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On the Transverse Vibrations of Strings and Beams on Semi-Infinite Domains

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Abstract

In this paper, we study the transverse vibrations of a string and of a beam which are infinitely long in one direction. These vibration problems can be used as a toy model for rain-wind induced oscillations of cables. In order to suppress undesired vibrations in the string (or beam), dampers are used at the boundary. The main aim of this paper is to show how solutions for these string and beam problems on a semi-infinite domain can be computed. We derive explicit solutions for a linear string problem which is attached to a mass-spring-dashpot system at $x = 0$ by using the D'Alembert method, and for a transversally vibrating beam problem which has a pinned, sliding, clamped or damping boundary, respectively, at $x = 0$ by using the method of Laplace transforms. It will be shown how waves are reflected for different types of boundaries.

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Keywords: Boundary damper ; strings ; beams ; D'Alembert Methods ; The method of Laplace transforms

1. Introduction

In recent decades, among both applied mathematicians and engineers, research in the field of the vibrations of the stay cables of cable stayed bridges has received a lot of attention. Usually inclined stay cables of bridges are attached to a pylon at one end and to the bridge deck at the other end. Due to low structural damping of the bridge, a wind-field containing raindrops may excite a galloping type of vibrations. For example, one can refer to the Erasmus bridge in Rotterdam, which started to swing under mild wind conditions, shortly after it was opened to the traffic in 1996. To suppress the undesired oscillations of the bridge, dampers were installed, as can be seen in Figure 1. As has been observed from engineering wind-tunnel experiments, raindrops hitting the inclined stay cable cause the generation of one or more rivulets on the surface of the cable. The presence of flowing water on the cable changes the mass of the bridge system that can lead to instabilities, which are not fully understood. The vibrations of the bridge cables can be described mathematically by string-like or beam-like problems. Models for such cables can be found in [1,2]. In order to stabilize the problem, boundary damping is taken into account. The aim of this paper is to provide an understanding

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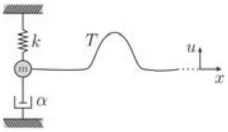
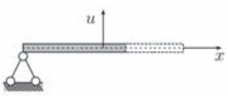
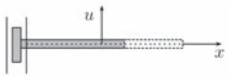
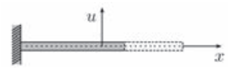
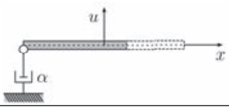
Fig. 1: Used new dampers to the Erasmus bridge to prevent vibrations. Photo: courtesy of TU Delft.

of how effective boundary damping is for string and beam equations. The reflection and damping properties of waves propagating at a non-classical boundary for wave equations on a semi-infinite interval was studied in[3] by using the D'Alembert formula.

Table 1: Boundary condition(s) for a semi-infinite string (Model 1) and a beam (Model 2).

Type of system	Left end condition	Boundary conditions at $x = 0$
Model 1		
Mass-spring-dashpot[3]		$mu_{tt} = Tu_x - ku - \alpha u_t$
Model 2		
Pinned		$u = 0, \quad u_{xx} = 0$
Sliding		$u_x = 0, \quad u_{xxx} = 0$
Clamped		$u = 0, \quad u_x = 0$
Damper		$u_{xx} = 0, EIu_{xxx} = \alpha u_t$

2. The Transverse Vibrations of The String-problem (Model 1)

In this section, we will consider the perfectly flexible string of infinite extension in the positive x-direction. It is assumed that gravity and other external forces can be neglected. The vertical transversal displacement $u(x, t)$ along a string, where x is the position along the string and t is the time, satisfies the following differential equation which can be obtained by using Hamilton's principle[4]:

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \tag{1}$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \le x < \infty, \tag{2}$$

where $c = \sqrt{T/\rho}$ is the wave speed, T is the tension and ρ is the mass density of the string. Here, the initial displacement and initial velocity of the string are $f(x)$ and $g(x)$, respectively, $f \in C^2$ and $g \in C^1$. For the mass-spring-dashpot system, we have the following boundary condition

$$m\, u_{tt}(0,t) = T u_x(0,t) - k u(0,t) - \alpha u_t(0,t), \quad \text{if } m \neq 0. \tag{3}$$

