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# Martingale decompositions and weak differential subordination in UMD Banach spaces

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In this paper, we consider Meyer–Yoeurp decompositions for UMD Banach space-valued martingales. Namely, we prove that  $X$  is a UMD Banach space if and only if for any fixed  $p \in (1, \infty)$ , any  $X$ -valued  $L^p$ -martingale  $M$  has a unique decomposition  $M = M^d + M^c$  such that  $M^d$  is a purely discontinuous martingale,  $M^c$  is a continuous martingale,  $M_0^c = 0$  and

$$\mathbb{E}\|M_\infty^d\|^p + \mathbb{E}\|M_\infty^c\|^p \leq c_{p,X}\mathbb{E}\|M_\infty\|^p.$$

An analogous assertion is shown for the Yoeurp decomposition of a purely discontinuous martingales into a sum of a quasi-left continuous martingale and a martingale with accessible jumps.

As an application, we show that  $X$  is a UMD Banach space if and only if for any fixed  $p \in (1, \infty)$  and for all  $X$ -valued martingales  $M$  and  $N$  such that  $N$  is weakly differentially subordinated to  $M$ , one has the estimate  $\mathbb{E}\|N_\infty\|^p \leq C_{p,X}\mathbb{E}\|M_\infty\|^p$ .

*Keywords:* accessible jumps; Brownian representation; Burkholder function; canonical decomposition of martingales; continuous martingales; differential subordination; Meyer–Yoeurp decomposition; purely discontinuous martingales; quasi-left continuous; stochastic integration; UMD Banach spaces; weak differential subordination; Yoeurp decomposition

## 1. Introduction

It is well known from the fundamental paper of Itô [20] on the real-valued case, and several works [1,2,5,13,32] on the vector-valued case, that for any Banach space  $X$ , any centered  $X$ -valued Lévy process has a unique decomposition  $L = W + \tilde{N}$ , where  $W$  is an  $X$ -valued Wiener process, and  $\tilde{N}$  is an  $X$ -valued weak integral with respect to a certain compensated Poisson random measure. Moreover,  $W$  and  $\tilde{N}$  are independent, and therefore since  $W$  is symmetric, for each  $1 < p < \infty$  and  $t \geq 0$ ,

$$\mathbb{E}\|\tilde{N}_t\|^p \leq \mathbb{E}\|L_t\|^p. \tag{1.1}$$

The natural generalization of this result to general martingales in the real-valued setting was provided by Meyer in [29] and Yoeurp in [44]. Namely, it was shown that any real-valued martingale  $M$  can be uniquely decomposed into a sum of two martingales  $M^d$  and  $M^c$  such that  $M^d$  is purely discontinuous (i.e., the quadratic variation  $[M^d]$  has a pure jump version), and  $M^c$  is

continuous with  $M_0^c = 0$ . The reason why they needed such a decomposition is a further decomposition of a semimartingale, and finding an exponent of a semimartingale (we refer the reader to [23] and [44] for the details on this approach). In the present article, we extend Meyer–Yoeurp theorem to the vector-valued setting, and provide extension of (1.1) for a general martingale (see Section 3.1). Namely, we prove that for any UMD Banach space  $X$  and any  $1 < p < \infty$ , an  $X$ -valued  $L^p$ -martingale  $M$  can be uniquely decomposed into a sum of two martingales  $M^d$  and  $M^c$  such that  $M^d$  is purely discontinuous (i.e.,  $\langle M^d, x^* \rangle$  is purely discontinuous for each  $x^* \in X^*$ ), and  $M^c$  is continuous with  $M_0^c = 0$ . Moreover, then for each  $t \geq 0$ ,

$$(\mathbb{E}\|M_t^d\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|M_t^c\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \tag{1.2}$$

where  $\beta_{p,X}$  is the UMD $_p$  constant of  $X$  (see Section 2.1). Theorem 3.33 shows that such a decomposition together with  $L^p$ -estimates of type (1.2) is possible if and only if  $X$  has the UMD property.

The purely discontinuous part can be further decomposed: in [44] Yoeurp proved that any real-valued purely discontinuous  $M^d$  can be uniquely decomposed into a sum of a purely discontinuous quasi-left continuous martingale  $M^q$  (analogous to the ‘‘compensated Poisson part’’, which does not jump at predictable stopping times), and a purely discontinuous martingale with accessible jumps  $M^a$  (analogous to the ‘‘discrete part’’, which jumps only at certain predictable stopping times). In Section 3.2, we extend this result to a UMD space-valued setting with appropriate estimates. Namely, we prove that for each  $1 < p < \infty$  the same type of decomposition is possible and unique for an  $X$ -valued purely discontinuous  $L^p$ -martingale  $M^d$ , and then for each  $t \geq 0$ ,

$$(\mathbb{E}\|M_t^q\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t^d\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|M_t^a\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|M_t^d\|^p)^{\frac{1}{p}}. \tag{1.3}$$

Again as Theorem 3.33 shows, the (1.3)-type estimates are a possible only in UMD Banach spaces.

Even though the Meyer–Yoeurp and Yoeurp decompositions can be easily extended from the real-valued case to a Hilbert space case, the author could not find the corresponding estimates of type (1.2)–(1.3) in the literature, so we wish to present this special issue here. If  $H$  is a Hilbert space,  $M : \mathbb{R}_+ \times \Omega \rightarrow H$  is a martingale, then there exists a unique decomposition of  $M$  into a continuous part  $M^c$ , a purely discontinuous quasi-left continuous part  $M^q$ , and a purely discontinuous part  $M^a$  with accessible jumps. Moreover, then for each  $1 < p < \infty$ , and for  $i = c, q, a$ ,

$$(\mathbb{E}\|M_t^i\|^p)^{\frac{1}{p}} \leq (p^* - 1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \tag{1.4}$$

where  $p^* = \max\{p, \frac{p}{p-1}\}$ . Notice that though (1.4) follows from (1.2)–(1.3) since  $\beta_{p,H} = p^* - 1$ , it can be easily derived from the differential subordination estimates for Hilbert space-valued martingales obtained by Wang in [38].

Both the Meyer–Yoeurp and Yoeurp decompositions play a significant rôle in stochastic integration: if  $M = M^c + M^q + M^a$  is a decomposition of an  $H$ -valued martingale  $M$  into continuous, purely discontinuous quasi-left continuous and purely discontinuous with accessible jumps parts, and if  $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$  is elementary predictable for some UMD Banach space  $X$ ,

then the decomposition  $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$  of a stochastic integral  $\Phi \cdot M$  is a decomposition of the martingale  $\Phi \cdot M$  into continuous, purely discontinuous quasi-left continuous and purely discontinuous with accessible jumps parts, and for any  $1 < p < \infty$  we have that

$$\mathbb{E}\|(\Phi \cdot M)_\infty\|^p \approx_{p,X} \mathbb{E}\|(\Phi \cdot M^c)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^q)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^a)_\infty\|^p.$$

The corresponding Itô isomorphism for  $\Phi \cdot M^c$  for a general UMD Banach space  $X$  was derived by Veraar and the author in [37], while Itô isomorphisms for  $\Phi \cdot M^q$  and  $\Phi \cdot M^a$  have been shown by Dirksen and the author in [14] for the case  $X = L^r(S)$ ,  $1 < r < \infty$ .

The major underlying techniques involved in the proofs of (1.2) and (1.3) are rather different from the original methods of Meyer in [29] and Yoeurp in [44]. They include the results on the differentiability of the Burkholder function of any finite dimensional Banach space, which have been proven recently in [41] and which allow us to use Itô formula in order to show the desired inequalities in the same way as it was demonstrated by Wang in [38].

The main application of the Meyer–Yoeurp decomposition are  $L^p$ -estimates for weakly differentially subordinated martingales. The weak differential subordination property was introduced by the author in [41], and can be described in the following way: an  $X$ -valued martingale  $N$  is weakly differentially subordinated to an  $X$ -valued martingale  $M$  if for each  $x^* \in X^*$  a.s.  $|\langle N_0, x^* \rangle| \leq |\langle M_0, x^* \rangle|$  and for each  $t \geq s \geq 0$

$$[\langle N, x^* \rangle]_t - [\langle N, x^* \rangle]_s \leq [\langle M, x^* \rangle]_t - [\langle M, x^* \rangle]_s.$$

If both  $M$  and  $N$  are purely discontinuous, and if  $X$  is a UMD Banach space, then by [41], for each  $1 < p < \infty$  we have that  $\mathbb{E}\|N_\infty\|^p \leq \beta_{p,X}^p \mathbb{E}\|M_\infty\|^p$ . Section 4 is devoted to the generalization of this result to continuous and general martingales. There we show that if both  $M$  and  $N$  are continuous, then  $\mathbb{E}\|N_\infty\|^p \leq c_{p,X}^p \mathbb{E}\|M_\infty\|^p$ , where the least admissible  $c_{p,X}$  is within the interval  $[\beta_{p,X}, \beta_{p,X}^2]$ . Furthermore, using the Meyer–Yoeurp decomposition and estimates (1.2) we show that for general  $X$ -valued martingales  $M$  and  $N$  such that  $N$  is weakly differentially subordinated to  $M$  the following holds

$$(\mathbb{E}\|N_\infty\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\beta_{p,X} + 1) (\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}.$$

The weak differential subordination as a stronger version of the differential subordination is of interest in Harmonic Analysis. For instance, it was shown in [41] that sharp  $L^p$ -estimates for weakly differentially subordinated purely discontinuous martingales imply sharp estimates for the norms of a broad class of Fourier multipliers on  $L^p(\mathbb{R}^d; X)$ . Also there is a strong connection between the weak differential subordination of continuous martingales and the norm of the Hilbert transform on  $L^p(\mathbb{R}; X)$  (see [41] and Remark 4.6).

Alternative approaches to Fourier multipliers for functions with values in UMD spaces have been constructed from the differential subordination for purely discontinuous martingales (see Bañuelos and Bogdan [4], Bañuelos, Bogdan and Bielaszewski [3], and recent work [41]), and for continuous martingales (see McConnell [26] and Geiss, Montgomery-Smith and Saksman [18]). It remains open whether one can combine these two approaches using the general weak differential subordination theory.

## 2. Preliminaries

In the sequel, we will omit proofs of some statements marked with a star (e.g., Lemma\*, Theorem\*, etc.). Please find the corresponding proofs in the Supplement [43].

We set the scalar field to be  $\mathbb{R}$ . We will use the *Kronecker symbol*  $\delta_{ij}$ , which is defined in the following way:  $\delta_{ij} = 1$  if  $i = j$ , and  $\delta_{ij} = 0$  if  $i \neq j$ . For each  $p \in (1, \infty)$  we set  $p' \in (1, \infty)$  and  $p^* \in [2, \infty)$  to be such that  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $p^* = \max\{p, p'\}$ . We set  $\mathbb{R}_+ := [0, \infty)$ .

### 2.1. UMD Banach spaces

A Banach space  $X$  is called a *UMD space* if for some (equivalently, for all)  $p \in (1, \infty)$  there exists a constant  $\beta > 0$  such that for every  $n \geq 1$ , every martingale difference sequence  $(d_j)_{j=1}^n$  in  $L^p(\Omega; X)$ , and every  $\{-1, 1\}$ -valued sequence  $(\varepsilon_j)_{j=1}^n$  we have

$$\left( \mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j d_j \right\|^p \right)^{\frac{1}{p}} \leq \beta \left( \mathbb{E} \left\| \sum_{j=1}^n d_j \right\|^p \right)^{\frac{1}{p}}.$$

The least admissible constant  $\beta$  is denoted by  $\beta_{p,X}$  and is called the *UMD constant*. It is well known (see [19], Chapter 4) that  $\beta_{p,X} \geq p^* - 1$  and that  $\beta_{p,H} = p^* - 1$  for a Hilbert space  $H$ . We refer the reader to [10,19,30,33] for details.

The following proposition is a vector-valued version of [11], Theorem 4.1.

**Proposition 2.1.** *Let  $X$  be a Banach space,  $p \in (1, \infty)$ . Then  $X$  has the UMD property if and only if there exists  $C > 0$  such that for each  $n \geq 1$ , for every martingale difference sequence  $(d_j)_{j=1}^n$  in  $L^p(\Omega; X)$ , and every sequence  $(\varepsilon_j)_{j=1}^n$  such that  $\varepsilon_j \in \{0, 1\}$  for each  $j = 1, \dots, n$  we have*

$$\left( \mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j d_j \right\|^p \right)^{\frac{1}{p}} \leq C \left( \mathbb{E} \left\| \sum_{j=1}^n d_j \right\|^p \right)^{\frac{1}{p}}.$$

*If this is the case, then the least admissible  $C$  is in the interval  $[\frac{\beta_{p,X}-1}{2}, \beta_{p,X}]$ .*

### 2.2. Martingales and stopping times in continuous time

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space with a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  which satisfies the usual conditions. Then  $\mathbb{F}$  is right-continuous, and the following proposition holds (see [41]).

