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Integer packing sets form a well-quasi-ordering

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Abstract

An integer packing set is a set of non-negative integer vectors with the property that, if a vector x is in the set, then every non-negative integer vector y with $y \leq x$ is in the set as well. The main result of this paper is that integer packing sets, ordered by inclusion, form a well-quasi-ordering. This result allows us to answer a recently posed question: the k-aggregation closure of any packing polyhedron is again a packing polyhedron.

Key words: Well-quasi-ordering; k-aggregation closure; polyhedrality; packing polyhedra.

1 Introduction

In order theory, a *quasi-order* is a binary relation $\preceq$ over a set X that is *reflexive:* $\forall a \in X$, $a \preceq a$, and *transitive:* $\forall a, b, c \in X$, $a \preceq b$ and $b \preceq c$ imply $a \preceq c$. A quasi-order $\preceq$ is a *well-quasi-order* (*wqo*) if for any infinite sequence $x_1, x_2, \ldots$ of elements from X there are indices $i < j$ such that $x_i \preceq x_j$.

A classic example of a quasi-order over the set of graphs is given by the graph minor relation. The Robertson-Seymour Theorem (also known as the graph minor theorem) essentially states that the set of finite graphs is well-quasi-ordered by the graph minor relation. This fundamental result is the culmination of twenty papers written as part of the Graph Minors Project [14]. Interested readers may find more examples and characterizations in the comprehensive survey paper by Kruskal [11]. The main result of this paper is that a quasi-order arising from Integer Optimization is a well-quasi-order.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers and let $[n] = \{1, 2, \ldots, n\}$ for any $n \in \mathbb{N}$. We define an *integer packing set* in $\mathbb{R}^n$ as a subset Q of $\mathbb{N}^n$ with the property that: if $x \in Q$, $y \in \mathbb{N}^n$ and $y \leq x$, then $y \in Q$. Note that the relation $\subseteq$ is a quasi-order over the set of integer packing sets. We are now ready to state our main result.

Theorem 1. *The set of integer packing sets in $\mathbb{R}^n$ is well-quasi-ordered by the relation $\subseteq$.*

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Integer packing sets appear naturally in Integer Optimization. A *packing polyhedron* is a set of the form $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geq 0\}$ where the data $A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^m$ is non-negative. Clearly, for any packing polyhedron P, the set $P \cap \mathbb{Z}^n$, is an integer packing set. However, note that not all integer packing sets are of this form. This connection between packing polyhedra and integer packing sets allows us to employ Theorem 1 to answer a recently posed open question in Integer Optimization.

In [3], the authors introduce the concept of k-aggregation closure for packing and covering polyhedra. Given a packing polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geq 0\}$, and a positive integer k, the *k-aggregation closure* of P is defined by

$$\mathcal{A}_k(P) := \bigcap_{\lambda^1,\ldots,\lambda^k \in \mathbb{R}^m_+} \operatorname{conv}(\{x \in \mathbb{N}^n \mid (\lambda^j)^\mathsf{T} Ax \leq (\lambda^j)^\mathsf{T} b,\ \forall j \in [k]\}).$$

The set $\mathcal{A}_k(P)$ is defined as the intersection of an infinite number of sets, each of which is the convex hull of an integer packing set. A natural question, posed in [3], is whether the set $\mathcal{A}_k(P)$ is polyhedral. The authors provide a partial answer to this question by showing that $\mathcal{A}_k(P)$ is a polyhedron, provided that every entry of A is positive. As a consequence of Theorem 1, we give a complete answer to the posed question.

Theorem 2. *For any packing polyhedron P and any $k \geq 1$, the set $\mathcal{A}_k(P)$ is a packing polyhedron.*

The generality of our proof techniques allows us to provide a generalization of Theorem 2 to the setting where the given set is a downset of $\mathbb{R}^n_+$ instead of a polyhedron. We recall that a *downset* of $\mathbb{R}^n_+$ is a subset D of $\mathbb{R}^n_+$ with the property that, if $x \in D$, $y \in \mathbb{R}^n_+$ and $y \leq x$, then $y \in D$. Clearly, a packing polyhedron in $\mathbb{R}^n$ is a downset of $\mathbb{R}^n_+$, but not all downsets are polyhedral. Our generalization relies on a natural extension of the definition of k-aggregation closure to downsets of $\mathbb{R}^n_+$. For any downset D of $\mathbb{R}^n_+$, we denote by

$$\Lambda(D) := \{f \in \mathbb{R}^n \mid \sup\{f^\mathsf{T} x \mid x \in D\} < \infty\}.$$

