# Tackling Different Velocity Borne Challenges of the Elastodynamic Marchenko Method

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**DOI**
10.3997/2214-4609.201901517

**Publication date**
2019

**Document Version**
Accepted author manuscript

**Published in**
81st EAGE Conference and Exhibition 2019

# Tackling different velocity borne challenges of the elastodynamic Marchenko method

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## Summary

The elastodynamic Marchenko method removes overburden interactions obscuring the target information. This method either relies on separability of the so-called focusing and Green's functions or requires an accurate initial estimate of the focusing and Green's function overlap. Hitherto, $\mathbf{F}_1^-$ and $\mathbf{G}^{(-,+)}$ have been assumed separable, whereas $\mathbf{F}_1^+$ and $\left(\mathbf{G}^{(-,-)}\right)^*$ share an unavoidable overlap, which has been considered understood but hard to predict without knowing the model. However, velocity differences between P- and S-waves cause so far unexplored fundamental challenges for elastodynamic Marchenko autofocusing. These challenges are analysed for horizontally-layered media. First, the $\mathbf{F}_1^-/\mathbf{G}^{(-,+)}$ separability assumption can be violated depending on the medium, the redatuming depth and the angle of incidence. Second, the initial estimate of the said unavoidable overlap can be even more complicated than originally thought, including some of the internal multiples. We propose a strategy where we trade-off this sophisticated initial estimate with a trivial one at the cost of a more restrictive $\mathbf{F}_1^-/\mathbf{G}^{(-,+)}$ separability assumption, or at the cost of introducing an overlap between $\mathbf{F}_1^-$ and $\mathbf{G}^{(-,+)}$ instead. The proposed method finds the desired solutions convolved by an unknown matrix which we can hope to remove by exploiting energy conservation and minimum-phase properties of the focusing functions.

## Introduction

Suppose a target buried under an overburden is examined using a single-sided reflection response. Interactions with the overburden, especially internal multiple scattering and elastic mode conversions, obscure the desired information. Thus, for a strongly scattering overburden, correct prediction and/or subtraction of internal multiples become crucial. Layer-stripping approaches such as Jakubowicz (1998) predict (acoustic) internal multiples for each reflector separately. In contrast, the Marchenko method is able to predict internal multiples associated with an entire group of reflectors (Wapenaar, 2014). Both strategies depend upon wavefield separation in the time domain.

Elastodynamic Marchenko autofocusing requires that representation theorem wavefields (focusing and Green's functions) are temporally separable, or that we know their overlaps. So far, it has been assumed that the $\mathbf{F}_1^-/\mathbf{G}^{-,+}$ overlap was zero, i.e. that $\mathbf{F}_1^-$ and $\mathbf{G}^{-,+}$ are separable. Besides, it has been recognised that the unavoidable $\mathbf{F}_1^+/(\mathbf{G}^{-,-})^*$ overlap (the initial estimate) is given by an (inverse) multiple-free transmission, containing all forward-scattered and converted waves.

We found that for realistic $V_p$-to-$V_s$ ratios ($\sim 1.5-2$), we may encounter two (unanticipated) temporal overlaps between the focusing and Green's functions in both representation theorems. We demonstrate that the aforementioned initial estimate is not always sufficient and that the general initial estimate can be even more complicated than originally thought due to fast multiples that partially predate the forward-scattered waves.

We propose a strategy for horizontally-layered media to trade the usually unavailable initial estimate of the elastodynamic Marchenko method for a trivial one. This trade-off results in a stricter $\mathbf{F}_1^-/\mathbf{G}^{-,+}$ separability condition (or an overlap) and leads to the desired solutions convolved by an unknown matrix. This is a generalisation of augmented Marchenko redatuming (Dukalski et al., 2019), which here addresses a fundamental problem rather than one caused by band-limitation of our measurements.