**Proposition 2.2.** *Let  $X$  be a Banach space. Then any martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  has a càdlàg version*

Let  $1 \leq p \leq \infty$ . A martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  is called an  $L^p$ -martingale if  $M_t \in L^p(\Omega; X)$  for each  $t \geq 0$ , there exists an a.s. limit  $M_\infty := \lim_{t \rightarrow \infty} M_t$ ,  $M_\infty \in L^p(\Omega; X)$  and  $M_t \rightarrow M_\infty$  in  $L^p(\Omega; X)$  as  $t \rightarrow \infty$ . We will denote the space of all  $X$ -valued  $L^p$ -martingales on  $\Omega$  by  $\mathcal{M}_X^p(\Omega)$ . For brevity, we will use  $\mathcal{M}_X^p$  instead. Notice that  $\mathcal{M}_X^p$  is a Banach space with the given norm:  $\|M\|_{\mathcal{M}_X^p} := \|M_\infty\|_{L^p(\Omega; X)}$  (see [21,23] and [19], Chapter 1).

**Proposition\* 2.3.** *Let  $X$  be a Banach space with the Radon–Nikodým property (e.g., reflexive),  $1 < p < \infty$ . Then  $(\mathcal{M}_X^p)^* = \mathcal{M}_{X^*}^{p'}$ , and  $\|M\|_{(\mathcal{M}_X^p)^*} = \|M\|_{\mathcal{M}_{X^*}^{p'}}$  for each  $M \in \mathcal{M}_X^p$ .*

A random variable  $\tau : \Omega \rightarrow \mathbb{R}_+$  is called an *optional stopping time* (or just a *stopping time*) if  $\{\tau \leq t\} \in \mathcal{F}_t$  for each  $t \geq 0$ . With an optional stopping time  $\tau$ , we associate a  $\sigma$ -field  $\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t, t \in \mathbb{R}_+\}$ . Note that  $M_\tau$  is strongly  $\mathcal{F}_\tau$ -measurable for any local martingale  $M$ . We refer to [23], Chapter 7, for details.

Due to the existence of a càdlàg version of a martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$ , we can define an  $X$ -valued random variables  $M_{\tau-}$  and  $\Delta M_\tau$  for any stopping time  $\tau$  in the following way:  $M_{\tau-} = \lim_{\varepsilon \rightarrow 0} M_{(\tau-\varepsilon) \vee 0}$ ,  $\Delta M_\tau = M_\tau - M_{\tau-}$ .

### 2.3. Quadratic variation

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space with a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  that satisfies the usual conditions,  $H$  be a Hilbert space. Let  $M : \mathbb{R}_+ \times \Omega \rightarrow H$  be a local martingale. We define a *quadratic variation* of  $M$  in the following way:

$$[M]_t := \mathbb{P} - \lim_{\text{mesh} \rightarrow 0} \sum_{n=1}^N \|M(t_n) - M(t_{n-1})\|^2, \tag{2.1}$$

where the limit in probability is taken over partitions  $0 = t_0 < \dots < t_N = t$ . Note that  $[M]$  exists and is nondecreasing a.s. The reader can find more on quadratic variations in [27,28] for the vector-valued setting, and in [23,28,31] for the real-valued setting.

For any martingales  $M, N : \mathbb{R}_+ \times \Omega \rightarrow H$  we can define a *covariation*  $[M, N] : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  as  $[M, N] := \frac{1}{4}([M + N] - [M - N])$ . Since  $M$  and  $N$  have càdlàg versions,  $[M, N]$  has a càdlàg version as well (see [22] and [27], Theorem I.4.47).

**Remark 2.4 ([27]).** The process  $\langle M, N \rangle - [M, N]$  is a local martingale.

### 2.4. Continuous martingales

Let  $X$  be a Banach space. A martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  is called *continuous* if  $M$  has continuous paths.

**Remark 2.5 ([23,28]).** If  $X$  is a Hilbert space,  $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$  are continuous martingales, then  $[M, N]$  has a continuous version.

Let  $1 \leq p \leq \infty$ . We will denote the linear space of all continuous  $X$ -valued  $L^p$ -martingales on  $\Omega$  which start at zero by  $\mathcal{M}_X^{p,c}(\Omega)$ . For brevity we will write  $\mathcal{M}_X^{p,c}$  instead of  $\mathcal{M}_X^{p,c}(\Omega)$  since  $\Omega$  is fixed. Analogously to [23], Lemma 17.4, by applying Doob's maximal inequality [19], Theorem 3.2.2, one can show the following proposition.

**Proposition 2.6.** *Let  $X$  be a Banach space,  $p \in (1, \infty)$ . Then  $\mathcal{M}_X^{p,c}$  is a Banach space with the following norm:  $\|M\|_{\mathcal{M}_X^{p,c}} := \|M_\infty\|_{L^p(\Omega; X)}$ .*

## 2.5. Purely discontinuous martingales

An increasing càdlàg process  $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  is called *pure jump* if a.s. for each  $t \geq 0$ ,  $A_t = A_0 + \sum_{s=0}^t \Delta A_s$ . A local martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  is called *purely discontinuous* if  $[M]$  is a pure jump process. The reader can find more on purely discontinuous martingales in [22,23]. We leave the following evident lemma without proof.

**Lemma 2.7.** *Let  $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$  be an increasing adapted càdlàg process such that  $A_0 = 0$ . Then there exist unique up to indistinguishability increasing adapted càdlàg processes  $A^c, A^d : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$  such that  $A^c$  is continuous a.s.,  $A^d$  is pure jump a.s.,  $A_0^c = A_0^d = 0$  and  $A = A^c + A^d$ .*

**Remark 2.8.** According to the works [29] by Meyer and [44] by Yoeurp (see also [23], Theorem 26.14), any martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  can be uniquely decomposed into a sum of a purely discontinuous local martingale  $M^d$  and a continuous local martingale  $M^c$  such that  $M_0^c = 0$ . Moreover,  $[M]^c = [M^c]$  and  $[M]^d = [M^d]$ , where  $[M]^c$  and  $[M]^d$  are defined as in Lemma 2.7.

**Corollary 2.9.** *Let  $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  be a martingale which is both continuous and purely discontinuous. Then  $M = M_0$  a.s.*

**Proposition\* 2.10.** *A martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  is purely discontinuous if and only if  $MN$  is a martingale for any continuous bounded martingale  $N : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  with  $N_0 = 0$ .*

Note that some authors take this equivalent condition as the definition of a purely discontinuous martingale, see, for example, [22], Definition I.4.11, and [21], Chapter I.

**Definition 2.11.** Let  $X$  be a Banach space,  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  be a local martingale. Then  $M$  is called *purely discontinuous* if for each  $x^* \in X^*$  the local martingale  $\langle M, x^* \rangle$  is purely discontinuous.

**Remark 2.12.** Let  $X$  be finite dimensional. Then similarly to Remark 2.8 any martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  can be uniquely decomposed into a sum of a purely discontinuous local martingale  $M^d$  and a continuous local martingale  $M^c$  such that  $M_0^c = 0$ .

**Remark 2.13.** Analogously to Proposition 2.10, a martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  is purely discontinuous if and only if  $\langle M, x^* \rangle N$  is a martingale for any  $x^* \in X^*$  and any continuous bounded martingale  $N : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  such that  $N_0 = 0$ .

Let  $p \in [1, \infty]$ . We will denote the linear space of all purely discontinuous  $X$ -valued  $L^p$ -martingales on  $\Omega$  by  $\mathcal{M}_X^{p,d}(\Omega)$ . Since  $\Omega$  is fixed, we will use  $\mathcal{M}_X^{p,d}$  instead. The scalar case of the next result have been presented in [21], Lemme I.2.12.

**Proposition 2.14.** *Let  $X$  be a Banach space,  $p \in (1, \infty)$ . Then  $\mathcal{M}_X^{p,d}$  is a Banach space with a norm defined as follows:  $\|M\|_{\mathcal{M}_X^{p,d}} := \|M_\infty\|_{L^p(\Omega; X)}$ .*

**Proof.** Let  $(M^n)_{n \geq 1}$  be a sequence of purely discontinuous  $X$ -valued  $L^p$ -martingales such that  $(M_\infty^n)_{n \geq 1}$  is a Cauchy sequence in  $L^p(\Omega; X)$ . Let  $\xi \in L^p(\Omega; X)$  be such that  $\lim_{n \rightarrow \infty} M_\infty^n = \xi$ . Define a martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  as follows:  $M = (M_s)_{s \geq 0} = (\mathbb{E}(\xi | \mathcal{F}_s))_{s \geq 0}$ . Let us show that  $M \in \mathcal{M}_X^{p,d}$ . First, notice that  $\|M_\infty\|_{L^p(\Omega; X)} = \|\xi\|_{L^p(\Omega; X)} < \infty$ . Further for each  $x^* \in X^*$  by [21], Lemme I.2.12, we have that  $\langle M, x^* \rangle$  as a limit of real-valued purely discontinuous martingales  $(\langle M^n, x^* \rangle)_{n \geq 1}$  in  $\mathcal{M}_\mathbb{R}^p$  is purely discontinuous. Therefore,  $M$  is purely discontinuous by the definition. □

**Lemma 2.15.** *Let  $X$  be a Banach space,  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  be a martingale such that  $M$  is both continuous and purely discontinuous. Then  $M = M_0$  a.s.*

**Proof.** Follows analogously Corollary 2.9. □

## 2.6. Time-change

A nondecreasing, right-continuous family of stopping times  $\tau = (\tau_s)_{s \geq 0}$  is called a *random time-change*. If  $\mathbb{F}$  is right-continuous, then according to [23], Lemma 7.3, the same holds true for the induced filtration  $\mathbb{G} = (\mathcal{G}_s)_{s \geq 0} = (\mathcal{F}_{\tau_s})_{s \geq 0}$  (see more in [23], Chapter 7). Let  $X$  be a Banach space. A martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  is said to be  $\tau$ -continuous if  $M$  is an a.s. constant on every interval  $[\tau_{s-}, \tau_s]$ ,  $s \geq 0$ , where we let  $\tau_{0-} = 0$ .

**Theorem\* 2.16.** *Let  $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$  be a strictly increasing continuous predictable process such that  $A_0 = 0$  and  $A_t \rightarrow \infty$  as  $t \rightarrow \infty$  a.s. Let  $\tau = (\tau_s)_{s \geq 0}$  be a random time-change defined as  $\tau_s := \{t : A_t = s\}$ ,  $s \geq 0$ . Then  $(A \circ \tau)(t) = (\tau \circ A)(t) = t$  a.s. for each  $t \geq 0$ . Let  $\mathbb{G} = (\mathcal{G}_s)_{s \geq 0} = (\mathcal{F}_{\tau_s})_{s \geq 0}$  be the induced filtration. Then  $(A_t)_{t \geq 0}$  is a random time-change with respect to  $\mathbb{G}$  and for any  $\mathbb{F}$ -martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  the following holds*

- (i)  $M \circ \tau$  is a continuous  $\mathbb{G}$ -martingale if and only if  $M$  is continuous, and
- (ii)  $M \circ \tau$  is a purely discontinuous  $\mathbb{G}$ -martingale if and only if  $M$  is purely discontinuous.



### 2.7. Stochastic integration

Let  $X$  be a Banach space,  $H$  be a Hilbert space. For each  $h \in H, x \in X$  we denote the linear operator  $g \mapsto (g, h)x, g \in H$ , by  $h \otimes x$ . The process  $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$  is called *elementary progressive* with respect to the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  if it is of the form

$$\Phi(t, \omega) = \sum_{k=1}^K \sum_{m=1}^M \mathbf{1}_{(t_{k-1}, t_k] \times B_{mk}}(t, \omega) \sum_{n=1}^N h_n \otimes x_{kmn}, \quad t \geq 0, \omega \in \Omega, \tag{2.2}$$

where  $0 \leq t_0 < \dots < t_K < \infty$ , for each  $k = 1, \dots, K$  the sets  $B_{1k}, \dots, B_{Mk}$  are in  $\mathcal{F}_{t_{k-1}}$  and the vectors  $h_1, \dots, h_N$  are orthogonal. Let  $M : \mathbb{R}_+ \times \Omega \rightarrow H$  be a martingale. Then we define the *stochastic integral*  $\Phi \cdot M : \mathbb{R}_+ \times \Omega \rightarrow X$  of  $\Phi$  with respect to  $M$  as follows:

$$(\Phi \cdot M)_t = \sum_{k=1}^K \sum_{m=1}^M \mathbf{1}_{B_{mk}} \sum_{n=1}^N ((M(t_k \wedge t) - M(t_{k-1} \wedge t)), h_n) x_{kmn}, \quad t \geq 0. \tag{2.3}$$

We will need the following lemma on stochastic integration (see [41]).

