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OBSERVING BANDLIMITED GRAPH PROCESSES FROM SUBSAMPLED MEASUREMENTS

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ABSTRACT

This work merges tools from graph signal processing and linear systems theory to propose sampling strategies for observing the initial state of a process evolving over a graph. The proposed method is ratified by a mathematical analysis that provides insights on the role played by the different actors, such as the graph topology, the process bandwidth, and the sampling strategy. Moreover, conditions when the graph process is observable from a few samples and (sub)optimal sampling strategies that jointly exploit the nature of the graph structure and graph process are proposed. Finally, numerical tests are conducted to illustrate the benefits of the proposed approach.

1. INTRODUCTION

A key aspect in graph signal processing (GSP) [1, 2] is the sampling of bandlimited graph signals, i.e., signals that are sparse in a well-defined graph Fourier domain [3–5]. Examples of the latter include temperature measurements, where adjacent sensors measure similar values, and data from networks that exhibit a clustering behavior such as opinion networks. Such a graph prior has in fact been exploited by a number of recent works [3–7] to propose graph signal reconstruction strategies from a few measurements.

Differently from the above works, where the signal is considered time invariant, we here generalize the sampling of graph signals to the observation of the initial state of a time-varying graph signal named *a graph process*. These processes are often encountered in consecutive sensor measurements, biological signal evolution prone to stimuli, and information diffusion over networks. The importance of temporal GSP has been recently recognized by a number of works. The authors in [8, 9] focus on harmonic analysis of time-varying graph signals, while [10–12] on graph-time filters. In [13, 14], on the other hand, the graph-bandlimited prior is exploited for prediction and tracking of graph processes.

From a different perspective, yet related to this paper, are the works in [15–17]. The authors in [15, 16] studied the observability of network processes for sensor placement, while [16] focused on designing observable network topologies. While these findings are of particular importance, these works do not consider sampling strategies for observing the network process. Differently, we exploit GSP tools and in particular the bandlimited prior to bring the graph sampling theory into the temporal dimension.

More specifically, this paper aims at answering the following research questions: $(Q1)$ *Under which conditions is a bandlimited graph process observable from a few measurements?* $(Q2)$ *When and where should we collect measurements to estimate the network state up to a desired accuracy?*

Next, we present the background material while the methods employed to answer the above questions are provided in Section 3. Section 4 instead presents the numerical results and Section 5 the paper conclusions.

2. BACKGROUND

GSP. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{S})$ be an undirected graph with $\mathcal{V}$ and $\mathcal{E}$ the node and edge set, respectively, and with graph shift operator $\mathbf{S}$ (e.g., adjacency matrix, or graph Laplacian). We denote with $\mathbf{x}$ a signal residing on the vertices of $\mathcal{G}$. Following [1], the graph Fourier transform (GFT) of $\mathbf{x}$ is

$$\hat{\mathbf{x}} = \mathbf{U}^{\mathsf{H}}\mathbf{x}, \tag{1}$$

where $\mathbf{U}$ is a unitary matrix obtained from the eigendecomposition of $\mathbf{S}$, i.e., $\mathbf{S} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathsf{H}}$. The diagonal matrix $\boldsymbol{\Lambda}$ contains the eigenvalues of $\mathbf{S}$ and in analogy with the classical harmonic analysis they are referred to as the graph frequencies [2]. Given a subset of graph frequency indices $\mathcal{F} \subseteq \{1, \ldots, N\}$, $\mathbf{x}$ is said to be perfectly localized over $\mathcal{F}$ (or $\mathcal{F}-$bandlimited) if

$$\mathbf{U}_{\mathcal{F}}\mathbf{U}_{\mathcal{F}}^{\mathsf{H}}\mathbf{x} = \mathbf{x}, \tag{2}$$

where $\mathbf{U}_{\mathcal{F}} \in \mathbb{C}^{N\times|\mathcal{F}|}$ contains the columns of $\mathbf{U}$ relative to the set $\mathcal{F}$. Similarly, if $\mathcal{S} \subseteq \mathcal{V}$ is a subset of vertices and $\mathbf{C}_{\mathcal{S}} = \mathrm{diag}(\mathbf{1}_{\mathcal{S}})$ the respective set projection matrix, $\mathbf{x}$ is said to be perfectly localized on $\mathcal{S}$ if $\mathbf{C}_{\mathcal{S}}\mathbf{x} = \mathbf{x}$.

