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Graph product Khintchine inequalities and Hecke C*-algebras: Haagerup inequalities, (non)simplicity, nuclearity and exactness

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ABSTRACT

Graph products of groups were introduced by Green in her thesis [32]. They have an operator algebraic counterpart introduced and explored in [14]. In this paper we prove Khintchine type inequalities for general C*-algebraic graph products which generalize results by Ricard and Xu [50] on free products of C*-algebras. We apply these inequalities in the context of (right-angled) Hecke C*-algebras, which are deformations of the group algebra of Coxeter groups (see [22]). For these we deduce a Haagerup inequality which generalizes results from [33]. We further use this to study the simplicity and trace uniqueness of (right-angled) Hecke C*-algebras. Lastly we characterize exactness and nuclearity of general Hecke C*-algebras.

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0. Introduction

A graph product of groups, first introduced in [32], is a group theoretic construction that generalizes both free products and Cartesian products of groups. It associates to a simplicial graph with discrete groups as vertices a new group by taking the free product of the vertex groups, with added relations depending on the graph. The construction preserves many important group theoretical properties (see [2], [3], [13], [17], [18], [19], [32], [34], [44]) and covers for instance right-angled Coxeter groups and right-angled Artin groups.

In [14] Fima and the first-named author explored the operator algebraic counterpart of graph products and developed the theory of reduced and universal graph products of C*-algebras as well as graph products of von Neumann algebras and quantum groups. The construction generalizes operator algebraic free products and - just as for groups - preserves properties like exactness for reduced graph products of C*-algebras and the Haagerup property for von Neumann algebras. Important examples of graph products are right-angled Hecke algebras (see below), special cases of mixed q-Gaussian algebras [8] as well as several group C*-algebras that have been studied in the literature, see in particular [28] and Section 5.4. In a different direction, C*-algebras associated to graph products of groups with distinguished positive cones were studied in [21].

In this paper we prove a Khintchine inequality for general C*-algebraic graph products. These are inequalities which estimate the operator norm of a reduced operator of a given length with the norm of certain Haagerup tensor products of column and row Hilbert spaces. In the case of free groups and words of length 1 these inequalities go back to Haagerup's fundamental paper [33]. In the case of general free products and arbitrary length a Khintchine type inequality has been proven by Ricard and Xu in [50, Section 2]. In the current paper we obtain a Khintchine inequality for general graph products. We do this by introducing an intertwining technique between graph products and free products.

Theorem 0.1. *Let Γ be a finite simplicial graph. Consider a graph product $(A, \varphi) = *_{v,\Gamma}(A_v, \varphi_v)$ of unital C*-algebras with* GNS-*faithful states φ_v. Denote by χ_d the word length projection of length d. Then for every $d \in \mathbb{N}_{\geq 1}$ there exists some operator space X_d and maps*

$$j_d\colon \chi_d(A) \to X_d, \qquad \pi_d\colon Dom(\pi_d) \subseteq X_d \to \chi_d(A),$$

with $Dom(\pi_d) = j_d(\chi_d(A))$ such that the following statements hold:

(i) X_d is a direct sum of Haagerup tensor products of column and row Hilbert spaces and is defined in (2.14);
(ii) $\pi_d \circ j_d$ is the identity on $\chi_d(A)$;
(iii) $\|\pi_d\colon Dom(\pi_d) \to A\|_{cb} \leq Cd$ for some (explicit) constant C, depending only on the graph Γ.

In Theorem 0.1 the bound on π_d in property (iii) should be regarded as the Khintchine inequality. Inequalities of this kind have a wide range of applications. We refer in particular to the weak amenability results by Ricard and Xu [50], the early connections to Coxeter groups by Bożejko and Speicher [8] and Nou's result on non-injectivity of q-Gaussians [41]. The second half of this paper gives further applications in the case of (right-angled) Hecke C*-algebras.

Hecke algebras are deformations of group algebras of a Coxeter group W depending on a multi-parameter q and have been studied since the 1950s. Their development played an important role in representation theory of algebraic groups (see e.g. [35], [5]). Hecke algebras naturally act on the Hilbert space $\ell^2(W)$ and thus complete to C*-algebras (resp. von Neumann algebras) denoted by $C^*_{r,q}(W)$ (resp. $\mathcal{N}_q(W)$) which carry a canonical tracial state. In the case of spherical or affine Coxeter groups these operator algebras have been studied early in [39], [38], [36]. Much later, motivated by the study of weighted L^2-cohomology of Coxeter groups, the study of general (non-affine) Hecke von Neumann algebras was initiated by Dymara in [26] (see also [22]). Other relevant references are [23], [31], [12], [15] and [45].

As observed in [12], Hecke C*-algebras (resp. Hecke-von Neumann algebras) of right-angled Coxeter systems (W, S) can be realized as graph products of finite-dimensional abelian C*-algebras (resp. von Neumann algebras). Applying Theorem 0.1 we deduce a Haagerup inequality for right-angled Hecke algebras that generalizes Haagerup's inequality for free groups, see [33] and [43, Section 9.6]. These are inequalities that estimate the operator norm of an operator of length d with the L^2-norm up to some polynomial bound.

Theorem 0.2. *Let (W, S) be a right-angled Coxeter group with finite generating set S. Then for every multi-parameter q and $x \in \chi_d\left(C^*_{r,q}(W)\right), d \in \mathbb{N}_{\geq 1}$ we have*

$$\|x\| \leq Cd\,\|x\|_2$$

for some (explicit) constant depending only on q and the graph Γ.

The main motivation for the second part of this paper is the main result in [31] where it was shown that single parameter Hecke-von Neumann algebras of irreducible right-angled Coxeter systems (W, S) with $|S| \geq 3$ are factors, up to a 1-dimensional direct summand.

Theorem 0.3. *([31, Garncarek]) Let (W, S) be an irreducible right-angled Coxeter system with $|S| \geq 3$. Then the single parameter Hecke-von Neumann algebra $\mathcal{N}_q(W)$ is a factor if and only if $q \in \left[\rho, \rho^{-1}\right]$ where ρ is the radius of convergence of the spherical growth series $\sum_{w \in W} z^{|w|}$ of W. Moreover, for q outside this interval, $\mathcal{N}_q(W)$ is a direct sum of a factor and $\mathbb{C}$.*

We take this theorem as a motivation to study the simplicity and the uniqueness of the tracial state of (right-angled) Hecke C*-algebras. By applying Theorem 0.2 in combination with an averaging argument inspired by [47] we are able to give some partial answers. Note that in the single parameter case $q = 1$ this has been done in [24] (see also [30] and [20]).

Theorem 0.4. *Let (W, S) be an irreducible Coxeter system and q some multi-parameter.*

(1) If (W, S) is of spherical or affine type, then $C^*_{r,q}(W)$ is not simple and does not have a unique tracial state for any choice of the parameter q;
(2) If (W, S) is of non-affine type and the multi-parameter $(q_s^{\epsilon_s})_{s\in S}$ with

$$\epsilon_s := \begin{cases} +1, & \text{if } q_s \leq 1 \\ -1, & \text{if } q_s > 1 \end{cases}$$

lies in the closure of the region of convergence of the multivariate growth series of W, then $C^*_{r,q}(W)$ is not simple and does not have unique tracial state;
(3) If (W, S) is right-angled with $|S| \geq 3$, then there exists an open neighborhood $\mathcal{U}$ of the multi-parameter $1 \in \mathbb{R}^S_{>0}$ such that $C^*_{r,q}(W)$ is simple for all $q \in \mathcal{U}$.

In the case of free products of abelian Coxeter groups we further give a full answer to the simplicity question by making use of Dykema's results [28] on the simplicity of free products of finite dimensional abelian C*-algebras. If the Hecke C*-algebra has a unique trace then these simplicity results imply factoriality of the Hecke-von Neumann algebra, see the Remarks 5.8 and 5.11. We thus extend the range of q for which factoriality of $\mathcal{N}_q(W)$ holds. In particular we obtain first results in the multi-parameter case which gives a partial answer to Question 2 in [31]. In the case of free products of abelian Coxeter groups our answer fully settles this question.

In the final part of this paper, we prove that Hecke C*-algebras are exact and characterize their nuclearity in terms of the properties of the underlying group. In particular, we show the following.

Theorem 0.5. *Let (W, S) be a Coxeter system and q some multi-parameter. Then $C^*_{r,q}(W)$ is exact. Further, $C^*_{r,q}(W)$ is nuclear if and only if (W, S) is of spherical or affine type.*

Structure. In Section 1 we recall the general theory of graph products, Coxeter groups and introduce multi-parameter Hecke algebras (resp. their operator algebras). In Section 2 we obtain the Khintchine inequality from Theorem 0.1. The consequences of this will be collected in Section 3 where we reformulate this Khintchine inequality in the setting of right-angled Hecke C*-algebras and deduce the Haagerup inequality from Theorem 0.2. Section 4 discusses isomorphism properties of Hecke algebras which are known to experts,

see for instance [39], [42]. In Section 5 we will then study the simplicity and the uniqueness of the tracial state of Hecke C*-algebras. In the case of spherical and affine type Coxeter systems and for free products of abelian groups we give full answers, whereas in the case of irreducible right-angled Coxeter systems we give partial answers (see Theorem 0.4) by making use of the results in Section 3. Finally, Section 6 characterizes exactness and nuclearity of Hecke C*-algebras (see Theorem 0.5) and we give (counter)examples regarding the isomorphism properties from Section 4.

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1. Preliminaries and notation

1.1. General notation

We use the notation $\mathbb{N}_{\geq 1} := \{1, 2, \ldots\}$ and $\mathbb{N}_{\geq 0} := \{0, 1, 2, \ldots\}$. By $M_{k,l}(\mathbb{C})$ we denote the k times l matrices and we let $M_k(\mathbb{C}) := M_{k,k}(\mathbb{C})$. We let $\mathbb{Z}_2$ be the group with two elements. We write $\delta(\mathcal{P})$ for the function that is 1 if a statement $\mathcal{P}$ is true and which is 0 otherwise. We write $\mathcal{B}(\mathcal{H})$ for the bounded operators on a Hilbert space $\mathcal{H}$.

For standard results on von Neumann algebras we refer to [51]. For a tracial von Neumann algebra (M, τ) with faithful τ we denote the norm induced by the tracial state by $\|x\|_2 := \tau(x^*x)^{\frac{1}{2}}$. The following lemma is standard and shall be used several times.

Lemma 1.1. *Let A and B be unital C*-algebras with respective faithful traces τ_A and τ_B. Let $A_0 \subseteq A$ and $B_0 \subseteq B$ be dense $*$-subalgebras of A and B and let $\pi : A_0 \to B_0$ be a $*$-isomorphism such that $\tau_B \circ \pi = \tau_A$. Then π extends to a $*$-isomorphism $A \to B$ as well as $\pi_{\tau_A}(A)'' \to \pi_{\tau_B}(B)''$ where π_{τ_A} and π_{τ_B} are the GNS-representations.*

Proof. This is essentially [40, Theorem 5.1.4]. Without loss of generality we may assume that A and B are represented on their GNS-spaces $L^2(A, \tau_A)$ and $L^2(B, \tau_B)$ with cyclic vectors Ω_A and Ω_B. Then $U : L^2(A, \tau_A) \to L^2(B, \tau_B) : a\Omega_A \mapsto \pi(a)\Omega_B, a \in A_0$ is unitary and $\pi(a) = UaU^*$. Therefore π extends to a map $A \to B$ as well as $A'' \to B''$. $\square$

We use the symbol $\otimes$ for the reduced tensor product of C*-algebras (through Takesaki's theorem [40, Section 6.4] also known as the spatial or minimal tensor product) and $\overline{\otimes}$ for the tensor product of von Neumann algebras. We denote the Haagerup tensor product of operator spaces by $\otimes_h$ (see below).

1.2. Column and row Hilbert spaces

For the theory of operator spaces we refer to [29] and [43]. The column Hilbert space of dimension k will be denoted by C_k and the row Hilbert space of dimension k will be

denoted by R_k. C_k is the operator space spanned by the matrix units $e_{i,0}, i = 1, \dots, k$ in $M_k(\mathbb{C})$. Its operator space structure is the restriction of the operator space structure of $M_k(\mathbb{C})$ as a C*-algebra. Concretely,

$$\|\sum_{i=1}^{k} x_i \otimes e_{i,0}\|_{M_l(\mathbb{C})\otimes \mathrm{C}_k} = \|\sum_{i=1}^{k} x_i^* x_i\|^{\frac{1}{2}}, \qquad x_i \in M_l(\mathbb{C}), l \in \mathbb{N}_{\geq 1}.$$

Similarly, R_k is the operator space spanned by matrix units $e_{0,i}, i = 1, \dots, k$ in $M_k(\mathbb{C})$ and it inherits the operator space structure of $M_k(\mathbb{C})$. We have,

$$\|\sum_{i=1}^{k} x_i \otimes e_{0,i}\|_{M_l(\mathbb{C})\otimes \mathrm{R}_k} = \|\sum_{i=1}^{k} x_i x_i^*\|^{\frac{1}{2}}, \qquad x_i \in M_l(\mathbb{C}), l \in \mathbb{N}_{\geq 1}.$$

For the Haagerup tensor product of operator spaces we refer to [29, Section 9]. We shall mainly need the completely isometric identifications (see [29, Proposition 3.9.4 and 3.9.5]) for $k, l \in \mathbb{N}_{\geq 1}$,

$$\mathrm{C}_k \otimes_h \mathrm{C}_l \simeq \mathrm{C}_{k+l}, \qquad \mathrm{R}_k \otimes_h \mathrm{R}_l \simeq \mathrm{R}_{k+l}, \qquad \mathrm{C}_k \otimes_h \mathrm{R}_l \simeq M_{k,l}(\mathbb{C}).$$

1.3. Graphs

Let Γ be a **simplicial graph** with vertex set $V\Gamma$ and edge set $E\Gamma \subseteq V\Gamma \times V\Gamma$. Simplicial means that Γ has no double edges and that $(v, v) \notin E\Gamma$ for any $v \in V\Gamma$. We will always assume that Γ is finite and undirected. For $v \in V\Gamma$ we denote by $\mathrm{Link}\,(v)$ the set of all $w \in V\Gamma$ such that $(v, w) \in E\Gamma$. For $X \subseteq V\Gamma$ we shall write $\mathsf{Link}(X) := \cap_{v\in X}\mathsf{Link}(v)$ for the common link. Further, we define $\mathsf{Link}(\emptyset) := V\Gamma$ (all points). We may view $\mathsf{Link}(X)$ as a subgraph of Γ by declaring the edge set of $\mathsf{Link}(X)$ to be $\{(v, w) \in \mathsf{Link}(X) \times \mathsf{Link}(X) \mid (v, w) \in E\Gamma\}$, so that it inherits precisely the edges of Γ.

A clique in the graph Γ is a subgraph $\Gamma_0 \subseteq \Gamma$ in which every two vertices share an edge. We write Cliq or $\mathsf{Cliq}(\Gamma)$ for the set of **cliques** in Γ and $\mathsf{Cliq}(l)$ or $\mathsf{Cliq}(\Gamma, l)$ for the set of cliques with l vertices. We will always assume that the empty graph is in Cliq. $\mathrm{Comm}\,(\Gamma_0)$ is the set of pairs $(\Gamma_1, \Gamma_2) \in \mathrm{Link}\,(\Gamma_0) \times \mathrm{Link}\,(\Gamma_0)$ such that $\Gamma_1, \Gamma_2 \in \mathrm{Cliq}$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

In this paper a **word** refers to an expression $\mathbf{v} = v_1 v_2 \cdots v_n$ with $v_i \in V\Gamma$, i.e. a concatenation of elements in $V\Gamma$ which we call the **letters**. Words will be denoted by bold face. We say that two words are **shuffle equivalent** (also known as **II-equivalent**) if they are in the same equivalence class of the equivalence relation generated by

- $v_1 \cdots v_{i-1} v_i v_{i+1} v_{i+2} \cdots v_n \sim v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_n$ if $(v_i, v_{i+1}) \in E\Gamma$.

We say that two words are equivalent, denoted by the symbol $\simeq$, if they are equivalent through shuffle equivalence and the additional relation:

- $v_1 \cdots v_i v_{i+1} v_{i+2} \cdots v_n \sim v_1 \cdots v_i v_{i+2} \cdots v_n$ if $v_i = v_{i+1}$.

A word $v_1 \cdots v_n$ is called **reduced** if whenever $v_i = v_j, i < j$ then there is $i < k < j$ such that $(v_i, v_k), (v_j, v_k) \notin E\Gamma$. If $\mathbf{v}$ and $\mathbf{w}$ are reduced words and $\mathbf{v} \simeq \mathbf{w}$ then necessarily $\mathbf{v}$ and $\mathbf{w}$ are shuffle equivalent. For a word $\mathbf{w}$ we define its **length** $|\mathbf{w}|$ as the number of letters in the shortest representative of $\mathbf{w}$ up to equivalence. We say that a word $\mathbf{v}$ **starts with** $v \in V\Gamma$ if it is equivalent to a reduced word $vv_1 \cdots v_n, v_i \in V\Gamma$. Similarly we say that a word $\mathbf{v}$ **ends with** $v \in V\Gamma$ if it is equivalent to a reduced word $v_1 \cdots v_n v, v_i \in V\Gamma$.