Here, mass m, the stiffness of the spring k, and the damping coefficient of the dashpot α are all positive constants. The wave propagates between $x = 0$ and $x = \infty$ as shown in Model 1 in Table 1. For more information, the readers are referred to[5]. For a non-dimensional form the following dimensionless quantities are used:

$$u(x,t) = \frac{u^*(x^*,t^*)}{\mathrm{L}_*}, \quad x = \frac{x^*}{\mathrm{L}_*}, \quad t = \frac{t^*}{\mathrm{T}_*}, \quad f(x) = \frac{f^*(x^*)}{\mathrm{L}_*}, \quad g(x) = g^*(x^*)\,\frac{\mathrm{T}_*}{\mathrm{L}_*},$$

where L_* and time T_* are some dimensional characteristic quantities for the length and the time respectively, and by inserting these non-dimensional quantities into Eqs. (1)-(3), we obtain

$$u_{tt} - u_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \tag{4}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leq x < \infty, \tag{5}$$

$$u_{tt}(0,t) = \eta u_x(0,t) - \mu u(0,t) - \psi u_t(0,t), \quad t \geq 0, \tag{6}$$

where $c^2 = \mathrm{L}_*^2/\mathrm{T}_*^2$, $\eta = T\,\mathrm{T}_*^2/m\,\mathrm{L}_* > 0$, $\mu = k\,\mathrm{T}_*^2/m \geq 0$ and $\psi = \alpha\,\mathrm{T}_*/m \geq 0$. In Eqs. (4)-(6), the non-dimensional quantities are used. As initial conditions, we will consider $u(x,0) = f(x) = \sin^2(x)$ for $\pi < x < 2\pi$ and zero elsewhere, and $u_t(x,0) = -f'(x)$, which implies that we initially only have waves traveling to the boundary at $x = 0$.

2.1. Application of the D'Alembert Method

In this section, we will determine the solution of Eqs. (4)-(6). The general solution of the one-dimensional wave equation is given by[6]

$$u(x,t) = F(x-t) + G(x+t). \tag{7}$$

Here the functions F and G represent propagating disturbances, and by using the initial conditions, we obtain

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2}\int_0^x g(s)\,\mathrm{d}s - \frac{K}{2} \quad \text{and} \quad G(x) = \frac{1}{2} f(x) + \frac{1}{2}\int_0^x g(s)\,\mathrm{d}s + \frac{K}{2}, \tag{8}$$

where K is a constant of integration. The well-known D'Alembert formula for $u(x,t)$ is obtained by substituting Eq. (8) into the general solution Eq. (7),yielding

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,\mathrm{d}s. \tag{9}$$

In Eq. (9), $f(x-t)$ and $g(s)$ are not yet defined for $x < t$ and $x - t < s < 0$, respectively. This "freedom" in f and in g will be used to satisfy the boundary condition Eq. (6). Substituting Eq. (9) into Eq. (6) gives

$$f''(t)+f''(-t)+g'(t)-g'(-t) = \eta\Big[f'(-t)+f'(t)-g(-t)+g(t)\Big]-\mu\Big[f(-t)+f(t)+\int_{-t}^{t} g(s)\,\mathrm{d}s\Big]-\psi\Big[-f'(-t)+f'(t)+g(t)+g(-t)\Big], \tag{10}$$

where f and g can be chosen independently. For $g \equiv 0$,

$$f''(-t) - (\eta+\psi)f'(-t) + \mu f(-t) = -f''(t) + (\eta-\psi)f'(t) - \mu f(t). \tag{11}$$

Eq. (11) may be rewritten with respect to the unknown function $f(-t) = y(t)$, $f'(-t) = -y'(t)$, and $f''(-t) = y''(t)$. Then, it follows that

$$y''(t) + \theta\, y'(t) + \mu\, y(t) = -f''(t) + (\eta-\psi)f'(t) - \mu f(t), \tag{12}$$

where, $\theta = (\eta+\psi)$. Similarly, $g(-t)$ can be obtained when $f \equiv 0$. The characteristic equation corresponding to Eq. (12) is given by

$$\lambda^2 + \theta\lambda + \mu = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-\theta \pm \sqrt{\Delta}}{2}, \tag{13}$$

where $\Delta = \theta^2 - 4\mu$. Solving the characteristic equation will give two roots, λ_1 and λ_2, which determine the qualitative behavior of the system according to two real roots, one real root, or two complex valued roots.