Then, we recall the following result from [5]. An $\mathcal{F}-$bandlimited graph signal $\mathbf{x}$ can be perfectly recovered from samples collected over the set $\mathcal{S}$ if and only if

$$\|\mathbf{C}_{\mathcal{S}^c}\mathbf{U}_{\mathcal{F}}\| < 1, \tag{3}$$

i.e., there are no $\mathcal{F}-$bandlimited graph signals perfectly localized on the complementary vertex set $\mathcal{S}^c = \mathcal{V}\backslash\mathcal{S}$. Here, $\mathbf{C}_{\mathcal{S}^c} = \mathbf{I}_N - \mathbf{C}_{\mathcal{S}}$ is the projection matrix onto $\mathcal{S}^c$.

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In the sequel, we will extend condition (3) to the observability of graph processes, which also corresponds to the answer of $(Q1)$.

Systems on graphs. Consider the $N-$state discrete linear time-varying system

$$\mathbf{x}_t = \mathbf{A}_{t-1}\mathbf{x}_{t-1} + \mathbf{B}_{t-1}\mathbf{u}_{t-1} \quad (4a)$$
$$\mathbf{y}_t = \mathbf{C}_{\mathcal{S}_t}(\mathbf{x}_t + \mathbf{v}_t), \quad (4b)$$

where $\mathbf{x}_t$ is the state vector containing the graph signal at time t and $\mathbf{u}_t$ is the input signal. $\mathbf{A}_t$ and $\mathbf{B}_t$ are the time-varying state-transition and input matrices, respectively, that are related with the graph structure (see Assumption 1 in the sequel). $\mathbf{y}_t \in \mathbb{C}^N$ is the measurement vector and $\mathbf{C}_{\mathcal{S}_t} = \text{diag}(c_{t,1}, \ldots, c_{t,N})$ is the sampling matrix related to the instantaneous sampling set $\mathcal{S}_t$, i.e., $c_{t,n} = 1$ if $n \in \mathcal{S}_t$ and $c_{t,n} = 0$ otherwise. $\mathbf{v}_t$ is white zero-mean noise with covariance matrix $\boldsymbol{\Sigma}_v = \sigma_v^2\mathbf{I}_N$. Such a model captures the evolution of different network processes including the heat diffusion [18], the discretized wave equation on graphs [19], and ARMA graph processes [20]. In this work, we consider the graph process to be $\mathcal{F}-$bandlimited as defined next.

Definition 1. *A graph process $\mathbf{x}_t$ with instantaneous GFT $\hat{\mathbf{x}}_t = \mathbf{U}^{\mathsf{H}}\mathbf{x}_t$ is $\mathcal{F}-$bandlimited if $\hat{\mathbf{x}}_t$ has non-zero frequency content only on a subset of graph frequency indices $\mathcal{F}$.*

The set $\mathcal{F} = \{n \in \{1, \ldots, N\} | \hat{x}_{t,n} \neq 0, t \geq 0\}$ consists of the union of all instantaneous sets $\mathcal{F}_t = \{n \in \{1, \ldots, N\} | \hat{x}_{t,n} \neq 0\}$. We can therefore write

$$\mathbf{x}_t = \mathbf{U}_{\mathcal{F}}\tilde{\mathbf{x}}_t, \quad (5)$$

where $\tilde{\mathbf{x}}_t \in \mathbb{C}^{|\mathcal{F}|}$ is the vector containing the entries of $\hat{\mathbf{x}}_t$ indicated by $\mathcal{F}$. We will further assume the following.

