}^{\alpha_x} \tilde{\eta}(x,i)\,,$$

the intertwining operator Λ is defined as

$$\Lambda\,\varphi(\tilde{\eta}) = \varphi(\pi(\tilde{\eta}))\,, \quad \tilde{\eta} \in \tilde{\mathcal{X}}\,. \tag{3.86}$$

The intertwining relation then reads, for all $\varphi : \mathcal{X} \to \mathbb{R}$, as

$$\tilde{L}\,\Lambda\,\varphi(\tilde{\eta}) = \Lambda\,L\,\varphi(\tilde{\eta})\,,$$

for $\tilde{\eta} \in \tilde{\mathcal{X}}$. In view of Theorem 3.19 and the above intertwining relation, from a self-duality function $D(\xi,\eta)$ for L we can build a duality function

$$D'(\xi,\tilde{\eta}) = \Lambda_{\text{right}} D(\xi,\tilde{\eta})$$

for L and $\tilde{L}$ and, furthermore, a self-duality function

$$D''(\tilde{\xi},\tilde{\eta}) = \Lambda_{\text{left}}\Lambda_{\text{right}} D(\tilde{\xi},\tilde{\eta})$$

for $\tilde{L}$.

"Inverse" stochastic intertwiner. We ask whether there exists an "inverse" intertwining relation, i.e. an operator $\tilde{\Lambda}$ for which the following intertwining relation

$$\tilde{\Lambda}\,\tilde{L}\,\tilde{\varphi}(\eta) = L\,\tilde{\Lambda}\,\tilde{\varphi}(\eta)\,, \quad \eta \in \mathcal{X} \tag{3.87}$$

holds, for all $\tilde{\varphi} : \tilde{\mathcal{X}} \to \mathbb{R}$. We answer to this question in the following proposition and provide the explicit expression of this inverse stochastic intertwiner. In what follows, we say that $\tilde{\eta} \in \tilde{\mathcal{X}}$ is *compatible* with $\eta \in \mathcal{X}$ or, shortly, $\tilde{\eta} \rhd \eta$, if $\pi(\tilde{\eta}) = \eta$.

Proposition 3.30 (INVERSE INTERTWINER). *The operator $\tilde{\Lambda}$ given by*

$$\tilde{\Lambda}\,\tilde{\varphi}(\eta) = \left(\prod_{x \in V} \frac{1}{\binom{\alpha_x}{\eta(x)}}\right) \sum_{\tilde{\eta} : \tilde{\eta} \rhd \eta} \tilde{\varphi}(\tilde{\eta})\,, \quad \eta \in \mathcal{X}\,, \tag{3.88}$$

satisfies the intertwining relation (3.87). *Moreover, this intertwining operator is* *stochastic.*

Proof. Without loss of generality, we consider $V = \{x, y\}$. By expanding the l.h.s. of (3.87) with $\tilde{\Lambda}$ as in (3.88), we obtain four terms:

$$\ell_1 = -\frac{1}{\binom{\alpha_x}{\eta(x)}} \frac{1}{\binom{\alpha_y}{\eta(y)}} \sum_{\tilde{\eta}:\tilde{\eta}\triangleright\eta} \sum_{i=1}^{\alpha_x} \sum_{j=1}^{\alpha_y} \tilde{\eta}(x,i)\,(1-\tilde{\eta}(y,j))\,\tilde{\varphi}(\tilde{\eta})$$

$$\ell_2 = \frac{1}{\binom{\alpha_x}{\eta(x)}} \frac{1}{\binom{\alpha_y}{\eta(y)}} \sum_{\tilde{\eta}:\tilde{\eta}\triangleright\eta} \sum_{i=1}^{\alpha_x} \sum_{j=1}^{\alpha_y} \tilde{\eta}(x,i)\,(1-\tilde{\eta}(y,j))\,\tilde{\varphi}(\tilde{\eta}^{(x,i),(y,j)})$$

$$\ell_3 = -\frac{1}{\binom{\alpha_x}{\eta(x)}} \frac{1}{\binom{\alpha_y}{\eta(y)}} \sum_{\tilde{\eta}:\tilde{\eta}\triangleright\eta} \sum_{i=1}^{\alpha_x} \sum_{j=1}^{\alpha_y} \tilde{\eta}(y,j)\,(1-\tilde{\eta}(x,i))\,\tilde{\varphi}(\tilde{\eta})$$

$$\ell_4 = \frac{1}{\binom{\alpha_x}{\eta(x)}} \frac{1}{\binom{\alpha_y}{\eta(y)}} \sum_{\tilde{\eta}:\tilde{\eta}\triangleright\eta} \sum_{i=1}^{\alpha_x} \sum_{j=1}^{\alpha_y} \tilde{\eta}(y,j)\,(1-\tilde{\eta}(x,i))\,\tilde{\varphi}(\tilde{\eta}^{(y,j),(x,i)})\,.$$

By doing the same thing with the r.h.s., we obtain:

$$r_1 = -\frac{1}{\binom{\alpha_x}{\eta(x)}} \frac{1}{\binom{\alpha_y}{\eta(y)}}\, \eta(x)\,(\alpha_y - \eta(y)) \sum_{\tilde{\eta}:\tilde{\eta}\triangleright\eta} \tilde{\varphi}(\tilde{\eta})$$

$$r_2 = \frac{1}{\binom{\alpha_x}{\eta(x)-1}} \frac{1}{\binom{\alpha_y}{\eta(y)+1}}\, \eta(x)\,(\alpha_y - \eta(y)) \sum_{\tilde{\eta}:\tilde{\eta}\triangleright\eta^{x,y}} \tilde{\varphi}(\tilde{\eta})$$

$$r_3 = -\frac{1}{\binom{\alpha_x}{\eta(x)}} \frac{1}{\binom{\alpha_y}{\eta(y)}}\, \eta(y)\,(\alpha_x - \eta(x)) \sum_{\tilde{\eta}:\tilde{\eta}\triangleright\eta} \tilde{\varphi}(\tilde{\eta})$$

$$r_4 = \frac{1}{\binom{\alpha_x}{\eta(x)+1}} \frac{1}{\binom{\alpha_y}{\eta(y)-1}}\, \eta(y)\,(\alpha_x - \eta(x)) \sum_{\tilde{\eta}:\tilde{\eta}\triangleright\eta^{y,x}} \tilde{\varphi}(\tilde{\eta})\,.$$

Note that $\ell_1 = r_1$ because, for all $\tilde{\eta} \triangleright \eta$,

$$\sum_{i=1}^{\alpha_x} \sum_{j=1}^{\alpha_y} \tilde{\eta}(x,i)\,(1-\tilde{\eta}(y,j)) = \eta(x)\,(\alpha_y - \eta(y))\,,$$

and similarly for $\ell_3 = r_3$. For $\ell_2 = r_2$ it is enough to verify that, for all

$\tilde{\eta}_* \rhd \eta^{x,y}$,

$$\sum_{\tilde{\eta}:\tilde{\eta}\rhd\eta} \sum_{i=1}^{\alpha_x} \sum_{j=1}^{\alpha_y} \tilde{\eta}(x,i)\,(1-\tilde{\eta}(y,j))\,\mathbf{1}_{\{\tilde{\eta}^{(x,i),(y,j)}=\tilde{\eta}_*\}} \;=\; (\eta(y)+1)\,(\alpha_x-\eta(x)+1)\,.$$

This last identity indeed holds, as the configurations $\tilde{\eta} \rhd \eta$ can be obtained from $\tilde{\eta}_*$ by picking one of the $\eta(y)+1$ particles on $y \in V$ and putting it back on one of the $\alpha_x - \eta(x) + 1$ holes of $x \in V$. Analogously for $\ell_4 = r_4$. □

A second proof of jointly factorized self-duality for SEP($\boldsymbol{\alpha}$). As a consequence of this proposition, from self-duality functions of LSEP($\boldsymbol{\alpha}$) we can produce duality functions between LSEP($\boldsymbol{\alpha}$) and SEP($\boldsymbol{\alpha}$) as well as self-duality functions for SEP($\boldsymbol{\alpha}$).

Actually, we have already obtained (Section 3.4.4 and Appendix 3.a) a characterization of all jointly factorized self-duality functions for SEP($\boldsymbol{\alpha}$) with single-site self-duality functions satisfying condition (3.5). By dropping this latter condition, the author in [125] provides a full characterization of jointly factorized self-duality functions for SEP(1), characterization which turns useful for the ladder SEP($\boldsymbol{\alpha}$) in V if viewed as a particular instance of SEP(1) in $\tilde{V}$. We report this result (Theorem 3.31) and further use it in Theorem 3.32 below to obtain jointly factorized self-duality functions for SEP($\boldsymbol{\alpha}$) for which condition (3.5) on the single-site self-duality functions does not necessarily hold.

Theorem 3.31 ([125, THEOREM 2.8]). SEP(1) *on* $\tilde{V}$ *is self-dual w.r.t. the duality function*

$$\tilde{D}(\tilde{\xi},\tilde{\eta}) \;=\; \prod_{(x,i)\in\tilde{V}} (a+b\,\tilde{\eta}(x,i))^{u+v\tilde{\xi}(x,i)}\,, \quad \tilde{\xi}\,,\,\tilde{\eta}\in\tilde{\mathcal{X}}\,, \tag{3.89}$$

for all a, b, u *and* $v \in \mathbb{R}$.

Now, we apply the intertwining operator $\tilde{\Lambda}$ first on the right and then on the left variables of $\tilde{D}$ above. What we obtain is a class of jointly factorized self-duality functions for SEP($\boldsymbol{\alpha}$). As expected, the jointly factorized self-duality functions found differ from those found in Sections 3.3.1, 3.4.4 and Appendix 3.a only by factors which depend only on conserved quantities – the total number of particles – of the process $\{\eta_t,\ t \geq 0\}$ and its dual.

Theorem 3.32. *All self-duality functions for* SEP($\boldsymbol{\alpha}$) *derived from self-duality functions of* LSEP($\boldsymbol{\alpha}$) *as in* (3.89) *are all in jointly factorized form, i.e.*

$$D(\xi,\eta) \;=\; \tilde{\Lambda}_{\text{left}}\tilde{\Lambda}_{\text{right}}\tilde{D}(\xi,\eta) \;=\; \prod_{x\in V} d_x^{a,b,u,v}(\xi(x),\eta(x))\,.$$

Moreover, the single-site self-duality functions $d_x^{a,b,u,v}(k,n)$, $k,n \in \{0,\dots,\alpha_x\}$, *are in either one of the following forms: the classical polynomials*

$$d_x^{0,b,0,v}(k,n) \;=\; (b^v)^k\,\frac{(\alpha_x-k)!}{\alpha_x!}\,\frac{n!}{(n-k)!}\,\mathbf{1}_{\{k\le n\}}\,,$$

the orthogonal polynomials

$$d_x^{a,b,u,v}(k,n) \;=\; (-1)^{vk}\,a^{u\alpha_x-un+vk}(a+b)^{un}\,{}_2F_1\!\left[\begin{matrix}-k \quad -n\\ -\alpha_x\end{matrix};1-\left(1+\frac{b}{a}\right)^{v}\right],$$

or other degenerate functions:

$$\begin{aligned}
d_x^{a,b,u,0}(k,n) &= (a+b)^{un}\,a^{u(\alpha_x-n)}\\
d_x^{0,b,u,v}(k,n) &= b^{u\alpha_x+vk}\,\mathbf{1}_{\{n=\alpha_x\}}\\
d_x^{a,0,u,v}(k,n) &= a^{u\alpha_x+vk}\\
d_x^{a,-a,u,v}(k,n) &= a^{u\alpha_x+vk}\,\mathbf{1}_{\{n=0\}}\,.
\end{aligned}$$

Proof. First thing to note is that the factorized structure of D is preserved under $\tilde{\Lambda}$. Indeed, if we use the notation

$$\delta(k,n) \;=\; (a+bn)^{u+vk}\,,$$

then we obtain

$$\tilde{\Lambda}_{\text{right}}D(\tilde{\xi},\eta) \;=\; \prod_{x\in V}\left(\frac{1}{\binom{\alpha_x}{\eta(x)}}\sum_{\tilde{\eta}:|\tilde{\eta}(x,\cdot)|=\eta(x)}\prod_{i=1}^{\alpha_x}\delta(\tilde{\xi}(x,i),\tilde{\eta}(x,i))\right).$$

As a second and final step, we compute only what is inside the parenthesis (which we will see that does depend on $\tilde{\xi}(x,\cdot)$ only through $\sum_{i=1}^{\alpha_x}\tilde{\xi}(x,i)$):

$$\frac{(a+b)^{u\eta(x)}\,a^{u(\alpha_x-\eta(x))}}{\binom{\alpha_x}{\eta(x)}}\times\sum_{\tilde{\eta}:|\tilde{\eta}(x,\cdot)|=\eta(x)}\prod_{i=1}^{\alpha_x}(a+b\tilde{\eta}(x,i))^{v\tilde{\xi}(x,i)}\,. \qquad (3.90)$$

The last summation depends on $\tilde{\xi}(x,\cdot)$ only through $\xi(x) = |\tilde{\xi}(x,\cdot)|$ and the following expression taken from (3.90)

$$\frac{1}{\binom{\alpha_x}{\eta(x)}} \sum_{\tilde{\eta}:|\tilde{\eta}(x,\cdot)|=\eta(x)} \prod_{i=1}^{\alpha_x} (a + b\tilde{\eta}(x,i))^{v\tilde{\xi}(x,i)}$$

equals

$$\frac{1}{\binom{\alpha_x}{n}} \sum_{\ell=0}^{k} \binom{k}{k-\ell} \binom{\alpha_x - k}{n-(k-\ell)} (a+b)^{v\,(k-\ell)}\, a^{v\ell}\,, \tag{3.91}$$

where $n = \eta(x)$ and $k = \xi(x)$. If $v = 0$, this last expression in (3.91) by Chu-Vandermonde identity equals 1, hence

$$d_x^{a,b,u,0}(k,n) \;=\; (a+b)^{un}\, a^{u\,(\alpha_x - n)}\,.$$

If $v \neq 0$ and $a = 0$, expression (3.91) rewrites as

$$\frac{1}{\binom{\alpha_x}{n}} \binom{\alpha_x - k}{n-k} b^{vk}\, \mathbf{1}_{\{k \le n\}} \;=\; (b^v)^k \frac{(\alpha_x - k)!}{\alpha_x!} \frac{n!}{(n-k)!} \mathbf{1}_{\{k \le n\}}\,,$$

and hence, for $u = 0$, (3.90) becomes

$$d_x^{0,b,0,v}(k,n) \;=\; (b^v)^k \frac{(\alpha_x - k)!}{\alpha_x!} \frac{n!}{(n-k)!} \mathbf{1}_{\{k \le n\}}\,,$$

i.e. the classical single-site self-duality functions (see also (3.35) in Section 3.3.1), while, for $u \neq 0$,

$$d_x^{0,b,u,v}(k,n) \;=\; b^{u\alpha_x + vk}\, \mathbf{1}_{\{n=\alpha_x\}}\,.$$

If $v \neq 0$ and $a \neq 0$ and $b = 0$, then again we get some trivial expression:

$$d_x^{a,0,u,v}(k,n) \;=\; a^{u\alpha_x + vk}\,.$$

The most interesting case is when $v \neq 0$, $a \neq 0$, $b \neq 0$ and $a \neq -b$. In this case the quantity in (3.91) equals

$$(a+b)^{vk} \frac{1}{\binom{\alpha_x}{n}} \sum_{\ell=0}^{k} \binom{k}{k-\ell} \binom{\alpha_x - k}{n-(k-\ell)} \left(\frac{a}{a+b}\right)^{v\ell}\,,$$

which rewrites, by using the "known relation" and the "transformation" in

[110, p. 51], as

$$(-a)^{vk}\,{}_2F_1\left[\begin{matrix}-n & -k\\ & -\alpha_x\end{matrix};1-\left(1+\frac{b}{a}\right)^{v}\right],$$

leading to

$$d_x^{a,b,u,v}(k,n) \;=\; (-1)^{vk}\,a^{u\alpha_x-un+vk}(a+b)^{un}\,{}_2F_1\left[\begin{matrix}-n & -k\\ & -\alpha_x\end{matrix};1-\left(1+\frac{b}{a}\right)^{v}\right],$$

i.e. we recover the orthogonal polynomial single-site self-duality functions for the SEP(α), namely families of Kravchuk polynomials (see also (3.36) in Section 3.3.1). If $a=-b$, we have

$$d_x^{a,-a,u,v}(k,n) \;=\; a^{u\alpha_x+vk}\,\mathbf{1}_{\{n=0\}}\,.$$

This concludes the proof. □

Duality and eigenfunctions

As we have already seen in the previous chapters, duality is a technique to connect two Markov processes via a so-called duality function. This connection – interesting in its own right – turns out to be extremely useful when the duality function carries information about the original process and the dual process is more tractable than the original one (cf. Chapter 1 and the beginning of Chapter 3 for further information on applications and instances of duality).

Finding and characterizing duality relations. Part of the research about duality deals with the problem of finding and characterizing duality functions relating two given Markov processes. This means that, for a given pair of Markov generators, one wants to find all duality functions or, alternatively, a basis of the linear space of duality functions. See, for instance, in this direction [106] in the context of population genetics, while for particle systems the works [13], [19], [55], [117] for symmetric and [21], [22], [124] for asymmetric processes. For Markov processes, algebraic constructions of duality relations for specific classes of models have also been provided (see e.g. [13], [21], [56], [70], [91], [126]).

Duality as a spectral relation. In this chapter we show that, viewing a duality relation as a spectral relation among the associated Markov generators, duality functions can be obtained from linear combinations of products of eigenfunctions associated to a common eigenvalue. We establish this connection with the general aim of characterizing all possible dualities in terms of the eigenfunctions of the generators involved. To this purpose, our discussion mainly focuses on continuous-time Markov chains on a finite state space for which no reversibility is assumed but canonical eigendecompositions of Jordan-type of the generators are available.

We emphasize that this connection between duality and eigenfunctions goes both ways: not only eigenfunctions of a shared spectrum give rise to duality functions, but also the existence of duality relations carries information about the spectrum of the generators. Here we can already see a clear distinction between the notion of *self-duality* and *integrability*: knowing certain linear combinations of products of eigenfunctions (self-duality) rather than knowing the eigenfunctions themselves (integrability).

Duality in the finite setting: matrix notation. In this chapter, we reformulate the definition of duality w.r.t. a function given in Section 3.1.1 of Chapter 3 to the finite setting (see also Remark 3.18). Let $\mathcal{X}$ be a finite state space with cardinality $|\mathcal{X}| = n$. We consider an irreducible continuous-time Markov process on finite state space $\mathcal{X}$ with generator L given by

$$Lf(x) = \sum_{y \in \mathcal{X}} \ell(x,y)\,(f(y) - f(x))\,,$$

where $f : \mathcal{X} \to \mathbb{R}$ is a real-valued function and $\ell : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ gives the transition rates. For $x \in \mathcal{X}$, we define the exit rate from $x \in \mathcal{X}$ as

$$\ell(x) = \sum_{y \in \mathcal{X} \setminus \{x\}} \ell(x,y)\,.$$

In the finite context we can identify L with the matrix, still denoted by L, given by

$$L(x,y) = \ell(x,y) \text{ for } x \neq y\,, \quad L(x,x) = -\ell(x)\,.$$

Indeed, given two state spaces $\mathcal{X}$, $\widehat{\mathcal{X}}$ of cardinalities $|\mathcal{X}| = n$, $|\widehat{\mathcal{X}}| = \widehat{n}$, and two Markov processes with generators L, $\widehat{L}$, we say that they are *dual* w.r.t. the *duality function* $D : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ if, for all $x \in \mathcal{X}$ and $\widehat{x} \in \widehat{\mathcal{X}}$, we have

$$\widehat{L}_{\text{left}} D(\widehat{x}, x) = L_{\text{right}} D(\widehat{x}, x)\,, \tag{4.1}$$

where "left", resp. "right", refers to action on the left, resp. right, variable. If the laws of the two processes coincide, we speak about *self-duality*.

The same notion in terms of matrix multiplication, where D also denotes the matrix with entries $\{D(\widehat{x}, x),\ \widehat{x} \in \widehat{\mathcal{X}},\ x \in \mathcal{X}\}$, is expressed as

$$\sum_{\widehat{y} \in \widehat{\mathcal{X}}} \widehat{L}(\widehat{x}, \widehat{y})\, D(\widehat{y}, x) \;=\; \sum_{y \in \mathcal{X}} L(x, y)\, D(\widehat{x}, y)\,,$$

or, shortly, as

$$\widehat{L}\, D \;=\; D\, L^{\mathsf{T}}\,, \tag{4.2}$$

where the symbol $^{\mathsf{T}}$ denotes *matrix transposition*, i.e., for a matrix A,

$$(A^{\mathsf{T}})(x, y) \;=\; A(y, x)\,, \quad x, y \in \mathcal{X}\,.$$

More generally, we define two operators $\widehat{L}$ and L *dual* w.r.t. the duality function D if relation (4.1), or equivalently (4.2) in matrix notation, holds.

Organization of the rest of the chapter. The rest of the chapter is organized as follows. After an introductory study of self-duality and duality in the reversible setting in Sections 4.1 and 4.2, in Section 4.3, via Jordan canonical decompositions, we make precise to which extent spectrum and eigenstructure of generators in duality are shared. In Sections 4.1 and 4.2 we further investigate the connection between eigenfunctions and particular instances of dualities that typically appear in the context of interacting particle systems, see e.g. [55, 117]. In Section 4.4 we provide an alternative way of proving and characterizing Siegmund duality in the finite context (see e.g. [128, 75]).

4.1 Self-duality and eigenfunctions: reversible case

Let $\mathcal{X}$ be a finite set of cardinality $|\mathcal{X}| = n$ and let L be a generator of an irreducible *reversible* Markov process on $\mathcal{X}$ w.r.t. the positive measure μ. This measure then satisfies the detailed balance condition

$$\mu(x)\, L(x, y) \;=\; \mu(y)\, L(y, x)\,, \tag{4.3}$$

for all $x, y \in \mathcal{X}$. This relation can be rewritten as a self-duality relation w.r.t. the so-called “cheap” self-duality function:

$$D_{\text{cheap}}(x, y) \;=\; \tfrac{1}{\mu(y)}\, \mathbf{1}_{\{x=y\}}\,. \tag{4.4}$$

The reversibility of μ implies that L is self-adjoint in L^2_μ and, as a consequence, there exists a basis $\{\psi_1, \dots, \psi_n\}$ of eigenfunctions of L with $\psi_1(x) = 1/\sqrt{n}$ corresponding to eigenvalue zero and $\{\psi_1, \dots, \psi_n\}$ orthonormal, i.e.

$$\langle \psi_i, \psi_j \rangle_\mu = \mathbf{1}_{\{i=j\}},$$

where $\langle \cdot, \cdot \rangle_\mu$ denotes inner product in L^2_μ. We denote by $\{\lambda_1, \dots, \lambda_n\}$ the corresponding real eigenvalues with

$$0 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n .$$

The following proposition shows how to obtain and characterize self-duality functions of the Markov generator L in terms of this orthonormal system. The last statement recovers an earlier result from [56].

Proposition 4.1. (i) *For $a_1, a_2 \dots, a_n \in \mathbb{R}$, the function*

$$D(x,y) = \sum_{i=1}^{n} a_i \, \psi_i(x) \, \psi_i(y) \tag{4.5}$$

is a self-duality function.

(ii) *Every self-duality function has a unique decomposition of the form*

$$D(x,y) = \sum_{i,j:\lambda_i=\lambda_j} a_{i,j} \, \psi_i(x) \, \psi_j(y), \tag{4.6}$$

with $\{a_{i,j}, \ i,j \in \{1, \dots, n\}\} \subset \mathbb{R}$.

(iii) *If a function of the form*

$$D(x,y) = f(x) \, g(y)$$

is a non-zero self-duality function, then f and g are eigenfunctions corresponding to the same eigenvalue.

(iv) *The inner product $\langle \cdot, \cdot \rangle_\mu$ of self-duality functions produces self-duality functions, i.e., if D and D' are self-duality functions, then*

$$D''(x,x') := \langle D(x,\cdot), D'(x',\cdot) \rangle_\mu \tag{4.7}$$

is a self-duality function.

Proof. For (i), by definition of eigenfunction $L\psi_i = \lambda_i\psi_i$ with $\lambda_i \in \mathbb{R}$, we obtain

$$\begin{aligned} L_{\text{left}}D(x,y) &= \sum_{i=1}^{n} a_i\, L\psi_i(x)\,\psi_i(y) = \sum_{i=1}^{n} a_i\, \lambda_i\psi_i(x)\,\psi_i(y) \\ &= \sum_{i=1}^{n} a_i\, \psi_i(x)\,\lambda_i\psi_i(y) = \sum_{i=1}^{n} a_i\, \psi_i(x)\,L\psi_i(y) = L_{\text{right}}D(x,y)\,, \end{aligned}$$

hence (4.1).

For (ii), we start by noticing that every function $D : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ can be written in a unique way as

$$D(x,y) = \sum_{i,j=1}^{n} a_{i,j}\,\psi_i(x)\,\psi_j(y)\,,$$

Now using the duality relation (4.1), it follows that

$$\sum_{i,j} a_{i,j}\,\lambda_i\psi_i(x)\,\psi_j(y) = \sum_{i,j} a_{i,j}\,\psi_i(x)\,\lambda_j\psi_j(y)\,,$$

which implies that, for all $i,j = 1,\dots,n$,

$$a_{i,j}\,\lambda_i = a_{i,j}\,\lambda_j\ .$$

For item (iii), we first write

$$f(x)\,g(y) = \sum_{i,j=1}^{n} a_{i,j}\,\psi_i(x)\,\psi_j(y)\,.$$

Then we find $a_{i,j} = \langle f,\psi_i\rangle_\mu\langle g,\psi_j\rangle_\mu =: \alpha_i\beta_j$. From self-duality we conclude, for all $i,j = 1,\dots,n$,

$$\alpha_i\beta_j\,(\lambda_i - \lambda_j) = 0\ .$$

Now use that $f(x)\,g(y)$ is not identically zero to conclude that there exists i with $\alpha_i \neq 0$. Then if $\lambda_j \neq \lambda_i$ we conclude $\beta_j = 0$, which implies that g is an eigenfunction with eigenvalue λ_i. Because g is not identically zero, we can reverse the argument and conclude.

For (iv), by exchanging order of summations and using $\langle\psi_j,\psi_\ell\rangle_\mu = \mathbf{1}_{\{j=\ell\}}$, the l.h.s. of (4.7) reads

$$\sum_{y\in\mathcal{X}} D(x,y)\, D(x',y)\, \mu(y)$$

$$= \sum_{y\in\mathcal{X}} \left(\sum_{i,j:\lambda_i=\lambda_j} a_{i,j}\, \psi_i(x)\, \psi_j(y) \right) \left(\sum_{k,\ell:\lambda_k=\lambda_\ell} a_{k,\ell}\, \psi_k(x')\, \psi_\ell(y) \right) \mu(y)$$

$$= \sum_{j=1}^{n} \left(\sum_{i:\lambda_i=\lambda_j} a_{i,j}\, \psi_i(x) \right) \left(\sum_{k:\lambda_k=\lambda_j} a_{k,j}\, \psi_k(x') \right) .$$

By noting that, for all $j = 1, \ldots, n$, the function $\psi_j' = \sum_{i:\lambda_i=\lambda_j} a_{i,j}\psi_i$ is either vanishing or is an eigenfunction of L associated to λ_j, the proof is concluded. □

In the next propositions we study particular instances of self-duality functions. More precisely, by using Proposition 4.1, we recover the cheap self-duality function in (4.4), while in Proposition 4.3 we characterize orthogonal self-duality functions (cf. (4.11)–(4.12) below).

Proposition 4.2 (CHEAP SELF-DUALITY). (i) *For the choice* $a_1 = a_2 = \ldots = a_n = 1$ *in* (4.5), *we obtain the cheap self-duality function:*

$$D_{\text{cheap}}(x,y) \;=\; \tfrac{1}{\mu(y)}\mathbf{1}_{\{x=y\}} \;=\; \sum_{i=1}^{n} \psi_i(x)\, \psi_i(y)\, . \tag{4.8}$$

(ii) *Conversely, if* $\{\phi_1, \ldots, \phi_n\}$ *is a basis of* L^2_μ *and satisfies*

$$\sum_{i=1}^{n} \phi_i(x)\, \phi_i(y) \;=\; \tfrac{1}{\mu(y)}\mathbf{1}_{\{x=y\}} \tag{4.9}$$

for all $x, y \in \mathcal{X}$, *then* $\{\phi_1, \ldots, \phi_n\}$ *is an orthonormal basis of* L^2_μ.

Proof. To show (4.8), by the positivity of μ, we need to show that, for all $f : \mathcal{X} \to \mathbb{R}$ and $x \in \mathcal{X}$,

$$\sum_{y\in\mathcal{X}} \sum_{i=1}^{n} \psi_i(x)\, \psi_i(y)\, \mu(y)\, f(y) \;=\; f(x)\, .$$

Now note, by interchanging the sum over i with the sum over y, that the l.h.s. equals

$$\sum_{i=1}^{n} \psi_i(x)\,\langle \psi_i, f\rangle_\mu \;=\; f(x)\,,$$

and hence we obtain (i).