## Challenges of the elastodynamic Marchenko method

Consider a horizontally-layered lossless elastic medium, in $(x,z)$ space with a reflection-free boundary at the surface $z_0$. Similar to prior work (Wapenaar, 2014), we exploit lateral invariance to decouple elastodynamic wavefield propagation and scattering to a set of 1D problems per horizontal-slowness $s_x$ associated with the intercept time $\tau$ and describe elastodynamic waves by power-flux normalised P- and S-modes. For a better analysis of fundamental challenges, we ignore band-limitation. The reflection response is recorded at $z_0$. After applying a Fourier transform along the intercept time $\tau$, we sort each discrete frequency component in a $2\times 2$ matrix $\mathbf{R}$. The columns are associated with P- and S-wave sources and the rows are associated with P- and S-wave recordings.

The convolution- and correlation-type reciprocity theorems establish mutual relations between the up-(-) and down(+)going focusing functions $\mathbf{F}_1^{\pm}$, that are recorded at $z_0$ and focus at $z_i$ at time zero, and the Green's functions $\mathbf{G}^{-,\pm}$ recorded at $z_0$ and associated with up- and downgoing virtual sources (second superscript) at $z_i$ (Wapenaar, 2014). The respective convolution- and correlation-type representation theorems are,

$$\mathbf{G}^{-,+}\mathbf{B}+\mathbf{F}_1^-\mathbf{B}=\mathbf{R}\mathbf{F}_1^+\mathbf{B} \;\text{ and }\; \left(\mathbf{G}^{-,-}\right)^*\mathbf{B}+\mathbf{F}_1^+\mathbf{B}=\mathbf{R}^{\mathrm{H}}\mathbf{F}_1^-\mathbf{B}, \tag{1}$$

where, without loss of generality, we use the freedom to post-multiply with a frequency-dependent matrix $\mathbf{B}$. The superscript "$*$" indicates complex conjugation and the superscript "H" expresses a complex-conjugate transpose. Moreover, we neglect evanescent waves.

Since both the focusing and the Green's functions are unknown the representation theorems cannot be solved without further constraints. Suppose two projectors $\mathbf{P}^+$ and $\mathbf{P}^-$ exist which allow for temporal separation of a-priori unknown fields convolved with an unknown $\mathbf{B}$ according to $\mathbf{P}^-\left[\mathbf{F}_1^-\mathbf{B}\right]=\mathbf{F}_1^-\mathbf{B}$, $\mathbf{P}^-\left[\mathbf{G}^{-,+}\mathbf{B}\right]=\mathbf{O}$, $\mathbf{P}^+\left[\mathbf{F}_1^+\mathbf{B}\right]=\mathbf{F}_1^+\mathbf{B}$ and $\mathbf{P}^+\left[(\mathbf{G}^{-,-})^*\mathbf{B}\right]=\chi$. If such a separation exists the representation theorems can be reduced to the coupled Marchenko equations, which can be solved recursively,

$$\mathbf{F}_1^+\mathbf{B}=\sum_{k=0}^{\infty}\Xi_k[x] \;\text{ with }\; \Xi_k[x]=\mathbf{P}^+\left[\mathbf{R}^{\mathrm{H}}\mathbf{P}^-\left[\mathbf{R}\Xi_{k-1}[x]\right]\right], \tag{2}$$

with $\Xi_0[x]=-\chi$ as the aforementioned initial estimate. If the assumptions for Eq. 2 are violated