In particular, note that $f^\mathsf{T} x \leq \beta$ is valid for D if and only if $f \in \Lambda(D)$ and $\beta \geq \sup\{f^\mathsf{T} x \mid x \in D\}$. Then, the *$k$-aggregation closure* of D is defined by

$$\tilde{\mathcal{A}}_k(D) := \bigcap_{f^1,\ldots,f^k \in \Lambda(D)} \operatorname{conv}(\{x \in \mathbb{N}^n \mid (f^j)^\mathsf{T} x \leq \sup\{(f^j)^\mathsf{T} d \mid d \in D\},\ \forall j \in [k]\}).$$

The next observation shows that $\tilde{\mathcal{A}}_k$ is indeed a generalization of $\mathcal{A}_k$.

Observation 1. *For any packing polyhedron P and any $k \geq 1$, we have $\tilde{\mathcal{A}}_k(P) = \mathcal{A}_k(P)$.*

Proof. Let $P = \{x \in \mathbb{R}^n_+ \mid Ax \leq b,\ x \geq 0\}$ be a packing polyhedron in $\mathbb{R}^n$. It is simple to show that $\tilde{\mathcal{A}}_k(P) \subseteq \mathcal{A}_k(P)$. To see this, consider an inequality $\lambda^\mathsf{T} Ax \leq \lambda^\mathsf{T} b$, for $\lambda \in \mathbb{R}^m_+$, in the definition of $\mathcal{A}_k(P)$. Then $\lambda^\mathsf{T} Ax \leq \lambda^\mathsf{T} b$ is valid for P. Thus $\lambda^\mathsf{T} A \in \Lambda(D)$, and $\sup\{\lambda^\mathsf{T} Ad \mid d \in P\} \leq \lambda^\mathsf{T} b$. Hence, the inequality $\lambda^\mathsf{T} Ax \leq \sup\{\lambda^\mathsf{T} Ad \mid d \in P\}$ in the definition of $\tilde{\mathcal{A}}_k(P)$ implies the original inequality $\lambda^\mathsf{T} Ax \leq \lambda^\mathsf{T} b$.

Next, we show $\tilde{\mathcal{A}}_k(P) \supseteq \mathcal{A}_k(P)$. Consider an inequality $f^\mathsf{T} x \leq \sup\{f^\mathsf{T} d \mid d \in P\}$, for $f \in \Lambda(P)$, in the definition of $\tilde{\mathcal{A}}_k(P)$. This inequality is valid for P. From Farkas' lemma we know that there exist some $\lambda \in \mathbb{R}^m_+$ and $\gamma \in \mathbb{R}^n_+$ such that $\lambda^\mathsf{T} A - \gamma^\mathsf{T} I = f^\mathsf{T}$ and $\lambda^\mathsf{T} b \leq \sup\{f^\mathsf{T} d \mid$

$d \in P\}$. Note that the inequality $\lambda^\mathsf{T} Ax \leq \lambda^\mathsf{T} b$ is valid for P. Furthermore, it dominates the inequality $f^\mathsf{T} x \leq \sup\{f^\mathsf{T} d \mid d \in P\}$ in the nonnegative orthant. This is because, whenever $x \geq 0$,

$$f^\mathsf{T} x = \lambda^\mathsf{T} Ax - \gamma^\mathsf{T} Ix \leq \lambda^\mathsf{T} Ax \leq \lambda^\mathsf{T} b \leq \sup\{f^\mathsf{T} d \mid d \in P\}.$$

We have shown $\tilde{\mathcal{A}}_k(P) \supseteq \mathcal{A}_k(P)$, which completes the proof of the observation. □

We now state our generalizations of Theorem 2 to downsets of $\mathbb{R}^n_+$.