For (ii), we need to show that for all $f : \mathcal{X} \to \mathbb{R}$ and $x \in \mathcal{X}$

$$f(x) \;=\; \sum_{i=1}^{n} \phi_i(x)\,\langle \phi_i, f\rangle_\mu \;=\; \sum_{i=1}^{n}\sum_{y\in\mathcal{X}} \phi_i(x)\,\phi_i(y)\,f(y)\,\mu(y)\,. \tag{4.10}$$

We conclude that, by interchanging the order of the two summations in the r.h.s. above and using (4.9), we indeed obtain (4.10). □

Remark that the cheap self-duality function is the only, up to multiplicative constants, diagonal self-duality, and that it is *orthogonal* in the sense that, for all $x, x' \in \mathcal{X}$,

$$\langle D_{\text{cheap}}(x,\cdot), D_{\text{cheap}}(x',\cdot)\rangle_\mu \;=\; \langle D_{\text{cheap}}(x,\cdot), D_{\text{cheap}}(x,\cdot)\rangle_\mu \mathbf{1}_{\{x=x'\}}\,, \tag{4.11}$$

and similarly, for all $y, y' \in \mathcal{X}$,

$$\langle D_{\text{cheap}}(\cdot,y), D_{\text{cheap}}(\cdot,y')\rangle_\mu \;=\; \langle D_{\text{cheap}}(\cdot,y), D_{\text{cheap}}(\cdot,y)\rangle_\mu \mathbf{1}_{\{y=y'\}}\,. \tag{4.12}$$

The next proposition shows how to find all orthogonal self-duality functions.

Proposition 4.3 (ORTHOGONAL SELF-DUALITY). (i) *If* $\{\widetilde{\psi}_1, \ldots, \widetilde{\psi}_n\}$ *is an orthonormal system in* L^2_μ *of eigenfunctions of* L*, corresponding to the same eigenvalues* $\{\lambda_1, \ldots, \lambda_n\}$*, then*

$$D(x,y) \;=\; \sum_{i=1}^{n} \widetilde{\psi}_i(x)\,\psi_i(y) \tag{4.13}$$

is an orthogonal self-duality function. More precisely, for all $x, x' \in \mathcal{X}$,

$$\langle D(x,\cdot), D(x',\cdot)\rangle_\mu \;=\; \tfrac{1}{\mu(x)} \mathbf{1}_{\{x=x'\}}\,. \tag{4.14}$$

(ii) *The self-duality functions of the form* (4.13) *are the only, up to a multiplicative factor, orthogonal self-duality functions.*

Proof. For (i), we compute, for all $k = 1, \ldots, n$ and $x \in \mathcal{X}$, the following quantity

$$\sum_{x' \in \mathcal{X}} \langle D(x,\cdot), D(x',\cdot)\rangle_\mu \, \widetilde{\psi}_k(x')\, \mu(x') \, .$$

By $\langle \psi_i, \psi_j \rangle_\mu = \langle \widetilde{\psi}_i, \widetilde{\psi}_j \rangle_\mu = \mathbf{1}_{\{i=j\}}$, the line above rewrites as follows:

$$\begin{aligned}
&\sum_{x' \in \mathcal{X}} \sum_{y \in \mathcal{X}} \left(\sum_{i=1}^n \widetilde{\psi}_i(x)\, \psi_i(y) \right) \left(\sum_{j=1}^n \widetilde{\psi}_j(x')\, \psi_j(y) \right) \mu(y)\, \widetilde{\psi}_k(x')\, \mu(x') \\
= \;& \sum_{i=1}^n \sum_{j=1}^n \widetilde{\psi}_i(x) \left(\sum_{y \in \mathcal{X}} \psi_i(y)\, \psi_j(y)\, \mu(y) \right) \left(\sum_{x' \in \mathcal{X}} \widetilde{\psi}_j(x')\, \widetilde{\psi}_k(x')\, \mu(x') \right) \\
= \;& \sum_{i=1}^n \sum_{j=1}^n \widetilde{\psi}_i(x)\, \mathbf{1}_{\{i=j\}}\, \mathbf{1}_{\{j=k\}} \;=\; \widetilde{\psi}_k(x) \, .
\end{aligned}$$

This together with Proposition 4.2 concludes the proof of part (i).

For (ii), by starting from a general self-duality function

$$D(x,y) \;=\; \sum_{i,j : \lambda_i = \lambda_j} a_{i,j}\, \psi_i(x)\, \psi_j(y) \, ,$$

the l.h.s. of (4.14) rewrites as

$$\sum_{j=1}^n \psi_j'(x)\, \psi_j'(x') \, ,$$

where $\{\psi_1', \ldots, \psi_n'\}$ are defined as

$$\psi_j'(x) \;=\; \sum_{i : \lambda_i = \lambda_j} a_{i,j}\, \psi_i(x) \, .$$

By remarking that either $\psi_j' = 0$ or ψ_j' is an eigenfunction of L associated to λ_j and applying Proposition 4.2, we have that

$$\langle \psi_i', \psi_j' \rangle_\mu \;=\; \mathbf{1}_{\{i=j\}} \, ,$$

and that the self-duality function D has the form (4.13) with $\widetilde{\psi}_i = \psi_i'$. □

4.2 Duality and eigenfunctions: reversible case

Now we consider two generators $L, \widehat{L}$ on the same finite state space $\mathcal{X}$ with reversible measures μ, $\widehat{\mu}$, respectively, and orthonormal systems of eigenfunctions $\{\psi_1, \ldots, \psi_n\}$, $\{\widehat{\psi}_1, \ldots, \widehat{\psi}_n\}$ corresponding to the *same* real eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, i.e. we assume that L and $\widehat{L}$ are self-adjoint in L^2_μ, resp. in $L^2_{\widehat{\mu}}$, and that they are isospectral.

In what follows we state – without proofs – analogous relations between duality functions and orthonormal systems of eigenfunctions of L and $\widehat{L}$.

Proposition 4.4. (i) *For $a_1, \ldots, a_n \in \mathbb{R}$ the function*

$$D(\widehat{x}, x) \; = \; \sum_{i=1}^{n} a_i \, \widehat{\psi}_i(\widehat{x}) \, \psi_i(x)$$

is a duality function for duality between $\widehat{L}$ and L.

(ii) *Every duality function has a unique decomposition of the form*

$$D(\widehat{x}, x) \; = \; \sum_{i,j : \lambda_i = \lambda_j} a_{i,j} \, \widehat{\psi}_i(\widehat{x}) \, \psi_j(x) \; .$$

(iii) *If a function of the form $D(\widehat{x}, x) = f(\widehat{x}) \, g(x)$ is a non-zero duality function, then f and g are eigenfunctions of $\widehat{L}$, resp. L, corresponding to the same eigenvalue.*

(iv) *The inner products $\langle \cdot, \cdot \rangle_\mu$ and $\langle \cdot, \cdot \rangle_{\widehat{\mu}}$ of duality functions produce self-duality functions, i.e., if D and D' are duality functions, then*

$$\widehat{D}(\widehat{x}, \widehat{x}') \; := \; \langle D(\widehat{x}, \cdot), D'(\widehat{x}', \cdot) \rangle_\mu$$

defines a self-duality function $\widehat{D}$ for $\widehat{L}$ and, similarly,

$$\widetilde{D}(x, x') \; := \; \langle D(\cdot, x), D'(\cdot, x') \rangle_{\widehat{\mu}}$$

determines a self-duality function $\widetilde{D}$ for L.

Proposition 4.5 (ORTHOGONAL DUALITY).

(i) *If* $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_n\}$ *is an orthonormal system in* $L^2_{\widehat{\mu}}$ *of eigenfunctions of* $\widehat{L}$ *corresponding to the same eigenvalues* $\{\lambda_1, \ldots, \lambda_n\}$, *then*

$$D(\widehat{x}, x) \;=\; \sum_{i=1}^{n} \tilde{\psi}_i(\widehat{x})\, \psi_i(x)$$

is an orthogonal duality function, i.e.

$$\langle D(\widehat{x}, \cdot), D(\widehat{x}', \cdot) \rangle_{\mu} \;=\; \tfrac{1}{\widehat{\mu}(\widehat{x}')} \mathbf{1}_{\{\widehat{x}=\widehat{x}'\}}$$

and

$$\langle D(\cdot, x), D(\cdot, x') \rangle_{\widehat{\mu}} \;=\; \tfrac{1}{\mu(x')} \mathbf{1}_{\{x=x'\}} \;.$$

(ii) *These above are the only, up to multiplicative constants, orthogonal dualities between* $\widehat{L}$ *and* L.

4.3 Duality and eigenfunctions: non-reversible case

Working in the *non-reversible* context, i.e. whenever there does not exist a probability measure μ on $\mathcal{X}$ for which the generator L is self-adjoint in L^2_μ, a spectral decomposition of the generator in terms of real non-positive eigenvalues and orthonormal real eigenfunctions is typically lost. In recent years, the study of the eigendecomposition of non-reversible generators has received an increasing attention (see e.g. [26], [27], [28], [112], [138]) and duality-related notions – intertwining relations – have been introduced to relate spectral information of one process, typically a reversible one, to another, typically non-reversible (see e.g. [52], [104]).

Regardless of the spectral eigendecomposition of the generators, in principle interesting dualities can still be constructed from eigenfunctions, either real or complex, and generalized eigenfunctions of the generators involved. The key on which this relation builds up, in the finite context, is the *Jordan canonical decomposition* of the generators. We remark that relations between duality and the Jordan canonical decomposition have already been investigated for a particular instance in the context of models of population dynamics in [106].

Below, before studying the most general result that exploits the Jordan form of the generators, we treat some special cases reminiscent of the previous sections. In the sequel, for a function $\psi : \mathcal{X} \to \mathbb{C}$, we denote by $\psi^* : \mathcal{X} \to \mathbb{C}$

its complex conjugate.

4.3.1 Duality and complex eigenfunctions

A first feature that typically drops as soon as one moves to the non-reversible situation is the appearance of only real eigenvalues. Indeed, given a non-reversible generator L of an irreducible Markov process on $\mathcal{X}$, pairs of complex conjugates eigenvalues $\{\lambda, \lambda^*\}$ and eigenfunctions $\{\psi, \psi^*\}$ may arise as in the following example.

Example 4.6 (COMPLEX CONJUGATE EIGENVALUES). *The continuous-time Markov chain on the state space $\mathcal{X} = \{1, 2, 3\}$ and described by the generator L, which, viewed as a matrix, reads*

$$L = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

represents a basic example of this situation. Indeed, the Markov chain is irreducible, the eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ are

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3^* = -\frac{3}{2} + i\frac{\sqrt{3}}{2},$$

while the associated eigenfunctions $\{\psi_1, \psi_2, \psi_3\}$ are, for $x \in \{1, 2, 3\}$,

$$\psi_1(x) = \frac{1}{\sqrt{3}}, \quad \psi_2(x) = \psi_3^*(x) = e^{(i\frac{2}{3}\pi)x}.$$

Let us, thus, consider two irreducible non-reversible generators $L, \widehat{L}$ on the same state space $\mathcal{X}$. We investigate the situation in which there exist $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and functions $\psi, \widehat{\psi} : \mathcal{X} \to \mathbb{C}$ such that

$$L\psi = \lambda\psi, \qquad \widehat{L}\widehat{\psi} = \lambda\widehat{\psi}. \tag{4.15}$$

Remark that, as $L, \widehat{L}$ are real operators, this implies that

$$L\psi^* = \lambda^*\psi^*, \qquad \widehat{L}\widehat{\psi}^* = \lambda^*\widehat{\psi}^*. \tag{4.16}$$

A real duality function arising from a shared pair of complex eigenvalues is obtained in the following proposition.

Proposition 4.7. *For $a \in \mathbb{C}$, the function*

$$D(\widehat{x}, x) = a\,\widehat{\psi}(\widehat{x})\,\psi(x) + a^*\,\widehat{\psi}^*(\widehat{x})\,\psi^*(x)$$

takes values in $\mathbb{R}$ and is a duality function for $\widehat{L}$ and L.

Proof. It is clear that $D(\widehat{x}, x)$ is in $\mathbb{R}$. Then, by using (4.15) and (4.16), we obtain

$$\begin{aligned}
\widehat{L}_{\text{left}} D(\widehat{x}, x) &= a\,(\widehat{L}\widehat{\psi})(\widehat{x})\,\psi(x) + a^*\,(\widehat{L}\widehat{\psi}^*)(\widehat{x})\,\psi^*(x) \\
&= a\,\lambda\widehat{\psi}(\widehat{x})\,\psi(x) + a^*\,\lambda^*\,\widehat{\psi}^*(\widehat{x})\,\psi^*(x) \\
&= a\,\widehat{\psi}(\widehat{x})\,\lambda\psi(x) + a^*\,\widehat{\psi}^*(\widehat{x})\,\lambda^*\psi^*(x) \\
&= a\,\widehat{\psi}(\widehat{x})\,(L\psi)(x) + a^*\,\widehat{\psi}^*(\widehat{x})\,(L\psi^*)(x) = L_{\text{right}} D(\widehat{x}, x)\,.
\end{aligned}$$

□

4.3.2 Duality and generalized eigenfunctions

A second feature that may be lacking is the existence of a linear independent system of eigenfunctions. However, if L is an irreducible non-reversible generator on the state space $\mathcal{X}$ with real non-negative eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, there always exists a linearly independent system of so-called *generalized eigenfunctions*, i.e., for each eigenvalue λ_i, there exists a set of linearly independent functions $\{\psi_i^{(1)}, \ldots, \psi_i^{(m_i)}\}$ such that $m_i \leq n$,

$$L\psi_i^{(1)} = \lambda_i \psi_i^{(1)}$$

and, for $1 < k \leq m_i$,

$$L\psi_i^{(k)} = \lambda_i \psi_i^{(k)} + \psi_i^{(k-1)}\,.$$

We refer to $\psi_i^{(k)}$ as the *k-th order generalized eigenfunction* associated to λ_i. Moreover, if $\lambda_i \neq \lambda_j$, then the set $\{\psi_i^{(1)}, \ldots, \psi_i^{(m_i)}, \psi_j^{(1)}, \ldots, \psi_j^{(m_j)}\}$ is linearly independent and any arbitrary function $f : \mathcal{X} \to \mathbb{R}$ can be written as linear combination of functions in $\{\psi_i^{(k)},\ i = 1, \ldots, n;\ k = 1, \ldots, m_i\}$.

Example 4.8 (GENERALIZED EIGENFUNCTIONS)**.** *The irreducible generator L on* *the state space $\mathcal{X} = \{1,2,3,4\}$ given by*

$$L = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix},$$

represents a basic example of this situation. Indeed, the eigenvalue $\lambda = -1$ has $\psi^{(1)}$, given by

$$\psi^{(1)}(x) = \frac{(-1)^x}{2}, \quad x \in \{1,2,3,4\},$$

as eigenfunction and

$$\psi^{(2)}(x) = \cos\left(\frac{\pi}{2}(x+1)\right), \quad x \in \{1,2,3,4\},$$

as a second order generalized eigenfunction, i.e.

$$L\psi^{(2)} = -\psi^{(2)} + \psi^{(1)}.$$

In this situation, in case of two generators $L, \widehat{L}$ sharing a real eigenvalue λ with associated generalized eigenfunctions $\{\psi^{(1)}, \ldots, \psi^{(m)}\}$, $\{\widehat{\psi}^{(1)}, \ldots, \widehat{\psi}^{(m)}\}$, the main idea is that a duality function is readily constructed from sums of products of generalized eigenfunctions whose order is, nevertheless, reversed. This connection is the content of the following proposition.

Proposition 4.9. *The function*

$$D(\widehat{x}, x) = \sum_{k=1}^{m} \widehat{\psi}^{(k)}(\widehat{x})\, \psi^{(m+1-k)}(x)$$

is a duality function between $\widehat{L}$ and L.

Proof. By using the definition of k-th order generalized eigenfunction and reordering summations, we obtain

$$\begin{aligned}
\widehat{L}_{\text{left}} D(\widehat{x}, x) &= \sum_{k=1}^{m} (\widehat{L}\widehat{\psi}^{(k)})(\widehat{x})\, \psi^{(m+1-k)}(x) \\
&= \sum_{k=1}^{m} \lambda\, \widehat{\psi}^{(k)}(\widehat{x})\, \psi^{(m+1-k)}(x) + \sum_{k=2}^{m} \widehat{\psi}^{(k-1)}(\widehat{x})\, \psi^{(m+1-k)}(x) \\
&= \sum_{k=1}^{m} \lambda\, \widehat{\psi}^{(k)}(\widehat{x})\, \psi^{(m+1-k)}(x) + \sum_{k=1}^{m-1} \widehat{\psi}^{(k)}(\widehat{x})\, \psi^{(m-k)}(x) \\
&= \sum_{k=1}^{m} \widehat{\psi}^{(k)}(\widehat{x})\, (L\psi^{(m+1-k)})(x) \;=\; L_{\text{right}} D(\widehat{x}, x)\,.
\end{aligned}$$

This concluses the proof. □

4.3.3 Duality and the Jordan canonical decomposition

In this section we provide a general framework that allows us to cover all instances of duality encountered so far in the finite setting. The standard strategy of decomposing generators – viewed as matrices – into their Jordan canonical form builds a bridge between dualities and spectral information of the generators involved. In particular, this linear algebraic approach is useful for the problem of *existence* and *characterization* of duality functions: on one side, the existence of a Jordan canonical decomposition for any generator leads, for instance, to the existence of self-dualities; on the other side, dualities between generators carry information about a common, at least partially, spectral structure of the generators.

Before stating the main result, we introduce some notation. Given a generator L on the state space $\mathcal{X}$ with cardinality $|\mathcal{X}| = n$, L is in *Jordan canonical form* if it can be written as

$$L \;=\; UJU^{-1}\,,$$

where $J \in \mathbb{C}^{n\times n}$ is the *unique* – up to permutations – *Jordan matrix* (cf. [74, Definition 3.1.1]) associated to L and $U \in \mathbb{C}^{n\times n}$ is an invertible matrix. Recall that columns $\{\psi_1, \ldots, \psi_n\}$ of U consists of (possibly generalized) eigenfunctions of L, while the rows $\{v_1, \ldots, v_n\}$ of U^{-1} the (possibly generalized) eigenfunctions of $L^\top$, chosen in such a way that

$$\langle \nu_i, \psi_j \rangle \;:=\; \sum_{x \in \mathcal{X}} \nu_i(x)\, \psi_i^*(x) \;=\; \mathbf{1}_{\{i=j\}}\;.$$

For all Jordan matrices $J \in \mathbb{C}^{n\times n}$ of the form

$$J \;=\; \begin{pmatrix} J_{m_1}(\lambda_1) & & \cdots & 0 \\ & J_{m_2}(\lambda_n) & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & J_{m_k}(\lambda_k) \end{pmatrix},$$

with $m_1 + \ldots + m_k = n$ and *Jordan blocks* $J_m(\lambda)$ of size m associated to eigenvalue $\lambda \in \mathbb{C}$, we define the matrix $B_J \in \mathbb{R}^{n\times n}$ as follows

$$B_J \;=\; \begin{pmatrix} H_{m_1} & & \cdots & 0 \\ & H_{m_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & H_{m_k} \end{pmatrix},$$

where, for all $m \in \mathbb{N}$, the matrix $H_m \in \mathbb{R}^{m\times m}$ is defined as

$$H_m \;=\; \begin{pmatrix} 0 & \cdots & & 1 \\ \vdots & & \iddots & \\ & \iddots & & \vdots \\ 1 & & \cdots & 0 \end{pmatrix},$$

i.e. in such a way that $B_J^\top = B_J^{-1} = B_J$ and $JB_J = B_J J^\top$.

Moreover, we say that two matrices $L \in \mathbb{R}^{n\times n}$, $\widehat{L} \in \mathbb{R}^{\widehat{n}\times\widehat{n}}$ are *r-similar* for some $r = 1, \ldots, \min\{n, \widehat{n}\}$ if there exist Jordan canonical forms

$$L \;=\; UJU^{-1}\,, \qquad \widehat{L} \;=\; \widehat{U}\widehat{J}\widehat{U}^{-1}\,, \tag{4.17}$$

matrices $S_r \in \mathbb{R}^{\widehat{n}\times n}$ and $I_r \in \mathbb{R}^{r\times r}$ of the form

$$S_r \;=\; \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad I_r \;=\; \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix},$$

and permutation matrices $\widehat{P} \in \mathbb{R}^{\widehat{n}\times\widehat{n}}$ and $P \in \mathbb{R}^{n\times n}$ such that

$$T_r = \widehat{P} S_r P$$

and

$$\widehat{J} T_r = T_r J \,. \tag{4.18}$$

Of course, if two matrices are r-similar, then they are necessarily r'-similar, for all $r' = 1, \dots, r$ and if $r = n = \widehat{n}$ then we simply say that they are *similar*.

In the following theorem we establish a general connection between duality relations and Jordan canonical forms for generators L, $\widehat{L}$.

Theorem 4.10. *The following statements are equivalent:*

(i) *There exists a duality function $D(\widehat{x}, x)$ of rank r between $\widehat{L}$ and L.*

(ii) *L and $\widehat{L}$ are r-*similar .

If either condition holds, any duality function is of the form

$$D = \widehat{U} T_r B_J U^\top \,. \tag{4.19}$$

In particular if $L = \widehat{L}$, for any $r = 1, \dots, n$, there always exists a self-duality function D of rank r and it must be of the form (4.19).

Proof. We start by proving that (ii) implies (i). By using the property of r-similarity (4.18) with Jordan decompositions as in (4.17), with the choice (4.19) of the candidate duality function D, we obtain

$$\widehat{L}\widehat{U} T_r B_J U^\top = \widehat{U}\widehat{J} T_r B_J U^\top = \widehat{U} T_r J B_J U^\top = \widehat{U} T_r B_J J^\top U^\top = \widehat{U} T_r B_J U^\top L^\top \,,$$

i.e. the duality relation (4.2) in matrix form.

For the other implication, as the matrices U, $\widehat{U}$ in (4.17) and B_J are invertible, the following chains of identities are equivalent:

$$\begin{aligned} \widehat{L} D = D L^\top \iff& \widehat{U}\widehat{J}\widehat{U}^{-1} D = D (U^{-1})^\top J^\top U^\top \\ \iff& \widehat{J}\widehat{U}^{-1} D (U^{-1})^\top = \widehat{U}^{-1} D (U^{-1})^\top J^\top \\ \iff& \widehat{J}\widehat{U}^{-1} D (U^{-1})^\top B_J = \widehat{U}^{-1} D (U^{-1})^\top B_J J \,. \end{aligned}$$

Moreover, if D has rank r, then $\widehat{U}^{-1} D (U^{-1})^\top B_J$ must have rank r as well. The last relation is of the form

$$\widehat{J} A = A J \,,$$

where $A = \widehat{U}^{-1} D (U^{-1})^{\mathsf{T}} B_J$ is a matrix of rank r. Therefore, we conclude that there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\widehat{J} S_r \;=\; S_r P J P^{-1}\,,$$

i.e. L and $\widehat{L}$ are r-similar according to the Jordan canonical decompositions

$$L \;=\; \widetilde{U}\widetilde{J}\widetilde{U}^{-1}\,, \qquad \widehat{L} \;=\; \widehat{U}\widehat{J}\widehat{U}^{-1}\,,$$

with $\widetilde{U} = UP^{-1}$ and $\widetilde{J} = PJP^{-1}$. □

In words, the theorem above states that there exists a rank-r duality matrix if and only if the generators $\widehat{L}$ and L have r eigenvalues (with multiplicities) in common with "compatible" structure of eigenspaces. Additionally, equation (4.19) provides the most general form of the duality function D in terms of matrices U, $\widehat{U}$. In particular, if J is *diagonal* (i.e., B_J is the identity matrix) all duality functions $D(\widehat{x}, x)$ of rank r read as

$$D(\widehat{x}, x) \;=\; \sum_{i=1}^{r} a_i\, \widehat{\psi}_i(\widehat{x})\, \psi_i(x)\,,$$

for $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$, given $\{\psi_1, \ldots, \psi_n\}$, $\{\widehat{\psi}_1, \ldots, \widehat{\psi}_{\widehat{n}}\}$ are the columns of U, $\widehat{U}$, invertible matrices in the Jordan decompositions (4.17) satisfying (4.18) with $T_r = S_r$. Note the analogy with the duality function described in Propositions 4.1, 4.4 and 4.7. If J is *non-diagonal*, all duality functions D have a similar form up to some index permutations as in Proposition 4.9.

Remark 4.11 (CONSTANT DUALITY FUNCTIONS). *We note that the* constant duality function *is always a trivial duality function between any two generators L, $\widehat{L}$ on $\mathcal{X}$, $\widehat{\mathcal{X}}$. Indeed, $\lambda = 0$ is always an eigenvalue for both L and $\widehat{L}$ with associated constant eigenfunctions $\psi : \mathcal{X} \to \mathbb{R}$, $\widehat{\psi} : \widehat{\mathcal{X}} \to \mathbb{R}$, i.e., for all $x \in \mathcal{X}$ and $\widehat{x} \in \widehat{\mathcal{X}}$,*

$$\psi(x) \;=\; 1\,, \quad \widehat{\psi}(\widehat{x}) \;=\; 1\,,$$

are eigenfunctions for L, $\widehat{L}$ associated to $\lambda = 0$.

Remark 4.12 (SELF-DUALITY & MATRIX SELF-SIMILARITY). *Another consequence, as already mentioned in [62], is that in the finite context* self-duality *functions* always *exist. In fact, a generator L, viewed as a matrix, is always* similar *to itself. Hence, viewing duality relations between generators as similarity relations among* *matrices allows one to transfer statements about* existence *of Jordan canonical decompositions to statements regarding the* existence *of duality relations, even when neither any explicit formula of the duality functions nor reversible measures for the processes are known. However, Theorem 4.10 above provides information on how to construct any self-duality matrix. Indeed, given any two Jordan decompositions of L, say*

$$LU = UJ\,, \qquad L\widetilde{U} = \widetilde{U}J\,,$$

the matrix D constructed from $U, \widetilde{U}$ *and J as in* (4.19), *namely*

$$D = \widetilde{U} B_J U^\mathsf{T}\,, \tag{4.20}$$

turns out to be a self-duality function for L and, viceversa, any self-duality matrix D for L is of the form (4.20).

4.3.4 Duality and time-reversal

We can now provide an analogue of Proposition 4.2 beyond the reversible context. To fix notation, let L be a generator on $\mathcal{X}$, with $|\mathcal{X}| = n$. Lacking reversibility, we have seen that complex eigenvalues and generalized eigenfunctions of the generator may arise. However, in the irreducible case, i.e. in case there exists a unique stationary measure $\mu > 0$ for which the adjoint of L in L^2_μ, say $L^\dagger$, is itself a generator, a trivial duality relation between L and $L^\dagger$ is available. Indeed, from the adjoint relation

$$\langle L^\dagger f, g\rangle_\mu = \langle f, L\,g\rangle_\mu\,, \quad f, g : \mathcal{X} \to \mathbb{R}\,,$$

it follows that the diagonal function $D : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ given by

$$D(x,y) = \tfrac{1}{\mu(y)} \mathbf{1}_{\{x=y\}}\,, \quad x, y \in \mathcal{X}\,, \tag{4.21}$$

is a duality function for $L^\dagger$, L. In analogy with (4.4), we refer to it as *cheap duality function*, also $D = D_{\text{cheap}}$.

From Theorem 4.10, the above duality tells us that, beside the fact that the generators L and $L^\dagger$ are indeed similar as matrices, the cheap duality function D_{cheap} in (4.21) should be represented in terms of functions $\{\psi_1, \ldots, \psi_n\}$ and $\{\widetilde{\psi}_1, \ldots, \widetilde{\psi}_n\}$, which, up to suitably reordering, are indeed the generalized eigenfunctions of L and $L^\dagger$, respectively.

As a consequence of the following lemma, which we use in the proof of Theorem 4.17, we obtain that a relation of *bi-orthogonality* w.r.t. μ among the generalized eigenfunctions of L and those of $L^\dagger$ can be derived from the duality w.r.t. D_{cheap}. For the proof, we refer back to the proof of Proposition 4.2.

Proposition 4.13. *Let L be a generator, μ a positive measure on $\mathcal{X}$ (not necessarily stationary for L) and let $L^\dagger$ be the adjoint operator of L in L^2_μ. Let the spans of the generalized eigenfunctions of L and $L^\dagger$, say $\{\psi_1, \ldots, \psi_n\}$ and $\{\widetilde{\psi}_1, \ldots, \widetilde{\psi}_n\}$, both coincide with L^2_μ. Then the following statements are equivalent:*

(i) Cheap duality from generalized eigenfunctions. *For $x, y \in \mathcal{X}$,*

$$\sum_{i=1}^{n} \widetilde{\psi}_i(x)\, \psi_i(y) \;=\; \tfrac{1}{\mu(y)} \mathbf{1}_{\{x=y\}} \,.$$

(ii) Bi-orthogonality of generalized eigenfunctions. *For all $i, j = 1, \ldots, n$,*

$$\langle \widetilde{\psi}_i, \psi_j^* \rangle_\mu \;=\; \sum_{x' \in \mathcal{X}} \widetilde{\psi}_i(x')\, \psi_j(x')\, \mu(x') \;=\; \mathbf{1}_{\{i=j\}} \,. \qquad (4.22)$$

Two families $\{\psi_1^*, \ldots, \psi_n^*\}$, $\{\widetilde{\psi}_1, \ldots, \widetilde{\psi}_n\}$ satisfying condition (4.22) *are also said to be* bi-orthogonal *w.r.t. the measure μ.*

4.3.5 From eigenfunctions to duality: a first example

Typically, to find the eigenvalues and eigenfunctions of the generator associated to a Markov chain is a much more challenging task than establishing duality relations. However, we have seen that the knowledge of the eigenfunctions leads to a full characterization of duality and/or self-duality functions. This is, indeed, the case of the example below, in which we exploit the knowledge of eigenfunctions of two generators to characterize the family of self-duality and duality functions.