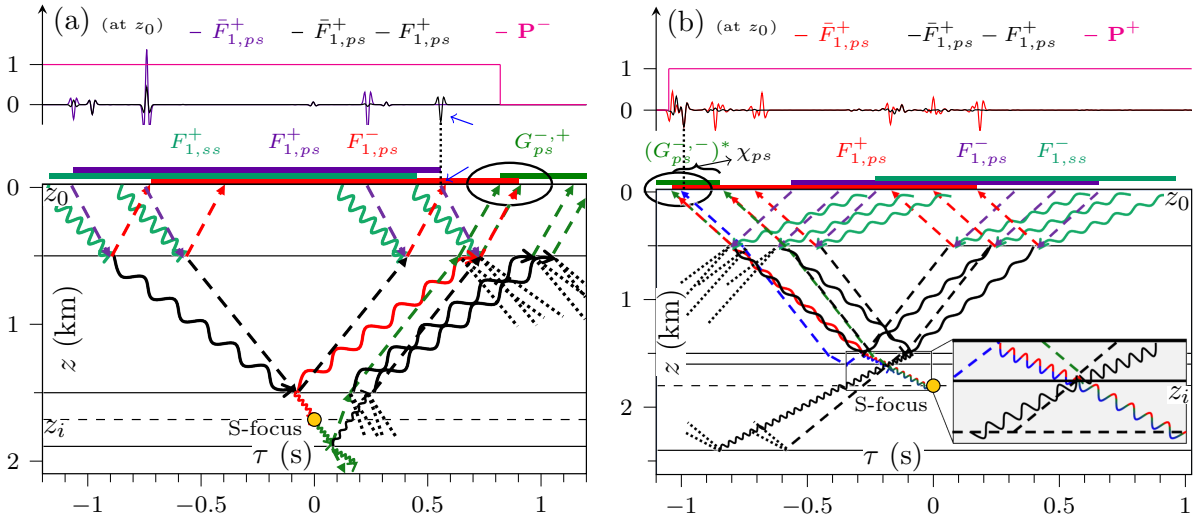


Figure 1: (a) Convolution-type representation theorem (ps component) for a 4-layer model (see $\tau - z$ figure). Dashed and sinusoidal lines represent P- and S-waves, respectively. The temporal extent of each wavefield is illustrated by colour-coded bars. The retrieved focusing function $\bar{\mathbf{F}}_1^+$ and the projector $\mathbf{P}^-$ are shown in the top trace which we convolved with a 30 Hz Ricker wavelet for illustrational purpose. (b) As (a) but for the correlation-type representation theorem and a 5-layer time-reversing model (to account for $\mathbf{R}^{\mathrm{H}}$ in Eq. 1). Due to the small temporal separation between P- and S-waves propagating through the thin layer (see close-up window) all rays above the thin layer are actually two nearly coinciding events. The projector $\mathbf{P}^+$ is shown in the top trace.

we obtain an incorrect solution $\bar{\mathbf{F}}_1^+$ instead of $\mathbf{F}_1^+\mathbf{B}$. The quantity $\chi$ is the overlap between $\mathbf{F}_1^+\mathbf{B}$ and $(\mathbf{G}^{-,-})^*\mathbf{B}$ which is unavoidable because the focus of $\mathbf{F}_1^+$ at $z_i$ at time zero becomes the virtual source of $(\mathbf{G}^{-,-})^*$. The smaller the overlap the less prior knowledge is required to solve the representation theorems via Marchenko equations. In prior work we considered $\mathbf{B}$ as an identity ($\mathbf{B} = \mathbf{I}$) and made two assumptions which we found out to be invalid in some cases.