Theorem 3. *For any downset D of $\mathbb{R}^n_+$ and any $k \geq 1$, the set $\tilde{\mathcal{A}}_k(D)$ is a packing polyhedron.*

In the special case $k = 1$, the k-aggregation closure is also known as the *aggregation closure*. In the recent unpublished manuscript [13], the authors independently show that the aggregation closure of a packing or covering rational polyhedron P is polyhedral. The main differences with our Theorem 3 are the following: (i) The result in [13] holds for both packing and covering polyhedra, while our Theorem 2 only deals with the packing case; (ii) The result in [13] requires the given set to be a polyhedron, while in our case the given set can be a general downset of $\mathbb{R}^n_+$; (iii) The proof in [13] is direct, while our Theorem 2 is a consequence of Theorem 1; (iv) In [13] the authors only discuss in detail the aggregation closure, and claim that an analogous proof can be obtained for the k-aggregation closure, while in this paper we directly consider the k-aggregation closure.

Our techniques also allow us to obtain the following result.

Theorem 4. *For any closed convex downset D of $\mathbb{R}^n_+$, the set* $\operatorname{conv}(D \cap \mathbb{Z}^n)$ *is a packing polyhedron.*

Our work sheds light onto the connection between Order Theory and polyhedrality of closures in Integer Optimization. Only few papers so far have explored this connection. In [2], Averkov exploits the Gordan-Dickson lemma to show the polyhedrality of the closure of a rational polyhedron obtained via disjunctive cuts from a family of lattice-free rational polyhedra with bounded max-facet-width. In the paper [7], Dash et al. consider fairly well-ordered qosets to extend the result of Averkov. In particular, the authors prove the polyhedrality of the closure of a rational polyhedron with respect to any family of t-branch sets, where each set is the union of t polyhedral sets that have bounded max-facet-width. Other recent polyhedrality results in Integer Optimization include [1, 4, 9, 6, 5].

The organization of this paper is as follows: In Section 2 we present some preliminaries and notations from Order Theory that will be used in our proofs. In Section 3 we show Theorem 1, while in Section 4 we provide a proof of Theorem 2. In Section 5 we turn our attention to non-polyhedral sets and prove Theorem 3 and Theorem 4.

2 Preliminaries in Order Theory

Recall that a *quasi-order* is a binary relation $\preceq$ over a set X that is reflexive and transitive. If $a \preceq b$, we also write $b \succeq a$. If $a \preceq b$ or $b \preceq a$, the elements a and b are said to be *comparable*. If both $a \preceq b$ and $b \preceq a$, then we write $a \sim b$ (which is an equivalence relation). A sequence $x_1, x_2, \dots$ of elements from X is said to be *increasing* if $x_1 \preceq x_2 \preceq \dots$ and *decreasing* if $x_1 \succeq x_2 \succeq \dots$.

Most quasi-orders in this paper will in fact be *partial orders*, that is, they are *antisymmetric*: $a \preceq b$ and $b \preceq a$ imply $a = b$. In particular, we will consider the subset relation on $\mathbb{R}^n$ (and the induced partial order on integer packing sets), and the partial order on $(\mathbb{N}^n, \leq)$ given by the component-wise comparison: $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in [n]$.

A quasi-order $(X, \preceq)$ is a *well-quasi-order* (*wqo*) if for any infinite sequence of elements $x_1, x_2, \ldots$ from X there are indices $i < j$ such that $x_i \preceq x_j$. A quasi-order $(X, \preceq)$ is said to have the *finite basis property* if for all $X' \subseteq X$, there exists a finite subset $B \subseteq X'$ such that for every $x \in X'$ there is a $b \in B$ such that $b \preceq x$. The next result provides us with characterizations of well-quasi-orders.

Lemma 1 ([10, Theorem 2.1])**.** *Let $(X, \preceq)$ be a quasi-order. The following statements are equivalent:*

(i) *$(X, \preceq)$ is a wqo;*

(ii) *$(X, \preceq)$ has the finite basis property;*

(iii) *every infinite sequence of elements from X has an infinite increasing subsequence.*

Given two quasi-orders $(X_1, \preceq_1)$ and $(X_2, \preceq_2)$, the *product quasi-order* is $(X_1 \times X_2, \preceq)$ where $(x_1, x_2) \preceq (y_1, y_2)$ if and only if $x_1 \preceq_1 y_1$ and $x_2 \preceq_2 y_2$.