Example 4.14 (ONE-DIMENSIONAL SYMMETRIC RANDOM WALKS ON A FINITE GRID). *Let us introduce the symmetric random walk on $\mathcal{X} = \{1, \ldots, n\}$* reflected on the left *and* absorbed on the right. *We describe the action of the generator L on functions $f : \mathcal{X} \to \mathbb{R}$ as*

$$Lf(x) \;=\; (f(x+1) - f(x)) + (f(x-1) - f(x))\,, \quad x \in \mathcal{X} \setminus \{1, n\}\,,$$

while for $x \in \{1, n\}$ we have

$$Lf(1) \;=\; 2(f(2) - f(1))\,, \qquad Lf(n) \;=\; 0\,.$$

Similarly, we denote by $\widehat{L}$ *the generator of the symmetric random walk on* $\mathcal{X}$ reflected on the right *and* absorbed on the left. *Namely,*

$$\widehat{L}f(x) = (f(x+1) - f(x)) + (f(x-1) - f(x)), \quad x \in \mathcal{X} \setminus \{1, n\},$$

and

$$\widehat{L}f(1) = 0, \qquad \widehat{L}f(n) = 2(f(n-1) - f(n)).$$

As an application of Theorem 4.10, we prove the following dualities: self-duality *of* L, self-duality *of* $\widehat{L}$ *and* duality *between* L *and* $\widehat{L}$. *The key is to explicitly find eigenvalues and eigenfunctions of the generators. Indeed, the eigenvalues* $\{\lambda_1, \ldots, \lambda_n\}$ *of* L *and* $\widehat{L}$ *read as follows:*

$$\lambda_1 = 0, \quad \lambda_i = 2(\cos(\theta_i) - 1), \quad \theta_i = \frac{i - \frac{1}{2}}{n-1}\pi, \quad i = 2, \ldots, n. \tag{4.23}$$

The eigenfunctions $\{\psi_1, \ldots, \psi_n\}$ *of* L *are, for* $x \in \mathcal{X}$,

$$\psi_1(x) = \frac{1}{\sqrt{n}}, \quad \psi_i(x) = \frac{1}{\sqrt{n}} \cos(\theta_i(x-1)), \quad i = 2, \ldots, n,$$

while the eigenfunctions $\{\widehat{\psi}_1, \ldots, \widehat{\psi}_n\}$ *of* $\widehat{L}$ *are, for* $x \in \mathcal{X}$,

$$\widehat{\psi}_1(\widehat{x}) = \frac{1}{\sqrt{n}}, \quad \widehat{\psi}_i(\widehat{x}) = \frac{1}{\sqrt{n}} \sin(\theta_i(\widehat{x}-1)), \quad i = 2, \ldots, n.$$

Hence, we conclude the following:

(a) Self-duality functions for L. *For all values* $a_1, \ldots, a_n \in \mathbb{R}$, *the function*

$$\begin{aligned} D(x,y) &= \sum_{i=1}^{n} a_i\, \psi_i(x)\, \psi_i(y) \\ &= \frac{a_1}{n} + \sum_{i=2}^{n} \frac{a_i}{n} \cos(\theta_i(x-1)) \cos(\theta_i(y-1)) \end{aligned} \tag{4.24}$$

is a self-duality function for L *and all self-duality functions are of this form.*

(b) Self-duality functions for $\widehat{L}$. *For all* $a_1, \ldots, a_n \in \mathbb{R}$,

$$\widehat{D}(\widehat{x}, \widehat{y}) = \sum_{i=1}^{n} a_i \, \widehat{\psi}_i(\widehat{x}) \, \widehat{\psi}_i(\widehat{y})$$

$$= \frac{1}{n} + \sum_{i=2}^{n} \frac{a_i}{n} \sin(\theta_i(\widehat{x}-1)) \sin(\theta_i(\widehat{y}-1)) \quad (4.25)$$

is a self-duality function for $\widehat{L}$ *and all self-duality functions are of this form.*

(c) Duality functions between L and $\widehat{L}$. *For all* $a_1, \ldots, a_n \in \mathbb{R}$,

$$D'(\widehat{x}, x) = \frac{a_1}{n} + \sum_{i=2}^{n} \frac{a_i}{n} \sin(\theta_i(\widehat{x}-1)) \cos(\theta_i(x-1)) \quad (4.26)$$

is a duality function between L *and* $\widehat{L}$ *and all duality functions are of this form.* □

4.3.6 Intertwining, duality and generalized eigenfunctions

Symmetries of the generators or, more generally, *intertwining* relations have proved to be useful in producing new duality relations from existing ones, e.g. cheap dualities (see e.g. [21], [117] and Section 3.4 of Chapter 3 of this thesis). Here, we analyze this technique and revisit Theorem 3.19 (also [117, Theorem 5.1]) in the finite setting from the point of view of generalized eigenfunctions.

Theorem 4.15 (INTERTWINING RELATIONS, DUALITY AND GENERALIZED EIGENFUNCTIONS). *Let* L, $\widetilde{L}$ *and* $\widehat{L}$ *be three generators on* $\mathcal{X}$, $\widetilde{\mathcal{X}}$ *and* $\widehat{\mathcal{X}}$ *respectively. We assume that* L *and* $\widetilde{L}$ *are* intertwined, *i.e. there exists a linear operator* $\Lambda : L^2(\mathcal{X}) \to L^2(\widetilde{\mathcal{X}})$ *such that, for all* $f \in L^2(\mathcal{X})$, *we have*

$$\widetilde{L}\Lambda f = \Lambda L f \, . \quad (4.27)$$

Moreover, we assume that L *and* $\widehat{L}$ *are dual with duality function* $D : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$, *i.e.*

$$\widehat{L}_{\text{left}} D(\widehat{x}, x) = L_{\text{right}} D(\widehat{x}, x) \, .$$

Then, the function $\Lambda_{\text{right}} D : \widehat{\mathcal{X}} \times \widetilde{\mathcal{X}} \to \mathbb{R}$ is a duality function for $\widetilde{L}$ and $\widehat{L}$, i.e.

$$\widehat{L}_{\text{left}} \Lambda_{\text{right}} D(\widehat{x}, \widetilde{x}) \;=\; \widetilde{L}_{\text{right}} \Lambda_{\text{right}} D(\widehat{x}, \widetilde{x}) \,.$$

Proof. We observe that the intertwining operator Λ maps eigenspaces of L to eigenspaces of $\widetilde{L}$. More precisely, if there exists a subset $\{\psi^{(1)}, \ldots, \psi^{(m)}\}$ of $L^2(\mathcal{X})$ such that, for some $\lambda \in \mathbb{C}$,

$$L\psi^{(1)} \;=\; \lambda\psi^{(1)}\,, \quad L\psi^{(k)} \;=\; \lambda\psi^{(k)} + \psi^{(k-1)}\,, \quad k = 2, \ldots, m\,, \tag{4.28}$$

then, by (4.27), the subset $\{\Lambda\psi^{(1)}, \ldots, \Lambda\psi^{(m)}\}$ in $L^2(\widetilde{\mathcal{X}})$ satisfy the same identities as in (4.28) up to replace L by $\widetilde{L}$:

$$\widetilde{L}\Lambda\psi^{(1)} \;=\; \lambda\Lambda\psi^{(1)}\,, \quad \widetilde{L}\Lambda\psi^{(k)} \;=\; \lambda\Lambda\psi^{(k)} + \Lambda\psi^{(k-1)}\,, \quad k = 2, \ldots, m\,. \tag{4.29}$$

By Theorem 4.10, the duality function is given by

$$D(\widehat{x}, x) \;=\; \sum_{i=1}^{n} \widehat{\psi}_i(\widehat{x})\, \psi_i(x)\,,$$

where $\{\psi_1, \ldots, \psi_n\}$, $\{\widehat{\psi}_1, \ldots, \widehat{\psi}_n\}$ are sets of (possibly generalized) eigenfunctions of L, $\widehat{L}$. Then, by applying the intertwining operator Λ on the right variables, we obtain

$$\Lambda_{\text{right}} D(\widehat{x}, \widetilde{x}) \;=\; \sum_{i=1}^{n} \widehat{\psi}_i(\widehat{x})\, (\Lambda\psi_i)(\widetilde{x})\,.$$

We conclude from the considerations in (4.29), (4.28) and Theorem 4.10. □

Typical examples of intertwining relations occur when either Λ is a *symmetry* of a generator, i.e. $\widetilde{L} = L$ in (4.27) (see e.g. [21]) or when Λ is a *positive contractive operator* such that $\Lambda 1 = 1$, i.e. viewed as a matrix, it is a stochastic matrix from the space $\widetilde{\mathcal{X}}$ to $\mathcal{X}$ (see e.g. [75]). A particular instance, which recovers the so-called *lumpability*, of this last situation is when Λ is a "deterministic" stochastic kernel, i.e. induced by a map from $\widetilde{\mathcal{X}}$ to $\mathcal{X}$.

4.4 Siegmund duality and eigenfunctions

This connection between duality functions and eigenfunctions enables us to recover another special instance of duality, the so-called *Siegmund duality*. Siegmund duality, which arises in the context of totally ordered state spaces $\mathcal{X} = \widehat{\mathcal{X}}$, was first established by Siegmund [128] for pairs of absorbed/reflected-at-0 processes on the positive real line and on the positive integers. Further applications and generalizations of Siegmund dualities were studied by many authors, see for instance [88], [95], [98].

What we focus here on is a finite-context characterization of Siegmund duality already obtained via an intertwining relation in [75]. However, by using a representation of duality in terms of generalized eigenfunctions of the generators, the characterization result of Siegmund duality that we obtain, besides simplifying the proof of an analogous result in [128, Theorem 3], adds spectral information to the proof in [75].

Moreover, as Siegmund duality can be seen as a full-rank duality between two processes in view of Theorem 4.10), a spectral approach guarantees the existence of other duality relations in presence of Siegmund duality. If, in addition, the eigenfunctions of the generators are explicitly known, then all duality functions can be explicitly recovered.

4.4.1 Siegmund duality

On the totally ordered state space $\mathcal{X} = \{1, \ldots, n\}$, two generators L, $\widehat{L}$ are said to be *Siegmund dual* if

$$\widehat{L}_{\text{left}} D_{\text{S}}(x,y) \;=\; L_{\text{right}} D_{\text{S}}(x,y)\,, \tag{4.30}$$

with duality function $D_{\text{S}} : \mathcal{X} \times \mathcal{X} \to [0,1]$ given by

$$D_{\text{S}}(x,y) \;=\; \mathbf{1}_{\{x \geq y\}}\,. \tag{4.31}$$

Note that the duality relation (4.30) with duality function D_{S} (4.31) reads out

$$\sum_{x'=y}^{n} \widehat{L}(x,x') \;=\; \sum_{y'=1}^{x} L(y,y')\,. \tag{4.32}$$

From (4.32), a *necessary* relation between two Siegmund dual generators L and $\widehat{L}$ reads as follows:

$$L(y,x) \;=\; \sum_{x'=y}^{n} \widehat{L}(x,x') - \widehat{L}(x-1,x')\,, \quad x,y\in\mathcal{X}\,, \tag{4.33}$$

with the convention $\widehat{L}(0,\cdot) = 0$. As (4.33) implies (4.32), this condition is indeed also *sufficient*.

Remark 4.16 (SUB-GENERATORS AND MONOTONICITY). *If we require that only $\widehat{L}$ is a generator, the operator L as defined in* (4.33) *is not necessarily a generator. However, the following implications hold:*

(a) *If $\widehat{L}$ is a generator and $L(y,x)\geq 0$ for all $x\neq y$, then L is a* sub-generator *on $\mathcal{X}$, i.e.*

$$L(y,x) \;\geq\; 0\,,\; x\neq y \quad \text{and} \quad \sum_{x=1}^{n} L(y,x) \;\leq\; 0\,,\; y\in\mathcal{X}\,. \tag{4.34}$$

The proof goes as follows:

$$\begin{aligned}\sum_{x=1}^{n} L(y,x) \;&=\; \sum_{x'=y}^{n}\sum_{x=1}^{n} \widehat{L}(x,x') - \widehat{L}(x-1,x')\\ &=\; \sum_{x'=y}^{n} \widehat{L}(n,x') \;\leq\; \sum_{x'=1}^{n} \widehat{L}(n,x') \;=\; 0\,,\end{aligned}$$

where we used (4.33) *in the first equality and the last inequality is a consequence of $\widehat{L}$ being a generator.*

(b) *Note that, by [84, Theorem 2.1],*

$$\sum_{x'=y}^{n} \widehat{L}(x,x') - \widehat{L}(x-1,x') \;\geq\; 0\,, \quad x\neq y\,, \tag{4.35}$$

is equivalent to require that the continuous-time Markov chain with generator $\widehat{L}$ is monotone *(cf. [98]).*

As a consequence, L is a sub-generator if and only if $\widehat{L}$ is associated to a monotone process on $\mathcal{X}$.

4.4.2 From Siegmund duality to eigenfunctions and back

In the following theorem, we study the relation between eigenfunctions of Siegmund dual (sub-)generators and how the Siegmund duality function D_{S} in (4.31) is constructed from the eigenfunctions.

Theorem 4.17. (i) *Let L and $\widehat{L}$ be Siegmund dual (sub-)generators in the sense of* (4.30). *If $\widehat{v}$ is a k-th order generalized eigenfunction of $\widehat{L}^{\mathsf{T}}$ associated to eigenvalue λ, then*

$$\psi(x) \;=\; \sum_{y=x}^{n} \widehat{v}(y)\,, \qquad x \in \mathcal{X}\,, \tag{4.36}$$

is a k-th order generalized eigenfunction of L associated to the eigenvalue λ.

(ii) *In the same context of item* (i), *given a set $\{\widehat{v}_1, \ldots, \widehat{v}_n\}$ of (generalized) eigenfunctions of $\widehat{L}^{\mathsf{T}}$ whose span coincides with $L^2(\mathcal{X})$, if $\{\widehat{\psi}_1, \ldots, \widehat{\psi}_n\}$ are (generalized) eigenfunctions of $\widehat{L}$ such that*

$$\langle \widehat{v}_i, \widehat{\psi}_j^* \rangle \;=\; \sum_{x=1}^{n} \widehat{v}_i(x)\,\widehat{\psi}_j(x) \;=\; \mathbf{1}_{\{i=j\}}\,, \tag{4.37}$$

and $\{\psi_1, \ldots, \psi_n\}$ are defined in terms of $\{\widehat{v}_1, \ldots, \widehat{v}_n\}$ as in (4.36), *then the function*

$$D(x,y) \;=\; \sum_{i=1}^{n} \widehat{\psi}_i(x)\,\psi_i(n)\,, \qquad x,y \in \mathcal{X}\,,$$

is the Siegmund duality function D_{S}.

(iii) *Let L and $\widehat{L}$ be (sub-)generators on $\mathcal{X}$. If for any k-th order generalized eigenfunction $\widehat{v}$ of $\widehat{L}^{\mathsf{T}}$ associated to eigenvalue λ, ψ as defined in* (4.36) *is a k-th order generalized eigenfunction of L associated to the same eigenvalue λ, then L and $\widehat{L}$ are Siegmund dual and D_{S} is obtained as in item* (ii).

Proof. Let $\widehat{v}$ and ψ be as in item (i). Then,

$$\begin{aligned}\sum_{x=1}^{n} L(y,x)\,\psi(x) &= \sum_{x=1}^{n}\left(\sum_{x'=y}^{n}\widehat{L}(x,x') - \widehat{L}(x-1,x')\right)\psi(x)\\ &= \sum_{x'=y}^{n}\sum_{x=1}^{n}\left(\widehat{L}^{\mathsf{T}}(x',x)\,\psi(x) - \widehat{L}^{\mathsf{T}}(x',x-1)\,\psi(x)\right),\end{aligned}$$

which, by noting that $\widehat{v}(n) = \psi(n)$, reads as

$$\sum_{x'=y}^{n}\sum_{x=1}^{n}\widehat{L}^{\mathsf{T}}(x',x)\,\widehat{v}(x) = \sum_{x'=y}^{n}\lambda\widehat{v}(x') = \lambda\sum_{x'=y}^{n}\widehat{v}(x') = \lambda\psi(y)\,,$$

thus, ψ is eigenfunction with eigenvalue λ. For the generalized eigenfunctions, the proof follows the same line.

For item (ii) and (iii), from the sets $\{\widehat{v}_1,\ldots,\widehat{v}_n\}$ and $\{\psi_1,\ldots,\psi_n\}$ of generalized eigenfunctions of $\widehat{L}^{\mathsf{T}}$ and L related as in (4.36), by Theorem 4.10, the function

$$D(x,y) = \sum_{i=1}^{n}\widehat{\psi}_i(x)\,\psi_i(y) = \sum_{i=1}^{n}\widehat{\psi}_i(x)\sum_{x'=y}^{n}\widehat{v}_i(x') = \sum_{x'=y}^{n}\sum_{i=1}^{n}\widehat{v}_i(x')\,\widehat{\psi}_i(x) \tag{4.38}$$

is a full-rank duality for L and $\widehat{L}$. By condition (4.37) and Proposition 4.13, we obtain

$$\sum_{i=1}^{n}\widehat{v}_i(x')\,\widehat{\psi}_i(x) = \mathbf{1}_{\{x=x'\}}\,,$$

and hence the function $D(x,y)$ in (4.38) writes as

$$D(x,y) = \sum_{x'=y}^{n}\mathbf{1}_{\{x=x'\}} = \mathbf{1}_{\{x\geq y\}} = D_{\mathrm{S}}(x,y)\,.$$

□

4.4.3 From eigenfunctions to duality: a second example

In this final example, by using item (iii) of Theorem 4.17, we show how to obtain Siegmund duality from the knowledge of eigenvalues and eigenfunctions of (sub-)generators. The example we consider here concerns two symmetric simple random walks on $\mathcal{X} = \{1, \ldots, n\}$.

Example 4.18 (BLOCKED VS ABSORBED ONE-DIMENSIONAL RANDOM WALKS). *The first symmetric nearest-neighbor random walk is* blocked at the boundaries, *namely the generator $\widehat{L}$ is described, for $f : \mathcal{X} \to \mathbb{R}$, as*

$$\widehat{L}f(x) \ = \ (f(x+1) - f(x)) + (f(x-1) - f(x)) \,, \quad x \in \mathcal{X} \setminus \{1, n\} \,,$$

and, on the boundaries,

$$\widehat{L}f(1) \ = \ f(2) - f(1) \,, \qquad \widehat{L}f(n) \ = \ f(n-1) - f(n) \,.$$

The second random walk is absorbed at the boundaries, *i.e. it is a sub-Markov process on $\mathcal{X} = \{1, \ldots, n\}$ with sub-generator L which acts on functions $f : \mathcal{X} \to \mathbb{R}$ as*

$$Lf(x) \ = \ \widehat{L}f(x) \,, \quad x \in \mathcal{X} \setminus \{1, n\} \,,$$

and

$$Lf(1) \ = \ 0 \,, \qquad Lf(n) \ = \ f(n-1) - 2f(n) \,,$$

i.e. $x = 1$ is an absorbing point, while at $x = n$ the random walk either jumps to the left at rate 1 or "exits the system" at rate 1.

To explicitly obtain eigenfunctions and eigenvalues in this setting we use the following ansatz*:*

$$f_{a,b,c,\theta}(x) \ = \ a\cos(\theta x + c) + b \sin(\theta x + c) \,, \quad x \in \mathcal{X} \,,$$

where a, b, c and $\theta \in \mathbb{R}$ are the parameters to be determined. Regarding the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, in both cases we have

$$\lambda_1 \ = \ 0 \,, \quad \lambda_i \ = \ 2(\cos(\theta_i) - 1) \,, \quad \theta_i \ = \ \frac{i-1}{n}\pi \,, \quad i = 2, \ldots, n \,. \qquad (4.39)$$

Hence, all eigenvalues are distinct. The eigenfunctions $\{\widehat{\psi}_1, \ldots, \widehat{\psi}_n\}$ of $\widehat{L}$ are, for

$x \in \{1, \ldots, n\}$ and $i = 2, \ldots, n$,

$$\begin{aligned}\widehat{\psi}_1(x) &= \frac{1}{Z_1}\,,\\ \widehat{\psi}_i(x) &= \frac{1}{Z_i}\left\{-\sin(\theta_i)\cos(\theta_i(x-1)) + (1-\cos(\theta_i))\sin(\theta_i(x-1))\right\}\,,\end{aligned} \tag{4.40}$$

where $Z_i = \sqrt{n(1-\cos(\theta_i))}$ for all $i = 1, \ldots, n$. The eigenfunctions $\{\psi_1, \ldots, \psi_n\}$ of L are given, for $x \in \{1, \ldots, n\}$ and $i = 2, \ldots, n$, by

$$\psi_1(x) = \frac{n+1-x}{Z_1}\,, \quad \psi_i(x) = \frac{1}{Z_i}\sin(\theta_i(x-1))\,. \tag{4.41}$$

Hence, we note that:

(a) *By Theorem 4.10, L and $\widehat{L}$ are dual and any duality function is of the form*

$$D(x,y) = \sum_{i=1}^{n} a_i\, \widehat{\psi}_i(x)\, \psi_i(y)\,, \tag{4.42}$$

for $a_1, \ldots, a_n \in \mathbb{R}$.

(b) *By denoting by μ the counting measure on $\mathcal{X} = \{1, \ldots, n\}$, the generator $\widehat{L}$ is self-adjoint in L^2_μ and is, as a matrix, symmetric, i.e. $\widehat{L}^\top = \widehat{L}$. As a consequence, $\{\widehat{\psi}_1, \ldots, \widehat{\psi}_n\}$ are eigenfunctions of both $\widehat{L}$ and $\widehat{L}^\top$.*

(c) *For all $i = 1, \ldots, n$,*

$$\psi_i(x) = \sum_{y=x}^{n} \widehat{\psi}_i(y)\,, \quad x \in \mathcal{X}\,,$$

i.e. the eigenfunctions $\{\psi_1, \ldots, \psi_n\}$ in (4.41) are related to $\{\widehat{\psi}_1, \ldots, \widehat{\psi}_n\}$ in (4.40) as in (4.36).

(d) *The eigenfunctions $\widehat{\psi}_1, \ldots, \widehat{\psi}_n$ are normalized in L^2_μ, i.e., for all $i, j = 1, \ldots, n$,*

$$\langle \widehat{\psi}_i, \widehat{\psi}_j \rangle_\mu = \mathbf{1}_{\{i=j\}}\,.$$

As a consequence, by Theorem 4.17, for the choice $a_1 = \ldots = a_n = 1$, the duality function $D(x,y)$ in (4.42) is the Siegmund duality function $D_S(x,y)$ in (4.31), *namely, for all* $x, y \in \mathcal{X}$,

$$\frac{n+1-y}{n} + \sum_{i=2}^{n} \frac{\sin(\theta_i(y-1))}{n(1-\cos(\theta_i))} \times$$
$$\times \{-\sin(\theta_i)\cos(\theta_i(x-1)) + (1-\cos(\theta_i))\sin(\theta_i(x-1))\} = \mathbf{1}_{\{x \ge y\}} .$$

As a final remark, we note that, by adding the cemetery state $\Delta = \{n+1\}$ *accessible at rate* 1 *only from the state* $\{n\}$, *the absorbed sub-Markov random walk associated to L becomes a proper Markov process with* $\{1\}$ *and* $\{n+1\}$ *as absorbing states. If we denote by* L^{ext} *the generator on the extended space* $\mathcal{X}^{\text{ext}} = \mathcal{X} \cup \Delta$, *it follows that the spectrum (with multiplicities) of* L^{ext}, *say* $\Sigma(L^{\text{ext}})$, *equals the spectrum (with multiplicities) of L with an additional eigenvalue zero, i.e.*

$$\Sigma(L^{\text{ext}}) = \Sigma(L) \cup \{0\} ,$$

while the new eigenfunctions $\{\psi_1^{\text{ext}}, \ldots, \psi_n^{\text{ext}}, \psi_{n+1}^{\text{ext}}\}$ *are such that*

$$\psi_{n+1}^{\text{ext}}(x) = 1 , \quad x \in \mathcal{X}^{\text{ext}} ,$$

and, for all $i = 1, \ldots, n$,

$$\psi_i^{\text{ext}}(n+1) = 0 , \qquad \psi_i^{\text{ext}}(x) = \psi_i(x) , \quad x \in \mathcal{X} .$$

Hence, also the function

$$D_{\mathrm{S}}^{\text{ext}}(x,y) = \sum_{i=1}^{n} \widehat{\psi}_i(x)\, \psi_i^{\text{ext}}(y) , \quad x \in \mathcal{X} , \quad y \in \mathcal{X}^{\text{ext}} ,$$

equals $\mathbf{1}_{\{x \ge y\}}$.

4.5 Spectral self-duality for finite conservative particle systems

In view of this linear algebraic point of view on duality, we return to the problem of finding self-duality relations for (finite) conservative interacting particle systems, with the goal of verifying what we called in Section 1.2 "spectral self-duality". In words, this notion of self-duality for conservative particle systems requires that the spectra (with multiplicities) of the generators associated to systems with different total numbers of particles are "nested" one into each other.

This section will be divided in two parts. In the first part, we prove that, for all finite set of sites $(V, \sim)$ equipped with conductances $\boldsymbol{c}$, the particle systems SEP(α), IRW(α) and SIP(α) introduced in Chapter 3, besides being self-dual with "non-trivial" jointly factorized self-duality function, are also spectrally self-dual. We recover this result by means of intertwining relations – which, in turn, are derived from well-known symmetries of the infinitesimal generators (cf. e.g. [62]). In particular, we remark that we achieve this without any explicit study of neither eigenvalues nor eigenfunctions of the generators.

In the second part, we consider some specific examples of conservative particle systems evolving on *two sites* only – being a finite set $V = \{x, y\}$ with two sites the simplest non-trivial set on which interacting particles may hop. Here, the aim consists in proving – or disproving – spectral self-duality by deriving explicitly the eigenvalues of the generators.

In what follows, we first introduce setting and notations and, in particular, define the notion of *spectral self-duality* for conservative particle systems.

Setting. If we denote by $\mathcal{X} \subset \mathbb{N}_0^V$ the space of "admissible" particle configurations on the finite set V, for all $n \in \mathbb{N}_0$, $\mathcal{X}_n \subset \mathcal{X}$ stands for the subset of configurations whose number of particles sums up to exactly n. Note that $\bigsqcup_{n \in \mathbb{N}_0} \mathcal{X}_n = \mathcal{X}$, where possibly $\mathcal{X}_n = \emptyset$ for some $n \in \mathbb{N}_0$.

On these (finite) configuration spaces $\{\mathcal{X}_n,\ n \in \mathbb{N}_0\}$, we define continuous-time Markovian dynamics described by the infinitesimal generators $\{L_n,\ n \in \mathbb{N}_0\}$, where, for all $n \in \mathbb{N}_0$, L_n acts on $L^2(\mathcal{X}_n)$. The spaces $\{\mathcal{X}_n,\ n \in \mathbb{N}_0\}$ and operators $\{L_n,\ n \in \mathbb{N}_0\}$ define an abstract conservative particle system. In the following definition, we introduce the notion of spectral self-duality for such conservative particle systems. For a motivation behind this definition, we refer to Theorem 4.10 and its consequences.

Definition 4.19 (SPECTRAL SELF-DUALITY). *We say that the conservative particle system $\{L_n,\ n \in \mathbb{N}_0\}$ is* spectrally self-dual *if, for all $n, m \in \mathbb{N}_0$, either one of the following inclusions*

$$\Sigma(L_n) \subset \Sigma(L_m)\,, \quad \Sigma(L_m) \subset \Sigma(L_n)\,,$$

holds, where $\Sigma(L_n)$ denotes the spectrum (with multiplicities) of L_n.

Remark 4.20 (SPECTRAL SELF-DUALITY AND MAXIMAL SIMILARITY). *Note that, if we adopt the same terminology as in Theorem 4.10, without the requirement* *that the conservative particle system is reversible, spectral self-duality does not necessarily imply $r_{n,m}$-similarity of the generators L_n and L_m, where $r_{n,m} := \max\{|\mathcal{X}_n|, |\mathcal{X}_m|\}$. However, in what follows, we will need only the converse statement; namely, that $r_{n,m}$-similarity for all $n, m \in \mathbb{N}_0$ always implies spectral self-duality.*

4.5.1 Spectral self-duality for SEP, IRW and SIP

In view of the setting described above, we revisit the three interacting particle systems introduced in Section 3.1.4 and further generalized in Section 3.a. More in details, given conductances $\boldsymbol{c} = \{c(\{x,y\}),\ x,y \in V \text{ with } x \sim y\}$ and site parameters $\boldsymbol{\alpha} = \{\alpha_x,\ x \in V\}$, for all $\sigma \in \{-1,0,1\}$, we denote by

$$\begin{aligned} L\varphi(\eta) \;=\; \sum_{x\sim y} c(\{x,y\})\, \big\{\eta(x)\,(\alpha_y + \sigma\eta(y))\,(\varphi(\eta^{x,y}) - \varphi(\eta)) \\ + \eta(y)\,(\alpha_x + \sigma\eta(x))\,(\varphi(\eta^{y,x}) - \varphi(\eta))\big\}\ , \end{aligned} \tag{4.43}$$

where $\varphi : \mathcal{X} \to \mathbb{R}$ is a bounded function, the infinitesimal generator of either SEP($\boldsymbol{\alpha}$), IRW($\boldsymbol{\alpha}$) or SIP($\boldsymbol{\alpha}$) on $(V,\sim)$ – corresponding to either $\sigma = -1, 0$ or 1, respectively. For all $n \in \mathbb{N}_0$, we denote by L_n the generator L in (4.43) restricted to functions $\varphi \in L^2(\mathcal{X}_n)$, i.e. functions defined only on configurations with n particles.

