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A multiplicative best–worst method for multi-criteria decision making

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ABSTRACT

This communication examines the best–worst method for multi-criteria decision making from a more mathematical perspective. The central part of this manuscript is the introduction of a new metric into the framework of the best–worst method. This alternative metric does not change the original idea behind the best–worst method and yet it can be shown that it is not only mathematically more sound but also that it ultimately leads to an optimization problem which can be simply linearized and thus solved.

1. Introduction and preliminaries

The best–worst method is a methodology which was recently introduced to deal with pairwise comparisons in multi-criteria decision making problems [10,11]. Since its inception, it has been used to solve a number of real-world problems, in technology innovation analysis [7], environmental management [6] and supply chain management [1], just to cite few applications. Given the quantity and the relevance of its applications, it is reasonable to expect that formal research on the method may progress at the same pace. The scope of this paper is to propose an alternative formulation of the best–worst method and show that it is both mathematically sound and easier to solve.

Hereafter, we consider a set of entities $X = \{x_1, \dots, x_n\}$ – often criteria and alternatives – and a set of pairwise comparisons on them. We denote a_{ij} the value of the comparison between x_i and x_j. Pairwise comparisons approximate ratios between the 'weights' of the entities, e.g. $a_{ij} \approx w_i/w_j$, where w_i and w_j are the weights of x_i and x_j, respectively. Since pairwise comparisons are ratios between weights we call this the multiplicative approach. Furthermore, we call E the set of pairs of indices for which the pairwise comparisons between elements of X have been elicited and we say that a set of pairwise comparisons is *consistent* if and only if there exists a weight vector $\mathbf{w} = (w_1, \dots, w_n)$ such that $a_{ij} = w_i/w_j \ \forall (i,j) \in E$. The best–worst method assumes an *a priori* knowledge of the 'best' and the 'worst' elements of X, here denoted as x_B and x_W respectively. Its final goal is to simplify the analysis by requiring the decision maker to elicit only the values a_{ij}'s comparing x_B to all the other elements of X and comparing all the other elements of X to x_W. Formally, in the best–worst method the subset of pairs of indices of the elements of X for which the judgments are explicitly elicited is

$$E = \{(i,j) | (i = B \lor j = W) \land i \neq j\} .$$

In the best–worst method, the optimization problem used to estimate a suitable weight vector $\mathbf{w} = (w_1, \dots, w_n)$ was originally formulated as

$$\begin{aligned} &\underset{w_1,\dots,w_n}{\text{minimize}} && \max_{(i,j)\in E} \left| a_{ij} - \frac{w_i}{w_j} \right| \\ &\text{subject to} && w_i > 0 \ \forall i, \quad w_1 + \cdots + w_n = 1. \end{aligned} \tag{1.1}$$

and then rewritten as

$$\begin{aligned} &\underset{\xi, w_1,\dots,w_n}{\text{minimize}} && \xi \\ &\text{subject to} && \xi \geq \left| a_{ij} - \frac{w_i}{w_j} \right| \quad \forall (i,j) \in E \\ & && w_i > 0 \ \forall i, \quad w_1 + \cdots + w_n = 1. \end{aligned} \tag{1.2}$$

The terms $\left| a_{ij} - w_i/w_j \right|$ in optimization problem (1.1) and its equivalent form (1.2) are analogous to the terms in the optimization problem of the Least Worst Absolute Error which was studied by Choo and Wedley [4] and described as "difficult to solve". Rezaei [11] proposed to linearize the previous optimization problem by replacing the terms $|a_{ij} - w_i/w_j|$ with $|w_j a_{ij} - w_i|$ and

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thus solve a different optimization problem; that is,

$$\begin{aligned} \underset{\xi, w_1,\ldots,w_n}{\text{minimize}} \quad & \xi \\ \text{subject to} \quad & \xi \geq \left| w_j a_{ij} - w_i \right| \;\; \forall (i,j) \in E \\ & w_i > 0 \;\forall i, \quad w_1 + \cdots + w_n = 1. \end{aligned} \tag{1.3}$$