Lemma 2. *Let $(X_1, \preceq_1)$ and $(X_2, \preceq_2)$ be wqo's. Then the product quasi-order is a wqo.*

The proof of this well-known fact follows easily from the equivalence of (i) and (iii) in Lemma 1: given an infinite sequence of elements in $X_1 \times X_2$, we can find an infinite subsequence for which the components in X_1 form an increasing sequence, and then a further subsequence in which also the components in X_2 form an increasing sequence. The resulting subsequence is an increasing subsequence in $X_1 \times X_2$.

Since $(\mathbb{N}, \leq)$ is a wqo, the lemma implies that for any positive n the set $\mathbb{N}^n$ is a wqo under the usual component-wise comparison.

Lemma 3 (Gordan-Dickson, [8])**.** *The poset $(\mathbb{N}^n, \leq)$ is a wqo.*

Given a quasi order $(X, \preceq)$ we denote by X^* the set of all finite sequences of elements from X. We define a quasi order $\preceq^*$ on X^* by setting $(x_1, \ldots, x_n) \preceq^* (y_1, \ldots, y_m)$ if and only if there is a strictly increasing function $f : [n] \to [m]$ such that $x_i \preceq y_{f(i)}$ for all $i \in [n]$ (in particular, we require $n \leq m$). In this paper we will need the following generalization of the Gordon-Dickson lemma.

Lemma 4 (Higman's lemma, [10])**.** *Let $(X, \preceq)$ be a wqo. Then $(X^*, \preceq^*)$ is a wqo as well.*

3 Integer packing sets are well-quasi-ordered

In this section we prove our main result that integer packing sets in $\mathbb{R}^n$ form a wqo under inclusion. The proof is based on the following lemma.

Lemma 5. *Let $(X, \preceq)$ be a wqo. Define X^{**} to be the set of decreasing sequences in X:*

$$X^{**} = \{(x_0, x_1, \ldots) \in X^{\mathbb{N}} : x_0 \succeq x_1 \succeq \cdots\}.$$

For $x, y \in X^{**}$ set $x \preceq^{**} y$ if $x_i \preceq y_i$ for all $i \in \mathbb{N}$. Then $(X^{**}, \leq^{**})$ is a wqo.

Proof. We start with the following claim.

Claim 1. *Let $x \in X^{**}$. There is a $k \in \mathbb{N}$ such that $x_k \sim x_\ell$ for all $\ell \geq k$.*

Proof of claim. Since X is a wqo, it follows by Lemma 1 that there is a finite subset $I \subseteq \mathbb{N}$ of indices such that for any $\ell \in \mathbb{N}$ there is an $i \in I$ with $x_\ell \succeq x_i$. Let k be the largest index in I. Consider any $\ell \geq k$. Since $x_0, x_1, \ldots$ is decreasing, we have $x_\ell \preceq x_k$, but also $x_\ell \succeq x_i \succeq x_k$ for some $i \in I$. Hence, $x_\ell \sim x_k$. ◇

We call the smallest k as in the claim the *tail* of x. By Higman's lemma, it follows that the product quasi-order $\preceq'$ on $X^* \times X$ is a wqo. Let $\phi : X^{**} \to X^* \times X$ be defined by $\phi(x) = ((x_0, \ldots, x_{k-1}), x_k)$, where k is the tail of x. Let $x, y \in X^{**}$ and suppose that $\phi(x) \preceq' \phi(y)$. To complete the proof, it suffices to show that $x \preceq^{**} y$.

Let k and ℓ be the tails of x and y, respectively. Since $\phi(x) \preceq' \phi(y)$ we have a strictly increasing function $f : \{0, \ldots, k-1\} \to \{0, \ldots, \ell-1\}$ such that $x_i \preceq y_{f(i)}$ for all $i \in \{0, \ldots, k-1\}$.

Since $y_0 \succeq y_1 \succeq \cdots$ and $f(i) \geq i$ for all $i \in \{0, \ldots, k-1\}$, we have $x_i \preceq y_{f(i)} \preceq y_i$ for all $i \in \{0, \ldots, k-1\}$. Since $x_k \preceq y_\ell$ and $y_\ell \leq y_j$ for all $j \in \mathbb{N}$ (as ℓ is the tail of y), it follows that for all $i \geq k$ we have $x_i \leq x_k \leq y_\ell \leq y_i$. We conclude that $x_i \leq y_i$ for all $i \in \mathbb{N}$ and therefore that $x \preceq^{**} y$. □

We will now prove Theorem 1: the set of integer packing sets in $\mathbb{R}^n$ is a wqo under inclusion.