(i) The case $\Delta > 0$.

Two different real roots give the following homogeneous solution

$$y_h(t) = b_1 e^{\lambda_1 t} + b_2 e^{\lambda_2 t}, \tag{14}$$

where b_1 and b_2 are constants. By applying the method of variations of parameters[7], we obtain the solution of Eq. (12), which satisfies the conditions $y(0) = f(0)$, $y'(0) = -f'(0)$:

$$f(-t) = -f(t) + f(0)\left[(e^{\lambda_1 t} + e^{\lambda_2 t}) + \frac{(\eta-\psi)}{(\lambda_2-\lambda_1)}(e^{\lambda_1 t} - e^{\lambda_2 t})\right] + \frac{e^{\lambda_1 t}}{(\lambda_2-\lambda_1)}\left[(\lambda_1^2 - \lambda_1(\eta-\psi) + \mu)\int_0^t e^{-\lambda_1 s} f(s) ds\right]$$
$$- \frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)}\left[(\lambda_2^2 - \lambda_2(\eta-\psi) + \mu)\int_0^t e^{-\lambda_2 s} f(s) ds\right]. \tag{15}$$

(ii) The case $\Delta = 0$.

There is a double root λ, which is real. In this case, with only one root λ, the homogeneous solution of Eq. (12) is given by

$$y_h(t) = (b_3 + b_4 t) e^{-\theta t/2}, \tag{16}$$

where b_3 and b_4 are constants. Completely similar to the previous case, $y(t) = f(-t)$ can be computed.

(iii) The case $\Delta < 0$.

In this case λ is complex valued, the homogeneous solution is given by

$$y_h(t) = e^{-\theta t/2}\left[b_5 \cos\left(\frac{\sqrt{-\Delta} t}{2}\right) + b_6 \sin\left(\frac{\sqrt{-\Delta} t}{2}\right)\right], \tag{17}$$

where b_5 and b_6 are constants. For further information on the extension of $f(t)$ for negative arguments the reader is referred to[3]. For existence of these solutions, the following equation should be satisfied

$$f''(0) = \eta f'(0) - \mu f(0) - \psi g(0), \quad \eta > 0, \ \mu > 0, \ \psi > 0, \tag{18}$$

where $u \in C^2$, $f \in C^2$ and $g \in C^1$. We generally consider f and g as being independent functions, then

$$\begin{cases} g(0) = 0, \text{and} \\ f''(0) = \eta f'(0) - \mu f(0). \end{cases} \tag{19}$$

For fixed tension coefficient η and damping coefficient ψ, and varying stiffness coefficient μ, Fig. 2 shows some reflected waves in cases (i), (ii) and (iii), when $f(x) = \sin^2(x)$, and $g(x) = -f'(x)$, for $\pi < x < 2\pi$, and $f(x) = 0$ elsewhere. The reflected wave for $\mu = 1$ (case (ii)) is more or less in between the reflected waves for $\mu = 1/2$ (case (i)) and $\mu = 7/5$ (case (iii)), respectively. Moreover, it is also shown in Fig. 2a and Fig. 2b how the damping coefficient ψ influences the amplitude of the reflected wave, that is, the amplitudes for $\psi = 0$ are higher than those for $\psi \neq 0$.

3. The Transvers Vibratons of The Beam-problem (Model 2)

In the second model, we intend to solve exactly the boundary value problem for a beam on a semi-infinite interval. Different types of boundary conditions will be considered, and Green's functions will be constructed by using the Laplace transform method. The function $u(x,t)$ is the vertical deflection of the beam, where x is the position along the beam, and t is the time. Gravity is neglected. The equation describing the vertical displacement of the Euler-Bernoulli beam is given by

$$u_{tt} + a^2 u_{xxxx} = \frac{q}{\rho A}, \quad 0 < x < \infty, \quad t > 0, \tag{20}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x < \infty, \tag{21}$$