1.4. Graph products of groups

As before let Γ be a simplicial graph. The following construction goes back to Green's thesis [32]. For each $v \in V\Gamma$ let G_v be a discrete group. Then the **graph product group** $*_{v,\Gamma} G_v$ is the discrete group obtained from the free product of G_v, $v \in V\Gamma$ by adding additional relations $[s, t] = 1$ for all $s \in G_v$, $t \in G_w$ with $(v, w) \in E\Gamma$. Special (extremal) cases of graph products are free products (graph with no edges) and Cartesian products (complete graphs). Right-angled Coxeter and right-angled Artin groups are natural examples of graph products. A right-angled Coxeter group can be seen as a graph product where all groups $G_v, v \in V\Gamma$ are equal to $\mathbb{Z}_2$. A right-angled Artin group is a graph product with $G_v = \mathbb{Z}, v \in V\Gamma$.

1.5. Graph products of operator algebras

In [14] the concept of graph products has been translated to the operator algebraic setting. The construction is a generalization of free products by adding commutation relations depending on the underlying graph. We present a slightly different viewpoint of [14] by identifying Hilbert spaces up to shuffle equivalence. This makes the notation much shorter and yields the same construction.

Let Γ be a finite simplicial graph. For $v \in V\Gamma$ let A_v be a unital C*-algebra. Let φ_v be a GNS-faithful state on A_v, meaning that the GNS-representation of φ_v is faithful. Set A_v° as the space of $a \in A_v$ with $\varphi_v(a) = 0$. For $a \in A_v$ set

$$a^\circ = a - \varphi_v(a)1 \in A_v^\circ.$$

Let $L^2(A_v^\circ, \varphi_v)$ be the closure of A_v° in the GNS-space $L^2(A_v, \varphi_v)$ of φ_v. We let $\Omega_v \in L^2(A_v, \varphi_v)$ be the unit of A_v. For a reduced word $\mathbf{v} = v_1 \cdots v_n$ let

$$\mathcal{H}_{\mathbf{v}} = L^2(A_{v_1}^\circ, \varphi_{v_1}) \otimes \cdots \otimes L^2(A_{v_n}^\circ, \varphi_{v_n}).$$

We have the convention $\mathcal{H}_\emptyset := \mathbb{C}\Omega$ where Ω is a unit vector called the **vacuum vector**. If $\mathbf{v} = v_1 \cdots v_n$ and $\mathbf{w} = w_1 \cdots w_n$ are reduced equivalent (hence shuffle equivalent) words then $\mathcal{H}_{\mathbf{v}} \simeq \mathcal{H}_{\mathbf{w}}$ naturally by applying flip maps to the vectors dictated by the shuffle

equivalence. More precisely, by [14, Lemma 1.3] if $\mathbf{v}$ and $\mathbf{w}$ are equivalent reduced words, there exists a unique permutation σ of the numbers $1, \dots, n$ such that $v_{\sigma(i)} = w_i$ and such that if $i < j$ and $v_i = v_j$ then also $\sigma(i) < \sigma(j)$. Then there exists a unitary map $\mathcal{Q}_{\mathbf{v},\mathbf{w}} : \mathcal{H}_{\mathbf{v}} \to \mathcal{H}_{\mathbf{w}}$ which sends $\xi_1 \otimes \cdots \otimes \xi_n$ to $\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}$. From now on, we will omit $\mathcal{Q}_{\mathbf{v},\mathbf{w}}$ in the notation and identify the spaces $\mathcal{H}_{\mathbf{v}}$ and $\mathcal{H}_{\mathbf{w}}$ through $\mathcal{Q}_{\mathbf{v},\mathbf{w}}$. This significantly simplifies our notation compared to [14] and it is directly verifyable that our constructions below agree with [14].

Let I be a set of **representatives** of all reduced words modulo shuffle equivalence. Set $\mathcal{H} = \oplus_{\mathbf{v} \in I} \mathcal{H}_{\mathbf{v}}$. We represent each $A_v, v \in V\Gamma$ on $\mathcal{H}$ as follows. Take $x \in A_v$. Take $\xi_1 \otimes \cdots \otimes \xi_d \in \mathcal{H}_{\mathbf{w}}$ with $\mathbf{w} = w_1 \dots w_d$ a reduced word. If $\mathbf{w}$ does not start with v then we set

$$x \cdot (\xi_1 \otimes \cdots \otimes \xi_d) = x^\circ \Omega_v \otimes \xi_1 \otimes \cdots \otimes \xi_d + \varphi_v(x) \xi_1 \otimes \cdots \otimes \xi_d.$$

If $\mathbf{w}$ starts with v we may assume w.l.o.g. (by shuffling the letters if necessary and identifying corresponding Hilbert spaces as described above) that $w_1 = v$ and we set

$$x \cdot (\xi_1 \otimes \cdots \otimes \xi_d) = (x\xi_1 - \langle x\xi_1, \Omega_v \rangle \Omega_v) \otimes \xi_2 \otimes \cdots \otimes \xi_d + \langle x\xi_1, \Omega_v \rangle \xi_2 \otimes \cdots \otimes \xi_d.$$

This defines a faithful $*$-representation of A_v on $\mathcal{H}$. We set the **(reduced) graph product C*-algebra**,

$$(A, \varphi) := *_{v,\Gamma}(A_v, \varphi_v)$$

as the C*-algebra generated by all $A_v, v \in V\Gamma$ acting on $\mathcal{H}$, with faithful **graph product state**

$$\varphi(x) := \langle x\Omega, \Omega \rangle.$$

If for every $v \in V\Gamma$, A_v is a von Neumann algebra and φ_v is moreover faithful then we define the **graph product von Neumann algebra** as $(*_{v,\Gamma}(A_v, \varphi_v))''$. We will usually write

$$L^2(A, \varphi) := \mathcal{H},$$

and call this the (graph product) **Fock space**. The notation is justified as in [14] it is shown that $\mathcal{H}$ is the Hilbert space of the standard form of the von Neumann algebraic graph product. An operator $a_1 \cdots a_n$ with $a_i \in A_{v_i}^\circ$ and $\mathbf{v} = v_1 \cdots v_n$ a reduced word is called a **reduced operator of type v**. We refer to n as the **length** of the operator. We define P_v as the orthogonal projection of $\mathcal{H}$ onto $\oplus_{\mathbf{v} \in I_v} \mathcal{H}_{\mathbf{v}}$ where I_v are representatives of all words that start with v up to shuffle equivalence. For $d \in \mathbb{N}_{\geq 0}$, let

$$\chi_d : A \to A : a_1 \cdots a_r \mapsto \delta(r = d) a_1 \cdots a_r, \tag{1.1}$$

where $a_1 \cdots a_r$ is a reduced operator. So χ_d is the word length projection of length d.

1.6. Coxeter groups

A **Coxeter group** W is a group that is freely generated by a set S subject to relations $(st)^{m_{s,t}} = 1$ for exponents $m_{s,t} \in \{1, 2, \ldots, \infty\}$ where $m_{s,s} = 1$, $m_{s,t} \geq 2$ for all $s \neq t$ and $m_{s,t} = m_{t,s}$. The condition $m_{s,t} = \infty$ means that no relation of the form $(st)^m = 1, m \in \mathbb{N}_{\geq 1}$ is imposed. The pair (W, S) is called a **Coxeter system**. The data of (W, S) can be encoded in its **Coxeter diagram**[1] with vertex set S and edge set $\{(s,t) \mid m_{s,t} \geq 3\}$ where every edge between two vertices $s, t \in S$ is labeled by the corresponding exponent $m_{s,t}$.

A **special subgroup** of W is one that is generated by a subset S_0 of S and which will be denoted by W_{S_0}. Special subgroups are also Coxeter groups with the same exponents as W [22, Theorem 4.1.6]. We briefly write $W_s = W_{\{s\}}$ for $s \in S$ and note that by the relation $s^2 = e$ we have $W_s = \mathbb{Z}_2$, the group of two elements. We call the Coxeter system (W, S) *irreducible* if its Coxeter diagram is connected. This is equivalent to W not having a non-trivial decomposition into a direct product of special subgroups. Irreducible Coxeter groups fall into three classes.

Definition 1.2. Let (W, S) be an irreducible Coxeter system.

- It is of **spherical type** if it is locally finite, i.e. every finitely generated subgroup of W is finite.
- It is of **affine type** if it is finitely generated, infinite and virtually abelian.
- It is of **non-affine type** if it is neither spherical nor affine.

An irreducible Coxeter group is non-amenable if and only if it is of non-affine type, see [22, Theorem 14.1.2 and Proposition 17.2.1].

1.7. Right-angled Coxeter groups as graph products

A Coxeter group (or Coxeter system) is called **right-angled** if $m_{s,t} \in \{1, 2, \infty\}$ for all $s, t \in S$. So we have that for $s \in S, s^2 = e$ and two letters $s, t \in S, s \neq t$ either commute (case $m_{s,t} = 2$) or they are free (case $m_{s,t} = \infty$). Suppose that (W, S) is a right-angled Coxeter system. Set Γ as the graph with vertices $V\Gamma = S$ and edge set $E\Gamma = \{(s,t) \mid m_{s,t} = 2\}$. Then we find an isomorphism of groups

$$W \simeq *_{s,\Gamma} W_s \simeq *_{s,\Gamma} \mathbb{Z}_2, \tag{1.2}$$

sending $s_1 \cdots s_n \in W, s_i \in S$ to $s_1 \cdots s_n \in *_{s,\Gamma} W_s$ (i.e. our notation coincides). Indeed, this follows from the fact that the defining (universal) properties of a right-angled Coxeter group and a graph product over the special subgroups $W_s \simeq \mathbb{Z}_2$ are the same.

[1] We emphasize that we refer to this as the *diagram* of the Coxeter group, which is different from the graph defined in Section 1.7.

1.8. Words in Coxeter groups

Let $\mathbf{w} \in W$ with $\mathbf{w} = s_1 \cdots s_n$ where $s_1, \ldots, s_n \in S$. We call this expression **reduced** if it has minimal length, i.e. $n \leq m$ for every other representation $\mathbf{w} = t_1 \cdots t_m, t_1, \ldots, t_m \in S$. The set of letters that can occur in a reduced expression is independent of the choice of the reduced expression, [22, Proposition 4.1.1]. By $|\mathbf{w}| := n$ we define a **word-length** on W. If $\left|\mathbf{v}^{-1}\mathbf{w}\right| = |\mathbf{w}| - |\mathbf{v}|$ (resp. $\left|\mathbf{w}\mathbf{v}^{-1}\right| = |\mathbf{w}| - |\mathbf{v}|$) for $\mathbf{v} \in W$ we say that $\mathbf{w}$ **starts** (resp. **ends**) **with** $\mathbf{v}$. Note that in the right-angled case these definitions agree with the definitions for words and lengths on graphs under the isomorphism (1.2).

Coxeter groups satisfy three important conditions on words in S of which we will make use implicitly. We use the usual convention that $\widehat{s}$ means that s is removed from an expression.

Theorem 1.3 *([22], Theorem 3.2.16 and Theorem 3.3.4). Let (W, S) be a Coxeter system, $\mathbf{w} = s_1 \cdots s_n$ an expression for an element $\mathbf{w} \in W$ and $s, t \in S$. Then the following (equivalent) conditions hold:*

- **Deletion condition***: If $s_1 \cdots s_n$ is not a reduced expression for $\mathbf{w}$, then there exist $i < j$ such that $s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n$ is also an expression for $\mathbf{w}$.*
- **Exchange condition***: If $\mathbf{w} = s_1 \cdots s_n$ is reduced, then either $|s\mathbf{w}| = n + 1$ or there exists $1 \leq i \leq n$ with $s\mathbf{w} = s_1 \cdots \widehat{s_i} \cdots s_n$.*
- **Folding condition***: If $|s\mathbf{w}| = |\mathbf{w}| + 1$ and $|\mathbf{w}t| = |\mathbf{w}| + 1$, then either $|s\mathbf{w}t| = |\mathbf{w}| + 2$ or $|s\mathbf{w}t| = |\mathbf{w}|$.*

In the right-angled case, if we have cancellation of the form $s_1 \cdots s_n = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_n$ for $s_1, \ldots, s_n \in S$, then $s_i = s_j$ and s_i commutes with $s_{i+1}, \ldots, s_{j-1}$.

1.9. Multi-parameter Hecke algebras

Von Neumann algebraic closures of Hecke algebras were first studied in [26], [23], see also [22, Section 19]. Note that we use a different normalization of the generators so that our notation coincides with Garncarek's [31] (as well as [12] and [15]).

Let (W, S) be a Coxeter system and let $\mathbb{R}_{>0}^{(W,S)}$ (resp. $\mathbb{C}^{(W,S)}$ and $\{-1, 1\}^{(W,S)}$) be the set of tuples $q := (q_s)_{s \in S}$ in $\mathbb{R}_{>0}^{S}$ (resp. in $\mathbb{C}^S$ and $\{-1, 1\}^S$) of positive real numbers (resp. complex numbers and numbers -1 or 1) with the property that $q_s = q_t$ whenever s and t are conjugate in W.

Take $q := (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}$. For every reduced expression $\mathbf{w} = s_1 \cdots s_n$ of $\mathbf{w} \in W$ define

$$q_{\mathbf{w}} := q_{s_1} \cdots q_{s_n}, \qquad p_s(q) := q_s^{-\frac{1}{2}} (q_s - 1).$$

It follows from [22, Proposition 19.1.1][2] that there exists a unique $*$-algebra $\mathbb{C}_q[W]$ that is spanned by a linear basis $\{T_{\mathbf{w}}^{(q)} \mid \mathbf{w} \in W\}$ and such that for every $s \in S$ and $\mathbf{w} \in W$ we have,

$$T_s^{(q)} T_{\mathbf{w}}^{(q)} = \begin{cases} T_{s\mathbf{w}}^{(q)} & \text{if } |s\mathbf{w}| > |\mathbf{w}|, \\ T_{s\mathbf{w}}^{(q)} + p_s(q) T_{\mathbf{w}}^{(q)} & \text{if } |s\mathbf{w}| < |\mathbf{w}|, \end{cases} \tag{1.3}$$

and

$$\left(T_{\mathbf{w}}^{(q)}\right)^* = T_{\mathbf{w}^{-1}}^{(q)}. \tag{1.4}$$

Denote by $\ell^2(W)$ the Hilbert space of square-summable functions on W with canonical orthonormal basis $(\delta_{\mathbf{w}})_{\mathbf{w} \in W}$. For every $s \in S$ the element $T_s^{(q)}$ acts boundedly on $\ell^2(W)$ by

$$T_s^{(q)} \delta_{\mathbf{w}} := \begin{cases} \delta_{s\mathbf{w}} & \text{if } |s\mathbf{w}| > |\mathbf{w}| \\ \delta_{s\mathbf{w}} + p_s(q)\delta_{\mathbf{w}} & \text{if } |s\mathbf{w}| < |\mathbf{w}| \end{cases}. \tag{1.5}$$

Moreover, this action extends to a faithful $*$-representation $\mathbb{C}_q[W] \to \mathcal{B}(\ell^2(W))$. We will usually identify $\mathbb{C}_q[W]$ with its image in $\mathcal{B}(\ell^2(W))$.

The unital $*$-algebra $\mathbb{C}_q[W]$ is called the (Iwahori) **Hecke algebra** of (W, S) with parameter q. We set the **reduced Hecke C*-algebra** $C_{r,q}^*(W)$ as the norm closure of $\mathbb{C}_q[W]$ in $\mathcal{B}\left(\ell^2(W)\right)$ using the representation (1.5). Then set the **Hecke-von Neumann algebra** $\mathcal{N}_q(W) = C_{r,q}^*(W)''$. For $q_s = 1, s \in S$ we get that $\mathbb{C}_q[W] = \mathbb{C}[W], C_{r,q}^*(W) = C_r^*(W)$ and $\mathcal{N}_q(W) = \mathcal{L}(W)$ are respectively the group algebra, reduced group C*-algebra and group von Neumann algebra of W. Hence Hecke algebras are q-deformations of group (C*- and von Neumann) algebras and the deformation (in principle) depends on q; we comment on this further in Section 4. For every q the vector state

$$\tau(x) = \langle x\delta_e, \delta_e \rangle, \qquad x \in \mathcal{B}(\ell^2(W)),$$

restricts to a **tracial state** τ_q on $C_{r,q}^*(W)$ and $\mathcal{N}_q(W)$ with $\tau_q(T_w^{(q)}) = 0$ for all $w \in W \setminus \{e\}$. Finally, define P_s, $s \in S$ to be the projection of $\ell^2(W)$ onto

$$\overline{\operatorname{span}\{\delta_{\mathbf{v}} \mid \mathbf{v} \in W \text{ with } |s\mathbf{v}| < |\mathbf{v}|\}} \subseteq \ell^2(W). \tag{1.6}$$

Note that if the Coxeter group is right-angled then under the graph product identification (1.2) this notation coincides with the notation of Section 1.5.