Now, the crucial point is that the optimization problem (1.3) is substantially different from (1.2) and thus it leads to different results. Moreover, the expression $|w_j a_{ij} - w_i|$ can be easily linearized but remains an heuristic and lacks a strong mathematical justification. In fact, in their study on methods for deriving weight vectors, Choo and Wedley [4] called $\max_{i,j} |w_j a_{ij} - w_i|$ Preference Weighted Least Worst Absolute Error and stated that "[it] is not a true distance function".

2. A new proposal

The goal of this paper is to introduce an alternative objective function and eventually replace (1.1) with the following optimization problem:

$$\begin{aligned} \underset{w_1,\ldots,w_n}{\text{minimize}} \quad & \max_{(i,j)\in E} \max \left\{ a_{ij} \Big/ \frac{w_i}{w_j}, \frac{w_i}{w_j} \Big/ a_{ij} \right\} \\ \text{subject to} \quad & w_i > 0 \;\forall i, \quad w_1 + \cdots + w_n = 1. \end{aligned} \tag{2.1}$$

Hereafter we will show that, in spite of being seemingly less intuitive, this alternative objective function has two advantages when compared to the state of the art in the best–worst method: it (i) uses a notion of distance which is more suitable to pairwise comparisons expressed as ratios and (ii) allows to formulate an equivalent optimization problem which can be solved as a linear programming problem.

2.1. A metric for Abelian linearly ordered groups

The idea behind a suitable metric/distance is that it should be coherent with the group (in the algebraic sense) on which it is applied [3]. The concept of group is a cornerstone of abstract algebra

Definition 1 (*Group [5]*). A *group* is a set G equipped with a binary operation $\odot$ such that

- $\odot : G \times G \to G$ (closure w.r.t. $\odot$)
- $a \odot (b \odot c) = (a \odot b) \odot c$ (associativity)
- $\exists e \in G$ such that $a \odot e = a \;\forall a \in G$ (neutral element)
- $\forall a \in G \;\exists a^{-1} \in G$ such that $a \odot a^{-1} = e$ (inverse element)

In addition, a group is *Abelian* if the operation $\odot$ is commutative in G and *linearly ordered* if there is an order relation $\leq$ on G. We call $\mathcal{G} = (G, \odot, \leq)$ a generic Abelian linearly ordered group. Research in the field of preference relations [2,3] has shown that, if we consider $x, y \in G$ to be numerical judgments expressed on an Abelian linearly ordered group $\mathcal{G} = (G, \odot, \leq)$, then

$$d_{\mathcal{G}}(x, y) = \max \{x \div y, y \div x\} \tag{2.2}$$

where $\div$ is the inverse of $\odot$, is a suitable distance function ($\mathcal{G}$-distance).

Example 1. For the group $(\mathbb{R}, +, \leq)$, i.e. the real line equipped with addition (additive approach), the $\mathcal{G}$-distance (2.2) collapses into

$$\max\{x \div y, y \div x\} = \max\{x - y, y - x\} = |x - y|. \tag{2.3}$$

which is nothing else but the absolute value of the difference between two real numbers.

Example 2. For the group $(\mathbb{R}_>, \cdot, \leq)$, i.e. the positive reals equipped with multiplication (multiplicative approach), the $\mathcal{G}$-distance (2.2) becomes

$$\max\{x \div y, y \div x\} = \max \left\{ \frac{x}{y}, \frac{y}{x} \right\} \tag{2.4}$$

Since the multiplicative group is the group we are dealing with when we consider pairwise comparisons expressed as ratios, when we define the distance between an entry a_{ij} and its theoretical value w_i/w_j we should use (2.4) which, in our context, becomes

$$\max \left\{ a_{ij} \div \frac{w_i}{w_j}, \frac{w_i}{w_j} \div a_{ij} \right\} = \max \left\{ a_{ij} \Big/ \frac{w_i}{w_j}, \frac{w_i}{w_j} \Big/ a_{ij} \right\} \tag{2.5}$$