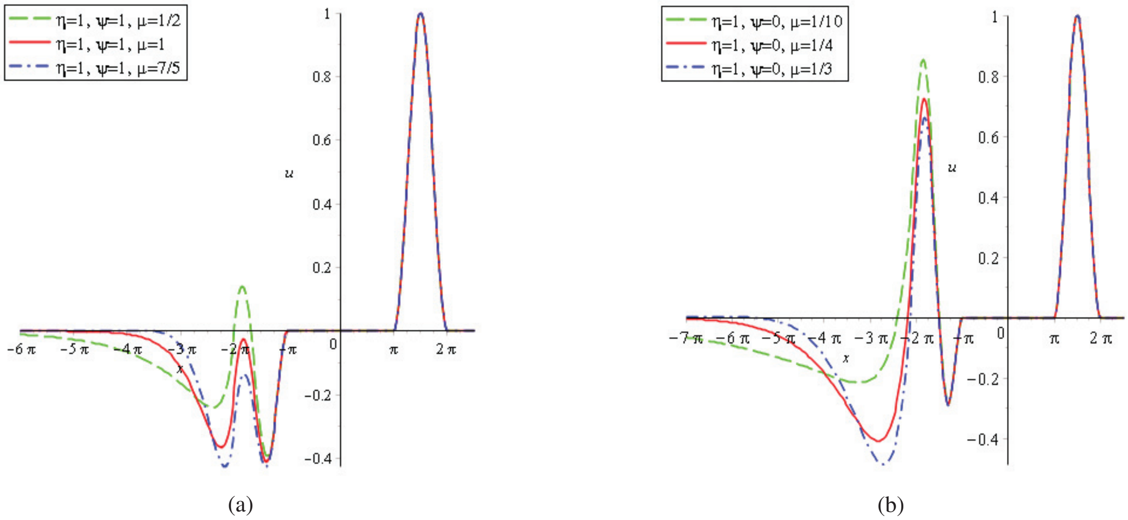


Fig. 2: Some reflected waves for η and ψ fixed, and varying stiffness coefficient μ. (a) $\eta = \psi = 1$, and for various μ; (b) $\eta = 1$, $\psi = 0$, and for various μ.

where $a^2 = (EI/\rho A) > 0$, E is Young's modulus of elasticity, I is the moment of inertia of the cross-section, ρ is the density, A is the area of the cross-section, and $q(x,t)$ is an external load. Here, $f(x)$ represents the initial deflection and $g(x)$ the initial velocity.

Studying the Euler-Bernoulli beam equation on an infinite domain, Guenther and Lee [8] solved the initial value problem by using Fourier transforms obtaining

$$u(x,t) = \int_{-\infty}^{\infty} [K(y-x,t)f(y) + L(y-x,t)g(y)]\,\mathrm{d}y, \tag{22}$$

where

$$K(x,t) = \frac{1}{\sqrt{4\pi a\,t}} \sin\left(\frac{x^2}{4at} + \frac{\pi}{4}\right), \tag{23}$$

and

$$L(x,t) = \frac{1}{\pi a}\left\{\frac{\pi x}{2\sqrt{2\pi}}\left[\int_0^{\frac{x^2}{4at}} \frac{\sin(s)}{\sqrt{s}} ds - \int_0^{\frac{x^2}{4at}} \frac{\cos(s)}{\sqrt{s}} ds\right] + \sqrt{\pi at}\sin\left(\frac{x^2}{4at} + \frac{\pi}{4}\right)\right\}. \tag{24}$$

In order to put the Eq. (20) and Eq. (21) in a non-dimensional form the following dimensionless quantities are used:

$$u(x,t) = \frac{u^*(x^*,t^*)}{\mathrm{L}_*},\quad x = \frac{x^*}{\mathrm{L}_*},\quad t = \frac{\kappa t^*}{\mathrm{L}_*},\quad \kappa = \frac{1}{\mathrm{L}_*}\sqrt{\frac{EI}{\rho A}},\ f(x) = \frac{f^*(x^*)}{\mathrm{L}_*},\ g(x) = \frac{g^*(x^*)}{\kappa},\quad q(x,t) = \frac{q^*(x^*,t^*)\rho A\kappa^2}{\mathrm{L}_*},$$

where L_* is the dimensional characteristic quantity for the length, and by inserting these non-dimensional quantities into Eqs. (20)-(21), we obtain

$$u_{tt} + u_{xxxx} = q,\quad 0 < x < \infty,\quad t > 0, \tag{25}$$

$$u(x,0) = f(x),\quad u_t(x,0) = g(x),\quad 0 \le x < \infty. \tag{26}$$

In the following subsections, it will be shown how solutions can be obtained in explicit form for semi-infinite beams with boundary conditions given at $x = 0$.

