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# THE IN-PLANE STEADY-STATE RESPONSE OF A RING IN RELATIVE MOTION TO A CONSTANT LOAD

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**Keywords:** Rotating ring, Moving load, High-order model, Steady-state response, Resonance.

**Abstract.** *Ring-like structures are very commonly used in civil, mechanical and aerospace engineering. Typical examples of such structures are components in turbomachinery, compliant gears, conventional pneumatic tires and more recent non-pneumatic tires, to name a few. In this paper, a ring on elastic foundation is considered. The foundation, modelled as distributed springs, connects the inner surface of the ring to an immovable axis. Focus is placed on the in-plane response of the ring subjected to in-plane load only. A high-order ring model, which accounts for the through-thickness variations of displacements is adopted for the study. Two loading situations of a ring structure are of interest in practice: (i) a stationary ring subjected to a circumferentially moving constant load; and (ii) a rotating ring under a stationary constant load. For the first situation, it is well-known that resonances occur when the rotational speeds of the load satisfy certain conditions. In a series of recent investigations, such resonance speeds have been predicted for a rotating ring subjected to a stationary load. In this paper the case of the rotating ring under a stationary constant load and that of a stationary ring subjected to a moving load are compared in terms of their resonance speeds, as well as the steady-state responses for various parameters. It is found that these two cases are distinguishable even for system parameters which result at similar critical speeds.*

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## 1 INTRODUCTION

The in-plane vibration of rings is a classical problem in solid mechanics due to the broad applications of ring-like structures in practice. It is usually the case that the ring and the load acting on it are in relative motion. Two particular situations, namely a stationary ring subjected to a circumferentially moving load (hereafter it is termed as moving load case) and a rotating ring under a stationary load (hefeafter it is termed as rotating ring case) are commonly encountered in engineering. Despite the absence of consensus on the existence of resonances of a rotating ring subjected to a stationary load with constant magnitude [1, 2], a seemingly conclusive result has been obtained in [3] according to which resonance can occur in rotating rings. Modes which are stationary as observed in a space-fixed reference system, are excited by the load [4] when a ring rotates at high speeds, resulting in a steady-state response which is time-invariant to a space-fixed observer. The experimental evidence of such a response is the occurrence of the so-called “standing waves” in rolling tires [5]. Similar wave phenomena have been reported in soft calenders of paper machines [6]. On the contrary, there is no doubt in the literature that resonances of a stationary ring subjected to a circumferentially moving constant load occur when the rotational speeds of the load equal to one of the natural frequencies divided by the corresponding mode number. Investigations of the steady-state responses in such a case can be found in [6, 7].

To what extent the moving load case and the rotating ring situation can be treated as equal is of interest and some deliberations on the topic can be found in the literature. It is concluded in [6] that the effect of rotation is negligible and the rotating ring under stationary load and stationary ring under moving load can be treated as equal. In this work, the steady-state deflection patterns of the ring when the speed of the relative motion is lower and higher than the minimum resonance speed are investigated for both moving load and rotating ring cases. The high-order model adopted from [3] is employed to simulate the in-plane response of rings on elastic foundation in relative motion with a load. The primary aim is to critically study the earlier suggested equivalence between the two cases. The responses of the two cases are compared in terms of their resonance speeds, as well as the steady-state responses for various parameters. It is shown that these two cases need to be distinguished especially when the foundation stiffness is relatively large.

## 2 HIGH-ORDER RING MODEL

A ring on elastic foundation subjected to a point load is shown in Fig. 1. The ring and the load are in relative motion with $\Omega_r$ being the rotating speed of the ring and $\Omega_p$ the velocity of the load. The inner surface of the ring is connected to an immovable axis by distributed radial springs $k_r$ and circumferential springs $k_c$. The in-plane radial and circumferential displacements of the ring are designated by $w(z, \theta, t)$ and $u(z, \theta, t)$. A space-fixed coordinate system $(r, \theta)$ is adopted to describe the motions of the ring. It is assumed that the mean radius of the ring is $R$. An auxiliary coordinate $z$ is introduced as $z = r - R$. The material properties of the ring are: the density $\rho$, Young’s modulus $E$, shear modulus $G$, Poisson’s ratio $\nu$, Lamé constants $\lambda$ and $\mu$. The dissipation in the ring material is considered by replacing $E$ by $E^* = E(1 + \zeta \mathrm{d}/\mathrm{d}t)$ in which $\zeta$ is a loss factor of the material. In addition, $A$ is the cross-sectional area, $I$ is the cross-sectional moment of inertia, $b$ is the width of the ring. A constant point load $P(t) = P_0$ is applied on the outer surface of the ring.