Proof of Theorem 1. The proof is by induction on n. The case $n = 1$ follows directly from the fact that $(\mathbb{N}, \leq)$ is a wqo. For the induction step, we associate to any integer packing set $S \subseteq \mathbb{R}^{n+1}$ a sequence $(S_0, S_1, \ldots)$ of 'slices' by setting

$$S_i = \{(x_1, \ldots, x_n) \in \mathbb{N}^n : (x_1, \ldots, x_n, i) \in S\}.$$

As S is an integer packing set, it follows that the S_i are integer packing sets in $\mathbb{R}^n$ and that $S_0 \supseteq S_1 \supseteq \cdots$. For two packing sets S, T in $\mathbb{R}^{n+1}$ we have $S \subseteq T$ if and only if for the corresponding slices we have $S_i \subseteq T_i$ for all $i \in \mathbb{N}$. Hence, the well-quasi-ordering of integer packing sets in $\mathbb{R}^{n+1}$ follows from that of integer packing sets in $\mathbb{R}^n$ by Lemma 5. □

As a consequence to Theorem 1 we obtain the following structural result about integer packing sets. An *n-dimensional block* is a set of the form $X_1 \times \cdots \times X_n$, where each X_i is equal to $\mathbb{N}$ or to $[m]$ for some $m \in \mathbb{N}$.

Corollary 1. *Let Q be an integer packing set in $\mathbb{R}^n$. Then Q is the union of finitely many n-dimensional blocks.*

Proof. The proof is by induction on n. If $n = 1$, then any integer packing set in $\mathbb{R}^n$ is an n-dimensional block. Now suppose that the statement holds for a given n and consider an integer packing set Q in $\mathbb{R}^{n+1}$. Define the n-dimensional slices $Q_i = \{(x_1, \ldots, x_n) : (x_1, \ldots, x_n, i) \in Q\}$ for all $i \in \mathbb{N}$. Then $Q_0, Q_1, \ldots$ is a decreasing sequence of integer packing sets in $\mathbb{R}^n$. Hence, by Theorem 1, there is a $k \in \mathbb{N}$ such that $Q_k = Q_\ell$ for all $\ell \geq k$. By assumption, each set Q_i is a union of finitely many n-dimensional blocks. Hence $Q_i \times \{0, 1, \ldots, i\}$ is a union of finitely many $n+1$-dimensional block for any $i = 0, 1, \ldots, k-1$, and also $Q_k \times \mathbb{N}$ is a union of finitely many $n+1$-dimensional blocks. Since

$$Q = (Q_k \times \mathbb{N}) \cup \bigcup_{i=0}^{k-1} Q_i \times \{0, 1, \ldots, i\},$$

the result follows. □

4 Polyhedrality of the k-aggregation closure

In this section we prove that the k-aggregation closure of a packing polyhedron is itself a packing polyhedron (Theorem 2). We will use some standard notation from polyhedral theory. In particular, given $A \subseteq \mathbb{R}^n$, we denote by $\text{conv}(A)$ the convex hull of A, and given a polyhedron $P \subseteq \mathbb{R}^n$, we denote by $P_I = \text{conv}(P \cap \mathbb{Z}^n)$ the integer hull of P. Given $a \in \mathbb{R}^n$, we define $a_+ \in \mathbb{R}^n$ by $(a_+)_i := \max\{0, a_i\}$ for all $i \in [n]$.

Lemma 6. *Let D be a downset of $\mathbb{R}^n_+$ and let $a^\mathsf{T} x \le \beta$ be a valid inequality for D. Then $a_+^\mathsf{T} x \le \beta$ is valid for D.*

Proof. Let $x \in D$ and let $x' \in \mathbb{R}^n$ be defined by $x'_i = x_i$ if $a_i \ge 0$ and $x'_i = 0$ if $a_i < 0$. Since D is a downset, we have $x' \in D$. Hence, $a_+^\mathsf{T} x = a^\mathsf{T} x' \le \beta$. □

Lemma 7. *A polyhedron is a downset of $\mathbb{R}^n_+$ if and only if it is a packing polyhedron.*

Proof. It is simple to see that every packing polyhedron is a downset of $\mathbb{R}^n_+$. For the converse implication, let P be a polyhedron that is a downset of $\mathbb{R}^n_+$. Then P can be written in the form

$$P = \{x \in \mathbb{R}^n \mid x \ge 0, \quad (a^i)^\mathsf{T} x \le b_i, \quad i \in [m]\}.$$

Consider any inequality $(a^i)^\mathsf{T} x \le b_i$. Since P is a downset, it follows from Lemma 6 that $(a^i_+)^\mathsf{T} x \le b_i$ is valid for P. Moreover, the inequality $(a^i)^\mathsf{T} x \le b_i$ is implied by the inequalities $(a^i_+)^\mathsf{T} x \le b_i$ and $x \ge 0$.

