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## THE SUM OF DIGITS FUNCTION OF THE BASE PHI EXPANSION OF THE NATURAL NUMBERS

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### Abstract

In the base phi expansion any natural number is written uniquely as a sum of powers of the golden mean with digits 0 and 1, where one requires that the product of two consecutive digits is always 0. In this paper we show that the sum of digits function modulo 2 of these expansions is a morphic sequence. In particular we prove that — like for the Thue-Morse sequence — the frequency of 0's and 1's in this sequence is equal to 1/2.

### 1. Introduction

Base phi representations were introduced by George Bergman in 1957 [1]. Base phi representations are also known as beta-expansions of the natural numbers, with  $\beta = (1 + \sqrt{5})/2 =: \varphi$ , the golden mean. A natural number  $N$  is written in base phi if  $N$  is represented as

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

with digits  $d_i = 0$  or 1, and where  $d_i d_{i+1} = 11$  is not allowed. We write these expansions as

$$\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R.$$

Ignoring leading and trailing 0's, the base phi representation of a number  $N$  is unique, as shown by Bergman.

Let for  $N \geq 0$

$$s_\beta(N) := \sum_{k=R}^{k=L} d_k(N)$$

be the sum of digits function of the base phi expansions. We have

$$(s_\beta(N)) = 0, 1, 2, 2, 3, 3, 3, 2, 3, 4, 4, 5, 4, 4, 4, 5, 4, 4, 2, 3, 4, 4, 5, 5, 5, 4, 5, 6, 6, 7, 5, \dots$$

In this paper we study the base phi analogue of the Thue-Morse sequence (where the base equals 2), i.e., the sequence

$$(x_\beta(N)) := (s_\beta(N) \bmod 2) = 0, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, \dots$$

Recall that a morphism is a map from the set of infinite words over an alphabet to itself, respecting the concatenation operation. The Thue Morse sequence is the fixed point starting with 0 of the morphism  $0 \rightarrow 01, 1 \rightarrow 10$ .

**Theorem 1.** *The sequence  $x_\beta$  is a morphic sequence, i.e., the letter-to-letter image of the fixed point of a morphism.*

This theorem permits us to answer a number of natural questions one may ask about  $x_\beta$ , for example: will a word 00000 ever occur? What are the frequencies of 0 and 1?

We end this introduction by mentioning some related work. In [2] asymptotic expressions for  $\sum_{N < x} s_\beta(N)$  as  $x \rightarrow \infty$  were obtained. In [7], so-called  $\alpha$ -irreducibles were introduced, which might serve as building blocks for  $s_\beta(N)$ . An  $\alpha$ -irreducible is a natural number  $N$ , such that if  $\beta(N) = \beta(N') + \beta(N'')$  with  $N' < N''$ , then  $N' = 0$  and  $N'' = N$ . The first twelve  $\alpha$ -irreducibles are 1, 2, 3, 5, 6, 7, 12, 13, 14, 16, 17, 18. Grabner and Prodinger give a detailed asymptotic description of the counting function  $A$ , where  $A(n)$  is the number of  $\alpha$ -irreducibles among  $1, 2, \dots, n$ . From their Theorem 1, and Lemma 1 and Lemma 2 in the next section, one can obtain new insights in  $A$ . Let  $(L_n)$  be the Lucas numbers. The even Lucas intervals  $[L_{2n}, L_{2n+1}]$  will contain no  $\alpha$ -irreducibles, with exception of  $N = L_{2n}$ . The odd Lucas intervals  $[L_{2n+1} + 1, L_{2n+2} - 1]$ , with  $N = L_{2n+2}$  added, will contain two shifted copies of the  $\alpha$ -irreducibles in the previous (extended) odd Lucas interval. Since  $L_{2n+1} \sim \varphi^{2n+1}$ , this directly implies the crude asymptotics of the counting function:  $A(n) \asymp n^\rho$ , with  $\rho = \log 2 / \log \varphi^2$ .