The high-order model of the ring developed in [3] is employed. Plane strain configuration is assumed for the model. The external load is incorporated in the governing equations by the

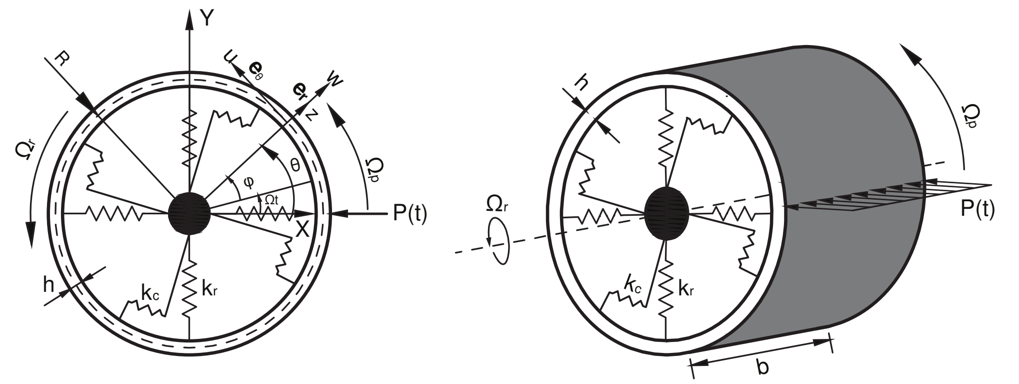


Figure 1: An elastic ring on elastic foundation in relative motion with a point (line) load. $\Omega_r$ and $\Omega_p$ are the rotational speeds of the ring and the load, respectively: left figure for front view; right figure for side view.

Hamilton's principle. One is referred to [8] for details of the derivation procedure. According to [3], the displacement fields are expressed as polynomial functions of the ring thickness $z$:

$$w(z,\theta,t) = \sum_{l=0}^{l=N_1} w_l(\theta,t)\, z^l,\; u(z,\theta,t) = \sum_{q=0}^{q=N_2} u_q(\theta,t)\, z^q \tag{1}$$

in which $l$, $q$ are integers and $l \geq 0$, $q \geq 0$. $N_1$ and $N_2$ are the orders of the polynomials of the displacement fields. Assuming a radial point load of constant amplitude is applied on the outer surface of a rotating ring, the equations of motion that govern the small vibrations of the ring around the static equilibrium in the radial direction are:

$$\begin{aligned} &\int_{\frac{-h}{2}}^{\frac{h}{2}} (\mathrm{I}_1^{\mathrm{lin}}\, z^l)\, \mathrm{d}z + \rho \int_{\frac{-h}{2}}^{\frac{h}{2}} \left( r(\dot{v}_1 + \Omega_r\, v_1' - \Omega_r\, v_2) z^l \right) \mathrm{d}z + \left( f_1^{\mathrm{lin}} - f_2^{\mathrm{lin}}(-1)^l \right) \left(\frac{h}{2}\right)^l \\ &= -\left(\frac{h}{2}\right)^l P(t)\delta(\theta) = -\left(\frac{h}{2}\right)^l P_0\, \delta(\theta), \quad (l = 0, 1, 2, 3 ... N_1). \end{aligned} \tag{2}$$

The dimension of $P_0$ is N.m$^{-1}$ and $\delta$ is the Dirac delta function.