3.1. Pinned boundary conditions, $u = u_{xx} = 0$

In this section, we consider a semi-infinite beam equation, when the displacement and the bending moment are specified at $x = 0$, i.e. $u(0,t) = u_{xx}(0,t) = 0$, and of infinite extension in the positive x-direction. We assume $u(x,t) = 0$ for $t < 0$. The Green's function $G_\xi(x,t)$, $\xi > 0$, represents the displacements along this beam produced by a concentrated instantaneous force at the point $x = \xi$ at time $t = 0$, which amounts to putting $q(x,t) = \delta(x-\xi) \otimes \delta(t)$, δ being Dirac's function.

3.1.1. Application of the Laplace transform method

The Laplace transform g_ξ of G_ξ with respect to t is given by

$$g_\xi(x,p) = \mathcal{L}\{G_\xi(x,t)\} = \int_0^\infty \mathrm{e}^{-pt} G_\xi(x,t)\mathrm{d}t. \tag{27}$$

Then g_ξ is the Green's function of the differential operator $(\mathrm{d}^4/\mathrm{d}x^4) + p^2$ in the interval $(0,\infty)$. The function g_ξ has to satisfy:

$$g_\xi^{(iv)} + p^2\, g_\xi = 0, \quad 0 < x < \infty, \quad x \neq \xi, \quad g_\xi(0) = g_\xi''(0) = 0. \tag{28}$$

Moreover, g_ξ has to be bounded and twice continuously differentiable in $x = \xi$, and

$$\lim_{\epsilon \to 0} [g_\xi'''(\xi+\epsilon) - g_\xi'''(\xi-\epsilon)] = 1. \tag{29}$$

By using these requirements, g_ξ is uniquely determined, and we obtain

$$g_\xi = \frac{1}{8\beta^3}\{\mathrm{e}^{-\beta|x-\xi|}[\cos\beta(x-\xi) + \sin\beta|x-\xi|] + \mathrm{e}^{-\beta(x+\xi)}[-\cos\beta(x+\xi) - \sin\beta(x+\xi)]\}, \tag{30}$$

where $\beta^2 = p/2$.

3.1.2. Inversion of the Laplace transform

In order to invert the Laplace transform, we use the formula (see[9], page 93) and (see[10], page 279)

$$\mathcal{L}^{-1}\left[(p)^{-1}\phi(p)\right] = \int_0^t \mathcal{L}^{-1}\{\phi(\tau)\}\,\mathrm{d}\tau, \tag{31}$$

$$\mathcal{L}^{-1}\left[p^{-1/2}\mathrm{e}^{-\sqrt{pz}}\cos(\sqrt{pz})\right] = \frac{1}{\sqrt{\pi t}}\cos\left(\frac{z}{2t}\right) \quad \text{and} \quad \mathcal{L}^{-1}\left[p^{-1/2}\mathrm{e}^{-\sqrt{pz}}\sin(\sqrt{pz})\right] = \frac{1}{\sqrt{\pi t}}\sin\left(\frac{z}{2t}\right), \tag{32}$$

where $z = \frac{|x\pm\xi|}{\sqrt{2}}$. The Green's function yields

$$G_\xi(x,t) = -\int_0^t \left[K(x-\xi,\tau) - K(x+\xi,\tau)\right]\mathrm{d}\tau, \tag{33}$$

where the kernel function is defined by

$$K(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\sin\left(\frac{x^2}{4\tau} + \frac{\pi}{4}\right). \tag{34}$$