It follows that

$$P = \{x \in \mathbb{R}^n \mid x \ge 0, \quad (a^i_+)^\mathsf{T} x \le b_i, \quad i \in [m]\}.$$

Since $0 \in P$, it follows that $b_i \ge 0$ for every $i \in [m]$. We conclude that P is a packing polyhedron. □

Lemma 8. *Let P be a packing polyhedron. Then the integer hull P_I is also a packing polyhedron.*

Note that in Lemma 8 we do not require the polyhedron P to be rational.

Proof. We can write $P = \{x \in \mathbb{R}^n \mid x \ge 0, \ (a^i)^\mathsf{T} x \le b_i, \ i \in [m]\}$ where the a^i and b_i are nonnegative. Consider any of the inequalities $(a^i)^\mathsf{T} x \le b_i$. We claim that, for all $i \in [m]$, there exist a nonnegative rational vector $c^i \le a^i$ and a rational number $d_i \ge b_i$ such that

$$\{x \in \mathbb{N}^n \mid (a^i)^\mathsf{T} x \le b_i\} = \{x \in \mathbb{N}^n \mid (c^i)^\mathsf{T} x \le d_i\}. \tag{1}$$

To show the claim, we observe that since a^i is nonnegative, the set $\{(a^i)^\mathsf{T} x \mid x \in \mathbb{N}^n\} \cap (b_i, b_i + 1)$ is finite. Hence, there is a number $b' > b_i$ such that for all $x \in \mathbb{N}^n$ we have either $(a^i)^\mathsf{T} x \le b_i$ or $(a^i)^\mathsf{T} x \ge b'$.

Let $\epsilon \in (0, 1)$ be such that $(1 + \epsilon) b_i < b'(1 - \epsilon)$. We can choose c^i to be a rational vector with $(1 - \epsilon) a^i \le c^i \le a^i$, and d_i to be a rational number with $b_i \le d_i \le (1 + \epsilon) b_i$. Now, the inclusion $\subseteq$ in equality (1) is clear. The converse inclusion follows since for any $x \in \mathbb{N}^n$ we have

$$(c^i)^\mathsf{T} x \le d_i \implies (1 - \epsilon)(a^i)^\mathsf{T} x \le (1 + \epsilon) b_i \implies (a^i)^\mathsf{T} x < b' \implies (a^i)^\mathsf{T} x \le b_i.$$

This concludes the proof of the claim.

Let $P' = \{x \in \mathbb{R}^n \mid x \geq 0,\ (c^i)^\mathsf{T} x \leq d_i,\ i \in [m]\}$. Then $P \cap \mathbb{N}^n = P' \cap \mathbb{N}^n$ and hence $P_I = P'_I$. By Meyer's theorem [12], the integer hull of a rational polyhedron is itself a polyhedron. Hence, $P_I = P'_I$ is a polyhedron.

It is clear that the polyhedron P_I is a downset of $\mathbb{R}^n_+$. Hence, by Lemma 7, P_I is a packing polyhedron. □

Now we are ready to present the proof of Theorem 2.

Proof of Theorem 2. Let P be a packing polyhedron defined by $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geq 0\}$, and let k be a positive integer. Denote by $\mathcal{P}$ the collection of polyhedra of the form

$$\{x \in \mathbb{R}^n \mid x \geq 0, \quad (\lambda^j)^\mathsf{T} Ax \leq (\lambda^j)^\mathsf{T} b, \quad \forall j \in [k]\},$$

for all possible $\lambda^1, \ldots, \lambda^k \in \mathbb{R}^m_+$.