Assumption 1. *The system evolution matrices $\mathbf{A}_t$ and $\mathbf{B}_t$ share the eigenvectors with the graph shift operator $\mathbf{S}$.*

This assumption focuses our attention to linear time-varying systems on graphs that are a function of the graph shift operator. This is the case for all network processes listed earlier. With this in place, (4) can be written as

$$\tilde{\mathbf{x}}_t = \tilde{\mathbf{A}}_{t-1}\tilde{\mathbf{x}}_{t-1} + \tilde{\mathbf{B}}_{t-1}\tilde{\mathbf{u}}_{t-1} \quad (6a)$$
$$\mathbf{y}_t = \mathbf{C}_{\mathcal{S}_t}(\mathbf{U}_{\mathcal{F}}\tilde{\mathbf{x}}_t + \mathbf{v}_t), \quad (6b)$$

where $\tilde{\mathbf{A}}_t = \mathbf{U}_{\mathcal{F}}^{\mathsf{H}}\mathbf{A}_t\mathbf{U}_{\mathcal{F}}$ and $\tilde{\mathbf{B}}_t = \mathbf{U}_{\mathcal{F}}^{\mathsf{H}}\mathbf{B}_t\mathbf{U}_{\mathcal{F}}$ are diagonal matrices containing the in-band spectrum of $\mathbf{A}_t$ and $\mathbf{B}_t$, respectively and $\tilde{\mathbf{u}}_t = \mathbf{U}_{\mathcal{F}}^{\mathsf{H}}\mathbf{u}_t$. For short, we will refer to systems that can be written in the form (6) as $\mathcal{F}-$bandlimited systems on graphs.

3. METHODS

We here presents the theoretical results of the paper and generalize condition (3) from the reconstruction of an $\mathcal{F}-$ bandlimited graph signal to the observation of an $\mathcal{F}-$bandlimited graph process. To this aim, we adapt the definition of observability [21] to our context.

Definition 2. *An $\mathcal{F}-$bandlimited system on a graph is observable over the set $\mathcal{S}_{0:T} = \bigcup_{t=0}^{T} \mathcal{S}_t = \{n \in \{1, \ldots, N\}; t \in \{0, \ldots, T\} | \ c_{t,n} = 1\}$ if for any $\mathcal{F}-$bandlimited initial state $\mathbf{x}_0$ and some final time T, the initial state $\mathbf{x}_0$ can be uniquely determined in the absence of noise by the knowledge of the input $\mathbf{u}_t$ and measurement $\mathbf{y}_t$ for all $t \in \{0, \ldots, T\}$.*

The set $\mathcal{S}_{0:T}$ contains all graph-time locations where and when the nodes are sampled in the interval $\{0, \ldots, T\}$. Using recursion (6), we can now express $\mathbf{y}_t$ as

$$\mathbf{y}_t = \mathbf{C}_{\mathcal{S}_t}\mathbf{U}_{\mathcal{F}}\tilde{\mathbf{A}}_{t,0}\tilde{\mathbf{x}}_0 + \mathbf{C}_{\mathcal{S}_t}\mathbf{U}_{\mathcal{F}}\sum_{\tau=0}^{t-1}\tilde{\mathbf{A}}_{t,\tau+1}\tilde{\mathbf{B}}_\tau\tilde{\mathbf{u}}_\tau + \mathbf{C}_{\mathcal{S}_t}\mathbf{v}_t, \quad (7)$$

with

$$\tilde{\mathbf{A}}_{t,\tau} = \begin{cases} \tilde{\mathbf{A}}_{t-1}\tilde{\mathbf{A}}_{t-2}\ldots\tilde{\mathbf{A}}_\tau, & t > \tau \\ \mathbf{I}_{|\mathcal{F}|}, & t = \tau \\ \mathbf{0}_{|\mathcal{F}|}\mathbf{0}_{|\mathcal{F}|}^{\mathsf{T}}, & t < \tau. \end{cases} \quad (8)$$

Then, by setting $\mathbf{y}_{0:T} = [\mathbf{y}_0^{\mathsf{T}}, \mathbf{y}_1^{\mathsf{T}}, \ldots, \mathbf{y}_T^{\mathsf{T}}]^{\mathsf{T}}$, we can write

$$\mathbf{y}_{0:T} = \mathbf{O}_{0:T}\tilde{\mathbf{x}}_0 + \mathbf{J}_{0:T}\mathbf{u}_{0:T-1} + \mathbf{C}_{\mathcal{S}_{0:T}}\mathbf{v}_{0:T}, \quad (9)$$

where $\mathbf{O}_{0:T} = \mathbf{C}_{\mathcal{S}_{0:T}}(\mathbf{I}_{T+1} \otimes \mathbf{U}_{\mathcal{F}})\tilde{\mathbf{A}}_{0:T}$, with