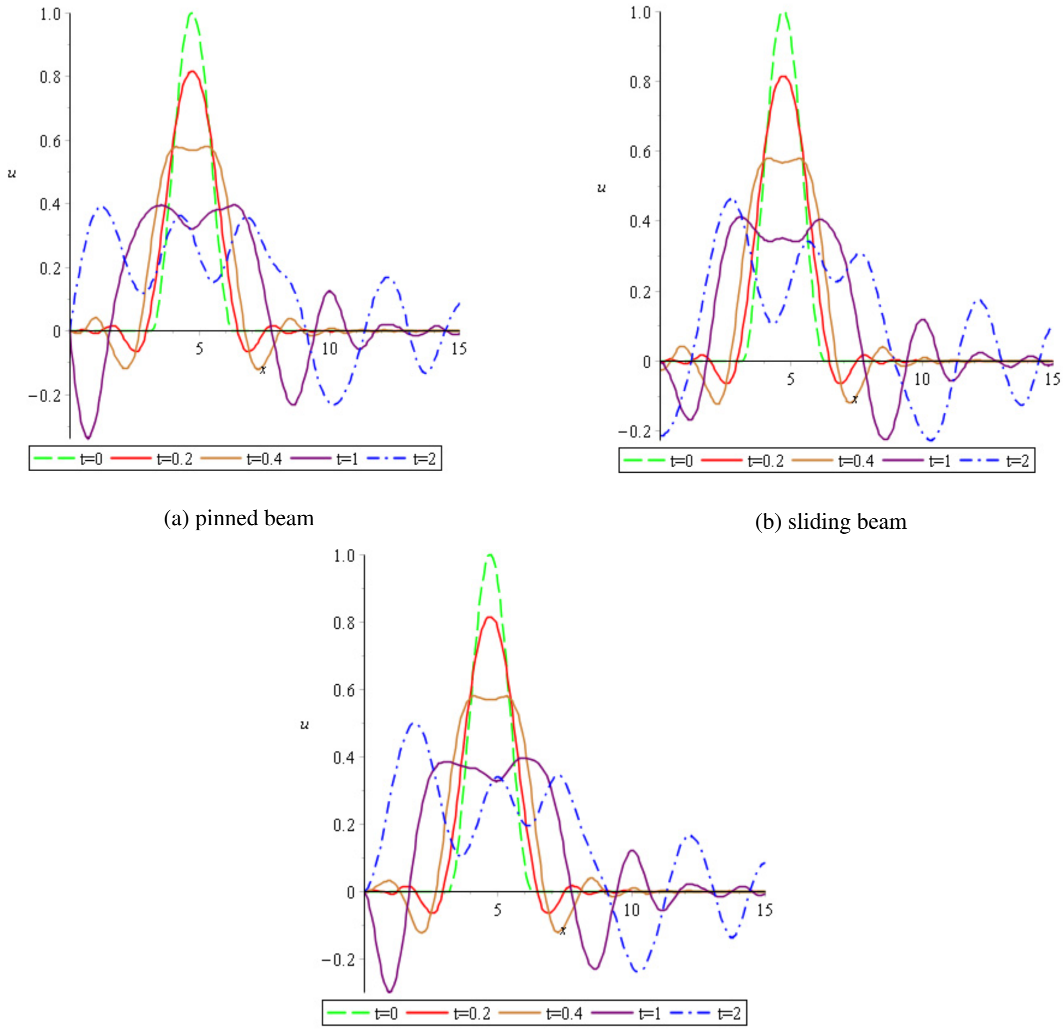
3.1.3. The fundamental solution

The transverse displacement $u(x,t)$ of the beam can be represented in terms of the Green's function as

$$u(x,t) = \frac{\partial}{\partial t}\int_0^\infty f(\xi)\,G(x,\xi,t)\,\mathrm{d}\xi + \int_0^\infty g(\xi)\,G(x,\xi,t)\,\mathrm{d}\xi + \int_0^t\int_0^\infty q(\xi,\tau)\,G(x,\xi,t-\tau)\mathrm{d}\xi\,\mathrm{d}\tau. \tag{35}$$

If we suppose that the initial velocity and external force are $g(x) = q(x,t) = 0$, we obtain finally

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_0^\infty f(\xi)\left[\sin\left(\frac{(x-\xi)^2}{4t} + \frac{\pi}{4}\right) - \sin\left(\frac{(x+\xi)^2}{4t} + \frac{\pi}{4}\right)\right]\mathrm{d}\xi. \tag{36}$$



(a) pinned beam

(b) sliding beam

(c) clamped beam

Fig. 3: Some reflected waves for a semi-infinite one-sided beam with $f(x) = \sin^2(x)$ for $\pi < x < 2\pi$, and $f(x) = 0$ elsewhere, and $g(x) = 0$: (a) pinned beam; (b) sliding beam and (c) clamped beam.