Since A is nonnegative, for every $Q \in \mathcal{P}$ the set $Q \cap \mathbb{N}^n$ is an integer packing set. By Theorem 1, the set of integer packing sets in $\mathbb{R}^n$ is a wqo under inclusion. Hence, it follows from the finite basis property that there is a finite subset $\mathcal{P}' \subseteq \mathcal{P}$ such that for any $Q \in \mathcal{P}$ there is a $Q' \in \mathcal{P}'$ such that $Q' \cap \mathbb{N}^n \subseteq Q \cap \mathbb{N}^n$, and hence also that $Q'_I \subseteq Q_I$. We conclude that

$$\mathcal{A}_k(P) = \bigcap \big\{Q_I : Q \in \mathcal{P}\big\} = \bigcap \big\{Q_I : Q \in \mathcal{P}'\big\}.$$

Since by Lemma 8 the integer hull Q_I is a packing polyhedron for every $Q \in \mathcal{P}'$, and the intersection of finitely many packing polyhedra is again a packing polyhedron, it follows that $\mathcal{A}_k(P)$ is a packing polyhedron. □

5 Generalization to non-polyhedral sets

In this section, we provide the proofs of our generalizations of Theorem 2 to non-polyhedral sets. In particular, we give the proofs of Theorem 3 and of Theorem 4.

Proof of Theorem 3. Define $\Lambda^+(D) := \Lambda(D) \cap \mathbb{R}^n_+$. We first show that in the definition of $\tilde{\mathcal{A}}_k(D)$ we can replace $\Lambda(D)$ with $\Lambda^+(D)$, i.e.,

$$\tilde{\mathcal{A}}_k(D) = \bigcap_{f^1,\ldots,f^k \in \Lambda^+(D)} \operatorname{conv}(\{x \in \mathbb{N}^n \mid (f^j)^\mathsf{T} x \leq \sup\{(f^j)^\mathsf{T} d \mid d \in D\},\ \forall j \in [k]\}).$$

The containment $\subseteq$ is trivial, thus we only need to show the containment $\supseteq$. Let $f \in \Lambda(D)$, and consider the associated valid inequality for D given by $f^\mathsf{T} x \leq \sup\{f^\mathsf{T} d \mid d \in D\}$. Since D is a downset of $\mathbb{R}^n_+$, we know from Lemma 6 that $(f_+)^\mathsf{T} x \leq \sup\{f^\mathsf{T} d \mid d \in D\}$ is also valid for D, and dominates the original inequality in $\mathbb{R}^n_+$. In particular, this implies that $\sup\{(f_+)^\mathsf{T} d \mid d \in D\} \leq \sup\{f^\mathsf{T} d \mid d \in D\}$, hence $f_+ \in \Lambda^+(D)$ since the latter supremum is finite by assumption. Hence, we have shown that $(f_+)^\mathsf{T} x \leq \sup\{(f_+)^\mathsf{T} d \mid d \in D\}$ dominates the original inequality $f^\mathsf{T} x \leq \sup\{f^\mathsf{T} d \mid d \in D\}$ in $\mathbb{R}^n_+$. We have therefore proven the containment $\supseteq$.

Lastly, we follow almost the exact same argument as in the proof of Theorem 2, except now we consider the collection $\mathcal{P}$ of polyhedra of the form

$$\{x \in \mathbb{R}^n \mid x \geq 0,\ (f^j)^\mathsf{T} x \leq \sup\{(f^j)^\mathsf{T} d \mid d \in D\},\ \forall j \in [k]\},$$

for all possible $f^1, \ldots, f^k \in \Lambda^+(D)$. □

We now turn our attention to Theorem 4.

Proof of Theorem 4. For any $f \in \Lambda(D)$ we denote $\beta_f := \max\{f^\mathsf{T} d \mid d \in D\}$ and define $\Lambda^+(D) := \Lambda(D) \cap \mathbb{R}^n_+$ as in the previous proof. We obtain

$$D = \{x \in \mathbb{R}^n_+ \mid f^\mathsf{T} x \leq \beta_f,\ \forall f \in \Lambda^+(D)\},$$
$$\operatorname{conv}(D \cap \mathbb{Z}^n) = \operatorname{conv}(\{x \in \mathbb{N}^n \mid f^\mathsf{T} x \leq \beta_f,\ \forall f \in \Lambda^+(D)\}).$$