First, Wapenaar (2014) demonstrates that the representation theorems can be solved accurately, under the assumption that analogous to the acoustic case the focusing and Green's functions are separable. In Fig. 1a, we show an example in which $\mathbf{F}_1^-$ and $\mathbf{G}^{-,+}$ are *not* separable in the time domain: We show a ps component associated with an S-wave focus at $z_i$ and P-wave recordings at $z_0$ (the overall picture is similar for the other elastic components). To create an S-wave focus at $z_i$, we inject P- and S-waves of the focusing function $\mathbf{F}_1^+$ at $z_0$ (see $\tau$-$z$ panel). At each interface the injected wavefield transmits and reflects as P- and S-waves. Upgoing waves at $z_0$ associated with travel paths going through the focal point belong to $\mathbf{F}_1^-$. The remaining upgoing waves at $z_0$ belong to $\mathbf{G}^{-,+}$. The last event of the focusing function $\mathbf{F}_1^-$ (red path from $z_i$ to $z_0$) reaches the surface $z_0$ after the first event of the Green's function $\mathbf{G}^{-,+}$ (green path from $z_i$ to $z_0$). Due to this overlap (see ellipse) the projector $\mathbf{P}^-$ (see top trace), which we define to just mute $\mathbf{G}^{-,+}$, erroneously removes the last event of $\mathbf{F}_1^-$. As a consequence, the last event of $\mathbf{F}_1^+$ (see blue arrows) is superfluous and not part of the retrieved solution $\bar{\mathbf{F}}_1^+$ (see top trace). To compensate for this missing event the Marchenko series (see Eq. 2) introduces several additional artefacts. Besides, we define the projector $\mathbf{P}^+$ to just protect $\mathbf{F}_1^+$ (see Fig. 1b). Note that, the illustrations of $\mathbf{P}^\pm$ refer to two different media. The cut-off times of the projectors $\mathbf{P}^+$ are often assumed to be identical, except for a minus sign. However, this assumption may increase the overlap between $\mathbf{F}_1^-$ and $\mathbf{G}^{-,+}$, leading to a more severe error. The travel paths in Fig. 1a indicate that the choice of focusing depth is crucial for accurate elastodynamic Marchenko redatuming. Increasing width of the focusing layer moves $\mathbf{F}_1^-$ and $\mathbf{G}^{-,+}$ apart eventually making them separable. Bear in mind that this effect is a function of horizontal-slowness.

Second, Wapenaar (2014) recognised that the overlap $\chi$ is not simply a direct P- and a direct S-wave propagating from $z_i$ to $z_0$ but consists of the forward-scattered part of the Green's function $(\mathbf{G}^{-,-})^*$, i.e. all forward-transmitted waves including conversions. Unfortunately, in some cases the overlap can complicate further, independent of the $\mathbf{F}_1^-/\mathbf{G}^{-,+}$ separability. If the overburden contains at least one

sufficiently thin layer, mode conversions enable multiples of the Green's function $(\mathbf{G}^{-,-})^*$ to partially overtake the forward-scattered waves. In Fig. 1b, we illustrate such a scenario: We inject P- and S-waves of the focusing function $\mathbf{F}_1^-$ at $z_0$ to create an S-wave focus at $z_i$ (see $\tau$-$z$ panel). The overlap $\chi$ is bounded by the fastest event of the Green's function $(\mathbf{G}^{-,-})^*$ (green path from $z_i$ to $z_0$) and the fastest event of the focusing function $\mathbf{F}_1^+$ (red path from $z_i$ to $z_0$). Here, the overlap includes internal multiples of $(\mathbf{G}^{-,-})^*$ (the blue path from $z_i$ to $z_0$ highlights the strongest internal multiple in $\chi$). In this case, approximating the initial estimate by the forward-scattered part of the Green's function $(\mathbf{G}^{-,-})^*$ results in an incorrect solution $\bar{\mathbf{F}}_1^+$ (see top trace), even though the projectors $\mathbf{P}^\pm$ correctly separate the wavefields in Eq. 1 meaning that $\mathbf{F}_1^-$ and $\mathbf{G}^{-,+}$ are separable. For example, the neglected internal multiple associated with the blue path is not retrieved (see vertical dotted line).

**Reduction of required prior knowledge: A trade-off**

Now we trade the nearly unpredictable initial estimate for a trivial one. To this end, we demand that $\mathbf{B}$ is no longer an identity but an unknown operator that turns the overlap $\chi$ into an identity matrix, $\mathbf{P}^+\left[(\mathbf{G}^{-,-})^*\mathbf{B}\right] = \chi = \mathbf{I}$. Hence, $\mathbf{B}$ can have as much temporal support as the overlap between $(\mathbf{G}^{-,-})^*$ and $\mathbf{F}_1^+$. The multiplication with $\mathbf{B}$ removes an overall time shift, some of the multiple scattering and forward-conversions, similar to van der Neut and Wapenaar (2016) removing the overall time shift and Dukalski et al. (2019) accounting for the overall time shift as well as some of the multiple scattering. In addition, to satisfy the above defined wavefield separation the projectors $\mathbf{P}^\pm$ must be modified appropriately. Using these modifications, we revisit the example of Fig. 1b. The retrieved solution $\mathbf{F}_1^+\mathbf{B}$ is the desired focusing function convolved by an unknown operator $\mathbf{B}$.