**Lemma 2.17.** *Let  $d$  be a natural number,  $H$  be a  $d$ -dimensional Hilbert space,  $p \in (1, \infty)$ ,  $M, N : \mathbb{R}_+ \times \Omega \rightarrow H$  be  $L^p$ -martingales,  $F : H \rightarrow H$  be a measurable function such that  $\|F(h)\| \leq C\|h\|^{p-1}$  for each  $h \in H$  and some  $C > 0$ . Define  $N_- : \mathbb{R}_+ \times \Omega \rightarrow H$  by  $(N_-)_t = N_{t-}, t \geq 0$ . Then  $F(N_-) \cdot M$  is a martingale and for each  $t \geq 0$*

$$\mathbb{E} |(F(N_-) \cdot M)_t| \lesssim_{p,d} C (\mathbb{E} \|N_t\|^p)^{\frac{p-1}{p}} (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}. \tag{2.4}$$

### 2.8. Multidimensional Wiener process

Let  $d$  be a natural number.  $W : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$  is called a *standard  $d$ -dimensional Wiener process* if  $\langle W, h \rangle$  is a standard Wiener process for each  $h \in \mathbb{R}^d$  such that  $\|h\| = 1$ . The following lemma is a multidimensional variation of [24], (3.2.19).

**Lemma 2.18.** *Let  $X = \mathbb{R}, d \geq 1, W$  be a standard  $d$ -dimensional Wiener process,  $\Phi, \Psi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R})$  be elementary progressive. Then for all  $t \geq 0$  a.s.*

$$[\Phi \cdot W, \Psi \cdot W]_t = \int_0^t \langle \Phi^*(s), \Psi^*(s) \rangle ds.$$

The reader can find more on stochastic integration with respect to a Wiener process in the Hilbert space case in [12], in the case of Banach spaces with a martingale type 2 in [7], and in the UMD case in [35]. Notice that the last mentioned work provides sharp  $L^p$ -estimates for stochastic integrals for the broadest till now known class of spaces.

### 2.9. Brownian representation

The following theorem can be found in [24], Theorem 3.4.2 (see also [34,39]).

**Theorem 2.19.** *Let  $d \geq 1$ ,  $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$  be a continuous martingale such that  $[M]$  is a.s. absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}_+$ . Then there exist an enlarged probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  with an enlarged filtration  $\tilde{\mathbb{F}} = (\tilde{F}_t)_{t \geq 0}$ , a  $d$ -dimensional standard Wiener process  $W : \mathbb{R}_+ \times \tilde{\Omega} \rightarrow \mathbb{R}^d$  which is defined on the filtration  $\tilde{\mathbb{F}}$ , and an  $\tilde{\mathbb{F}}$ -progressively measurable  $\Phi : \mathbb{R}_+ \times \tilde{\Omega} \rightarrow \mathcal{L}(\mathbb{R}^d)$  such that  $M = \Phi \cdot W$ .*

### 2.10. Lebesgue measure

Let  $X$  be a finite dimensional Banach space. Then according to Theorem 2.20 and Proposition 2.21 in [16] there exists a unique translation-invariant measure  $\lambda_X$  on  $X$  such that  $\lambda_X(\mathbb{B}_X) = 1$  for the unit ball  $\mathbb{B}_X$  of  $X$ . We will call  $\lambda_X$  the *Lebesgue measure*.

## 3. UMD Banach spaces and martingale decompositions

Let  $X$  be a Banach space,  $1 < p < \infty$ . In this section, we will show that the Meyer–Yoeurp and Yoeurp decompositions for  $X$ -valued  $L^p$ -martingales take place if and only if  $X$  has the UMD property.

### 3.1. Meyer–Yoeurp decomposition in UMD case

This subsection is devoted to the generalization of Meyer–Yoeurp decomposition (see Remark 2.8) to the UMD Banach space case:

**Theorem 3.1 (Meyer–Yoeurp decomposition).** *Let  $X$  be a UMD Banach space,  $p \in (1, \infty)$ ,  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  be an  $L^p$ -martingale. Then there exist unique martingales  $M^d, M^c : \mathbb{R}_+ \times \Omega \rightarrow X$  such that  $M^d$  is purely discontinuous,  $M^c$  is continuous,  $M_0^c = 0$  and  $M = M^d + M^c$ . Moreover, then for all  $t \geq 0$*

$$(\mathbb{E} \|M_t^d\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E} \|M_t^c\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}. \tag{3.1}$$

The proof of the theorem consists of several steps. First we introduce the main tool of our proof – the Burkholder function.

**Definition 3.2.** Let  $E$  be a linear space with a scalar field  $\mathbb{R}$ .

- (i) A function  $f : E \times E \rightarrow \mathbb{R}$  is called *biconcave* if for each  $x, y \in E$  one has that the mappings  $e \mapsto f(x, e)$  and  $e \mapsto f(e, y)$  are concave.
- (ii) A function  $f : E \times E \rightarrow \mathbb{R}$  is called *zigzag-concave* if for each  $x, y \in E$  and  $\varepsilon \in \mathbb{R}$  such that  $|\varepsilon| \leq 1$ , the function  $z \mapsto f(x + z, y + \varepsilon z)$  is concave.

The following theorem is a small variation of [9] and [19], Theorem 4.5.6, and has been proven in [41].

**Theorem 3.3 (Burkholder).** *For a Banach space  $X$  the following are equivalent*

1.  $X$  is a UMD Banach space;
2. for each  $p \in (1, \infty)$  there exists a constant  $\beta$  and a zigzag-concave function  $U : X \times X \rightarrow \mathbb{R}$  such that

$$U(x, y) \geq \|y\|^p - \beta^p \|x\|^p, \quad x, y \in X. \tag{3.2}$$

The smallest admissible  $\beta$  for which such  $U$  exists is  $\beta_{p,X}$ .

**Remark 3.4.** Fix a UMD space  $X$  and  $p \in (1, \infty)$ . A special zigzag-concave function  $U$  from Theorem 3.3 have been obtained in [19], Theorem 4.5.6. We will call this function *the Burkholder function*. For the convenience of the reader we leave out the construction of the Burkholder function. The following properties of the Burkholder function  $U$  were demonstrated in [41], Section 3:

- (A)  $U(\alpha x, \alpha y) = |\alpha|^p U(x, y)$  for all  $x, y \in X, \alpha \in \mathbb{R}$ .
- (B)  $U(x, \alpha x) \leq 0$  for all  $x \in X, \alpha \in [-1, 1]$ .
- (C)  $U$  is continuous.

**Remark 3.5.** Fix a UMD space  $X$  and  $p \in (1, \infty)$ . Let the Burkholder function  $U$  be as in Remark 3.4. Then there exists a biconcave function  $V : X \times X \rightarrow \mathbb{R}$  such that

$$V(x, y) = U\left(\frac{x - y}{2}, \frac{x + y}{2}\right), \quad x, y \in X. \tag{3.3}$$

In [41], Section 3, the following properties of  $V$  have been explored:

- (A) For each  $x, y \in X$  and  $a, b \in \mathbb{R}$  such that  $|a + b| \leq |a - b|$  one has that the function

$$z \mapsto V(x + az, y + bz) = U\left(\frac{x - y}{2} + \frac{(a - b)z}{2}, \frac{x + y}{2} + \frac{(a + b)z}{2}\right)$$

is concave.

- (B)  $V$  is continuous.
- (C) Let  $X$  be finite dimensional. Then  $x \mapsto V(x, y)$  and  $y \mapsto V(x, y)$  are a.s. Fréchet-differentiable with respect to the Lebesgue measure  $\lambda_X$ , and for a.a.  $(x, y) \in X \times X$  for each  $u, v \in X$  there exists the directional derivative  $\frac{\partial V(x+tu, y+tv)}{\partial t}$ . Moreover,

$$\frac{\partial V(x + tu, y + tv)}{\partial t} = \langle \partial_x V(x, y), u \rangle + \langle \partial_y V(x, y), v \rangle, \tag{3.4}$$

where  $\partial_x V$  and  $\partial_y V$  are the corresponding Fréchet derivatives with respect to the first and the second variable.

(D) Let  $X$  be finite dimensional. Then for a.e.  $(x, y) \in X \times X$ , for all  $z \in X$  and real-valued  $a$  and  $b$  such that  $|a + b| \leq |a - b|$

$$\begin{aligned} V(x + az, y + bz) &\leq V(x, y) + \frac{\partial V(x + atz, y + btz)}{\partial t} \\ &= V(x, y) + a\langle \partial_x V(x, y), z \rangle + b\langle \partial_y V(x, y), z \rangle. \end{aligned} \tag{3.5}$$

(E) Let  $X$  be finite dimensional. Then there exists  $C > 0$  which depends only on  $V$  such that for a.e. pair  $x, y \in X$ ,  $\|\partial_x V(x, y)\|, \|\partial_y V(x, y)\| \leq C(\|x\|^{p-1} + \|y\|^{p-1})$ .

**Definition 3.6.** Let  $d$  be a natural number,  $E$  be a  $d$ -dimensional linear space,  $(e_n)_{n=1}^d$  be a basis of  $E$ . Then  $(e_n^*)_{n=1}^d \subset E^*$  is called the *corresponding dual basis* of  $(e_n)_{n=1}^d$  if  $\langle e_n, e_m^* \rangle = \delta_{nm}$  for each  $m, n = 1, \dots, d$ .

Note that the corresponding dual basis is uniquely determined. Moreover, if  $(e_n^*)_{n=1}^d$  is the corresponding dual basis of  $(e_n)_{n=1}^d$ , then, the other way around,  $(e_n)_{n=1}^d$  is the corresponding dual basis of  $(e_n^*)_{n=1}^d$  (here we identify  $E^{**}$  with  $E$  in the natural way).

**Lemma\* 3.7.** Let  $d$  be a natural number,  $E$  be a  $d$ -dimensional linear space. Let  $V : E \times E \rightarrow \mathbb{R}$  and  $W : E^* \times E^* \rightarrow \mathbb{R}$  be two bilinear functions. Then the expression

$$\sum_{n,m=1}^d V(e_n, e_m) W(e_n^*, e_m^*) \tag{3.6}$$

does not depend on the choice of basis  $(e_n)_{n=1}^d$  of  $E$  (here  $(e_n^*)_{n=1}^d$  is the corresponding dual basis of  $(e_n)_{n=1}^d$ ).

The following Itô formula is a version of [23], Theorem 26.7, that does not use the Euclidean structure of a finite dimensional Banach space. The proof can be found in [41].

**Theorem 3.8 (Itô formula).** Let  $d$  be a natural number,  $X$  be a  $d$ -dimensional Banach space,  $f \in C^2(X)$ ,  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  be a martingale. Let  $(x_n)_{n=1}^d$  be a basis of  $X$ ,  $(x_n^*)_{n=1}^d$  be the corresponding dual basis. Then for each  $t \geq 0$

$$\begin{aligned} f(M_t) &= f(M_0) + \int_0^t \langle \partial_x f(M_{s-}), dM_s \rangle \\ &\quad + \frac{1}{2} \int_0^t \sum_{n,m=1}^d f_{x_n, x_m}(M_{s-}) d[[M, x_n^*], [M, x_m^*]]_s^c \\ &\quad + \sum_{s \leq t} (\Delta f(M_s) - \langle \partial_x f(M_{s-}), \Delta M_s \rangle). \end{aligned} \tag{3.7}$$

**Proposition 3.9.** *Let  $X$  be a finite dimensional Banach space,  $p \in (1, \infty)$ . Let  $Y = X \oplus \mathbb{R}$  be a Banach space such that  $\|(x, r)\|_Y = (\|x\|_X^p + |r|^p)^{\frac{1}{p}}$ . Then  $\beta_{p,Y} = \beta_{p,X}$ . Moreover, if  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  is a martingale on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ , then there exists a sequence  $(M^m)_{m \geq 1}$  of  $Y$ -valued martingales on an enlarged probability space  $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$  with an enlarged filtration  $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t \geq 0}$  such that*

1.  $M_t^m$  has absolutely continuous distributions with respect to the Lebesgue measure on  $Y$  for each  $m \geq 1$  and  $t \geq 0$ ;
2.  $M_t^m \rightarrow (M_t, 0)$  pointwise as  $m \rightarrow \infty$  for each  $t \geq 0$ ;
3. if for some  $t \geq 0$   $\mathbb{E}\|M_t\|^p < \infty$ , then for each  $m \geq 1$  one has that  $\mathbb{E}\|M_t^m\|^p < \infty$  and  $\mathbb{E}\|M_t^m - (M_t, 0)\|^p \rightarrow 0$  as  $m \rightarrow \infty$ ;
4. if  $M$  is continuous, then  $(M^m)_{m \geq 1}$  are continuous as well,
5. if  $M$  is purely discontinuous, then  $(M^m)_{m \geq 1}$  are purely discontinuous as well.

**Proof.** The proof of (1)–(3) follows from [41], while (4) and (5) follow from the construction of  $M^m$  and  $N^m$  given in [41]. □

**Remark 3.10.** Notice that the construction in [41] also allows us to sum these approximations for different martingales. Namely, if  $M$  and  $N$  are two  $X$ -valued martingales, then we can construct the corresponding  $Y$ -valued martingales  $(M^m)_{m \geq 1}$  and  $(N^m)_{m \geq 1}$  as in Proposition 3.9 in such a way that  $M_t^m + N_t^m$  has an absolutely continuous distribution for each  $t \geq 0$  and  $m \geq 1$ .