The linearised equations of motion of a rotating ring in the circumferential direction are:

$$\begin{aligned} &\int_{\frac{-h}{2}}^{\frac{h}{2}} (\mathrm{I}_2^{\mathrm{lin}}\, z^q)\, \mathrm{d}z + \rho \int_{\frac{-h}{2}}^{\frac{h}{2}} \left( r(\dot{v}_2 + \Omega_r\, v_2' + \Omega_r\, v_1) z^q \right) \mathrm{d}z + \left( f_3^{\mathrm{lin}} - f_4^{\mathrm{lin}}(-1)^q \right) \left(\frac{h}{2}\right)^q = 0, \\ &(q = 0, 1, 2, 3 ... N_2). \end{aligned} \tag{3}$$

The details of the governing equations including the expressions for $\mathrm{I}_1^{\mathrm{lin}}$, $\mathrm{I}_2^{\mathrm{lin}}$, $f_1^{\mathrm{lin}} - f_4^{\mathrm{lin}}$ and the velocities $v_1$ and $v_2$ of a differential element of the ring in radial and circumferential directions in the left-hand side of Eqs. (2-3) can be found in [8]. For the case of a stationary ring subjected to a circumferentially moving load of constant amplitude, the governing equations can be obtained by setting $\Omega_r$ and the static equilibrium to zero on the left side of Eqs. (2-3) and altering $P_0\, \delta(\theta)$ to $P_0\, \delta(\theta - \Omega_p t)$ on the right-hand side of Eq. (2).

The following dimensionless parameters are introduced [3]:

$$k = \sqrt{I/A},\; \bar{k} = k/R,\; \bar{\gamma} = n\, \bar{k},\; \bar{\omega} = \omega\, k/c_0,\; \bar{v}_{(r,p)} = R\, \Omega_{(r,p)}/c_0,\; \bar{k}_{(r,c)} = k_{(r,c)} k^2/(Eh), \tag{4}$$

where $c_0 = \sqrt{E/\rho}$ is the speed of the longitudinal wave in the rod, $I = bh^3/12$ is the cross section area moment of inertia and $\bar{k}$ is the non-dimensional radius of gyration. $\bar{\theta} = \theta/\bar{k}$ and $\tau = c_0 t/k$ are the dimensionless angle and temporal variables, respectively. $\bar{P} = P_0\, \bar{k}/(Eh)$ is the dimensionless load amplitude.

## 3 MOVING LOAD VERSUS ROTATING RING

The load speeds causing resonance of a stationary ring subjected to a constant point load moving circumferentially are well known [7], namely $\Omega_p = \omega_n/n$ in which $n$ is the circumferential mode number and $\omega_n$ is the $n$th natural frequency of the ring. The minimum resonance speed (the critical speed) is the lowest value of $\omega_n/n$. Resonance speeds of a rotating ring subjected to a stationary constant load satisfy the condition $\bar{\omega}_n = 0$ in which $\bar{\omega}_n$ is the natural frequency calculated in a space-fixed reference system [3]. By substituting $\bar{\omega}_n = 0$ into the frequency equation [3], one can solve for resonance speeds for each circumferential wavenumber.

Theoretically, resonances of the moving load case always exist. However, in practice the load speed can not always reach the resonance speeds, especially for stiff rings since their resonance speeds are high. On the contrary, resonance occurs only for certain parameters of a rotating ring subjected to a stationary load of constant magnitude. Specifically, resonances of a rotating ring may only occur for relatively soft rings. For example; for a ring made of steel, resonance speeds do not exist for the rotating ring case whereas resonance speeds of such a ring for the moving load case do exist but are extremely large. One needs to bear in mind that if a ring is stiff, the responses of the ring for the rotating ring and moving load cases are similar. The reason is that in the operational speed range, the translational rigid-body like motion governs the response in both cases. Thus, only soft rings are considered in this section because resonance speeds exist for such rings for both moving load and rotating ring cases.