## 2. Properties of the Base Phi Representation

The Lucas numbers  $(L_n) = (2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \dots)$  are defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2.$$

The Lucas numbers have a particularly simple base phi representation: from the well-known formula  $L_{2n} = \varphi^{2n} + \varphi^{-2n}$ , and the recursion  $L_{2n+1} = L_{2n} + L_{2n-1}$ , we have for all  $n \geq 1$

$$\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1, \quad \beta(L_{2n+1}) = 1(01)^n \cdot (01)^n.$$

The properties of base phi expansion of the natural numbers are intrinsically determined by the *Lucas intervals*:

$$\Lambda_{2n} := [L_{2n}, L_{2n+1}], \quad \Lambda_{2n+1} := [L_{2n+1} + 1, L_{2n+2} - 1].$$

When we add  $\Lambda_0 := [0, 1]$ , these intervals partition the natural numbers as  $n = 0, 1, 2, \dots$ . The partition elements correspond to the lengths of the expansions:

if  $\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R$ , then the left most index  $L = L(N)$  and the right most index  $R = R(N)$  satisfy

$$\begin{aligned} L(N) = 2n + 1, R(N) = -2n & \text{ if and only if } N \in \Lambda_{2n}, \\ L(N) = 2n + 2 = -R(N) & \text{ if and only if } N \in \Lambda_{2n+1}. \end{aligned}$$

This is not hard to see from the simple expressions we have for the  $\beta$ -expansions of the Lucas numbers, see also Theorem 1 in [6].

For two expansions  $\beta(N)$  and  $\beta(N')$ , we write  $\beta(N) + \beta(N')$  for the digit-wise addition of these expansions, tacitly assuming that 0's have been added to the left and/or right of these expansions to make this well-defined. Since  $\beta(L_{2n})$  consists of only 0's between the exterior 1's, the following lemma is obvious.

**Lemma 1.** ([3]) *For all  $n \geq 1$  and  $k = 0, \dots, L_{2n-1}$  one has  $\beta(L_{2n} + k) = \beta(L_{2n}) + \beta(k)$ .*

This gives a recursive relation for the expansions in the Lucas interval  $\Lambda_{2n}$ . To obtain recursive relations for the interval  $\Lambda_{2n+1}$ , this interval has to be divided into three subintervals. These three intervals are

$$\begin{aligned} I_n &:= [L_{2n+1} + 1, L_{2n+1} + L_{2n-2} - 1], \\ J_n &:= [L_{2n+1} + L_{2n-2}, L_{2n+1} + L_{2n-1}], \\ K_n &:= [L_{2n+1} + L_{2n-1} + 1, L_{2n+2} - 1]. \end{aligned}$$

To formulate the next lemma, it is notationally convenient to extend the semi-group of words to the free group of words. For example, one has  $110^{-1}01^{-1}00 = 100$ .

**Lemma 2.** ([11], [3])<sup>1</sup> *For all  $n \geq 2$  and  $k = 1, \dots, L_{2n-2} - 1$ ,*

$$\begin{aligned} I_n : \quad \beta(L_{2n+1} + k) &= 1000(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}1001, \\ K_n : \quad \beta(L_{2n+1} + L_{2n-1} + k) &= 1010(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}0001. \end{aligned}$$

Moreover, for all  $n \geq 2$  and  $k = 0, \dots, L_{2n-3}$ ,

$$J_n : \quad \beta(L_{2n+1} + L_{2n-2} + k) = 10010(10)^{-1}\beta(L_{2n-2} + k)(01)^{-1}001001.$$

### 3. The Sequence $x_\beta$ is Morphic

If  $V = [K, K + 1, \dots, L]$  is an interval of natural numbers, then we write

$$x_\beta(V) := [x_\beta(K), x_\beta(K + 1), \dots, x_\beta(L)]$$

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<sup>1</sup>See [4] for a comprehensive proof of Lemma 2

for the consecutive sums of digits modulo 2 of these numbers.

Since  $x_\beta(L_{2n}) = 0$  and  $x_\beta(0) = 0$ , Lemma 1 implies directly the following lemma.