**Proof of Theorem 3.1.** *Step 1: finite dimensional case.* Let  $X$  be finite dimensional. Then  $M^d$  and  $M^c$  exist due to Remark 2.12. Without loss of generality  $\mathcal{F}_t = \mathcal{F}_\infty$ ,  $M_t^d = M_\infty^d$  and  $M_t^c = M_\infty^c$ . Let  $d$  be the dimension of  $X$ .

Let  $\|\cdot\|$  be a Euclidean norm on  $X$ . Then  $(X, \|\cdot\|)$  is a Hilbert space, and by Remark 2.5 the quadratic variation  $[M^c]$  exists and has a continuous version. Let us show that without loss of generality we can suppose that  $[M^c]$  is a.s. absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}_+$ . Let  $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$  be as follows:  $A_t = [M^c]_t + t$ . Then  $A$  is strictly increasing continuous,  $A_0 = 0$  and  $A_\infty = \infty$  a.s. Let the time-change  $\tau = (\tau_s)_{s \geq 1}$  be defined as in Theorem 2.16. Then by Theorem 2.16,  $M^c \circ \tau$  is a continuous martingale,  $M^d \circ \tau$  is a purely discontinuous martingale,  $(M^c \circ \tau)_0 = 0$ ,  $(M^d \circ \tau)_0 = M_0^d$  and due to the Kazamaki theorem [23], Theorem 17.24,  $[M^c \circ \tau] = [M^c] \circ \tau$ . Therefore for all  $t > s \geq 0$  by Theorem 2.16 and the fact that  $\tau_t \geq \tau_s$  a.s.

$$\begin{aligned} [M^c \circ \tau]_t - [M^c \circ \tau]_s &= [M^c]_{\tau_t} - [M^c]_{\tau_s} \leq [M^c]_{\tau_t} - [M^c]_{\tau_s} + (\tau_t - \tau_s) \\ &= ([M^c]_{\tau_t} + \tau_t) - ([M^c]_{\tau_s} + \tau_s) \\ &= A_{\tau_t} - A_{\tau_s} = t - s. \end{aligned}$$

Hence  $[M^c \circ \tau]$  is a.s. absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}_+$ . Moreover,  $(M^i \circ \tau)_\infty = M_\infty^i$ ,  $i \in \{c, d\}$ , so this time-change argument does not affect (3.1). Hence we can redefine  $M^c := M^c \circ \tau$ ,  $M^d := M^d \circ \tau$ ,  $\mathbb{F} = (\mathcal{F}_s)_{s \geq 0} := \mathbb{G} = (\mathcal{F}_{\tau_s})_{s \geq 0}$ .

Since  $[M^c]$  is a.s. absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}_+$  and thanks to Theorem 2.19, we can extend  $\Omega$  and find a  $d$ -dimensional Wiener process  $W$  :

$\mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$  and a stochastically integrable progressively measurable function  $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^d, X)$  such that  $M^c = \Phi \cdot W$ .

Let  $U : X \times X \rightarrow \mathbb{R}$  be the Burkholder function that was discussed in Remark 3.4 and Remark 3.5. Let us show that  $\mathbb{E}U(M_t, M_t^d) \leq 0$ .

Due to Proposition 3.9 and Remark 3.10 we can assume that  $M_s^c, M_s^d$  and  $M_s = M_s^d + M_s^c$  have absolutely continuous distributions with respect to the Lebesgue measure  $\lambda_X$  on  $X$  for each  $s \geq 0$ . Let  $(x_n)_{n=1}^d$  be a basis of  $X$ ,  $(x_n^*)_{n=1}^d$  be the corresponding dual basis of  $X^*$  (see Definition 3.6). By the Itô formula (3.7),

$$\begin{aligned} \mathbb{E}U(M_t, M_t^d) &= \mathbb{E}U(M_0, M_0^d) + \mathbb{E} \int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d), dM_s \rangle \\ &\quad + \mathbb{E} \int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d), dM_s^d \rangle + \mathbb{E}I_1 + \mathbb{E}I_2, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} I_1 &= \sum_{0 < s \leq t} [\Delta U(M_s, M_s^d) - \langle \partial_x U(M_{s-}, M_{s-}^d), \Delta M_s \rangle - \langle \partial_y U(M_{s-}, M_{s-}^d), \Delta M_s^d \rangle], \\ I_2 &= \frac{1}{2} \int_0^t \sum_{i,j=1}^d U_{x_i, x_j}(M_{s-}, M_{s-}^d) d[\langle M, x_i^* \rangle, \langle M, x_j^* \rangle]_s^c \\ &= \frac{1}{2} \int_0^t \sum_{i,j=1}^d U_{x_i, x_j}(M_{s-}, M_{s-}^d) \langle \Phi^*(s)x_i^*, \Phi^*(s)x_j^* \rangle ds. \end{aligned}$$

(Recall that by (3.3) and Remark 3.5(C),  $U$  is Fréchet-differentiable a.s. on  $X \times X$ , hence  $\partial_x U$  and  $\partial_y U$  are well-defined. Moreover,  $U$  is zigzag-concave, so  $U$  is concave in the first variable, and therefore the second-order derivatives  $U_{x_i, x_j}$  in the first variable are well-defined and exist a.s. on  $X \times X$  by the Alexandrov theorem [15], Theorem 6.4.1.) The last equality holds due to Theorem 3.8 and the fact that by Lemma 2.18 for all  $s \geq 0$  a.s.

$$\begin{aligned} [\langle M, x_i^* \rangle, \langle M, x_j^* \rangle]_s^c &= [\langle \Phi \cdot W, x_i^* \rangle, \langle \Phi \cdot W, x_j^* \rangle]_s \\ &= [(\Phi^* x_i^*) \cdot W, (\Phi^* x_j^*) \cdot W]_s \\ &= \int_0^s \langle \Phi^*(r)x_i^*, \Phi^*(r)x_j^* \rangle dr. \end{aligned}$$

Let us first show that  $I_1 \leq 0$  a.s. Indeed, since  $M^d$  is a purely discontinuous part of  $M$ , then by Definition 2.11  $\langle M^d, x^* \rangle$  is a purely discontinuous part of  $\langle M, x^* \rangle$ , and due to Remark 2.8 a.s. for each  $t \geq 0$

$$\Delta |\langle M^d, x^* \rangle|_t^2 = \Delta [\langle M^d, x^* \rangle]_t = \Delta [\langle M, x^* \rangle]_t = \Delta |\langle M, x^* \rangle|_t^2$$

for each  $x^* \in X^*$ . Thus for each  $s \geq 0$  by (3.4) and (3.5)  $\mathbb{P}$ -a.s.

$$\begin{aligned} & \Delta U(M_s, M_s^d) - \langle \partial_x U(M_{s-}, M_{s-}^d), \Delta M_s \rangle - \langle \partial_y U(M_{s-}, M_{s-}^d), \Delta M_s^d \rangle \\ &= V(M_{s-} + M_{s-}^d + 2\Delta M_s, M_{s-}^d - M_{s-}) - V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}) \\ & \quad - \langle \partial_x V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), 2\Delta M_s \rangle \\ & \leq 0, \end{aligned}$$

so  $I_1 \leq 0$  a.s., and  $\mathbb{E}I_1 \leq 0$ . Now we show that

$$\mathbb{E} \left( \int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d), dM_s \rangle + \int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d), dM_s^d \rangle \right) = 0.$$

Indeed,

$$\begin{aligned} & \int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d), dM_s \rangle + \int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d), dM_s^d \rangle \\ &= \int_0^t \langle \partial_x V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), d(M_s + M_s^d) \rangle \\ & \quad + \int_0^t \langle \partial_y V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), d(M_s^d - M_s) \rangle \end{aligned}$$

so by Lemma 2.17 and Remark 3.5(E) it is a martingale which starts at zero, hence its expectation is zero.

Finally, let us show that  $I_2 \leq 0$  a.s. Fix  $s \in [0, t]$  and  $\omega \in \Omega$ . Then  $x^* \mapsto \|\Phi^*(s, \omega)x^*\|^2$  defines a nonnegative definite quadratic form on  $X^*$ , and since any nonnegative quadratic form defines a Euclidean seminorm, there exists a basis  $(\tilde{x}_n^*)_{n=1}^d$  of  $X^*$  and a  $\{0, 1\}$ -valued sequence  $(a_n)_{n=1}^d$  such that

$$\langle \Phi^*(s, \omega)\tilde{x}_n^*, \Phi^*(s, \omega)\tilde{x}_m^* \rangle = a_n \delta_{mn}, \quad m, n = 1, \dots, d.$$

Let  $(\tilde{x}_n)_{n=1}^d$  be the corresponding dual basis of  $X$  as it is defined in Definition 3.6. Then due to Lemma 3.7 and the linearity of  $\Phi$  and directional derivatives of  $U$  (we skip  $s$  and  $\omega$  for the simplicity of the expressions)

$$\begin{aligned} \sum_{i,j=1}^d U_{x_i, x_j}(M_{s-}, M_{s-}^d) \langle \Phi^* x_i^*, \Phi^* x_j^* \rangle &= \sum_{i,j=1}^d U_{\tilde{x}_i, \tilde{x}_j}(M_{s-}, M_{s-}^d) \langle \Phi^* \tilde{x}_i^*, \Phi^* \tilde{x}_j^* \rangle \\ &= \sum_{i=1}^d U_{\tilde{x}_i, \tilde{x}_i}(M_{s-}, M_{s-}^d) \|\Phi^* \tilde{x}_i^*\|^2. \end{aligned}$$

Recall that  $U$  is zigzag-concave, so  $t \mapsto U(x + t\tilde{x}_i, y)$  is concave for each  $x, y \in X, i = 1, \dots, d$ . Therefore  $U_{\tilde{x}_i, \tilde{x}_i}(M_{s-}, M_{s-}^d) \leq 0$  a.s., and a.s.

$$\sum_{i=1}^d U_{\tilde{x}_i, \tilde{x}_i}(M_{s-}(\omega), M_{s-}^d(\omega)) \|\Phi^*(s, \omega)\tilde{x}_i^*\|^2 \leq 0.$$

Consequently,  $I_2 \leq 0$  a.s., and by (3.8), Remark 3.4(B) and the fact that  $M_0^d = M_0$

$$\mathbb{E}U(M_t, M_t^d) \leq \mathbb{E}U(M_0, M_0) \leq 0.$$

By (3.2),  $\mathbb{E}\|M_t^d\|^p - \beta_{p,X}^p \mathbb{E}\|M_t\|^p \leq \mathbb{E}U(M_t, M_t^d) \leq 0$ , so the first part of (3.1) holds.

The second part of (3.1) follows from the same machinery applied for  $V$ . Namely, one can analogously show that

$$\mathbb{E}\|M_t^c\|^p - \beta_{p,X}^p \mathbb{E}\|M_t\|^p \leq \mathbb{E}U(M_t, M_t^c) = \mathbb{E}V(M^d + 2M^c, -M^d) \leq 0$$

by using a  $V$ -version of (3.8), inequality (3.5), and the fact that  $V$  is concave in the first variable a.s. on  $X \times X$ .

*Step 2: general case.* Without loss of generality, we set  $\mathcal{F}_\infty = \mathcal{F}_t$ . Let  $M_t = \xi$ . If  $\xi$  is a simple function, then it takes its values in a finite dimensional subspace  $X_0$  of  $X$ , and therefore  $(M_s)_{s \geq 0} = (\mathbb{E}(\xi|\mathcal{F}_s))_{s \geq 0}$  takes its values in  $X_0$  as well, so the theorem and (3.1) follow from Step 1.

Now let  $\xi$  be general. Let  $(\xi_n)_{n \geq 1}$  be a sequence of simple  $\mathcal{F}_t$ -measurable functions in  $L^p(\Omega; X)$  such that  $\xi_n \rightarrow \xi$  as  $n \rightarrow \infty$  in  $L^p(\Omega; X)$ . For each  $n \geq 1$  define  $\mathcal{F}_t$ -measurable  $\xi_n^d$  and  $\xi_n^c$  such that

$$\begin{aligned} M^{d,n} &= (M_s^{d,n})_{s \geq 0} = (\mathbb{E}(\xi_n^d|\mathcal{F}_s))_{s \geq 0}, \\ M^{c,n} &= (M_s^{c,n})_{s \geq 0} = (\mathbb{E}(\xi_n^c|\mathcal{F}_s))_{s \geq 0} \end{aligned} \tag{3.9}$$

are the respectively purely discontinuous and continuous parts of martingale  $M^n = (\mathbb{E}(\xi_n|\mathcal{F}_s))_{s \geq 0}$  as in Remark 2.12. Then due to Step 1 and (3.1),  $(\xi_n^d)_{n \geq 1}$  and  $(\xi_n^c)_{n \geq 1}$  are Cauchy sequences in  $L^p(\Omega; X)$ . Let  $\xi^c := L^p - \lim_{n \rightarrow \infty} \xi_n^c$  and  $\xi^d := L^p - \lim_{n \rightarrow \infty} \xi_n^d$ . Define the  $X$ -valued  $L^p$ -martingales  $M^d$  and  $M^c$  by

$$M^d = (M_s^d)_{s \geq 0} := (\mathbb{E}(\xi^d|\mathcal{F}_s))_{s \geq 0}, \quad M^c = (M_s^c)_{s \geq 0} := (\mathbb{E}(\xi^c|\mathcal{F}_s))_{s \geq 0}.$$

Thanks to Proposition 2.14,  $M^d$  is purely discontinuous, and due to Proposition 2.6  $M^c$  is continuous and  $M_0^c = 0$ , so  $M = M^d + M^c$  is the desired decomposition.