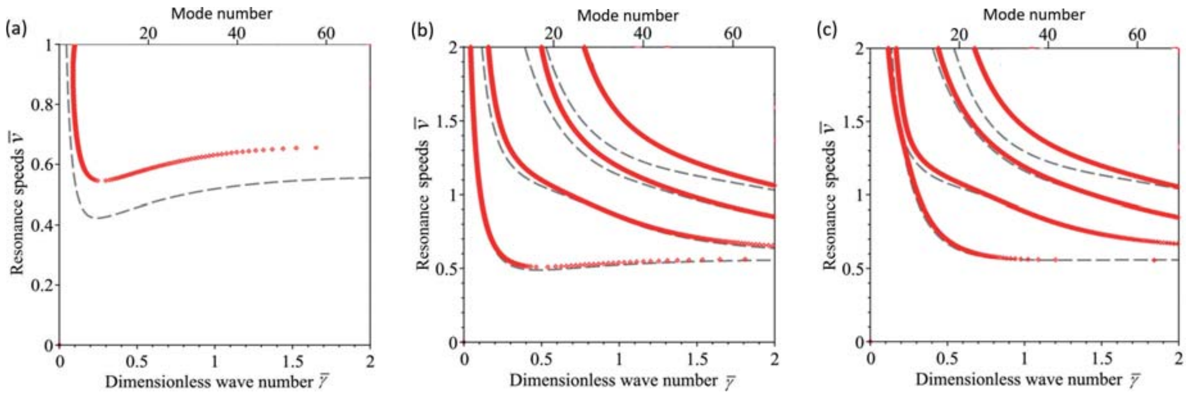


Figure 2: Comparison of resonance speeds, $h/R = 0.1, \bar{k}_c = 0.1$ using the high-order model with increasing stiffness of radial springs. Grey dashed line for moving load case; Red dotted lines for rotating ring case: (a) $\bar{k}_r = 0.001$; (b) $\bar{k}_r = 0.01$; (c) $\bar{k}_r = 0.1$.

### 3.1 Comparison of resonance speeds between both cases for relatively soft rings

Fig. 2 shows the comparisons of resonance speeds, as functions of the mode number, between moving load and rotating ring case with different values of the foundation stiffness. Since high-order theory is used, there are several curves representing higher order motions of the ring. In the figure, the lower abscissa in each plot is the dimensionless wavenumber, whereas the upper abscissa is the corresponding discrete circumferential mode number $n$. All the chosen parameters represent relatively stiff foundation (soft ring) configuration since only in this case resonance speeds of rotating rings exist.

In Fig. 2 the upper limit of the plots is set at $\bar{v} = 1$ and therefore only the lowest branch of resonance speeds is shown. The reason is that at higher speeds, the predictions of other branches are not accurate since the rotation-induced hoop tension is approaching unrealistically high value. Generally, rotation stiffens the ring, therefore, the resonance speeds of rotating rings are larger than those in the moving load case as shown in Fig. 2(a). With increasing $\bar{k}_r$, the resonance speeds for the two lower branches of both cases become close as shown in Figs. 2(b)(c). For the two higher branches of resonance speeds, the differences are still large. The minimum resonance speed in Fig. 2 is a critical speed at which a wave-like steady-state deformation pattern is initiated. For the parameters shown in Figs. 2(b)(c), the critical speed converges to the Rayleigh wave speed with increasing wavenumber and a Rayleigh wave resonance is expected when a stationary constant load is applied [6].

### 3.2 Steady-state response for soft rings on stiff elastic foundation

The steady-state responses of a ring in relative motion to a constant point load is investigated in this section for a soft ring. The parameters are chosen the same as in Fig. 2(b). For this set of parameters, resonance speeds in the two cases are close, especially for the lower order motion. The dynamic responses are derived using the so-called ”method of the images” described in [7, 8].

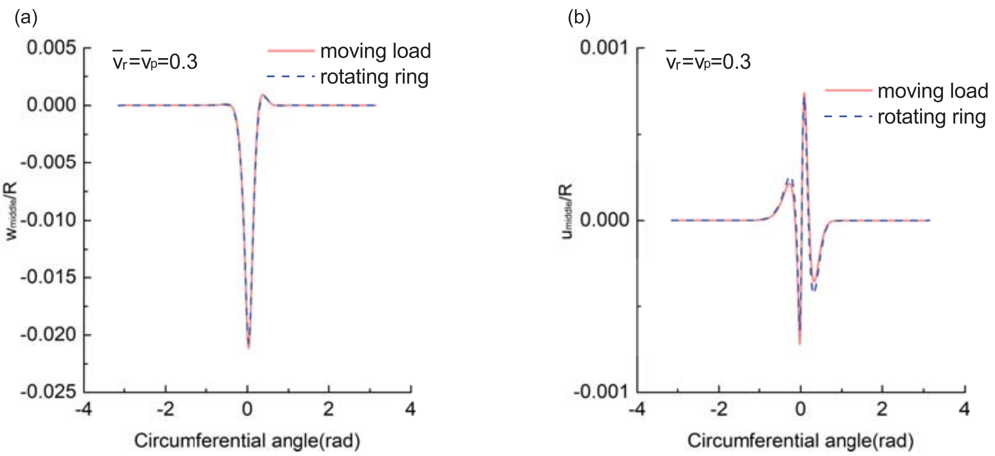


Figure 3: Displacements at the middle surface of the ring for $\bar{v}_r = \bar{v}_p = 0.3$, $\bar{k}_r = 0.01$, $\bar{k}_c = 0.1$, $\zeta = 0.002$, $\bar{P} = 0.002$: (a) Normalised radial displacement; (b) Normalised circumferential displacement.