$$\mathbf{C}_{\mathcal{S}_{0:T}} = \text{blkdiag}(\mathbf{C}_{\mathcal{S}_0}, \ldots, \mathbf{C}_{\mathcal{S}_T}),$$

$$\tilde{\mathbf{A}}_{0:T} = \left[\mathbf{I}_{|\mathcal{F}|}, \tilde{\mathbf{A}}_{1,0}^{\mathsf{T}}, \ldots, \tilde{\mathbf{A}}_{T,0}^{\mathsf{T}}\right]^{\mathsf{T}},$$

and where $\mathbf{u}_{0:T-1} = [\mathbf{u}_0^{\mathsf{T}}, \mathbf{u}_1^{\mathsf{T}}, \ldots, \mathbf{u}_{T-1}^{\mathsf{T}}]^{\mathsf{T}}$, and $\mathbf{v}_{0:T} = [\mathbf{v}_0^{\mathsf{T}}, \mathbf{v}_1^{\mathsf{T}}, \ldots, \mathbf{v}_T^{\mathsf{T}}]^{\mathsf{T}}$. $\mathbf{J}_{0:T}$ is the input evolution matrix in the interval $\{0, \ldots, T\}$ whose expression is not required for our derivations, but can be obtained from (7).

Given then $\mathbf{C}_{\mathcal{S}_{0:T}}$, the set projection matrix over the set $\mathcal{S}_{0:T}$, system (6) is observable over $\mathcal{S}_{0:T}$ iff the observability matrix $\mathbf{O}_{0:T}$ is full column-rank [21], i.e., $\text{rank}(\mathbf{O}_{0:T}) = |\mathcal{F}|$. Then, we have

$$\tilde{\mathbf{x}}_0^o = \mathbf{O}_{0:T}^{\dagger}\left(\mathbf{y}_{0:T} - \mathbf{J}_{0:T}\mathbf{u}_{0:T-1}\right), \quad (10)$$

which is also the least squares (LS) estimate of $\tilde{\mathbf{x}}_0$ in the presence of noise. Given the above expression, we can present our first result.

Proposition 1. *An $\mathcal{F}-$bandlimited system on a graph is observable over the set $\mathcal{S}_{0:T}$ only if at least $|\mathcal{F}|$ graph-time samples are taken in the time interval $\{0, \ldots, T\}$. These samples can be taken by $|\mathcal{F}|$ nodes at a fixed time instant, by one node in $|\mathcal{F}|$ time instants, or by a combination of the two.*

Proof. By applying the rank inequality

$$\text{rank}(\mathbf{AB}) \leq \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\} \quad (11)$$

to $\mathbf{O}_{0:T} = \mathbf{C}_{\mathcal{S}_{0:T}}(\mathbf{I}_{T+1} \otimes \mathbf{U}_{\mathcal{F}})\tilde{\mathbf{A}}_{0:T}$, we have that $\mathbf{O}_{0:T}$ can be full column-rank $|\mathcal{F}|$ only if $\text{rank}\left(\mathbf{C}_{\mathcal{S}_{0:T}}\right) \geq |\mathcal{F}|$, which from the structure of $\mathbf{C}_{\mathcal{S}_{0:T}}$ is always true when the claimed conditions are satisfied. □

In other words, the result of Proposition 1 is equivalent to

$$|\mathcal{S}_{0:T}| \geq |\mathcal{F}|, \tag{12}$$

i.e., the cardinality of the sampling set must be greater than or equal to the process bandwidth. However, (12) is only a necessary condition for observability. In fact, $\mathbf{O}_{0:T}$ may be easily ill-conditioned depending on the particular location of these samples and the spectral support of $\tilde{\mathbf{x}}_0$. It is therefore paramount to carefully pick the samples in a graph-time fashion such that $\mathbf{O}_{0:T}$ is of full column-rank $|\mathcal{F}|$, and in the presence of noise $\mathbf{v}_t \neq 0$, possibly also well-conditioned. That is, from $\text{rank}(\mathbf{O}^{\dagger}_{0:T}) = |\mathcal{F}|$ [cf. (10)], $\mathcal{S}_{0:T}$ should satisfy