The uniqueness of the decomposition follows from Lemma 2.15. For estimates (3.1), we note that by Step 1, (3.1) applied for Step 1, and [19], Proposition 4.2.17, for each  $n \geq 1$

$$(\mathbb{E}\|\xi_n^d\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|\xi_n^c\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n\|^p)^{\frac{1}{p}},$$

and it remains to let  $n \rightarrow \infty$ . □



**Remark 3.11.** Let  $X$  be a UMD Banach space,  $1 < p < \infty$ ,  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  be continuous (resp. purely discontinuous)  $L^p$ -martingale. Then there exists a sequence  $(M^n)_{n \geq 1}$  of continuous (resp. purely discontinuous)  $X$ -valued  $L^p$ -martingales such that  $M^n$  takes its values in a finite dimensional subspace of  $X$  for each  $n \geq 1$  and  $M^n_\infty \rightarrow M_\infty$  in  $L^p(\Omega; X)$  as  $n \rightarrow \infty$ . Such a sequence can be provided e.g. by (3.9).

We have proven the Meyer–Yoeurp decomposition in the UMD setting. Next, we prove a converse result which shows the necessity of the UMD property.

**Theorem 3.12.** Let  $X$  be a finite dimensional Banach space,  $p \in (1, \infty)$ ,  $\delta \in (0, (\beta_{p,X} - 1) \wedge 1)$ . Then there exist a purely discontinuous martingale  $M^d : \mathbb{R}_+ \times \Omega \rightarrow X$ , a continuous martingale  $M^c : \mathbb{R}_+ \times \Omega \rightarrow X$  such that  $\mathbb{E}\|M^d_\infty\|^p, \mathbb{E}\|M^c_\infty\|^p < \infty$ ,  $M^d_0 = M^c_0 = 0$ , and for  $M = M^d + M^c$  and  $i \in \{c, d\}$  the following hold

$$(\mathbb{E}\|M^i_\infty\|^p)^{\frac{1}{p}} \geq \left(\frac{\beta_{p,X} - 1}{2} - \delta\right) (\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{3.10}$$

Recall that by [19], Proposition 4.2.17,  $\beta_{p,X} \geq \beta_{p,\mathbb{R}} = p^* - 1 \geq 1$  for any UMD Banach space  $X$  and  $1 < p < \infty$ .

**Definition 3.13.** A random variable  $r : \Omega \rightarrow \{-1, 1\}$  is called a *Rademacher variable* if  $\mathbb{P}(r = 1) = \mathbb{P}(r = -1) = \frac{1}{2}$ .

**Lemma\* 3.14.** Let  $\varepsilon > 0$ ,  $p \in (1, \infty)$ . Then there exists a continuous martingale  $M : [0, 1] \times \Omega \rightarrow [-1, 1]$  with a symmetric distribution such that  $\text{sign } M_1$  is a Rademacher random variable and

$$\|M_1 - \text{sign } M_1\|_{L^p(\Omega)} < \varepsilon. \tag{3.11}$$

We will need a definition of a Paley–Walsh martingale.

**Definition 3.15 (Paley–Walsh martingales).** Let  $X$  be a Banach space. A discrete  $X$ -valued martingale  $(f_n)_{n \geq 0}$  is called a *Paley–Walsh martingale* if there exist a sequence of independent Rademacher variables  $(r_n)_{n \geq 1}$ , a function  $\phi_n : \{-1, 1\}^{n-1} \rightarrow X$  for each  $n \geq 2$  and  $\phi_1 \in X$  such that  $df_n = r_n \phi_n(r_1, \dots, r_{n-1})$  for each  $n \geq 2$  and  $df_1 = r_1 \phi_1$ .

**Remark 3.16.** Let  $X$  be a UMD space,  $1 < p < \infty$ ,  $\delta > 0$ . Then using Proposition 2.1 one can construct a martingale difference sequence  $(d_j)_{j=1}^n \in L^p(\Omega; X)$  and a  $\{-1, 1\}$ -valued sequence  $(\varepsilon_j)_{j=1}^n$  such that

$$\left(\mathbb{E}\left\|\sum_{j=1}^n \frac{\varepsilon_j \pm 1}{2} d_j\right\|^p\right)^{\frac{1}{p}} \geq \frac{\beta_{p,X} - \delta - 1}{2} \left(\mathbb{E}\left\|\sum_{j=1}^n d_j\right\|^p\right)^{\frac{1}{p}}.$$

**Proof of Theorem 3.12.** Denote  $\frac{\beta_{p,X} - \delta - 1}{2}$  by  $\gamma_{p,X}^\delta$ . By Proposition 2.1, there exists a natural number  $N \geq 1$ , a discrete  $X$ -valued martingale  $(f_n)_{n=0}^N$  such that  $f_0 = 0$ , and a sequence of scalars  $(\varepsilon_n)_{n=1}^N$  such that  $\varepsilon_n \in \{0, 1\}$  for each  $n = 1, \dots, N$ , such that

$$\left( \mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n df_n \right\|^p \right)^{\frac{1}{p}} \geq \gamma_{p,X}^\delta (\mathbb{E} \|f_N\|^p)^{\frac{1}{p}}. \tag{3.12}$$

According to [19], Theorem 3.6.1, we can assume that  $(f_n)_{n=0}^N$  is a Paley–Walsh martingale. Let  $(r_n)_{n=1}^N$  be a sequence of Rademacher variables and  $(\phi_n)_{n=1}^N$  be a sequence of functions as in Definition 3.15, that is, be such that  $f_n = \sum_{k=2}^n r_k \phi_k(r_1, \dots, r_{k-1}) + r_1 \phi_1$  for each  $n = 1, \dots, N$ . Without loss of generality, we assume that

$$(\mathbb{E} \|f_N\|^p)^{\frac{1}{p}} \geq 2. \tag{3.13}$$

For each  $n = 1, \dots, N$  define a continuous martingale  $M^n : [0, 1] \times \Omega \rightarrow [-1, 1]$  as in Lemma 3.14, that is, a martingale  $M^n$  with a symmetric distribution such that  $\text{sign } M_1^n$  is a Rademacher variable and

$$\|M_1^n - \text{sign } M_1^n\|_{L^p(\Omega)} < \frac{\delta}{KL}, \tag{3.14}$$

where  $K = \beta_{p,X} N \max\{\|\phi_1\|, \|\phi_2\|_\infty, \dots, \|\phi_N\|_\infty\}$ , and  $L = 2\beta_{p,X}$ . Without loss of generality, suppose that  $(M^n)_{n=1}^N$  are independent. For each  $n = 1, \dots, N$  set  $\sigma_n = \text{sign } M_1^n$ . Define a martingale  $M : [0, N + 1] \times \Omega \rightarrow X$  in the following way:

$$M_t = \begin{cases} 0, & \text{if } 0 \leq t < 1; \\ M_{n-} + M_{t-n}^n \phi_n(\sigma_1, \dots, \sigma_{n-1}), & \text{if } t \in [n, n + 1) \text{ and } \varepsilon_n = 0; \\ M_{n-} + \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}), & \text{if } t \in [n, n + 1) \text{ and } \varepsilon_n = 1. \end{cases}$$

Let  $M = M^d + M^c$  be the decomposition of Theorem 3.1. Then

$$M_{N+1}^c = \sum_{n=1}^N M_1^n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \mathbf{1}_{\varepsilon_n=0},$$

$$M_{N+1}^d = \sum_{n=1}^N \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \mathbf{1}_{\varepsilon_n=1} = \sum_{n=1}^N \varepsilon_n \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}).$$

Notice that  $(\sigma_n)_{n=1}^N$  is a sequence of independent Rademacher variables, so by (3.12) and the discussion thereafter

$$\left( \mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}} \geq \gamma_{p,X}^\delta \left( \mathbb{E} \left\| \sum_{n=1}^N \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}}. \tag{3.15}$$

Let us first show (3.10) with  $i = d$ . Note that by the triangle inequality, (3.13) and (3.14)

$$\begin{aligned}
 (\mathbb{E}\|M_{N+1}\|^p)^{\frac{1}{p}} &\geq (\mathbb{E}\|f_N\|^p)^{\frac{1}{p}} - \sum_{n=1}^N (\mathbb{E}\|(M_1^n - \sigma_n)\phi_n(\sigma_1, \dots, \sigma_{n-1})\|^p)^{\frac{1}{p}} \\
 &\geq 2 - \frac{\delta}{KL} \cdot N \cdot \max\{\|\phi_1\|, \|\phi_2\|_\infty, \dots, \|\phi_N\|_\infty\} > 1.
 \end{aligned}
 \tag{3.16}$$

Therefore,

$$\begin{aligned}
 (\mathbb{E}\|M_{N+1}^d\|^p)^{\frac{1}{p}} &= \left( \mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}} \\
 &\stackrel{(i)}{\geq} \gamma_{p,X}^\delta \left( \mathbb{E} \left\| \sum_{n=1}^N \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}} \\
 &\stackrel{(ii)}{\geq} \gamma_{p,X}^\delta \left( \mathbb{E} \left\| \sum_{n=1}^N \mathbf{1}_{\varepsilon_n=1} \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right. \right. \\
 &\quad \left. \left. + \sum_{n=1}^N \mathbf{1}_{\varepsilon_n=0} M_1^n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}} \\
 &\quad - \gamma_{p,X}^\delta \sum_{n=1}^N (\mathbb{E}\|(M_1^n - \sigma_n)\phi_n(\sigma_1, \dots, \sigma_{n-1})\|^p)^{\frac{1}{p}} \\
 &\stackrel{(iii)}{\geq} \gamma_{p,X}^\delta (\mathbb{E}\|M_{N+1}\|^p)^{\frac{1}{p}} - \frac{\delta}{L} \stackrel{(iv)}{\geq} \left( \frac{\beta_{p,X} - 1}{2} - \delta \right) (\mathbb{E}\|M_{N+1}\|^p)^{\frac{1}{p}},
 \end{aligned}$$

where (i) follows from (3.15), (ii) holds by the triangle inequality, (iii) holds by (3.14), and (iv) follows from (3.16). By the same reason and Remark 3.16, (3.10) holds for  $i = c$ .  $\square$

Let  $p \in (1, \infty)$ . Recall that  $\mathcal{M}_X^p$  is a space of all  $X$ -valued  $L^p$ -martingales,  $\mathcal{M}_X^{p,d}, \mathcal{M}_X^{p,c} \subset \mathcal{M}_X^p$  are its subspaces of purely discontinuous martingales and continuous martingales that start at zero respectively (see Sections 2.2, 2.4, and 2.5).

**Theorem\* 3.17.** *Let  $X$  be a Banach space. Then  $X$  is UMD if and only if for some (or, equivalently, for all)  $p \in (1, \infty)$ , for any probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with any filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  that satisfies the usual conditions,  $\mathcal{M}_X^p = \mathcal{M}_X^{p,d} \oplus \mathcal{M}_X^{p,c}$ , and there exist projections  $A^d, A^c \in \mathcal{L}(\mathcal{M}_X^p)$  such that  $\text{ran } A^d = \mathcal{M}_X^{p,d}$ ,  $\text{ran } A^c = \mathcal{M}_X^{p,c}$ , and for any  $M \in \mathcal{M}_X^p$  the decomposition  $M = A^d M + A^c M$  is the Meyer–Yoeurp decomposition from Theorem 3.1. If this is the case, then*

$$\|A^d\| \leq \beta_{p,X} \quad \text{and} \quad \|A^c\| \leq \beta_{p,X}.
 \tag{3.17}$$

Moreover, there exist  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  such that

$$|A^d|, |A^c| \geq \frac{\beta_{p,X} - 1}{2} \vee 1. \quad (3.18)$$

**Corollary 3.18.** *Let  $X$  be a UMD Banach space,  $p \in (1, \infty)$ . Let  $i \in \{c, d\}$ . Then  $(\mathcal{M}_X^{p,i})^* \simeq \mathcal{M}_{X^*}^{p',i}$ , and for each  $M \in \mathcal{M}_{X^*}^{p',i}$  and  $N \in \mathcal{M}_X^{p,i}$*

$$\langle M, N \rangle := \mathbb{E}\langle M_\infty, N_\infty \rangle, \quad \|M\|_{(\mathcal{M}_X^{p,i})^*} \widetilde{\simeq}_{p,X} \|M\|_{\mathcal{M}_{X^*}^{p',i}}.$$

To prove the corollary above we will need the following lemma.