Let us also use a concrete example to corroborate the use of this metric. Unlike $|a_{ij} - w_i/w_j|$, the metric (2.5) considers the distance between the values 2 and 3 to be greater than the distance between the values 100 and 101. In fact,

$$\max \left\{ \frac{2}{3}, \frac{3}{2} \right\} > \max \left\{ \frac{100}{101}, \frac{101}{100} \right\}$$

In the multiplicative framework, this translates into evaluating the distance between the statements 'x_i is 2 times as heavy as x_j' and 'x_i is 3 times as heavy as x_j' to be greater than the distance between the statements 'x_i is 100 times as heavy as x_j' and 'x_i is 101 times as heavy as x_j'. As also claimed by other researchers [8], this property is reasonable for pairwise comparisons representing ratios.

To summarize, the absolute value of a difference, is a suitable metric when we operate in an 'additive' framework, and its corresponding suitable metric in a 'multiplicative' framework is (2.4). In the best–worst method, which is based on the multiplicative framework, this latter metric can be rewritten as (2.5).

2.2. Linearization of the model

In the previous subsection we argued that the optimization model proposed in this manuscript, i.e. (2.1), is more suitable than the optimization model originally proposed for the best–worst method, i.e. (1.1). Nevertheless, it remains the problem of solving a nonlinear problem. Also in this case, group theory helps.

First of all, we can use the logarithmic transformation $b_{ij} := \ln a_{ij}$, which is a group isomorphism, to pass from the multiplicative to the additive representation of preferences. Any logarithmic function, regardless of its base, acts as a group isomorphism between the positive reals and the real line. Its inverse isomorphism is the corresponding exponential function. It follows that the relation $a_{ij} = w_i/w_j$ is transformed into $b_{ij} = v_i - v_j$ where $v_1, \ldots, v_n$ are 'additive' weights, i.e. their relation with the 'multiplicative' weights is $v_k = \ln w_k \;\forall k$. Furthermore, since we are now operating in the additive group $(\mathbb{R}, +, \leq)$ the distance becomes the absolute value of differences, i.e. (2.3). Hence, in the additive setting, the optimization problem (2.1) has its equivalent in:

$$\begin{aligned} \underset{v_1,\ldots,v_n}{\text{minimize}} \quad & \max_{(i,j)\in E} |b_{ij} - (v_i - v_j)| \\ \text{subject to} \quad & v_1 + \cdots + v_n = 0. \end{aligned}$$

With the addition of auxiliary variables, this last optimization problem can easily be linearized to obtain,

$$\begin{aligned} \text{minimize} \quad & \xi \\ \text{subject to} \quad & b_{ij} - (v_i - v_j) = d_{ij}^+ - d_{ij}^- \;\; \forall (i,j) \in E \\ & \xi \geq d_{ij}^+ + d_{ij}^- \;\; \forall (i,j) \in E \\ & d_{ij}^+, d_{ij}^- \geq 0 \;\; \forall (i,j) \in E \\ & v_1 + \cdots + v_n = 0. \end{aligned} \tag{2.6}$$

In a nutshell, it is sufficient to transform the original pairwise comparisons into their additive form by means of the logarithmic

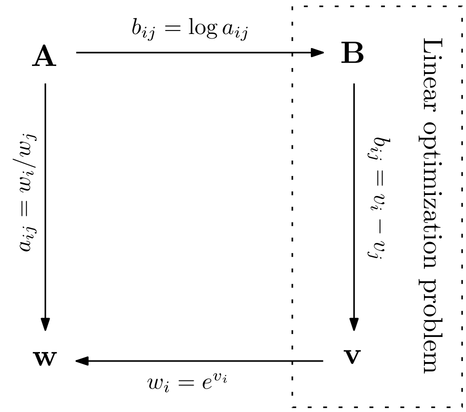

Fig. 1. Graphical description of the procedure developed in this manuscript.

transformation, and then use the optimization problem (2.6) to find the additive weights $v_1, \ldots, v_n$. To recover the multiplicative weights it is sufficient to apply the exponential transformation $w_i = \exp(v_i)$ and divide each weight by the total sum of the weights to normalize them so that their sum equals 1.