We interpret $\mathbf{B}$ as a wavefield associated with a source at $z_0$ and a receiver at $z_i$. We assume that the fastest and slowest events of $\mathbf{B}$ follow the travel paths of the fastest and slowest forward-scattered waves from $z_0$ to $z_i$ (see second column in Fig. 2). The presented scheme relies on the separability of the wavefields $\mathbf{F}_1^-\mathbf{B}$ and $\mathbf{G}^{-,+}\mathbf{B}$, i.e. the last event of $\mathbf{F}_1^-\mathbf{B}$ must reach the surface $z_0$ before the first event of $\mathbf{G}^{-,+}\mathbf{B}$. We depict these travel paths in a cartoon in Fig. 2. The cartoon illustrates that the proposed change reduces the temporal separation between the focusing function $\mathbf{F}_1^-\mathbf{B}$ and the Green's function $\mathbf{G}^{-,+}\mathbf{B}$ by the temporal width of the wavefield $\mathbf{B}$. Therefore, the separability of the convolution-type representation theorem becomes stricter. Compared to the original scheme (Wapenaar, 2014), the proposed strategy is a trade-off and we name $\mathbf{B}$ the trade-off operator.

In practice, the trade-off operator ought to be removed from the solutions to obtain the redatumed responses $\mathbf{G}^{-,\pm}$ that are needed to remove the overburden interactions, e.g. via multidimensional deconvolution. From the reciprocity theorem of the correlation-type it follows that the focusing functions conserve energy, which allows us to determine the normal product of the trade-off operator,

$$\left(\mathbf{F}_1^+\right)^{\mathrm{H}}\mathbf{F}_1^+ - \left(\mathbf{F}_1^-\right)^{\mathrm{H}}\mathbf{F}_1^- = \mathbf{I} \;\rightarrow\; \left(\mathbf{F}_1^+\mathbf{B}\right)^{\mathrm{H}}\mathbf{F}_1^+\mathbf{B} - \left(\mathbf{F}_1^-\mathbf{B}\right)^{\mathrm{H}}\mathbf{F}_1^-\mathbf{B} = \mathbf{B}^{\mathrm{H}}\mathbf{B}. \tag{3}$$

Note that, the normal product can be considered as a generalisation of a single trace amplitude spectrum to a matrix. Next, we take the inverse of Eq. 3 and multiply the result by $\mathbf{F}_1^+\mathbf{B}$ from the left and by $\left(\mathbf{F}_1^+\mathbf{B}\right)^{\mathrm{H}}$ from the right. As a result we obtain the normal product of the desired focusing function $\mathbf{F}_1^+\left(\mathbf{F}_1^+\right)^{\mathrm{H}}$. By applying this strategy to the example in Fig. 1b we accurately retrieve the normal product of the desired focusing function (the relative error is of the order of 15 parts per million).

The desired focusing function $\mathbf{F}_1^+$ (a matrix) is stable, has a stable inverse, namely the transmission response of the overburden, and its determinant is stable and causal with a stable and causal inverse. Thus, $\mathbf{F}_1^+$ possesses a minimum-phase behaviour (Silvia and Robinson, 1979). Details about minimum-phase properties as well as stability and causality of this class of matrices are beyond the scope of this abstract. For minimum-phase scalar functions, the phase spectrum can be uniquely reconstructed from its amplitude spectrum via the Kolmogorov method. However, for matrix-valued functions the phase-amplitude relation is more elaborate. Tunnicliffe-Wilson (1972) demonstrates how to retrieve a minimum-phase matrix from its normal product. This method requires an initial estimate of the minimum-phase matrix as well as a projector that applies a temporal mute. So far, we are capable to reconstruct with numerical precision the solution $\mathbf{F}_1^+\mathbf{B}$ (with $\chi = \mathbf{I}$) from its normal product using an identity as initial estimate and a projector that is nearly identical to the Marchenko projector $\mathbf{P}^+$ (the one that just preserves $\mathbf{F}_1^+\mathbf{B}$, only differs by a factor $\frac{1}{2}$ at time zero on the diagonal elements). Nevertheless, we aim to retrieve $\mathbf{F}_1^+$ from