**Lemma 3.19.** *Let  $X$  be a UMD Banach space,  $p \in (1, \infty)$ ,  $M \in \mathcal{M}_X^{p,d}$ ,  $N \in \mathcal{M}_{X^*}^{p',c}$ . Then  $\mathbb{E}\langle M_\infty, N_\infty \rangle = 0$ .*

**Proof.** First, suppose that  $N_\infty$  takes its values in a finite dimensional subspace  $Y$  of  $X^*$ . Let  $d \geq 1$  be the dimension of  $Y$ ,  $(y_k)_{k=1}^d$  be the basis of  $Y$ . Then there exist  $N^1, \dots, N^d \in \mathcal{M}_{\mathbb{R}}^{p',c}$  such that  $N = \sum_{k=1}^d N^k y_k$ . Hence,

$$E\langle M_\infty, N_\infty \rangle = E\left\langle M_\infty, \sum_{k=1}^d N_\infty^k y_k \right\rangle = \sum_{k=1}^d \mathbb{E}\langle M_\infty, y_k \rangle N_\infty^k \stackrel{(*)}{=} 0, \quad (3.19)$$

where  $(*)$  holds due to Proposition 2.10.

Now turn to the general case. By Remark 3.11 for each  $N \in \mathcal{M}_{X^*}^{p',c}$  there exists a sequence  $(N^n)_{n \geq 1}$  of continuous martingales such that each of  $N^n$  is in  $\mathcal{M}_{X^*}^{p',c}$  and takes its values in a finite dimensional subspace of  $X^*$ , and  $N_\infty^n \rightarrow N_\infty$  in  $L^{p'}(\Omega; X^*)$  as  $n \rightarrow \infty$ . Then due to (3.19),  $E\langle M_\infty, N_\infty \rangle = \lim_{n \rightarrow \infty} E\langle M_\infty, N_\infty^n \rangle = 0$ , so the lemma holds.  $\square$

**Proof of Corollary 3.18.** We will show only the case  $i = d$ , the case  $i = c$  can be shown analogously.

$\mathcal{M}_{X^*}^{p',d} \subset (\mathcal{M}_X^{p,d})^*$  and  $\|M\|_{(\mathcal{M}_X^{p,d})^*} \leq \|M\|_{\mathcal{M}_{X^*}^{p',d}}$  for each  $M \in \mathcal{M}_{X^*}^{p',d}$  thanks to the Hölder inequality. Now let us show the inverse. Let  $f \in (\mathcal{M}_X^{p,d})^*$ . Since due to Proposition 2.14  $\mathcal{M}_X^{p,d}$  is a closed subspace of  $\mathcal{M}_X^p$ , by the Hahn-Banach theorem and Proposition 2.3 there exists  $L \in \mathcal{M}_{X^*}^{p'}$  such that  $\mathbb{E}\langle L_\infty, N_\infty \rangle = f(N)$  for any  $N \in \mathcal{M}_X^{p,d}$ , and  $\|L\|_{\mathcal{M}_{X^*}^{p'}} = \|f\|_{(\mathcal{M}_X^{p,d})^*}$ . Let  $L = L^d + L^c$  be the Meyer–Yoeurp decomposition of  $L$  as in Theorem 3.1. Then by (3.1)

$$\|L^d\|_{\mathcal{M}_{X^*}^{p',d}} \lesssim_{p,X} \|L\|_{\mathcal{M}_{X^*}^{p'}} = \|f\|_{(\mathcal{M}_X^{p,d})^*}$$

and  $\mathbb{E}\langle L_\infty^d, N_\infty \rangle = \mathbb{E}\langle L_\infty, N_\infty \rangle$ , so the theorem holds.  $\square$

### 3.2. Yoeurp decomposition of purely discontinuous martingales

As Yoeurp shown in [44], one can provide further decomposition of a purely discontinuous martingale into two parts: a martingale with accessible jumps and a quasi-left continuous martingale. This subsection is devoted to the generalization of this result to a UMD case.

**Definition 3.20.** Let  $\tau$  be a stopping time. Then  $\tau$  is called a *predictable stopping time* if there exists a sequence of stopping times  $(\tau_n)_{n \geq 1}$  such that  $\tau_n < \tau$  a.s. on  $\{\tau > 0\}$  for each  $n \geq 1$  and  $\tau_n \nearrow \tau$  a.s.

**Definition 3.21.** Let  $\tau$  be a stopping time. Then  $\tau$  is called a *totally inaccessible stopping time* if  $\mathbb{P}\{\tau = \sigma < \infty\} = 0$  for each predictable stopping time  $\sigma$ .

**Definition 3.22.** Let  $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  be an adapted càdlàg process.  $A$  has *accessible jumps* if  $\Delta A_\tau = 0$  a.s. for any totally inaccessible stopping time  $\tau$ .  $A$  is called *quasi-left continuous* if  $\Delta A_\tau = 0$  a.s. for any predictable stopping time  $\tau$ .

For the further information on the definitions given, we refer the reader to [23].

**Remark 3.23.** According to [23], Proposition 25.17, one can show that for any pure jump increasing adapted càdlàg process  $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  there exist unique increasing adapted càdlàg processes  $A^a, A^q : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  such that  $A^a$  has accessible jumps,  $A^q$  is quasi-left continuous,  $A_0^q = 0$  and  $A = A^a + A^q$ .

The following decomposition theorem was shown by Yoeurp in [44] (see also [23], Corollary 26.16):

**Theorem 3.24.** Let  $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  be a purely discontinuous martingale. Then there exist unique purely discontinuous martingales  $M^a, M^q : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  such that  $M^a$  is has accessible jumps,  $M^q$  is quasi-left continuous,  $M_0^q = 0$  and  $M = M^a + M^q$ . Moreover, then  $[M^a] = [M]^a$  and  $[M^q] = [M]^q$ .

**Corollary 3.25.** Let  $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  be a purely discontinuous martingale which is both with accessible jumps and quasi-left continuous. Then  $M = M_0$  a.s.

**Proof.** Without loss of generality, we can set  $M_0 = 0$ . Then  $M = M + 0 = 0 + M$  are decompositions of  $M$  into a sum of a martingale with accessible jumps and a quasi-left continuous martingale. Since by Theorem 3.24 this decomposition is unique,  $M = 0$  a.s.  $\square$

**Proposition\* 3.26.** Let  $1 < p < \infty$ ,  $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  be a purely discontinuous  $L^p$ -martingale. Let  $(M^n)_{n \geq 1}$  be a sequence of purely discontinuous martingales such that  $M_\infty^n \rightarrow M_\infty$  in  $L^p(\Omega)$ . Then the following assertions hold

- (a) if  $(M^n)_{n \geq 1}$  have accessible jumps, then  $M$  has accessible jumps as well;
- (b) if  $(M^n)_{n \geq 1}$  are quasi-left continuous martingales, then  $M$  is quasi-left continuous as well.

**Definition 3.27.** Let  $X$  be a Banach space. A martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  has *accessible jumps* if  $\Delta M_\tau = 0$  a.s. for any totally inaccessible stopping time  $\tau$ . A martingale  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  is called *quasi-left continuous* if  $\Delta M_\tau = 0$  a.s. for any predictable stopping time  $\tau$ .

**Lemma\* 3.28.** Let  $X$  be a reflexive Banach space,  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  be a purely discontinuous martingale.

- (i)  $M$  has accessible jumps if and only if for each  $x^* \in X^*$  the martingale  $\langle M, x^* \rangle$  has accessible jumps;
- (ii)  $M$  is quasi-left continuous if and only if for each  $x^* \in X^*$  the martingale  $\langle M, x^* \rangle$  is quasi-left continuous.

**Definition 3.29.** Let  $X$  be a Banach space,  $p \in (1, \infty)$ . Then we define  $\mathcal{M}_X^{p,q} \subset \mathcal{M}_X^{p,d}$  as a linear space of all  $X$ -valued purely discontinuous quasi-left continuous  $L^p$ -martingales which start at 0. We define  $\mathcal{M}_X^{p,a} \subset \mathcal{M}_X^{p,d}$  as a linear space of all  $X$ -valued purely discontinuous  $L^p$ -martingales with accessible jumps.

**Proposition\* 3.30.** Let  $X$  be a Banach space,  $1 < p < \infty$ . Then  $\mathcal{M}_X^{p,q}$  and  $\mathcal{M}_X^{p,a}$  are closed subspaces of  $\mathcal{M}_X^{p,d}$ .

The following lemma follows from Corollary 3.25.

**Lemma\* 3.31.** Let  $X$  be a Banach space,  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  be a purely discontinuous martingale. Let  $M$  be both with accessible jumps and quasi-left continuous. Then  $M = M_0$  a.s. In other words,  $\mathcal{M}_X^{p,q} \cap \mathcal{M}_X^{p,a} = 0$ .

The main theorem of this subsection is the following UMD variant of Theorem 3.24.

**Theorem 3.32.** Let  $X$  be a UMD Banach space,  $M : \mathbb{R}_+ \times \Omega \rightarrow X$  be a purely discontinuous  $L^p$ -martingale. Then there exist unique purely discontinuous martingales  $M^a, M^q : \mathbb{R}_+ \times \Omega \rightarrow X$  such that  $M^a$  has accessible jumps,  $M^q$  is quasi-left continuous,  $M_0^q = 0$  and  $M = M^a + M^q$ . Moreover, if this is the case, then for  $i \in \{a, q\}$

$$(\mathbb{E} \|M_\infty^i\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_\infty\|^p)^{\frac{1}{p}}. \tag{3.20}$$

**Proof.** *Step 1: finite dimensional case.* First, assume that  $X$  is finite dimensional. Then  $M^a$  and  $M^q$  exist and unique due to coordinate-wise applying of Theorem 3.24. Let  $M = M^a + M^q$ ,  $N = M^a$ . Then for any  $x^* \in X^*$ ,  $t \geq 0$  by Theorem 3.24 and Lemma 3.28 a.s.

$$[\langle M, x^* \rangle]_t = [\langle M, x^* \rangle]_t^a + [\langle M, x^* \rangle]_t^q = [\langle M^a, x^* \rangle]_t + [\langle M^q, x^* \rangle]_t,$$

and

$$[\langle N, x^* \rangle]_t = [\langle N, x^* \rangle]_t^a + [\langle N, x^* \rangle]_t^q = [\langle M^a, x^* \rangle]_t.$$

Therefore, a.s.

$$[(N, x^*)]_t - [(N, x^*)]_s \leq [(M, x^*)]_t - [(M, x^*)]_s, \quad 0 \leq s < t.$$

Moreover  $M_0 = N_0$ . Hence,  $N$  is weakly differentially subordinated to  $M$  (see Section 4), and (3.20) for  $i = a$  follows from [41]. By the same reason and since  $M_0^q = 0$ , (3.20) holds true for  $i = q$ .

*Step 2: general case.* Now let  $X$  be general. Let  $\xi = M_\infty$ . Without loss of generality, we set  $\mathcal{F}_\infty = \mathcal{F}_t$ . Let  $(\xi_n)_{n \geq 1}$  be a sequence of simple  $\mathcal{F}_t$ -measurable functions in  $L^p(\Omega; X)$  such that  $\xi_n \rightarrow \xi$  as  $n \rightarrow \infty$  in  $L^p(\Omega; X)$ . For each  $n \geq 1$  define  $\mathcal{F}_t$ -measurable  $\xi_n^d$  and  $\xi_n^c$  such that  $M^{d,n} = (\mathbb{E}(\xi_n^d | \mathcal{F}_s))_{s \geq 0}$  and  $M^{c,n} = (\mathbb{E}(\xi_n^c | \mathcal{F}_s))_{s \geq 0}$  are respectively, purely discontinuous and continuous parts of a martingale  $(\mathbb{E}(\xi_n | \mathcal{F}_s))_{s \geq 0}$  as in Remark 2.12. Then thanks to Theorem 3.1,  $\xi_n^d \rightarrow \xi$  and  $\xi_n^c \rightarrow 0$  in  $L^p(\Omega; X)$  as  $n \rightarrow \infty$  since  $M$  is purely discontinuous.

Since for each  $n \geq 1$  the random variable  $\xi_n^d$  takes its values in a finite dimensional space, by Theorem 3.24 there exist  $\mathcal{F}_t$ -measurable  $\xi^a, \xi^q \in L^p(\Omega; X)$  such that purely discontinuous martingales  $M^{a,n} = (\mathbb{E}(\xi_n^a | \mathcal{F}_s))_{s \geq 0}$  and  $M^{q,n} = (\mathbb{E}(\xi_n^q | \mathcal{F}_s))_{s \geq 0}$  are respectively with accessible jumps and quasi-left continuous,  $\mathbb{E}(\xi_n^q | \mathcal{F}_0) = 0$ , and the decomposition  $M^{d,n} = M^{a,n} + M^{q,n}$  is as in Theorem 3.24. Since  $(\xi_n^d)_{n \geq 1}$  is a Cauchy sequence in  $L^p(\Omega; X)$ , by Step 1 both  $(\xi_n^a)_{n \geq 1}$  and  $(\xi_n^q)_{n \geq 1}$  are Cauchy in  $L^p(\Omega; X)$  as well. Let  $\xi^a$  and  $\xi^q$  be their limits. Define martingales  $M^a, M^q : \mathbb{R}_+ \times \Omega \rightarrow X$  in the following way:

$$M_s^a := \mathbb{E}(\xi^a | \mathcal{F}_s), \quad M_s^q := \mathbb{E}(\xi^q | \mathcal{F}_s), \quad s \geq 0.$$

By Proposition 3.30  $M^a$  is a martingale with accessible jumps,  $M^q$  is quasi-left continuous,  $M_0^q = 0$  a.s., and therefore  $M = M^a + M^q$  is the desired decomposition. Moreover, by Step 1 for each  $n \geq 1$  and  $i \in \{a, q\}$ ,  $(\mathbb{E}\|\xi_n^i\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E}\|\xi_n^d\|^p)^{\frac{1}{p}}$ , and hence the estimate (3.20) follows by letting  $n$  to infinity.