**Lemma 3.** (EVEN) *For  $n \geq 1$  one has  $x_\beta(\Lambda_{2n}) = x_\beta([0, L_{2n-1}])$ .*

The mirror morphism on  $\{0, 1\}$  is defined by  $\bar{0} = 1, \bar{1} = 0$ .

We obtain from Lemma 2 with  $x_\beta(I_n) = x_\beta(K_n) = \overline{x_\beta(\Lambda_{2n-1})}$ , and  $x_\beta(J_n) = x_\beta(\Lambda_{2n-2})$  the following.

**Lemma 4.** (ODD) *For  $n \geq 1$  one has  $x_\beta(\Lambda_{2n+1}) = \overline{x_\beta(\Lambda_{2n-1})}x_\beta(\Lambda_{2n-2})\overline{x_\beta(\Lambda_{2n-1})}$ .*

We illustrate the base phi expansions with the following table.

$N$	$\beta(N)$	$x_\beta(N)$	Lucas interval
0	0	0	$\Lambda_0$
1	1	1	$\Lambda_0$
2	10 · 01	0	$\Lambda_1$
3	100 · 01	0	$\Lambda_2$
4	101 · 01	1	$\Lambda_2$
5	1000 · 1001	1	$\Lambda_3$
6	1010 · 0001	1	$\Lambda_3$
7	10000 · 0001	0	$\Lambda_4$
8	10001 · 0001	1	$\Lambda_4$
9	10010 · 0101	0	$\Lambda_4$
10	10100 · 0101	0	$\Lambda_4$
11	10101 · 0101	1	$\Lambda_4$
12	100000 · 101001	0	$\Lambda_5$

Let  $\tau$  be the morphism on the alphabet  $A := \{1, \dots, 8\}$  defined by

$$\begin{aligned} \tau(1) &= 12, & \tau(2) &= 312, & \tau(3) &= 47, & \tau(4) &= 8312, \\ \tau(5) &= 56, & \tau(6) &= 756, & \tau(7) &= 83, & \tau(8) &= 4756. \end{aligned}$$

Define the mirroring morphism  $\mu$  on  $A$  by

$$\mu : 1 \rightarrow 5, 2 \rightarrow 6, 3 \rightarrow 7, 4 \rightarrow 8, 5 \rightarrow 1, 6 \rightarrow 2, 7 \rightarrow 3, 8 \rightarrow 4.$$

Then  $\tau$  is mirror invariant:  $\tau\mu = \mu\tau$ .

**Theorem 2.** *Let  $x_\beta$  be the sum of digits function of the base phi expansions of the natural numbers. Let  $\lambda : A^* \rightarrow \{0, 1\}$  be the letter-to-letter morphism given by*

$$\lambda(1) = \lambda(3) = \lambda(6) = \lambda(8) = 0, \text{ and } \lambda(2) = \lambda(4) = \lambda(5) = \lambda(7) = 1.$$

*Then  $x_\beta = \lambda(t)$ , where  $t = 1231247123\dots$  is the fixed point of  $\tau$  starting with 1.*

Theorem 2 is a direct consequence of the following result. Note that  $\overline{\lambda\tau} = \lambda\tau\mu$ .

**Proposition 1.** *For  $n = 1, 2, \dots$  one has  $x_\beta(\Lambda_{2n}) = \lambda(\tau^n(1))$ , and  $x_\beta(\Lambda_{2n+1}) = \lambda(\tau^n(3))$ .*

*Proof.* By induction. For  $n = 1$  one has  $x_\beta(\Lambda_2) = 01 = \lambda(12) = \lambda(\tau(1))$ , and  $x_\beta(\Lambda_3) = 11 = \lambda(47) = \lambda(\tau(3))$ . From Lemma 3 and the induction hypothesis we have

$$\begin{aligned} x_\beta(\Lambda_{2n+2}) &= x_\beta([0, L_{2n-1}])x_\beta([L_{2n-1} + 1, L_{2n} - 1])x_\beta([L_{2n}, L_{2n+2}]) \\ &= \lambda(\tau^n(1))\lambda(\tau^{n-1}(3))\lambda(\tau^n(1)) \\ &= \lambda(\tau^{n-1}(12312)) = \lambda(\tau^{n+1}(1)). \end{aligned}$$