For any $f \in \Lambda^+(D)$ let $S_f := \{x \in \mathbb{N}^n \mid f^\mathsf{T} x \leq \beta_f\}$. Then S_f is an integer packing set in $\mathbb{R}^n$. By Theorem 1, the set of integer packing sets in $\mathbb{R}^n$ is a wqo under inclusion. Hence, it follows from the finite basis property that there is a finite subset $B \subseteq \Lambda^+(D)$ such that for every $f \in \Lambda^+(D)$ there is a $f' \in B$ for which $S_{f'} \subseteq S_f$. It follows that

$$\operatorname{conv}(D \cap \mathbb{Z}^n) = \operatorname{conv}(\{x \in \mathbb{N}^n \mid f^\mathsf{T} x \leq \beta_f,\ \forall f \in B\}).$$

By Lemma 8, it follows that $\operatorname{conv}(D \cap \mathbb{Z}^n)$ is a packing polyhedron. □

Theorem 4 can be used to prove the following result about the natural extension of $\tilde{\mathcal{A}}_k$ to $k = \infty$ defined by

$$\tilde{\mathcal{A}}_\infty(D) := \operatorname{conv}(\{x \in \mathbb{N}^n \mid f^\mathsf{T} x \leq \sup\{f^\mathsf{T} d \mid d \in D\},\ \forall f \in \Lambda(D)\}).$$

Corollary 2. *For any downset D of $\mathbb{R}^n_+$, the set $\tilde{\mathcal{A}}_\infty(D)$ is a packing polyhedron.*

Proof. From Lemma 6, $\overline{\operatorname{conv}}(D)$ is a closed convex downset, and we can write

$$\overline{\operatorname{conv}}(D) = \{x \in \mathbb{R}^n_+ \mid f^\mathsf{T} x \leq \sup\{f^\mathsf{T} d \mid d \in D\},\ \forall f \in \Lambda(D)\}.$$

Hence $\tilde{\mathcal{A}}_\infty(D) = \operatorname{conv}(\overline{\operatorname{conv}}(D) \cap \mathbb{N}^n)$. The corollary then follows from Theorem 4. □

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References

[1] K. Andersen, Q. Louveaux, and R. Weismantel. An analysis of mixed integer linear sets based on lattice point free convex sets. *Mathematics of Operations Research*, 35(1):233–256, 2010.

[2] G. Averkov. On finitely generated closures in the theory of cutting planes. *Discrete Optimization*, 9(4):209–215, 2012.

[3] M. Bodur, A. Del Pia, S. Dey, M. Molinaro, and S. Pokutta. Aggregation-based cutting-planes for packing and covering integer programs. *Mathematical Programming*, pages 1–29, 2017.

[4] D. Dadush, S. Dey, and J. Vielma. On the Chvátal-Gomory closure of a compact convex set. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 130–142. Springer, 2011.

[5] S. Dash, O. Günlük, and D. Lee. Generalized Chvátal-Gomory closures for integer programs with bounds on variables. *Mathematical Programming*, pages 1–33, 2020.

[6] S. Dash, O. Günlük, and D. Morán. On the polyhedrality of cross and quadrilateral closures. *Mathematical Programming*, 160(1-2):245–270, 2016.

[7] S. Dash, O. Günlük, and D. Morán. On the polyhedrality of closures of multibranch split sets and other polyhedra with bounded max-facet-width. *SIAM Journal on Optimization*, 27(3):1340–1361, 2017.

[8] L. Dickson. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *American Journal of Mathematics*, 35(4):413–422, 1913.

[9] J. Dunkel and A. Schulz. The Gomory-Chvátal closure of a nonrational polytope is a rational polytope. *Mathematics of Operations Research*, 38(1):63–91, 2013.

[10] G. Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, 3(1):326–336, 1952.

[11] J. Kruskal. The theory of well-quasi-ordering: A frequently discovered concept. *Journal of Combinatorial Theory, Series A*, 13(3):297–305, 1972.

[12] Robert R Meyer. On the existence of optimal solutions to integer and mixed-integer programming problems. *Mathematical Programming*, 7(1):223–235, 1974.

[13] K. Pashkovich, L. Poirrier, and H. Pulyassary. The aggregation closure is polyhedral for packing and covering integer programs. *arXiv preprint arXiv:1910.03404v1*, 2019.

[14] N. Robertson and P. Seymour. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357, 2004.