3.2. Sliding boundary conditions, $u_x = u_{xxx} = 0$

In this section, we consider a semi-infinite beam equation, when the bending slope and the shear force are specified at $x = 0$, i.e. $u_x(0, t) = u_{xxx}(0, t) = 0$, and of infinite extension in the positive x-direction. We apply the same method for the pinned boundary condition, and the Green's function is defined by

$$G_\xi(x, t) = -\int_0^t \left[-K(x - \xi, \tau) - K(x + \xi, \tau)\right] \mathrm{d}\tau, \tag{37}$$

and the transverse displacement $u(x, t)$ of the beam is given by

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty f(\xi) \left[\sin\left(\frac{(x - \xi)^2}{4t} + \frac{\pi}{4}\right) + \sin\left(\frac{(x + \xi)^2}{4t} + \frac{\pi}{4}\right)\right] \mathrm{d}\xi. \tag{38}$$

3.3. Clamped boundary conditions, $u = u_x = 0$

In this section, we consider a semi-infinite beam equation, when the deflection and the slope are specified at $x = 0$, i.e. $u(0, t) = u_x(0, t) = 0$, and of infinite extension in the positive x-direction. We construct the same explicit Green's function as in[11]. For more information on clamped boundary condition for semi-infinite beams the reader is refered to[11]. For pinned, sliding and clamped boundary condition at $x = 0$, Fig. 3 demonstrates some initial phase of the "reflected" wave and the fading-out wave for the initial values with $f(x) = \sin^2(x)$ and $g(x) = 0$ for $\pi < x < 2\pi$, and $f(x) = g(x) = 0$ elsewhere. It can be seen that the height of the reflected waves are the same until $t = 0.4$. After $t = 1$ the deflection curves, which illustrate the fast oscillations in the beginning of the movement, start to dissipate rapidly.

3.4. Damper boundary conditions, $u_{xx} = 0,\ EIu_{xxx} = \alpha u_t$

In this section, we consider a semi-infinite beam equation, when the bending moment is zero and the shear force is proportional to the velocity (damper) at $x = 0$, i.e. $u_{xx} = 0$, $EIu_{xxx} = \alpha u_t$, and of infinite extension in the positive x-direction. After applying the dimensionless quantities to the damper boundary conditions, it follows that $u_{xxx} = \gamma u_t$, $u_{xx} = 0$. The results for this problem will be given in a forthcoming research paper, and can be obtained in a similar way as shown in the previous cases and the transverse displacement $u(x, t)$ of the beam is given by

$$u(x,t) = \int_0^\infty \frac{f(\xi)}{\sqrt{\pi t}} \sin\left(\frac{(x\xi)}{2t}\right) \sin\left(\frac{(x^2+\xi^2)}{4t} + \frac{\pi}{4}\right) d\xi + \int_0^\infty f(\xi) \int_0^t 2\cos\left(\frac{x\xi}{4(t-r)}\right) \sin\left(\frac{(x^2+\xi^2)}{8(t-r)} + \frac{\pi}{4}\right) \left[e^{-(x+\xi)^2/8r} \frac{(x+\xi-4\gamma r)}{4\pi r\sqrt{r(t-r)}} + \frac{\gamma^2\sqrt{2}}{\sqrt{\pi(t-r)}} e^{\gamma(x+\xi)+2\gamma^2 r} \operatorname{Erfc}\left(\frac{x+\xi+4\gamma r}{2\sqrt{2r}}\right)\right] dr\, d\xi. \tag{39}$$

4. Conclusion

This paper provides an understanding of how waves are damped and reflected for semi-infinite string and beam equations for different boundary conditions. The "reflected" waves for semi-infinite elastic strings and beams are obtained by using the D'Alembert method and the method of Laplace transforms, respectively.

The results, as given in this paper, can be used in determinig the optimal position and characteristics of the dampers. For instance, in[12] the authors modelled the dampers as discrete dampers in a finite element model to show the time-displacement diagram of a single cable. In this paper, we computed the exact reflected waves of the system with and without dampers.

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