As shown in Fig. 2(b), the minimum resonance speed in both cases is the same, namely $\bar{v}_{cr} \approx 0.5$. Therefore, two velocities of the relative motion are chosen: $\bar{v}_r = \bar{v}_p = 0.3$ as a sub-critical speed and $\bar{v}_r = \bar{v}_p = 0.7$ as a super-critical one. Fig. 3 shows the displacements (normalized with respect to the ring radius at the middle surface) at the middle surface of the ring for the two cases. It can be seen that the responses are very close. In this case, the stiffening effect due to rotation for the rotating ring case is not obvious. When the speeds increase, for the rotating ring case, the static expansion due to rotation becomes large, resulting in high hoop tension. This tension stiffens the ring, especially it suppresses the radial displacement of the rotating ring comparing to the moving load case as shown in Fig. 4(a) for the super-critical case $\bar{v}_r = \bar{v}_p = 0.7$. However, the stiffening effect is not obvious on the circumferential displacement

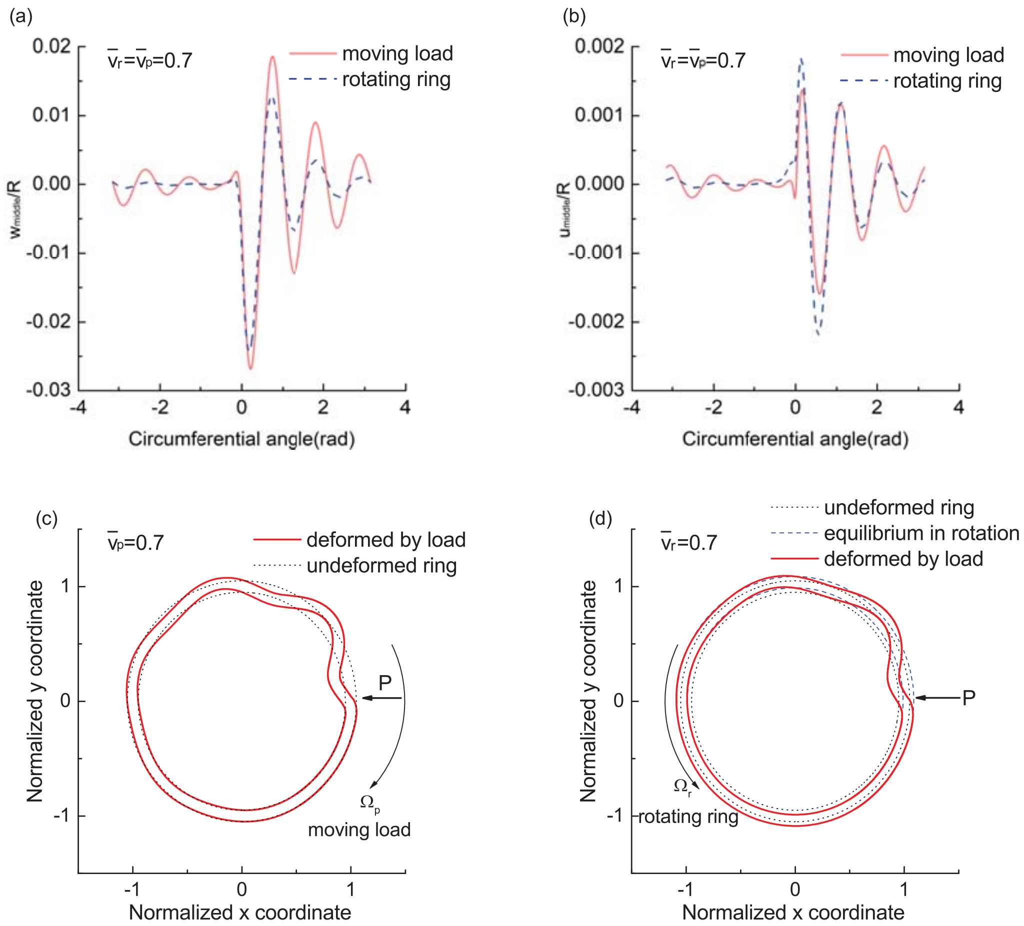


Figure 4: Displacements at the middle surface of the ring for $\bar{v}_r = \bar{v}_p = 0.7$, $\bar{k}_r = 0.01$, $\bar{k}_c = 0.1$, $\zeta = 0.002$, $\bar{P} = 0.002$: (a) Normalized radial displacement; (b) Normalized circumferential displacement; (c) Ring deformation, moving load; (d) Ring deformation, rotating ring. The ring deformations are scaled by 5.

(Fig. 4(b)). In Figs. 4(c)(d), the ring deformations in both cases are illustrated. Besides the different patterns, the rotating ring shows static radial expansion caused by rotation. In addition, the effect of damping is more pronounced in the rotating ring case as the displacements decay fast along the circumference away from the loading point.