[2] If $\widetilde{T}_{\mathbf{w}}^{(q)}$ are the generators from [22, Proposition 19.1.1] then take normalized elements $T_{\mathbf{w}}^{(q)} = q_{\mathbf{w}}^{-1/2} \widetilde{T}_{\mathbf{w}}^{(q)}$ to get the relations (1.3) and (1.4). This convention is consistent with [12], [15] and [31].

1.10. Hecke C-algebras as graph products*

Let (W, S) be a right-angled Coxeter system with graph Γ as defined in Section 1.7. In [12, Corollary 3.4][3] it is proved that for every $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ there exists an isomorphism,

$$(C^*_{r,q}(W), \tau_q) \xrightarrow{\simeq} *_{s,\Gamma}(C^*_{r,q_s}(W_s), \tau_{q_s}). \tag{1.7}$$

Moreover, the isomorphism is given by the canonical map $T^{(q)}_{\mathbf{v}} \to T^{(q_{s_1})}_{s_1} \cdots T^{(q_{s_d})}_{s_d}$ where $\mathbf{v} = s_1 \cdots s_d$ is reduced. For $q = 1$ this coincides with the corresponding result on Coxeter groups.

2. A graph product Khintchine inequality

The aim of this section is to prove a Khintchine inequality for general graph products. Such a Khintchine inequality estimates the operator norm of a reduced operator of length d with the norm of certain Haagerup tensor products of column and row Hilbert spaces. This estimate of norms holds up to a bound that is polynomial in d. We make this more precise in the current section.

Remark 2.1. In the context of free products a Khintchine inequality was already obtained by Ricard and Xu in [50, Section 2]. If in the current paper we would only treat the case of free Coxeter systems and their Hecke deformations (i.e. $m(s,t) = \infty, s \neq t$) the results from [50] would be sufficient. Here however, we want a more general theorem. One of the problems that arises while proving such a theorem is that the analogue of [50, Lemma 2.3] in its form fails in a general graph product setting. We remedy this problem by using maps which intertwine graph products with free products.

Let us now prepare for the proof of the main theorem of this section. We fix notation for both a graph product and a free product. As before, let Γ be a finite simplicial graph and let I be a set of representatives of equivalence classes of reduced words with letters in $V\Gamma$. Let $A_v, v \in V\Gamma$ be unital C*-algebras with GNS-faithful states φ_v. Let

$$(A, \varphi) := *_{v,\Gamma}(A_v, \varphi_v)$$

be its graph product with vacuum vector Ω. We also set the free product (i.e. the graph product over Γ with all edges removed)

$$(A_f, \varphi_f) := *_v(A_v, \varphi_v),$$

[3] The corollary is only stated for a single parameter $q_s = q$ but it also holds in the multi-parameter case with the same proof.

with vacuum vector Ω_f. We shall view $A_v, v \in V\Gamma$ as a C*-subalgebra of A and A_f. We let P_v be the graph product projection onto words that start with v (as before) and we let P_v^f be the free product projection onto words which start with v. Let $\Gamma_0 \in \mathsf{Cliq}(\Gamma, l)$ and let $\mathbf{w} = w_1 \cdots w_l \in I$ be the word consisting of all letters in $V\Gamma_0$. We write

$$P_{\Gamma_0}^f \tag{2.1}$$

for the projection of $L^2(A_f, \varphi_f)$ onto the direct sum of all $\mathcal{H}_{\mathbf{v}}$ where $\mathbf{v}$ starts with $\mathbf{w}$. Recall that A_v° is the set of $a \in A_v$ with $\varphi_v(a) = 0$.

View A_v as a subalgebra of A and then let $\Sigma_1 := \operatorname{span}\{A_v^\circ \mid v \in V\Gamma\}$, and, for $d \in \mathbb{N}_{\geq 1}$,

$$\Sigma_d := \{a_1 \otimes \cdots \otimes a_d \mid a_i \in A_{v_i}^\circ \text{ and } v_1 \cdots v_d \text{ reduced}\} \subseteq \Sigma_1^{\otimes d},$$

where the latter is the d-fold algebraic tensor product.

Our first aim is to show that reduced operators in A of length d can be decomposed as sums of creation operators, annihilation operators and diagonal operators in a sense to be made precise below. Crucial is that first the annihilation operators act, then the diagonal operators and then the creation operators. That is, we shall be looking for an analogue of the decomposition [50, Fact 2.6].

Remark 2.2. Definition 2.3 below formally defines the following permutation. Let $\mathbf{v} = v_1 \cdots v_d$ for $v_i \in V\Gamma$, be a reduced word. Let $d \in \mathbb{N}_{\geq 1}$, $0 \leq l \leq d$, $0 \leq k \leq d - l$, $\Gamma_0 \in \mathsf{Cliq}(\Gamma, l)$ and $(\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)$. Then, if possible, we permute the letters of $\mathbf{v}$ through shuffle equivalence in the form:

$$\begin{aligned}&(v_{\sigma(1)} \cdots v_{\sigma(k)})(v_{\sigma(k+1)} \cdots v_{\sigma(k+l)})(v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}) \\ &\simeq \overbrace{(* \cdots * \, V\Gamma_1)}^{k \text{ letters}} \overbrace{(\, V\Gamma_0 \,)}^{l \text{ letters}} \overbrace{(V\Gamma_2 \diamond \cdots \diamond)}^{d-l-k \text{ letters}},\end{aligned} \tag{2.2}$$

where $*$ and $\diamond$ are the remaining letters and each of the 3 respective terms in between brackets are shuffle equivalent themselves. This means that in between the first brackets there is a word of length k that ends on the clique $V\Gamma_1$, in between the second brackets there is the clique of length l given by $V\Gamma_0$, and at the end there is a word of length $d - k - l$ that starts with the clique $V\Gamma_2$.

Moreover, we want that $* \cdots *$ does not end on letters commuting with $V\Gamma_0$ and $V\Gamma_1$ and $\diamond \cdots \diamond$ does not start with letters commuting with $V\Gamma_0$ and $V\Gamma_2$. This means that the cliques Γ_1 and Γ_2 are maximal for the property that a decomposition like (2.2) exists.

If moreover we demand that $v_{\sigma(1)} \cdots v_{\sigma(k)}$, $v_{\sigma(k+1)} \cdots v_{\sigma(k+l)}$ and $v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}$ are in I, then there can be at most one such permutation coming from shuffle equivalences.

Of course not for every $d, l, k, \Gamma_0, \Gamma_1, \Gamma_2$ and $\mathbf{v}$ this permutation exists since $\mathbf{v}$ cannot always be written in the form (2.2).

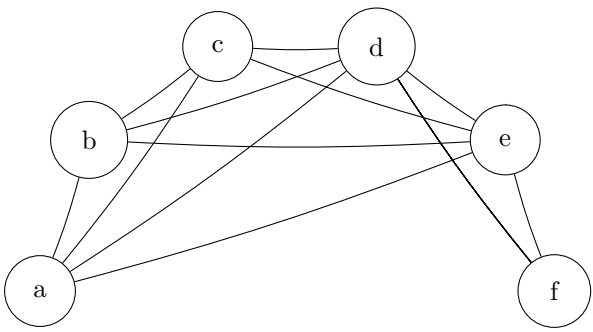
Definition 2.3. Let $d \in \mathbb{N}_{\geq 1}$. Suppose that $0 \leq l \leq d$, $0 \leq k \leq d-l$, $\Gamma_0 \in \mathsf{Cliq}(\Gamma, l)$ and $(\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)$. So if $l = 0$ we have that Γ_0 is the empty clique, and Condition (2) below vanishes. Take a reduced word $\mathbf{v} = v_1 \cdots v_d, v_i \in V\Gamma$. If existent, define $\sigma (= \sigma^{\mathbf{v}}_{l,k,\Gamma_0,\Gamma_1,\Gamma_2})$ as the permutation of indices $1, \ldots, d$ that satisfies:

(1) $v_1 \cdots v_d = v_{\sigma(1)} \cdots v_{\sigma(d)}$;
(2) $\{v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\} = V\Gamma_0$;
(3) $|v_{\sigma(1)} \cdots v_{\sigma(k)} s| = k-1$ whenever $s \in V\Gamma_1$;
(4) $|v_{\sigma(1)} \cdots v_{\sigma(k)} s| = k+1$ whenever $s \in \mathsf{Link}(\Gamma_0) \backslash V\Gamma_1$;
(5) $|s v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}| = d-k-l-1$ whenever $s \in V\Gamma_2$;
(6) $|s v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}| = d-k-l+1$ whenever $s \in \mathsf{Link}(\Gamma_0) \backslash V\Gamma_2$.

We shall assume moreover that $v_{\sigma(1)} \cdots v_{\sigma(k)}$, $v_{\sigma(k+1)} \cdots v_{\sigma(k+l)}$ and $v_{\sigma(k+l+1)} \cdots v_{\sigma(d)}$ are in I (i.e. they are the representatives of their equivalence class) and if $v_i = v_j, i < j$ then $\sigma(i) < \sigma(j)$ so that σ comes from a shuffle equivalence. Then σ is unique if it exists.

The permutation σ of Definition 2.3 does not necessarily exist. All expressions below in which a non-existing σ occurs need to be interpreted as 0 and we shall recall this at the relevant places.

Example 2.4. Consider the following graph:



This is the complete graph K_5 consisting of vertices a, b, c, d, e together with an extra vertex f that is connected only to d and e. Say that a word is in I (i.e. is a representative) if it is minimal in alphabetical order amongst all equivalent words. Now suppose that we have a reduced word:

$$abcdef.$$

- *Example 1.* Take $\Gamma_0 = \{a, b, c\}$, $\Gamma_1 = \{d, e\}$ and $\Gamma_2 = \emptyset$. Set $l = 3, k = 2$. Then σ as in Definition 2.3 exists and it is the permutation moving the word $abcdef$ to

$$(de)(abc)f$$

(since every letter occurs uniquely it is clear what the permutation is). Indeed: abc forms a clique; de ends on Γ_1 and there is no other letter commuting with Γ_1 at the end of de; f has no letters commuting with abc at the start.

- *Example 2.* Take $\Gamma_0 = \{a, b\}$, $\Gamma_1 = \{d\}$ and $\Gamma_2 = \{c\}$. Set $l = 2, k = 2$. Then σ as in Definition 2.3 does not exist. Indeed, by the choice of k, l and Γ_0 if σ exists there must be a word equivalent to $abcdef$ of the form ($*$ and $\diamond$ being undetermined letters):

$$** (ab) \diamond \diamond.$$

By the choice of Γ_1 we see that $**$ ends on d and there are no other letters in $\mathsf{Link}(\Gamma_0)$ at the end of $**$. So we must have

$$*d(ab) \diamond \diamond.$$

However, there is no choice for the letter $*$ (e is not allowed as it is in $\mathsf{Link}(\Gamma_0)$ and f cannot be moved past ab).

Now we find the following decomposition.

Lemma 2.5. *Let $a_1 \cdots a_d$ be a reduced operator of type $\mathbf{v} = v_1 \cdots v_d$ in the graph product (A, φ). Suppose that we have a non-zero expression of the form*

$$Q^{(1)}_{v_1} a_1 Q^{(2)}_{v_1} \cdots Q^{(1)}_{v_d} a_d Q^{(2)}_{v_d}, \tag{2.3}$$

in $\mathcal{B}(L^2(A, \varphi))$, where $Q^{(j)}_{v_i}$ equals either P_{v_i} or $P^{\perp}_{v_i}$. Then, for some permutation α of the indices $1, \ldots, d$ coming from a shuffle equivalence and for some $0 \leq r \leq d$ and $0 \leq m \leq d - r$ we have that (2.3) equals

$$\begin{aligned}(P_{v_{\alpha(1)}} a_{\alpha(1)} P^{\perp}_{v_{\alpha(1)}}) \cdots (P_{v_{\alpha(r)}} a_{\alpha(r)} P^{\perp}_{v_{\alpha(r)}})(P_{v_{\alpha(r+1)}} a_{\alpha(r+1)} P_{v_{\alpha(r+1)}}) \cdots (P_{v_{\alpha(m)}} a_{\alpha(m)} P_{v_{\alpha(m)}}) \\ \times \quad (P^{\perp}_{v_{\alpha(m+1)}} a_{\alpha(m+1)} P_{v_{\alpha(m+1)}}) \cdots (P^{\perp}_{v_{\alpha(d)}} a_{\alpha(d)} P_{v_{\alpha(d)}}).\end{aligned} \tag{2.4}$$

Further, for a non-zero expression of the form (2.4) we have that $v_{\alpha(r+1)} \cdots v_{\alpha(m)}$ is in a clique.

Proof. We prove this in a series of claims. To avoid cumbersome notation we shall not write the permutation of the shuffle equivalences in the proof.

Claim 1. The expression (2.3) is up to shuffle equivalence equal to:

$$Q^{(1)}_{v_1} a_1 Q^{(2)}_{v_1} \cdots Q^{(1)}_{v_m} a_m Q^{(2)}_{v_m} (P^{\perp}_{v_{m+1}} a_{m+1} P_{v_{m+1}}) \cdots (P^{\perp}_{v_d} a_d P_{v_d}). \tag{2.5}$$

Moreover, the tail of annihilation operators is maximal in the sense that if for some $i \leq m$ we have $Q_{v_i}^{(2)} = P_{v_i}$ then $Q_{v_i}^{(1)} = P_{v_i}$.

Proof of Claim 1. Suppose that we are given an expression as in (2.5). Suppose that for some $i < m$ we have $Q_{v_i}^{(1)} = P_{v_i}^{\perp}, Q_{v_i}^{(2)} = P_{v_i}$. Then we need to show that v_i commutes with $v_{i+1} \cdots v_m$. To do so we may suppose the index i was chosen maximally. Suppose that v_i and $v_{i+1} \cdots v_m$ do not commute and let v_k be the first letter in $v_{i+1} \cdots v_m$ that does not commute with v_i. Our choice of i yields that $Q_{v_k}^{(1)} = P_{v_k}$. Indeed if $Q_{v_k}^{(1)}$ were to be $P_{v_k}^{\perp}$ then (2.5) is 0 in case $Q_{v_k}^{(2)} = P_{v_k}^{\perp}$ and in case $Q_{v_k}^{(2)} = P_{v_k}$ this would contradict maximality of i. But then (2.5) contains a factor $P_{v_i} P_{v_k} = 0$ which means that (2.5) would be zero which in turn is a contradiction.

Claim 2. The expression (2.3) is up to shuffle equivalence equal to:

$$\begin{aligned} &Q_{v_1}^{(1)} a_1 Q_{v_1}^{(2)} \cdots Q_{v_r}^{(1)} a_r Q_{v_r}^{(2)} (P_{v_{r+1}} a_{r+1} P_{v_{r+1}}) \cdots (P_{v_m} a_m P_{v_m}) \\ &\quad \times \quad (P_{v_{m+1}}^{\perp} a_{m+1} P_{v_{m+1}}) \cdots (P_{v_d}^{\perp} a_d P_{v_d}). \end{aligned} \tag{2.6}$$

Moreover, the tail of annihilation and diagonal operators is maximal in the sense that if for some $i \leq r$ we have $Q_{v_i}^{(1)} = P_{v_i}$ then $Q_{v_i}^{(2)} = P_{v_i}^{\perp}$.

Proof of Claim 2. Suppose that we are given a non-zero expression as in (2.6). Suppose that for some $i < r$ we have $Q_{v_i}^{(1)} = P_{v_i}, Q_{v_i}^{(2)} = P_{v_i}$. Then we need to show that v_i commutes with $v_{i+1} \cdots v_r$. To do so we may suppose the index $i < r$ was chosen maximally. Suppose that v_i and $v_{i+1} \cdots v_r$ do not commute and let v_k be the first letter in $v_{i+1} \cdots v_r$ that does not commute with v_i. We claim that our choice of i yields that $Q_{v_k}^{(2)} = P_{v_k}^{\perp}$. Indeed, suppose that $Q_{v_k}^{(2)} = P_{v_k}$. Then if $Q_{v_k}^{(1)} = P_{v_k}$ this contradicts maximality of i and if $Q_{v_k}^{(1)} = P_{v_k}^{\perp}$ it would contradict Claim 1. From $Q_{v_k}^{(2)} = P_{v_k}^{\perp}$ we find that $Q_{v_k}^{(1)} = P_{v_k}$ since if $Q_{v_k}^{(1)} = P_{v_k}^{\perp}$ then $P_{v_k}^{\perp} a_k P_{v_k}^{\perp} = 0$. But then (2.6) contains the factor $P_{v_i} P_{v_k} = 0$ with v_i and v_k non-commuting, which means that (2.6) would be zero. As this is a contradiction the claim follows.

Claim 3. The expression (2.3) is up to shuffle equivalence equal to:

$$\begin{aligned} &(P_{v_1} a_1 P_{v_1}^{\perp}) \cdots (P_{v_r} a_r P_{v_r}^{\perp}) (P_{v_{r+1}} a_{r+1} P_{v_{r+1}}) \cdots (P_{v_m} a_m P_{v_m}) \\ &\quad \times \quad (P_{v_{m+1}}^{\perp} a_{m+1} P_{v_{m+1}}) \cdots (P_{v_d}^{\perp} a_d P_{v_d}). \end{aligned} \tag{2.7}$$

Moreover $v_{r+1} \cdots v_m$ forms a clique.