$$\text{rank}\Big(\sum_{t=0}^{T} \tilde{\mathbf{A}}^{\mathsf{H}}_{t,0}\mathbf{U}^{\mathsf{H}}_{\mathcal{F}}\mathbf{C}_{\mathcal{S}_t}\mathbf{U}_{\mathcal{F}}\tilde{\mathbf{A}}_{t,0}\Big) = |\mathcal{F}|, \tag{13}$$

where the single shot graph signal reconstruction [5] is the special case $T = 0$. The following theorem generalizes (3) to a necessary and sufficient condition for the observability of an $\mathcal{F}$-bandlimited graph process over a sampling set.

Theorem 1. *An $\mathcal{F}-$bandlimited system on a graph is observable over the set $\mathcal{S}_{0:T}$ if and only if*

$$\|\mathbf{C}_{\mathcal{S}^c_{0:T}}(\mathbf{I}_{T+1}\otimes\mathbf{U}_{\mathcal{F}})\| < \frac{s^2_{min}(\tilde{\mathbf{A}}_{0:T})}{s^2_{max}(\tilde{\mathbf{A}}_{0:T})}, \tag{14}$$

where $\mathbf{C}_{\mathcal{S}^c_{0:T}} = \mathbf{I}_{N(T+1)} - \mathbf{C}_{\mathcal{S}_{0:T}}$ is the operator that projects onto the complementary set $\mathcal{S}^c_{0:T} = \{n \in \{1,\ldots,N\}; t \in \{0,\ldots,T\} \mid c_{t,n} = 0\}$ and $s_{min}(\tilde{\mathbf{A}}_{0:T})$, $s_{max}(\tilde{\mathbf{A}}_{0:T})$ indicate the minimum and maximum singular values of $\tilde{\mathbf{A}}_{0:T}$, respectively.

Proof. The proof is provided in [22]. □

Condition (14) (analogous to (3)) is again related to the localization properties of graph signals involving also the evolution model of the latter. It implies that in their *evolution* there are no $\mathcal{F}$-bandlimited graph processes perfectly localized on the complementary set $\mathcal{S}^c_{0:T}$. The single shot condition (3) is obtained for $T = 0$.

Mean square error. We here quantify how the sampling set $\mathcal{S}_{0:T}$ affects the MSE of the LS estimate (10). The latter will be then used as a criterion to select good graph-time samples. The MSE expression is given by the following proposition.

Proposition 2. *Given an $\mathcal{F}-$bandlimited graph process following the model* (6) *and assuming the result of Theorem 1 holds. Then, the MSE of the LS observed signal $\tilde{\mathbf{x}}^o_0$ is*

$$\begin{aligned} MSE &= \mathbb{E}\left\{\|\tilde{\mathbf{x}}^o_0 - \tilde{\mathbf{x}}_0\|^2\right\} = \mathbb{E}\left\{\text{tr}\left[(\tilde{\mathbf{x}}^o_0 - \tilde{\mathbf{x}}_0)(\tilde{\mathbf{x}}^o_0 - \tilde{\mathbf{x}}_0)^{\mathsf{H}}\right]\right\} \\ &= \sigma^2_v \text{tr}\left\{\left[\tilde{\mathbf{A}}^{\mathsf{H}}_{0:T}(\mathbf{I}_{T+1}\otimes\mathbf{U}_{\mathcal{F}})^{\mathsf{H}}\mathbf{C}_{\mathcal{S}_{0:T}}(\mathbf{I}_{T+1}\otimes\mathbf{U}_{\mathcal{F}})\tilde{\mathbf{A}}_{0:T}\right]^{-1}\right\}. \end{aligned} \tag{15}$$

Proof. The claim follows from the covariance matrix of the LS estimator [23]. □

Besides characterizing the impact of the graph-time samples on the MSE[1], expression (15) shows that not only the number of selected samples plays a role, but also their location in graph and time. In the sequel, we show how to select these samples such that a target MSE is guaranteed, which also answers question $(Q2)$.