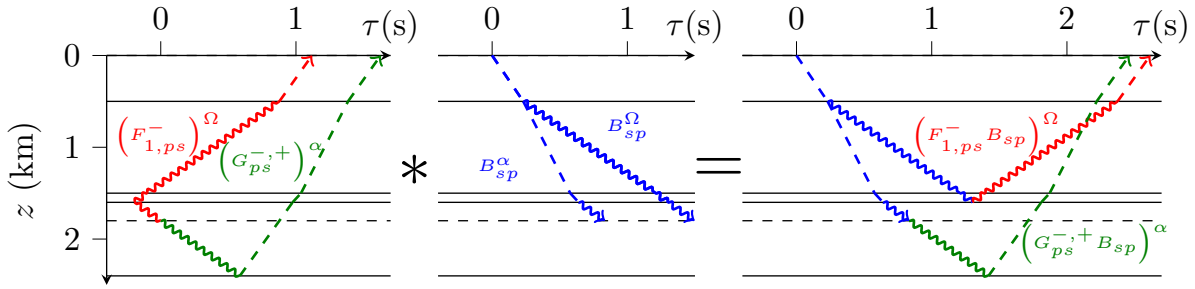


Figure 2: We depict the first (superscript $\alpha$) and last (superscript $\Omega$) events of $\mathbf{F}_1^-$, $\mathbf{G}^{-,+}$ and $\mathbf{B}$ in a cartoon fashion where dashed and sinusoidal lines represent P- and S-waves, respectively. From these events we construct the last event of $\left(\mathbf{F}_1^-\mathbf{B}\right)_{pp}$ and the first event of $\left(\mathbf{G}^{-,+}\mathbf{B}\right)_{pp}$. The operator $\mathbf{B}$ reduces the temporal separation between the $\mathbf{F}_1^-$ and $\mathbf{G}^{-,+}$. Compared to the model in Fig. 1b we reduced the S-wave velocity of the second layer to show a case where multiplication by $\mathbf{B}$ makes originally separable wavefields inseparable, violating one assumption of Eq. 2.

its normal product which again requires a suitable projector as well as an initial estimate. Currently, we are still investigating how to simplify the initial estimate for the scheme of Tunnicliffe-Wilson (1972).

## Conclusions

We demonstrate that P- and S-wave velocity differences cause challenges for the elastodynamic Marchenko method. First, the separability of the convolution-type representation theorem, which is a fundamental assumption for the Marchenko method, can be violated due the mode conversions. As a result the choice of focusing depth is limited. Second, the initial estimate can be even more sophisticated than originally assumed. We traded the sophisticated initial estimate for a trivial one. Since this change imposes stricter separability requirements it remains a trade-off. The solutions are the desired ones convolved with an unknown filter. We foresee that this filter can be removed by augmenting the Marchenko equations with energy conservation and minimum-phase constraints in a matrix sense. Although implementing the latter constraint still requires further investigation, minimum-phase retrieval for matrices appears to be possible. We suggest that the elastodynamic Marchenko method, and elastic multiple elimination in general, is fundamentally different compared to the acoustic case.

## Acknowledgements

We thank Mike Jervis for his comments. In addition, we are thankful to the European Union's Horizon 2020 research and innovation programme (Marie Sklodowska-Curie grant agreement No 641943) and the European Research Council (grant agreement No 742703) for the funding of this research.

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