Proof of Claim 3. This is obvious now from Claim 2 and the fact that $P_{v_i}^{\perp} a_{v_i} P_{v_i}^{\perp} = 0$. As $P_{v_i} P_{v_j}$ is non-zero only if v_i and v_j commute we must have that $v_{r+1} \cdots v_m$ forms a clique.

Proof of the lemma. We may now directly conclude the lemma from Claim 3. The permutation α is then the composition of the shuffle equivalences coming from claims 1, 2 and 3. $\square$

Proposition 2.6. *Let $a_1 \cdots a_d$ be a reduced operator of type $\mathbf{v} = v_1 \cdots v_d$ in the graph product (A, φ). We have the following equality of operators in $\mathcal{B}(L^2(A, \varphi))$*

$$\begin{aligned} a_1 \cdots a_d = \sum_{l=0}^{d} \sum_{k=0}^{d-l} \sum_{\Gamma_0 \in \mathsf{Cliq}(\Gamma, l)} \sum_{(\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)} & (P_{v_{\sigma(1)}} a_{\sigma(1)} P^{\perp}_{v_{\sigma(1)}}) \cdots (P_{v_{\sigma(k)}} a_{\sigma(k)} P^{\perp}_{v_{\sigma(k)}}) \\ & \times (P_{v_{\sigma(k+1)}} a_{\sigma(k+1)} P_{v_{\sigma(k+1)}}) \cdots (P_{v_{\sigma(k+l)}} a_{\sigma(k+l)} P_{v_{\sigma(k+l)}}) \\ & \times (P^{\perp}_{v_{\sigma(k+l+1)}} a_{\sigma(k+l+1)} P_{v_{\sigma(k+l+1)}}) \cdots (P^{\perp}_{v_{\sigma(d)}} a_{\sigma(d)} P_{v_{\sigma(d)}}), \end{aligned} \tag{2.8}$$

where σ (changing over the summation) is as in Definition 2.3. If such σ does not exist then the summand is understood as 0.

Proof. We first note that we may decompose,

$$a_1 \cdots a_d = (P_{v_1} + P^{\perp}_{v_1}) a_1 (P_{v_1} + P^{\perp}_{v_1}) \cdots (P_{v_d} + P^{\perp}_{v_d}) a_d (P_{v_d} + P^{\perp}_{v_d}). \tag{2.9}$$

What we showed in Lemma 2.5 and (2.9) is that the product $a_1 \cdots a_d$ decomposes as a sum of operators of the form (2.4) (where α depends on the summand). The proof is finished by arguing that the summation in (2.8) runs exactly over all these summands.

It is clear that each (non-zero) summand in (2.8) is an expression of the form (2.4) with $r = k$ and $l = m - r$. Conversely, take an expression of the form (2.4), then the letters $v_{\alpha(r+1)} \cdots v_{\alpha(m)}$ form a clique Γ_0 by Lemma 2.5. Set $l = \#V\Gamma_0$ and $k = r$. Now in $v_{\alpha(1)}, \ldots, v_{\alpha(r)}$ there may be letters at the end that commute with Γ_0. Let Γ_1 be the clique of letters that appear at the end of $v_{\alpha(1)}, \ldots, v_{\alpha(r)}$ that commute with Γ_0 that is maximal in the following sense: there are no letters s appearing at the end of $v_{\alpha(1)} \cdots v_{\alpha(r)}$ that commute with Γ_0 and Γ_1. Such a clique is unique since if both Γ_1 and Γ_1' would be such cliques, then so is $\Gamma_1 \cup \Gamma_1'$ and hence $\Gamma_1 = \Gamma_1'$ by maximality. Similarly we may let Γ_2 be a clique of letters that appear at the start of $v_{\alpha(m+1)} \cdots v_{\alpha(d)}$ that commute with Γ_0 and that is maximal. Then for this choice of $\Gamma_0, \Gamma_1, \Gamma_2$ we have that σ satisfying (1) - (6) exists and it is moreover the only choice for which it exists. This shows that each non-zero expression (2.4) occurs exactly once in the summation (2.8). $\square$

In order to prove our Khintchine inequality we introduce the necessary notation. We define as in [50, Section 2] the following subspaces of $\mathcal{B}(L^2(A_f, \varphi_f))$,

$$L_1 = \operatorname{span}\{P^f_v a_v P^{f\,\perp}_v \mid v \in V\Gamma, a \in A_v\}, \qquad K_1 = L_1^*.$$

It is proved in [50, Lemma 2.3] that

$$L_1 \simeq \left(\oplus_{v\in V\Gamma} L^2(A_v^\circ, \varphi_v)\right)_{\mathrm{C}}, \qquad K_1 \simeq \left(\oplus_{v\in V\Gamma} L^2(A_v^\circ, \varphi_v)\right)_{\mathrm{R}} \tag{2.10}$$

completely isometrically, where the subscript C (resp. R) denotes the column (resp. row) Hilbert space structure. In particular, if each A_v° is one dimensional (as is the case for right-angled Hecke algebras) we have that L_1 (resp. K_1) is completely isometrically isomorphic to the column Hilbert space $\mathrm{C}_{\#V\Gamma}$ (resp. row Hilbert space $\mathrm{R}_{\#V\Gamma}$). We set k-fold Haagerup tensor products,

$$L_k = L_1^{\otimes_h k}, \qquad K_k = K_1^{\otimes_h k}.$$

Fix $\Gamma_0 \in \mathsf{Cliq}(l)$. For $a_1, \dots, a_l$ with $a_i \in A_{v_i}$ and $\mathbf{v} = v_1 \cdots v_l$ a reduced word (for the graph product) in I consisting precisely of all letters of Γ_0 we define an element of $\mathcal{B}(L^2(A_f, \varphi_f))$ by setting for $r > l$,

$$\mathrm{Diag}(a_1, \dots, a_l) : b_1 \cdots b_l b_{l+1} \cdots b_r \Omega_f \mapsto \overbrace{a_1 b_1}^{\circ} \cdots \overbrace{a_l b_l}^{\circ} b_{l+1} \cdots b_r \Omega_f, \tag{2.11}$$

where $b_i \in A_{w_i}^\circ$ with $w_i \neq w_{i+1}$ (so $b_1 \cdots b_r$ is a reduced word in the free product). If $r < l$ then the image in (2.11) is 0.

Lemma 2.7. *The operator defined in* (2.11) *is bounded.*

Proof. Let $r \in \mathbb{N}_{\geq 1}$. Fix a word $\mathbf{w} = w_1 \cdots w_r$ with $w_i \neq w_{i+1}$ for all $1 \leq i < r$. Then,

$$L^2(A_{w_1}^\circ, \varphi_{w_1}) \otimes \cdots \otimes L^2(A_{w_r}^\circ, \varphi_{w_r})$$

is an invariant subspace for the action (2.11). Moreover, note that (2.11) is for $r \geq l$ just the tensor product operator

$$P_{w_1} a_1 P_{w_1} \otimes \cdots \otimes P_{w_l} a_l P_{w_l} \otimes 1^{\otimes r-l}, \tag{2.12}$$

where $P_{w_i} a_i P_{w_i}$ acts on $L^2(A_{w_i}^\circ, \varphi_{w_i})$. Clearly this operator is bounded. □

Set the diagonal space

$$A_{\Gamma_0} \subseteq \mathcal{B}(L^2(A_f, \varphi_f)) \tag{2.13}$$

to be the linear span of all operators of the form (2.11). A_{Γ_0} inherits the operator space structure of A_f. Set for $d \in \mathbb{N}_{\geq 1}$,

$$X_d = \bigoplus_{l=0}^{d} \bigoplus_{k=0}^{d-l} \bigoplus_{\Gamma_0 \in \mathrm{Cliq}(\Gamma, l)} \bigoplus_{(\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)} L_k \otimes_h A_{\Gamma_0} \otimes_h K_{d-k-l}. \tag{2.14}$$

In case $A_v^\circ, v \in V\Gamma$ are all 1-dimensional, the space X_d can also be understood in terms of bounded operators on a Hilbert space. Indeed, the remarks after (2.10) and [6, Proposition 3.5] give the first two completely isometric isomorphisms of:

$$\begin{aligned} L_k \otimes_h A_{\Gamma_0} \otimes_h K_{d-k-l} \simeq & \mathrm{C}_k \otimes_h A_{\Gamma_0} \otimes_h \mathrm{R}_{d-k-l} \\ \simeq & M_{k,d-k-l}(A_{\Gamma_0}) \simeq M_{k,d-k-l}(\mathbb{C}) \otimes_{\min} A_{\Gamma_0}. \end{aligned} \tag{2.15}$$

The third completely isometric isomorphism of (2.15) holds by definition of the operator space structure on A_{Γ_0} as part of the C*-algebra A_f. Next consider the following embedding

$$j_d : \Sigma_d \to X_d,$$

where the image of $a_1 \otimes \cdots \otimes a_d$ is given as follows: Consider a summand of X_d indexed by $(l, k, \Gamma_0, \Gamma_1, \Gamma_2)$ with $0 \leq l \leq d, 0 \leq k \leq d - l$ and $\Gamma_0 \in \mathsf{Cliq}(\Gamma, l), (\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)$. Then the restriction of the image of j_d to this summand is given by

$$\begin{aligned} j_d(a_1 \otimes \cdots \otimes a_d)|_{X_d} = & (P^f_{v_{\sigma(1)}} a_{\sigma(1)} P^{f\,\perp}_{v_{\sigma(1)}}) \otimes \cdots \otimes (P^f_{v_{\sigma(k)}} a_{\sigma(k)} P^{f\,\perp}_{v_{\sigma(k)}}) \\ & \otimes \mathrm{Diag}(a_{\sigma(k+1)}, \ldots, a_{\sigma(k+l)}) \\ & \otimes (P^{f\,\perp}_{v_{\sigma(k+l+1)}} a_{\sigma(k+l+1)} P^f_{v_{\sigma(k+l+1)}}) \otimes \cdots \otimes (P^{f\,\perp}_{v_{\sigma(d)}} a_{\sigma(d)} P^f_{v_{\sigma(d)}}), \end{aligned} \tag{2.16}$$

with σ given by Definition 2.3; if such σ is non-existent then the image of $j_d(a_1 \otimes \cdots \otimes a_d)$ in the summand of X_d corresponding to $(l, k, \Gamma_0, \Gamma_1, \Gamma_2)$ is 0. Let

$$\pi^f_d : X_d \to \mathcal{B}(L^2(A_f, \varphi_f))$$

be the direct sums of product maps. For a 5-tuple $(l, k, \Gamma_0, \Gamma_1, \Gamma_2)$ as above, let

$$\pi^f_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2} : L_k \otimes_h A_{\Gamma_0} \otimes_h K_{d-k-l} \to \mathcal{B}(L^2(A_f), \varphi_f) \tag{2.17}$$

be the product map π^f_d restricted to the corresponding summand of X_d. This map is completely bounded as follows from the definition of the Haagerup tensor product. Consequently, $\pi^f_d : X_d \to \mathcal{B}(L^2(A_f, \varphi_f))$ is completely bounded by the number of summands of X_d, i.e.

$$\|\pi^f_d\|_{cb} \leq (\#\mathsf{Cliq}(\Gamma))^3 d.$$

Definition of two partial isometries. Given a 5-tuple $(l, k, \Gamma_0, \Gamma_1, \Gamma_2)$ as in the previous paragraph, we define two partial isometries.

- We define a partial isometry,

$$\mathcal{Q}_{l,k,\Gamma_0,\Gamma_1,\Gamma_2} : L^2(A,\varphi) \to L^2(A_f,\varphi_f), \tag{2.18}$$

as follows. Consider a reduced operator $b_1 \cdots b_n \in A$ of type $\mathbf{w}$. We need to define a permutation $\sigma_{\mathcal{Q}} = \sigma^{\mathbf{w}}_{\mathcal{Q},l,k,\Gamma_0,\Gamma_1,\Gamma_2}$ coming from a shuffle equivalence satisfying (1) – (4) of Definition 2.3 and the additional relation that $|sw_{\sigma_{\mathcal{Q}}(k+l+1)} \cdots w_{\sigma_{\mathcal{Q}}(n)}| = n-k-l+1$ whenever $s \in V\Gamma_2$. Moreover we assume that this $\sigma_{\mathcal{Q}}$ is chosen such that each of the expressions $w_{\sigma_{\mathcal{Q}}(k)} \cdots w_{\sigma_{\mathcal{Q}}(1)}$ (decreasing indices), $w_{\sigma_{\mathcal{Q}}(k+1)} \cdots w_{\sigma_{\mathcal{Q}}(k+l)}$ and $w_{\sigma_{\mathcal{Q}}(k+l+1)} \cdots w_{\sigma_{\mathcal{Q}}(n)}$ are in I. If $\sigma_{\mathcal{Q}}$ exists it is unique and we set

$$\mathcal{Q}_{l,k,\Gamma_0,\Gamma_1,\Gamma_2} b_1 \cdots b_n \Omega = b_{\sigma_{\mathcal{Q}}(1)} \cdots b_{\sigma_{\mathcal{Q}}(n)} \Omega_f.$$

If $\sigma_{\mathcal{Q}}$ does not exist we set $\mathcal{Q}_{l,k,\Gamma_0,\Gamma_1,\Gamma_2} b_1 \cdots b_n \Omega = 0$.

- We define the partial isometry,

$$\mathcal{R}_{l,k,\Gamma_0,\Gamma_1} : L^2(A,\varphi) \to L^2(A_f,\varphi_f), \tag{2.19}$$

as follows. Consider a reduced operator $b_1 \cdots b_n \in A$ of type $\mathbf{w}$. We need to define a permutation $\sigma_{\mathcal{R}} = \sigma^{\mathbf{w}}_{\mathcal{R},l,k,\Gamma_0,\Gamma_1}$ coming from a shuffle equivalence satisfying (1) – (4) of Definition 2.3. Moreover we assume that this $\sigma_{\mathcal{R}}$ is chosen such that each of the expressions $w_{\sigma_{\mathcal{R}}(1)} \cdots w_{\sigma_{\mathcal{R}}(k)}$, $w_{\sigma_{\mathcal{R}}(k+1)} \cdots w_{\sigma_{\mathcal{R}}(k+l)}$ and $w_{\sigma_{\mathcal{R}}(k+l+1)} \cdots w_{\sigma_{\mathcal{R}}(n)}$ are in I. If $\sigma_{\mathcal{R}}$ exists then it is unique and we set

$$\mathcal{R}_{l,k,\Gamma_0,\Gamma_1} b_1 \cdots b_n \Omega = b_{\sigma_{\mathcal{R}}(1)} \cdots b_{\sigma_{\mathcal{R}}(n)} \Omega_f.$$

If $\sigma_{\mathcal{R}}$ does not exist we set $\mathcal{R}_{l,k,\Gamma_0,\Gamma_1} b_1 \cdots b_n \Omega = 0$.

The maps $\mathcal{Q}_{l,k,\Gamma_0,\Gamma_1,\Gamma_2}$ and $\mathcal{R}_{l,k,\Gamma_0,\Gamma_1}$ preserve orthogonality and inner products and are therefore partial isometries.

Proposition 2.8. *Let $x = a_1 \otimes \cdots \otimes a_d \in \Sigma_d$ be of type $\mathbf{v} = v_1 \cdots v_d$ and let $x_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2}$ with $0 \le l \le d$, $0 \le k \le d-l$, $\Gamma_0 \in \mathsf{Cliq}(\Gamma, l)$ and $(\Gamma_1,\Gamma_2) \in \mathsf{Comm}(\Gamma_0)$ be the corresponding summands of $j_d(x)$ in X_d as in (2.16). We have*

$$\begin{aligned}
&\mathcal{R}^*_{l,k,\Gamma_0,\Gamma_1} \pi^f_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2}(x_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2}) \mathcal{Q}_{l,d-l-k,\Gamma_0,\Gamma_2,\Gamma_1} \\
=&(P_{v_{\sigma(1)}} a_{\sigma(1)} P^{\perp}_{v_{\sigma(1)}}) \cdots (P_{v_{\sigma(k)}} a_{\sigma(k)} P^{\perp}_{v_{\sigma(k)}}) \\
&\times \quad (P_{v_{\sigma(k+1)}} a_{\sigma(k+1)} P_{v_{\sigma(k+1)}}) \cdots (P_{v_{\sigma(k+l)}} a_{\sigma(k+l)} P_{v_{\sigma(k+l)}}) \\
&\times \quad (P^{\perp}_{v_{\sigma(k+l+1)}} a_{\sigma(k+l+1)} P_{v_{\sigma(k+l+1)}}) \cdots (P^{\perp}_{v_{\sigma(d)}} a_{\sigma(d)} P_{v_{\sigma(d)}}),
\end{aligned} \tag{2.20}$$

where σ is defined as in (1) – (6) of Definition 2.3 and the right hand side should be understood as 0 otherwise.