We further note that, while for IRW($\boldsymbol{\alpha}$) and SIP($\boldsymbol{\alpha}$) the space of admissible configurations $\mathcal{X}_n$ is given by

$$\mathcal{X}_n \;=\; \{\eta \in \mathbb{N}_0^V \,:\, |\eta| := \sum_{x\in V} \eta(x) = n\}\,,$$

for SEP($\boldsymbol{\alpha}$) we must further impose that at each site $x \in V$ there can sit at most $\alpha_x \in \mathbb{N}$ particles:

$$\mathcal{X}_n \;=\; \{\eta \in \mathbb{N}_0^V \,:\, |\eta| = n \text{ and } \eta(x) \in \{0,\ldots,\alpha_x\} \text{ for all } x \in V\}\,.$$

As a consequence of the discussion in Sections 3.1.3–3.1.4, we have that, for all $n \in \mathbb{N}_0$ and $\sigma \in \{-1,0,1\}$, the particle systems with infinitesimal generator L_n defined above admit a unique reversible measure – which we denote by μ_n – given:

(a) For the symmetric exclusion process SEP($\boldsymbol{\alpha}$) by

$$\mu_n(\eta) \;=\; \frac{1}{\binom{|\alpha|}{n}} \prod_{x\in V} \binom{\alpha_x}{\eta(x)}\,, \qquad \eta \in \mathcal{X}_n\,. \tag{4.44}$$

(b) For the system of independent random walkers IRW($\boldsymbol{\alpha}$) by

$$\mu_n(\eta) \;=\; \frac{n!}{|\alpha|^n} \prod_{x\in V} \frac{\alpha_x^{\eta(x)}}{\eta(x)!}\,, \qquad \eta \in \mathcal{X}_n\,. \tag{4.45}$$

(c) For the symmetric inclusion process SIP($\boldsymbol{\alpha}$) by

$$\mu_n(\eta) \;=\; \frac{n!\,\Gamma(|\boldsymbol{\alpha}|)}{\Gamma(|\boldsymbol{\alpha}|+n)} \prod_{x\in V} \frac{\Gamma(\alpha_x+\eta(x))}{\eta(x)!\,\Gamma(\alpha_x)}\,, \qquad \eta \in \mathcal{X}_n\,. \tag{4.46}$$

As a consequence, for all three particle system, for all $n \in \mathbb{N}_0$, the generator L_n is a self-adjoint operator in $L^2_{\mu_n}(\mathcal{X}_n)$ and, thus, admits in that Hilbert space a basis of μ_n-orthonormal eigenfunctions.

Symmetries. As part of the Lie algebraic approach to duality for Markov processes (see e.g. [62]), it has been derived that, for all $\sigma \in \{-1,0,1\}$, the following operators

$$\begin{aligned} K^-\varphi(\eta) &= \sum_{x\in V} \eta(x)\,\varphi(\eta-\delta_x) \\ K^+\varphi(\eta) &= \sum_{x\in V} (\alpha_x+\sigma\eta(x))\,\varphi(\eta+\delta_x)\,, \qquad \eta\in\mathbb{N}_0^V\,, \end{aligned} \tag{4.47}$$

are *symmetries* for the generator L as defined in (4.43), i.e., for all $\sigma \in \{-1,0,1\}$,

$$\begin{aligned} K^-L &= LK^- \\ K^+L &= LK^+\,. \end{aligned} \tag{4.48}$$

These operators arise in e.g. [62], [126] from generating elements of discrete representations of suitable co-product Lie algebras – SU(2) for SEP, Heisenberg for IRW and SU(1, 1) for SIP. Within this framework, the Markov generator L adopts the interpretation – up to some “additive constant” – of *central* (=commuting with all elements) element of the corresponding co-product Lie algebra representation (see e.g. [62] for further details). This construction enables, in particular, to derive in an elegant way the commutation relations in (4.48).

Particle removal and addition operators as intertwiners. When restricting the dynamics to configurations with a fixed total number of particles, the above operators K^- and K^+ acquire an interesting probabilistic interpretation. Indeed, on the one side, a suitable normalization gives rise to stochastic operators; on the other side, the commutation relations (4.48) yield intertwining relations between generators of systems with different number of particles.

More precisely, if we introduce, for all $n \in \mathbb{N}$ and $\sigma \in \{-1, 0, 1\}$,

$$\begin{aligned} K_n^- &: L^2(\mathcal{X}_{n-1}) \longrightarrow L^2(\mathcal{X}_n) \\ K_{n-1}^+ &: L^2(\mathcal{X}_n) \longrightarrow L^2(\mathcal{X}_{n-1}) \end{aligned}$$

given by

$$K_n^- \varphi(\eta) = \frac{1}{n} \sum_{x \in V} \eta(x)\, \varphi(\eta - \delta_x)\,, \quad \eta \in \mathcal{X}_n \tag{4.49}$$

$$K_{n-1}^+ \varphi(\eta) = \frac{1}{|\alpha| + \sigma n} \sum_{x \in V} (\alpha_x + \sigma \eta(x))\, \varphi(\eta + \delta_x)\,, \quad \eta \in \mathcal{X}_{n-1}\,, \tag{4.50}$$

then we obtain stochastic operators, i.e., for all $n \in \mathbb{N}_0$, $K_{n+1}^- 1_{\mathcal{X}_n} = K_n^+ 1_{\mathcal{X}_n} = 1_{\mathcal{X}_{n-1}}$, and, as a consequence of (4.48), we get the following intertwining relations: for all $n \in \mathbb{N}$,

$$\begin{aligned} K_n^- L_{n-1} &= L_n K_n^- \\ K_{n-1}^+ L_n &= L_{n-1} K_n^+ \,. \end{aligned} \tag{4.51}$$

In what follows, we call $\{K_n^-,\ n \in \mathbb{N}\}$ (resp. $\{K_n^+,\ n \in \mathbb{N}_0\}$) *particle removal* (resp. *addition*) *operators*.

Proof of spectral self-duality. Here we decide to pursue further the linear algebraic implications – rather than the probabilistic ones, which will be subject of future studies – of the intertwining relations in (4.51). In particular, we ask ourselves whether the operators $\{K_n^-,\ n \in \mathbb{N}\}$ and $\{K_n^+,\ n \in \mathbb{N}\}$ are "full-rank". In particular, we will need that, for all $n \in \mathbb{N}$, either one of K_n^- and K_{n-1}^+ is injective. As a consequence, either eigenfunctions of L_{n-1} are mapped into eigenfunctions of L_n via K_n^- or viceversa via K_{n-1}^+. We summarize our findings in the following theorem, whose proof requires a lemma.

Lemma 4.21 (PARTICLE REMOVAL OPERATORS). (i) *The particle removal operators for* IRW($\boldsymbol{\alpha}$) *and* SIP($\boldsymbol{\alpha}$) *are injective for all* $n \in \mathbb{N}$.

(ii) *The particle removal operators for* SEP($\boldsymbol{\alpha}$) *are injective for all* $n \leq \frac{|\alpha|+1}{2}$.

Proof. For (i), we fix $n \in \mathbb{N}$ and prove that

$$K_n^- \varphi(\eta) \;=\; 0 \;\text{ for all }\; \eta \in \mathcal{X}_n\,, \tag{4.52}$$

implies $\varphi = 0 \in L^2(\mathcal{X}_{n-1})$. Indeed, (4.52) with $\eta = n\delta_x \in \mathcal{X}_n$ yields, by definition (4.49),

$$\varphi((n-1)\delta_x) \;=\; 0\,, \tag{4.53}$$

for all $x \in V$. By removing one out of the n particles at $x \in V$ and placing it at $y \in V$, we obtain $\eta = (n-1)\delta_x + \delta_y \in \mathcal{X}_n$. With this configuration, in view of (4.52), (4.53) and (4.49), we get

$$\varphi((n-2)\delta_x + \delta_y) \;=\; 0\,.$$

Note that this holds for all $x, y \in V$. By iterating this argument for all sites in V, we conclude this part of the proof.

For (ii), we cannot use the argument above because, in general, $n\delta_x \notin \mathcal{X}_n$ may occur for some $x \in V$. Instead, we use the intertwinings with the ladder symmetric exclusion process LSEP($\boldsymbol{\alpha}$) studied in Section 3.b. Indeed, if we denote the "ladder" n-particle configuration space by $\tilde{\mathcal{X}}_n$ (see (3.85)) and define the "ladder" particle removal operator for $n \in \{0, \ldots, |\boldsymbol{\alpha}|\}$ as follows:

$$\mathscr{K}_n^- \tilde{\varphi}(\tilde{\eta}) \;=\; \frac{1}{n} \sum_{x \in V} \sum_{i=1}^{\alpha_x} \tilde{\eta}(x,i)\, \tilde{\varphi}(\tilde{\eta} - \delta_{(x,i)})\,, \quad \tilde{\eta} \in \tilde{\mathcal{X}}_n\,,$$

we obtain the following properties (recall the definitions of intertwiners Λ and $\tilde{\Lambda}$ in (3.86) and (3.88), respectively; here, as usual, Λ_n and $\tilde{\Lambda}_n$ denote restrictions to $L^2(\tilde{\mathcal{X}}_n)$): for all $n \in \{0, \ldots, |\boldsymbol{\alpha}|\}$,

(1) $\tilde{\Lambda}_n \Lambda_n = \mathbf{1}_{\tilde{\mathcal{X}}_n}$.

(2) $\mathscr{K}_n^- \operatorname{Im}(\Lambda_{n-1}) \subset \operatorname{Im}(\Lambda_n)$.

(3) $K_n^- = \tilde{\Lambda}_n \mathscr{K}_n^- \Lambda_{n-1}$.

While, for $n \leq \frac{|\alpha|+1}{2}$,

(4) $\mathscr{K}_n^-$ is injective.

While (1)–(3) follow at once from the definitions, for (4) we first observe that

$$|\tilde{\mathcal{X}}_{n-1}| = \binom{|\alpha|}{n-1} \leq \binom{|\alpha|}{n} = |\tilde{\mathcal{X}}_n|$$

if and only if $n \leq \frac{|\alpha|+1}{2}$. Then, from the observation that the removal of a single particle from any two different configurations, say $\tilde{\eta} \neq \tilde{\xi}$, in $\tilde{\mathcal{X}}_n$ yields (at least) two different configurations, say $\tilde{\eta}' \neq \tilde{\xi}'$, in $\tilde{\mathcal{X}}_{n-1}$, we conclude that the $\binom{|\alpha|}{n}$ equations arising from $\mathscr{K}_n^- \varphi = 0$ are all independent.

As a consequence of (3)–(2), (1), (4) and (1), we obtain the following chain of implications:

$$\begin{aligned} K_n^- \varphi = 0 \quad &\Longleftrightarrow \quad \tilde{\Lambda}_n \mathscr{K}_n^- \Lambda_{n-1} \varphi = 0 \\ &\Longleftrightarrow \quad \mathscr{K}_n^- \Lambda_{n-1} \varphi = 0 \\ &\Longleftrightarrow \quad \Lambda_{n-1} \varphi = 0 \\ &\Longleftrightarrow \quad \varphi = 0\,. \end{aligned}$$

This concluses the proof.

□

Remark 4.22 (PARTICLE ADDITION OPERATORS). *Given Lemma 4.21, we do not need to prove directly any non-degeneracy properties of the particle addition operators* $\{K_n^+,\ n \in \mathbb{N}_0\}$. *Indeed, for all three particle systems* SEP(α), IRW(α) *and* SIP(α), *for all* $n \in \mathbb{N}$, $\varphi \in L^2(\mathcal{X}_n)$ *and* $\phi \in L^2(\mathcal{X}_{n-1})$, *we have*

$$\langle K_{n-1}^+ \varphi, \phi \rangle_{\nu_{n-1}} = \langle \varphi, K_n^- \phi \rangle_{\nu_n}\,.$$

As a consequence, K_{n-1}^+ *is surjective, resp. injective, if and only if* K_n^- *is injective, resp. surjective.*

Theorem 4.23 (SPECTRAL SELF-DUALITY FOR SEP, IRW & SIP). *For all quenched random environments* $(\boldsymbol{c}, \boldsymbol{\alpha})$, SEP($\boldsymbol{\alpha}$), IRW($\boldsymbol{\alpha}$) *and* SIP($\boldsymbol{\alpha}$) *are spectrally self-dual conservative particle systems. More specifically:*

(i) *For* IRW($\boldsymbol{\alpha}$) *and* SIP($\boldsymbol{\alpha}$), *we have*

$$\Sigma(L_{n-1}) \subset \Sigma(L_n)\,,$$

for all $n \in \mathbb{N}$. In particular, K_n^- maps eigenfunctions of L_{n-1} into eigenfunctions of L_n, for all $n \in \mathbb{N}$.

(ii) *For* SEP($\boldsymbol{\alpha}$)*, we have $\Sigma(L_n) = \Sigma(L_{|\alpha|-n})$ and, moreover,*

$$\Sigma(L_{n-1}) \subset \Sigma(L_n)$$

for all $n \leq \frac{|\alpha|+1}{2}$. As a consequence,

$$\Sigma(L_{n-1}) \supset \Sigma(L_n)$$

for all $\frac{|\alpha|+1}{2} < n \leq |\alpha|$. In particular, if $n \leq \frac{|\alpha|+1}{2}$, eigenfunctions of L_{n-1} are mapped into eigenfunctions of L_n via K_n^-, while, if $\frac{|\alpha|+1}{2} < n \leq |\alpha|$, the converse holds via K_{n-1}^+.

Proof. By Lemma 4.21 and the intertwining relations with K_n^- in (4.51), we get (i) for all $n \in \mathbb{N}$ and the first part of (ii) for $n \leq \frac{|\alpha|+1}{2}$.

For the second part of (ii), we use the well-known particle-hole symmetry for the exclusion process (see e.g. [34]) to obtain that $\Sigma(L_n) = \Sigma(L_{|\alpha|-n})$ for all $n \in \{0, \dots, |\alpha|\}$. □

We conclude this section with the following proposition which, roughly speaking, states that the so-called "classical" jointly factorized self-duality functions for SEP, IRW and SIP (see e.g. Section 3.1.4) are "full-rank" because obtained from the composition of "full-rank" intertwinings and "full-rank" cheap self-duality functions.

Proposition 4.24. *For all $n > m$ and for all $\xi \in \mathcal{X}_m$ and $\eta \in \mathcal{X}_n$, we have*

$$(K_n^- \dots K_{m+1}^- \mathbf{1}_{\{\cdot=\xi\}})(\eta) = \frac{1}{\binom{n}{m}} \prod_{x \in V} \frac{\eta(x)!}{\xi(x)!(\eta(x)-\xi(x))!} \mathbf{1}_{\{\xi(x) \leq \eta(x)\}} .$$

As a consequence of Lemma 4.21 and reversibility of SEP, SIP *and* IRW *w.r.t. the "canonical" measures $\{\mu_n,\ n \in \mathbb{N}\}$ on $\mathcal{X}_n$ given in* (4.44)–(4.45)–(4.46)*, the "classical" jointly factorized self-duality functions*

$$D_{m,n}(\xi, \eta) = \frac{1}{\binom{n}{m}} \frac{1}{\mu_m(\xi)} \prod_{x \in V} \frac{\eta(x)!}{\xi(x)!(\eta(x)-\xi(x))!} \mathbf{1}_{\{\xi(x) \leq \eta(x)\}} ,$$

namely

$$D_{m,n}(\xi,\eta) \;=\; \frac{|\boldsymbol{\alpha}|!\,(n-m)!}{(|\boldsymbol{\alpha}|-m)!\,n!}\prod_{x\in V}\frac{(\alpha_x-\xi(x))!}{\alpha_x!}\,\frac{\eta(x)!}{(\eta(x)-\xi(x))!}\,\mathbf{1}_{\{\xi(x)\le\eta(x)\}}\,,$$

for SEP($\boldsymbol{\alpha}$),

$$D_{m,n}(\xi,\eta) \;=\; \frac{(n-m)!\,|\boldsymbol{\alpha}|^m}{n!}\prod_{x\in V}\frac{1}{\alpha_x^{\xi(x)}}\,\frac{\eta(x)!}{(\eta(x)-\xi(x))!}\,\mathbf{1}_{\{\xi(x)\le\eta(x)\}}\,,$$

for IRW($\boldsymbol{\alpha}$) *and*

$$D_{m,n}(\xi,\eta) \\ \;=\; \frac{\Gamma(|\boldsymbol{\alpha}|+m)\,(n-m)!}{\Gamma(|\boldsymbol{\alpha}|)\,n!}\prod_{x\in V}\frac{\Gamma(\alpha_x)}{\Gamma(\alpha_x+\xi(x))}\,\frac{\eta(x)!}{(\eta(x)-\xi(x))!}\,\mathbf{1}_{\{\xi(x)\le\eta(x)\}}\,,$$

for SIP($\boldsymbol{\alpha}$), *obtained from*

$$D_{m,n} \;=\; (K_n^-\ldots K_{m+1}^-)_{\text{left}}D_{\text{cheap},m}(\xi,\eta)\,,$$

are "full-rank".

4.5.2 Spectral self-duality on two sites: examples

In this section, we take under consideration some examples of conservative particle systems hopping on a set consisting of only two sites, say $V=\{x,y\}$. This simplified underlying geometrical structure allows us to explicitly compute the spectrum of the Markov generators and, in turn, to draw some interesting conclusions regarding the (im)possibility to find informative self-duality functions for the same systems on more complex geometries.

SEP, IRW **and** SIP. We revisit SEP($\boldsymbol{\alpha}$), IRW($\boldsymbol{\alpha}$) and SIP($\boldsymbol{\alpha}$) on two sites $V=\{x,y\}$. From the previous section, we know that, for all $n\in\mathbb{N}$ for IRW($\boldsymbol{\alpha}$) and SIP($\boldsymbol{\alpha}$) and for all $n\le\frac{|\boldsymbol{\alpha}|+1}{2}$ for SEP($\boldsymbol{\alpha}$), $\Sigma(L_{n-1})\subset\Sigma(L_n)$ and that, in the specific instance of two-site systems which we are here considering,

$$|\Sigma(L_n)\setminus\Sigma(L_{n-1})| \;=\; 1\,,$$

i.e. the spectrum (with multiplicities) $\Sigma(L_n)$ contains only one more eigenvalue than $\Sigma(L_{n-1})$. Our goal is to reconstruct the spectra $\{\Sigma(L_n),\ n \in \mathbb{N}_0\}$ by finding this "extra" eigenvalue.

To this purpose, we use the particle addition operators $\{K^+_{n-1},\ n \in \mathbb{N}\}$. Indeed, by Lemma 4.21 and Remark 4.22, we get

$$\dim(\mathrm{Ker}(K^+_{n-1})) = 1 ,$$

which yields, as a consequence of the intertwining relations (4.51), that there exists a function $0 \neq \psi_n \in L^2(\mathcal{X}_n)$ and a value $\lambda_n \in (-\infty, 0]$ such that

$$K^+_{n-1}\psi_n = 0 \quad \text{and} \quad L_n\psi_n = \lambda_n\psi_n . \tag{4.54}$$

This $\lambda_n \in (-\infty, 0]$ is the "extra" eigenvalue for which

$$\Sigma(L_{n-1}) \cup \{\lambda_n\} = \Sigma(L_n) .$$

From the definition of K^+_{n-1} in (4.50) and by solving (4.54), we get:

(a) For SEP($\boldsymbol{\alpha}$), for all $n \leq \frac{|\boldsymbol{\alpha}|+1}{2}$,

$$\psi_n(\eta^{x,y}) = -\tfrac{\alpha_x-(\eta(x)-1)}{\alpha_y-\eta(y)}\,\psi_n(\eta) \quad \text{and} \quad \lambda_n = -n\,(|\boldsymbol{\alpha}| - (n-1)) .$$

(b) For IRW($\boldsymbol{\alpha}$), for all $n \in \mathbb{N}_0$,

$$\psi_n(\eta^{x,y}) = -\tfrac{\alpha_x}{\alpha_y}\,\psi_n(\eta) \quad \text{and} \quad \lambda_n = -n|\boldsymbol{\alpha}| .$$

(c) For SIP($\boldsymbol{\alpha}$), for all $n \in \mathbb{N}_0$,

$$\psi_n(\eta^{x,y}) = -\tfrac{\alpha_x+(\eta(x)-1)}{\alpha_y+\eta(y)}\,\psi_n(\eta) \quad \text{and} \quad \lambda_n = -n\,(|\boldsymbol{\alpha}| + (n-1)) .$$

We remark that the functions ψ_n above are proper eigenfunctions, i.e. they may be chosen such that $\psi_n \neq 0$. Moreover, we observe that the eigenvalues of SEP($\boldsymbol{\alpha}$), IRW($\boldsymbol{\alpha}$) and SIP($\boldsymbol{\alpha}$) found above, namely

$$\Sigma(L_n) = \{\lambda_0, \ldots, \lambda_n\} , \quad \lambda_k = -k\,(|\boldsymbol{\alpha}| + \sigma(k-1))$$

for $\sigma = -1, 0$ and 1, respectively, satisfy the following:

$$\lambda_1 = -|\boldsymbol{\alpha}| \geq \lambda_k ,$$

for all $k \in \mathbb{N}_0$ if $\sigma \in \{0,1\}$ and for all $k \in \{1,\dots,n\}$ if $\sigma = -1$. In particular, for all three systems, the spectral gap λ_1 associated to an n-particle system coincides with the spectral gap of a one-particle system.

While this fact is known to hold for any finite set of sites $(V,\sim)$ and quenched random environment $(\boldsymbol{c},\boldsymbol{\alpha})$ both for IRW($\boldsymbol{\alpha}$) due to the independence of the walkers, see e.g. [96], and SEP($\boldsymbol{\alpha}$) [17], whether it holds for SIP($\boldsymbol{\alpha}$) on more general geometries is nowadays an open problem. These considerations lead us to the following conjecture.

Conjecture 4.1 (ALDOUS'S SPECTRAL GAP CONJECTURE FOR SIP)**.** *Given any finite set of sites $(V,\sim)$ and quenched random environment $(\boldsymbol{c},\boldsymbol{\alpha})$, the spectral gap of the symmetric inclusion process* SIP($\boldsymbol{\alpha}$) *and the one of the symmetric random walk coincide.*

Zero-range process with constant rate. We consider the zero-range process with constant rate, see e.g. [32, p. 147], [50], [51], whose configuration space coincides with $\mathbb{N}_0^V$ and infinitesimal generator is given by

$$L\varphi(\eta) \;=\; \mathbf{1}_{\{\eta(x)\geq 1\}}\left(\varphi(\eta^{x,y})-\varphi(\eta)\right) \;+\; \mathbf{1}_{\{\eta(y)\geq 1\}}\left(\varphi(\eta^{y,x})-\varphi(\eta)\right),$$

i.e. corresponding to the following choice for the interaction functions:

$$g_x(n) \;=\; g_y(n) \;=\; \mathbf{1}_{\{n\geq 0\}} \quad \text{and} \quad h_x(n) \;=\; h_y(n) \;=\; 1\,, \quad n\in\mathbb{N}_0\,.$$

While the same process when considered on $(V,\sim)=\mathbb{Z}$ (with nearest-neighbor interactions and unit conductances $c(\{x,x+1\})=1$) is known to be isomorphic to a symmetric exclusion process on $\mathbb{Z}$ with a marked particle [32], on two sites $V=\{x,y\}$ the process with $n\in\mathbb{N}$ particles is isomorphic to a symmetric simple random walk on $\{0,\dots,n\}$ which is "blocked" at the boundaries $\{0,n\}$, i.e. whose generator acting on functions $f:\{0,\dots,n\}\to\mathbb{R}$ is given by

$$A_nf(x) \;=\; \left(f(x+1)+f(x-1)-2f(x)\right), \quad x\in\{1,\dots,n-1\}\,,$$

and

$$A_nf(0) \;=\; \left(f(1)-f(0)\right), \quad A_nf(n) \;=\; \left(f(n-1)-f(n)\right).$$

We note that we have already encountered such a random walk in Example 4.18 and computed the spectrum of the generator (there the infinitesimal generator has been denoted by $\widehat{L}$). Hence, from (4.39), we obtain, for all $n \in \mathbb{N}$,

$$\Sigma(L_n) \;=\; \Sigma(A_n) \;=\; \{\lambda_0^{(n)}, \ldots, \lambda_n^{(n)}\}\,,$$

with

$$\lambda_i^{(n)} \;=\; 2(\cos(\theta_i^{(n)}) - 1)\,, \quad \theta_i^{(n)} \;=\; \frac{i}{n+1}\pi\,, \quad i = 0, \ldots, n\,.$$

We remark that $|\Sigma(L_{n-1})| < |\Sigma(L_n)|$ for all $n \in \mathbb{N}$. Moreover, for all $n, m \in \mathbb{N}_0$ for which $\frac{m}{n} \notin \mathbb{N}$, there exists $i \in \{1, \ldots, n\}$ such that $\lambda_i^{(n)} = 2(\cos(\theta_i^{(n)} - 1) \notin \Sigma(A_m)$. As a consequence of these two facts, it follows that there exist $n, m \in \mathbb{N}_0$ for which neither of

$$\Sigma(L_n) \subset \Sigma(L_m) \quad \text{and} \quad \Sigma(L_n) \supset \Sigma(L_m)$$

holds true, i.e. the zero-range process with constant rate on two sites is *not* spectrally self-dual.

Remark 4.25 (ZERO-RANGE PROCESS WITH CONSTANT RATE ON N SITES)**.** *In view of the isomorphism between an n-particle zero-range process on* $V = \{1, \ldots, N\}$*, whose generator is given by*

$$\begin{aligned} L_n\varphi(\eta) \;&=\; \sum_{x=1}^{N-1} L_{\{x,x+1\}}\varphi(\eta) \\ &=\; \sum_{x=1}^{N-1} \mathbf{1}_{\{\eta(x)\geq 1\}}\,(\varphi(\eta^{x,x+1}) - \varphi(\eta)) \;+\; \mathbf{1}_{\{\eta(x+1)\geq 1\}}\,(\varphi(\eta^{x+1,x}) - \varphi(\eta))\,, \end{aligned}$$

and a symmetric simple exclusion process with $N-1$ *particles on* $\widehat{V} = \{0, \ldots, N + n - 1\}$ *– both with unit nearest-neighbor interactions – we conjecture that, even in this case, spectral self-duality does not hold.*

General zero-range process. We address the question whether spectral self-duality may hold for more general zero-range processes. We provide an answer by comparing only systems with one and two particles. Indeed, by relating the spectrum of two-by-two and three-by-three matrices (corresponding to the generators L_1 and L_2, respectively), we obtain some necessary conditions for spectral self-duality. As a consequence, we obtain that the class of spectrally self-dual symmetric zero-range processes on two sites is rather limited.

More precisely, if we introduce, for all $n \in \mathbb{N}$, the n-particle infinitesimal generator

$$L_n\varphi(\eta) \;=\; g_x(\eta(x))\,\alpha_y\,(\varphi(\eta^{x,y})-\varphi(\eta)) + g_y(\eta(y))\,\alpha_x\,(\varphi(\eta^{y,x})-\varphi(\eta))\,, \quad \eta \in \mathcal{X}_n\,,$$

where $\alpha_x, \alpha_y > 0$ and $g_x(n), g_y(n) > 0$ for all $n \in \mathbb{N}$, we obtain that

$$\Sigma(L_1) \;\subset\; \Sigma(L_2)$$

if and only if

$$g_x(2)\,g_y(2) \;=\; g_x(2)\,g_y(1) \;+\; g_x(1)\,g_y(2)\,. \tag{4.55}$$

In particular, (4.55) yields

$$g_x(2) \;=\; \gamma_x\,g_x(1) \quad \text{and} \quad g_y(2) \;=\; \tfrac{\gamma_x}{\gamma_x-1}\,g_y(1)\,,$$

for some $\gamma_x > 1$.

If, additionally, we restrict our analysis to *homogeneous* zero-range processes, i.e. $g_x = g_y = g$, we get

$$g(2) \;=\; 2\,g(1)\,,$$

corresponding to a zero-range process in which one and two particle systems hop *independently* on two sites $\{x, y\}$ with attraction parameters $\{\alpha_x, \alpha_y\}$, respectively.

Generalized immediate exchange models and their symmetries

The immediate exchange model and its discrete counterpart. The *immediate exchange model* (IEM) is a model of wealth distribution, introduced in [73], further studied in [83], generalized and investigated from the viewpoint of processes with duality in [65].

In words, it is a model in which two agents at random exponential event times each split their wealth – a non-negative real quantity – into two parts, uniformly. Then, they exchange the "top parts" and add the two parts again to obtain their updated wealth. The model conserves the total wealth and is reminiscent of models of statistical mechanics such as the KMP model [86] and its generalizations [20]. Moreover, it has reversible product measures of type $\text{Gamma}(2, \lambda)$.

The authors in [65] showed that, if the splitting is done according to a $\text{Beta}(\alpha, \beta)$ distribution in place of a uniform splitting, then the model has reversible product measures of type $\text{Gamma}(\alpha + \beta, \lambda)$. This was established by using a duality with a discrete model of the same type, where discrete mass is redistributed in an analogous way and the splitting procedure is using a Beta Binomial distribution. There, it is proved that this discrete model – which we call *discrete immediate exchange model* (IEM_d) – is self-dual and has reversible product measures with $\text{Gamma}_\text{d}(\alpha + \beta, \lambda)$ – discrete analogous of the Gamma distribution – as marginals. Moreover, by considering a many-particle limit of this discrete model, one recovers the original continuum model, as well as the duality between these two models.

Wealth redistribution as splitting, exchange and addition. In this chapter, we give a new perspective on the IEM, by viewing the *splitting part* of the dynamics as a "thermalization" of suitable conservative Markov processes, the *exchange part* as a permutation of indexes and the *addition part* as a lumping of Markov processes.