Sampling strategy. Given (15), we follow the sparse sensing approach in [24] to design the sampling set $\mathcal{S}_{0:T}$ such that a target MSE estimation performance is guaranteed. The latter is achieved as the solution of the convex problem

$$\begin{aligned} \underset{\mathbf{c}_{0:T}}{\text{minimize}} \quad & \mathbf{1}^{\mathsf{T}}_{N\times(T+1)}\mathbf{c}_{0:T} \\ \text{subject to} \quad & \text{tr}\left[\left(\boldsymbol{\Psi}^{\mathsf{H}}_{0:T}\mathbf{C}_{\mathcal{S}_{0:T}}\boldsymbol{\Psi}_{0:T}\right)^{-1}\right] \leq \frac{\gamma}{\sigma^2_v}, \\ & \mathbf{C}_{\mathcal{S}_{0:T}} = \text{diag}(\mathbf{c}_{0:T}), \\ & \boldsymbol{\Psi}_{0:T} = (\mathbf{I}_{T+1}\otimes\mathbf{U}_{\mathcal{F}})\tilde{\mathbf{A}}_{0:T}, \\ & 0 \leq c_{0:T,i} \leq 1, \quad i = 1,\ldots,N(T+1), \end{aligned} \tag{16}$$

where the objective function is the l_1-surrogate of the l_0-norm and imposes sparsity in $\mathcal{S}_{0:T}$; the constant $\gamma > 0$ imposes a target MSE performance; and the last constraint is the relaxation of the Boolean constraint $c_{0:T,i} \in \{0,1\}$ to the box one. Alternatively, one can adopt a greedy approach similar to [25] for building $\mathcal{S}_{0:T}$. Obviously, we can also consider the opposite problem where the aim is to minimize the MSE, while imposing a fixed budget on the selected number of samples. The latter translates as well into a convex problem.

4. NUMERICAL RESULTS

We evaluate the proposed findings using the Molene weather data set[2]. In these simulations, we made use of the GSP box [26] and CVX [27].

The data set consists of $R = 744$ hourly temperature recordings collected in January 2014 over 32 cities in the region of Brest, France. The graph is a k-nearest neighbour (kNN) [26] graph with $k = 3$. For every recording[3] $\mathbf{r}_\tau$, we diffuse it following the heat diffusion model $\mathbf{x}_t = e^{-w\mathbf{L}_d}\mathbf{r}_\tau$, with $w = 1.5$ and $T = 10$.

First, we analyze the effect of the sampling set $\mathcal{S}_{0:T}$ when the graph process is perfectly $\mathcal{F}-$bandlimited. In this regard, we consider $|\mathcal{F}| = N = 32$ (i.e., the entire bandwidth) and corrupt the measurements with a zero-mean Gaussian noise with $\sigma^2_v = 10^{-1}$, which corresponds to an average signal-to-noise ratio (SNR) of 19.3dB computed as

$$\overline{\text{SNR}} = 10\log_{10}\left[\frac{\sum_{\tau=1}^{R}\|\mathbf{r}_\tau\|^2_2}{NR\sigma^2_v}\right]. \tag{17}$$

[1]The absence of model noise in (4a) allows us to find a closed-form expression for the MSE, rather than an upper bound. It also matches perfectly the considered models.

[2]Data publicly available at https://donneespubliques.meteofrance.fr/donnees_libres/Hackathon/RADOMEH.tar.gz.

[3]The graph signal consists of the measured temperature after subtracting their average value.

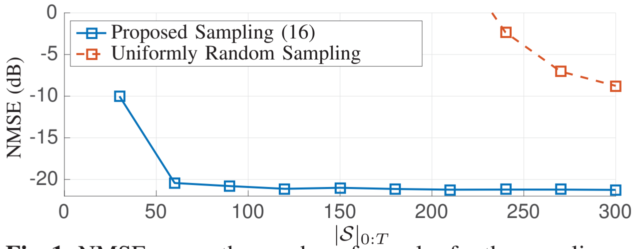

Fig. 1: NMSE versus the number of samples for the sampling algorithm (16) and uniformly random sampling. The process bandwidth is $|\mathcal{F}| = N = 32$.