Proof. Note that both sides of (2.20) equal 0 if a σ as in the statement of the proposition does not exist, cf. the definition of j_d. So from now on we assume that σ exists and that the right hand side of (2.20) is non-zero for some elementary tensor product $a_1 \otimes \cdots \otimes a_d \in \Sigma_d$ of type $v_1 \cdots v_d$.

We argue first that without loss of generality we may assume that the permutation σ on the right hand side of (2.20) is trivial. Indeed, if σ is non-trivial then we may work with the element $x' = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(d)} \in \Sigma_d$ instead of x. Then note that the left hand side of (2.20) for x and x' are the same. Similarly the right hand side of (2.20) is the same for x and x'.

So assume that σ is trivial. Now take a reduced operator $b_1 \cdots b_r, r \geq 0$ of type $\mathbf{w} = w_1 \cdots w_r$. We first prove the proposition for the case that the permutation $\sigma_{\mathcal{Q}} := \sigma^{\mathbf{w}}_{\mathcal{Q},l,d-l-k,\Gamma_0,\Gamma_2,\Gamma_1}$ exists. In that case set,

$$\mathcal{Q}_{l,d-l-k,\Gamma_0,\Gamma_2,\Gamma_1}(b_1 \cdots b_r \Omega) = b'_1 \cdots b'_r \Omega_f,$$

with $b'_i = b_{\sigma_{\mathcal{Q}}(i)}, w'_i = w_{\sigma_{\mathcal{Q}}(i)}$. Further,

$$w'_1 \cdots w'_r \simeq \overbrace{(* \cdots * V\Gamma_2)}^{d-l-k}(V\Gamma_0)(\diamond \cdots \diamond),$$

where $(\diamond \cdots \diamond)$ has no letters in $V\Gamma_1$ at the start. We have by definition of $\pi^f_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2}$,

$$\begin{aligned}
&\pi^f_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2}(x_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2}) \\
=&(P^f_{v_1} a_1 P^{f\perp}_{v_1}) \cdots (P^f_{v_k} a_k P^{f\perp}_{v_k}) \mathrm{Diag}(a_{k+1}, \cdots, a_{k+l}) \\
&\times (P^{f\perp}_{v_{k+l+1}} a_{k+l+1} P^f_{v_{k+l+1}}) \cdots (P^{f\perp}_{v_d} a_d P^f_{v_d}).
\end{aligned}$$

Then, for the left hand side of (2.20),

$$\begin{aligned}
&\pi^f_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2}(x_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2}) \mathcal{Q}_{l,d-l-k,\Gamma_0,\Gamma_2,\Gamma_1}(b_1 \cdots b_r \Omega) \\
=&\langle a_d b'_1 \Omega_f, \Omega_f \rangle \cdots \langle a_{k+l+1} b'_{d-k-l} \Omega_f, \Omega_f \rangle a_1 \cdots a_k \\
&\times \overbrace{(a_{k+1} b'_{d-k-l+1})}^{\circ} \cdots \overbrace{(a_{k+l} b'_{d-k})}^{\circ} b'_{d-k+1} \cdots b'_r \Omega_f.
\end{aligned} \tag{2.21}$$

Now, for the right hand side of (2.20) we consider an expression,

$$\begin{aligned}
&(P_{v_1} a_1 P^{\perp}_{v_1}) \cdots (P_{v_k} a_k P^{\perp}_{v_k}) (P_{v_{k+1}} a_{k+1} P_{v_{k+1}}) \cdots (P_{v_{k+l}} a_{k+l} P_{v_{k+l}}) \\
&\times (P^{\perp}_{v_{k+l+1}} a_{k+l+1} P_{v_{k+l+1}}) \cdots (P^{\perp}_{v_d} a_d P_{v_d}) b_1 \cdots b_r \Omega.
\end{aligned} \tag{2.22}$$

The assumption that σ is trivial yields that $v_{k+l+1} \cdots v_d$ starts with $V\Gamma_2$, that the letters $v_k, \ldots, v_{k+l}$ exhaust $V\Gamma_0$ and that the letters at the end of $v_1 \cdots v_k$ that commute with Γ_0 are precisely given by $V\Gamma_1$. If (2.22) is non-zero then let us argue that there exists a

word $w_1' \cdots w_r'$ as defined above. Indeed, if (2.22) is non-zero, then we may shuffle $b_1 \cdots b_r$ into an operator $b_1' \cdots b_r'$ of type $w_1' \cdots w_r'$ such that: $w_1' \cdots w_{d-k-l}'$ equals $v_d \cdots v_{k+l+1}$ and ends with $V\Gamma_2$; the letters $w_{d-k-l+1}' \cdots w_{d-k}'$ exhaust $V\Gamma_0$; $w_{d-k+1}' \cdots w_r'$ does not have a letter of $V\Gamma_1$ up front (because if that happens then applying $P_{v_i}^{\perp}, i \leq k$ will give zero). So we conclude that (2.22) can only be non-zero if there exists $w_1' \cdots w_r'$ as defined above, in which case

$$\begin{aligned}(2.22) =& (P_{v_1} a_1 P_{v_1}^{\perp}) \cdots (P_{v_k} a_k P_{v_k}^{\perp})(P_{v_{k+1}} a_{k+1} P_{v_{k+1}}) \cdots (P_{v_{k+l}} a_{k+l} P_{v_{k+l}}) \\ & \times (P_{v_{k+l+1}}^{\perp} a_{k+l+1} P_{v_{k+l+1}}) \cdots (P_{v_d}^{\perp} a_d P_{v_d}) b_1' \cdots b_r' \Omega \\ =& \langle a_d b_1' \Omega, \Omega \rangle \cdots \langle a_{k+l+1} b_{d-k-l}' \Omega, \Omega \rangle a_1 \cdots a_k \overbrace{(a_{k+1} b_{d-k-l+1}')}^{\circ} \cdots \overbrace{(a_{k+l} b_{d-k}')}^{\circ} \\ & \times b_{d-k+1}' \cdots b_r' \Omega.\end{aligned} \tag{2.23}$$

If one of the terms with a $\overset{\circ}{\frown}$ is zero, then also this term was zero in (2.21) and the proposition is proved. If none of these terms are zero, then the image of (2.23) under $\mathcal{R}_{l,k,\Gamma_0,\Gamma_1}$ equals (2.21) and (2.22) is in $\ker(\mathcal{R}_{l,k,\Gamma_0,\Gamma_1})^{\perp}$. This concludes the proposition in case $\sigma_{\mathcal{Q}}$ exists.

If $\sigma_{\mathcal{Q}}$ does not exist, then

$$\mathcal{Q}_{l,d-l-k,\Gamma_0,\Gamma_2,\Gamma_1}(b_1 \cdots b_r \Omega) = 0.$$

On the other hand we already noted that (2.23) can only be nonzero if a permutation $\sigma_{\mathcal{Q}}$ exists. So if $\sigma_{\mathcal{Q}}$ is non-existent then also (2.23) is zero, yielding the proposition. □

Set the product map

$$\rho_d : \Sigma_d \to \mathcal{B}(L^2(A, \varphi)) : a_1 \otimes \cdots \otimes a_d \mapsto a_1 \cdots a_d.$$

We define the map for $d \in \mathbb{N}_{\geq 1}$,

$$\pi_d : j_d(\Sigma_d) \to \mathcal{B}(L^2(A, \varphi)) : j_d(x) \mapsto \rho_d(x), \tag{2.24}$$

so that by definition $\pi_d \circ j_d = \rho_d$. Now the crucial part is to show that the map π_d is well-defined and completely bounded with linear bound in d. This is where we use the announced intertwining argument between graph products and free products.

Now we are ready for the main theorem of this section. Recall that the word length projection χ_d was defined in (1.1).

Theorem 2.9 *(Graph product Khintchine inequality). Let Γ be a finite simplicial graph and consider a graph product $(A, \varphi) = *_{v,\Gamma}(A_v, \varphi_v)$ of unital C^*-algebras A_v with GNS-faithful states φ_v. Then for every $d \in \mathbb{N}_{\geq 1}$ there exist maps*

$$j_d : \chi_d(A) \to X_d, \qquad \pi_d : \mathrm{Dom}(\pi_d) \subseteq X_d \to \chi_d(A),$$

with $\mathrm{Dom}(\pi_d) = j_d(\chi_d(A))$ *and where* X_d *is defined in* (2.14) *and* (2.15) *such that:*

(i) $\pi_d \circ j_d$ *is the identity on* $\chi_d(A)$*;*
(ii) $\|\pi_d : \mathrm{Dom}(\pi_d) \to A\|_{cb} \leq (\#\mathsf{Cliq}(\Gamma))^3 d$.

Proof. Part (i) follows from Proposition 2.6. From Proposition 2.8 we see that on the domain $j_d(\chi_d(A))$ the map π_d is given by the direct sum of the maps

$$\mathcal{R}^*_{l,k,\Gamma_0,\Gamma_1} \pi^f_{d,l,k,\Gamma_0,\Gamma_1,\Gamma_2}(\,\cdot\,) \mathcal{Q}_{l,d-l-k,\Gamma_0,\Gamma_2,\Gamma_1}.$$

In particular, π_d is well-defined. As each of these summands is completely contractive and there are at most $(\#\mathsf{Cliq}(\Gamma))^3 d$ summands, we see that π_d is completely bounded with the desired complete bound. □

Remark 2.10. It is possible that a clever refinement of the present methods or perhaps an entirely different proof could yield an improvement of the constant $(\#\mathsf{Cliq}(\Gamma))^3 d$ in the Khintchine inequality. For the applications we have in mind, the current method cannot be altered without affecting some of the results in the remainder of the paper as they depend on the explicit form of the map j_d and the space X_d. We leave the question of whether an improved constant can be attained to future work.

3. Khintchine and Haagerup inequalities for right-angled Hecke C*-algebras

In this section we make the Khintchine inequality from Section 2 explicit in the case of Hecke algebras. As a consequence we derive a Haagerup inequality for right-angled Coxeter groups and their Hecke deformations. Such a Haagerup inequality shows that the L^∞-norm of an operator of length d can be estimated with the L^2-norm up to a polynomial bound $Q(d)$. It is a generalization of Haagerup's inequality for free groups $\mathbb{F}_n$, see [33] or also [43, Section 9.6]. It entails that there exists a constant C such that for every $x \in \mathbb{C}[\mathbb{F}_n]$ supported on group elements of length $d \in \mathbb{N}_{\geq 1}$ we have

$$\|x\| \leq Cd\|x\|_2.$$

In particular, here $Q(d) = d$ so that we have a linear estimate in the length d. Haagerup and Khintchine inequalities have found a wide range of applications in operator theory. We will give further applications to C*-simplicity problems in Section 5.3.

Let (W, S) be a finitely generated right-angled Coxeter system and let $q = (q_s)_{s\in S} \in \mathbb{R}^{(W,S)}_{>0}$ be a multi-parameter set. Let Γ be the graph associated to (W, S) defined in Section 1.7. From (1.7) we see that we have a canonical isomorphism

$$(C^*_{r,q}(W), \tau_q) \simeq *_{s,\Gamma}(C^*_{r,q_s}(W_s), \tau_{q_s}).$$

Furthermore, we may specialize Section 2 to $*_{s,\Gamma}(C^*_{r,q_s}(W_s), \tau_{q_s})$. In particular this defines the space of diagonal operators A_{Γ_0} as in (2.13) and the operator space X_d of (2.14). We first observe that A_{Γ_0} simplifies. Recall that $P_s, s \in S$ and $P^f_{\Gamma_0}, \Gamma_0 \in \mathsf{Cliq}(\Gamma)$ were defined in (1.6) and (2.1).

Lemma 3.1. *For a right-angled Coxeter group (W, S) we have $A_{\Gamma_0} = \mathbb{C}P^f_{\Gamma_0}$ for any $\Gamma_0 \in$ Cliq(Γ, l). Moreover, for $\mathbf{v} = s_1 \cdots s_l, s_i \in S$ a reduced word in I with $\{s_1, \ldots, s_l\} = V\Gamma_0$ we have,*

$$\mathrm{Diag}\left(P_{s_1}T^{(q)}_{s_1}P_{s_1}, \ldots, P_{s_l}T^{(q)}_{s_l}P_{s_l}\right) = (\prod_{s\in V\Gamma_0} p_s(q))P^f_{\Gamma_0}.$$

Proof. For $s \in S$ we have that $P_sT^{(q)}_sP_s = p_s(q)P_s$ and $C^*_{r,q}(W_s)$ is a two dimensional C*-algebra spanned by the identity and $T^{(q)}_s$ (see the comments after [12, Corollary 3.3]). Now let $\mathbf{v} = s_1 \cdots s_l$ be as in the lemma. Let $A_s = C^*_{r,q_s}(W_s)$. For operators $a_i \in A_{s_i}, 0 \leq i \leq l$, we then have that $P_{s_i}a_iP_{s_i}$ is a scalar multiple of P_{s_i} so that $\mathrm{Diag}(a_1, \ldots, a_l)$ is a scalar multiple of $P^f_{\Gamma_0}$, see (2.12). If $a_i = T^{(q)}_s$ the scalar multiple is $\prod_{s\in V\Gamma_0} p_s(q)$. $\square$

Lemma 3.1 shows that we may identify $A_{\Gamma_0} \simeq \mathbb{C}$ completely isometrically. Then in (2.15) note that $M_{k,d-k-l}(\mathbb{C}) \otimes_h \mathbb{C} = M_{k,d-k-l}(\mathbb{C})$. So for a right-angled Coxeter system we get by Lemma 3.1, (2.14) and (2.15),

$$X_d = \bigoplus_{l=0}^{d} \bigoplus_{k=0}^{d-l} \bigoplus_{\Gamma_0\in\mathsf{Cliq}(\Gamma,l)} \bigoplus_{(\Gamma_1,\Gamma_2)\in\mathsf{Comm}(\Gamma_0)} M_{k,d-k-l}(\mathbb{C}). \tag{3.1}$$

Let $p_{l,k,\Gamma_0,\Gamma_1,\Gamma_2}$ be the projection of X_d onto the summand $M_{k,d-k-l}(\mathbb{C})$ indexed by $(l, k, \Gamma_0, \Gamma_1, \Gamma_2)$. We equip $M_{k,d-k-l}(\mathbb{C})$ with the inner product

$$\langle x, y\rangle_{\mathrm{Tr}} = (\mathrm{Tr}_{d-k-l})(y^*x),$$

where Tr_{d-k-l} is the non-normalized trace that takes the value 1 on rank 1 projections. We further equip X_d with the direct sum of these inner products. For $x \in X_d$ we let

$$\|x\|_{2,\mathrm{Tr}} = \langle x, x\rangle^{\frac{1}{2}}_{\mathrm{Tr}}.$$

Then, as for any finite dimensional type I von Neumann algebra, we have

$$\|x\| \leq \|x\|_{2,\mathrm{Tr}}. \tag{3.2}$$

By Theorem 2.9 we obtain maps

$$j_d : \chi_d(C^*_{r,q}(W)) \to X_d, \qquad \text{and} \qquad \pi_d : \mathrm{Dom}(\pi_d) \subseteq X_d \to \chi_d(C^*_{r,q}(W)),$$

with $\mathrm{Dom}(\pi_d) = j_d(\chi_d(C^*_{r,q}(W)))$ such that $\pi_d \circ j_d$ is the identity on $C^*_{r,q}(W)$ and

$$\|\pi_d : \mathrm{Dom}(\pi_d) \to X_d\|_{cb} \leq \#\mathsf{Cliq}(\Gamma)^3 d.$$

We now have the following orthogonality lemma.

Lemma 3.2. *Let $d \in \mathbb{N}_{\geq 1}, 0 \leq l \leq d, 0 \leq k \leq d-l, \Gamma_0 \in \mathrm{Cliq}(\Gamma, l), (\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)$. Let $\mathbf{v}, \mathbf{w} \in W$ be reduced of length d. If the permutation $\sigma^{\mathbf{v}}$ given in Definition 2.3 exists we have*

$$\langle p_{l,k,\Gamma_0,\Gamma_1,\Gamma_2} j_d(T^{(q)}_{\mathbf{v}}), j_d(T^{(q)}_{\mathbf{w}})\rangle_{\mathrm{Tr}} = \delta(\mathbf{v} \simeq \mathbf{w}) \prod_{s \in V\Gamma_0} p_s(q)^2. \tag{3.3}$$

If such $\sigma^{\mathbf{v}}$ does not exist then $p_{l,k,\Gamma_0,\Gamma_1,\Gamma_2} j_d(T^{(q)}_{\mathbf{v}}) = 0$.

Proof. The final claim of the statement follows from the definition of j_d. So it remains to prove (3.3) and we assume that $\sigma^{\mathbf{v}}$ as in Definition 2.3 exists. Since the right hand side of (3.3) has the term $\delta(\mathbf{v} \simeq \mathbf{w})$ we may assume that also $\sigma^{\mathbf{w}}$ exists. Moreover, by shuffling the letters of $\mathbf{v}$ and $\mathbf{w}$ if necessary, which does not change the operators $T^{(q)}_{\mathbf{v}}$ and $T^{(q)}_{\mathbf{w}}$, we may assume that $\sigma^{\mathbf{v}}$ and $\sigma^{\mathbf{w}}$ are the identity permutation.