In particular, we carry out this analysis in details for the discrete immediate exchange model $\mathrm{IEM_d}$, for which the conservative Markov process that undergoes this thermalization procedure happens to be the symmetric inclusion process – and, more precisely, SIP(1). This immediately leads to symmetries of the splitting part. We then show that these symmetries are permutation invariant, and therefore also commute with the exchange part of the dynamics. Remarkably, these symmetries can be pulled through also the addition part because symmetries of SIP(1) have a natural additive structure in the parameter labeling their representation.

Symmetries and (self-)dualities. This connection with an underlying interacting particle system such as SIP(1) allows us to recover in a much more elegant way the full SU(1, 1) symmetry of the $\mathrm{IEM_d}$ model, where the parameter of the discrete representation is $\alpha = 2 = 1 + 1$, arising as the addition of the parameters of the representations of the underlying symmetric inclusion process with parameter $\alpha = 1$. This picture can then be immediately transferred to inhomogeneous variants of the discrete immediate exchange model and opens many possibilities of further generalizations to other splitting mechanisms based on different thermalizations (e.g. SEP instead of SIP corresponding to "maximal wealth" restrictions).

This self-duality property is of great use if one wants to analyze the multi-agent model because the time dependent expectation of a multivariate polynomial of degree k in the wealth of the different agents will be linked to the evolution of the total wealth of at most k "dual units" – which is, of course, much simpler: e.g. the expected wealth of one agent can simply be understood from the initial condition and a single continuous-time random walk moving on the set of agents. Moreover, self-duality allows a quite complete characterization of the invariant measures of infinite systems (e.g. the continuous IEM), using properties – so-called existence of a successful coupling – of the finite system (e.g. the discrete $\mathrm{IEM_d}$) only (see also Section 3.2 in Chapter 3 for results of this sort). Finally, taking a many-particle limit where the wealth of agent x scales as $\lfloor N\zeta(x)\rfloor$, with $N \to \infty$ (see also e.g. Section 3.1.6), one recovers from self-duality of the discrete models duality between the discrete and the continuous immediate exchange models.

Organization of the rest of the chapter. The rest of the chapter is organized as follows. In Section 5.1 we start by giving a new perspective on the $\mathrm{IEM_d}$, by viewing its dynamics as a composition of splitting, exchange and addition. In Sections 5.1.1–5.1.2, we generalize our definition of discrete immediate exchange model covering also inhomogeneous models and connect the splitting mechanism of these models to a thermalization procedure of the symmetric inclusion process. In Section 5.1.3, in view of this connection, we introduce a class of $SU(1,1)$ symmetries for the splitting part and study its relation with the other parts of the dynamics, namely the exchange and the addition mechanisms. Lastly, in Section 5.1.4, we derive symmetries for the discrete immediate exchange model and provide a class of jointly factorized self-duality functions for these models. We remark that we postpone to Section 5.3 the proof of reversibility of the immediate exchange models. Indeed, there, in a more general context, we show how to recover reversible measures of models consisting of splitting, exchange and addition mechanisms from reversible measures of the splitting part only. For the inhomogeneous $\mathrm{IEM_d}$ considered, reversible measures for the splitting mechanisms coincide with those of the SIP used in the thermalization procedure. In Section 5.2 we present further generalizations of immediate exchange models in which the splitting mechanism is obtained from thermalization of different underlying systems, yielding exchange models of a different nature. More specifically, in Sections 5.2.1 and 5.2.2 we study discrete immediate exchange models based on SEP, resp. IRW, thermalization, while in Section 5.2.3 we study and revisit continuum inhomogeneous IEMs, either splitting their wealth deterministically or according to a fraction Beta distributed. For all these models we recover their full symmetry structures and prove self-duality as well as duality relations between discrete and continuum IEMs. As we will only partially use the abstract Lie algebraic framework linked to interacting particle systems, for the reader interested in further details on the underlying algebras, their representations and their connection with Markov processes, symmetries and (self-)duality, we refer to e.g. [19] and [62].

Remark 5.1 (FROM TWO-AGENT TO MANY-AGENT MODELS). *In this chapter we restrict to models with two agents. However, all the results – symmetries and self-dualities – straightforwardly generalize to a many-agent model, where the network of agents is described by a finite set V equipped with a nearest-neighboring relation "$\sim$", each agent being identified with a site $x \in V$. Then, independently and at* *exponential times of rate* $c(\{x,y\}) > 0$, *the nearest-neighboring agents* x *and* y *update their wealth according to the redistribution rule, which depends only on the wealth of the two agents. The jointly factorized form of the (self-)duality functions allows these extensions.*

We further mention that similar situations have been addressed e.g. in Remark 3.1 in Chapter 3 as well as in [65].

5.1 A new perspective on the discrete immediate exchange model

We start by reconsidering the dual immediate exchange model ($\text{IEM}_\text{d}((1,1),(1,1))$ or, shortly, IEM_d), introduced in [65] and which is also a natural discrete analogue of the continuous immediate exchange model (IEM) (see e.g. [73]). Here we have two agents, which we call "agent x" and "agent y", with initial wealths $n_x, n_y \in \mathbb{N}_0$, where $\mathbb{N}_0 = \{0,1,2,\ldots\}$ denotes the set of non-negative integers. In what follows we will denote by $\mathbb{V}_m$ the vector space of functions $\varphi : \mathbb{N}_0^m \to \mathbb{R}$.

Our model is then described as follows: at discrete (or, alternatively, at the event times of a mean-one Poisson process), the wealth is updated according to the following *splitting*, *exchange* and *addition* mechanisms.

(a) *Splitting*. In the first step, the wealth of both agents is split according to $n_x \mapsto (k_x, n_x - k_x)$ and $n_y \mapsto (k_y, n_y - k_y)$, where k_x (resp. k_y) are independent discrete uniform on $\{0,\ldots,n_x\}$ (resp. $\{0,\ldots,n_y\}$). After this splitting we call k_x (resp. k_y) the "top part" of the wealth of agent x (resp. agent y); the remaining ones, i.e. $n_x - k_x$ and $n_y - k_y$, are called the "bottom parts".

Let us denote by $\mathcal{W}_{n_x,n_y}^{(1,1),(1,1)}$ the $\mathbb{N}_0^4$-valued random variable with distribution $(k_x, n_x - k_x, k_y, n_y - k_y)$ just described. Note that here the upper index "$(1,1),(1,1)$" refers to the choice of discrete uniforms for both k_x and k_y. This will be generalized later, where the distribution of k_x can be chosen to be Beta Binomial with parameters $(n_x, (\alpha, \alpha')$. We remark that $\alpha = \alpha' = 1$ corresponds to the present uniform choice.

The splitting part of the dynamics can then simply be seen as the update from (n_x, n_y) to the four component random variable $\mathcal{W}_{n_x,n_y}^{(1,1),(1,1)}$.

(b) *Exchange*. In the second step, the top parts of both agents are exchanged, i.e. $(k_x, n_x - k_x, k_y, n_y - k_y)$ goes to $(k_y, n_x - k_x, k_x, n_y - k_y)$. This corresponds to the action of what we call the *exchange map*:

$$\mathscr{E} \;:\; \mathbb{N}_0^4 \longrightarrow \mathbb{N}_0^4 : \; (k_x, \ell_x, k_y, \ell_y) \longmapsto (k_y, \ell_x, k_x, \ell_y)\,,$$

to which we associate a corresponding so-called *exchange operator* on functions $\varphi \in \mathbb{V}_4$:

$$\mathscr{E}(\varphi) \;:=\; \varphi \circ \mathscr{E}. \tag{5.1}$$

(c) *Addition*. At last, both parts of the wealth of each agent are added again, i.e. the final new wealths of both agents are

$$(k_y + n_x - k_x, k_x + n_y - k_y)\,.$$

This corresponds to the surjective map

$$\gamma \;:\; \mathbb{N}_0^4 \longrightarrow \mathbb{N}_0^2 : \; (k_x, \ell_x, k_y, \ell_y) \longmapsto (k_x + \ell_x, k_y + \ell_y)$$

and its corresponding *addition operator* $\Gamma : \mathbb{V}_2 \to \mathbb{V}_4$ mapping functions from two variables to functions of four variables as follows:

$$\Gamma\varphi \;:=\; \varphi \circ \gamma\,. \tag{5.2}$$

More explicitly, for all $\varphi \in \mathbb{V}_2$, we define $\Gamma\varphi \in \mathbb{V}_4$ as follows:

$$\Gamma\varphi(k_x, \ell_x, k_y, \ell_y) \;:=\; \varphi(k_x + \ell_x, k_y + \ell_y)\,.$$

We note that if a function $\psi \in \mathbb{V}_4$ of four variables is in the image of Γ, i.e. it is of the form $\Gamma\varphi$ with $\varphi \in \mathbb{V}_2$, then on that function we can of course define Γ^{-1} via $\Gamma^{-1}\psi := \varphi$ with $\Gamma^{-1}\Gamma = \mathbf{1}_{\mathbb{V}_2}$, the identity on $\mathbb{V}_2$. The extension of Γ^{-1} to the whole $\mathbb{V}_4$ is not unique. Hence, when we will want to define Γ^{-1} on $\mathbb{V}_4$ in Definition 5.8 below, our choice will be in accordance with the redistribution of mass operator P (see (5.4) below).

With the notation introduced so far, we can describe one update in the $\mathrm{IEM_d}((1,1),(1,1))$ model as replacing the initial wealth distribution of the two agents – $(n_x, n_y) \in \mathbb{N}_0^2$, say – by $\gamma(\mathscr{E}(\mathcal{W}_{n_x,n_y}^{(1,1),(1,1)}))$. We can then write the *transition operator* of the discrete immediate exchange model Π on $\mathbb{V}_2$ as given by

$$\Pi\varphi(n_x, n_y) \;:=\; \mathbb{E}\left[\varphi(\gamma(\mathscr{E}(\mathcal{W}_{n_x,n_y}^{(1,1),(1,1)})))\right]. \tag{5.3}$$

Furthermore, we introduce what we call the *splitting operator* P acting on functions $\varphi \in \mathbb{V}_4$ as follows:

$$P\varphi(k_x, \ell_x, k_y, \ell_y) \;=\; \sum_{k_x=0}^{n_x} \sum_{k_y=0}^{n_y} \frac{1}{n_x+1} \frac{1}{n_y+1}\, \varphi(k_x, n_x - k_x, k_y, n_y - k_y)\;, \tag{5.4}$$

with $n_x = k_x + \ell_x$, $n_y = k_y + \ell_y$. Notice that $P : \mathbb{V}_4 \to \mathbb{V}_4$ maps a function $\Gamma\phi$ with $\phi \in \mathbb{V}_2$ into a function of the form $\Gamma\psi$, for some $\psi \in \mathbb{V}_2$. Indeed, it is clear that the r.h.s. of (5.4) only depends on $\gamma(k_x, \ell_x, k_y, \ell_y) = (n_x, n_y)$, provided $\varphi(k_x, n_x - k_x, k_y, n_y - k_y) = \phi(n_x, n_y)$ for some $\phi \in \mathbb{V}_2$. Moreover, via the operators P, $\mathscr{E}$ and Γ, an equivalent form for the transition operator Π in (5.3) is deduced:

$$\Pi\varphi \;=\; \Gamma^{-1}(P(\varphi \circ \gamma \circ \mathscr{E})) \;=\; \Gamma^{-1} P \mathscr{E} \Gamma \varphi\;. \tag{5.5}$$

In what follows, we will see that we can view P as a thermalization of two SIP(1) processes and, as a consequence, the operator Π will have symmetries (=commuting operators) arising from the "addition" (or "lumping", see Section 5.1.3) of the symmetries of these two SIP(1) generators. These "added" symmetries will correspond to the symmetries of a SIP(2) process.

5.1.1 Splitting mechanism as thermalization of SIP

In order to find relevant symmetries of Π in (5.3), it is now useful to understand the connection between the splitting operator P and the symmetric inclusion process, via thermalization (see e.g. [20] and [62] for more details on the notion of "thermalization" in the context of models of heat conduction). The idea is to view the splitting of the wealth of each agent as "running a SIP(1) process for infinite time" (= thermalization of SIP(1)) as we will now explain.

We recall here that SIP(1) on two sites is the Markov jump process that performs jumps from state $(k, \ell) \in \mathbb{N}_0^2$ towards $(k-1, \ell+1) \in \mathbb{N}_0^2$ at rate $k(1+\ell)$ and towards $(k+1, \ell-1) \in \mathbb{N}_0^2$ at rate $\ell(1+k)$; i.e. the process on $\mathbb{N}_0^2$ with generator

$$\begin{aligned} L^{\mathrm{SIP}(1)}\varphi(k, \ell) \;&=\; k\,(1+\ell)\,(\varphi(k-1, \ell+1) - \varphi(k, \ell)) \\ &\quad+\; \ell\,(1+k)\,(\varphi(k+1, \ell-1) - \varphi(k, \ell))\,, \quad (k, l) \in \mathbb{N}_0^2\,. \end{aligned} \tag{5.6}$$

As a consequence of the fact that stationary measures of SIP(1) are products of geometric distributions with equal success probability $\lambda \in (0,1)$, i.e.

$$\mu_\lambda(k,\ell) \;=\; \nu_\lambda(k)\,\nu_\lambda(\ell) \;=\; \lambda^{k+\ell}\,(1-\lambda)^2\,, \quad (k,l) \in \mathbb{N}_0^2\,,$$

when started from an initial state (k,ℓ), due to conservation of particles, SIP(1) converges in the course of time to $(k', k+\ell-k')$ with k' uniformly distributed on $\{0,\ldots,k+\ell\}$.

In our context, we may view agent x as starting with a total wealth equal to $n_x \in \mathbb{N}_0$, placing an arbitrary fraction of its wealth in its top "pocket", running from that configuration the SIP(1) between its top and bottom pockets for "infinite time", yielding the splitting part described by the slitting operator P in (5.4). In that way, the splitting is done according to a uniform distribution which depends only on the total wealth of the agent, not on details of how the wealth had been previously organized between the top and bottom pockets.

By performing this operation for both agents x and y *independently*, we obtain the following rewriting of the splitting operator P given in (5.4):

$$\begin{aligned} &P\varphi(k_x,\ell_x,k_y,\ell_y) \\ &\qquad = \lim_{t\to\infty} \left(\mathbb{E}^{\mathrm{SIP}(1)}_{(k_x,\ell_x)} \otimes \mathbb{E}^{\mathrm{SIP}(1)}_{(k_y,\ell_y)}\right) \left[\varphi(k_x(t),\ell_x(t),k_y(t),\ell_y(t))\right]\,. \qquad (5.7)\end{aligned}$$

5.1.2 Inhomogeneous $\mathrm{IEM_d}$

In what follows, we denote by $x_\mathfrak{t}$ (resp. $x_\mathfrak{b}$) the "top" (resp. "bottom") pocket of agent x. We use an analogous notation for agent y.

Building up on the connection between the splitting mechanism described by the operator in (5.4) and SIP(1) via a thermalization procedure as in (5.7), a first natural generalization is to consider inhomogeneous immediate exchange models as arising from thermalizations of inhomogeneous SIP($\boldsymbol{\alpha}$), i.e., for some $\boldsymbol{\alpha}_x = (\alpha_{x_\mathfrak{t}}, \alpha_{x_\mathfrak{b}})$ and $\boldsymbol{\alpha}_y = (\alpha_{y_\mathfrak{t}}, \alpha_{y_\mathfrak{b}}) \subset (0,\infty)^2$,

$$\begin{aligned} &P\varphi(k_x,\ell_x,k_y,\ell_y) \\ &\qquad = \lim_{t\to\infty} \left(\mathbb{E}^{\mathrm{SIP}(\boldsymbol{\alpha}_x)}_{(k_x,\ell_x)} \otimes \mathbb{E}^{\mathrm{SIP}(\boldsymbol{\alpha}_y)}_{(k_y,\ell_y)}\right) \left[\varphi(k_x(t),\ell_x(t),k_y(t),\ell_y(t))\right]\,. \qquad (5.8)\end{aligned}$$

We recall from Section 3.a the definition of SIP($\boldsymbol{\alpha}_x$) as the process on $\mathbb{N}_0^2$ with generator

$$\begin{aligned} L^{\mathrm{SIP}(\alpha_x)}\varphi(k,\ell) &= k\,(\alpha_{x_\mathfrak{b}}+\ell)\,(\varphi(k-1,\ell+1)-\varphi(k,\ell)) \\ &+\ \ell\,(\alpha_{x_\mathfrak{t}}+k)\,(\varphi(k+1,\ell-1)-\varphi(k,\ell))\,, \quad (k,\ell)\in\mathbb{N}_0^2\,, \end{aligned}$$

and reversible product measures given in terms of "inhomogeneous" products of discrete Gamma distributions with equal scale parameter $\lambda \in (0,1)$, namely

$$\begin{aligned} \mu_\lambda(k,\ell) &= \nu_{x,\lambda}(k)\,\nu_{y,\lambda}(\ell) \\ &= \frac{\Gamma(\alpha_{x_\mathfrak{t}}+k)}{\Gamma(\alpha_{x_\mathfrak{t}})}\,\frac{\Gamma(\alpha_{x_\mathfrak{b}}+\ell)}{\Gamma(\alpha_{x_\mathfrak{b}})}\,\frac{\lambda^{k+\ell}}{k!\,\ell!}\,(1-\lambda)^{\alpha_{x_\mathfrak{t}}+\alpha_{x_\mathfrak{b}}}\,, \quad (k,\ell)\in\mathbb{N}_0^2\,. \end{aligned}$$

Hence, the splitting mechanism produced by $\mathrm{SIP}(\boldsymbol{\alpha}_x)$ is governed by an inhomogeneous $\mathrm{BetaBinomial}(n_x,(\alpha_{x_\mathfrak{t}},\alpha_{x_\mathfrak{b}}))$ if agent x starts from a wealth equal to $n_x \in \mathbb{N}_0$. Indeed, given two independent random variables X and Y distributed, respectively, as $\mathrm{Gamma}_\mathrm{d}(\alpha,\lambda)$ and $\mathrm{Gamma}_\mathrm{d}(\alpha',\lambda)$, the first random variable has distribution $\mathrm{BetaBinomial}(n,(\alpha,\alpha'))$ conditioned on $X+Y=n\in\mathbb{N}_0$.

If we compose this more general splitting mechanism with the exchange and addition mechanisms described at the beginning of Section 5.1, we obtain an inhomogeneous version of the discrete immediate exchange model IEM_d, which we denote by $\mathrm{IEM}_\mathrm{d}(\boldsymbol{\alpha})$ if $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_x,\boldsymbol{\alpha}_y)$ and whose transition operator Π reads

$$\Pi^{\boldsymbol{\alpha}} = \mathit{\Gamma}^{-1}P^{\boldsymbol{\alpha}}\mathscr{E}\mathit{\Gamma}\,, \tag{5.9}$$

where the splitting operator reads

$$\begin{aligned} &P^{\boldsymbol{\alpha}}\varphi(k_x,\ell_x,k_y,\ell_y) \\ &\quad= \sum_{k_x=0}^{n_x}\sum_{k_y=0}^{n_y} w_{n_x,\boldsymbol{\alpha}_x}(k_x)\,w_{n_y,\boldsymbol{\alpha}_y}(k_y)\,\varphi(k_x,n_x-k_x,k_y,n_y-k_y)\,, \end{aligned} \tag{5.10}$$

with $n_x = k_x+\ell_x$ and $n_y = k_y+\ell_y$ and

$$w_{n,(\alpha,\alpha')}(k) = \frac{\Gamma(\alpha+k_x)}{\Gamma(\alpha)\,k!}\,\frac{\Gamma(\alpha'+n-k)}{\Gamma(\alpha')\,(n-k)!}\,\frac{\Gamma(\alpha+\alpha')\,n!}{\Gamma(\alpha+\alpha'+n)}\,.$$

We remark that a similar model has already been studied in [65] and that, by setting $\alpha_{x_\mathfrak{t}} = \alpha_{x_\mathfrak{b}} = \alpha_{y_\mathfrak{t}} = \alpha_{y_\mathfrak{b}} = 1$ in $\mathrm{IEM}_\mathrm{d}(\boldsymbol{\alpha})$, one recovers IEM_d =

$\mathrm{IEM_d}((1,1),(1,1))$.

5.1.3 Symmetries of the splitting part

Viewing the splitting mechanism as thermalization of symmetric inclusion processes has the advantage of generating a whole class of symmetries for the splitting operator. Moreover, if the parameters $\boldsymbol{\alpha}$ are chosen accordingly, these symmetries yields symmetries for the full transition operator Π obtained from the splitting, the exchange and the addition of wealth. In this section, we give a full account on symmetries which arise for the generalized discrete immediate exchange models presented so far.

We reconsider the operators K, given in (4.47), for the choices

$$\sigma = 1 \quad \text{and} \quad \boldsymbol{\alpha} = ((\alpha_{x_\mathfrak{t}}, \alpha_{x_\mathfrak{b}}), (\alpha_{y_\mathfrak{t}}, \alpha_{y_\mathfrak{b}}))\,, \tag{5.11}$$

in terms of the "single-site" operators $\mathcal{K}$, which we now introduce. Indeed, to the "particle removal" (or "annihilation") and "addition" (or "creation")-type of operators K^- and K^+, we associate, respectively, the following operators $\mathcal{K}^{-,\alpha}$ and $\mathcal{K}^{+,\alpha}$ ($\alpha > 0$), acting on functions $\varphi \in \mathbb{V}_1$ as follows:

$$\begin{aligned}\mathcal{K}^{-,\alpha}\varphi(n) &= n\,\varphi(n-1)\\ \mathcal{K}^{+,\alpha}\varphi(n) &= (\alpha+n)\,\varphi(n+1)\,.\end{aligned} \tag{5.12}$$

We introduce also a "number"-type of single-site operator $\mathcal{K}^{\circ,\alpha}$:

$$\mathcal{K}^{\circ,\alpha}\varphi(n) = (\tfrac{\alpha}{2}+n)\,\varphi(n)\,, \quad n \in \mathbb{N}_0\,. \tag{5.13}$$

We recall from e.g. [19] that, for all $\alpha > 0$, the operators $\{\mathcal{K}^{-\alpha}, \mathcal{K}^{+,\alpha}, \mathcal{K}^{\circ,\alpha}\}$ in (5.12)–(5.13) form a (left) discrete representation of the $\mathrm{SU}(1,1)$ algebra, i.e. they generate the algebra and satisfy the commutation relations

$$\begin{aligned}[\mathcal{K}^+, \mathcal{K}^-] &= 2\mathcal{K}^\circ\\ [\mathcal{K}^\pm, \mathcal{K}^\circ] &= \pm\mathcal{K}^\pm\,.\end{aligned} \tag{5.14}$$

In this representation, the generator of $\mathrm{SIP}(\boldsymbol{\alpha}_x)$ on $\{x_\mathfrak{t}, x_\mathfrak{b}\}$ is given by

$$L^{\mathrm{SIP}(\boldsymbol{\alpha}_x)} = \mathcal{K}^{-,\alpha_{x_\mathfrak{t}}}_{x_\mathfrak{t}}\mathcal{K}^{+,\alpha_{x_\mathfrak{b}}}_{x_\mathfrak{b}} + \mathcal{K}^{+,\alpha_{x_\mathfrak{t}}}_{x_\mathfrak{t}}\mathcal{K}^{-,\alpha_{x_\mathfrak{b}}}_{x_\mathfrak{b}} - 2\mathcal{K}^{\circ,\alpha_{x_\mathfrak{t}}}_{x_\mathfrak{t}}\mathcal{K}^{\circ,\alpha_{x_\mathfrak{b}}}_{x_\mathfrak{b}} + \frac{\alpha_{x_\mathfrak{t}}\alpha_{x_\mathfrak{b}}}{2}\mathbf{1}_{x_\mathfrak{t}}\mathbf{1}_{x_\mathfrak{b}}\,, \tag{5.15}$$

where $\mathcal{K}^{\bullet,\alpha}_z$ denotes $\mathcal{K}^{\bullet,\alpha}$ working on the $z \in \{x_\mathfrak{t}, x_\mathfrak{b}\}$ variable ($\bullet \in \{-,+,\circ\}$)

and $\mathbb{1}$ stands for the identity operator. The form (5.15) is called the "abstract" form of the SIP generator. From this form, one easily infers well-known commutation properties for $L^{\mathrm{SIP}(\alpha_x)}$, namely that it commutes with

$$\mathcal{K}_{x_{\mathrm{t}}}^{\bullet,\alpha_{x_{\mathrm{t}}}} + \mathcal{K}_{x_{\mathrm{b}}}^{\bullet,\alpha_{x_{\mathrm{b}}}} , \quad \bullet \in \{-,+,\circ\} ,$$

(cf. e.g. [62] for further details on this).

As a consequence of (5.8), also the splitting operator P, viewed as a simultaneous thermalization of two independent SIP($\boldsymbol{\alpha}_x$) and SIP($\boldsymbol{\alpha}_y$) on sites $\{x_{\mathrm{t}}, x_{\mathrm{b}}\}$ and $\{y_{\mathrm{t}}, y_{\mathrm{b}}\}$, respectively, commutes with

$$K^{\bullet,\boldsymbol{\alpha}} := \mathcal{K}_{x_{\mathrm{t}}}^{\bullet,\alpha_{x_{\mathrm{t}}}} + \mathcal{K}_{x_{\mathrm{b}}}^{\bullet,\alpha_{x_{\mathrm{b}}}} + \mathcal{K}_{y_{\mathrm{t}}}^{\bullet,\alpha_{y_{\mathrm{t}}}} + \mathcal{K}_{y_{\mathrm{b}}}^{\bullet,\alpha_{y_{\mathrm{b}}}} , \quad \bullet \in \{-,+,\circ\} . \tag{5.16}$$

Moreover, because these "symmetries" of P are sum of four parametrized copies of the same operators, $K^{\bullet,\boldsymbol{\alpha}}$ is permutation invariant if the parameters of the copies interchanged are the same. These findings are formalized in the following lemma.

Lemma 5.2 (SYMMETRIES AND EXCHANGE). *Let, for* $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_x, \boldsymbol{\alpha}_y) = ((\alpha_{x_{\mathrm{t}}}, \alpha_{x_{\mathrm{b}}}), (\alpha_{y_{\mathrm{t}}}, \alpha_{y_{\mathrm{b}}})$ *and* $\bullet \in \{-,+,\circ\}$, *the operators* $\mathscr{E}$, $P^{\boldsymbol{\alpha}}$ *and* $K^{\bullet,\boldsymbol{\alpha}}$ *be given as in* (5.1), (5.10) *and* (5.16), *respectively. Then,*

(a) $P^{\boldsymbol{\alpha}}$ *and* $K^{\bullet,\boldsymbol{\alpha}}$ *commute.*

(b) $K^{\bullet,\boldsymbol{\alpha}}$ *and* $\mathscr{E}$ *commute if* $\alpha_{x_{\mathrm{t}}} = \alpha_{y_{\mathrm{t}}}$.

Proof. Item (a) follows from the commutation of $L^{\mathrm{SIP}(\boldsymbol{\alpha}_x)} + L^{\mathrm{SIP}(\boldsymbol{\alpha}_y)}$ with $K^{\bullet,\boldsymbol{\alpha}}$ (see e.g. [19]), relation (5.10) and the fact that symmetries for a generator are also symmetries for the corresponding "thermalized" transition operator (see e.g. [20]).

We show item (b) for $\bullet = +$, the other two cases being analogous. First, it suffices to check the commutation relation for function $\varphi \in \mathbb{V}_4$ in product form, i.e. $\varphi = \varphi_{x_{\mathrm{t}}} \otimes \varphi_{x_{\mathrm{b}}} \otimes \varphi_{y_{\mathrm{t}}} \otimes \varphi_{y_{\mathrm{b}}}$, where $\varphi_z \in \mathbb{V}_1$. Then, we observe that, acting $\mathscr{E}$ as the identity w.r.t. to the second and forth coordinates, we have

$$[K^{+,\boldsymbol{\alpha}}\mathscr{E}, \mathscr{E}K^{+,\boldsymbol{\alpha}}]\,\varphi = [K^{+,\boldsymbol{\alpha}}\mathscr{E}, \mathscr{E}K^{+,\boldsymbol{\alpha}}](\varphi_{x_{\mathrm{t}}} \otimes \varphi_{y_{\mathrm{t}}}) .$$

In conclusion, we get

$$
\begin{aligned}
&[K^{+,\alpha}\mathscr{E}, \mathscr{E}K^{+,\alpha}](\varphi_{x_t} \otimes \varphi_{y_t})(k_x, k_y)\\
&= (\alpha_{x_t} + k_x)\, \varphi_{x_t}(k_y)\, \varphi_{y_t}(k_x + 1) \;+\; (\alpha_{y_t} + k_y)\, \varphi_{x_t}(k_y + 1)\, \varphi_{y_t}(k_x)\\
&- (\alpha_{x_t} + k_y)\, \varphi_{x_t}(k_y + 1)\, \varphi_{y_t}(k_x) \;-\; (\alpha_{y_t} + k_x)\, \varphi_{x_t}(k_y)\, \varphi_{y_t}(k_x + 1)\,,
\end{aligned}
$$

which always vanishes if $\alpha_{x_t} = \alpha_{y_t}$. □

The above operators $K^{\bullet,\alpha}$ in (5.16) have a natural additive structure which is expressed in terms of an intertwining relation in the following lemma.