From Lemma 4 and the induction hypothesis we have

$$\begin{aligned} x_\beta(\Lambda_{2n+3}) &= \overline{x_\beta(\Lambda_{2n+1})}x_\beta(\Lambda_{2n})\overline{x_\beta(\Lambda_{2n+1})} \\ &= \overline{\lambda(\tau^n(3))\lambda(\tau^n(1))(\lambda(\tau^n(3)))} \\ &= \lambda(\tau^n(7))\lambda(\tau^n(1))\lambda(\tau^n(7)) \\ &= \lambda(\tau^n(717)) = \lambda(\tau^n(47)) = \lambda(\tau^{n+1}(3)). \quad \square \end{aligned}$$

Since  $\tau$  is mirror invariant, the letters  $a$  and  $\mu(a)$  have the same frequency for  $a \in A$ . As  $\overline{\lambda} = \lambda\mu$ , this implies the following.

**Proposition 2.** *The letters 0 and 1 have frequency  $\frac{1}{2}$  in  $x_\beta$ .*

It is well-known that the words of length 2 in the Thue-Morse sequence have frequencies  $\frac{1}{6}$  for 00 and 11, and  $\frac{1}{3}$  for 01 and 10. Here is the corresponding result for the golden mean sum of digits function.

**Proposition 3.** *In  $x_\beta$  the words 00 and 11 have frequency  $\frac{1}{10}\sqrt{5}$ , and the words 01 and 10 have frequency  $\frac{1}{2} - \frac{1}{10}\sqrt{5}$ .*

*Proof.* As in [10] we compute the frequencies  $\nu[ab]$  of the words  $ab$  of length 2 occurring in the fixed point  $t$  of the morphism  $\tau$  by using the 2-block substitution  $\tau^{[2]}$ . The words of length 2 occurring in the fixed point  $t$  of the morphism  $\tau$  are

$$12, 23, 24, 28, 31, 35, 47, 56, 64, 67, 68, 71, 75, 83.$$

When we code the 14 words of length 2 by  $\ell_1, \dots, \ell_{14}$ , in the order given above, then  $\tau^{[2]}$  is given for the letters  $\ell_1, \dots, \ell_7$  by

$$\ell_1 \rightarrow \ell_1\ell_2, \ell_2 \rightarrow \ell_5\ell_{13}, \ell_3 \rightarrow \ell_5\ell_{14}, \ell_4 \rightarrow \ell_5\ell_{13}, \ell_5 \rightarrow \ell_7\ell_{12}, \ell_6 \rightarrow \ell_7\ell_{13}, \ell_7 \rightarrow \ell_{14}\ell_5\ell_{14}.$$

The  $\tau^{[2]}$ -images of  $\ell_8, \dots, \ell_{14}$  follow from this by mirror-symmetry. The first 7 components of the normalized eigenvector of the incidence matrix of the morphism  $\tau^{[2]}$  are given by

$$\left[ \frac{1}{4} - \frac{1}{20}\sqrt{5}, \frac{1}{2} - \frac{1}{5}\sqrt{5}, \frac{3}{20} - \frac{1}{20}\sqrt{5}, \frac{1}{5}\sqrt{5} - \frac{2}{5}, \frac{1}{10}, \frac{3}{20} - \frac{1}{20}\sqrt{5}, \frac{3}{20}\sqrt{5} - \frac{1}{4} \right].$$

This means that, e.g.,  $\nu[12] = \frac{1}{4} - \frac{1}{20}\sqrt{5}$ , and  $\nu[31] = \frac{1}{10}$ . The frequency of 00 equals  $\mu[00] = \nu[13] + \nu[68] + \nu[83] = \frac{1}{10}\sqrt{5}$ .  $\square$

**Remark.** Christian Mauduit with Michael Drmota and Joël Rivat proved that the Thue-Morse sequence is normal along squares (see [5]). We conjecture that this also holds for the sum of digits function modulo 2 of the basis phi expansion of the natural numbers, i.e., for  $(x_\beta(n^2))$ .

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