For $s \in S$ set

$$e_s = P^f_s T^{(q)}_s P^{f,\perp}_s \quad \text{and} \quad e'_s = P^{f,\perp}_s T^{(q)}_s P^f_s.$$

These form an orthonormal basis of the respective column Hilbert space L_1 and row Hilbert space K_1. Now write a reduced expression $\mathbf{v} = s_1 \cdots s_d, s_i \in S$. By assumption that $\sigma^{\mathbf{v}}$ was trivial we see that $s_{k+1}, \ldots, s_{k+l}$ commute and form a clique Γ_0 in Γ. From Lemma 3.1 we find that

$$\mathrm{Diag}\left(P_{s_{k+1}} T^{(q)}_{s_{k+1}} P_{s_{k+1}}, \ldots, P_{s_{k+l}} T^{(q)}_{s_{k+l}} P_{s_{k+l}}\right) = (\prod_{s \in V\Gamma_0} p_s(q)) P^f_{\Gamma_0}.$$

It now follows from the definition of j_d that

$$p_{l,k,\Gamma_0,\Gamma_1,\Gamma_2} j_d(T^{(q)}_{\mathbf{v}}) = \left(\prod_{s \in V\Gamma_0} p_s(q)\right) e_{s_1} \otimes \cdots \otimes e_{s_k} \otimes e'_{s_{k+l+1}} \otimes \cdots \otimes e'_{s_d}.$$

From this we can directly conclude (3.3). □

Theorem 3.3 *(Khintchine inequality for right-angled Hecke C^*-algebras). Let (W, S) be a right-angled Coxeter system with finite generating set S and graph Γ. Let $q = (q_s)_{s \in S} \in \mathbb{R}^{(W,S)}_{>0}$ be a multi-parameter. Then for every $d \in \mathbb{N}_{\geq 1}$ there exist maps*

$$j_d : \chi_d(C^*_{r,q}(W)) \to X_d, \qquad \pi_d : \mathrm{Dom}(\pi_d) \subseteq X_d \to \chi_d(C^*_{r,q}(W)),$$

with $j_d(\chi_d(C^*_q(W)) = \mathrm{Dom}(\pi_d)$ *and* X_d *defined in* (3.1) *such that:*

(i) $\pi_d \circ j_d$ *is the identity on* $\chi_d(C^*_{r,q}(W))$*;*
(ii) $\|\pi_d : \mathrm{Dom}(\pi_d) \to C^*_{r,q}(W)\|_{cb} \leq (\#\mathsf{Cliq}(\Gamma))^3 d$*;*
(iii) j_d *extends to a bounded map*

$$L^2(\chi_d(C^*_{r,q}(W)), \tau_q) \to L^2(X_d, \mathrm{Tr}),$$

with bound majorized by $\prod_{s\in S} p_s(q)$*.*

Proof. Statements (i) and (ii) are immediate from (1.7) and Theorem 2.9. We thus prove (iii). Let $\mathbf{v} \in I$ have length d. Since $\|T^{(q)}_{\mathbf{v}}\Omega\|_2 = 1$ we find from (3.3) that

$$\|j_d : L^2(\chi_d(C^*_{r,q}(W)), \tau_q) \to L^2(X_d, \mathrm{Tr}_{X_d})\| \leq \sup_{\mathbf{v}\in W, |\mathbf{v}|=d} \|j_d(T^{(q)}_{\mathbf{v}})\|_{2,\mathrm{Tr}} \leq \prod_{s\in S} p_s(q).$$

This completes the proof. □

Theorem 3.4 *(Haagerup inequality for right-angled Hecke C*-algebras). Let* (W, S) *be a right-angled Coxeter group with finite generating set* S*. Let* $q = (q_s)_{s\in S} \in \mathbb{R}^{(W,S)}_{>0}$ *be a multi-parameter. Then for each* $d \in \mathbb{N}_{\geq 1}$ *and* $x \in \chi_d(C^*_{r,q}(W))$ *we have*

$$\|x\| \leq d(\#\mathsf{Cliq}(\Gamma))^3 (\prod_{s\in S} p_s(q)) \|x\|_2.$$

Proof. By respectively Theorem 3.3 (i), (3.2) and Theorem 3.3 (ii), (iii), we get for every $x \in \chi_d(C^*_{r,q}(W))$,

$$\begin{aligned}\|x\| =& \|(\pi_d \circ j_d)(x)\| \leq \|\pi_d\| \|j_d(x)\| \leq \|\pi_d\| \|j_d(x)\|_{2,\mathrm{Tr}} \\ \leq& d(\#\mathsf{Cliq}(\Gamma))^3 \|j_d(x)\|_{2,\mathrm{Tr}} \leq d(\#\mathsf{Cliq}(\Gamma))^3 (\prod_{s\in S} p_s(q)) \|x\|_2.\end{aligned}$$

This completes the proof. □

4. On isomorphisms of Hecke algebras and their C*- and von Neumann algebras

In this section we discuss isomorphism properties of Hecke algebras and in particular their dependence on q. These properties are well understood for finite Coxeter systems, which we summarize as follows.

Remark 4.1. By Tits's deformation theorem, the Hecke algebras $\mathbb{C}_q[W]$ of a given finite Coxeter system (W, S) are pairwise isomorphic to each other for different $q \in \mathbb{R}^{(W,S)}_{>0}$, see

[11, Proposition 10.11.2]. The argument is based on the classification of finite-dimensional semi-simple algebras and the isomorphism is not explicit.

To the knowledge of the authors there is no general statement for arbitrary Coxeter systems (W, S) about the dependence on $q \in \mathbb{R}_{>0}^{(W,S)}$ of the isomorphism class of their Hecke deformations. However, in the case of right-angled (not necessarily finite) Coxeter systems one can still prove that all Hecke deformations are isomorphic; even with an explicit isomorphism. See e.g. [39, (2.1.13)], [42, Corollary 9.7] for this result which we present in an alternative way and which is suited for the next sections.

Proposition 4.2. *Let (W, S) be a right-angled Coxeter system and $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$. Then the map $\pi_{q,1}\colon \mathbb{C}_1[W] \to \mathbb{C}_q[W]$ given by*

$$1 \mapsto 1 \quad \text{and} \quad T_s^{(1)} \mapsto \frac{1-q_s}{1+q_s} + \frac{2\sqrt{q_s}}{1+q_s} T_s^{(q)} \tag{4.1}$$

for $s \in S$ defines an isomorphism of $*$-algebras.

Proof. Set $\alpha_s(q) := (1-q_s)/(1+q_s)$ and $\beta_s(q) := 2\sqrt{q_s}/(1+q_s)$. Being its own inverse, for $s \in S$ the expression $\alpha_s(q) + \beta_s(q) T_s^{(q)} \in \mathbb{C}_q[W]$ is invertible. Hence we get a map $S \to \mathbb{C}_q[W]^\times$, $s \mapsto \alpha_s(q) + \beta_s(q) T_s^{(q)}$ that uniquely extends to a group homomorphism ϕ on the free group $F(S)$ in S. Since (W, S) is right-angled one easily checks that $\phi(st)^{m_{s,t}} = (\phi(s)\phi(t))^{m_{s,t}}$ for all $s, t \in S$. This implies that ϕ induces a group homomorphism $\phi' : W \to \mathbb{C}_q[W]^\times$ with $(\phi'(\mathbf{w}))^* = \phi(\mathbf{w}^{-1})$ for every $\mathbf{w} \in W$. The universal property of the group algebra $\mathbb{C}_1[W]$ then implies the existence of the unital $*$-algebra homomorphism $\pi_{q,1}$. It is clearly surjective. The injectivity follows from the universal property of the Hecke algebra $\mathbb{C}_1[W]$. $\square$

Remark 4.3. The homomorphism prescribed by (4.1) does not necessarily exist if (W, S) is not right-angled. This already fails for the Coxeter system (W, S) with $S = \{s, t\}$ and $m_{s,s} = m_{t,t} = 2$, $m_{s,t} = 3$ and points out an inaccuracy in [22, Section 19, Note 19.2 on p. 358].

Remark 4.4. For a right-angled Coxeter system (W, S) with $|S| \geq 2$ the isomorphism of Proposition 4.2 does not extend to an isomorphism $C^*_{r,1}(W) \to C^*_{r,q}(W)$ for all $q \in \mathbb{R}_{>0}^{(W,S)}$. Indeed, in Example 6.3 we show that this cannot be the case. In the case $|S| \geq 3$ this can also be proved through the simplicity of $C^*_{r,q}(W)$ in the same way as Remark 4.5, see Section 5.

Remark 4.5. Garncarek's factoriality result of Theorem 0.3 illustrates that the isomorphism of Proposition 4.2 does not necessarily extend to an isomorphism of the corresponding Hecke-von Neumann algebras. Indeed, for $q \notin [\rho, \rho^{-1}]$ the Hecke-von Neumann algebra $\mathcal{N}_q(W)$ is not a factor whereas $\mathcal{N}_1(W)$ is a factor. Hence there can

be no isomorphism at all between $\mathcal{N}_q(W)$ and $\mathcal{N}_1(W)$. This was already observed in [22, Section 19, Note 19.2 on p. 358]. The situation is even more delicate, see the next Remark 4.6.

Remark 4.6 *(Free factor problem).* Consider the right-angled Coxeter group $W = (\mathbb{Z}_2)^{*l}, l \geq 3$. By Theorem 0.3 we see that for the single parameter $q \in [\frac{1}{l-1}, 1]$ we have that $\mathcal{N}_q(W)$ is a II_1-factor. Moreover, [27] together with a calculation in [31, Section 6] shows that for $q \in [\frac{1}{l-1}, 1]$ we have that $\mathcal{N}_q(W) \simeq \mathcal{L}(\mathbb{F}_{2lq(1+q)^{-2}})$ where $\mathcal{L}(\mathbb{F}_t), t \in \mathbb{R}_{>1}$ is the interpolated free group factor, cf. [27], [49]. By [27], [49], the interpolated free group factors are either all isomorphic or they are all non-isomorphic. The problem which of the two in this dichotomy is true is known as the famous free factor problem. Hence, solving the isomorphism question of $\mathcal{N}_q(W)$ for different $q \in [\frac{1}{l-1}, 1]$ is equivalent to the free factor problem. In Section 5 we shall show that for $q \in [\frac{1}{l-1}, 1]$ we have that $C^*_{r,q}(W)$ has unique trace; therefore if any two $C^*_{r,q}(W)$ with $q \in [\frac{1}{l-1}, 1]$ are isomorphic we get by Lemma 1.1 that two of the von Neumann algebras $\mathcal{N}_q(W), q \in [\frac{1}{l-1}, 1]$ would be isomorphic. Since solving the free factor problem using these C*-algebraic methods seems unrealistic[4] we believe that all $C^*_{r,q}(W)$ with $q \in [\frac{1}{l-1}, 1]$ are non-isomorphic.

We finish this section with the following lemma which is well-known on the algebraic level, see for instance [42, Section 9]. For convenience of the reader we include the proof for the associated C*- and von Neumann algebras here.

Proposition 4.7. *Let (W, S) be a Coxeter system and $q = (q_s)_{s\in S} \in \mathbb{R}^{(W,S)}_{>0}$, $\epsilon = (\epsilon_s)_{s\in S} \in \{-1, 1\}^{(W,S)}$. Then $C^*_{r,q}(W) \simeq C^*_{r,q'}(W)$ via $T^{(q)}_s \mapsto \epsilon_s T^{(q')}_s$ where $q' := (q_s^{\epsilon_s})_{s\in S} \in \mathbb{R}^{(W,S)}_{>0}$.*

Proof. Note that $\epsilon_s p_s(q') = p_s(q)$. Then from the defining properties of a Hecke algebra (1.3) and (1.4) we have that $T^{(q)}_s \mapsto \epsilon_s T^{(q')}_s, s \in S$ determines a $*$-isomorphism $\pi_{q',q} : \mathbb{C}_q[W] \to \mathbb{C}_{q'}[W]$. Moreover,

$$\tau_{q'} \circ \pi_{q',q}(T^{(q)}_{\mathbf{w}}) = \epsilon_{\mathbf{w}} \tau_{q'}(T^{(q')}_{\mathbf{w}}) = \epsilon_{\mathbf{w}} \delta(\mathbf{w} = \emptyset) = \delta(\mathbf{w} = \emptyset) = \tau_q(T^{(q)}_{\mathbf{w}}),$$

so that $\pi_{q',q}$ is trace preserving. By Lemma 1.1 $\pi_{q',q}$ extends to a $*$-isomorphism $C^*_{r,q}(W) \to C^*_{r,q'}(W)$. $\square$

5. Simplicity of Hecke C*-algebras

A C*-algebra is called **simple** if it does not contain any non-trivial closed two-sided ideal. In this section we investigate the (non-)simplicity of Hecke C*-algebras.

[4] And solving it in the affirmative using C*-algebras seems even more unrealistic.

Remark 5.1. Note that in order to study the simplicity of Hecke C*-algebras it suffices to consider irreducible Coxeter systems. Indeed, if (W, S) is a Coxeter system that is not irreducible, it admits a non-trivial decomposition of the form $(W, S) = (W_T \times W_{T'}, T \cup T')$. For every $q \in \mathbb{R}_{>0}^{(W,S)}$ this induces a decomposition of the corresponding Hecke algebra into an algebraic tensor product

$$\mathbb{C}_q[W] \simeq \mathbb{C}_q[W_T] \odot \mathbb{C}_q[W_{T'}].$$

By Lemma 1.1 this extends to the C*- and von Neumann algebraic level

$$C^*_{r,q}(W) \simeq C^*_{r,q}(W_T) \otimes C^*_{r,q}(W_{T'}), \mathcal{N}_q(W) \simeq \mathcal{N}_q(W_T) \overline{\otimes} \mathcal{N}_q(W_{T'}).$$

Since the (spatial) tensor product of two C*-algebras is simple if and only if both C*-algebras are simple, it suffices to look at irreducible Coxeter systems.

Recall Garncarek's characterization of factoriality of the single-parameter Hecke-von Neumann algebras of right-angled Coxeter groups, see Theorem 0.3. Another result related to the simplicity of Hecke C*-algebras appears in [24] (see also [30] and [20]).

Theorem 5.2. *An irreducible Coxeter group which is neither of spherical nor affine type is C*-simple and has unique tracial state.*

In the remaining sections we give partial answers to the question for a characterization of those Hecke C*-algebras that are simple and have unique tracial state. In the case of a free product of abelian Hecke C*-algebras we obtain a complete answer.

5.1. Non-simplicity of Hecke C-algebras*

Let (W, S) be a Coxeter system with $|S| < \infty$ and $z := (z_s)_{s \in S} \in \mathbb{C}^{(W,S)}$. For every reduced expression $\mathbf{w} = s_1 ... s_n$ of $\mathbf{w} \in W$ define $z_{\mathbf{w}} := z_{s_1} ... z_{s_n}$. The growth series of W is the power series in z defined by

$$W(z) := \sum_{\mathbf{w} \in W} z_{\mathbf{w}}.$$

We denote its region of convergence by $\mathcal{R} := \left\{ z \in \mathbb{C}^{(W,S)} \mid W(z) \text{ converges} \right\}$. For more information on the growth series see [22, Chapter 17]. We further set

$$\mathcal{R}' := \left\{ (q_s^{\epsilon_s})_{s \in S} \mid q \in \mathcal{R} \cap \mathbb{R}_{>0}^{(W,S)}, \epsilon \in \{-1, 1\}^{(W,S)} \right\}. \tag{5.1}$$

Denote the closure of $\mathcal{R}'$ in $\mathbb{R}_{>0}^{(W,S)}$ by $\overline{\mathcal{R}'}$.

Lemma 5.3. *Let (W, S) be a Coxeter system with $|S| < \infty$ and $q \in \overline{\mathcal{R}'}$. Then there exists a character on $C^*_{r,q}(W)$. In particular $C^*_{r,q}(W)$ is not simple and does not have unique tracial state.*

Proof. First assume that $q \in \mathcal{R}'$ with $q_s \leq 1$ for all $s \in S$. As shown in the proof of [31, Theorem 5.3], there exists a central projection $E_q \in \mathcal{N}_q(W)$ with $E_q T^{(q)}_{\mathbf{w}} = q^{1/2}_{\mathbf{w}} E_q$ for every $\mathbf{w} \in W$ (compare also with [22, Lemma 19.2.5] but note that our notational conventions differ slightly). Hence the map $\chi_q(\,\cdot\,) := \tau_q(\,\cdot\, E_q) / \|E_q\|_2$ is a character on $C^*_{r,q}(W)$ with $\chi_q(T^{(q)}_{\mathbf{w}}) = q^{1/2}_{\mathbf{w}}$, $\mathbf{w} \in W$. Its kernel is a non-trivial maximal ideal in $C^*_{r,q}(W)$.

