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New Mass-Lumped Tetrahedral Elements for 3D Wave Propagation Modelling

S. Geevers[1,*]

Abstract

We present a new accuracy condition for constructing mass-lumped elements. This condition is less restrictive than the one previously used and enabled us to construct new mass-lumped tetrahedral elements for 3D wave propagation modelling. The new degree-2 and degree-3 elements require significantly fewer nodes than previous versions and mass-lumped tetrahedral elements of higher degree had not been found before. We also present a new accuracy condition for evaluating the stiffness matrix-vector product. This enabled us to obtain tailored quadrature rules for the new elements that further reduce the computational cost.

Keywords: mass lumping, tetrahedral element, spectral element, wave equation

1 Introduction

Finite element methods offer a good alternative to finite difference methods for wave propagation modelling when the geometry of objects or the topography of seismic models need to be accurately modelled. Tetrahedral elements are particularly suitable for complex 3D models due to their geometric flexibility. A major drawback of the classical finite element method, however, is that, at each time step, it requires solving a large system of equations of the form $Mx = y$, with M the stiffness matrix, which is large and sparse, but not diagonal. The high computational cost required for solving this linear problem is avoided by lumping the mass matrix into a diagonal matrix. This is accomplished by approximating the matrix using a quadrature rule and by placing the basis function nodes at the quadrature points.

For first-order elements, mass-lumping is accomplished by placing the quadrature points and nodes at the vertices. Higher-order triangular and tetrahedral mass-lumped elements are obtained by enriching the element space with higher-degree bubble functions [1–3, 5]. For tetrahedra, only elements up to degree 3 [1, 5] were available. By deriving a new and less restrictive accuracy condition for the quadrature rule, we could obtain mass-lumped tetrahedral elements of degree 4 and new elements of degree 2 and 3 with significantly fewer nodes.

Table 1: Element spaces $\hat{U}$ of mass-lumped tetrahedral elements of degree p with # nodes. Asterisks indicate new elements.

p	#	$\hat{U}$
2*	15	$P_2 + F + I$
2	23	$P_2 + FP_1 + I$
3*	32	$P_3 + FP_1 + IP_1$
3	50	$P_3 + FP_2 + IP_2$
4*	60	$P_4 + FP_2 + I(P_2 + F)$
4*	61	$P_4 + FP_2 + I(P_2 + F + I)$
4*	65	$P_4 + F(P_2 + F) + I(P_2 + F + I)$

To further improve the efficiency of the numerical scheme, we also derived a new accuracy condition for evaluating the stiffness matrix-vector product and obtained new tailored quadrature rules with less points than rules previously available. It is known that, for higher-degree elements, computing the stiffness matrix-vector product on the fly is more efficient than using a pre-assembled matrix and recently, we showed that the new quadrature rules also outperform exact integration algorithms [4].

2 New mass-lumped elements

Previously, the accuracy condition imposed on the quadrate rule for the mass matrix was exactness for polynomials of degree $p+p'-2$, with p the polynomial degree of the element and p' the highest polynomial degree of the enriched element space. We proved in [3] that, to obtain optimal convergence rates, it is sufficient if the quadrature rule is exact for $\hat{U} \otimes P_{p-2}$, where $\hat{U}$ denotes the element space and P_{p-2} the polyno-

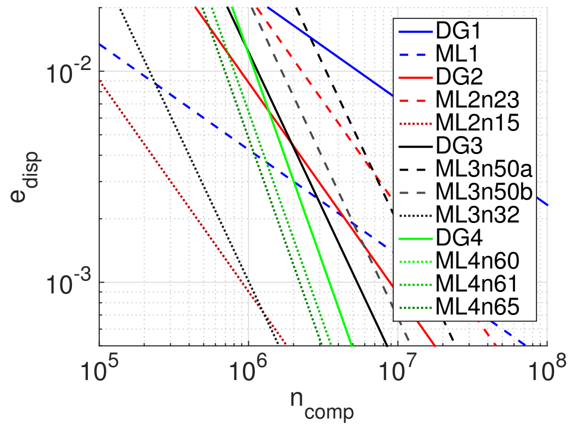

Figure 1: Dispersion error of the symmetric interior penalty DG method (solid), the former mass-lumped element method (dashed), and the new mass-lumped elements (dotted).

mials of degree at most $p-2$.

Using this new accuracy condition, we were able to construct new mass-lumped tetrahedral elements. An overview of the different elements is given in Table 1. There, F denotes the set of degree-3 face bubble functions and I denotes the degree-4 interior bubble function.

The efficiency of the new and former mass-lumped elements, together with symmetric interior penalty discontinuous Galerkin methods, have also been compared using a dispersion analysis. The dispersion error, defined as the relative error of the numerical wave propagation speed, is computed for travelling plane waves moving through a regular tetrahedral mesh and is plotted in Figure 1 against the estimated computational cost, which is based on the size of the stiffness matrix and the number of time steps. The figure shows that the new mass-lumped methods are more efficient than the previous ones and the DG methods, especially for $p=2$ and $p=3$.

3 Tailored quadrature for the stiffness matrix

To obtain optimal convergence rates, it is sufficient to approximate the stiffness matrix-vector product using a quadrature rule that is exact for $D\hat{U} \otimes P_{p-1}$ [4], with $D\hat{U}$ the space of all partial derivatives of all functions in $\hat{U}$. With this condition, we obtained tailored quadrature rules for the new elements. A comparison with other available quadrature rules is given in Table 2.

Table 2: Number of quadrature points of the new tailored rules, order-$(p+p'-2)$, and exact (order-$(2p'-2)$) rules, for computing the stiffness matrix-vector product of the new n-node degree-p mass-lumped tetrahedral elements.

p–n	new	order-$(p+p'-2)$	exact
2–15	14	14	24
3–32	21	24	46
4–60	51	59	127
4–61	60	79	194
4–65	60	79	194

Especially the tailored rules for the degree-4 elements have significantly less points than other rules available in the literature.

Various numerical tests also confirm the efficiency of the new mass-lumped elements.

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