Lemma 5.3 (SYMMETRIES AND ADDITION). *Recall the definitions of the addition operator Γ in (5.2). Then, for all $\bullet \in \{-,+,\circ\}$ and $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_x, \boldsymbol{\alpha}_y) = ((\alpha_{x_t}, \alpha_{x_b}), (\alpha_{y_t}, \alpha_{y_b}))$, we have, for all $\varphi \in \mathbb{V}_2$,*

$$
K^{\bullet,\boldsymbol{\alpha}}\Gamma\varphi \;=\; \Gamma K^{\bullet,\widehat{\boldsymbol{\alpha}}}\varphi\,, \tag{5.17}
$$

where $\widehat{\boldsymbol{\alpha}} = (|\boldsymbol{\alpha}_x|, |\boldsymbol{\alpha}_y|) = (\alpha_{x_t} + \alpha_{x_b}, \alpha_{y_t} + \alpha_{y_b})$ and $K^{\bullet,\widehat{\boldsymbol{\alpha}}}$ is given by

$$
K^{\bullet,\widehat{\boldsymbol{\alpha}}}\varphi(n_x, n_y) \;=\; \mathcal{K}_x^{\bullet,\alpha_{x_t}+\alpha_{x_b}}\varphi(n_x, n_y) \;+\; \mathcal{K}_y^{\bullet,\alpha_{y_t}+\alpha_{y_b}}\varphi(n_x, n_y)\,. \tag{5.18}
$$

Proof. We show (5.17) for $\bullet = +$, the other cases being analogous. Hence, for $\varphi \in \mathbb{V}_2$ and $k_x, \ell_x, k_y, \ell_y \in \mathbb{N}_0$, from the definition of $K^{+,\boldsymbol{\alpha}}$, we obtain

$$
\begin{aligned}
K^{\bullet,\boldsymbol{\alpha}}\Gamma\varphi(k_x, \ell_x, k_y, \ell_y) \;&= (\alpha_{x_t} + k_x)\, \Gamma\varphi(k_x + 1, \ell_x, k_y, \ell_y)\\
&+ (\alpha_{x_b} + \ell_x)\, \Gamma\varphi(k_x, \ell_x + 1, k_y, \ell_y)\\
&+ (\alpha_{y_t} + k_y)\, \Gamma\varphi(k_x, \ell_x, k_y + 1, \ell_y)\\
&+ (\alpha_{y_b} + \ell_y)\, \Gamma\varphi(k_x, \ell_x, k_y, \ell_y + 1)\,,
\end{aligned}
$$

which, from the definition of Γ, if $n_x = k_x + \ell_x$ and $n_y = k_y + \ell_y$, rewrites as

$$
\begin{aligned}
K^{\bullet,\boldsymbol{\alpha}}\Gamma\varphi(k_x, \ell_x, k_y, \ell_y) \;&= (\alpha_{x_t} + \alpha_{x_b} + n_x)\, \varphi(n_x + 1, n_y)\\
&+ (\alpha_{y_t} + \alpha_{y_b} + n_y)\, \varphi(n_x, n_y + 1)\,,
\end{aligned}
$$

i.e. the r.h.s. in (5.17).

□

5.1.4 Self-duality

In the previous section, via the connection with a thermalized version of SIP(α), on the one hand we obtained symmetries $\{K^{-,\alpha}, K^{+,\alpha}, K^{\circ,\alpha}\}$ for the splitting and addition operators – P^α and $\mathscr{E}$, respectively – (Lemma 5.2). On the other hand, these symmetries have an additive structure "compatible" with the addition operator Γ (Lemma 5.3).

In this section, we use this information to find symmetries for the discrete immediate exchange model transition operator Π^α in (5.9). The knowledge of both symmetries and reversible measures for Π^α generate – in the same spirit of the so-called Lie algebraic method for duality, see e.g. Section 1.2 and, further, [62] – self-duality relations. In our case, both symmetries and reversible measures are in product form, thus, yielding jointly factorized self-duality functions for $\mathrm{IEM}_\mathrm{d}(\alpha)$ as those appearing in Chapter 3 for the symmetric inclusion process.

Before presenting the theorem, we observe that generalized discrete immediate exchange models satisfying the assumptions in the statement below admit a one-parameter family of reversible product measures, which are, in turn, related to the reversible product measures of the symmetric inclusion process which governs the splitting mechanism.

In the particular instance of the discrete immediate exchange model $\mathrm{IEM}_\mathrm{d}((1,1),(1,1))$, the reversible measures have been found in [65] by means of a direct detailed balance computation and are given by products of $\mathrm{Gamma}_\mathrm{d}(2,\lambda)$ (negative binomial with parameters $(2,\lambda)$) distributions. More precisely, the one-parameter family of product probability measures $\{\mu_\lambda = \nu_{x,\lambda} \otimes \nu_{y,\lambda},\ \lambda \in (0,1)\}$ with marginals given by

$$\nu_{\cdot,\lambda}(n) \;=\; (n+1)\,\lambda^n\,(1-\lambda)^2\,, \quad n \in \mathbb{N}_0\,,$$

are reversible for $\mathrm{IEM}_d((1,1),(1,1))$ with transition operator Π given in (5.5).

Similar detailed balance computations yield related reversible measures in a similar form. However, we will present in Section 5.3 a more constructive strategy to obtain reversible measures for the transition operators Π^α from those of the splitting operator P^α. Hence, we postpone the actual proof of reversibility (item (b) in the theorem below) to that section. There, we will apply this method to models with different splitting mechanisms as introduced in Section 5.2.

We note that part of this theorem was already proved in [65] with the help of direct – somewhat tedious – computations with hypergeometric functions and that $\mathrm{IEM_d}$ is recovered by choosing $\boldsymbol{\alpha} = ((1,1),(1,1))$.

Theorem 5.4 (self-duality for discrete immediate exchange models). *For all* $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_x, \boldsymbol{\alpha}) = ((\alpha_{x_t}, \alpha_{x_b}), (\alpha_{y_t}, \alpha_{y_b}))$*, let* Π^{α} *be the transition operator of* $\mathrm{IEM_d}(\boldsymbol{\alpha})$ *as defined in* (5.9)*. Recall the notation* $\widehat{\boldsymbol{\alpha}} = (|\boldsymbol{\alpha}_x|, |\boldsymbol{\alpha}_y|) = (\alpha_{x_t} + \alpha_{x_b}, \alpha_{y_t} + \alpha_{y_b})$*.*

Then, if $\alpha_{x_t} = \alpha_{y_t}$*, we have:*

(a) *For* $\bullet \in \{-,+,\circ\}$*, the operators* $K^{\bullet,\widehat{\alpha}}$ *in* (5.18) *and* Π^{α} *commute.*

(b) *For all* $\lambda \in (0,1)$*, the following product measures* $\widehat{\mu}_\lambda$ *on* $\mathbb{N}_0^2$

$$\widehat{\mu}_\lambda = \mathrm{Gamma_d}(|\boldsymbol{\alpha}_x|, \lambda) \otimes \mathrm{Gamma_d}(|\boldsymbol{\alpha}_y|, \lambda)$$

are reversible measures for Π^{α}*.*

As a consequence of (a) *and* (b)*, we get:*

(c) *The process with transition operator* Π^{α} *is self-dual with jointly factorized self-duality functions*

$$D(\xi(x), \eta(x)) = d_x(\xi(x), \eta(x)) \cdot d_y(\xi(y), \eta(y)),$$

for $\xi, \eta \in \mathbb{N}_0^{\{x,y\}}$*, where the single-site self-duality function* d_x *takes either one of the following forms (cf. also* (3.80)*):*

$$d_x(k,n) = \frac{\Gamma(|\boldsymbol{\alpha}_x|)}{\Gamma(|\boldsymbol{\alpha}_x| + k)} \frac{n!}{(n-k)!} \mathbf{1}_{\{k \le n\}} \tag{5.19}$$

$$d_x(k,n) = {}_2F_1\left[\begin{matrix} -k \quad -n \\ |\boldsymbol{\alpha}_x| \end{matrix}; -c\right], \quad c \in \mathbb{R}, \tag{5.20}$$

and, analogously, for d_y*.*

Proof. The commutation in (a) is a consequence of Lemma 5.3, the assumption $\alpha_{x_t} = \alpha_{y_t}$ and Lemma 5.2:

$$\begin{aligned} \Pi^{\alpha} K^{\bullet,\widehat{\alpha}} &\overset{\star}{=} \Gamma^{-1} P^{\alpha} \mathscr{E} \Gamma \Gamma^{-1} K^{\bullet,\alpha} \Gamma = \Gamma^{-1} P^{\alpha} \mathscr{E} K^{\bullet,\alpha} \Gamma \\ &= \Gamma^{-1} K^{\bullet,\alpha} P^{\alpha} \mathscr{E} \Gamma \overset{\text{✡}}{=} \Gamma^{-1} K^{\bullet,\alpha} \Gamma^{-1} \Gamma P^{\alpha} \mathscr{E} \Gamma = K^{\bullet,\widehat{\alpha}} \Pi^{\alpha}, \end{aligned}$$

where $\star$ and ✡ follow, respectively, from Lemma 5.3 – and, in particular, from $\Gamma \Gamma^{-1} K^{\bullet,\alpha} \Gamma = K^{\bullet,\alpha} \Gamma$ – and $\Gamma^{-1} \Gamma P^{\alpha} \mathscr{E} \Gamma = P^{\alpha} \mathscr{E} \Gamma$, i.e. the fact that the composition of the splitting and exchange mechanisms depends only on the total wealths of each agent.

Item (b) may be checked directly by detailed balance computation. However, we will provide another proof in Section 5.3.

The fact that these symmetries lead to the self-duality function (5.19) follows from the general strategy for obtaining self-duality functions from symmetries in [18], [19] and [62]. In particular, the self-duality function (5.19), resp. (5.20), arises by acting with the symmetry $\exp K^{+,\widehat{\alpha}}$, resp. unitary symmetries as in [18], on the cheap self-duality function coming from the reversible measures $\widehat{\mu}_\lambda$, yielding item (c). □

5.2 Further generalizations of immediate exchange models

In this section, we present further generalizations of models with splitting, exchange and addition mechanisms and obtain reversible measures and self-duality relations. We base our discussion on selecting different splitting mechanisms related to thermalization of those conservative interacting particle systems and conservative interacting diffusion processes thoroughly studied in Chapter 3.

5.2.1 Models based on thermalization of SEP

We consider the following splitting mechanism in these models.

Af first, the initial wealth of each agent is redistributed – independently for the two agents – over its two "top" and "bottom pockets" having both a maximal capacity, i.e. the pockets contain a fixed number of "slots" in which only one "coin" at the time can fit. Given this "pocket-structure", each agent places one coin at the time in one of the non-occupied slots, uniformly chosen among the two pockets. Therefore, the model has four positive integers as parameters, the maximal capacities of the pockets, which we denote by $\alpha_{x_\mathrm{t}}, \alpha_{x_\mathrm{b}}, \alpha_{y_\mathrm{t}}$ and $\alpha_{y_\mathrm{b}} \in \mathbb{N}$.

More precisely, for agent x, say, $n_x \in \mathbb{N}_0$ coins are redistributed over two pockets with maximal capacities α_{x_t} and $\alpha_{x_\mathrm{b}} \in \mathbb{N}$ according to a hypergeometric distribution with parameters $n_x, (\alpha_{x_\mathrm{t}}, \alpha_{x_\mathrm{b}})$, i.e. $n_x \in \{0, \ldots, \alpha_{x_\mathrm{t}} + \alpha_{x_\mathrm{b}}\}$ is split in $(k_x, n_x - k_x)$ with k_x having distribution

$$w_{n,(\alpha,\alpha')}(k) \;=\; \frac{\binom{\alpha}{k}\binom{\alpha'}{n-k}}{\binom{\alpha+\alpha'}{n}}\,\mathbf{1}_{\{k\le\alpha\}}\,, \tag{5.21}$$

with $n = n_x$ and $(\alpha, \alpha') = (\alpha_{x_{\mathfrak{t}}}, \alpha_{x_{\mathfrak{b}}})$.

We read out of this splitting mechanism just described a thermalization procedure of two independent symmetric exclusion processes SEP($\boldsymbol{\alpha}_x$) and SEP($\boldsymbol{\alpha}_y$), with $\boldsymbol{\alpha}_x = (\alpha_{x_{\mathfrak{t}}}, \alpha_{x_{\mathfrak{b}}})$ and $\boldsymbol{\alpha}_y = (\alpha_{y_{\mathfrak{t}}}, \alpha_{y_{\mathfrak{b}}})$, whose infinitesimal generators – we recall from e.g. Section 3.a – are given by

$$\begin{aligned} L^{\mathrm{SEP}(\boldsymbol{\alpha}_x)}\varphi(k_x,\ell_x) &= k_x\,(\alpha_{x_{\mathfrak{b}}}-\ell_x)\,(\varphi(k_x-1,\ell_x+1)-\varphi(k_x,\ell_x))\\ &\quad+\ell_x\,(\alpha_{x_{\mathfrak{b}}}-k_x)\,(\varphi(k_x+1,\ell_x-1)-\varphi(k_x,\ell_x))\,, \end{aligned}$$

for $\varphi \in \mathbb{V}_2$ and $k_x \in \{0, \ldots, \alpha_{x_{\mathfrak{t}}}\}$, $\ell_x \in \{0, \ldots, \alpha_{x_{\mathfrak{b}}}\}$. An analogous expression holds for the generator of SEP($\boldsymbol{\alpha}_y$).

The connection is motivated from the fact that reversible measures of SEP($\boldsymbol{\alpha}_x$) are product of Binomial distributions with parameters $\alpha_{x_{\mathfrak{t}}}$, $\alpha_{x_{\mathfrak{b}}}$ and with equal success probability, i.e.

$$\mu_\lambda \;=\; \mathrm{Binomial}(\alpha_{x_{\mathfrak{t}}}, \tfrac{\lambda}{1+\lambda}) \otimes \mathrm{Binomial}(\alpha_{x_{\mathfrak{b}}}, \tfrac{\lambda}{1+\lambda})\,, \quad \lambda \in (0,\infty)\,,$$

which, conditioned on total number of particles being equal to $n_x \in \{0, \ldots, |\boldsymbol{\alpha}_x|\}$, yields a Hypergeometric distribution as in (5.21) for the first coordinate.

As a consequence, we can write the corresponding splitting operator $P^{\boldsymbol{\alpha}}$ – in analogy with (5.9) – as follows:

$$\begin{aligned} &P\varphi(k_x,\ell_x,k_y,\ell_y)\\ &\quad= \lim_{t\to\infty}\left(\mathbb{E}^{\mathrm{SEP}(\boldsymbol{\alpha}_x)}_{(k_x,\ell_x)}\otimes\mathbb{E}^{\mathrm{SEP}(\boldsymbol{\alpha}_y)}_{(k_y,\ell_y)}\right)\left[\varphi(k_x(t),\ell_x(t),k_y(t),\ell_y(t))\right]\,, \end{aligned} \tag{5.22}$$

or, equivalently, if we use the definition of the functions $w_{n,(\alpha,\alpha')}$ in (5.21),

$$\begin{aligned} &P\varphi(k_x,n_x-k_x,k_y,n_y-k_x)\\ &\quad= \sum_{k_x=0}^{n_x}\sum_{k_y=0}^{n_y} w_{n_x,(\alpha_{x_{\mathfrak{t}}},\alpha_{x_{\mathfrak{b}}})}(k_x)\,w_{n_y,(\alpha_{y_{\mathfrak{t}}},\alpha_{y_{\mathfrak{b}}})}(k_y)\,\varphi(k_x,n_x-k_x,k_y,n_y-k_y)\,, \end{aligned} \tag{5.23}$$

for all $\varphi \in \mathbb{V}_4$, and compose this splitting mechanism with exchange and addition as in the previous section obtaining the transition operator of a SEP($\boldsymbol{\alpha}$)-based discrete immediate exchange model:

$$\Pi^{\boldsymbol{\alpha}} = \Gamma^{-1} P^{\boldsymbol{\alpha}} \mathscr{E} \Gamma \,. \tag{5.24}$$

In analogy with what has been discussed in Section 5.1.3, we have symmetries for the splitting operator given by

$$\begin{aligned} K^{-,\boldsymbol{\alpha}} &= \mathcal{K}_{x_{\mathrm{t}}}^{-,\alpha_{x_{\mathrm{t}}}} + \mathcal{K}_{x_{\mathrm{b}}}^{-,\alpha_{x_{\mathrm{b}}}} + \mathcal{K}_{y_{\mathrm{t}}}^{-,\alpha_{y_{\mathrm{t}}}} + \mathcal{K}_{x_{\mathrm{b}}}^{-,\alpha_{y_{\mathrm{b}}}} \\ K^{+,\boldsymbol{\alpha}} &= \mathcal{K}_{x_{\mathrm{t}}}^{+,\alpha_{x_{\mathrm{t}}}} + \mathcal{K}_{x_{\mathrm{b}}}^{+,\alpha_{x_{\mathrm{b}}}} + \mathcal{K}_{y_{\mathrm{t}}}^{+,\alpha_{y_{\mathrm{t}}}} + \mathcal{K}_{x_{\mathrm{b}}}^{+,\alpha_{y_{\mathrm{b}}}} \\ K^{\circ,\boldsymbol{\alpha}} &= \mathcal{K}_{x_{\mathrm{t}}}^{\circ,\alpha_{x_{\mathrm{t}}}} + \mathcal{K}_{x_{\mathrm{b}}}^{\circ,\alpha_{x_{\mathrm{b}}}} + \mathcal{K}_{y_{\mathrm{t}}}^{\circ,\alpha_{y_{\mathrm{t}}}} + \mathcal{K}_{x_{\mathrm{b}}}^{\circ,\alpha_{y_{\mathrm{b}}}} \,, \end{aligned}$$

where we have adopted the same notation of Section 5.1.3 with single-site operators $\mathcal{K}^{\bullet,\alpha}$ defined, for all $\alpha \in \mathbb{N}$ and $\varphi \in \mathbb{V}_1$, as

$$\begin{aligned} \mathcal{K}^{-,\alpha}\varphi(n) &= n\,\varphi(n-1) \\ \mathcal{K}^{+,\alpha}\varphi(n) &= (\alpha - n)\,\varphi(n+1) \\ \mathcal{K}^{\circ,\alpha}\varphi(n) &= (\tfrac{\alpha}{2} - n)\,\varphi(n) \,. \end{aligned}$$

We remark that the operators $\{\mathcal{K}^{-,\alpha}, \mathcal{K}^{+,\alpha}, \mathcal{K}^{\circ,\alpha}\}$ are, for all $\alpha \in \mathbb{N}$, generators of a (left) discrete representation of the SU(2) algebra.

These symmetries $K^{\bullet,\boldsymbol{\alpha}}$, with $\bullet \in \{-,+,\circ\}$, commute with the exchange operator $\mathscr{E}$ if and only if $\alpha_{x_{\mathrm{t}}} = \alpha_{y_{\mathrm{t}}}$, i.e. the parameters of the representations of SU(2) for the sites where the exchange takes place have to be the same. Moreover, these operators have the same additive structure of the symmetries considered in the previous section, i.e.

$$K^{\bullet,\boldsymbol{\alpha}}\Gamma = \Gamma K^{\bullet,\widehat{\alpha}} \,,$$

where $\widehat{\alpha} = (|\boldsymbol{\alpha}_x|, |\boldsymbol{\alpha}_y|) = (\alpha_{x_{\mathrm{t}}} + \alpha_{x_{\mathrm{b}}}, \alpha_{y_{\mathrm{t}}} + \alpha_{y_{\mathrm{b}}})$ and

$$\begin{aligned} &K^{\bullet,\widehat{\alpha}}\varphi(n_x, n_y) \\ &\quad = (\alpha_{x_{\mathrm{t}}} + \alpha_{x_{\mathrm{b}}} - n_x)\,\varphi(n_x+1, n_y) + (\alpha_{y_{\mathrm{t}}} + \alpha_{y_{\mathrm{b}}} - n_y)\,\varphi(n_x, n_y+1) \,, \end{aligned} \tag{5.25}$$

for $\varphi \in \mathbb{V}_2$. As a consequence, we obtain the following analogue of Theorem 5.4.

Theorem 5.5. *For all* $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_x, \boldsymbol{\alpha}) = (\alpha_{x_{\mathrm{t}}}, \alpha_{x_{\mathrm{b}}}, \alpha_{y_{\mathrm{t}}}, \alpha_{y_{\mathrm{b}}}) \in (0,\infty)^4$, *let* $\Pi^{\boldsymbol{\alpha}}$ *be the* *transition operator of the* SEP($\boldsymbol{\alpha}$)*-based immediate exchange model as defined in* (5.24).

Then, if $\alpha_{x_t} = \alpha_{y_t}$*, we have:*

(a) *For* $\bullet \in \{-,+,\circ\}$*, the operators* $K^{\bullet,\widehat{\alpha}}$ *in* (5.25) *and* Π^{α} *commute.*

(b) *For all* $\lambda \in (0,\infty)$*, the following product measures* $\widehat{\mu}_\lambda$ *on* $\{0,\ldots,|\boldsymbol{\alpha}_x|\} \times \{0,\ldots,|\boldsymbol{\alpha}_y|\}$

$$\widehat{\mu}_\lambda = \text{Binomial}(|\boldsymbol{\alpha}_x|, \tfrac{\lambda}{1+\lambda}) \otimes \text{Binomial}(|\boldsymbol{\alpha}_y|, \tfrac{\lambda}{1+\lambda})$$

are reversible measures for Π^{α}.

As a consequence of (a) *and* (b)*, we get:*

(c) *The process with transition operator* Π^{α} *is self-dual with jointly factorized self-duality functions*

$$D(\xi(x),\eta(x)) = d_x(\xi(x),\eta(x)) \cdot d_y(\xi(y),\eta(y)),$$

for $\xi,\eta \in \{0,\ldots,|\boldsymbol{\alpha}_x|\} \times \{0,\ldots,|\boldsymbol{\alpha}_y|\}$*, where the single-site self-duality function* d_x *takes either one of the following forms (cf. also* (3.80)*):*

$$d_x(k,n) = \frac{(|\boldsymbol{\alpha}_x| - k)!}{|\boldsymbol{\alpha}_x|!} \frac{n!}{(n-k)!} \mathbf{1}_{\{k \le n\}} \tag{5.26}$$

$$d_x(k,n) = {}_2F_1\left[\begin{matrix} -k \quad -n \\ -|\boldsymbol{\alpha}_x| \end{matrix}; c\right], \quad c \in \mathbb{R}, \tag{5.27}$$

and, analogously, for d_y.

5.2.2 Models based on thermalization of IRW

We follow the same ideas presented in the previous section to generalize discrete immediate exchange models to models where the splitting mechanism arises from thermalization of independent random walkers.

More precisely, given the parameters $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_x, \boldsymbol{\alpha}_y) = ((\alpha_{x_t}, \alpha_{x_b}), (\alpha_{y_t}, \alpha_{y_b}))$, by "running for an infinite time" IRW($\boldsymbol{\alpha}_x$) between the top and bottom pockets of agent x and – independently and analogously – IRW($\boldsymbol{\alpha}_y$) for agent y, we obtain the following "Binomial" splitting rule, i.e.

$$P\varphi(k_x,\ell_x,k_y,\ell_y) = \lim_{t\to\infty}\left(\mathbb{E}^{\mathrm{IRW}(\boldsymbol{\alpha}_x)}_{(k_x,\ell_x)}\otimes\mathbb{E}^{\mathrm{IRW}(\boldsymbol{\alpha}_y)}_{(k_y,\ell_y)}\right)\left[\varphi(k_x(t),\ell_x(t),k_y(t),\ell_y(t))\right]\,, \quad (5.28)$$

where

$$P\varphi(k_x,\ell_x,k_y,\ell_y) = \sum_{k_x=0}^{n_x}\sum_{k_y=0}^{n_y} w_{n_x,(\alpha_{x_t},\alpha_{x_b})}(k_x)\, w_{n_y,(\alpha_{y_t},\alpha_{y_b})}(k_y)\,\varphi(k_x,n_x-k_x,k_y,n_y-k_y)\,, \quad (5.29)$$

with

$$w_{n,(\alpha,\alpha')}(k) = \binom{n}{k}\left(\frac{\alpha}{\alpha'}\right)^k\left(\frac{\alpha'}{\alpha+\alpha'}\right)^n\,, \quad k\in\{0,\ldots,n\}\,.$$

We note that this Binomial-type of splitting occurs because stationary distributions for IRW($\boldsymbol{\alpha}_x$) are products of Poisson distributions in the following form

$$\mu_\lambda = \mathrm{Poisson}(\alpha_{x_t}\lambda)\otimes\mathrm{Poisson}(\alpha_{x_b}\lambda)\,, \quad \lambda\in(0,\infty)\,.$$

Also in this case, a set of symmetries for the splitting operator arise from the abstract form of the IRW-generator. Namely,

$$\begin{aligned} K^{-,\boldsymbol{\alpha}} &= \mathcal{K}^{-,\alpha_{x_t}}_{x_t} + \mathcal{K}^{-,\alpha_{x_b}}_{x_b} + \mathcal{K}^{-,\alpha_{y_t}}_{y_t} + \mathcal{K}^{-,\alpha_{y_b}}_{x_b} \\ K^{+,\boldsymbol{\alpha}} &= \mathcal{K}^{+,\alpha_{x_t}}_{x_t} + \mathcal{K}^{+,\alpha_{x_b}}_{x_b} + \mathcal{K}^{+,\alpha_{y_t}}_{y_t} + \mathcal{K}^{+,\alpha_{y_b}}_{x_b}\,, \end{aligned}$$

where the single-site operators $\mathcal{K}^{\bullet,\alpha}$ are given, for all $\alpha>0$ and $\bullet\in\{-,+\}$, by

$$\begin{aligned} \mathcal{K}^{-,\alpha}\varphi(n) &= n\,\varphi(n-1) \\ \mathcal{K}^{+,\alpha}\varphi(n) &= \alpha\,\varphi(n+1)\,. \end{aligned}$$

We remark that, for all $\alpha>0$, the operators $\{\mathcal{K}^{-,\alpha},\mathcal{K}^{+,\alpha}\}$ are generators of a (left) discrete representation of the so-called Heisenberg algebra.

By similar arguments as in Section 5.1.3, we recover analogues of Lemmas

5.2–5.3, in which, provided that $\alpha_{x_t} = \alpha_{y_t}$, the operators

$$\begin{aligned} K^{-,\widehat{\alpha}} &= \mathcal{K}_x^{-,\alpha_{x_t}+\alpha_{x_b}} + \mathcal{K}_y^{-,\alpha_{y_t}+\alpha_{y_b}} \\ K^{+,\widehat{\alpha}} &= \mathcal{K}_x^{+,\alpha_{x_t}+\alpha_{x_b}} + \mathcal{K}_y^{+,\alpha_{y_t}+\alpha_{y_b}} \end{aligned} \tag{5.30}$$

commute with the splitting-exchange-addition transition operator Π^α defined in terms of P^α (cf. (5.28)) as

$$\Pi^\alpha = \Gamma^{-1} P^\alpha \mathscr{E} \Gamma \ . \tag{5.31}$$

We list below properties and self-duality of this type of models.

Theorem 5.6. *For all* $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_x, \boldsymbol{\alpha}) = (\alpha_{x_t}, \alpha_{x_b}, \alpha_{y_t}, \alpha_{y_b}) \in (0,\infty)^4$, *let* Π^α *be the transition operator of the* IRW($\boldsymbol{\alpha}$)*-based immediate exchange model as defined in* (5.31).

Then, if $\alpha_{x_t} = \alpha_{y_t}$, *we have:*

(a) *For* $\bullet \in \{-,+,\circ\}$, *the operators* $K^{\bullet,\widehat{\alpha}}$ *in* (5.30) *and* Π^α *commute.*

(b) *For all* $\lambda \in (0,\infty)$, *the following product measures* $\widehat{\mu}_\lambda$ *on* $\mathbb{N}_0^{\{x,y\}}$

$$\widehat{\mu}_\lambda = \text{Poisson}(|\boldsymbol{\alpha}_x|\lambda) \otimes \text{Poisson}(|\boldsymbol{\alpha}_y|\lambda)\ , \quad \lambda \in (0,\infty)\ ,$$

are reversible measures for Π^α.

As a consequence of (a) *and* (b), *we get:*

(c) *The process with transition operator* Π^α *is self-dual with jointly factorized self-duality functions*

$$D(\xi(x),\eta(x)) = d_x(\xi(x),\eta(x)) \cdot d_y(\xi(y),\eta(y))\ , \quad \xi,\eta \in \mathbb{N}_0^{\{x,y\}}\ ,$$

where the single-site self-duality function d_x *takes either one of the following forms (cf. also* (3.79)*):*

$$d_x(k,n) = \frac{1}{|\boldsymbol{\alpha}_x|^k} \frac{n!}{(n-k)!} \mathbf{1}_{\{k \le n\}} \tag{5.32}$$

$$d_x(k,n) = {}_2F_0\left[\begin{matrix} -k \quad -n \\ - \end{matrix} ; -\frac{c}{|\boldsymbol{\alpha}_x|}\right]\ , \quad c \in \mathbb{R}\ , \tag{5.33}$$

and, analogously, for d_y.

5.2.3 Models based on thermalization of interacting diffusions

We conclude this section by showing that continuum immediate exchange models with self-duality as well as duality with discrete counterparts may be constructed if the splitting part is modeled according to thermalization of those interacting diffusion processes presented in Section 3.1.6 of Chapter 3. We have seen that these interacting diffusions may arise as many-particle limits of their discrete analogues (cf. Section 3.1.6) as well as satisfy intertwining relations with the corresponding particle systems (cf. Propositions 3.20, 3.22 and 3.28 in Chapter 3).