Example 3. Consider the pairwise comparisons contained in the following matrix/table where x_2 and x_5 are the 'best' and 'worst' entities, respectively,

$$\mathbf{A} = \begin{array}{c} \\ x_1 \\ \boldsymbol{x_2} \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \boldsymbol{x_5} \end{array} \\ \begin{bmatrix} - & - & - & - & 4 \\ 2 & - & 4 & 3 & 8 \\ - & - & - & - & 2 \\ - & - & - & - & 3 \\ - & - & - & - & - \end{bmatrix} \end{array}$$

Note that $\mathbf{A}$ is the matrix representation of the judgments presented in Example 2 in the original paper by Rezaei [11]. With the transformation $b_{ij} := \ln a_{ij}$ we obtain the matrix/table

$$\mathbf{B} = \begin{array}{c} \\ x_1 \\ \boldsymbol{x_2} \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \boldsymbol{x_5} \end{array} \\ \begin{bmatrix} - & - & - & - & \ln 4 \\ \ln 2 & - & \ln 4 & \ln 3 & \ln 8 \\ - & - & - & - & \ln 2 \\ - & - & - & - & \ln 3 \\ - & - & - & - & - \end{bmatrix} \end{array}$$

Next, we solve the optimization problem (2.6) and obtain

$$\mathbf{v} \approx (0.1155, 1.0397, -0.1155, 0, -1.0397)$$

which can then be transformed and normalized to obtain

$$\mathbf{w} = (0.181176, 0.456536, 0.143806, 0.161413, 0.0570693)$$

The procedure proposed in this section is summarized in Fig. 1.

3. Consistency analysis and interval weights

The multiplicative best–worst method proposed in this paper retains all the basic principles that made the original one so appealing. The goal of this section is to show how features of the original can be extended to the revisited.

Inconsistency analysis aims at estimating the level of irrationality of pairwise comparisons and it has been implemented in the best–worst method too. In the original best–worst method inconsistency was estimated as the ratio between the optimal value of the minimization problem and its maximum possible value, given

Table 1
Values of $\xi_{\max}$ obtained by solving the optimization problem (2.6) with different values of a_{BW}.

a_{BW}	1	2	3	4	5	6	7	8	9
$\xi_{\max}$	0	0.2310	0.3662	0.4621	0.5365	0.5973	0.6486	0.6931	0.7324

the knowledge of the value a_{BW}. Following the same guidelines, in this revisited approach we define the inconsistency level of $\mathbf{A}$ as

$$I(\mathbf{A}) = \frac{\xi^\star(\mathbf{A})}{\xi_{\max}}$$

where $\xi^\star(\mathbf{A})$ is the optimal value of the linear optimization problem (2.6) and $\xi_{\max}$ the maximum possible value of the same optimization problem. As originally noted by Rezaei [10], $\xi_{\max}$ depends on the value of a_{BW}, since a_{BW} implicitly defines an upper bound for all other judgments. Rezaei also proposed to obtain the value $\xi_{\max}$ by solving the optimization problem with a j such that $a_{Bj} = a_{jW} = a_{BW}$. By adopting the same approach with the revisited best–worst method and the linear optimization problem (2.6) it is possible to determine the values reported in Table 1.

Example 4. By reprising Example 3, for the optimal solution we have that the value of the objective function was $\xi^\star(\mathbf{A}) = 0.03926$. Since for $\mathbf{A}$ in the same example $a_{BW} = 8$, we shall use $\xi_{\max} = 0.6931$ and the inconsistency level of $\mathbf{A}$ is $I(\mathbf{A}) = 0.03926/0.6931 = 0.0566$.