### 3.3 Maximum deflection of the ring versus velocity

The maximum displacement at the middle surface ($D_{\max}$ is defined as $\max\{\sqrt{{w_0}^2 + {u_0}^2}\}$ in which $w_0$ and $u_0$ are the radial and circumferential displacements at the middle surface, respectively) for a ring with the same parameters as used in section 3.2 is shown in Fig. 5(a). For the chosen parameters, the resonance speeds of both cases are quite similar as shown in Fig. 2(b). However, the responses are different under the same load as shown in Fig. 5(a), especially when the relative speeds between the load and the ring exceed the minimum resonance speed. For rotating ring case, the rotation of the ring stiffens the ring, resulting in smaller responses. In Figs. 5(b) and (c), the maximum middle surface displacements in the radial and circumferential directions are demonstrated, respectively. It can be seen that the responses are mainly governed by the radial motion. To conclude, for the moving load and rotating ring cases which have similar resonance speeds, their responses are in large disagreement at higher speeds of the

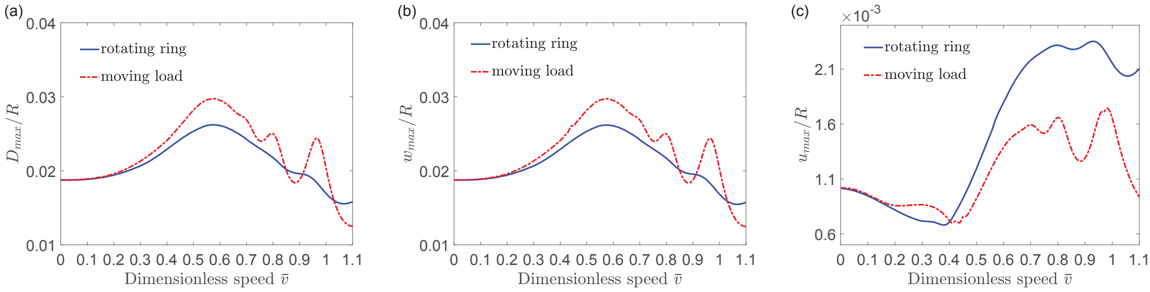


Figure 5: Comparison of maximum deflection at the middle surface versus velocity, $\bar{k}_r = 0.01$, $\bar{k}_c = 0.1$: (a) $D_{\max}$; (b) maximum radial displacement; (c) maximum circumferential displacement.

relative motion between the load and the ring.

## 4 CONCLUSIONS

The equivalence of the rotating ring under a stationary constant load case and a stationary subjected to a moving constant load case are discussed by comparing their resonance speeds, as well as the steady-state responses. It is found that these two cases need to be distinguished even for system parameters which result in similar critical speeds. The moving load on stationary ring and the rotating ring under stationary load cases can only be considered equivalent when the relative speeds between the ring and the load are low or the responses are mainly governed by the $n = 1$ mode. First, if resonance speeds exist, these can be very different in the two cases under consideration, which will result in different dynamic responses. Second, even for system parameters which result in similar critical speeds, the responses under the same load can be different due to the rotation effects.

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