The uniqueness of the decomposition follows from Lemma 3.31. □

The following theorem, as Theorem 3.12, illustrates that the decomposition in Theorem 3.32 takes place only in the UMD space case.

**Theorem 3.33.** *Let  $X$  be a finite dimensional Banach space,  $p \in (1, \infty)$ ,  $\delta \in (0, \frac{\beta_{p,X}-1}{2})$ . Then there exist purely discontinuous martingales  $M^a, M^q : \mathbb{R}_+ \times \Omega \rightarrow X$  such that  $M^a$  has accessible jumps,  $M^q$  is quasi-left continuous,  $\mathbb{E}\|M_\infty^a\|^p, \mathbb{E}\|M_\infty^q\|^p < \infty$ ,  $M_0^a = M_0^q = 0$ , and for  $M = M^a + M^q$  and  $i \in \{a, q\}$  the following holds*

$$(\mathbb{E}\|M_\infty^i\|^p)^{\frac{1}{p}} \geq \left( \frac{\beta_{p,X}-1}{2} - \delta \right) (\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{3.21}$$

For the proof, we will need the following lemma.

**Lemma 3.34.** *Let  $\varepsilon \in (0, \frac{1}{2})$ ,  $p \in (1, \infty)$ . Then there exist martingales  $M, M^a, M^q : [0, 1] \times \Omega \rightarrow [-1 - \varepsilon, 1 + \varepsilon]$  with symmetric distributions such that  $M^a$  is a martingale with accessible*

jumps,  $\|M_1^a\|_{L^p(\Omega)} < \varepsilon$ ,  $M^q$  is a quasi-left continuous martingale,  $M_0^q = 0$  a.s.,  $M = M^a + M^q$ ,  $\text{sign } M_1$  is a Rademacher random variable and

$$\|M_1 - \text{sign } M_1\|_{L^p(\Omega)} < \varepsilon. \tag{3.22}$$

**Proof.** Let  $N^+, N^- : [0, 1] \times \Omega \rightarrow \mathbb{R}$  be independent Poisson processes with the same intensity  $\lambda_\varepsilon$  such that  $\mathbb{P}(N_1^+ = 0) = \mathbb{P}(N_1^- = 0) < \frac{\varepsilon^p}{2^p}$  (such  $\lambda_\varepsilon$  exists since  $N_1^+$  and  $N_1^-$  have Poisson distributions, see [25]). Define a stopping time  $\tau$  in the following way:

$$\tau = \inf\{t : N_t^+ \geq 1\} \wedge \inf\{t : N_t^- \geq 1\} \wedge 1.$$

Let  $M_t^q := N_{t \wedge \tau}^+ - N_{t \wedge \tau}^-$ ,  $t \in [0, 1]$ . Then  $M^q$  is quasi-left continuous with a symmetric distribution. Let  $r$  be an independent Rademacher variable,  $M_t^a = \frac{\varepsilon}{2}r$  for each  $t \in [0, 1]$ . Then  $M^a$  is a martingale with accessible jumps and symmetric distribution, and  $\|M_1^a\|_{L^p(\Omega)} = \frac{\varepsilon}{2} < \varepsilon$ . Let  $M = M^a + M^q$ . Then a.s.

$$M_1 \in \left\{ -1 - \frac{\varepsilon}{2}, -1 + \frac{\varepsilon}{2}, -\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2} \right\}, \tag{3.23}$$

so  $\mathbb{P}(M_1 = 0) = 0$ , and therefore  $\text{sign } M_1$  is a Rademacher random variable. Let us prove (3.22). Notice that due to (3.23) if  $|M_1^q| = 1$ , then  $|M_1 - \text{sign } M_1| < \frac{\varepsilon}{2}$ , and if  $|M_1^q| = 0$ , then  $|M_1 - \text{sign } M_1| < 1$ . Therefore,

$$\begin{aligned} \mathbb{E}|M_1 - \text{sign } M_1|^p &= \mathbb{E}|M_1 - \text{sign } M_1|^p \mathbf{1}_{|M_1^q|=1} + \mathbb{E}|M_1 - \text{sign } M_1|^p \mathbf{1}_{|M_1^q|=0} \\ &< \frac{\varepsilon^p}{2^p} + \frac{\varepsilon^p}{2^p} < \varepsilon^p, \end{aligned}$$

so (3.22) holds. □

**Proof of Theorem 3.33.** The proof is analogous to the proof of Theorem 3.12, while one has to use Lemma 3.34 instead of Lemma 3.14. □

Theorem 3.33 yields the following characterization of the UMD property.

**Theorem 3.35.** *Let  $X$  be a Banach space. Then  $X$  is a UMD Banach space if and only if for some (equivalently, for all)  $p \in (1, \infty)$  there exists  $c_{p,X} > 0$  such that for any  $L^p$ -martingale  $M := \mathbb{R}_+ \times \Omega \rightarrow X$  there exist unique martingales  $M^c, M^q, M^a : \mathbb{R}_+ \times \Omega \rightarrow X$  such that  $M_0^c = M_0^q = 0$ ,  $M^c$  is continuous,  $M^q$  is purely discontinuous quasi-left continuous,  $M^a$  is purely discontinuous with accessible jumps,  $M = M^c + M^q + M^a$ , and*

$$\left(\mathbb{E}\|M_\infty^c\|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}\|M_\infty^q\|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}\|M_\infty^a\|^p\right)^{\frac{1}{p}} \leq c_{p,X} \left(\mathbb{E}\|M_\infty\|^p\right)^{\frac{1}{p}}. \tag{3.24}$$

If this is the case, then the least admissible  $c_{p,X}$  is in the interval  $\left[\frac{3\beta_{p,X}-3}{2} \vee 1, 3\beta_{p,X}\right]$ .



The decomposition  $M = M^c + M^q + M^a$  is called the *canonical decomposition* of the martingale  $M$  (see [14,23,44]).

**Proof.** The “if and only if” part follows from Theorem 3.17, Theorem 3.32 and Theorem 3.33. The estimate  $c_{p,X} \leq 3\beta_{p,X}$  follows from (3.1) and (3.20). The estimate  $c_{p,X} \geq \frac{3\beta_{p,X}-3}{2} \vee 1$  follows from (3.10) and (3.21). □

**Corollary 3.36.** *Let  $X$  be a Banach space. Then  $X$  is a UMD Banach space if and only if  $\mathcal{M}_X^{p,d} = \mathcal{M}_X^{p,a} \oplus \mathcal{M}_X^{p,q}$  and  $\mathcal{M}_X^p = \mathcal{M}_X^{p,c} \oplus \mathcal{M}_X^{p,q} \oplus \mathcal{M}_X^{p,a}$  for any filtration that satisfies the usual conditions.*

**Proof.** The corollary follows from Theorem 3.32, Theorem 3.33 and Theorem 3.35. □

### 3.3. Stochastic integration

The current subsection is devoted to application of Theorem 3.35 to stochastic integration with respect to a general martingale.

**Proposition\* 3.37.** *Let  $H$  be a Hilbert space,  $X$  be a Banach space,  $M : \mathbb{R}_+ \times \Omega \rightarrow H$  be a martingale,  $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$  be elementary progressive. Then*

- (i) *if  $M$  is continuous, then  $\Phi \cdot M$  is continuous;*
- (ii) *if  $M$  is purely discontinuous, then  $\Phi \cdot M$  is purely discontinuous;*
- (iii) *if  $M$  has accessible jumps, then  $\Phi \cdot M$  has accessible jumps;*
- (iv) *if  $M$  is quasi-left continuous, then  $\Phi \cdot M$  is quasi-left continuous.*

**Proposition 3.38.** *Let  $H$  be a Hilbert space,  $M : \mathbb{R}_+ \times \Omega \rightarrow H$  be a local martingale. Then there exist unique martingales  $M^c, M^q, M^a : \mathbb{R}_+ \times \Omega \rightarrow H$  such that  $M^c$  is continuous,  $M^q$  and  $M^a$  are purely discontinuous,  $M^q$  is quasi-left continuous,  $M^a$  has accessible jumps,  $M_0^c = M_0^q = 0$  a.s., and  $M = M^c + M^q + M^a$ .*

**Proof.** Analogously to Theorem 26.14 and Corollary 26.16 in [23]. □

**Theorem 3.39.** *Let  $H$  be a Hilbert space,  $X$  be a UMD Banach space,  $p \in (1, \infty)$ ,  $M : \mathbb{R}_+ \times \Omega \rightarrow H$  be a local martingale,  $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$  be elementary progressive. Let  $M = M^c + M^q + M^a$  be the canonical decomposition from Proposition 3.38. Then*

$$\mathbb{E}\|(\Phi \cdot M)_\infty\|^p \approx_{p,X} \mathbb{E}\|(\Phi \cdot M^c)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^q)_\infty\|^p + \mathbb{E}\|(\Phi \cdot M^a)_\infty\|^p \tag{3.25}$$

*and if  $(\Phi \cdot M)_\infty \in L^p(\Omega; X)$ , then  $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$  is the canonical decomposition from Theorem 3.35.*

**Proof.** The statement that  $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$  is the canonical decomposition follows from Proposition 3.37, Theorem 3.35 and the fact that a.s.  $(\Phi \cdot M)_0 = (\Phi \cdot M^c)_0 = (\Phi \cdot M^q)_0 = 0$ . (3.25) follows then from (3.24) and the triangle inequality. □

**Remark 3.40.** Notice that the Itô isomorphism for the term  $\Phi \cdot M^c$  from (3.25) was explored in [37]. It remains open what to do with the other two terms, but positive results in this direction were obtained in the case of  $X = L^q(S)$  in [14].

### 4. Weak differential subordination and general martingales

This subsection is devoted to the generalization of the main theorem in work [41]. Namely, here we show the  $L^p$ -estimates for general  $X$ -valued weakly differentially subordinated martingales.

**Definition 4.1.** Let  $X$  be a Banach space,  $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$  be local martingales. Then  $N$  is weakly differentially subordinated to  $M$  if  $[\langle M, x^* \rangle] - [\langle N, x^* \rangle]$  is an increasing process a.s. for each  $x^* \in X^*$ .

The following theorem have been proven in [41].

**Theorem 4.2.** Let  $X$  be a Banach space. Then  $X$  has the UMD property if and only if for some (equivalently, for all)  $p \in (1, \infty)$  there exists  $\beta > 0$  such that for each pair of purely discontinuous martingales  $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$  such that  $N$  is weakly differentially subordinated to  $M$  one has that

$$(\mathbb{E}\|N_\infty\|^p)^{\frac{1}{p}} \leq \beta(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}.$$

If this is the case, then the least admissible  $\beta$  is the UMD constant  $\beta_{p,X}$ .

The main goal of the current section is to prove the following generalization of Theorem 4.2 to the case of arbitrary martingales.

**Theorem 4.3.** Let  $X$  be a UMD Banach space,  $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$  be two martingales such that  $N$  is weakly differentially subordinated to  $M$ . Then for each  $p \in (1, \infty), t \geq 0$ ,

$$(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2(\beta_{p,X} + 1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{4.1}$$

The proof will be done in several steps. First, we show an analogue of Theorem 4.2 for continuous martingales.

**Theorem\* 4.4.** Let  $X$  be a Banach space. Then  $X$  is a UMD Banach space if and only if for some (equivalently, for all)  $p \in (1, \infty)$  there exists  $c > 0$  such that for any continuous martingales  $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$  such that  $N$  is weakly differentially subordinated to  $M, M_0 = N_0 = 0$ , one has that

$$(\mathbb{E}\|N_\infty\|^p)^{\frac{1}{p}} \leq c_{p,X}(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \tag{4.2}$$

If this is the case, then the least admissible  $c_{p,X}$  is in the segment  $[\beta_{p,X}, \beta_{p,X}^2]$ .

For the proof, we will need the following proposition, which demonstrates that one needs a slightly weaker assumption rather than in Theorem 4.4 so that the estimate (4.2) holds in a UMD Banach space.