For $q \in \overline{\mathcal{R}'} \setminus \mathcal{R}'$ with $q_s \leq 1$ for all $s \in S$ choose a sequence $(q_n)_{n \in \mathbb{N}_{\geq 1}} \subseteq \mathcal{R}$ with $q_n \to q$. The map $\chi\colon T^{(q)}_{\mathbf{w}} \mapsto q^{1/2}_{\mathbf{w}}$ defines a character on $\mathbb{C}_q[W]$. For the finite sums $x := \sum_{\mathbf{w} \in W} x(\mathbf{w}) T^{(q)}_{\mathbf{w}} \in \mathbb{C}_q[W]$ and $x_n := \sum_{\mathbf{w} \in W} x(\mathbf{w}) T^{(q_n)}_{\mathbf{w}} \in \mathbb{C}_{q_n}[W]$ we have $x_n \to x$ in $\mathcal{B}\left(\ell^2(W)\right)$ and $\chi_{q_n}(x_n) \to \chi(x)$ with χ_{q_n} defined as above. This implies

$$|\chi(x)| = \lim_{n \to \infty} |\chi_{q_n}(x_n)| \leq \lim_n \|x_n\| = \|x\|,$$

so χ extends to a character on $C^*_{r,q}(W)$. Hence $C^*_{r,q}(W)$ is not simple and does not have unique tracial state.

For general $q \in \overline{\mathcal{R}'}$ the statement follows from the above in combination with Proposition 4.7 □

It follows from Lemma 5.3 that Hecke C*-algebras coming from irreducible Coxeter systems of spherical or affine type are never simple and never have unique tracial state, see Corollary 5.4. We are aware of the fact that affine type Coxeter systems always give rise to type I Hecke C*-algebras (see [39, §4]), which implies non-simplicity. However, the proof of this statement makes use of the (non-trivial) characterization of affine Coxeter groups in terms of affine Weyl groups and a resulting description of the corresponding Hecke algebras. The proof we present here is elementary.

Corollary 5.4. *Let (W, S) be an irreducible Coxeter system of spherical or affine type. Then for any choice of parameter $q \in \mathbb{R}^{(W,S)}_{>0}$ there exists a character on the corresponding Hecke C*-algebra $C^*_{r,q}(W)$. In particular, $C^*_{r,q}(W)$ is not simple and does not have unique tracial state.*

Proof. First assume that $|S| < \infty$ and let $q \in \mathbb{R}^{(W,S)}_{>0}$ with $q_s \leq 1$ for every $s \in S$. Being of spherical or affine type the Coxeter group W is amenable. Hence by [22, Proposition 17.2.1] the radius of convergence of the power series $W(z)$ is one. But then $q \in \overline{\mathcal{R}} \cap \mathbb{R}^{(W,S)}_{>0}$, so $C^*_{r,q}(W)$ admits a character by Lemma 5.3. For general $q \in \mathbb{R}^{(W,S)}_{>0}$, the statement follows with Proposition 4.7.

Next assume that $|S| = \infty$ and so (W, S) is of spherical type. Again, the map $\chi\colon T^{(q)}_{\mathbf{w}} \mapsto q^{1/2}_{\mathbf{w}}$, $\mathbf{w} \in W$ defines a character on $\mathbb{C}_q[W]$. For every element $x := \sum_{\mathbf{w} \in W} x(\mathbf{w}) T^{(q)}_{\mathbf{w}} \in$

$\mathbb{C}_q[W]$ there exists a finite subset $T \subseteq S$ such that the support $\{\mathbf{w} \in W \mid x(\mathbf{w}) \neq 0\}$ of x is contained in the finite subgroup W_T of W generated by T. Now W_T is also a Coxeter group with the same exponents as W, see [22, Theorem 4.1.6 (i), Theorem 3.4.2 (i)]. As in [22, Lemma 19.2.2] one sees that $C^*_{r,q}(W_T)$ embeds into $C^*_{r,q}(W)$ canonically. Under this identification we have $x \in C^*_{r,q}(W_T)$. But the map $T^{(q)}_{\mathbf{w}} \mapsto q^{1/2}_{\mathbf{w}}$ is a character on the finite-dimensional C*-algebra $C^*_{r,q}(W_T)$, hence $\|\chi(x)\| \leq \|x\|$. As this holds for every $x \in \mathbb{C}_q[W]$, χ extends to a character on $C^*_{r,q}(W)$. $\square$

5.2. Averaging operators

To show simplicity properties of Hecke C*-algebras coming from right-angled, irreducible Coxeter systems with at least three generators (see subsection 5.3), we will make use of a method inspired by Power's averaging argument in [47]. This requires the introduction of suitable averaging operators on the corresponding Hecke algebra which average over a finite subset of the Coxeter group. The statements of this subsection apply in greater generality. Therefore we will formulate our approach for arbitrary discrete groups.

Let G be a discrete group. Denote the left regular representation of G by λ, let τ be the canonical tracial state on $C^*_r(G)$ and fix some finite set $F \subseteq G$. Every such F defines an averaging operator Φ on $C^*_r(G)$ given by

$$\Phi(x) := \frac{1}{|F|} \sum_{g \in F} \lambda_{g^{-1}} x \lambda_g.$$

The map Φ is trace-preserving, unital and completely positive. In particular it induces a bounded operator $\tilde{\Phi}$ on $\ell^2(G) \ominus \mathbb{C}\delta_e$ given by $\tilde{\Phi}(x\delta_e) := \Phi(x)\delta_e$ for all $x \in \mathbb{C}[G]$ with $\tau(x) = 0$.

Proposition 5.5. *Let G be a discrete group with the property that there exist three elements $g_1, g_2, g_3 \in G$ and a subset $D \subseteq G \setminus \{e\}$ such that $D \cup g_1 D g_1^{-1} = G \setminus \{e\}$ and the sets D, $g_2 D g_2^{-1}$ and $g_3 D g_3^{-1}$ are pairwise disjoint. Take $F := \{e, g_1, g_2, g_3\}$. Then the operator $\tilde{\Phi}$ on $\ell^2(G) \ominus \mathbb{C}\delta_e$ has norm strictly smaller than one.*

The proof of Proposition 5.5 is based on Ching's following variation of Pukánszky's 14ε-argument in [48, Lemma 10].

Lemma 5.6. *([16, Lemma 4]) Let G be a group as in Proposition 5.5. Then*

$$\|x - \tau(x)\|_2 \leq 14 \max_{i=1,2,3} \|x - \lambda_{g_i^{-1}} x \lambda_{g_i}\|_2$$

for every $x \in C^*_r(G)$.

Further, we shall need that for arbitrary vectors $\xi_0, \dots, \xi_n$ in a Hilbert space $\mathcal{H}$ we have the following equality, which for $n = 1$ is known as the parallelogram law:

$$\|\sum_{i=0}^{n} \xi_i\|^2 + \sum_{0 \le i < j \le n} \|\xi_i - \xi_j\|^2 = (n+1) \sum_{i=0}^{n} \|\xi_i\|^2. \tag{5.2}$$

Note that (5.2) can be verified directly by writing out all norms as inner products.

Proof of Proposition 5.5. The norm of $\tilde{\Phi} : \ell^2(G) \ominus \mathbb{C}\delta_e \to \ell^2(G) \ominus \mathbb{C}\delta_e$ is clearly majorized by 1. Now suppose that $\|\tilde{\Phi}\| = 1$. Take a sequence $x_k \in \mathbb{C}[G]$ with $\tau(x_k) = 0$ and $\|x_k \delta_e\| = 1$ such that $\|\tilde{\Phi}(x_k \delta_e)\| \nearrow 1$. Set $\xi_i^k = \lambda_{g_i^{-1}} x_k \lambda_{g_i} \delta_e$ with g_1, g_2, g_3 as in the proposition and g_0 the identity. By (5.2) we have

$$\sum_{0 \le i < j \le 3} \|\xi_i^k - \xi_j^k\|^2 = 4 \sum_{i=0}^{3} \|\xi_i^k\|^2 - \|\sum_{i=0}^{3} \xi_i^k\|^2 = 4^2 - \|4\tilde{\Phi}(x_k \delta_e)\|^2 \to 4^2 - 4^2 = 0.$$

Therefore each of the individual summands on the left hand side converge to 0 as $k \to \infty$. Since $\xi_0^k = x_k \delta_e$ and $\|y\|_2 = \|y\delta_e\|, y \in \mathbb{C}[G]$ we see from Lemma 5.6 that $\|x_k\|_2 = \|x_k - \tau(x_k)\|_2 \to 0$. This contradicts that $\|x_k \delta_e\| = 1$. We conclude that $\|\tilde{\Phi}\| < 1$. □

5.3. Simplicity in the right-angled case for q close to 1

Let us now bring together the statements from Theorem 3.4 and Section 5.2.

Theorem 5.7. *Let (W, S) be a right-angled, irreducible Coxeter system with $3 \le |S| < \infty$. Then there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{R}_{>0}^{(W,S)}$ of $1 = (1_s)_{s \in S}$ such that for all $q \in \mathcal{U}$ the C^*-algebra $C^*_{r,q}(W)$ is simple and has unique tracial state.*

Proof. As (W, S) is irreducible with $|S| \ge 3$ we find elements $s, t_0, \dots, t_n \in S$ with $n \ge 1$, $S = \{s, t_0, \dots, t_n\}$, $m_{s,t_0} = \infty$, $t_1 \ne s$ and $m_{t_i, t_{i+1}} = \infty$ for $i = 0, \dots, n-1$. Then $\mathbf{w}_1 := t_0 \cdots t_n \cdots t_0$, $\mathbf{w}_2 := s$, $\mathbf{w}_3 := t_1$ and $D := \{\mathbf{w} \in W \mid |t_0 \mathbf{w}| < |\mathbf{w}|\}$ satisfy the conditions from Proposition 5.5. For every reduced expression $\mathbf{w} = s_1 \cdots s_m$ in W consider the operator

$$\prod_{i=1}^{m} \left(\frac{1 - q_{s_i}}{1 + q_{s_i}} + \frac{2\sqrt{q_{s_i}}}{1 + q_{s_i}} T_{s_i}^{(q)} \right) \in C^*_{r,q}(W).$$

By the same arguments as in the proof of Proposition 4.2 this operator is unitary and does not depend on the reduced expression for $\mathbf{w}$. By abuse of notation we will denote it by $\pi_{q,1}\left(T_{\mathbf{w}}^{(1)}\right)$. Choose a positive integer d with $|\mathbf{w}| \le d$ for all $\mathbf{w}$ in $F = \{e, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ and define a *deformed* averaging operator Φ_q on $C^*_{r,q}(W)$ by

$$\Phi_q(x) = \frac{1}{|F|} \sum_{\mathbf{w} \in F} \pi_{q,1}\left(T^{(1)}_{\mathbf{w}^{-1}}\right) x \pi_{q,1}\left(T^{(1)}_{\mathbf{w}}\right).$$

Again, these maps are trace-preserving, unital, completely positive and they induce contractive linear operators $\tilde{\Phi}_q$ on $\ell^2(W) \ominus \mathbb{C}\delta_e$ via $\tilde{\Phi}_q(x\delta_e) := \Phi_q(x)\delta_e$ for $x \in \mathbb{C}_q[W]$. One easily checks that $\|\tilde{\Phi}_q\| \to \|\tilde{\Phi}\|$ for $q \to 1$. In particular, Proposition 5.5 implies that there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{R}^{(W,S)}_{>0}$ of $1 = (1_s)_{s \in S}$ such that $\tilde{\Phi}_q$ has norm strictly smaller than $1 \in \mathbb{R}$ for all $q \in \mathcal{U}$.

Denote by χ_d the word length projection on $C^*_{r,q}(W)$ from equation (1.1) and put $\chi_{\leq d} := \sum_{r=0}^{d} \chi_r$. Let $l \geq 1$ and $x \in \chi_{\leq d}(\mathbb{C}_q[W])$. Note that for every $\mathbf{w} \in F$ we have $\chi_r(\pi_{q,1}(T^{(1)}_{\mathbf{w}})) = 0$ for $r \geq d$. As Φ_q^l averages over $\left\{\pi_{q,1}(T^{(1)}_{\mathbf{w}}) \mid \mathbf{w} \in F\right\}$ and $\chi_r(x) = 0$ for $r \geq d$, we get that $\chi_r(\Phi_q^l(x)) = 0$ for $r \geq (2l+1)d$. In particular

$$\|\Phi_q^l(x) - \tau_q(x)\| \leq \sum_{r=1}^{(2l+1)d} \|\chi_r\left(\Phi_q^l(x)\right)\|$$

by the triangle inequality and with $C := (\#\mathsf{Cliq}(\Gamma))^3(\prod_{s \in S} p_s(q))$ we have

$$\|\Phi_q^l(x) - \tau_q(x)\| \leq \sum_{r=1}^{(2l+1)d} Cr\|\chi_r\left(\Phi_q^l(x)\right)\|_2$$

by Theorem 3.4. The Cauchy-Schwarz inequality then implies

$$\begin{aligned}
\|\Phi_q^l(x) - \tau_q(x)\| &\leq ((2l+1)d)^{\frac{1}{2}} \left(\sum_{r=1}^{(2l+1)d} C^2 r^2 \|\chi_r\left(\Phi_q^l(x)\right)\|_2^2\right)^{\frac{1}{2}} \\
&\leq C((2l+1)d)^{\frac{3}{2}} \left(\sum_{r=1}^{(2l+1)d} \|\chi_r\left(\Phi_q^l(x)\right)\|_2^2\right)^{\frac{1}{2}} \\
&= C((2l+1)d)^{\frac{3}{2}} \|\Phi_q^l(x) - \tau_q \circ \Phi_q^l(x)\|_2 \\
&\leq C((2l+1)d)^{\frac{3}{2}} \|\tilde{\Phi}_q\|^l \|x - \tau_q(x)\|_2.
\end{aligned}$$

For $q \in \mathcal{U}$ this converges to 0 as $l \to \infty$. The simplicity and the unique trace property of $C^*_{r,q}(W)$, $q \in \mathcal{U}$ now follow by a standard argument (see for instance [47]): Let I be a non-zero ideal in $C^*_{r,q}(W)$. Choose $0 \neq x \in I$ positive. For every $\varepsilon > 0$ we find $x_\varepsilon \in \mathbb{C}_q[W]$ with $\|x - x_\varepsilon\| < \varepsilon/3$. For l large enough this implies

$$\begin{aligned}
\|\Phi_q^l(x) - \tau_q(x)\| &\leq \|\Phi_q^l(x - x_\varepsilon)\| + \|\Phi_q^l(x_\varepsilon) - \tau_q(x_\varepsilon)\| + \|\tau_q(x_\varepsilon) - \tau_q(x)\| \\
&< \varepsilon, \qquad (5.3)
\end{aligned}$$

so $0 \neq \tau_q(x) \in I$. Hence I must be trivial and this shows that $C^*_{r,q}(W)$ is simple if $q \in \mathcal{U}$. Further, as $\Phi^l_q(x) \to \tau_q(x)$ by (5.3) and $\tau'\left(\Phi^l_q(x)\right) = \tau'(x)$ for every tracial state τ', $C^*_{r,q}(W)$ has τ_q as its unique tracial state. □

Remark 5.8. *(a)* Theorem 5.7 immediately implies that for $q \in \mathcal{U}$ the Hecke-von Neumann algebra $\mathcal{N}_q(W)$ is a II_1-factor by uniqueness of the trace and Lemma 1.1. This extends results by Garncarek [31] to the multi-parameter case; note that the proof from [31] does not trivially extend to the multi-parameter case. *Note:* After completion of this paper the factoriality question in the right-angled multi-parameter case was fully solved by Raum and Skalski in [45].
(b) Theorem 5.7 does not give any information about the simplicity of Hecke C*-algebras of infinitely generated Coxeter groups unless q equals 1.

5.4. Free products of Abelian Coxeter groups

In this section we want to investigate the maximum size of the open neighborhood $\mathcal{U} \subseteq \mathbb{R}^{(W,S)}_{>0}$ in Theorem 5.7. Though our aim is different, the approach is inspired by [31, Section 6] and translates the remarks made there into the C*-algebraic setting by using results from [28].

Let us assume that (W,S) is a Coxeter system where W is of the form $W = \mathbb{Z}_2^{k_1} * \cdots * \mathbb{Z}_2^{k_l}$ with $l, k_1 \geq 2$ and $k_2, \ldots, k_l \in \mathbb{N}_{\geq 1}$. For each $1 \leq m \leq l$ denote by $s_1^{(m)}, \ldots, s_{k_m}^{(m)}$ the mutually commuting generators corresponding to the component $\mathbb{Z}_2^{k_m}$ of W and set $S_m := \left\{s_1^{(m)}, \ldots, s_{k_m}^{(m)}\right\}$, so in particular $S = \bigcup_{m=1}^l S_m$. Let $q \in \mathbb{R}^{(W,S)}_{>0}$. The Hecke C*-algebra $C^*_{r,q}(W)$ decomposes as a reduced free product

$$\left(C^*_{r,q}(W), \tau_q\right) \simeq *_{m=1}^l \left(\bigotimes_{i=1}^{k_m} \left(C^*_{r,q}\left(W_{s_i^{(m)}}\right), \tau_q\right)\right)$$

over the canonical traces. In the notation of [28] we get using [31, Lemma 6.2] that

$$\left(C^*_{r,q}(W), \tau_q\right) \simeq *_{m=1}^l \left(C\left(\mathbb{Z}_2^{k_m}\right), \int (\cdot)\, \mu_m\right), \tag{5.4}$$

where $\mu_m(\mathbf{w}) = q_{\mathbf{w}} \prod_{i=1}^{k_m} \left(1 + q_{s_i^{(m)}}\right)^{-1}$ for $\mathbf{w} \in \mathbb{Z}_2^{k_m}$.