In particular, we focus on two (possibly degenerate) diffusion processes on two sites whose generators $\mathscr{L}$ are the following first and second-order differential operators, defined for all $\alpha, \alpha' > 0$ and smooth functions $\varphi \in \mathbb{V}_2$ (where we keep the notation $\mathbb{V}_m$ to denote now functions $\varphi : [0,\infty)^m \to \mathbb{R}$), respectively, as

$$\mathscr{L}^{\mathrm{IRW}(\alpha,\alpha')}\varphi(z,z') = -(\alpha' z - \alpha z')\,(\partial_z - \partial_{z'})\,\varphi(z,z') \tag{5.34}$$

and

$$\mathscr{L}^{\mathrm{SIP}(\alpha,\alpha')}\varphi(z,z') = -(\alpha' z - \alpha z')\,(\partial_z - \partial_z)\,\varphi(z,z') + z\,z'\,(\partial_z - \partial_{z'})^2\varphi(z,z')\,. \tag{5.35}$$

The diffusion processes associated to the first generator (5.34) is a deterministic process (= degenerate diffusion) on two sites whose absorbing stable point, if starting from the configuration $(z,z') \in [0,\infty)^2$, is given by

$$\left(\tfrac{\alpha}{\alpha+\alpha'}\,(z+z'),\, \tfrac{\alpha'}{\alpha+\alpha'}\,(z+z')\right)\,.$$

We recognize in the process associated to the second generator (5.35) the so-called BEP($\boldsymbol{\alpha}$) (see e.g. Section 3.1.6 for the homogeneous process while Proposition 3.28 for its inhomogeneous counterpart). For this process, the reversible measures are products of Gamma distributions with equal scale parameter, namely

$$\mu_\lambda = \mathrm{Gamma}(\alpha,\lambda) \otimes \mathrm{Gamma}(\alpha',\lambda)\,, \quad \lambda \in (0,\infty)\,,$$

where the explicit form of $\mathrm{Gamma}(\alpha,\lambda)$ may be found in (3.21).

By observing that for two independent random variables X and Y distributed according to $\mathrm{Gamma}(\alpha,\lambda)$ and $\mathrm{Gamma}(\alpha',\lambda)$, respectively, one obtains

$$\frac{X}{X+Y} \sim \text{Beta}(\alpha, \alpha') ,$$

we have all information needed to construct two continuum immediate exchange models from a thermalization procedure of the above diffusion processes: a first deterministic one, whose splitting mechanisms prescribing that the agents allocate a deterministic – fixed from the "pocket parameters " $\alpha_{x_\mathfrak{t}}, \alpha_{x_\mathfrak{b}}, \alpha_{y_\mathfrak{t}}$ and $\alpha_{y_\mathfrak{b}} \in (0, \infty)$ – fraction of their wealth to their top pockets; a second stochastic one, whose fraction of wealth assigned to the top pockets is stochastic and distributed according to Beta distributions.

Also in this case, symmetries for the splitting operators arise from coproducts of (right) continuum representations of Lie algebras (Heisenberg for the deterministic model and SU(1, 1) for the stochastic splitting model) and, under the assumption of choosing the pocket parameters accordingly, namely $\alpha_{x_\mathfrak{t}} = \alpha_{y_\mathfrak{t}}$, we obtain symmetries for the full splitting-exchange-addition transition operator (cf. e.g. [19, Section 5] for further details).

We further observe that from the knowledge of symmetries and reversible measures, we obtain self-duality relations for these continuum models of wealth/mass/energy exchange, while from the intertwining relations with the discrete models or from the many-particle limiting procedure and self-duality of the particle exchange models, duality between continuum and associated discrete immediate exchange models follows. Moreover, we note that for the choice $\alpha_{x_\mathfrak{t}} = \alpha_{x_\mathfrak{b}} = \alpha_{y_\mathfrak{t}} = \alpha_{y_\mathfrak{b}} = 1$ in the "Beta"-splitting model, the splitting mechanism corresponds to a thoroughly-studied model of heat conduction, the so-called KMP-model [86].

We summarize precise definitions and results for the above-mentioned stochastic model in the following theorem.

Theorem 5.7. *Let* $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_x, \boldsymbol{\alpha}_y) = ((\alpha_{x_\mathfrak{t}}, \alpha_{x_\mathfrak{b}}), (\alpha_{y_\mathfrak{t}}, \alpha_{y_\mathfrak{b}}))$. *We define a model of splitting, exchange and addition of continuous wealth as the process in which, at discrete (or mean-one exponential) times, agents x and y split, exchange and add-up their updated wealth according to the following transition operator*

$$\Pi^{\alpha} = \mathit{\Gamma}^{-1} P^{\alpha} \mathscr{E} \mathit{\Gamma} ,$$

where the operators $\mathscr{E}$, $\mathit{\Gamma}$ *and* $\mathit{\Gamma}^{-1}$ *are those defined in Section 5.1, while the splitting* *operator P^α is given for $\varphi \in \mathbb{V}_4$ by*

$$P^\alpha \varphi(z_x, u_x, z_y, u_y) = \int_0^1 \int_0^1 w_{(\alpha_{x_t}, \alpha_{x_b})}(\varepsilon_x)\, w_{(\alpha_{y_t}, \alpha_{y_b})}(\varepsilon_y) \times$$
$$\times \varphi(\varepsilon_x(z_x + u_x), (1-\varepsilon_x)(z_x+u_x), \varepsilon_y(z_y+u_y), (1-\varepsilon_y)(z_y+u_y))\, d\varepsilon_x\, d\varepsilon_y\,, \tag{5.36}$$

with

$$w_{(\alpha,\alpha')}(\varepsilon) = \frac{\Gamma(\alpha+\alpha')}{\Gamma(\alpha)\,\Gamma(\alpha')}\, \varepsilon^{\alpha-1}\,(1-\varepsilon)^{\alpha'-1}\,, \quad \varepsilon \in (0,1)\,.$$

Then, if $\alpha_{x_t} = \alpha_{y_t}$, we have:

(a) Π^α *has the following product measures*

$$\mu_\lambda = \mathrm{Gamma}(|\boldsymbol{\alpha}_x|, \lambda) \otimes \mathrm{Gamma}(|\boldsymbol{\alpha}_y|, \lambda), \qquad \lambda \in (0,\infty)\,,$$

as reversible measures.

(b) Π^α *is self-dual with jointly factorized self-duality function*

$$D(v,\zeta) = d_x(v(x), \zeta(x)) \cdot d_y(v(y), \zeta(y))\,, \quad v, \zeta \in [0,\infty)^{\{x,y\}}\,,$$

whose single-site self-duality functions are given, by (cf. e.g. (3.84))

$$d_x(v,z) = {}_0F_1\left[\begin{matrix} - \\ |\boldsymbol{\alpha}_x| \end{matrix}; -cvz\right], \quad c \in \mathbb{R}\,,$$

and, analogously, for d_y.

(c) *The $\boldsymbol{\alpha}$-discrete and $\boldsymbol{\alpha}$-continuum processes with transition operators Π^α given, respectively, in Theorem 5.4 and in the present theorem are dual with jointly factorized duality function*

$$D(\xi,\zeta) = d_x(\xi(x), \zeta(x)) \cdot d_y(\xi(y), \zeta(y))\,,$$

with $\xi \in \mathbb{N}_0^{\{x,y\}}$, $\zeta \in [0,\infty)^{\{x,y\}}$ and single-site duality functions given by

$$d_x(k,z) = {}_1F_1\left[\begin{matrix} -k \\ |\boldsymbol{\alpha}|_x \end{matrix}; cz\right], \quad c \in \mathbb{R}\,,$$

and, analogously, for d_y.

5.3 Reversible measures for generalized immediate exchange models

In this section, we discuss a more general strategy to obtain reversible measures for more abstract models with splitting, exchange and addition mechanisms from those of the splitting part of the dynamics only. This scheme exploits the form of the abstract transition operator Π, namely

$$\Pi \;=\; \Gamma^{-1} P \mathscr{E} \Gamma\,, \tag{5.37}$$

and its relation with the splitting operator P. In order to do so, we need to choose the generalized inverse $\Gamma^{-1} : \mathbb{V}_4 \to \mathbb{V}_2$ in a special form. To this purpose, we introduce the notion of μ-canonical generalized inverse of Γ in the following definition.

Definition 5.8 (μ-CANONICAL GENERALIZED INVERSE)**.** *For any measure μ on $\mathbb{N}_0^4$, the operator $\Gamma^{-1} : \mathbb{V}_4 \to \mathbb{V}_2$ in* (5.37) *is said to be μ-*canonical *if, for all $\varphi \in \mathbb{V}_4$,*

$$\Gamma^{-1}\varphi(n_x, n_y) \;:=\; \sum_{\substack{(k_x,\ell_x,k_y,\ell_y)\in\mathbb{N}_0^4\\ k_x+\ell_x=n_x\\ k_y+\ell_y=n_y}} \varphi(k_x,\ell_x,k_y,\ell_y)\cdot\frac{\mu(k_x,\ell_x,k_y,\ell_y)}{\widehat{\mu}(n_x,n_y)}\,,$$

where $\widehat{\mu} := \mu\circ\gamma$ is the image measure of μ under γ.

Remark that saying that Γ^{-1} is μ-canonical means

$$\Gamma^{-1}\varphi(n_x,n_y) \;=\; \mathbb{E}_\mu\left[\varphi(\cdot)\,\middle|\, k_x+\ell_x = n_x,\, k_y+\ell_y = n_y\right]\,. \tag{5.38}$$

We first prove some elementary properties of this version of Γ^{-1}.

Lemma 5.9. *Let $\Gamma : \mathbb{V}_2 \to \mathbb{V}_4$ be the operator in* (5.37) *and Γ^{-1} be μ-canonical for some μ. Then we have:*

(i) $\Gamma^{-1}\Gamma\varphi(n_x,n_y) \;=\; \varphi(n_x,n_y)$ *for* $\varphi\in\mathbb{V}_2$ *and* $(n_x,n_y)\in\mathbb{N}_0^2$.

(ii) $\int \Gamma\varphi\, d\mu \;=\; \int \varphi\, d\widehat{\mu}$ *for* $\varphi\in\mathbb{V}_2$.

(iii) $\int \Gamma^{-1}\varphi\, d\widehat{\mu} \;=\; \int\varphi\, d\mu$ *for* $\varphi\in\mathbb{V}_4$.

(iv) $\Gamma^{-1}(\Gamma\varphi\cdot\phi) \;=\; \varphi\cdot\Gamma^{-1}\phi$ *for* $\varphi\in\mathbb{V}_2$ *and* $\phi\in\mathbb{V}_4$.

Proof. Items (i) and (ii) follow from the definition. For item (iii), by definition of $\gamma : \mathbb{N}_0^4 \to \mathbb{N}_0^2$ and the law of total probability, we get

$$\begin{aligned}
&\int \mathit{\Gamma}_\varphi^{-1} \varphi \, d\widehat{\mu} \\
&\quad = \sum_{(n_x,n_y)\in\mathbb{N}_0^2} \mathbb{E}_\mu \left[\varphi \,\middle|\, k_x + \ell_x = n_x,\, k_y + \ell_y = n_y\right] \widehat{\mu}((n_x,n_y)) \\
&\quad = \sum_{(n_x,n_y)\in\mathbb{N}_0^2} \mathbb{E}_\mu \left[\varphi \,\middle|\, k_x + \ell_x = n_x,\, k_y + \ell_y = n_y\right] \mu(\gamma^{-1}\{(n_x,n_y)\}) \\
&\quad = \mathbb{E}_\mu [\varphi],
\end{aligned}$$

where we remind that

$$\gamma^{-1}(n_x,n_y) \;:=\; \{(k'_x,\ell'_x,k'_y,\ell'_y) :\; k'_x + \ell'_x = n_x,\, k'_y + \ell'_y = n_y\} \subset \mathbb{N}_0^4 \,.$$

For part (iv), for any $\varphi \in \mathbb{V}_2$ and $\phi \in \mathbb{V}_4$ we have

$$\begin{aligned}
&(\mathit{\Gamma}^{-1}(\mathit{\Gamma}\varphi \cdot \phi))\,(n_x,n_y) \\
&\quad = \mathbb{E}_\mu \left[(\mathit{\Gamma}\varphi)\cdot\phi \,\middle|\, k_x + \ell_x = n_x,\, k_y + \ell_y = n_y\right] \\
&\quad = \varphi(n_x,n_y) \cdot \mathbb{E}_\mu \left[\phi \,\middle|\, k_x + \ell_x = n_x,\, k_y + \ell_y = n_y\right] \\
&\quad = \varphi(n_x,n_y) \cdot \mathit{\Gamma}^{-1}\phi(n_x,n_y)\,,
\end{aligned}$$

where this last identity is a consequence of the μ-canonical form of $\mathit{\Gamma}^{-1}$, cf. (5.38). □

We discuss below a condition to recover reversibility of the process $\Pi = \mathit{\Gamma}^{-1}P\mathscr{E}\mathit{\Gamma}$ in terms of the reversible measure for P anytime the generalized inverse $\mathit{\Gamma}^{-1}$ is canonical w.r.t. the reversible measure for P.

Proposition 5.10. *Let μ be an probability measure on $\mathbb{N}_0^4$ invariant under the exchange map $\mathscr{E}$, reversible for the process P, and assume moreover that*

$$\mathit{\Gamma}^{-1}P\mathscr{E}\mathit{\Gamma} \;=\; \mathit{\Gamma}^{-1}\mathscr{E}P\mathit{\Gamma}\,, \tag{5.39}$$

with $\mathit{\Gamma}^{-1}$ being μ-canonical. Then $\widehat{\mu} := \mu \circ \gamma$ is a reversible measure for the operator in (5.39), *i.e.* Π *in* (5.37).

Proof. First note that for all $\varphi, \psi \in \mathbb{V}_4$,

$$\int \varphi\,(\mathscr{E}\psi)\,d\mu \;=\; \int (\mathscr{E}\varphi)\,\psi\,d\mu\,, \tag{5.40}$$

by invariance of μ under $\mathscr{E}$ and since $\mathscr{E}^{-1} = \mathscr{E}$. Therefore, for any $\varphi, \psi \in \mathbb{V}_2$, by Lemma 5.9, we have

$$\begin{aligned}
\int (\Pi\varphi)\,\psi\,d\widehat{\mu} \;&=\; \int (\Gamma^{-1}P\mathscr{E}\Gamma\varphi)\,\psi\,d\widehat{\mu} \\
&\overset{\text{(i)}}{=} \int (\Gamma^{-1}P\mathscr{E}\Gamma\varphi)\,(\Gamma^{-1}T_\gamma\psi)\,d\widehat{\mu} \overset{\text{(iv)}}{=} \int \Gamma^{-1}[(P\mathscr{E}\Gamma\varphi)\,(\Gamma\psi)]\,d\widehat{\mu} \\
&\overset{\text{(iii)}}{=} \int (P\mathscr{E}\Gamma\varphi)\,(\Gamma\psi)\,d\mu \overset{(5.40)}{=} \int (\Gamma\varphi)\,(\mathscr{E}P^*\Gamma\psi)\,d\mu\,.
\end{aligned}$$

By reversibility of P w.r.t. μ, i.e. $P^* = P$, we further get

$$\begin{aligned}
\int (\Pi\varphi)\,\psi\,d\widehat{\mu} \;&=\; \int (\Gamma\varphi)\,(\mathscr{E}P\Gamma\psi)\,d\mu \\
&\overset{\text{(iii)}}{=} \int \Gamma^{-1}[(T_\gamma\varphi)\,(\mathscr{E}P\Gamma\psi)]\,d\widehat{\mu} \overset{\text{(iii)}}{=} \int (\Gamma^{-1}\Gamma\varphi)\,(\Gamma^{-1}\mathscr{E}P\Gamma\psi)\,d\widehat{\mu} \\
&\overset{\text{(i)}}{=} \int \varphi\,(\Gamma^{-1}\mathscr{E}P\Gamma\psi)\,d\widehat{\mu} \overset{(5.39)}{=} \int \varphi\,(\Gamma^{-1}P\mathscr{E}\Gamma\psi)\,d\widehat{\mu} \;=\; \int \varphi\,(\Pi\psi)\,d\widehat{\mu}\,,
\end{aligned}$$

which concludes the proof. □

We conclude this discussion by providing a useful criterion for condition (5.39) to hold. This criterion is the key to obtain reversible measures for any generalized immediate exchange model presented in this chapter. We recall that

$$\Gamma^{-1}\Gamma\varphi \;=\; \varphi \tag{5.41}$$

for all $\varphi \in \mathbb{V}_2$, while, in general,

$$\Gamma\Gamma^{-1}\psi \neq \psi$$

for some $\psi \in \mathbb{V}_4$.

Proposition 5.11. *If the redistribution operator P is such that*

$$P \;=\; \Gamma\Gamma^{-1}\,, \tag{5.42}$$

then condition (5.39) *holds.*

Proof. The proof is straightforward by using (5.42) and (5.41):

$$\begin{aligned} \Gamma^{-1} P \mathscr{E} \Gamma &\overset{(5.42)}{=} \Gamma^{-1} \Gamma \Gamma^{-1} \mathscr{E} \Gamma \\ &\overset{(5.41)}{=} \Gamma^{-1} \mathscr{E} \Gamma \overset{(5.41)}{=} \Gamma^{-1} \mathscr{E} \Gamma \Gamma^{-1} \Gamma \overset{(5.42)}{=} \Gamma^{-1} \mathscr{E} P \Gamma \,. \end{aligned}$$

□

Proof of reversibility of models of Sections 5.1–5.2. In view of Propositions 5.10–5.11 and the definition of the splitting operators P^α in terms of conservative systems thermalizations, we may derive – without the need of carrying out detailed balance computations – all reversible product measures for all transition operators Π^α considered.

Indeed, we first observe that the thermalization procedure employed to define the splitting mechanism yields automatically (product) reversible measures

$$\mu_\lambda \;=\; \nu_{x_{\mathfrak{t}},\lambda} \otimes \nu_{x_{\mathfrak{b}},\lambda} \otimes \nu_{y_{\mathfrak{t}},\lambda} \otimes \nu_{y_{\mathfrak{b}},\lambda} \tag{5.43}$$

for the operator P^α. Secondly, the choice $\alpha_{x_{\mathfrak{t}}} = \alpha_{y_{\mathfrak{t}}}$ of the parameters and the product structure of the measures (5.43) ensures the invariance of μ w.r.t. the exchange operator $\mathscr{E}$. Furthermore, it follows from the definitions of the splitting operators P^α in (5.4), (5.10), (5.23), (5.29) and (5.36) that these splitting operators are in the form (5.42) with Γ^{-1} being μ_λ-canonical.

As a consequence of these considerations, Proposition 5.10 applies to all models considered and the reversible measures $\widehat{\mu}_\lambda = \mu_\lambda \circ \gamma$ for Π^α obtained maintain the product form, i.e.

$$\widehat{\mu}_\lambda \;=\; \widehat{\nu}_{x,\lambda} \otimes \widehat{\nu}_{y,\lambda} \;.$$

Moreover, due to the additive structure of $\mathrm{Gamma}_{\mathrm{d}}$, Binomial, Poisson and Gamma distributions, we get that

$$\nu_{x_{\mathfrak{t}},\lambda} \otimes \nu_{x_{\mathfrak{b}},\lambda} \;=\; \mathrm{Gamma}_{\mathrm{d}}(\alpha_{x_{\mathfrak{t}}}, \lambda) \otimes \mathrm{Gamma}_{\mathrm{d}}(\alpha_{x_{\mathfrak{b}}}, \lambda)\,, \qquad \lambda \in (0,1)\,,$$

yields

$$\widehat{\nu}_{x,\lambda} \;=\; \mathrm{Gamma}_{\mathrm{d}}(\alpha_{x_{\mathfrak{t}}} + \alpha_{x_{\mathfrak{b}}}, \lambda)\,,$$

and, similarly, for $\widehat{\nu}_{y,\lambda}$ and the remaining Binomial, Poisson and Gamma distributions.

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Summary

Within the mathematical statistical physics program of rigorously explaining the emergence of macroscopic phenomena in terms of the underlying microscopic dynamics, stochastic *interacting particle systems* (IPS) play an eminent role. Introduced in the early '70s as simplified stochastic "cartoons" of more realistic and complex microscopic Newtonian deterministic systems, IPS, on the one side, enable an extensive modeling flexibility – the possibility of describing particle interactions and, in turn, macroscopic systems of various nature, e.g. attraction as well as repulsion, independence as well as "non-linear" dependence. On the other side, they bear a significant reduction of the set of assumptions needed to investigate scaling limits, especially if compared to those required when studying more realistic Hamiltonian systems.

Interacting particle systems and, in particular, a subclass which we call *conservative factorized symmetric* IPS are the main objects of this thesis. More specifically, the first part is entirely dedicated to the derivation of the solution to a linear heat (or diffusion) equation from an underlying microscopic system modeled as a symmetric simple exclusion process in presence of dynamic random conductances. The second part focuses on *duality* and *self-duality*, useful mathematical tools in the context of Markov processes and, in particular, of IPS, as they typically reduce the study of observables of complicated processes to that of quantities of more tractable processes.

Duality plays a crucial role already in Chapter 2, in which we prove the hydrodynamic behavior in path space for the symmetric simple exclusion process in $\mathbb{Z}^d$ evolving on uniformly bounded time-dependent conductances. To achieve this result – the precise statement may be found in Theorem 2.3 – we require essentially two assumptions: firstly, the association of the initial conditions to a macroscopic profile, namely that the initial empirical density of particles "approximates in probability" a suitable macroscopic density; sec-

ondly, that the random walk evolving in the same dynamic environment diffusively rescales to a Brownian motion. Duality – which may be recognized in Sections 2.3.1–2.3.2 and, in particular, in Proposition 2.8 – ensures the connection between the evolution of the average empirical density of particles and the expected position of the above-mentioned random walk.

This approach via duality in the study of hydrodynamic limits – although it may not be considered as general and standard as other methods such as those referred to as "entropy" and "relative entropy" methods (cf. e.g. [85]) – has found several applications e.g. in [31] in the context of Glauber + Kawasaki dynamics as well as in e.g. [42], [108] for symmetric exclusion processes in an environment generated by static inhomogeneous conductances.

The main contribution presented in the chapter consists, first of all, in generalizing from the static to the dynamic context the techniques in [108], [42] to prove the hydrodynamic limit at fixed macroscopic times, i.e. the hydrodynamic behavior for finite-dimensional distributions. This extension consists, essentially, in observing that a form of duality still holds between the occupation variables of the particle system and suitable backward random walks and, additionally, in noting that, due to the Feller property of these random walks, an invariance principle for the random walks yields a convergence of both forward and backward semigroups uniformly on bounded intervals of time. This is, in brief, the content of Section 2.4.1.

Next, we push this convergence from convergence of finite-dimensional distributions to convergence of the trajectories by proving pathwise tightness of the empirical density fields. Indeed, while the decomposition (2.10) of the empirical density fields is a key step in our proof as it yields a "closed equation" for the empirical density fields, as a drawback this decomposition is incompatible with the classical Aldous-Rebolledo tightness criterion (see e.g. [85]), the latter being well-suited for semimartingale decomposition as in (2.9). Therefore, we develop a tightness criterion based on the notion of *uniform conditional stochastic continuity* of a process introduced in [137], which we show to apply to symmetric simple exclusion processes in $\mathbb{Z}^d$ equipped with uniformly bounded dynamic conductances. We present the tightness criterion in its general form in Section 2.c and apply it to our case in Section 2.4.2. We emphasize that the uniform convergence w.r.t. time of the random walk semigroups to those of Brownian motion plays in this criterion a crucial role.

The second part of the thesis deals with the problem of finding and characterizing dual Markov processes and duality functions. More in detail, in

Chapter 3 the focus is on conservative factorized symmetric IPS and duality functions in a special form, namely on duality functions which "jointly (=w.r.t. both – original and dual – configurations) factorize" over the sites (cf. Section 3.1). We refer to the factors of these jointly factorized duality functions as *single-site duality functions*. Particle systems such as symmetric exclusion, zero-range and symmetric inclusion processes belong to this class of IPS, while duality and self-duality functions in this form include those studied in e.g. [98, Chapter VIII] for the symmetric exclusion process (SEP), [31, §2.9.2] for independent random walkers (IRW) and [20], [62] for the symmetric inclusion process (SIP).

The first result of the chapter, to be found in Theorem 3.3, is a characterization result: within the class of conservative factorized symmetric IPS, "nontrivial" jointly factorized self-duality may be expected only for SEP, IRW and SIP. Furthermore, from this result we also obtain the most general expression of the "first" single-site self-duality function for these three particle systems.

As a second step, in Section 3.2, we study a general relation (3.23) between jointly factorized duality functions and stationary product measures for the original process. We use this relation for two purposes. First, in Section 3.2.3, as a criterion to determine whether ergodic measures for systems with jointly factorized duality are in product form. Then, in Section 3.3, as a method to construct all single-site (self-)duality functions from the knowledge of the first one only.

We combine this method with the explicit expression of the first single-site self-duality function found in Theorem 3.3 to recover all possible jointly factorized self-duality functions for SEP, IRW and SIP, as well as jointly factorized duality functions with their associated (possibly improper) interacting diffusions (see e.g. Section 3.1.6, where these interacting diffusions are introduced). We note that all jointly factorized (self-)duality functions belong either to the class of what we call "classical" duality (self-)functions – well-known in literature (see e.g. [62]) – or to the class of the so-called "orthogonal" (self-)duality functions – which we express as products of suitable hypergeometric functions – recently obtained also in [18] and [55]. We remark that, at this stage, we have recovered only "candidate" (self-)duality functions, as they have all been derived from the "candidate" first single-site self-duality functions of Theorem 3.3.

Building up on similar relations, we observe that these three particle systems and associated (im)proper interacting diffusions are *intertwined* by means of products of Poisson distributions (Proposition 3.20) and its inverse intertwiner (Proposition 3.22). As intertwining operators acting on duality functions yield other duality functions (Theorem 4.15 as well as Section 3.b for an application), self-duality relations for particle systems are transferred – via the Poissonan intertwining operator – to duality relations with their associated continuum processes and, further, to self-duality relations for the interacting diffusions themselves, while – via the inverse intertwiner – we get the inverse chain of implications. Moreover, due to the factorized form of both intertwiners and (self-)duality functions, the jointly factorized form of (self-)duality functions constructed in this way is preserved.

As a consequence, the action of all these intertwining relations may be checked site by site, reducing the construction of the (self-)duality functions to the determination of (exponential) generating functions of well-known hypergeometric functions. This program is explained via tables in Section 3.4.4. This procedure is then concluded if we ensure that at least one of – and, consequently, all – these "candidate" (self-)duality functions is an actual (self-)duality function. This final check is the content of Proposition 3.24, where we prove self-duality for the above-mentioned (im)proper interacting diffusions; this computation involves only first and second derivatives of simple hypergeometric functions.

The chapter ends introducing a suitable class of conservative factorized *inhomogeneous* IPS, which become central in the subsequent chapters and for which an analogous program of derivation of jointly factorized (self-)duality functions applies.

In Chapter 4, we relate duality relations for finite Markov processes to spectral properties of the corresponding generators. In particular, if two Markov generators share an eigenvalue, then the function constructed from the product of the two generators' eigenfunctions associated to that common eigenvalue is a duality function. From this linear algebraic consideration, we further explore the connection between special instances of (self-)duality, such as those involving "cheap" or "orthogonal" (self-)duality functions, with the structure of eigenfunctions of Markov generators for reversible (self-)dual Markov processes in Sections 4.1–4.2. In Section 4.3, we further extend this connection to the non-reversible case by means of the Jordan canonical form.

We employ this spectral point of view on duality to address two specific problems. First, in Theorem 4.17, we find a characterization of the so-called Siegmund duality [128] in terms of structural properties of the eigenfunctions of the Markov generators involved and their transpose. Then, in Section 4.5, we address the problem of self-duality for (finite) conservative particle systems – where by "self-duality for conservative particle systems" we now, more specifically, mean "duality between systems with different numbers of particles", cf. Section 1.2 in Chapter 1 for an organic discussion on this notion.

In particular, we focus on the class of conservative factorized IPS previously considered and, even though we have already characterized those particle systems which admit a "non-trivial" jointly factorized self-duality relation in Chapter 3, other instances of self-duality – with self-duality functions in a different form – may occur.

In the reversible case, an answer to this problem – in view of the characterization of duality provided by the Jordan canonical decomposition of the generators – is offered by the study of spectra of Markov generators. Hence, while in Section 4.5.1 we show by means of intertwiners between systems with different numbers of particles that SEP, IRW and SIP are self-dual also in this "spectral" sense, in Section 4.5.2, by direct inspection of the spectra of Markov generators, we prove – or disprove, in some other cases – spectral self-duality for some simplified concrete examples of conservative particles systems – including zero-range processes – evolving on two sites only.

The thesis ends, in Chapter 5, with an extension of the class of particle systems with duality studied in the previous chapters. Inspired by recent developments in models of heat conduction and mass transport [64], [86], [109], as well as in econophysics [24], we construct out of the wealth distribution model introduced in [73] – and later studied in [65], [83] – a wider class of "wealth" (or, interchangeably, "mass" or "energy") splitting and exchange models based on the so-called "instantaneous thermalization" [20], [62] of particle systems. From this connection with particle systems, we recover for these "immediate exchange models" reversible measures, symmetries and jointly factorized duality and self-duality functions compatible with the dynamics mechanisms of splitting, exchange and addition of wealth.

Samenvatting[1]

Binnen het programma van mathematische statistische fysica om op een rigoureuze manier te verklaren hoe mascroscopische phenomenen ontstaan uit microscopische dynamica, spelen *interacterende deeltjessystemen* (IPS) een voorname rol. IPS, geïntroduceerd in de vroege jaren '70 als gesimplificeerde stochastische "cartoons" van realistischere en complexere microscopische Newtoniaanse deterministische systemen, geven een uitgebreide flexibiliteit in modelleren – de mogelijkheid om interactie tussen deeltjes te beschrijven, en, op hun beurt, macroscopische systemen van verscheiden aard, bijvoorbeeld zowel aantrekking als afstoting en zowel onafhankelijkheid als "niet-lineaire" afhankelijkheid. Aan de andere kant geven ze een significante vermindering van het aantal benodigde aannames om schalingslimieten te onderzoeken, in het bijzonder als we het vergelijken met de aannames die nodig zijn voor het bestuderen van realistischere Hamiltoniaanse systemen.