**Proposition 4.5.** *Let  $X$  be a UMD Banach space,  $1 < p < \infty$ ,  $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$  be continuous  $L^p$ -martingales s.t.  $M_0 = N_0 = 0$  and for each  $x^* \in X^*$  a.s. for each  $t \geq 0$*

$$[[N, x^*]]_t \leq [[M, x^*]]_t. \tag{4.3}$$

Then for each  $t \geq 0$

$$(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}. \tag{4.4}$$

**Proof.** Without loss of generality by a stopping time argument, we assume that  $M$  and  $N$  are bounded and that  $M_\infty = M_t$  and  $N_\infty = N_t$ .

One can also restrict to a finite dimensional case. Indeed, since  $X$  is a separable reflexive space,  $X^*$  is separable as well. Let  $(Y_m)_{m \geq 1}$  be an increasing sequence of finite-dimensional subspaces of  $X^*$  such that  $\overline{\bigcup_m Y_m} = X^*$  and  $\|\cdot\|_{Y_m} = \|\cdot\|_{X^*|_{Y_m}}$  for each  $m \geq 1$ . Then for each fixed  $m \geq 1$  there exists a linear operator  $P_m : X \rightarrow Y_m^*$  of norm 1 defined as follows:  $\langle P_m x, y \rangle = \langle x, y \rangle$  for each  $x \in X, y \in Y_m$ . Therefore  $P_m M$  and  $P_m N$  are  $Y_m^*$ -valued martingales. Moreover, (4.3) holds for  $P_m M$  and  $P_m N$  since there exists  $P_m^* : Y_m \rightarrow X^*$ , and for each  $y \in Y_m$  we have that  $\langle P_m M, y \rangle = \langle M, P_m y \rangle$  and  $\langle P_m N, y \rangle = \langle N, P_m y \rangle$ . Since  $Y_m$  is a closed subspace of  $X^*$ , [19], Proposition 4.2.17, yields  $\beta_{p',Y_m} \leq \beta_{p',X^*}$ , consequently again by [19], Proposition 4.2.17,  $\beta_{p,Y_m^*} \leq \beta_{p,X^{**}} = \beta_{p,X}$ . So if we prove the finite dimensional version, then

$$(\mathbb{E}\|P_m N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,Y_m^*}^2 (\mathbb{E}\|P_m M_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E}\|P_m M_t\|^p)^{\frac{1}{p}},$$

and (4.4) with  $c_{p,X} = \beta_{p,X}^2$  will follow by letting  $m \rightarrow \infty$ .

Let  $d$  be the dimension of  $X$ ,  $\|\cdot\|$  be a Euclidean norm on  $X \times X$ . Let  $L = (M, N) : \mathbb{R}_+ \times \Omega \rightarrow X \times X$  be a continuous martingale. Since  $(X \times X, \|\cdot\|)$  is a Hilbert space,  $L$  has a continuous quadratic variation  $[L] : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$  (see Remark 2.5). Let  $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$  be such that  $A_s = [L]_s + s$  for each  $s \geq 0$ . Then  $A$  is continuous strictly increasing predictable. Define a random time-change  $(\tau_s)_{s \geq 0}$  as in Theorem 2.16. Let  $\mathbb{G} = (\mathcal{G}_s)_{s \geq 0} = (\mathcal{F}_{\tau_s})_{s \geq 0}$  be the induced filtration. Then thanks to the Kazamaki theorem [23], Theorem 17.24,  $\tilde{L} = L \circ \tau$  is a  $G$ -martingale, and  $[\tilde{L}] = [L] \circ \tau$ . Notice that  $\tilde{L} = (\tilde{M}, \tilde{N})$  with  $\tilde{M} = M \circ \tau, \tilde{N} = N \circ \tau$ , and since by Kazamaki theorem [23], Theorem 17.24,  $[M \circ \tau] = [M] \circ \tau, [N \circ \tau] = [N] \circ \tau$ , and  $(M \circ \tau)_0 = (N \circ \tau)_0 = 0$ , we have that by (4.3) for each  $x^* \in X^*$  a.s. for each  $s \geq 0$

$$[[\tilde{N}, x^*]]_s = [[N, x^*]]_{\tau_s} \leq [[M, x^*]]_{\tau_s} = [[\tilde{M}, x^*]]_s. \tag{4.5}$$

Moreover, for all  $0 \leq u < s$  we have that a.s.

$$\begin{aligned} [\tilde{L}]_s - [\tilde{L}]_u &= ([L] \circ \tau)_s - ([L] \circ \tau)_u \leq ([L] \circ \tau)_s + \tau_s - ([L] \circ \tau)_u - \tau_u \\ &= ([L]_{\tau_s} + \tau_s) - ([L]_{\tau_u} + \tau_u) = s - u. \end{aligned}$$

Therefore  $[\tilde{L}]$  is a.s. absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}_+$ . Consequently, due to Theorem 2.19, there exists an enlarged probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  with an enlarged filtration  $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_s)_{s \geq 0}$ , a  $2d$ -dimensional standard Wiener process  $W$ , which is defined on  $\tilde{\mathbb{G}}$ , and a stochastically integrable progressively measurable function  $f : \mathbb{R}_+ \times \tilde{\Omega} \rightarrow \mathcal{L}(\mathbb{R}^{2d}, X \times X)$  such that  $\tilde{L} = f \cdot W$ . Let  $f^M, f^N : \mathbb{R}_+ \times \tilde{\Omega} \rightarrow \mathcal{L}(\mathbb{R}^{2d}, X)$  be such that  $f = (f^M, f^N)$ . Then  $\tilde{M} = f^M \cdot W$  and  $\tilde{N} = f^N \cdot W$ . Let  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$  be an independent probability space with a filtration  $\bar{\mathbb{G}}$  and a  $2d$ -dimensional Wiener process  $\bar{W}$  on it. Denote by  $\bar{\mathbb{E}}$  the expectation on  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ . Then because of the decoupling theorem [19], Theorem 4.4.1, for each  $s \geq 0$

$$\begin{aligned} (\mathbb{E}\|\tilde{N}_s\|^p)^{\frac{1}{p}} &= (\mathbb{E}\|(f^N \cdot W)_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\bar{\mathbb{E}}\|(f^N \cdot \bar{W})_s\|^p)^{\frac{1}{p}}, \\ \frac{1}{\beta_{p,X}}(\mathbb{E}\bar{\mathbb{E}}\|(f^M \cdot \bar{W})_s\|^p)^{\frac{1}{p}} &\leq (\mathbb{E}\|(f^M \cdot W)_s\|^p)^{\frac{1}{p}} = (\mathbb{E}\|\tilde{M}_s\|^p)^{\frac{1}{p}}. \end{aligned} \tag{4.6}$$

Due to the multidimensional version of [23], Theorem 17.11, and (4.5) for each  $x^* \in X^*$  we have that

$$s \mapsto [([\tilde{M}, x^*])_s] - [([\tilde{N}, x^*])_s] = \int_0^s (|\langle x^*, f^M(r) \rangle|^2 - |\langle x^*, f^N(r) \rangle|^2) dr \tag{4.7}$$

is nonnegative and absolutely continuous a.s. Since  $X$  is separable, we can fix a set  $\tilde{\Omega}_0 \subset \tilde{\Omega}$  of full measure on which the function (4.7) is nonnegative for each  $s \geq 0$ .

Now fix  $\omega \in \tilde{\Omega}_0$  and  $s \geq 0$ . Let us prove that

$$\bar{\mathbb{E}}\|(f^N(\omega) \cdot \bar{W})_s\|^p \leq \bar{\mathbb{E}}\|(f^M(\omega) \cdot \bar{W})_s\|^p.$$

Since  $f^M(\omega)$  and  $f^N(\omega)$  are deterministic on  $\bar{\Omega}$ , and since due to (4.7) for each  $x^* \in X^*$

$$\begin{aligned} \bar{\mathbb{E}}|\langle (f^N(\omega) \cdot \bar{W})_s, x^* \rangle|^2 &= \int_0^s |\langle x^*, f^N(r, \omega) \rangle|^2 dr \\ &\leq \int_0^s |\langle x^*, f^M(r, \omega) \rangle|^2 dr = \bar{\mathbb{E}}|\langle (f^M(\omega) \cdot \bar{W})_s, x^* \rangle|^2, \end{aligned}$$

by [36], Corollary 4.4, we have that  $\bar{\mathbb{E}}\|(f^N(\omega) \cdot \bar{W})_s\|^p \leq \bar{\mathbb{E}}\|(f^M(\omega) \cdot \bar{W})_s\|^p$ . Consequently, due to (4.6) and the fact that  $\bar{\mathbb{P}}(\Omega_0) = 1$

$$(\mathbb{E}\|\tilde{N}_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\bar{\mathbb{E}}\|(f^N \cdot \bar{W})_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\bar{\mathbb{E}}\|(f^M \cdot \bar{W})_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2(\mathbb{E}\|\tilde{M}_s\|^p)^{\frac{1}{p}}.$$

Recall that  $\tilde{M}$  and  $\tilde{N}$  are bounded, so thanks to the dominated convergence theorem one gets (4.4) with  $c_{p,X} = \beta_{p,X}^2$  by letting  $s$  to infinity. □

**Proof of Theorem 4.4.** *The “only if” part & the upper bound of  $c_{p,X}$ :* The “only if” part and the estimate  $c_{p,X} \leq \beta_{p,X}^2$  follows from Proposition 4.5 since (4.3) holds for  $M$  and  $N$  because  $N$  is weakly differentially subordinated to  $M$ .

*The “if” part & the lower bound of  $c_{p,X}$ :* See the supplement [43]. □

**Remark 4.6.** Let  $X$  be a Banach space. Then according to [6,8,17] the Hilbert transform  $\mathcal{H}_X$  can be extended to  $L^p(\mathbb{R}; X)$  for each  $1 < p < \infty$  if and only if  $X$  is a UMD Banach space. Moreover, if this is the case, then

$$\sqrt{\beta_{p,X}} \leq \|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \leq \beta_{p,X}^2.$$

As it was shown in [41], the upper bound  $\beta_{p,X}^2$  can be also directly derived from the upper bound for  $c_{p,X}$  in Theorem 4.4. The sharp upper bound for  $\|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R}; X))}$  remains an open question (see [19], pp. 496–497), so the sharp upper bound for  $c_{p,X}$  is of interest.

**Lemma\* 4.7.** Let  $X$  be a Banach space,  $M^c, N^c : \mathbb{R}_+ \times \Omega \rightarrow X$  be continuous martingales,  $M^d, N^d : \mathbb{R}_+ \times \Omega \rightarrow X$  be purely discontinuous martingales,  $M_0^c = N_0^c = 0$ . Let  $M := M^c + M^d$ ,  $N := N^c + N^d$ . Suppose that  $N$  is weakly differentially subordinated to  $M$ . Then  $N^c$  is weakly differentially subordinated to  $M^c$ , and  $N^d$  is weakly differentially subordinated to  $M^d$ .

**Proof of Theorem 4.3.** By Theorem 3.1 there exist martingales  $M^d, M^c, N^d, N^c : \mathbb{R}_+ \times \Omega \rightarrow X$  such that  $M^d$  and  $N^d$  are purely discontinuous,  $M^c$  and  $N^c$  are continuous,  $M_0^c = N_0^c = 0$ , and  $M = M^d + M^c$  and  $N = N^d + N^c$ . By Lemma 4.7,  $N^d$  is weakly differentially subordinated to  $M^d$  and  $N^c$  is weakly differentially subordinated to  $M^c$ . Therefore, for each  $t \geq 0$

$$\begin{aligned} (\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} &\stackrel{(i)}{\leq} (\mathbb{E}\|N_t^d\|^p)^{\frac{1}{p}} + (\mathbb{E}\|N_t^c\|^p)^{\frac{1}{p}} \stackrel{(ii)}{\leq} \beta_{p,X}^2 (\mathbb{E}\|M_t^d\|^p)^{\frac{1}{p}} + \beta_{p,X} (\mathbb{E}\|M_t^c\|^p)^{\frac{1}{p}} \\ &\stackrel{(iii)}{\leq} \beta_{p,X}^2 (\beta_{p,X} + 1) (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \end{aligned}$$

where (i) holds thanks to the triangle inequality, (ii) follows from Theorem 4.2 and Theorem 4.4, and (iii) follows from (3.1). □

**Remark 4.8.** It is worth noticing that in a view of recent results the sharp constant in (3.1) and (3.20) can be derived and equals the  $UMD_p^{(0,1)}$  constant  $\beta_{p,X}^{(0,1)}$ . In order to show that this is the right upper bound one needs to use a  $\{0, 1\}$ -Burkholder function instead of the Burkholder function, while the sharpness follows analogously Theorem 3.12 and 3.33. See [40] for details.

**Remark 4.9.** In the recent paper, [42] the existence of the canonical decomposition of a general local martingale together with the corresponding weak  $L^1$ -estimates were shown. Again existence of the canonical decomposition of any  $X$ -valued martingale is equivalent to  $X$  having the UMD property.

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## Supplementary Material

**Some proofs** (DOI: [10.3150/18-BEJ1031SUPP](https://doi.org/10.3150/18-BEJ1031SUPP); .pdf). Recall that throughout the paper many technical proofs have been omitted. The reader can find those proofs in the supplementary file.

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