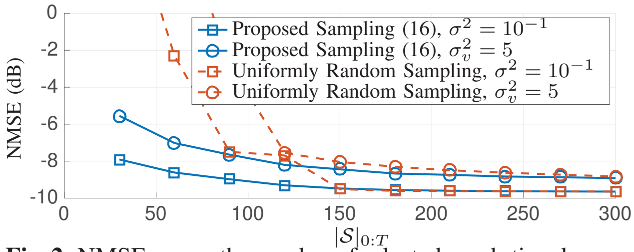

Fig. 2: NMSE versus the number of selected graph-time locations for the sampling algorithm (16) and a uniformly random sampling. The process bandwidth is $|\mathcal{F}| = 8$.

The $|\mathcal{S}_{0:T}|$ samples are chosen as the ones that minimize the MSE (15) in a sparse sense fashion. To evaluate the performance, we use the normalized MSE (NMSE) between the estimated (observed) τth recording $\mathbf{r}_\tau^o$ and the true one $\mathbf{r}_\tau$, defined as NMSE $= \sum_{\tau=1}^{R} \|\mathbf{r}_\tau^o - \mathbf{r}_\tau\|^2 / \sum_{\tau=1}^{R} \|\mathbf{r}_\tau\|^2$.

Fig. 1 shows the obtained NMSE as a function of $|\mathcal{S}_{0:T}|$. It can be seen that, in contrast to uniformly random sampling, even with 60 samples (out of 320) an NMSE of -20dB is achieved. *This finding suggests that the sparse observability approach can also be implemented for graph processes that have a contribution on the entire bandwidth.*

In the second scenario, we restrict the process bandwidth to the first $|\mathcal{F}| = 8$ graph frequencies and analyze two different noise variances $\sigma_v^2 = \{10^{-1}, 5\}$ ($\overline{\text{SNR}} = \{19.3\text{dB}, 2.3\text{dB}\}$). The sampling set is built similarly as in the previous scenario.

Fig. 2 depicts the average NMSE as a function of $|\mathcal{S}_{0:T}|$, where the proposed selection strategy outperforms again the uniformly random sampling. We further observe that the NMSE has a lower floor much higher than for the full bandwidth case and its value does not reduce even by increasing $|\mathcal{S}_{0:T}|$. We attribute this limitation to the restricted bandwidth, since the out-of-band signal contribution plays a role in further improving the performance.

In the third scenario, we analyze the effect of the signal bandwidth on the observability performance. We fix $|\mathcal{S}_{0:T}| = 60$ samples (i.e., almost twice the full bandwidth) and compute the NMSE for different values of $|\mathcal{F}|$ and σ_v^2. These results are shown in Fig. 3.

We observe an increasing trend of the NMSE in high noise regimes (i.e., $\sigma_v^2 = 5$). This suggests that the meaningful information is concentrated in the first few frequencies and, therefore, the graph process is bandlimited. *This result suggests that in the presence of noise the process bandwidth should not be determined solely by the signal energy, but also by the signal-to-noise ratio (SNR).* Indeed, a larger bandwidth (although the signal has energy content) degrades the overall $\overline{\text{SNR}}$. This finding is further reinforced in the low noise regime, where a larger bandwidth is preferred to exploit the SNR on the high frequencies for better observing the process.

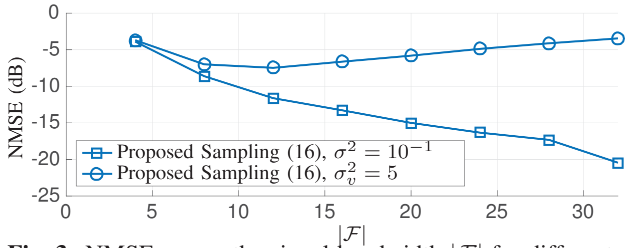

Fig. 3: NMSE versus the signal bandwidth $|\mathcal{F}|$ for different noise powers. The sampling set has cardinality $|\mathcal{S}_{0:T}| = 100$ chosen by minimizing the MSE (15).

5. CONCLUSIONS

We have generalized the sampling of bandlimited graph signals to time-varying graph processes that follow a predefined evolution model. We provided necessary and sufficient conditions to observe the initial state of the graph process and performed a detailed MSE analysis to highlight the role played by the underlying topology, the process dynamics, and the sampled nodes. Finally, we proposed a sparse sensing sampling strategy to select graph-time samples such that a target MSE performance is guaranteed.

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