The best–worst method focuses on the largest deviation discarding all the others, and can lead to multiple optimal solutions. This had been already noted in the literature [11] and, to mitigate this issue, it was suggested that interval weights be computed and compared to rank the elements of X. A similar approach can be implemented within the revisited best–worst method proposed in this paper. The upper and lower bounds of the generic additive weight v_i can be obtained by solving the following optimization problems, respectively,

$$\begin{aligned} v_i^+ = \ & \text{maximize} \quad v_i \\ & \text{subject to} \quad b_{ij} - (v_i - v_j) = d_{ij}^+ - d_{ij}^- \quad \forall (i,j) \in E \\ & \qquad \xi^\star \geq d_{ij}^+ + d_{ij}^- \quad \forall (i,j) \in E \\ & \qquad d_{ij}^+, d_{ij}^- \geq 0 \quad \forall (i,j) \in E \\ & \qquad v_1 + \cdots + v_n = 0. \end{aligned} \tag{3.1}$$

$$\begin{aligned} v_i^- = \ & \text{minimize} \quad v_i \\ & \text{subject to} \quad b_{ij} - (v_i - v_j) = d_{ij}^+ - d_{ij}^- \quad \forall (i,j) \in E \\ & \qquad \xi^\star \geq d_{ij}^+ + d_{ij}^- \quad \forall (i,j) \in E \\ & \qquad d_{ij}^+, d_{ij}^- \geq 0 \quad \forall (i,j) \in E \\ & \qquad v_1 + \cdots + v_n = 0. \end{aligned} \tag{3.2}$$

where the parameter $\xi^\star$ is the optimal value obtained by solving the optimization problem (2.6).

Example 5. We can still consider Example 3. Since the optimal value of the optimization problem was $\xi^\star(\mathbf{A}) = 0.03926$, we can use this parameter to solve the linear programs (3.1) and (3.2) for all $i = 1, \ldots, 5$ and obtain the additive interval weights $[v_i^-, v_i^+]$. Thanks to the exponential function and $w_i^- = exp(v_i^-)$ and $w_i^+ = exp(v_i^+)$ and the use of normalization [9] we can recover the interval-valued multiplicative weight vector

$$\bar{\mathbf{w}} = \begin{pmatrix} [0.22418, 0.23315] \\ [0.46266, 0.46999] \\ [0.11209, 0.11658] \\ [0.16039, 0.16293] \\ [0.05561, 0.05646] \end{pmatrix}$$

plotted in Fig. 2.

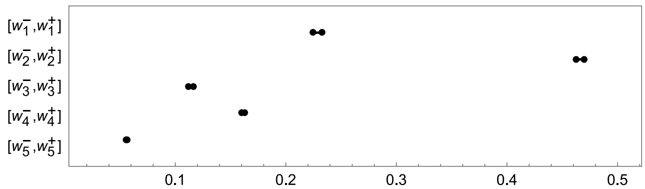

Fig. 2. Graphical representation of the interval-valued weight vector $\bar{\mathbf{w}} = ([w_i^-, w_i^+])_{i=1,\ldots,n}$.

Table 2
Summary of the main optimization problems presented in this paper.

	Optimization problem	Equivalent linear program
Original BWM	(1.1)	–
Multiplicative BWM	(2.1)	(2.6)

4. Conclusions

This communication adopts an algebraic approach to analyze the best–worst method. This approach suggests an alternative formulation of the best–worst method which uses a modified objective function. Not only the new objective function is mathematically more sound, but the entire optimization problem can be easily transformed into an equivalent linear program and solved quickly and without distortions. Table 2 presents a summary.

We have also shown that features of the best–worst method like consistency analysis and derivation of interval weights, can straightforwardly be implemented in the revisited best–worst method discussed in this manuscript.

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