Interacterende deeltjessystemen en, in het bijzonder, een subklasse die we *conservatieve gefactoriseerde symmetrische* IPS noemen, zijn de voornaamste onderzoeksobjecten van dit proefschrift. Specifieker is het eerste deel volledig gewijd aan het afleiden van de oplossing van een lineaire warmtevergelijking (of diffusievergelijking) vanuit een onderliggend microscopisch systeem dat gemodelleerd wordt door het symmetrische exclusieproces in de aanwezigheid van dynamische toevalsgeleidingen. Het tweede deel is gericht op *dualiteit* en *zelfdualiteit*, een nuttig wiskundig gereedschap in de context van Markovprocessen en, in het bijzonder, van IPS, aangezien het typisch het bestuderen van observabelen van gecompliceerde processes reduceert tot grootheden van meer handelbare processen.

Dualiteit speelt een cruciale rol in Hoofdstuk 2, waarin we het hydrodynamische gedrag in padenruimte van het symmetrische exclusieproces in $\mathbb{Z}^d$ met

[1] Translation from English to Dutch by Bart van Ginkel.

uniform begrensde tijdsafhankelijke geleidingen aantonen. Om dit resultaat te bereiken – de precieze formulering staat in Stelling 2.3 – hebben we essentieel twee aannames nodig: ten eerste de verbinding van de beginvoorwaarden met een macroscopisch profiel, namelijk dat de initiële empirische deeltjesdichtheden "in kans"een geschikte macroscopische dichtheid "benaderen"; ten tweede dat de toevalswandeling in dezelfde dynamische omgeving met een diffusieve schaling geschaald kan worden tot Brownse beweging. Zoals te herkennen is in Hoofdstuk 2.3.1–2.3.2 en, in het bijzonder, in Propositie 2.8, zorgt dualiteit voor de verbinding tussen de evolutie van de gemiddelde empirische dichtheid van de deeltjes en de verwachte positie van de hiervoor genoemde toevalswandeling.

Deze aanpak via dualiteit in de studie van hydrodynamische limieten – hoewel het als minder algemeen en standaard beschouwd kan worden dan andere methodes zoals de zogenaamde "entropie"- en "relatieve entropie"-methodes (zie bijv. [85]) – heeft meerdere toepassingen gevonden, bijvoorbeeld in [31] in de context van Glauber+Kawasaki dynamica en bijvoorbeeld in [42], [108] voor symmetrische exclusieprocessen in een omgeving die gegenereerd wordt door statische inhomogene geleidingen.

De voornaamste bijdrage die wordt gepresenteerd in het hoofdstuk bestaat in de eerste plaats uit het generaliseren van de technieken in [42], [108] om hydrodynamische limieten te bewijzen op vaste macroscopische tijden van de statische naar de dynamische context. Deze uitbreiding bestaat essentieel uit de observatie dat er nog steeds een vorm van dualiteit geldt tussen de bezettingsvariabelen van het deeltjessysteem en geschikte achterwaartse toevalswandelingen en, daarbij, uit het opmerken dat door de Fellereigenschap van deze toevalswandelingen een invariantieprincipe voor de toevalswandelingen de convergentie levert van zowel de voorwaartse als de achterwaartse halfgroepen, uniform op begrensde tijdsintervallen. Dit is in het kort de inhoud van Sectie 2.4.1.

Hierna verbeteren we de convergentie van eindig dimensionale verdelingen naar convergentie van de paden door de padsgewijze tightness van de empirische dichtheidsvelden aan te tonen. Hoewel de decompositie (2.10) van de empirische dichtheidsvelden een belangrijke stap vormt in ons bewijs, aangezien het een "gesloten vergelijking" voor de empirische dichtheidsvelden levert, heeft het als nadeel dat het niet verenigbaar is met het klassieke Aldous-Rebolledo tightnesscriterium (zie bijv. [85]) dat geschikt is voor semimartingaaldecomposities zoals in (2.9). Daarom ontwikkelen we een tightnesscriterium dat is gebaseerd op *uniforme conditionele stochastische continuïteit* van een proces, geïntroduceerd in [137], waarvoor we laten zien dat het toegepast kan worden op symmetrische exclusieprocessen in $\mathbb{Z}^d$ met uniform begrensde dynamische geleidingen. We presenteren het tightnesscriterium in zijn algemene vorm in Sectie 2.c and passen het toe op ons geval in Sectie 2.4.2. We benadrukken dat de uniforme convergentie met betrekking tot de tijd van halfgroepen van toevalswandelingen naar de halfgroep van Brownse beweging een cruciale rol speelt in dit criterium.

Het tweede deel van dit proefschrift behandelt het probleem van het vinden en karakteriseren van duale Markovprocessen en dualiteitsfuncties. Hoofdstuk 3 is gericht op conservatieve gefactoriseerde symmetrische IPS en dualiteitsfuncties met een speciale vorm, namelijk dualiteitsfuncties die "gemeenschappelijk factoriseren" (d.w.z. tegelijk de originele en de duale configuraties) over de posities (zie Sectie 3.1.2). We noemen de factoren van deze gemeenschappelijk gefactoriseerde dualiteitsfunctie *enkele-positie dualiteitsfuncties*. Deeltjessystemen zoals symmetrische exclusieprocessen, zerorangeprocessen en symmetrische inclusieprocessen behoren tot deze klasse van IPS, terwijl dualiteits- en zelfsdualiteitsfuncties in deze vorm de dualiteitsfuncties bevatten die bestudeerd worden in bijvoorbeeld [98, Hoofdstuk VIII] voor het symmetrische exclusieproces (SEP), [31, §2.9.2] voor onafhankelijke toevalswandelingen (IRW) en [20], [62] voor het symmetrische inclusieproces (SIP).

Het eerste resultaat van het hoofdstuk, beschreven in Stelling 3.3, is een karakteriseringsresultaat: binnen de klasse van conservatieve gefactoriseerde symmetrische IPS kunnen alleen "niet-triviale" gemeenschappelijk gefactoriseerde dualiteitsfuncties verwacht worden voor SEP, IRW en SEP. Uit dit resultaat kunnen we verder de meest algemene vorm afleiden van de "eerste" enkele-positie zelfdualiteitsfunctie voor deze drie deeltjessystemen.

Als tweede stap bestuderen we in Sectie 3.2 een algemene relatie (3.23) tussen gemeenschappelijk gefactoriseerde dualiteitsfuncties en stationaire productmaten van het oorspronkelijke proces. We gebruiken deze relaties voor twee doelen. Ten eerste gebruiken we het in Sectie 3.2.3 als criterium om te bepalen of ergodische maten voor systemen met gemeenschappelijk gefactoriseerde dualiteit een productvorm hebben. Vervolgens gebruiken we het in Sectie 3.3 als methode om alle enkele-positie zelfdualiteitsfuncties te construeren als we alleen de eerste kennen.

We combineren deze methode met de expliciete uitdrukking van de eerste enkele-positie zelfdualiteitsfunctie die we hebben gevonden in Stelling 3.3

om alle mogelijke gemeenschappelijk gefactoriseerde dualiteitsfuncties met de daaraan geassocieerde (mogelijk oneigenlijke) interacterende diffusies te vinden (zie bijv. Sectie 3.1.6 waar deze interacterende diffusies worden geïntroduceerd). We merken op dat alle gemeenschappelijk gefactoriseerde (zelf-)dualiteitsfuncties ofwel behoren tot de klasse van wat we “klassieke” (zelf-)dualiteitsfuncties noemen – bekend in de literatuur (zie bijv. [62]) – ofwel tot de klasse van de zogenaamde “orthogonale” (zelf-)dualiteitsfuncties – die we uitdrukken als producten van geschikte hypergeometrische functies – die recent ook gevonden zijn in [18] en [55]. Verder merken we op dat we in dit stadium alleen “kandidaat” (zelf-)dualiteitsfuncties hebben gevonden, aangezien ze allemaal zijn afgeleid van de “kandidaat” eerste enkele-positie zelfdualiteitsfuncties van Stelling 3.3.

Verder bouwend op gelijksoortige relaties, observeren we dat deze drie deeltjessystemen en de aan hen geassocieerde (on)eigenlijke diffusies *intertwined* zijn door producten van Poisson verdelingen (Propositie 3.20) en zijn inverse intertwiner (Propositie 3.22). Aangezien intertwined operatoren die werken op dualiteitsfuncties andere dualiteitsfuncties leveren (zie Stelling 4.15 en Sectie 3.b voor een toepassing), worden zelfdualiteitsrelaties voor deeltjessystemen overgedragen – via de Possion intertwining operator – naar dualiteitsrelaties met de aan hen geassocieerde continuumprocessen en, verder, naar zelfsdualiteitsrelaties voor de interacterende diffusies zelf, terwijl we – via de inverse intertwiner – de inverse keten van implicaties verkrijgen. Bovendien wordt door de gefactoriseerde vorm van zowel de intertwiners als de (zelf-)dualiteitsfuncties de gemeenschappelijk gefactoriseerde vorm behouden van de (zelf-)dualiteitsfuncties die op deze wijze worden geconstrueerd.

Als gevolg hiervan kan de werking van al deze intertwining relaties positiegewijs worden nagegaan, wat de constructie van (zelf-)dualiteitsfuncties reduceert to het bepalen van (exponentiële) genererende functies van bekende hypergeometrische functies. Dit programma wordt uitgelegd via de tabellen in Sectie 3.4.4. Deze procedure kan voltooid worden door na te gaan dat één van deze “kandidaat” (zelf-)dualiteitsfuncties (en daardoor allemaal) daadwerkelijk een (zelf-)dualiteitsfunctie is. Deze laatste check is de inhoud van Propositie 3.24, waar we zelfdualiteit aantonen voor de hiervoor genoemde (on)eigenlijke diffusies; deze berekening gebruikt alleen de eerste en tweede afgeleides van simpele hypergeometrische functies.

Het hoofdstuk eindigt met het introduceren van een geschikte klasse van conservatieve gefactoriseerde *inhomogene* IPS die centraal komen te staan in de volgende hoofdstukken en waarvoor een analoog programma voor het afleiden van gemeenschappelijk gefactoriseerde (zelf-)dualiteitsfuncties kan worden toegepast.

In Hoofdstuk 4 relateren we dualiteitsrelaties voor eindige Markovprocessen aan spectrale eigenschappen van de corresponderende generatoren. In het bijzonder, als twee Markovprocessen eenzelfde eigenwaarde hebben, dan is de functie die wordt geconstrueerd uit het product van de bij de eigenwaarde behorende eigenfuncties van de generatoren een dualiteitsfunctie. Vanuit deze lineaire algebraïsche overwegingen gaan we verder met het verkennen van de connectie tussen speciale instanties van (zelf-)dualiteit, zoals die met "goedkope" of "orthogonale" (zelf-)dualiteitsfuncties, en de structuur van eigenfuncties van Markovgeneratoren van reversibele (zelf-)duale Markovprocessen in Secties 4.1–4.2. In Sectie 4.3 breiden we deze connectie verder uit naar het niet-reversibele geval met behulp van de Jordan-normaalvorm.

We gebruiken dit spectrale perspectief op dualiteit voor het benaderen van twee specifieke problemen. Ten eerste vinden we in Stelling 4.17 een karakterisering van de zogenaamde Siegmunddualiteit [128], uitgedrukt in structurele eigenschappen van de eigenfuncties van de betrokken Markovgeneratoren en hun getransponeerden. Vervolgens richten we ons in Sectie 4.5 op het probleem van zelfdualiteit voor (eindige) conservatieve deeltjessystemen, waar we met "zelfdualiteit voor conservatieve deeltjessystemen" "dualiteit tussen systemen met verschillende aantallen deeltjes" bedoelen, zie Sectie 1.2 in Hoofdstuk 1 voor een organische bespreking van deze notie.

In het bijzonder richten we ons op de klasse van de eerder besproken conservatieve gefactoriseerde IPS en kunnen er, ondanks dat we de deeltjessystemen die een "niet-triviale" gemeenschappelijk gefactoriseerde zelfdualiteitsrelatie hebben al hebben gekarakteriseerd in Hoofdstuk 3, andere vormen van zelfdualiteit – met zelfdualiteitsfuncties in een andere vorm – voorkomen.

In het reversibele geval wordt – gezien de karakterisering van dualiteit door de Jordan-normaalvorm van de generatoren – een antwoord op dit probleem gegeven door het bestuderen van de spectra van Markovgeneratoren. Terwijl we in Sectie 4.5.1 met behulp van intertwiners tussen systemen met verschillende aantallen deeltjes laten zien dat SEP, IRW en SIP zelfduaal zijn in deze spectrale zin, bewijzen – of in andere gevallen weerleggen – we in Sectie 4.5.2 door directe inspectie van de spectra van Markovgeneratoren spectrale zelfdualiteit voor een aantal vereenvoudigde concrete voorbeelden van conservatieve deeltjessystemen – inclusief zero-rangeprocessen – die zich op slechts twee posities begeven.

Het proefschrift sluit in Hoofdstuk 5 af met een uitbreiding van de klasse van de deeltjessystemen met dualiteit die in de voorgaande hoofdstukken zijn bestudeerd. Geïnspireerd door recente ontwikkelingen in modellen van zowel warmtegeleiding en massatransport [64], [86], [109] als econophysica [24], construeren we uit het welvaartverdelingsmodel dat in [73] is geïntroduceerd – en dat later werd bestudeerd in [65], [83] – een grotere klasse van modellen van splitsing en uitwisseling van "welvaart" (dit kan ook "massa" of "energie" genoemd worden) gebaseerd op zogenaamde "instantane thermalisatie" [20], [62] van deeltjessystemen. Uit deze connectie met deeltjessystemen vinden we voor deze "directe uitwisselingsmodellen" reversibele maten, symmetrieën en gemeenschappelijk gefactoriseerde dualiteits- en zelfdualiteitsfuncties die corresponderen met de dynamische mechanismen van splitsing, uitwisseling en toename van welvaart.

Riassunto[2]

I *sistemi* stocastici *di particelle interagenti* (IPS) assumono un ruolo centrale in quel ramo della matematica che studia – nel contesto della fisica statistica – l'emergenza di fenomeni macroscopici da complesse dinamiche microscopiche sottostanti. Introdotti all'inizio del 1970, come caricature stocastiche di sistemi deterministici Newtoniani più realistici e complessi, IPS, da un lato, consentono una flessibilità modellistica maggiore, permettendo la descrizione di interazioni di natura diversa come, ad esempio, interazioni di tipo attrattivo o repulsivo e di indipendenza o di dipendenza "non lineare". Dall'altro, con l'utilizzo di IPS si ha un notevole alleggerimento delle assunzioni richieste per derivare limiti a grandi scale, specialmente se comparate a quelle utilizzate nel contesto di sistemi Hamiltoniani più realistici.

I sistemi di particelle interagenti – e, più specificamente, una sottoclasse di sistemi *conservativi fattorizzati simmetrici* – occupano un posto centrale in questa tesi. La prima parte della tesi è interamente dedicata all'approssimazione della soluzione dell'equazione lineare del calore (o di diffusione), partendo da un sistema microscopico sottostante modellizzato da un processo di esclusione semplice simmetrico in presenza di conduttanze dinamiche. La seconda parte si concentra su *dualità* e *autodualità*, strumenti matematici utilizzati nel contesto di processi di Markov ed, in particolare, di IPS, con l'obiettivo di ridurre lo studio di osservabili di processi complessi a osservabili di processi più facilmente trattabili.

La dualità entra in gioco fin da subito, al Capitolo 2, nel quale si determina il comportamento idrodinamico – a livello delle traiettorie – per il processo simmetrico di esclusione semplice in $\mathbb{Z}^d$ con evoluzione governata da conduttanze uniformemente limitate e dipendenti dal tempo. Tale risultato è conseguito – per l'enunciato preciso, si faccia riferimento al Teorema 2.3 – per

[2] Translated from English to Italian by Giulia Buriola.

mezzo di due assunzioni: in primo luogo, il fatto che le condizioni iniziali del sistema microscopico siano "associate" ad un profilo macroscopico; in altre parole, che la densità empirica iniziale delle particelle "approssimi in probabilità" una certa densità macroscopica. In secondo luogo, che la camminata aleatoria in movimento sullo stesso ambiente dinamico su cui il sistema di particelle evolve, se riscalata in maniera diffusiva, converga ad un moto Browniano. La dualità – che può essere riconosciuta nella Sezioni 2.3.1–2.3.2 ed, in particolare, in Proposizione 2.8 – è responsabile della connessione tra l'evoluzione della densità empirica media di particelle e la posizione media della passeggiata aleatoria sopra menzionata.

Questo approccio tramite dualità nello studio di limiti idrodinamici, benché non possa essere considerato generale e standard al pari di altri metodi (come quelli cosiddetti "di entropia" e "di entropia relativa" [85], per citarne alcuni), è stato più volte adottato in diversi scenari; ad esempio, in [31] per un processo con dinamiche di tipo Glauber+Kawasaki, oppure in e.g. [42], [108] per un processo simmetrico di esclusione semplice in un ambiente costituito da conduttanze inomogenee e statiche.

Il principale contributo all'interno del capitolo consiste, innanzitutto, nel generalizzare dal caso statico a quello dinamico le tecniche adottate in [108], [42] al fine di ottenere il limite idrodinamico per tempi macroscopici fissati, vale a dire il comportamento idrodinamico delle distribuzioni finito-dimensionali. Questa estensione si basa su due elementi: sull'osservazione che una forma di dualità persiste anche in questo contesto, collegando le variabili di occupazione del processo di particelle a delle passeggiate aleatorie che evolvono all'indietro nel tempo, e sul fatto che, essendo tali passeggiate aleatorie processi di Feller, un principio di invarianza per le passeggiate aleatorie induce la convergenza – uniforme su intervalli limitati di tempo – dei semigruppi associati sia alle passeggiate aleatorie in avanti sia a quelle all'indietro. A questo passaggio è dedicata la Sezione 2.4.1.

Inoltre, ci occupiamo di estendere tale convergenza delle distribuzioni finito-dimensionali alle traiettorie, dimostrando la proprietà di *tightness* per le traiettorie dei campi di densità empirica. Tale dimostrazione non si può basare su criteri di *tightness* già noti. Infatti, mentre la decomposizione (2.10) per le misure empiriche è chiave nella derivazione del limite idrodinamico – dal momento che permette di ottenere un'equazione "chiusa" per le misure empiriche –, tale decomposizione risulta essere incompatibile con il più classico criterio di Aldous-Rebolledo (e.g. [85]), essendo quest'ultimo particolarmente adatto a processi di semimartingala nella forma (2.9). Perciò, sviluppiamo un criterio di *tightness* basato sulla nozione di *continuità stocastica condizionata uniforme* di un processo – nozione introdotta in [137] – che viene applicato al processo simmetrico di esclusione semplice in $\mathbb{Z}^d$ con conduttanze dinamiche e uniformemente limitate. Il criterio di *tightness* è presentato nella sua forma più generale nella Sezione 2.c, mentre la sua applicazione è riportata nella Sezione 2.4.2. La convergenza uniforme rispetto al tempo dei semigruppi associati alle passeggiate aleatorie al semigruppo associato al moto Browniano gioca un ruolo cruciale nell'applicazione di tale criterio.

La seconda parte della tesi si occupa del problema di trovare e caratterizzare processi di Markov duali e le associate funzioni di dualità. Nel Capitolo 3, il focus verte su IPS conservativi fattorizzati simmetrici e funzioni di dualità esprimibili in una particolare forma, cosiddetta "congiuntamente (=rispetto ad entrambe le configurazioni, quelle del processo originario ed il suo duale) fattorizate" sui siti (cf. Sezione 3.1). Chiameremo i fattori all'interno di queste funzioni di dualità congiuntamente fattorizate *funzioni di dualità del singolo sito*. Processi come quello simmetrico di esclusione, a raggio nullo, e simmetrico di inclusione appartengono a tale classe di IPS, mentre le funzioni di dualità e di autodualità in questa forma includono funzioni già apparse e.g. in [98, Chapter VIII] per il processo di esclusione (SEP), in [31, §2.9.2] per sistemi di passeggiate aleatorie indipendenti (IRW), e in [20], [62] per il processo simmetrico di inclusione (SIP).

Il primo risultato del capitolo, riportato nel Teorema 3.3, è un risultato di caratterizzazione: all'interno della classe di IPS conservativi fattorizzati simmetrici, autodualità congiuntamente fattorizzate "non banali" possono insorgere solo nel caso di processi di tipo SEP, IRW, e SIP. Inoltre, dallo stesso risultato, deduciamo anche la forma più generale che le "prime" autofunzioni di dualità del singolo sito possono assumere in questi tre casi.

Inoltre, nella Sezione 3.2, prendiamo in considerazione una relazione generale (3.23) che vige tra le funzioni di dualità congiuntamente fattorizzate e le misure prodotto stazionarie per il processo originario. Utilizziamo tale relazione con due scopi precisi. In primo luogo, come criterio per determinare se le misure ergodiche di un sistema avente dualità congiuntamente fattorizzate sono prodotto (cf. Sezione 3.2.3). In secondo luogo, come metodo per construire tutte le funzioni di (auto)dualità del singolo sito, partendo dalla conoscenza della "prima" di tali funzioni (cf. Sezione 3.3).

Combinando questo metodo con l'espressione esplicita della prima funzione di autodualità trovata precedentemente nel Teorema 3.3, recuperiamo tutte le possibili funzioni congiuntamente fattorizzate per SEP, IRW, e SIP, oltre alle funzioni di dualità congiuntamente fattorizzate con i loro associati processi di diffusione (possibilmente impropri) interagenti, introdotti nella Sezione 3.1.6. Successivamente, notiamo che tutte le funzioni di (auto)dualità congiuntamente fattorizzate appartengono o alla classe di funzioni di autodualità che chiamiamo “classiche” – già note nella letteratura (e.g. [62]) – o alla classe delle cosiddette funzioni di autodualità “ortogonali”– le quali sono espresse in termini di appropriate funzioni ipergeometriche – recentemente ottenute in [18] e [55]. Osserviamo che, a questo punto, le funzioni ottenute sono solo delle possibili “candidate”funzioni di (auto)dualità, dal momento che esse sono state derivate dall’altrettanto “candidata” funzione di (auto)dualità ottenuta nel Teorema 3.3.

Nello sviluppare relazioni simili a quella studiata in (3.23), osserviamo che questi tre processi di particelle e i loro associati processi di diffusione interagenti (im)propri sono *intrecciati* tramite dei prodotti di distribuzioni di Poisson (Proposizione 3.20) e il suo intrecciatore inverso (Proposizione 3.22). Dal momento che, generalmente, gli operatori di intrecciamento, agendo su funzioni di dualità, producono altre funzioni di dualità (si veda Teorema 4.15, ma anche l’applicazione presentata in Sezione 3.b), le relazioni di autodualità per i sistemi di particelle possono essere trasferite – tramite gli intrecciatori Poissoniani – in relazioni di dualità con i processi di diffusione interagenti associati e, in aggiunta, in relazioni di autodualità per gli stessi processi di diffusione. Analogamente, tramite gli intrecciatori inversi, possiamo invertire questa catena di implicazioni. Oltretutto, grazie alla forma fattorizzata di entrambi gli intrecciatori e delle funzioni di (auto)dualità, la forma congiuntamente fattorizzata delle funzioni di (auto)dualità così costruite viene preservata in questa procedura.

Come conseguenza, l’azione di tutte queste relazioni di intrecciamento può essere studiata sito per sito, riducendo la costruzione di funzioni di (auto)dualità alla determinazione di funzioni generatrici (esponenziali) di note funzioni ipergeometriche. Questo procedimento è presentato, con l’aiuto di tabelle, nella Sezione 3.4.4 e termina con la dimostrazione che almeno una – e, quindi, tutte – le “candidate” funzioni di (auto)dualità sono effettivamente una funzione di (auto)dualità. Questa verifica è contenuta nella Proposizione 3.24, in cui la relazione di autodualità che dimostriamo direttamente è quella a livello dei processi (im)propri di diffusione interagenti sopra menzionati; in questo caso, i dettagli tecnici constano solo di derivate prime e seconde di semplici funzioni ipergeometriche.

Il capitolo si conclude introducendo un'appropriata classe di IPS inomogenei, che diventerà centrale nei capitoli successivi. Si mostrerà, inoltre, come sia possibile applicare a questa classe un procedimento analogo a quello sopra presentato, al fine di ottenere funzioni di (auto)dualità congiuntamente fattorizzate.

Nel Capitolo 4, colleghiamo relazioni di dualità per processi di Markov finiti alle proprietà spettrali dei generatori corrispondenti. In particolare, se due generatori associati a due processi di Markov posseggono un autovalore in comune, allora la funzione costruita dal prodotto delle rispettive autofunzioni è una funzione di dualità. A partire da questa considerazione di natura algebrico-lineare, nelle Sezioni 4.1–4.2, esploriamo ulteriormente questa connessione in situazioni topiche di (auto)dualità – come nel caso di funzioni di (auto)dualità "senza sforzo" o "ortogonali"– investigando la struttura delle autofunzioni di generatori associati a processi di Markov reversibili (auto)duali. Nella Sezione 4.3, estendiamo tale connessione al caso non reversibile, passando alla forma canonica di Jordan.

Adottiamo questo punto di vista spettrale sulla dualità per affrontare due problemi specifici. Innanzitutto, nel Teorema 4.17, troviamo una caratterizzazione della cosiddetta dualità di Siegmund [128] a partire dalle proprietà strutturali delle autofunzioni dei generatori di Markov, e i corrispettivi trasposti, coinvolti. Successivamente, nella Sezione 4.5, affrontiamo il problema di autodualità per sistemi conservativi (finiti) di particelle interagenti. Con la frase "autodualità per sistemi conservativi di particelle" intendiamo qui "dualità tra sistemi con un numero diverso di particelle". Si veda la Sezione 1.2, nel Capitolo 1, per una discussione più organica su tale nozione.

Il nostro interesse torna, ora, alla classe di IPS conservativi fattorizzati già considerati in precedenza e, benchè i sistemi di particelle che godono di relazioni di autodualità congiuntamente fattorizzata "non banale" siano già stati caratterizzati nel Capitolo 3, ciò non esclude che altre forme di autodualità possano ancora persistere con funzioni di autodualità in una forma differente.

Nel caso reversibile, una risposta a questo problema – proprio grazie alla caratterizzazione di dualità fornita dalla decomposizione canonica di Jordan dei generatori – è offerta dallo studio dello spettro dei generatori di Markov. Mentre nella Sezione 4.5.1, grazie all'uso di intrecciatori tra sistemi di particelle con un numero diverso di particelle, mostriamo che i processi di tipo SEP, IRW, e SIP sono autoduali anche in questo senso "spettrale", nella Sezione 4.5.2, attraverso una diretta ispezione dello spettro di generatori di Markov, dimostriamo – oppure confutiamo – autodualità spettrale per alcuni esempi concreti di sistemi conservativi di particelle, che includono processi a raggio nullo, su una geometria estremamente semplificata, ovvero consistente di due soli siti.

La tesi termina, nel Capitolo 5, con un'estensione della classe di sistemi di particelle con dualità considerati nei capitoli precedenti. Ispirati da sviluppi recenti nel campo della modellistica di fenomeni quali la conduzione di calore o il trasporto di massa [64], [86], [109], ma anche nell'econofisica [24], costruiamo dai modelli di distribuzione della ricchezza introdotti in [73] – e successivamente studiati in [65], [83] – una più ampia classe di modelli di divisione e scambio di "ricchezza" (o, intercambiabilmente, "massa" o "energia") basati sulla cosiddetta "termalizzazione istantanea" [20], [62] di sistemi di particelle. Da tale connessione con i sistemi di particelle, recuperiamo per questi "modelli di scambio immediato" misure reversibili, simmetrie e funzioni di dualità e autodualità congiuntamente fattorizzati compatibili con i meccanismi di divisione, scambio e addizione della ricchezza che compongono la dinamica dei suddetti modelli.

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Walking to the office 1.W.540 (and somewhere around it) in van Mourik Broekmanweg 6, Delft, thanks to Bart, Mario, Mikola, Rik, Antonio, Richard and Alessandra for sharing fun moments of mathematics (and more!) and to Eni, Andrea, Martina, Birbal and Bruno for entering – at different stages – as friends in each other’s lives. An incredible thanks go to Evelyn, Dorothee, Stefanie and Carl for your constant kindness, patience and helpfulness. A last and least thanks goes to Simone for being such an energetic daily student/supervisor/friend: “Grande!”

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Curriculum Vitae

Federico Sau was born in Vimercate, in the province of Milan, Italy, on the 5th of August 1991. In 2010, after a year of exchange in Whitefish, Montana, U.S.A., he completed his high-school studies at *Liceo Scientifico "A. Banfi"* in Vimercate - whose province, in the meanwhile, turned into *provincia di Monza e Brianza*.

F.S. acquired Bachelor and Master degrees in Mathematics in 2013 and 2015, respectively, both at the *Università degli Studi di Milano* and both *cum laude*, with a Master thesis entitled "Slow-Fast Stochastic Dynamics and Applications to Diploid Populations with Varying Size", written in Leiden under the joint supervision of Dr. Daniela Morale (*Università degli Studi di Milano*) and Prof. dr. Frank H.J. Redig (*TU Delft*).

In October 2015 F.S. started his Ph.D. research at *TU Delft* under the supervision of Prof. dr. Frank H.J. Redig. His research was financially supported by the Dutch Science Foundation NWO through the TOP-1 project n° 613.001.552 "Large Deviations and Gradient Flows: Beyond Equilibrium" with Prof. dr. Mark A. Peletier (*TU Eindhoven*) and Prof. dr. Frank H.J. Redig (*TU Delft*) as principal investigators.

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