Proposition 5.9. *Let (W,S) be a Coxeter system of the form $W = \mathbb{Z}_2^{k_1} * \cdots * \mathbb{Z}_2^{k_l}$ where $l, k_1, \ldots, k_l \in \mathbb{N}_{\geq 1}$ and $q \in \mathbb{R}^{(W,S)}_{>0}$. Then the growth series $W(z)$ of W is equal to the Taylor expansion of the multivariate function*

$$z \mapsto \left(\sum_{m=1}^l \prod_{i=1}^{k_m} \left(1 + z_{s_i^{(m)}}\right)^{-1} - (l-1)\right)^{-1}. \tag{5.5}$$

The series converges for every element in

$$\Omega := \left\{ (z_s)_{s\in S} \in \mathbb{C}^{(W,S)} \cap [0,1]^S \mid \sum_{m=1}^{l} \prod_{i=1}^{k_m} \left(1 + z_{s_i^{(m)}}\right)^{-1} > l-1 \right\},$$

i.e. $\Omega \subseteq \mathcal{R}$ *where* $\mathcal{R}$ *is the region of convergence of* $W(z)$.

Proof. It is clear that the multivariate function $z \mapsto \sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ converges absolutely on a domain $\mathcal{U}$ around 0 and that it defines a holomorphic function there. Put $\mathcal{A}_m := \left\{ \mathbf{w} \in W \mid \mathbf{w} \text{ starts in } \mathbb{Z}_2^{k_m} \right\}$ where we assume by convention that $e \in \mathcal{A}_m$. For every $z \in \mathcal{U}$ and $m = 1, \ldots, l$ we have

$$\begin{aligned}
W(z) &= 1 + \sum_{\mathbf{w}\in\mathcal{A}_m\setminus\{e\}} z_{\mathbf{w}} + \sum_{\mathbf{w}\in\mathcal{A}_m^c} z_{\mathbf{w}} \\
&= 1 + \sum_{\mathbf{v}\in\mathbb{Z}_2^{k_m}\setminus\{e\}} z_{\mathbf{v}} + \sum_{\mathbf{v}\in\mathbb{Z}_2^{k_m}\setminus\{e\}} z_{\mathbf{v}} \sum_{\mathbf{w}\in\mathcal{A}_m^c} z_{\mathbf{w}} + \sum_{\mathbf{w}\in\mathcal{A}_m^c} z_{\mathbf{w}} \\
&= \left(\sum_{\mathbf{v}\in\mathbb{Z}_2^{k_m}} z_{\mathbf{v}} \right) \left(1 + \sum_{\mathbf{w}\in\mathcal{A}_m^c} z_{\mathbf{w}} \right) \\
&= \left(1 + \sum_{\mathbf{w}\in\mathcal{A}_m^c} z_{\mathbf{w}} \right) \prod_{i=1}^{k_m} \left(1 + z_{s_i^{(m)}}\right).
\end{aligned}$$

We get, by using this in the third line of the next equalities, that

$$\begin{aligned}
W(z) &= 1 + \sum_{m=1}^{l} \sum_{\mathbf{w}\in\mathcal{A}_m\setminus\{e\}} z_{\mathbf{w}} \\
&= 1 + \sum_{m=1}^{l} \left(W(z) - 1 - \sum_{\mathbf{w}\in\mathcal{A}_m^c} z_{\mathbf{w}} \right) \\
&= 1 + \sum_{m=1}^{l} \left(1 - \prod_{i=1}^{k_m} \left(1 + z_{s_i^{(m)}}\right)^{-1} \right) W(z)
\end{aligned}$$

and hence

$$W(z) = \left(\sum_{m=1}^{l} \prod_{i=1}^{k_m} \left(1 + z_{s_i^{(m)}}\right)^{-1} - (l-1) \right)^{-1}. \tag{5.6}$$

This implies that the growth series $W(z)$ of W is equal to the Taylor expansion of (5.5) in 0. From (5.6) we also find that

$$W(z) = (1 - Q(z))^{-1} \prod_{s \in S} (1 + z_s) \tag{5.7}$$

with the polynomial

$$Q(z) := 1 + (l-1) \prod_{s \in S} (1 + z_s) - \sum_{m=1}^{l} \prod_{n \neq m} \prod_{j=1}^{k_n} \left(1 + z_{s_j^{(n)}}\right).$$

One has $Q(0) = 0$. With $D^\alpha Q$ denoting the higher order partial derivative of Q with respect to $0 \neq \alpha \in \{0,1\}^S$ we have

$$\begin{aligned} D^\alpha Q|_{z=0} &= \left[(l-1) \prod_{s \in S:\ \alpha_s = 0} (1 + z_s) - \sum_{m \in J} \prod_{n \neq m} \prod_{j \in K_n} \left(1 + z_{s_j^{(n)}}\right) \right]_{z=0} \\ &= (l-1) - \sum_{m \in J} 1 \\ &\geq 0, \end{aligned}$$

where $J := \{m \mid 1 \leq m \leq l \text{ with } \alpha_s = 0 \text{ for all } s \in S_m\}$ and

$$K_n := \left\{ j \mid 1 \leq j \leq k_m \text{ with } \alpha_{s_j^{(n)}} = 0 \right\}.$$

This implies that Q has only positive coefficients. Hence for $z \in \Omega$ we have $Q(z) \geq 0$. Moreover, for $z \in \Omega$ we have $W(z) > 0$ and so by (5.7) we have $0 \leq Q(z) < 1$.

Further, for any $z \in \mathbb{C}^{(W,S)}$ with $0 \leq Q(z) < 1$ we can expand and rearrange the terms in the series $\sum_{m=0}^{\infty} (Q(z))^m$ to get the (converging) Taylor series of $(1 - Q(z))^{-1}$. The same is true for the product $(1 - Q(z))^{-1} \prod_{s \in S} (1 + z_s)$. Combining this with the previous paragraph we get that the Taylor series of the function in (5.5) (which is the growth series $W(z)$) converges on Ω. $\square$

Taking into account [28, Corollary 4.10], Proposition 5.9 implies the following.

Theorem 5.10. *Let (W,S) be a Coxeter system of the form $W = \mathbb{Z}_2^{k_1} * \cdots * \mathbb{Z}_2^{k_l}$ where $l, k_1 \geq 2$ and let $q \in \mathbb{R}_{>0}^{(W,S)}$. Then the following statements are equivalent.*

(1) $C_{r,q}^*(W)$ is simple and has unique tracial state;
(2) $q \notin \overline{\mathcal{R}'}$, where $\mathcal{R}'$ is defined in (5.1).

Proof. (1) $\Rightarrow$ (2): Assume that $q \in \overline{\mathcal{R}'}$. Then $C_{r,q}^*(W)$ is not simple and does not have unique tracial state by Lemma 5.3.

(2) $\Rightarrow$ (1): Assume that $C_{r,q}^*(W)$ is not simple and does not have unique tracial state. Assume further that $0 < q_s \leq 1$ for every $s \in S$. Using [28, Proposition 4.10] in combination with (5.4) we get that the set

$$\left\{(\mathbf{w}_1, ..., \mathbf{w}_l) \in \prod_{m=1}^{l} \mathbb{Z}_2^{k_m} \mid l-1 \leq \sum_{m=1}^{l} q_{\mathbf{w}_m} \prod_{i=1}^{k_m} \left(1 + q_{s_i^{(m)}}\right)^{-1}\right\}$$

is not empty, so in particular

$$l-1 \leq \max_{(\mathbf{w}_1,...,\mathbf{w}_l) \in \prod_{m=1}^{l} \mathbb{Z}_2^{k_m}} \left(\sum_{m=1}^{l} q_{\mathbf{w}_m} \prod_{i=1}^{k_m} \left(1 + q_{s_i^{(m)}}\right)^{-1}\right) \leq \sum_{m=1}^{l} \prod_{i=1}^{k_m} \left(1 + q_{s_i^{(m)}}\right)^{-1}.$$

Comparing this with Proposition 5.9, we get that $q \in \overline{\mathcal{R}'}$. The general case of $q \in \mathbb{R}_{>0}^{(W,S)}$ follows by an application of Proposition 4.7. $\square$

Remark 5.11. Proposition 5.9 also implies that the reasoning in [31, Section 6] applies to the multi-parameter Hecke-von Neumann algebra $\mathcal{N}_q(W)$ with W as in Theorem 5.10. In other words, $\mathcal{N}_q(W)$ is a factor if and only if $q \notin \mathcal{R}'$. *Note:* After completion of this paper this result was extended to arbitrary irreducible right-angled Hecke von Neumann algebras with $|S| \geq 3$ in [45].

Remark 5.12. In the proof of Theorem 5.10 we only used the simplicity part of [28, Corollary 4.10]. For W as above the full statement of the corollary also provides a detailed description of the ideal structure of $C^*_{r,q}(W)$ for $q \in \overline{\mathcal{R}'}$. Further we conclude by [28, Corollary 4.10] that for $l \geq 3$ the Hecke C*-algebra $C^*_{r,q}(W)$ has stable rank 1 for every q.

In view of Lemma 5.3, Proposition 4.7, Theorem 5.7 and Theorem 5.10 it is a natural question whether or not the characterization above holds for general Coxeter systems.

Question 5.13. Let (W, S) be an irreducible Coxeter system and $q \in \mathbb{R}_{>0}^{(W,S)}$. Is it true that the Hecke C*-algebra $C^*_{r,q}(W)$ is simple and has unique tracial state if and only if $q \notin \overline{\mathcal{R}'}$ where $\mathcal{R}'$ is defined as in (5.1)?

Remark 5.14. Recently a new approach to C*-simplicity results was obtained through Furstenberg/Hamana boundaries, see [37], [9], [4]. In particular, in [4] the Furstenberg boundary of a general unitary representation of a discrete group was defined and investigated in relation to trace-uniqueness properties. Proposition 4.2 shows that Hecke C*-algebras of a right-angled Coxeter group are C*-algebras generated by such a unitary representation. It would be interesting to exploit this connection. However, in light of the results from [4], it is not clear how manageable the Furstenberg-Hamana boundary is.

6. Exactness and nuclearity of Hecke C*-algebras

The aim of this last section is to have a look at additional properties of Hecke C*-algebras such as exactness and nuclearity and give (counter-)examples regarding the extension of the isomorphism $\pi_{q,1}$ in Proposition 4.2 to the C*-algebraic level.

Theorem 6.1. *Let (W, S) be a Coxeter system and $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$. Then $C^*_{r,q}(W)$ is an exact C*-algebra.*

Proof. By [25] (see also [1]) the group W acts amenably on a compact space. Therefore, by [10, Theorem 5.1.7] W is exact. With P_s as defined in (1.6), we have $T_s^{(q)} = T_s^{(1)} + p_s(q)P_s$ for $s \in S$. The Hecke C*-algebra $C^*_{r,q}(W)$ is generated by $\left\{T_s^{(q)}\right\}_{s\in S}$ and we have $P_s \in \ell^\infty(W) \subseteq \mathcal{B}\left(\ell^2(W)\right)$. Hence $C^*_{r,q}(W)$ is contained in the uniform Roe algebra of W which is nuclear by [10, Theorem 5.1.6]. That implies the exactness of $C^*_{r,q}(W)$. □

The following theorem generalizes [12, Theorem 3.6].

Theorem 6.2. *Let (W, S) be an irreducible Coxeter system and $q \in \mathbb{R}_{>0}^{(W,S)}$. Then the following statements are equivalent:*

(1) (W, S) is of spherical or affine type;
(2) $C^*_{r,q}(W)$ is nuclear for all q;
(3) $C^*_{r,q}(W)$ is nuclear for some q;
(4) $\mathcal{N}_q(W)$ is injective for all q;
(5) $\mathcal{N}_q(W)$ is injective for some q.

Proof. (1) ⇒ (2): If (W, S) is of spherical type, then $C^*_{r,q}(W)$ is finite-dimensional or an inductive limit over finite-dimensional C*-algebras, and in either case nuclear. So let us assume that (W, S) is of affine type. It is well-known that affine Coxeter systems correspond to affine Weyl groups, see for example [7, Chapter VI]. From the discussion in [39, §4] it follows that the Hecke C*-algebra $C^*_{r,q}(W)$ must be of type I. In particular it is nuclear, see [46].

(2) ⇒ (3): Clear.

(3) ⇒ (5): If $C^*_{r,q}(W)$ is nuclear for some q, then the bicommutant $\mathcal{N}_q(W) = \left(C^*_{r,q}(W)\right)''$ must be injective [10, Exercise 3.6.4, Corollary 3.8.6 and Theorem 9.3.3].

(5) ⇒ (1): Let (W, S) be an irreducible Coxeter system of non-affine type. By [22, Proposition 17.2.1], W contains the free group on two generators. Denote the corresponding generators by $\mathbf{a}_1$, $\mathbf{a}_2$ and let $\mathcal{M}$ be the von Neumann subalgebra of $\mathcal{N}_q(W)$ generated by $T_{\mathbf{a}_1}^{(q)}$ and $T_{\mathbf{a}_2}^{(q)}$. Let further $\mathcal{M}_1$ be the von Neumann algebra generated by $T_{\mathbf{a}_1}^{(q)}$ and $\mathcal{M}_2$ the one generated by $T_{\mathbf{a}_2}^{(q)}$. Then $\mathcal{M}$ is isomorphic to the free product $(\mathcal{M}_1, \tau_{q,1}) * (\mathcal{M}_2, \tau_{q,2})$ over the canonical traces. The dimensions of $\mathcal{M}_1$, $\mathcal{M}_2$ are infinite, so $\mathcal{M}$ is non-injective by [52, Theorem 4.1 and Remark 4.2 (5)]. But there exists a

trace-preserving normal conditional expectation $E\colon \mathcal{N}_q(W) \twoheadrightarrow \mathcal{M}$ (see for example [10, Lemma 1.5.11] or [12, Corollary 3.3]), so $\mathcal{N}_q(W)$ must be non-injective as well.

(2) $\Leftrightarrow$ (4): Clear from the arguments above. $\square$

We have seen that for spherical type Coxeter systems the corresponding Hecke C*-algebras are independent of the choice of q. Further, if the Coxeter system (W, S) is right-angled and amenable, then the map $\pi_{q,1}$ from Proposition 4.2 extends to a surjection $C_r^*(W) \twoheadrightarrow C_{r,q}^*(W)$ for every q. It is a natural question whether or not this map is an isomorphism as well. This is not always the case as the following example illustrates.

Example 6.3. Let (W, S) be the Coxeter system generated by two elements s, t with $m_{s,s} = m_{t,t} = 2$, $m_{s,t} = \infty$. This is the infinite dihedral group which is the only irreducible right-angled Coxeter group of affine type. Let $C_{r,0}^*(W) \subseteq \mathcal{B}\left(\ell^2(W)\right)$ be the unital C*-algebra generated by P_s and P_t defined in (1.6). Assume that $\pi_{q,1}$ of Proposition 4.2 extends to a $*$-isomorphism for every $q \in \mathbb{R}_{>0}^{(W,S)}$. Then the map $P_u \mapsto (1 - T_u^{(1)})/2$, $u \in S$ extends to a $*$-isomorphism $\pi_{0,1}\colon C_{r,0}^*(W) \simeq C_r^*(W)$ as well. Indeed, using Proposition 4.2 one checks that as $q \downarrow 0$ we have

$$\pi_{q,1}^{-1}(x) \to \sum_{\mathbf{w} \in W} x(\mathbf{w})\, \pi_{0,1}^{-1}\left(T_{\mathbf{w}}^{(1)}\right) = \pi_{0,1}^{-1}(x)$$

in $\mathcal{B}\left(\ell^2(W)\right)$ for every $x := \sum_{\mathbf{w} \in W} x(\mathbf{w})\, T_{\mathbf{w}}^{(1)} \in \mathbb{C}_1[W]$. Hence,

$$\|\pi_{0,1}^{-1}(x)\| = \lim_{q \downarrow 0} \|\pi_{q,1}^{-1}(x)\| = \|x\|$$

as $\pi_{q,1}$ is isometric. Since the C*-algebra $C_{r,0}^*(W)$ is commutative, we have reached a contradiction.

The example illustrates that $\pi_{q,1}$ in general does not extend to an isomorphism of (reduced) C*-algebras. In the non-affine case it gets even worse. Here $\pi_{q,1}$ and $\pi_{q,1}^{-1}$ never extend to the reduced C*-algebra level for arbitrary choice of q.

Example 6.4. Let (W, S) be an irreducible right-angled Coxeter system of non-affine type. By Theorem 5.7 the reduced group C*-algebra $C_r^*(W)$ has unique tracial state. The map $\pi_{q,1}$ is not trace preserving with respect to the canonical trace. Hence it does not extend to the reduced C*-algebraic level. To see that $\pi_{q,1}^{-1}$ does in general not extend, one can use the same argument as in Example 6.3 or the trace-uniqueness in Theorem 5.7.

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